MODULI STACKS OF VECTOR BUNDLES AND FROBENIUS MORPHISMS

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ABSTRACT. We describe the action of the different Frobenius morphisms on the étale cohomology ring of the moduli stack of algebraic vector bundles of fixed rank and determinant on an algebraic curve over a finite field in characteristic p and analyse special situations like vector bundles on the projective line and relations with infinite Grassmannians.

INTRODUCTION

Suppose, $F : X \to X$ denotes the geometric Frobenius endomorphism of a smooth, projective algebraic curve of genus g over the finite field $\mathbb{F}_q$ of $q = p^s$ elements of characteristic $p > 0$. It is well known and has been studied quite often, that the pullback operation on vector bundles on X, induced by F, does not necessarily respect the stability of the bundle. If the genus of the curve X is greater than one, not much is known about this situation.

In particular, the pullback operation F^* does not in general induce a morphism of the coarse moduli scheme of bundles in the sense of Narasimhan and Seshadri, but only a rational map. It is only recently, that Y. Laszlo and C. Pauly have been able to write down such a rational map explicitly in the special case of rank two vector bundles for $p = 2$, $g = 2$, making use of the explicit knowledge of the coarse moduli scheme of (semi) stable bundles of rank 2 and degree 0 in the case $g = 2$ ([15] and [16] and for more results [13]).

It is therefore natural to consider the action of F^* or $\overline{F}^*$, extending F to the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, on the stack of bundles of rank n with trivial determinant as it is done in this paper.

Our contribution here consists of the observation (Prop. 2.4) that despite the fact that we do not know much about the action of F^*, it is possible to evaluate the action of F^* or $\overline{F}^*$ on the étale cohomology ring of the moduli stack of vector bundles.

Of course, the next question is then about the existence of a Lefschetz trace formula in this context, and we show by some easy examples that this can not work, at least not in a naive sense. In fact, already the case of the projective line $X = \mathbb{P}^1$ with $g = 0$ and for simplicity, $n = 2$, shows that

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there have to be modifications for such a trace formula, perhaps taking care of fixed points at infinity of the stack of bundles. Formally, one can force convergence of the corresponding infinite sums (the stack of bundles has cohomology in infinitely many dimensions!) by twisting with a high enough power of the arithmetic Frobenius of the stack of bundles. Nevertheless this convergent expression is not directly related to the set of fixed points, as is shown already by the example above. It should be mentioned at this point that a Lefschetz trace formula for stacks works quite well for actions of the powers of the arithmetic Frobenius of the stack of bundles alone (without the ! pullback $\overline{F}^*$) as is discussed in [3], [17]).

The organisation of this paper is as follows: the first section contains some general material concerning stacks of vector bundles and describes the different Frobenii occuring.

Section two contains the cohomological computations and evaluates in particular the action of $\overline{F}^*$ on the étale cohomology of the stack of bundles. Concerning the cohomology of the stack of bundles we have made use of work of Behrend [3] and in particular of the nice diploma thesis of J. Heinloth [12]. The procedure followed in their paper is parallel to the fundamental paper of Atiyah and Bott [2], but whereas Atiyah and Bott work in a differential-geometric context the procedure here is in the language of algebraic geometry and stacks. The decisive point is the cohomology of the gauge group which has to be treated here in a different way using older results of Harder and Narasimhan on the Tamagawa number of the special linear group.

The third section gives in part one the discussion of the example $X = \mathbb{P}^1$ of the projective line and bundles of rank $n = 2$. The interesting situation is here, that thanks to Grothendieck's splitting theorem for vector bundles, the operation of $\overline{F}^*$ on the stack is completely explicit. As is mentioned already, a trace formula in a naive sense does not hold. On the other hand we have the definite impression, that, if one could make sense of our approach in this case, this should generalize to the cases of arbitrary genus g, where not much is known about $\overline{F}^*$.

Part two of section three contains some further computations of the étale cohomology of moduli stacks of bundles and discusses the description of the stack of bundles as some kind of double quotient using the uniformisation theorem of Drinfeld and Simpson [7]. The ingredients of this description are the (infinite) affine Grassmannians from the theory of loop groups and an algebraic version of the gauge group. We indicate here a computation of the cohomology of both objects, evaluating along this a certain Leray spectral sequence. These considerations and the double coset description might be useful for extending our approach to a compactified situation.

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1. Stacks of vector bundles on curves and Frobenii

Let X denote a smooth, complete and irreducible algebraic curve of genus g over the finite field $\mathbb{F}_q$ with q elements. For any scheme S over $\mathbb{F}_q$, $\tilde{\mathcal{L}}(S)$ denotes the following category: the objects of $\tilde{\mathcal{L}}(S)$ are the rank n vector bundles over the scheme $X \times_{\mathbb{F}_q} S$, the morphisms of $\tilde{\mathcal{L}}(S)$ are just isomorphisms between such vector bundles.

Additionally, we consider the category $\mathcal{L}(S)$, whose objects are pairs (E, δ), where E ist a rank n vector bundle over $X \times_{\mathbb{F}_q} S$ and δ is an isomorphism of line bundles

$$\delta : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} S}$$

of the determinant bundle of E with the trivial line bundle. We always identify vector bundles and their locally free module sheaves of sections. The morphisms $u : (E, \delta) \to (E', \delta')$ in $\mathcal{L}(S)$ are isomorphisms of vector bundles

$$u : E \xrightarrow{\sim} E',$$

inducing commutative diagrams

$$\begin{array}{ccc} \det(E) & \xrightarrow{\det(u)} & \det(E') \\ {\scriptstyle\delta}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle\delta'} \\ \mathcal{O}_{X \times_{\mathbb{F}_q} S} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{X \times_{\mathbb{F}_q} S} \end{array}$$

It is well known, that the functor

$$S/\mathbb{F}_q \mapsto \tilde{\mathcal{L}}(S)$$

on the category of schemes over $\mathbb{F}_q$, defines an algebraic stack in the sense of M. Artin ([17], théorème 4.6.2.1). Due to the fact that X is a curve over $\mathbb{F}_q$, it follows, that $\mathcal{L}$ is a smooth algebraic stack. ([8], [12], 2.1.3, p.39).

Besides $\tilde{\mathcal{L}}$, we have also the stack $\mathcal{L}$, given by the functor $S/\mathbb{F}_q \mapsto \mathcal{L}(S)$, which is again smooth.

We will consider also the corresponding functors and stacks over $\overline{\mathbb{F}}_q$, denoted by $\overline{\tilde{\mathcal{L}}}$ and $\overline{\mathcal{L}}$ respectively.

