Part III

The cohomology of $\mathrm{Spin}(8, \mathbb{Z}[\frac{1}{2}])$

Chapter 4

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4.1 Application to even unimodular lattices

Let V be a vector space of dimension $n = 8k$ over $\mathbb{Q}$ and fix a basis $x_1, \ldots, x_n$ of V. Then consider the quadratic form $q : V \to \mathbb{Q}$ on V, which is defined by

$$q(\sum_{i=1}^{n} \lambda_i x_i) = \frac{1}{2}(\sum_{i=1}^{n} \lambda_i^2),$$

and let b denote the polar of q. Since $n \equiv 0 \mathrm{mod}\ 8$, it follows from the classification of quadratic forms over p-adic fields, that the quadratic space $V_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} V$ is hyperbolic of Witt-index $r = \frac{n}{2}$ for any prime p, cf. [Sch85] p. 189-190. Put $M_p := \mathbb{Z}_p \otimes (\bigoplus_{i=1}^{n} \mathbb{Z}x_i)$ for $p \neq 2$. This is a maximal $\mathbb{Z}_p$-lattice in the quadratic space V_p. Choose also a maximal $\mathbb{Z}_2$-lattice $M_2 \subset V_2$. Then M_p is a hyperbolic, hence regular quadratic module over $\mathbb{Z}_p$ for any p and it follows from the fundamental correspondence between global and local lattices ([Kne02] 21.5), that

$$\Lambda_0 := \bigcap_{p \text{ prime}} M_p \cap V$$

is a $\mathbb{Z}$-lattice in the positive definite quadratic space V, which is regular as a quadratic module over $\mathbb{Z}$. Such a lattice is called an even unimodular $\mathbb{Z}$-lattice and it is well known, that any even unimodular $\mathbb{Z}$-lattice is of this form, cf. [Kne02] 26.7, b. In particular, for any $n = 8k$ there is exactly one genus of even unimodular lattices.

Fix a prime number p. Now, consider the ring $\mathbb{Z}[\frac{1}{p}]$ of S-integers with $S = \{\infty, p\}$ and the $\mathbb{Z}[\frac{1}{p}]$-lattice $L[\frac{1}{p}] := \mathbb{Z}[\frac{1}{p}] \otimes \Lambda_0 \subset V$. Note, that $q(L[\frac{1}{p}]) \subset \mathbb{Z}[\frac{1}{p}]$ and that $(L[\frac{1}{p}], q)$ is a regular quadratic module over $\mathbb{Z}[\frac{1}{p}]$. The subject of the following is the cohomology of the group $\Gamma = \mathrm{Spin}(L[\frac{1}{p}], q)$.

Let X_p be the Bruhat-Tits building, which is associated to the quadratic space $V_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} V$. Sometimes it will also be convenient to consider the building $\tilde{X}_p$ of simple chains of lattices. Since V_p is a hyperbolic space, it follows, that X_p is a thick affine building of type D_r. Let $l : \mathcal{V}(X_p) \to \{0^+, 0^-, 2, \ldots, r-2, r^+, r^-\}$ and $\tilde{l} : \mathcal{V}(X_p) \to \{0, 1, \ldots, r-1, r\}$ denote the labellings of X_p and $\tilde{X}_p$ respectively defined in chapter 2.

By the definition of the Bruhat-Tits building, the vertices of X_p correspond to certain lattices in V_p and these are in bijection with their preimages in $L[\frac{1}{p}]$ under the canonical injection

$$L[\frac{1}{p}] \hookrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}[\frac{1}{p}]} L[\frac{1}{p}] = V_p,$$

by the fundamental correspondence of global and local lattices. In contrast to chapter 2 we will now denote by $\mathcal{V}$ (resp. $\tilde{\mathcal{V}}$) the set of $\mathbb{Z}$-lattices contained in $L[\frac{1}{p}]$, which correspond to the vertices of X_p (resp. $\tilde{X}_p$). Further, if we put

$$L^{(\sharp)} := ((\mathbb{Z}_p \otimes L)^{(\sharp)} \cap V) \cap \bigcap_{p' \neq p} (\mathbb{Z}_{p'} \otimes L \cap V)$$

for $L \in \tilde{\mathcal{V}}$, then $\mathbb{Z}_p \otimes L^{(\sharp)} = (\mathbb{Z}_p \otimes L)^{(\sharp)}$. In this special case,

$$L^{(\sharp)} = \{x \in V \mid b(x, L) \in \mathbb{Z}\}$$

is the usual dual lattice of L, because $L[\frac{1}{p}]$ is regular and V_p is hyperbolic. If we define for $L \in \tilde{\mathcal{V}}$:

$$\overline{V}_L := (\frac{1}{p}L/L)/\mathrm{Rad}(\frac{1}{p}L/L)$$

and

$$\overline{V}^{L^{(\sharp)}} := (L/pL)/\mathrm{Rad}(L/pL)$$

in analogy to section 2.3.3, then it is easy to check that

$$\overline{V}_L \cong \overline{V}_{\mathbb{Z}_p \otimes L} \text{ and } \overline{V}^{L^{(\sharp)}} \cong \overline{V}^{(\mathbb{Z}_p \otimes L)^{(\sharp)}}.$$

Finally, we get homomorphisms

$$\begin{aligned} r_L : &\quad \mathrm{O}(L, q) \to \mathrm{O}(\overline{V}_L, \overline{q}) \\ {r_L}^{(\sharp)} : &\quad \mathrm{O}(L, q) \to \mathrm{O}(\overline{V}^{L^{(\sharp)}}, \overline{q}) \end{aligned}$$

by restriction from the homomorphisms $r_L, {r_L}^{(\sharp)}$ defined in section 2.3.3.

Lemma 19. *1. Let $L \subset L[\frac{1}{p}]$ be a lattice, which represents a vertex of $\tilde{X}_p$ with label $l_0 \in \{1, \ldots, r\}$. Then $dL = p^{2l_0}$.*

2. *If L represents a vertex of label 0, then L is an even unimodular lattice in (V,q), whereas vertices of label r are represented by lattices, that are even unimodular with respect to the scaled quadratic form $\frac{1}{p}q$.*

Proof. 1. follows from $dL[\frac{1}{p}] = 1$ and $d(\mathbb{Z}_p \otimes L) = p^{2l_0}$, since $\mathbb{Z}[\frac{1}{p}] \cap \mathbb{Z}_p = \mathbb{Z}$.