Lemma 1.1. *One has canonical isomorphisms of stacks*

$$\begin{aligned} \tilde{\mathcal{L}} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q &\xrightarrow{\sim} \overline{\tilde{\mathcal{L}}}, \\ \mathcal{L} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q &\xrightarrow{\sim} \overline{\mathcal{L}}. \end{aligned}$$

Proof This is an immediate consequence of the definitions. □

On $X \times_{\mathbb{F}_q} \tilde{\mathcal{L}}$ we have the universal vector bundle E^{univ}, such that for any scheme S over $\mathbb{F}_q$ a morphism $\varphi : S \to \tilde{\mathcal{L}}$ induces the vector bundle $(\mathrm{id}_X \times \varphi)^*(E^{\mathrm{univ}})$ on $X \times_{\mathbb{F}_q} S$, which is exactly the element of $\tilde{\mathcal{L}}(S)$ given by φ.

In the same way, on $X \times_{\mathbb{F}_q} \mathcal{L}$, we have a universal pair $(E^{\mathrm{univ}}, \delta_{\mathrm{univ}})$, where $\delta_{\mathrm{univ}} : \det(E^{\mathrm{univ}}) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} \mathcal{L}}$. Similar considerations are true over the algebraic closure $\overline{\mathbb{F}}_q$.

Following for example ([2], [8], [12]), we will consider various open substacks of $\tilde{\mathcal{L}}$ resp. $\mathcal{L}$ and similarly for $\overline{\tilde{\mathcal{L}}}$ resp. $\overline{\mathcal{L}}$.

For an arbitrary vector bundle E over an algebraic curve X (over a field), $\mu(E)$ denotes the quotient

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)},$$

where $\deg(E)$ is the degree of the bundle E, $\mathrm{rk}(E)$ its rank.

Definition 1.2. *A vector bundle E is stable (resp. semistable) if for all proper subsheaves (resp. all subsheaves) E' of E the inequality $\mu(E') < \mu(E)$ (resp. $\mu(E') \leq \mu(E)$) holds.*

As is well known, any vector bundle E on a curve X has a canonical filtration, the *Harder- Narasimhan filtration*:

$$E_0 = (0) \subset E_1 \subset \ldots \subset E_r = E,$$

such that the following properties hold:

i) The successive quotients E_i/E_{i-1} are all semistable.
ii) Given E_i, E_{i+1} is a maximal subbundle of E with the additional property that
$$\mu(E_{i+1}/E_i) \geq \mu(E')$$
holds for all subbundles $E' \subset E/E_i$. Given E_i, this determines E_{i+1} uniquely.
iii) Dually, given E_i, then E_{i-1} is the smallest subbundle of E_i with the property, that
$$\mu(E_i/E_{i-1}) \leq \mu(E'')$$
for all quotient bundles E_i/E''.

The Harder-Narasimhan filtration is uniquely determined by the above properties.

We associate now with this filtration the polygonal function , with $\mathrm{rk}(E) = n$

$$p_E : [0, \mathrm{rk}(E)] \to \mathbb{R},$$

given as follows: $p_E(\mathrm{rk}(E_i)) = \deg(E_i)$ for $i = 0, \ldots, r$. At all other values $x \in [0, \mathrm{rk}(E)]$, the value $p_E(x)$ is obtained by linear interpolation using the values defined before. Obviously p_E is a concave function, that is, slopes are decreasing with growing x.

For any piecewise linear function $p : [0, \mathrm{rk}(E)] \to \mathbb{R}$, satisfying $p(\mathrm{rk}(E)) = \deg(E) = 0$ (we assume, that $\det(E) \cong \mathcal{O}_X$), we define substacks $\mathcal{L}_{\leq p}$ resp. $\mathcal{L}_{<p}$.

Definition 1.3. *For any scheme S over $\mathbb{F}_q$, $\mathcal{L}_{\leq p}(S)$ is the subcategory of $\mathcal{L}(S)$ of pairs (E, δ), E a bundle over $X \times_{\mathbb{F}_q} S$ of rank n, $\delta : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} S}$ a trivialisation of the determinant bundle, such that for all closed points $s \in S$, the Harder-Narasimhan polygon p_{E_s} of $E \times_{\mathbb{F}_q} s$ satisfies $p_{E_s} \leq p$. Similarly, $\mathcal{L}_{<p}(S)$ denotes the subcategory of pairs (E, δ), such that $p_{E_s} < p$ holds for all closed points $s \in S$.*

Proposition 1.4. *The stacks $\mathcal{L}_{<p} \hookrightarrow \mathcal{L}$ resp. $\mathcal{L}_{\leq p} \hookrightarrow \mathcal{L}$ are open substacks of $\mathcal{L}$. Similarly $\mathcal{L}_{<p} \hookrightarrow \mathcal{L}_{\leq p}$ is an open substack. $\mathcal{L}_p$ denotes the reduced closed substack $(\mathcal{L}_{\leq p} \setminus L_{<p})$.*

Proof: The proposition follows by using the semicontinuity theorem. For the details see [12], 2.1.10. □

Remark: Similarly we have the open substacks $\overline{\mathcal{L}}_{\leq p} = \mathcal{L}_{\leq p} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and $\overline{\mathcal{L}}_{<p} = \mathcal{L}_{<p} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ of $\overline{\mathcal{L}}$ as well as the closed substack $\overline{\mathcal{L}}_p$.

We repeat for completeness:

Definition 1.5. *$(E^{\mathrm{univ}}, \delta^{\mathrm{univ}})$ denotes the universal rank n vector bundle over $X \times_{\mathbb{F}_q} \mathcal{L}$ with trivial determinant $\delta^{\mathrm{univ}} : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} \mathcal{L}}$.*

We consider the following Frobenius morphisms in this context. First we have the geometric Frobenius endomorphism

$$F_X : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X),$$
$$F_X = \mathrm{id} \text{ (on X)} , F_X^*(f) = f^q \text{ (on sections of } \mathcal{O}_X).$$

Let $\overline{F} := F_X \times_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_q}$ denote the extension of F_X to an endomorphism of $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ over $\overline{\mathbb{F}}_q$.

If $S/\overline{\mathbb{F}}_q$ denotes a scheme over $\overline{\mathbb{F}}_q$, one has the pullback operation

$$\begin{aligned} \overline{\mathcal{L}}(S) &\to \overline{\mathcal{L}}(S) \\ (E, \delta) &\mapsto \big(\overline{F}^*(E), \overline{F}^*(\delta)\big). \end{aligned}$$

This induces an endomorphism of the stack $\overline{\mathcal{L}}$,

$$\varphi : \overline{\mathcal{L}} \to \overline{\mathcal{L}}.$$

Remark: As already outlined in the introduction, the nature of φ is very mysterious. In particular it does not respect the open substacks $\overline{\mathcal{L}}_{\leq p} \hookrightarrow \overline{\mathcal{L}}$ or $\overline{\mathcal{L}}_{<p} \hookrightarrow \overline{\mathcal{L}}$.

Besides φ we have also the geometric Frobenius morphism

$$F_{\mathcal{L}} : (\mathcal{L}, \mathcal{O}_{\mathcal{L}}) \to (\mathcal{L}, \mathcal{O}_{\mathcal{L}})$$

and its extension $F_{\mathcal{L}} \times_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_q}$ as an endomorphism of $\overline{\mathcal{L}}$ over $\overline{\mathbb{F}}_q$. Of course this endomorphism respects the open substacks $\mathcal{L}_{\leq p} \hookrightarrow \mathcal{L}$, $\mathcal{L}_{<p} \hookrightarrow \mathcal{L}$ and their extensions over $\overline{\mathbb{F}}_q$.
Actually we prefer to work in the next chapters with the arithmetic Frobenius morphism on the étale site $(\mathcal{L} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\text{ét}}$, given as $\psi := \mathrm{id}_{\mathcal{L}} \times \mathrm{Frob}\big(\overline{\mathbb{F}}_q/\mathbb{F}_q\big)$ which acts as an inverse to the action of $F_{\mathcal{L}} \times_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_q}$ on the étale site $(\mathcal{L} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\text{ét}}$.

Proposition 1.6. *There is a canonical isomorphism*

$$\big(\overline{F}_X \times \mathrm{id}_{\overline{\mathcal{L}}}\big)^*\big(E^{\mathrm{univ}}\big) \cong \big(\mathrm{id}_{\overline{X}} \times \varphi\big)^*\big(E^{\mathrm{univ}}\big).$$

Proof: This is immediate. For a stack $T/\overline{\mathbb{F}}_q$ a rank n vector bundle E on $\overline{X} \times T$ together with an isomorphism

$$\det(E) \xrightarrow{\sim} \mathcal{O}_{\overline{X} \times_{\mathbb{F}_q} T}$$

is given by a morphism $u : T \to \overline{\mathcal{L}}$, such that $E \cong (\mathrm{id}_{\overline{X}} \times u)^*(E^{\mathrm{univ}})$ and similarly for the determinant.
Applying this to the vector bundle $\big(\overline{F}_X \times_{\overline{\mathbb{F}}_q} \mathrm{id}_{\overline{\mathcal{L}}}\big)^*(E^{\mathrm{univ}})$, which defines $\varphi : \overline{\mathcal{L}} \to \overline{\mathcal{L}}$, and the proposition follows. □

2. Cohomology of the moduli stacks of vector bundles

In this section we describe the cohomology of the moduli stack of vector bundles. We collect the results from various different treatments in the literature. (see [12], [3], [5] and [2]).

We have the following well known description of the étale cohomology ring of the curve $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ over $\overline{\mathbb{F}}_q$, where $\overline{\mathbb{Q}}_l$ denotes the algebraic closure of $\mathbb{Q}_l$, namely

$$\begin{aligned} H^0(\overline{X}; \overline{\mathbb{Q}}_l) &= \overline{\mathbb{Q}}_l \cdot 1 \\ H^1(\overline{X}; \overline{\mathbb{Q}}_l) &= \bigoplus_{c=1}^{2g} \overline{\mathbb{Q}}_l \cdot \alpha_i \\ H^2(\overline{X}; \overline{\mathbb{Q}}_l) &= \overline{\mathbb{Q}}_l[\overline{X}], \end{aligned}$$

where $[\overline{X}]$ denotes the orientation class of $\overline{X}/\overline{\mathbb{F}}_q$ and where we can assume, that the $\{\alpha_i : i = 1, \dots, 2g\}$ are eigenclasses unter the action of the Frobenius morphism.

More precisely, for $\overline{F} = F_X \times_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_q}$ as before, we have explicitely that

$$\begin{aligned} \overline{F}^*(1) &= 1 \\ \overline{F}^*(\alpha_1) &= \lambda_i \alpha_i \quad (i = 1, \dots, 2g), \end{aligned}$$

where $\lambda_i \in \overline{\mathbb{Q}}_l$ is algebraic, such that $|\lambda_i| = q^{\frac{1}{2}}$ for any possible embedding of λ_i into the complex numbers $\mathbb{C}$. Finally $\overline{F}^*([\overline{X}]) = q \cdot [\overline{X}]$ for the orientation class $[\overline{X}]$.

We recall now some of the computations concerning the étale cohomology of the stack $\overline{\mathcal{L}}$ over $\overline{\mathbb{F}}_q$. The definition of étale cohomology of a stack $\mathcal{X}$ can be found in [17], chapter 12. Basically, the method is to consider a representation $X \to \mathcal{X}$ (say, over $\overline{\mathbb{F}}_q$) of the stack $\mathcal{X}$ by a scheme X and consider the associated simplicial object, resolving the stack $\mathcal{X}$

$$\mathcal{X} \leftarrow [X \underset{\leftarrow}{\overset{\leftarrow}{\leftarrow}} X \times_{\mathcal{X}} X \underset{\rightarrow}{\overset{\leftarrow}{\overset{\rightarrow}{\leftarrow}}} X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \dots]$$

and an appropriate étale or smooth site over it. For details, see chapters 12 and 18 of the book [17].

Given now a piecewise linear function $p : [0, n] \to \mathbb{R}$, satisfying $p(n) = 0$ and $p(i) \in \mathbb{Q}$ for $i \in \{0, \dots, n\}$ we have the closed embedding of stacks

$$i : \overline{\mathcal{L}}_p \hookrightarrow \overline{\mathcal{L}}_{\leq p}.$$

Proposition 2.1. *The Gysin sequence for the pair $(\overline{\mathcal{L}}_{\leq p}, \overline{\mathcal{L}}_p)$ splits into the short exact sequences*

$$0 \to H^{*-2r}(\overline{\mathcal{L}}_p); \overline{\mathbb{Q}}_l(r)) \xrightarrow{i_*} H^*(\overline{\mathcal{L}}_{\leq p}; \overline{\mathbb{Q}}_l) \to H^*(\overline{\mathcal{L}}_{<p}; \overline{\mathbb{Q}}_l) \to 0,$$

where r is the codimension of the stack $\overline{\mathcal{L}}_p$ in $\overline{\mathcal{L}}_{\leq p}$.