2. If L represents a vertex of label 0 (resp. r), then $q(\mathbb{Z}_p \otimes L) = \mathbb{Z}_p$ (resp. $q(\mathbb{Z}_p \otimes L) = p\mathbb{Z}_p$). Thus $q(L) \subset \mathbb{Z}_p \cap \mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ (resp. $q(L) \subset p\mathbb{Z}_p \cap \mathbb{Z}[\frac{1}{p}] = p\mathbb{Z}$) and the assertion follows from 1. □

Since $L[\frac{1}{p}]$ is a regular and positive definite $\mathbb{Z}[\frac{1}{p}]$-lattice, the spinor norm on $\mathrm{SO}(L[\frac{1}{p}], q)$ takes values in $\mathrm{Sq}^+(p) := \{x \in \mathbb{Z}[\frac{1}{p}]^\times \mid x > 0\}/(\mathbb{Z}[\frac{1}{p}]^\times)^2$ and this group is generated by $1, p \in \mathbb{Z}[\frac{1}{p}]$ and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Lemma 20. *The spinor norm θ maps $\mathrm{SO}(L[\frac{1}{p}], q)$ surjectively onto $\mathrm{Sq}^+(p)$. In particular, condition **(R)** of section 3.2.3 is satisfied.*

Proof. Clearly, 1 and p are contained in $q(\mathbb{R} \otimes \Lambda_0)$. Since for any prime p' the $\mathbb{Z}_{p'}$-module $\mathbb{Z}_{p'} \otimes \Lambda_0$ is hyperbolic, 1 and p are also contained in $q(\mathbb{Z}_{p'} \otimes \Lambda_0)$. This implies, that there are lattices $\Lambda(1)$ and $\Lambda(p)$ in the genus of Λ_0, such that $1 \in q(\Lambda(1))$ and $p \in q(\Lambda(p))$, cf. [Kne02] (22.1). But by proposition 14, in $L[\frac{1}{p}]$ there is contained a lattice of every class in the genus of Λ_0. Hence there are elements $x, y \in L[\frac{1}{p}]$ satisfying $q(x) = 1$, $q(y) = p$. Then the rotation $\tau_x \circ \tau_y \in \mathrm{SO}(L[\frac{1}{p}], q)$ has spinor norm p. □

Example: The Leech-lattice is an even unimodular lattice of dimension 24, which contains no element x with $q(x) = 1$.

By lemma 20, the exact sequence (1.7) of section 1.11 takes the form

$$1 \to \langle \pm 1 \rangle \to \mathrm{Spin}(L[\frac{1}{p}], q) \xrightarrow{\iota} \mathrm{SO}(L[\frac{1}{p}], q) \xrightarrow{\theta} \mathbb{Z}/2\mathbb{Z} \to 1.$$

Note, that $-\mathrm{id}_{L[\frac{1}{p}]} \in \mathrm{SO}(L[\frac{1}{p}], q)$ acts trivially on the set $\tilde{\mathcal{V}}$. The preimage of this element in the spin group is $\pm z$, where $z = 2^r \iota(x_1) \cdots \iota(x_n)$ in the notation of section 1.6. The element z is contained in the center of $\mathrm{Spin}(L[\frac{1}{p}], q)$ and $z^2 = 1$. Thus, the group $U := \langle -1, z \rangle$ is an elementary abelian subgroup of order 4 and is contained in the stabilizer of any $L \in \tilde{\mathcal{V}}$ in Γ. With $\mathrm{PSO}(M,q) := \mathrm{SO}(M,q)/\langle \pm \mathrm{id}_M \rangle$ for any quadratic module (M,q) of even rank over an integral domain, we get an exact sequence

$$1 \to U \to \mathrm{Spin}(L[\frac{1}{p}], q) \to \mathrm{PSO}(L[\frac{1}{p}], q) \xrightarrow{\theta} \mathbb{Z}/2\mathbb{Z} \to 1.$$

Let $L \in \tilde{\mathcal{V}}$ be a lattice with label 0. Then (L, q) is a regular quadratic module over $\mathbb{Z}$; and since V is positive definite, it follows, that the spinor norm is trivial on $\mathrm{SO}(L, q)$, i.e. $\theta(\mathrm{SO}(L, q)) = \{1\}$. Thus, the sequence

$$1 \to U \to \mathrm{Spin}(L, q) \to \mathrm{PSO}(L, q) \xrightarrow{\theta} 1$$

is exact. Since L is contained in the genus of Λ_0, it follows, that $\mathbb{Z}_p \otimes L$ is a hyperbolic module over $\mathbb{Z}_p$. This implies that, $L = L^{(\sharp)}$ and that $\overline{V}_L = \frac{1}{p}L/L$ is a hyperbolic quadratic space of dimension n over $\mathbb{F}_p$. In the following, some facts about such spaces are given (see [CCN+85] p. xi-xiii for a short survey of these facts, [Die63] for elementary proofs or the book [Car93] for a more algebraic exposition).

Remark 17. *Quadratic spaces over finite fields:*
Let $n = 2\nu$ be an even positive integer and $\mathbb{F}$ a finite field of order $\mathfrak{q}$. Then there are exactly two classes of regular quadratic spaces of dimension n over $\mathbb{F}$:

1. The space is hyperbolic. In this case the orthogonal (resp. special orthogonal) group is denoted by $\mathrm{O}^+(n, \mathbb{F})$ (resp. $\mathrm{SO}^+(n, \mathbb{F})$) as abstract group. If $n \geq 6$ and θ denotes the spinor norm, then

   $$\{x \in \mathrm{SO}^+(n, \mathbb{F}) \mid \theta(x) = 1\}/\{\pm \mathrm{id}\}$$

   is the finite simple group of Lie type D_ν, which is denoted by $D_\nu(\mathfrak{q})$ (resp. $O_n^+(\mathfrak{q})$ in [CCN+85]). Note, that if the characteristic of the field $\mathbb{F}$ is 2, then $\mathrm{SO}^+(n, \mathbb{F}) \cong D_\nu(\mathfrak{q})$ in a natural way. The order of $D_\nu(\mathfrak{q})$ is

   $$\mathfrak{q}^{\nu^2-\nu}(\mathfrak{q}^2 - 1)(\mathfrak{q}^4 - 1) \cdots (\mathfrak{q}^{2\nu-2} - 1)(\mathfrak{q}^\nu - 1).$$

2. The space is the orthogonal sum of a hyperbolic quadratic space of dimension $n-2$ and an anisotropic space of dimension 2. Then the orthogonal or special orthogonal group is said to be of isomorphism type $\mathrm{O}^-(n, \mathbb{F})$ (resp. $\mathrm{SO}^-(n, \mathbb{F})$). As above, if $n \geq 6$ and θ denotes the spinor norm, then

   $$\{x \in \mathrm{SO}^-(n, \mathbb{F}) \mid \theta(x) = 1\}/\{\pm \mathrm{id}\}$$

   is the finite simple group of Lie type ${}^2D_\nu$, which is denoted by ${}^2D_\nu(\mathfrak{q}^2)$ (resp. $O_n^-(\mathfrak{q})$ in [CCN+85]), and $\mathrm{SO}^-(n, \mathbb{F}) \cong {}^2D_\nu(\mathfrak{q}^2)$, if the characteristic of $\mathbb{F}$ is 2. The order of ${}^2D_\nu(\mathfrak{q}^2)$ is given by the formula

   $$\mathfrak{q}^{\nu^2-\nu}(\mathfrak{q}^2 - 1)(\mathfrak{q}^4 - 1) \cdots (\mathfrak{q}^{2\nu-2} - 1)(\mathfrak{q}^\nu + 1).$$

In particular, any regular quadratic space of even dimension > 2 is isotropic and the orthogonal sum of two anisotropic spaces of dimension 2 is hyperbolic. There are also spaces and groups in odd dimensions, but those are not used here.