Proof: See [12], 2.2.1 or the other references. The argument in [12] follows the corresponding argument of Atiyah-Bott [2] in the differential geometric context. The composition $i^* i_*(x)$ for $x \in H^{*-2r}(\overline{\mathcal{L}}_p; \overline{\mathbb{Q}}_l(r))$ is

$$i^* i_*(x) = c_r(N^*) \cup x,$$

the cup product with the r-th Chern class (the top class) of the conormal bundle N^* of $\overline{\mathcal{L}}_p$ in $\overline{\mathcal{L}}_{\leq p}$.

Obviously injectivity of this map is sufficient to obtain the required splitting of the Gysin sequence. Therefore it suffices to show that $c_r(N^*)$ is not a zero-divisor on $H^*(\overline{\mathcal{L}}_p; \overline{\mathbb{Q}}_l)$. □

Proposition 2.2. *The Poincaré series of the cohomology ring* $H^*(\overline{\mathcal{L}};\overline{\mathbb{Q}}_l)$, $\sum_{i\geq 0}\dim_{\overline{\mathbb{Q}}_l}(H^i(\overline{\mathcal{L}};\mathbb{Q}_l))t^i$, *is given as*

$$P(\overline{\mathcal{L}};t)=\frac{\prod_{i=1}^{n}(1+t^{2i-1})^{2g}}{\prod_{i=2}^{n}(1+t^{2i})\prod_{i=2}^{n}(1-t^{2i-2})}.$$

Proof: Again we can quote [12], Satz 2.2.6, but with the slight modification, that we are considering the case with trivial determinant. The proof is obtained using 2.1., the Lefschetz trace formula (see also [3]) and the Tamagawa number computations from [11]. For a different proof using only tools from algebraic geometry see [5]. □

The cohomology of $\overline{\mathcal{L}}$ can be obtained now as follows: we consider on $\overline{X}\times\overline{\mathcal{L}}$ the universal bundle E^{univ} and its Chern classes

$$c_i(E^{\mathrm{univ}})\in H^{2i}(\overline{X}\times\overline{\mathcal{L}};\overline{\mathbb{Q}}_l)\quad(i\geq 2).$$

Remark: The first Chern class $c_1(E^{\mathrm{univ}})$ vanishes because we have by definition the trivialisation $\delta^{\mathrm{univ}}:\det(E^{\mathrm{univ}})\xrightarrow{\sim}\mathcal{O}_{\overline{X}\times\overline{\mathcal{L}}}$.

Using Künneth's theorem, which also holds in this context, we can decompose these classes as follows:

$$c_i(E^{\mathrm{univ}})=1\otimes c_i+\sum_{j=1}^{2g}\alpha_j\otimes a_i^{(j)}+[X]\otimes b_{i-1}.$$

Here we have, for $i\geq 2$, $c_i\in H^{2i}(\overline{\mathcal{L}};\overline{\mathbb{Q}}_l)$, $a_i^{(j)}\in H^{2i-1}(\overline{\mathcal{L}};\overline{\mathbb{Q}}_l)$ for $j=1,\dots,2g$ and $b_{i-1}\in H^{2(i-1)}(\overline{\mathcal{L}};\overline{\mathbb{Q}}_l)$.

Proposition 2.3. *There is an isomorphism of graded* $\overline{\mathbb{Q}}_l$*-algebras*

$$H^*(\overline{\mathcal{L}};\overline{\mathbb{Q}}_l)\cong\overline{\mathbb{Q}}_l[c_2,\dots,c_n;b_1,\dots,b_{n-1}]\otimes\bigotimes_{i=1}^{n}\Lambda[a_i^{(1)},\dots,a_i^{(2g)}],$$

where $\mathbb{Q}_l[c_2,\dots,c_n;b_1,\dots,b_{n-1}]$ *is the (graded) polynomial algebra in the generaters* $c_2,\dots,c_n,b_1\dots b_{n-1}$ *and* $\Lambda[a^{(1)},\dots,a_i^{(2g)}]$ *is an exterior algebra with generators* $a_i^{(1)},\dots,a_i^{(2g)}$ *in degree* $(2i-1)$.

Proof: We refer again to the literature. The proof can be found in [12], Satz 2.2.8 or [5], again of course with the slight modification, that we are considering the case of trivial determinant. The strategy of the proof can be outlined as follows: first, upon restriction to the closed substack of $\mathcal{L}$, consisting of vector bundles, which are direct-sums of line bundles, one can show that the canonical map

$$\overline{\mathbb{Q}}_l[c_1,\dots,c_n;b_1,\dots,b_{n-1}]\otimes\bigotimes_{i=1}^{n}\Lambda[a_i^{(1)},\dots,a_i^{(2g)}]\to H^*(\overline{\mathcal{L}};\overline{\mathbb{Q}}_l),$$

defined by sending the generators $c_i, b_i, a_i^{(j)}$ to the corresponding cohomology classes in $H^*(\overline{\mathcal{L}}; \overline{\mathbb{Q}}_l)$, is injective.
The surjectivity follows then from comparing the Poincaré polynomials of the two graded algebras using Prop. 2.2. □

We will study now the action induced by the morphism $\varphi : \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ of stacks on cohomology.
By Proposition 1.6. we have the isomorphism of vector bundles on $\bar{X} \times \bar{\mathcal{L}}$,

$$(\overline{F}_X \times \mathrm{id}_{\bar{\mathcal{L}}})(E^{\mathrm{univ}}) \cong (\mathrm{id}_{\bar{X}} \times \varphi)^*(E^{\mathrm{univ}})$$

This induces the equalities of Chern classes ($i \geq 2$)

$$c_i((\overline{F}_X \times \mathrm{id}_{\bar{\mathcal{L}}})^*(E^{\mathrm{univ}})) = c_i((\mathrm{id}_{\bar{X}} \times \varphi)^*(E^{\mathrm{univ}}))$$

Using the functoriality of Chern classes, we obtain therefore

$$(F_{\bar{X}} \times \mathrm{id}_{\bar{\mathcal{L}}})^*(c_i(E^{\mathrm{univ}})) = (\mathrm{id}_{\bar{X}} \times \varphi)^*(c_i(E^{\mathrm{univ}}))$$