Therefore, $\mathrm{SO}(\overline{V}_L, \overline{q})$ is the finite group of Lie type, which is usually denoted by $\mathrm{SO}^+(n, \mathbb{F}_p)$. By the following lemma, an analogous statement holds for the lattices $L \in \tilde{\mathcal{V}}$ with label r and $\mathrm{SO}(\overline{V}^{L^{(\sharp)}}, \overline{q})$.

If (M, q) is a hyperbolic quadratic module over a ring R and $c \in R^\times$ is a unit of R, then the quadratic module $(M, c \cdot q)$ is isometric to (M, q). Therefore we have $(V_p, q) \cong (V_p, \frac{1}{p}q)$ and $(\mathbb{Z}_{p'} \otimes_{\mathbb{Z}[\frac{1}{p}]} L[\frac{1}{p}], \frac{1}{p}q) \cong (\mathbb{Z}_{p'} \otimes_{\mathbb{Z}[\frac{1}{p}]} L[\frac{1}{p}], q)$ for any prime $p' \neq p$. It follows that $(L[\frac{1}{p}], q)$ and $(L[\frac{1}{p}], \frac{1}{p}q)$ are contained in the same genus of $\mathbb{Z}[\frac{1}{p}]$-lattices and by the argument of strong approximation in chapter 3.2.3, they are even contained in the same class.

Now fix a $\mathbb{Z}[\frac{1}{p}]$-linear map

$$\varphi : L[\frac{1}{p}] \to L[\frac{1}{p}],$$

which gives an isometry $(L[\frac{1}{p}], q) \cong (L[\frac{1}{p}], \frac{1}{p}q)$.

Lemma 21. *The map*

$$\overline{\varphi} : \tilde{\mathcal{V}} \to \tilde{\mathcal{V}}, L \mapsto \varphi(L^{(\sharp)})$$

is well defined and gives an automorphism of the building $\tilde{X}_p$ with the properties

$$\begin{aligned} \tilde{l}(\overline{\varphi}(L)) &= r - \tilde{l}(L) \\ \overline{V}_{\overline{\varphi}L} &\cong \overline{V}^{L^{(\sharp)}} \\ \overline{V}^{\overline{\varphi}L^{(\sharp)}} &\cong \overline{V}_L \end{aligned}$$

for $L \in \tilde{\mathcal{V}}$. An analogous statement holds for the building X_p, clearly.

Proof. Write

$$\mathbb{Z}_p \otimes L = \bigoplus_{i=1}^{r} ((p\mathbb{Z}_p)^{\lceil \lambda_i \rceil} e_i \oplus (p\mathbb{Z}_p)^{\lceil -\lambda_i \rceil} e_{-i})$$

with a suitable canonical basis of V_p and $\lambda_1, \ldots, \lambda_r \in \frac{1}{2}\mathbb{Z}$ as in section 2.3.2. Then by lemma 4 we get

$$\mathbb{Z}_p \otimes L^{(\sharp)} = \bigoplus_{i=1}^{r} ((p\mathbb{Z}_p)^{\lceil \lambda_i - \frac{1}{2} \rceil} e_i \oplus (p\mathbb{Z}_p)^{\lceil -\lambda_i - \frac{1}{2} \rceil} e_{-i})$$

and the definition of $\overline{\varphi}$ yields

$$\mathbb{Z}_p \otimes \overline{\varphi}(L^{(\sharp)}) \cong \bigoplus_{i=1}^{r} ((p\mathbb{Z}_p)^{\lceil \lambda_i + \frac{1}{2} \rceil} e_i \oplus (p\mathbb{Z}_p)^{\lceil -\lambda_i - \frac{1}{2} \rceil} e_{-i}).$$

Hence, $\overline{\varphi}(L^{(\sharp)})$ is an element of $\tilde{\mathcal{V}}$. Using this formula, the properties of the map $\overline{\varphi}$ can be checked immediately. □

Further, since $\mathrm{O}(L[\frac{1}{p}], \frac{1}{p}q)$ and $\mathrm{O}(L[\frac{1}{p}], q)$ can be identified in a natural way, the map φ induces an automorphism of $\mathrm{O}(L[\frac{1}{p}], q)$, in particular φ gives isomorphisms

$$\underline{\mathrm{O}}_L(\mathbb{Z}) \cong \underline{\mathrm{O}}_{\overline{\varphi}(L)}(\mathbb{Z})$$
$$\underline{\mathrm{SO}}_L(\mathbb{Z}) \cong \underline{\mathrm{SO}}_{\overline{\varphi}(L)}(\mathbb{Z}).$$

4.1.1 The combinatorial structure of the building

In this section, concrete formulas are given for the number of chambers in the link of a simplex σ in X and for the Euler-Poincaré characteristic of the group Γ. We use the notations of remark 13.

Let W be the Weyl group of X, which is defined in terms of an apartment A_0 containing a chamber Ch_0, and let S be the set of canonical generators of W, which is associated with the chamber Ch_0. Since X (hence W) is of type D_r, all elements of S are conjugate in W and for $Y \subsetneq S$, the polynomial P_Y reduces to

$$P_Y(t) = \sum_{w \in \langle Y \rangle} t^{l_w},$$

where l_w denotes the length of a representation of w as reduced word by elements of S.

These polynomials are well known. If the finite Coxeter group $W_Y = \langle Y \rangle$ is reducible, i.e. if there are disjunct subsets $Y_1, Y_2 \subset Y$ with

$$Y = Y_1 \sqcup Y_2 \quad \text{and} \quad W_Y = \langle Y_1 \rangle \times \langle Y_2 \rangle,$$

then

$$P_Y = P_{Y_1} \cdot P_{Y_2}.$$

If W_Y is irreducible, then

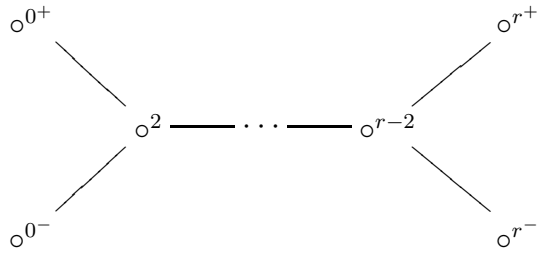
$$P_Y(t) = \prod_{i=1}^{s} (1 + t + t^2 + \ldots + t^{m_i}),$$

where $s := |Y|$ and $m_1, \ldots, m_s$ are the so called exponents of the Coxeter group W_Y, cf. [Hum90] 3.16. As a reference I give a table of the exponents of irreducible finite Coxeter groups.