From above we have the formula

$$c_i(E^{\mathrm{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}$$

for $i \geq 2$. Therefore we obtain the equality

$$\begin{aligned} & 1 \otimes c_i + \sum_{j=1}^{2g} \lambda_j \alpha_j \otimes a_i^{(j)} + q[X] \otimes b_{i-1} \\ = \quad & 1 \otimes \varphi^*(c_i) + \sum_{j=1}^{2g} \alpha_j \otimes \varphi^*(a_i^{(j)}) + [X] \otimes \varphi^*(b_{i-1}) \end{aligned}$$

This implies the following equalities

$$\begin{array}{ll} \varphi^*(c_i) = c_i & (i \geq 1) \\ \varphi^*(a_i^{(j)}) = \lambda_j a_i^{(j)} & (i \geq 1,\ j = 1, \ldots, 2g) \\ \varphi^*(b_i) = q\, b_i & (i \geq 1) \end{array}$$

and we have proved:

Proposition 2.4. *The morphism $\varphi : \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ of the stack of vector bundles of rank n on $\bar{X}$, given by the pullback operation induced by $\overline{F}_X : \bar{X} \to \bar{X}$, acts on the generating cohomology classes c_i, $a_i^{(j)}$, b_i of $H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$ as:*

$$\begin{array}{ll} \varphi^*(c_i) = c_i & (i \geq 1) \\ \varphi^*(a_i^{(j)}) = \lambda_j\, a_i^{(j)} & (i \geq 1,\ j = 1, \ldots, 2g) \\ \varphi^*(b_i) = q\, b_i & (i \geq 1) \end{array}$$

Remark: As mentioned earlier, it is somewhat surprising that one can determine the action of φ on the cohomology of $\mathcal{L}$ in such an explicit way, because otherwise the nature of φ is mysterious.

We will use now similar considerations to study the action of the genuine Frobenius endomorphism $\overline{F}_{\mathcal{L}} = F_{\mathcal{L}} \times \mathrm{id}_{\overline{\mathbb{F}}_q/\mathbb{F}_q}$ on the cohomology $H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$. To proceed further, we consider the classifying stacks of all rank n vector bundles $BGL(n)/\mathbb{F}_q$ and $BGL(n)/\mathbb{F}_q \times \overline{\mathbb{F}}_q$. As is well known, (see [3], Th. 2.3.2.) one has

$$H^*(BGL(n) \times \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l) \simeq \overline{\mathbb{Q}}_l[c_1, \ldots, c_n]$$

and the geometric Frobenius $\overline{F}_{BGL(n)/\mathbb{F}_q}$ acts as

$$(\overline{F}_{BGL(n)/\mathbb{F}_q})^*(c_i) = q^i c_i \text{ for } i \geq 1.$$

Using this we can obtain more information about the Frobenius action on the cohomology $H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$. We have

$$(\overline{F}_X \times \mathrm{id}_{\bar{\mathcal{L}}})^*(\mathrm{id}_{\bar{X}} \times \overline{F}_{\mathcal{L}})^*(c_i(E^{\mathrm{univ}})) \cong (\overline{F}_{X\times\mathcal{L}})^*(c_i(E^{\mathrm{univ}})) \cong (\overline{F}_{X\times\mathcal{L}})^* c_i(u^* \tilde{E}^{\mathrm{univ}}),$$

where $\tilde{E}^{\mathrm{univ}}$ denotes the universal vector bundle on the classifying stack $BGL(n)/\mathbb{F}_q$ and $u : X \times \mathcal{L} \to BGL(n)$ denotes the classifying morphism for E^{univ} on $(X \times \mathcal{L})$, such that

$$u^*(\tilde{E}^{\mathrm{univ}}) \cong E^{\mathrm{univ}}$$

with a canonical isomorphism. Furthermore we have the commutative diagram

$$\begin{array}{ccc} X \times \mathcal{L} & \xrightarrow{u} & BGL(n)/\mathbb{F}_q \\ {\scriptstyle F_{X\times\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle F_{BGL(n)/\mathbb{F}_q}} \\ X \times \mathcal{L} & \xrightarrow{u} & BGL(n)/\mathbb{F}_q \end{array}$$

and similarly after extension from $\mathbb{F}_q$ to $\overline{\mathbb{F}}_q$. Therefore we conclude

$$\begin{aligned} (\overline{F}_{X\times\mathcal{L}})^* c_i(\bar{u}^*(\tilde{E}^{\mathrm{univ}})) &= (\overline{F}_{X\times\mathcal{L}})^*(\bar{u})^* c_i(\tilde{E}^{\mathrm{univ}}) \\ &= (\bar{u})^*(\overline{F}_{BGL(n)/\mathbb{F}_q})^*(c_i(\tilde{E}^{\mathrm{univ}})) \\ &= q^i(\bar{u})^*(c_i(\tilde{E}^{\mathrm{univ}})) \\ &= q^i(c_i((\bar{u})^* \tilde{E}^{\mathrm{univ}})) \\ &= q^i c_i(E^{\mathrm{univ}}). \end{aligned}$$

On the other hand we obtain also

$$\begin{aligned} (\overline{F}_X \times \mathrm{id}_{\bar{\mathcal{L}}})^*(\mathrm{id}_{\bar{X}} \times \overline{F}_{\bar{\mathcal{L}}})^*(c_i(E^{\mathrm{univ}})) &= (\mathrm{id}_{\bar{X}} \times F_{\bar{\mathcal{L}}})^*(\overline{F}_X \times \mathrm{id}_{\bar{\mathcal{L}}})^*(c_i(E^{\mathrm{univ}})) \\ &= (\mathrm{id}_{\bar{X}} \times F_{\bar{\mathcal{L}}})^*(\overline{F}_X \times \mathrm{id}_{\bar{\mathcal{L}}})^*(1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}) \\ &= (\mathrm{id}_{\bar{X}} \times F_{\bar{\mathcal{L}}})^*(1 \otimes c_i + \sum_{j=1}^{2g} \lambda_j \alpha_j \otimes a_i^{(j)} + q[X] \otimes b_{i-1}) \end{aligned}$$

This implies immediately the following fundamental proposition

Proposition 2.5. *The action of the geometric Frobenius $\overline{F}_{\mathcal{L}}$ on the generating classes of the cohomology $H^*(\bar{\mathcal{L}};\overline{\mathbb{Q}}_l)$ of the stack $\mathcal{L}$ of rank n vector bundles on X is given as*

$$\begin{aligned}\overline{F}^*_{\mathcal{L}}(c_i) &= q^i c_i,\ (i\geq 2)\\ \overline{F}^*_{\mathcal{L}}(b_i) &= q^{i-1}b_i,\ (i\geq 1)\\ \overline{F}^*_{\mathcal{L}}(a_i^{(j)}) &= \lambda_j^{-1}q^i a_i^{(j)}\ (i\geq 2,\ j=1,\dots 2g).\end{aligned}$$

Corollary 2.6. *The action of the geometric Frobenius $F_{\bar{\mathcal{L}}}$ on the cohomology of the open substack $\bar{\mathcal{L}}_{\text{semistable}}$ of semistable bundles is semisimple.*

Proof: The restriction morphism $H^*(\bar{\mathcal{L}};\overline{\mathbb{Q}}_l)\to H^*(\bar{\mathcal{L}}_{\text{semistable}};\overline{\mathbb{Q}}_l)$ is surjective. As the cohomology $H^*(\bar{\mathcal{L}};\overline{\mathbb{Q}}_l)$ is a semisimple $\overline{\mathbb{Q}}_l[F_{\bar{\mathcal{L}}}]$-module, the same is also true for $H^*(\bar{\mathcal{L}}_{\text{semistable}};\overline{\mathbb{Q}}_l)$. □

Remark: We have considered above only the case of bundles with trivial determinant. The same technique can be applied however for the stack of bundles of rank n and degree d, where $\gcd(n,d)=1$, i.e. n and d are coprime numbers.

Proposition 2.7. *Suppose* $\gcd(n,d)=1$ *as above. Then the action of the Galois group* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ *resp. the arithmetic Frobenius ψ on the étale cohomology $H^*(M(n,d);\overline{\mathbb{Q}}_l)$ of the coarse moduli space $M(n,d)$ of stable vector bundles of rank n and degree d is semisimple.*

Proof: Because $\gcd(n,d)=1$, we have $\tilde{\mathcal{L}}(n;d)_{\text{semistable}}=\tilde{\mathcal{L}}(n,d)_{\text{stable}}$. Furthermore there is a canonical morphism

$$\tilde{\mathcal{L}}(n;d)_{\text{stable}}\to M(n,d),$$

given functorially for any scheme $S/\mathbb{F}_q$ by the obvious functor

$$\mathcal{L}(n,d)_{\text{stable}}(S)\to M(n,d)(S),$$

where $M(n,d)(S)$ denotes the discrete category of isomorphism classes of vector bundles of rank n and degree d (pointwise in S) on $X\times_{\mathbb{F}_q}S$.
The Leray spectral sequence for this morphism is degenerating, because we have an equality of Poincaré polynomials

$$P(\bar{\mathcal{L}}(n;d)_{\text{stable}};t)=P(\overline{M}(n,d);t)\cdot P(B\mathbb{G}_m;t),$$

where the morphism $\bar{\mathcal{L}}_{\text{stable}}\to\overline{M}(n;d)$ is a fibration with fibre the classifying stack $B\mathbb{G}_m$.
Therefore $H^*(\overline{M}(n;d);\overline{\mathbb{Q}}_l)$ is a submodule of $H^*(\bar{\mathcal{L}}_{\text{stable}};\overline{\mathbb{Q}}_l)$. As the last one is a semisimple module over $\overline{\mathbb{Q}}_l[\overline{F}_{\mathcal{L}}]$, $H^*(\overline{M}(n;d);\overline{\mathbb{Q}}_l)$ is a semisimple module over $\overline{\mathbb{Q}}_l[\overline{F}_{M(n;d)}]$, or equivalently the Frobenius ψ is acting as a semisimple endomorphism. □

Proposition 2.8. *Identifying the algebraic closure $\overline{\mathbb{Q}}_l$ with the field $\mathbb{C}$ of complex numbers, the expression for the formal trace*

$$\mathrm{tr}(\varphi^r \times \psi^s;\, H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(\varphi^r \times \psi^s; H^i(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l))$$

is absolutely convergent for $s > r$.

Proof: This is just an exercise in summing up geometric series. The formal series to be considered is given as

$$(\prod_{i=2}^{n} (\sum_{n=0}^{\infty} q^{-(si)})) \cdot (\prod_{k=1}^{n-1} \sum_{m=0}^{\infty} q^{m(r-ks)}) \cdot (\prod_{j=1}^{2g} \prod_{i=1}^{n} (1 + |\lambda_j|^{r+s} q^{-is}))$$

which implies the statement. □

Remark: The question is now of course, if there is a kind of Lefschetz trace formula in this general situation?

This is true at least in the case $r = 0$, $s = 1$, which was considered in [3], [12]. The following trace formula holds

$$q^{\dim(\mathcal{L})} \sum_{i \geq 0} (-1)^i \mathrm{tr}(\psi : H^i(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)) = \sum_{[E] \in \mathcal{L}(k)} \frac{1}{|\mathrm{Aut}^{(0)}(E)},$$

where

$$\mathrm{Aut}^{(0)}(E) := \{\alpha \in \mathrm{Aut}(E)(k) \mid \det(\alpha) = 1\}$$

and $\dim(\mathcal{L}) = n^2(g-1) + 1$.

But as mentioned already in the introduction, the situation in the cases $r = 0$ resp. $r > 0$ is rather different, because the proof in the case $r = 0$ works by studying $\bar{\mathcal{L}}$ and the traces above using the filtration by the $\bar{\mathcal{L}}_{\leq p}$ as done earlier. On the other hand the morphism $\varphi : \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ does not respect these open substacks.

3. Some complements and examples

Part 1: Vector bundles of rank 2 on the projective line

It seems worthwhile to study in detail the easiest example, namely the case of the projective line $X = \mathbb{P}^1$ and vector bundles of rank $n = 2$ with trivial determinant. We specialize our computations from section 2 to this case. By Proposition 2.3, we have in this special situation

$$H^*(\bar{\mathcal{L}}(\mathbb{P}^1; 2); \overline{\mathbb{Q}}_l) \cong \overline{\mathbb{Q}}_l[c_2, b_1],$$

where

$$b_1 \in H^2(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l),\ c_2 \in H^4(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$$

For the formal trace we obtain

$$\begin{aligned}\mathrm{tr}(\varphi^r \times \psi^s;\, H^*(\bar{\mathcal{L}}(\mathbb{P}^1;2);\overline{\mathbb{Q}}_l) &= \sum_{i\geq 0}(-1)^i \mathrm{tr}(\varphi^r \times \psi^s; H^i(\bar{\mathcal{L}}(\mathbb{P}^1;2);\overline{\mathbb{Q}}_l) \\ &= (\sum_{m=0}^{\infty} q^{-2sm})(\sum_{m=0}^{\infty} q^{(r-s)m}),\end{aligned}$$

which is convergent for $r < s$. We obtain therefore for $r < s$

$$\mathrm{tr}(\varphi^r \times \psi^s; H^*(\bar{\mathcal{L}}(\mathbb{P}^1;2);\overline{\mathbb{Q}}_l)) = (1-q^{-2s})^{-1}(1-q^{r-s})^{-1}$$