Type	$m_1, \ldots, m_s$
A_s	$1, 2, \ldots, s$
B_s	$1, 3, 5, \ldots, 2s-1$
D_s	$1, 3, 5, \ldots, 2s-3, s-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$
H_3	$1, 5, 9$
H_4	$1, 11, 19, 29$
$I_2(m)$	$1, m-1$

Remark 18. For the crystallographic Coxeter groups (i.e. the types $A_s, \ldots, G_2$) the exponents can easily be computed by the criterion of [Hum90] 3.20.

Since W is an irreducible Coxeter group, the chambers of X are simplices. Therefore the generators $s \in S$ can be indexed by the set $I_l = \{0^+, 0^-, 2, \ldots, r-2, r^+, r^-\}$ in such a way, that for $i \in I_l$ the generator $s_i \in S$ fixes the unique panel of Ch_0, which contains no vertex of label i. Considering the elements of S as vertices of the Coxeter graph of W, we get the following labelled graph



Now, to any simplex $\sigma \leq \mathrm{Ch}_0$ it is associated the set

$$Y_\sigma := \{s \in S \mid s\sigma = \sigma\} = \{s_i \mid i \in I_l \setminus l(\sigma)\}$$

and the Coxeter diagram of the group $\langle Y_\sigma \rangle$ is obtained by dropping the vertices, which are labelled by the same labels as the vertices of σ, and the touching edges from the diagram above. Then the number $n_\sigma^{\mathrm{Ch}_0}$ of chambers in the link of σ is $P_{Y_\sigma}(p)$. For example in the case $n = 8$ we get the following results

$l(\sigma)$	Type	$P_{Y_\sigma}(p)$
$\{0^+\},\{0^-\},\{4^+\},\{4^-\}$	D_4	$(1+p)(1+p+p^2+p^3)^2(1+p+p^2+p^3+p^4+p^5)$
$\{2\}$	$4A_1$	$(1+p)^4$
$\{0^+,0^-\},\{0^+,4^+\},\{0^+,4^-\},$ $\{0^-,4^+\},\{0^-,4^-\},\{4^+,4^-\}$	A_3	$(1+p)(1+p+p^2+p^3)(1+p+p^2+p^3+p^4+p^5)$
$\{0^+,2\},\{0^-,2\},\{4^+,2\},\{4^-,2\}$	$3A_1$	$(1+p)^3$
$\{0^+,0^-,4^+\},\{0^+,0^-,4^-\},$ $\{0^+,4^+,4^-\},\{0^-,4^+,4^-\}$	A_2	$(1+p)(1+p+p^2+p^3)$
$\{0^+,0^-,2\},\{0^+,4^+,2\},\{0^+,4^-,2\},$ $\{0^-,4^+,2\},\{0^-,4^-,2\},\{4^+,4^-,2\}$	$2A_1$	$(1+p)^2$
$\{0^+,0^-,4^+,4^-\},\{0^+,0^-,4^+,2\},\{0^+,0^-,$ $4^-,2\},\{0^+,4^+,4^-,2\},\{0^-,4^+,4^-,2\}$	A_1	$1+p$

As an application of the ideas in section 3.2.4 the Euler-Poincaré characteristic of Γ is computed in the following. Let, as above, $r=\frac{n}{2}$ denote the Witt-index of V_p and

$$m_1=1, m_2=3,\ldots,m_{r-1}=n-3, m_r=r-1$$

denote the exponents of the Weyl group of the building X, which is of type D_r. The lattice Λ_0 is a lattice of type 0 in X, hence p is an oriflamme place of Λ_0 and we have $|S_f|=N_{\Lambda_0}=1$. By lemma 20 condition **(R)** is satisfied and the $\mathbb{Z}[\frac{1}{p}]$-lattice $L[\frac{1}{p}]$ is regular. Therefore, theorem 3.16 and corollary 9 asserts, that the Euler-Poincaré characteristic of $\Gamma=\underline{\mathrm{Spin}}_{\Lambda_0}(\mathbb{Z}[\frac{1}{p}])$ is given by the formula

$$\chi(\Gamma)=M_X\cdot n_{\Lambda_0}^{\mathrm{Ch}_0}\cdot\tilde{K}_{\Lambda_0}.$$

The link of Λ_0 in X is a spherical building of type D_r and, as described above, the number of chambers in the link of Λ_0 is

$$n_{\Lambda_0}^{\mathrm{Ch}_0}=\prod_{i=1}^{r}(1+p+p^2+\ldots+p^{m_i}).$$

The constant

$$M_X:=\sum_{\sigma\in\Sigma_0}(-1)^{\dim\sigma}\frac{1}{n_\sigma^{\mathrm{Ch}_0}},$$

where Σ_0 denotes the set of faces of a fixed chamber Ch_0 of X, is by [Ser71] thm. 7

$$M_X=\prod_{i=1}^{r}\frac{1-p^{m_i}}{1+p+p^2+\ldots+p^{m_i}},$$

cf. remark 15. Hence

$$\chi(\Gamma)=\prod_{i=1}^{r}(1-p^{m_i})\cdot\tilde{K}_{\Lambda_0}.$$

But the Minkowski-Siegel mass-constant of an even unimodular lattice Λ_0 is well known to be

$$\tilde{K}_{\Lambda_0}=\frac{1}{2\pi}\prod_{m=3}^{n}\frac{\Gamma(\frac{m}{2}+1)}{m\,\pi^{m/2}}\cdot\prod_{i=1}^{r}\zeta(m_i+1),$$

where Γ denotes the Euler gamma-function and ζ the Riemann zeta-function, cf. [Kne02] §32-34.

For example, for $p = 2$ we get:

n=8:

$$\chi(\Gamma) = \frac{7 \cdot 31}{2^{14} \cdot 3^5 \cdot 5^2} \approx 2.18019 \cdot 10^{-6}$$

n=16:

$$\chi(\Gamma) = \frac{23 \cdot 31 \cdot 73 \cdot 89 \cdot 127^2 \cdot 691 \cdot 8191}{2^{30} \cdot 3^{10} \cdot 5^4 \cdot 11 \cdot 13} \approx 0.0746271$$

n=24:

$$\begin{aligned} \chi(\Gamma) &= \frac{7 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89^2 \cdot 127^2 \cdot 131 \cdot 151 \cdot 283 \cdot 337 \cdot 593 \cdot 617}{2^{45} \cdot 3^{17} \cdot 5^7 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19} \\ &\quad \underline{691^2 \cdot 3617 \cdot 8191 \cdot 43867 \cdot 131071 \cdot 524287} \\ &\approx 1.81152 \cdot 10^{25} \end{aligned}$$

Remark 19. We will see in the next section, that in dimension 8 and for $p = 2$ the space $\Gamma \setminus X$ consists of only one chamber. But for $n = 16$ and $n = 24$ there is very little known about the structure of the orbit space $\Gamma \setminus X$.

For $n = 24$ it is proven in [Nie73][1], that there are exactly 24 classes of even unimodular lattices of dimension 24. 23 of them contain some vector x with $q(x) = 1$ and are therefore fixed by the associated reflection τ_x. It follows from this, that for each such lattice L, the image of the spin group $\mathrm{Spin}(L, q)$ has index 2 in $\mathrm{O}(L, q)$. Therefore the class of L represents exactly one vertex of $\Gamma \setminus X$ of label 0^+ and 0^- respectively. The other class is given by the famous *Leech lattice* Λ_{24}. It is proven in [Con69] thm. 9, that the automorphism group of the Leech lattice is perfect (i.e. it has no abelian factor group). From this it follows, that the image of $\mathrm{Spin}(\Lambda_{24}, q)$ is exactly $\mathrm{O}(\Lambda_{24}, q)$. Thus, the class of Λ_{24} splits into two Γ-obits of type 0^+ and other two Γ-obits of type 0^-. Therefore $\Gamma \setminus X$ contains 25 vertices of type 0^+ and 0^- respectively. By symmetry the same is true for vertices of type 12^+ and 12^-.

For $n = 16$ and $p = 2$ it can be said a bit more. It is well known that there are exactly two classes of even unimodular lattices, whose automorphism groups are generated by reflections. This implies as for $n = 24$, that the space $\Gamma \setminus X$ contains exactly two vertices with label 0^+, 0^-, 8^+ and 8^- respectively. For $p = 2$ it is easy to check that any root lattice L_{4D_4} of type $4D_4$ contained in $L[\frac{1}{2}]$ represents a vertex of label 4 in X. The classes in the genus of L_{4D_4} are computed by [SV94]. The situation here is very similar to the case of even unimodular lattices in dimension 24 mentioned above. Indeed, the authors proved, that there are exactly 24 classes of lattices in the genus of L_{4D_4}, where two types can be distinguished. Any lattice

[1] A simplified proof can be found in [Ven99].

L which is contained in 23 of the 24 classes contains some elements x, y, called roots in [SV94], with $q(x) = 1$ and $q(y) = 2$ such that the reflections τ_x and τ_y are contained in $\mathrm{O}(L, q)$. It follows from this, that any of this 23 classes corresponds to an unique class of label 4 in $\Gamma \setminus X$. The remaining class is the class of the so called *Barnes-Wall lattice* Λ_{16} (see [CS99] ch. 4.10 for a short description and further references). This lattice can be constructed from the Leech lattice as a submodule in Λ_{24} fixed by some element σ of order 2 in $\mathrm{O}(\Lambda_{24}, q)$. The automorphism group of Λ_{16} in this construction is given by the centralizer of σ modulo $\langle\sigma\rangle$. It follows from this, that exactly as $\mathrm{O}(\Lambda_{24}, q)$, the group $\mathrm{O}(\Lambda_{16}, q)$ consists only of rotations with trivial spinor norm. Therefore, the class of Λ_{16} splits off into 4 Γ-orbits. Summing up, we have found that $\Gamma \setminus X$ contains exactly 27 vertices of label 4. Because of this small number it seems to be possible (and very interesting) to determine the space $\Gamma \setminus X$ completely.

4.2 The case of dimension 8

In this chapter, the case $\dim V = 8$ and $p = 2$ is studied in detail. So let $V, L[\frac{1}{2}], \Gamma, V_2, X = X_2, \tilde{X} = \tilde{X}_2$ etc. be as above. To compute the Γ-equivariant cohomology of X, we have to determine the orbit space $\Gamma \setminus X$ and the stabilizers Γ_σ in Γ of each simplex σ of X. But this special case is simplified by the following fact:

Theorem 12. *The group Γ acts transitively on the chambers of X.*

Therefore, the action of the group Γ on X is used by [Kan85] to construct certain finite geometries, called "geometries, that are almost buildings" (GAB). But the behavior of Γ described in the theorem is very singular. All S-arithmetic groups of this type are listed in [KLT87], cf. example 1 at the end of section 3.2.3. The proof of theorem 12 is closely related to the classical fact, that there is only one class of even unimodular lattices in dimension 8. This is proven by [Mor38] with reduction theory. Kneser found another proof with his method of neighbouring lattices [Kne57]. This proof is closely related with our situation, because it reflects exactly the geometry of the building. Indeed, let Λ be an even unimodular lattice, which is contained in $L[\frac{1}{2}]$ and therefore represents a vertex of X. Then another even unimodular lattice Λ' in V is a neighbour of Λ, if and only if Λ' is contained in $L[\frac{1}{2}]$ and the pair $\{\Lambda, \Lambda'\}$ represents a 1-simplex of X. On the other hand, the theorem implies, that the class number of any genus, that represents a type of vertices of X (or even $\tilde{X}$) is 1 and, by the geometry of the building, this is proved all at once.

In the following, I determine the stabilizers Γ_σ with $\sigma \in \Sigma(X)$ and, by the way, give a complete proof of theorem 12.

4.3 The stabilizers

Put

$$\Lambda_{0^+} := \{\sum_{i=1}^{8} \lambda_i x_i \mid \text{all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum_{i=1}^{8} \lambda_i \equiv 0 \bmod 2\}.$$

It is easy to check, that Λ_{0^+} is an even unimodular lattice, which is generated by the set $\Phi := \{x \in \Lambda_{0^+} \mid q(x) = 1\}$ and that this is a root system of type E_8 in the Euclidean space $(\mathbb{R} \otimes_{\mathbb{Q}} V, b)$. Hence $\mathrm{Aut}(\Lambda_{0^+}) = \underline{\mathrm{O}}_{\Lambda_{0^+}}(\mathbb{Z})$ is isomorphic to the group of automorphisms of Φ, which is a Coxeter group of type E_8. Therefore $\mathrm{Aut}(\Lambda_{0^+})$ is generated by the set $\{\tau_x \mid x \in \Phi\}$ of reflections, which are defined by the roots. For a more detailed description of the lattice Λ_{0^+} see [CS99] pp. 120-123.

Since ${\Lambda_{0^+}}^{(\sharp)} = \Lambda_{0^+}$, it follows, that $\overline{V}^{\Lambda_{0^+}} = (0)$ and $\overline{V}_{\Lambda_{0^+}} = \frac{1}{2}\Lambda_{0^+}/\Lambda_{0^+}$. Now it is easy to check, that under the natural projection

$$\rho = \rho_{\Lambda_{0^+}} : \frac{1}{2}\Lambda_{0^+} \to \overline{V}_{\Lambda_{0^+}}$$

the set $\frac{1}{2}\Phi = \{\frac{1}{2}x \mid x \in \Phi\}$ is mapped surjectively onto the set of anisotropic vectors of $\overline{V}_{\Lambda_{0^+}}$ in such a way, that if $x, x' \in \frac{1}{2}\Phi$ with $\rho(x) = \rho(x')$, then $x = \pm x'$.