Remark: In this special case the fixed points under the action of $(\varphi^r \times \psi^s)$ can be computed directly. The naive fixed point set can be computed as

$$(\bar{\mathcal{L}})^{\varphi^r \times \psi^s} = (\bar{\mathcal{L}}_{\mathrm{semistable}})^{\varphi^r \times \psi^s} = (BSL(2)/\mathbb{F}_q) \times \overline{\mathbb{F}}_q)^{\varphi^r \times \psi^s}$$

Here we have used the isomorphism of stacks

$$BSL(2)/\mathbb{F}_q \to \mathcal{L}(\mathbb{P}^1;2)_{\mathrm{semistable}},$$

given functorially for the categories of S-valued points, $S/\mathbb{F}_q$ a scheme, by

$$E \longmapsto pr_2^*(E),$$

where E is a vector bundle of rank two over S with trivial determinant and $pr_2^*(E)$ is the pullback to $\mathbb{P}^1 \times_{\mathbb{F}_q} S$ via the projection

$$pr_2 : \mathbb{P}^1 \times_{\mathbb{F}_q} S \to S$$

This implies immediately, that φ acts as the identity on $\mathcal{L}_{\mathrm{semistable}}$. But then we obtain

$$(\bar{\mathcal{L}})^{\varphi^r \times \psi^s} = (BSL(2)/\mathbb{F}_q) \times \overline{\mathbb{F}}_q)^{\psi^s}$$

This does not depend on r at all, which obviously does not fit with the formula obtained above.

Part 2: Affine Grassmannians and gauge groups

In this second part of section 3 we give a description of the moduli stack of vector bundles as a double quotient similar to the well known adelic description of vector bundles on a curve. These considerations seem to be useful for our goal to make sense of a general trace formula, in order to do our computations of section 2 for an appropriate compactification of the stack $\bar{\mathcal{L}}$ or $\mathcal{L}$.

For the following we consider the smooth projective curve X over $\mathbb{F}_q$ with a closed $\mathbb{F}_q$-rational point $\infty \in X$ (whose existence we assume for simplicity). $\mathcal{O}_{X,\infty}$ is the local ring of X at ∞, $\mathcal{O}_\infty$ its completion, F_∞ the completion of the function field $F = \mathbb{F}_q(X)$ at ∞. $X^{(\infty)} = X \setminus \{\infty\}$ is an open subscheme of X.

Let $\tilde{\mathcal{L}}(X;n)$ denote the algebraic stack of vector bundles of rank n on X, similarly $\tilde{\mathcal{L}}(X^{(\infty)};n)$ denotes the stack of vector bundles of rank n on $X^{(\infty)}$.

Finally, let $\mathcal{L}(X;n)$ resp. $\mathcal{L}(X^{(\infty)};n)$ denote the corresponding stacks with trivialized determinant.

There is an obvious morphism r of stacks

$$r : \mathcal{L}(X;n) \to \mathcal{L}(X^{(\infty)};n))$$

given by the restriction functor

$$E \in \mathcal{L}(X;n)(S) \longmapsto (E \mid X^{(\infty)} \times_{\mathbb{F}_q} S) \in \mathcal{L}(X^{(\infty)};n)(S),$$

where $S/\mathbb{F}_q$ is any scheme over $\mathbb{F}_q$.

The fibre over a closed point $(E^{(0)};\delta^0) \in \mathcal{L}(X^{(\infty)};n)$ can be described as the stack, given functorially by $S/\mathbb{F}_q \mapsto \underline{\underline{\mathrm{Gr}}}(E^{(0)})(S)$, where $\underline{\underline{\mathrm{Gr}}}(E^{(0)})(S)$ denotes the category of triples $((E,\delta),n)$, where $(E,\delta) \in \mathcal{L}(X;n)(S)$ that is, E is a rank n vector bundle over $X \times_{\mathbb{F}_q} S$, $\delta : \det(E) \xrightarrow{\sim} \mathcal{O}_{X\times_{\mathbb{F}_q}S}$ a trivialisation of the associated determinant bundle and u is an isomorphism of vector bundles

$$u : E\mid_{X^{(\infty)}\times_{\mathbb{F}_q}S} \widetilde{\longrightarrow} \mathrm{pr}_2^*(E^{(0)}),$$

where $\mathrm{pr}_2 : X^{(\infty)} \times_{\mathbb{F}_q} S \to S$ is the projection on the second factor. Additionally, the induced determinantal isomorphism $\det(u)$ has to make the obvious diagram of determinant bundles and trivialisations commutative.

Remark: For simplicity we have written $\underline{\underline{\mathrm{Gr}}}(E^{(0)})$ instead of $\underline{\underline{\mathrm{Gr}}}((E^{(0)};\delta^{(0)}))$.

As is known (see for example [10]), there is a purely local description of $\underline{\underline{\mathrm{Gr}}}(E^{(0)})$ as follows:

Consider the group-valued functors

$$\begin{array}{rcl} \underline{\text{affine schemes}}/\mathbb{F}_q & \longrightarrow & \underline{\text{groups}} \\ S & \longmapsto & \mathrm{SL}(n;\mathcal{O}_S[[t_\infty]]), \\ \text{resp.} \quad S & \longmapsto & \mathrm{SL}(n;\mathcal{O}_S((t_\infty))) \end{array}$$

Then the first functor is represented by an algebraic group (not of finite type) and the second functor is represented by an ind-algebraic group. We denote these two objects somewhat informally as $\underline{\underline{\mathrm{SL}}}(n;\mathcal{O}_\infty)$ resp. $\underline{\underline{\mathrm{SL}}}(n;F_\infty)$.

The quotient

$$\underline{\underline{\mathrm{SL}}}(n;F_\infty)/\underline{\underline{\mathrm{SL}}}(n;\mathcal{O}_\infty)$$

exists as a stack (or even an ind-algebraic scheme) and is isomorphic to the stack $\underline{\underline{\mathrm{Gr}}}(E^{(0)})$. (see [10] and [9].)