Now, consider the reduction map

$$r_{\Lambda_{0^+}} : \mathrm{O}(\Lambda_{0^+}, q) \to \mathrm{O}(\overline{V}_{\Lambda_{0^+}}, \overline{q}).$$

Since $\mathrm{O}(\overline{V}_{\Lambda_{0^+}}, \overline{q})$ is generated by the symmetries τ_y, where y runs through the anisotropic elements of $\overline{V}_{\Lambda_{0^+}}$ and $r_{\Lambda_{0^+}}(\tau_x) = \tau_{\rho(x)}$ for $x \in \frac{1}{2}\Phi$, it follows, that the reduction map gives exact sequences

$$\begin{aligned} 1 \to < \pm 1 > \to \quad & \mathrm{O}(\Lambda_{0^+}, q) \quad \xrightarrow{r_{\Lambda_{0^+}}} \mathrm{O}(\overline{V}_{\Lambda_{0^+}}, \overline{q}) \to 1 \\ 1 \to < \pm 1 > \to \quad & \mathrm{SO}(\Lambda_{0^+}, q) \quad \xrightarrow{r_{\Lambda_{0^+}}} \mathrm{SO}(\overline{V}_{\Lambda_{0^+}}, \overline{q}) \to 1. \end{aligned}$$

Thus, with z and $U = \langle -1, z\rangle \subset \mathrm{Spin}(\Lambda_{0^+}, q)$ as in section 4.1, we get the fundamental

Proposition 15. *There is an exact sequence of groups*

$$1 \to U \to \mathrm{Spin}(\Lambda_{0^+}, q) \to \mathrm{SO}(\overline{V}_{\Lambda_{0^+}}, \overline{q}) \to 1,$$

where U is an elementary abelian subgroup of order 4, which is contained in the center of $\mathrm{Spin}(\Lambda_{0^+}, q)$.

Remark 20. By symmetry, the same is true for any $\mathbb{Z}$-lattice in $\Lambda \subset L[\frac{1}{2}]$, with $(\Lambda, q) \cong (\Lambda_{0^+}, q)$ or $(\Lambda, \frac{1}{2}q) \cong (\Lambda_{0^+}, q)$.

Now, I proceed with the proof of theorem 12.

Proof. By proposition 15, the group $\Gamma_{0^+} := \mathrm{Spin}(\Lambda_{0^+}, q)$ considered as the stabilizer of the vertex $\{\Lambda_{0^+}\}$ of X in Γ acts as the full group $\mathrm{SO}(\overline{V}_{\Lambda_{0^+}}, \overline{q})$ on the link of $\{\Lambda_{0^+}\}$. Hence, Γ_{0^+} acts transitively on the set of chambers in X, which contain the vertex $\{\Lambda_{0^+}\}$.

Now, consider the lattice

$$\Lambda_{0^-} := \{\sum_{i=1}^{8} \lambda_i x_i \mid \text{all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum_{i=1}^{8} \lambda_i \equiv 2\lambda_8 \mathrm{mod}\ 2\}.$$

Let τ_{x_8} be the symmetry, which is defined by the basis element x_8. Then we get immediately $\tau_{x_8}\Lambda_{0^+} = \Lambda_{0^-}$, hence Λ_{0^-} is also a root lattice of type E_8, which is contained in $L[\frac{1}{2}]$. Thus Λ_{0^-} represents another vertex $\{\Lambda_{0^-}\}$ of X. By the remark after proposition 15, the stabilizer of this vertex acts also transitively on the chambers in the link $\mathrm{lk}(\{\Lambda_{0^-}\})$. Further, we have

$$\Lambda_1 := \Lambda_{0^+} \cap \Lambda_{0^-} = \{\sum_{i=1}^{8} \lambda_i x_i \mid \text{all } \lambda_i \in \mathbb{Z} \text{ and } \sum_{i=1}^{8} \lambda_i \equiv 0 \mathrm{mod}\ 2\}$$

and this lattice can be written in the form

$$\Lambda_1 = (\Lambda_{0^+} + \mathbb{Z}x_1)^{(\sharp)}.$$

But the image of x_1 in $\overline{V}_{\Lambda_{0^+}}$ is isotropic. Therefore, Λ_1 represents a vertex of label 1 in $\tilde{X}$ and it follows from the oriflamme construction, that the pair $\{\Lambda_{0^+}, \Lambda_{0^-}\}$ represents a 1-simplex in X. Hence, the theorem follows from the following, purely building-theoretical lemma. □

Lemma 22. *Let $(\Delta, \mathcal{A})$ be a building, which is a simplicial complex, and let G be a group of automorphisms of Δ. Assume, that x_+, x_- are the vertices of a 1-simplex of Δ and that the stabilizers G_{x_+}, G_{x_-} of x_+ and x_- act transitively on the chambers of the links of x_+ and x_-, respectively. Then G acts transitively on the set of chambers of X.*

Proof. Since the chambers are the maximal simplices of Δ, there is a chamber Ch that contains both x_+ and x_-. Assume, that G does not act transitively on the set of chambers of Δ. Then, choose a chamber Ch$'$ in the set of chambers, which are not contained in the G-orbit of Ch, such that the length d of a minimal gallery from Ch to Ch$'$ is minimal, and let

$$\mathrm{Ch} = \mathrm{Ch}_0, \mathrm{Ch}_1, \ldots, \mathrm{Ch}_d = \mathrm{Ch}'$$

be a gallery from Ch to Ch′ of minimal length. Since X is a simplicial complex, the common panel $\mathrm{Ch}_0 \cap \mathrm{Ch}_1$ of Ch_0 and Ch_1 contains at least one of the vertices x_+ and x_-, say x_+. But in G_{x_+}, there is an automorphism φ, which maps Ch_1 to Ch_0. It follows, that

$$\mathrm{Ch}_0 = \varphi\mathrm{Ch}_1, \varphi\mathrm{Ch}_2, \ldots, \varphi\mathrm{Ch}_d$$

is a gallery in X of length at most $d-1$. Hence, by the choice of Ch′, there is an automorphism $\psi \in G$ with $\psi\varphi\mathrm{Ch}' = \psi\varphi\mathrm{Ch}_d = \mathrm{Ch}$, contradiction. □

The lattice Λ_1 above is a so called root lattice of type D_8, i.e. the set $\{x \in \Lambda_1 \mid q(x) = 1\} = \{\pm x_i \pm x_j \mid i \neq j\}$ of short vectors in Λ_1 is a root system of type D_8 in $(\mathbb{R} \otimes_{\mathbb{Q}} V, b)$ and generates Λ_1. Since $q(x_1) = q(\frac{1}{2}(x_1 + \ldots + x_4)) = b(x_1, \frac{1}{2}(x_1 + \ldots + x_4)) = \frac{1}{2}$, the images of $x_1, \frac{1}{2}(x_1 + \ldots + x_4) \in \frac{1}{2}\Lambda_{0^+}$ generate a totally isotropic subspace of dimension 2 in $\overline{V}_{\Lambda_{0^+}}$. Hence,

$$\Lambda_2 := (\Lambda_{0^+} + \mathbb{Z}x_1 + \mathbb{Z}\frac{1}{2}(x_1 + \ldots + x_4))^{(\sharp)}$$

represents a vertex of label 2 in X. Note that

$$\Lambda_2 = \{\sum_{i=1}^{8} \lambda_i x_i \mid \text{all } \lambda_i \in \mathbb{Z},\ \sum_{i=1}^{4} \lambda_i \equiv 0 \bmod 2 \text{ and } \sum_{i=5}^{8} \lambda_i \equiv 0 \bmod 2\}$$

is a root lattice of type $2D_4$. Hence it follows from theorem 12

Corollary 10. *The resp. genera of root lattices of type E_8, D_8 and $2D_4$ have class number* 1.

It remains to give a description of the stabilizers Γ_σ for $\sigma \in \Sigma(X)$. Since $\Gamma_{g\sigma} = g\Gamma_\sigma g^{-1}$ for $g \in \Gamma$, it is enough to consider the groups Γ_σ, where σ runs through a set of representatives for the Γ-orbits in $\Sigma(X)$, and such a set is given by the set of faces of a fixed chamber Ch_0 of X by theorem 12. Put

$$\mathrm{Ch}_0 := \{\Lambda_{0^+}, \Lambda_{0^-}, \Lambda_2, \Lambda_{4^+}, \Lambda_{4^-}\},$$

where the vertices $\Lambda_i \in \mathcal{V}$ are indexed such that $l(\Lambda_i) = i$ for $i \in \{0^+, 0^-, 2, 4^+, 4^-\}$. Using lemma 21, we see, that the groups

$$\mathrm{SO}(\overline{V}_{\Lambda_{0^+}}, \overline{q}),\ \mathrm{SO}(\overline{V}_{\Lambda_{0^-}}, \overline{q}),\ \mathrm{SO}(\overline{V}^{\Lambda_{4^+}}, \overline{q}),\ \mathrm{SO}(\overline{V}^{\Lambda_{4^-}}, \overline{q})$$

are isomorphic groups of type $\mathrm{SO}^+(8, \mathbb{F}_2)$.

Now let σ be a face of Ch_0. Unless $\sigma = \{\Lambda_2\}$, σ contains one of the lattices $\Lambda_{0^+}, \Lambda_{0^-}, \Lambda_{4^+}, \Lambda_{4^-}$, say Λ.

If $\Lambda \in \{\Lambda_{0^+}, \Lambda_{0^-}\}$, then σ corresponds to an oriflamme flag F_σ of totally isotropic subspaces in the quadratic space $(\overline{V}_\Lambda, \overline{q})$. Let P_σ be the stabilizer of F_σ, which is canonically isomorphic to the group of $\mathbb{F}_2$-rational points of a $\mathbb{F}_2$-rational parabolic subgroup of the algebraic group $\underline{\mathrm{SO}}_\Lambda \times \mathrm{Spec}\mathbb{F}_2$, cf. section 1.13. Since the action of Γ preserves labels and $U \subset \Gamma$ acts trivially on X, we have $U \subset \Gamma_\sigma \subset \Gamma_\Lambda$ and the exact sequence of proposition 15 restricts to

$$1 \to U \to \Gamma_\sigma \to P_\sigma \to 1.$$

Analogously, if $\Lambda \in \{\Lambda_{4^+}, \Lambda_{4^-}\}$, then σ corresponds to an oriflamme flag of totally isotropic subspaces in $(\overline{V}^\Lambda, \overline{q})$ and we get a central extension

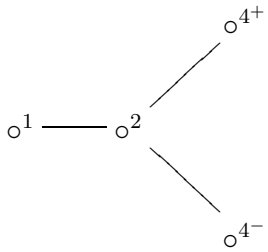
$$1 \to U \to \Gamma_\sigma \to P_\sigma \to 1$$

with a parabolic subgroup $P_\sigma \subset \mathrm{SO}(\overline{V}^\Lambda, \overline{q})$.

I continue to assume, that $\sigma \in \Sigma(X)$ is no vertex with label 2. As above we fix a vertex $\{\Lambda\}$ of σ of label $0^+, 0^-, 4^+$ or 4^- and write $\overline{V}$ for $\overline{V}_\Lambda$ resp. $\overline{V}^\Lambda$ such that $(\overline{V}, \overline{q})$ is a hyperbolic quadratic space of dimension 8 over the field $\mathbb{F}_2$ and Γ_Λ/U is isomorphic to $\mathrm{SO}(\overline{V}, \overline{q})$. Let $F := \{U_1, U_2, U_{4^+}, U_{4^-}\}$ be an oriflamme flag in $(\overline{V}, \overline{q})$ (which means that $U_1, U_2, U_{4^+}, U_{4^-}$ are totally isotropic subspaces of $(\overline{V}, \overline{q})$, $\dim U_1 = 1$, $\dim U_2 = 2$, $\dim U_{4^+} \cap U_{4^-} = 3$, $\dim U_{4^+} = \dim U_{4^-} = 4$ and $U_1 \subset U_2 \subset U_{4^+} \cap U_{4^-}$.)

Remark 21. Since $\mathrm{SO}(\overline{V}, \overline{q})$ is isomorphic to the (simple) Chevalley group of type $D_4(2)$, there is some more symmetry, called 'triality'. This is provided by the following more general principle. Let Φ be an irreducible root system, such that all roots of Φ have the same length. Let $\underline{\mathrm{G}}_\mathbb{Z}$ be the Chevalley group scheme over $\mathbb{Z}$, which is associated to Φ. Now fix a field K, a maximal K-split torus $\underline{\mathrm{T}}$ in $\underline{\mathrm{G}} := \underline{\mathrm{G}}_\mathbb{Z} \times \mathrm{Spec}\, K$ and a minimal K-parabolic subgroup $\underline{\mathrm{P}}$ of $\underline{\mathrm{G}}$ containing $\underline{\mathrm{T}}$. Then we can identify Φ with the root system $\Phi(\underline{\mathrm{T}}, \underline{\mathrm{G}})$ of $\underline{\mathrm{G}}$ associated with $\underline{\mathrm{T}}$. Let Φ^+ be the subset of positive roots in Φ, which is determined by $\underline{\mathrm{P}}$, and Δ the unique root basis in Φ^+. Then the vertices of the Dynkin diagram of Φ can be indexed by the elements of Δ and any automorphism π_0 of the Dynkin diagram induces an automorphism π of the root system Φ. It is well known, that for any such π_0 there is an automorphism x_π of the algebraic group $\underline{\mathrm{G}}$ defined over K, which fixes $\underline{\mathrm{T}}$ and $\underline{\mathrm{P}}$ and induces the map π on the root system $\Phi = \Phi(\underline{\mathrm{T}}, \underline{\mathrm{G}})$, [Car72] 12.2.3. Recall, that to any subset $S \subset \Delta$ it is associated a standard parabolic subgroup $\underline{\mathrm{P}}_S$ over K. It follows, that $\underline{\mathrm{P}}_S$ is isomorphic to $x_\pi \underline{\mathrm{P}}_S = \underline{\mathrm{P}}_{\pi S}$ over K. In particular, $\underline{\mathrm{P}}_S(K) \cong \underline{\mathrm{P}}_{\pi S}(K)$ as abstract groups.