Choosing a pair $(E^{(0)},\delta^{(0)}) \in \mathcal{L}(X^{(\infty)};n)(\mathbb{F}_q)$, for example the trivial bundle, we have the following diagram of stacks

$$\underline{\underline{\mathrm{Gr}}}(E^{(0)}) \to \mathcal{L}(X;n) \to \mathcal{L}(X^{(\infty)};n),$$

where $\underline{\underline{\mathrm{Gr}}}(E^{(0)}) \to \mathcal{L}(X;n)$ is the morphism of stacks, given functorially by

$$((E,\delta),u) \mapsto (E,\delta)$$

using the notations from above.

Now we indicate the computation of the étale cohomologies again.

Proposition 3.1. *The filtration of Section 2, Prop. 2.1. of the stack* $\mathcal{L}(X;n)$ *by the open substacks* $\mathcal{L}(X;n)_{\leq p}$ *induces a corresponding filtration* $\underline{\underline{\mathrm{Gr}}}(E^{(0)})_{\leq p}$ *of* $\underline{\underline{\mathrm{Gr}}}(E^{(0)})$.

This can be used to compute the étale cohomology of the infinite Grassmannian $\underline{\underline{\mathrm{Gr}}}(E^{(0)})$ as follows:

Proposition 3.2. *The cohomology ring* $H^*(\underline{\underline{\mathrm{Gr}}}(E^{(0)}) \times \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l)$ *is given as* $\mathbb{Q}_l[b_1, \ldots, b_{n-1}]$ *where the* b_i $(i = 1, \ldots, n-1)$ *are the restrictions of the corresponding classes* b_i *in the cohomology ring* $H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$.

Proof: We can consider the universal object

$$(E^{\mathrm{univ}}, \delta^{\mathrm{univ}}), u : E^{\mathrm{univ}} \mid_{X^{(\infty)} \times_{\mathbb{F}_q} \underline{\underline{\mathrm{Gr}}}(E^{(0)})} \widetilde{\longrightarrow} \mathrm{pr}_2^*(\mathcal{O}^n_{X^{(\infty)}}))$$

over $X \times_{\mathbb{F}_q} \underline{\underline{\mathrm{Gr}}}(E^{(0)})$. We can write down (for example by pullback) the Chern classes of E^{univ} as

$$c_i(E^{\mathrm{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}$$

for $i \geq 1$ in $H^{2i}((X \times_{\mathbb{F}_q} \underline{\underline{\mathrm{Gr}}}(E^{(0)})) \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l)$.
Here the $c_i, b_i, a_i^{(j)}$ are just the restrictions of the corresponding classes under the morphism $\underline{\underline{\mathrm{Gr}}}(E^{(0)}) \to \mathcal{L}$ considered above. On the other hand we have the identity

$$c_i(E^{\mathrm{univ}}) \mid (X^{(\infty)} \times \underline{\underline{\mathrm{Gr}}}(E^0)) \times \overline{\mathbb{F}}_q = \varphi^*(\mathrm{pr}_1^*(c_i(E^{(0)}))$$

by the very definition of $(E^{\mathrm{univ}}, \varphi^{\mathrm{univ}})$.
But $c_i(E^{(0)}) = 0$ for $i \geq 1$, therefore we obtain for the restrictions

$$\begin{aligned} c_i \mid \underline{\underline{\mathrm{Gr}}}(E^0) &= 0 \qquad (i \geq 1) \\ a_i^{(j)} \mid \underline{\underline{\mathrm{Gr}}}(E^0) &= 0 \qquad (i \geq 1,\ j = 1, \ldots, 2g) \end{aligned}$$

It remains to show that $b_1, \ldots, b_{n-1}$ are generating elements of the cohomology and are polynomially independent. This can be done using the stratification mentioned above in a similar treatment as in the corresponding statements for the étale cohomology of the stack $\bar{\mathcal{L}}$. □

Remark: In a similar way, using a yet to be defined version of étale cohomology appropriate for the stack $\mathcal{L}(X^{(\infty)} \times \overline{\mathbb{F}}_q; n)$, we could consider the morphism $r : \mathcal{L}(X;n) \to \mathcal{L}(X^{(\infty)}; n)$, given as restriction of bundles to $X^{(\infty)}$. For the universal bundles we have the pullback situation

$$r^*(E^{\mathrm{univ}}(\text{over } X^{(\infty)} \times \mathcal{L}(X^{(\infty)}; n)) = E^{\mathrm{univ}}(\text{over } X \times \mathcal{L}(X;n))$$

Again for E^{univ} over $X^{(\infty)} \times \mathcal{L}(X^{(\infty)}; n)$, we could consider the Chern classes and then obtain the following description

$$c_i(E^{\mathrm{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_j^{(i)},\ (1 \leq i \leq n)$$

because here we have $[X]|H^2(\bar{X}_\infty) = 0$.

Then the cohomology $H^*(\mathcal{L}(X^{(\infty)}; n) \times \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l)$ is given as

$$\overline{\mathbb{Q}}_l[c_1, \ldots, c_n] \otimes \bigotimes_{i=1}^{n} \Lambda[a_i^{(1)}, \ldots, a_i^{(2g)}]$$

in the sense of *Prop. 2.3.* and the Leray spectral sequence for the morphism of stacks

$$\bar{r} : \bar{\mathcal{L}}(X; n) \to \bar{\mathcal{L}}(X^{(\infty)}; n)$$

would degenerate.

We would get this description along the following lines. As the cohomology classes

$$\{c_i(\bar{\mathcal{L}}(X)), a_i^{(j)}(\bar{\mathcal{L}}(X)) \mid i = 1, \ldots, n,\ j = 1, \ldots, 2g\}$$
$$\{b_i(\bar{\mathcal{L}}(X)) \mid i = 1, \ldots, n-1\}$$

are independent in the sense of Prop. 2.3., the same is also true for their preimages

$$\{c_i(\bar{\mathcal{L}}(X^{(\infty)})); a_i^{(j)}(\bar{\mathcal{L}}(X^{(\infty)})) \mid i = 1, \ldots, n, \text{ and } j = 1, \ldots, 2g\}.$$

On the other hand comparing the Poincaré polynomials we would get

$$P(\mathcal{L}(X); t) = P(\underline{\underline{\mathrm{Gr}}}(E^{(0)}; t) P(\mathcal{L}(X^{(\infty)}); t)$$

From this and the corresponding computations of $H^*(\bar{\mathcal{L}}(X); \overline{\mathbb{Q}}_l)$ and the cohomology $H^*(\underline{\underline{\mathrm{Gr}}}(E^{(0)}); \overline{\mathbb{Q}}_l)$ we would then obtain the desired description of the cohomology rings and the Leray spectral sequence.

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