Applying this to the group $\mathrm{SO}(\overline{V},\overline{q})$ and its Dynkin diagram



we find for example, that the stabilizers of U_1, U_{4^+} and U_{4^-} are isomorphic. Analyzing all these symmetries we get

Proposition 16. *Let $\sigma \in \Sigma(X)$ be a simplex, which contains no vertex of label 2. Then, Γ_σ/U is isomorphic to the stabilizer in $\mathrm{SO}(\overline{V},\overline{q})$ of one of the following subflags of F.*

$\{U_1\},\{U_2\}$	*for* $dim(\sigma)=1$
$\{U_1,U_2\},\{U_1,U_{4^+}\}$	*for* $dim(\sigma)=2$
$\{U_1,U_2,U_{4^+}\}$	*for* $dim(\sigma)=3$
$\{U_1,U_2,U_{4^+},U_{4^-}\}$	*for* $dim(\sigma)=4$.

Finally, consider the vertex $\sigma=\{\Lambda_2\}$. Then, as we have seen, Λ_2 is a root lattice of type $2D_4$ and we can assume, that Λ_2 is given as in (4.3). The irreducible components of Λ_2 are the two submodules $\Lambda_2^{(1)}=\Lambda_2\cap\bigoplus_{i=1}^4\mathbb{Q}x_i$ and $\Lambda_2^{(2)}=\Lambda_2\cap\bigoplus_{i=5}^8\mathbb{Q}x_i$. Since $\Lambda_2^{(i)}$ is a root lattice of type D_4, the group $\mathrm{O}(\Lambda_2^{(i)},q)$ is isomorphic to the automorphism group of a root system of type D_4. This group can be described as extension of groups

$$1\to W_{D_4}\to \mathrm{O}(\Lambda_2^{(i)},q)\to S_3\to 1,$$

where W_{D_4} is the Weyl group of the root system and the symmetric group S_3 acts as automorphism group on the Coxeter diagram of the root system, cf. [CS99], p. 118. This extension splits, because the subgroup of $\mathrm{O}(\Lambda_2^{(1)},q)$, which is generated by the symmetries $\tau_{2x_1},\tau_{2x_4},\tau_{x_1+x_2+x_3+x_4}$, is mapped isomorphically onto the factor group S_3. (Alternatively, you find, that the set $\Phi_2^{(i)}=\{x\in\Lambda_2^{(i)}\mid q(x)=1 \text{ or } q(x)=2\}$ is a root system of type F_4. Therefore $\mathrm{O}(\Lambda_2^{(i)},q)$ can also be described as the group of automorphisms of this root system, which is a Coxeter group of type F_4. Hence we have

$$\mathrm{O}(\Lambda_2^{(i)},q)\cong W_{F_4}\cong W_{D_4}\rtimes S_3.)$$

Since the decomposition of a $\mathbb{Z}$-lattice into irreducible components is unique, [Kne02] (27.2), $\mathrm{O}(\Lambda_2,q)$ is an extension of groups

$$1\to \mathrm{O}(\Lambda_2^{(1)},q)\times\mathrm{O}(\Lambda_2^{(2)},q)\to \mathrm{O}(\Lambda_2,q)\to S_2\to 1, \tag{4.1}$$

where the quotient permutes the irreducible components. Clearly this extension splits and a splitting is given by the map, which maps the non trivial element of S_2 onto the element of $\mathrm{O}(\Lambda_2, q)$, which has the matrix

$$\begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}$$

with respect to the basis $x_1, \ldots, x_8$. As $\mathrm{O}(\Lambda_2, q)$ contains reflections, $\mathrm{SO}(\Lambda_2, q)$ is a subgroup of index 2, and since the root system $\Phi_2^{(1)} \sqcup \Phi_2^{(2)}$ contains "long" roots x with $q(x) = 2$ and "short" roots x with $q(x) = 1$, it follows that $\mathrm{SO}(\Lambda_2, q)$ consists of elements with spinor norm 1 and 2. Hence we get an exact sequence

$$1 \to \langle \pm 1 \rangle \to \Gamma_\sigma \to \mathrm{SO}(\Lambda_2, q) \xrightarrow{\theta} \mathbb{F}_2 \to 1$$

for $\Gamma_\sigma = \mathrm{Spin}((L[\frac{1}{2}], q), \Lambda_2)$.

Finally, we determine the orders of the stabilizers Γ_σ. Therefore, let σ be again an arbitrary face of the fixed chamber Ch_0. Since the action of Γ on X preserves the labelling of X, Γ_{Ch_0} is a subgroup of Γ_σ. Moreover, since Γ acts transitively on the chambers of X, the index $[\Gamma_\sigma : \Gamma_{\mathrm{Ch}_0}]$ is the number $n_\sigma^{\mathrm{Ch}_0}$ of chambers, which contain σ as a face, and this number can be computed as described in section 4.1.1. From proposition 15, it follows that $\Gamma_{\Lambda^{0+}}$ has order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ and the formula

$$|\Gamma_\sigma| = \frac{[\Gamma_\sigma : \Gamma_{\mathrm{Ch}_0}]}{[\Gamma_{\Lambda^{0+}} : \Gamma_{\mathrm{Ch}_0}]} = \frac{P_{Y_\sigma}(2)}{P_{\{0^+\}}(2)} |\Gamma_{\mathrm{Ch}_0}|$$

together with the table at the end of section 4.1.1 yields

$l(\sigma)$	$\sharp\Gamma_\sigma$
$\{0^+\}, \{0^-\}, \{4^+\}, \{4^-\}$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$\{2\}$	$2^{14} \cdot 3^4$
$\{0^+, 0^-\}, \{0^+, 4^+\}, \{0^+, 4^-\}, \{0^-, 4^+\},$ $\{0^-, 4^-\}, \{4^+, 4^-\}$	$2^{14} \cdot 3^2 \cdot 5 \cdot 7$
$\{0^+, 2\}, \{0^-, 2\}, \{4^+, 2\}, \{4^-, 2\}$	$2^{14} \cdot 3^3$
$\{0^+, 0^-, 4^+\}, \{0^+, 0^-, 4^-\}, \{0^+, 4^+, 4^-\},$ $\{0^-, 4^+, 4^-\}$	$2^{14} \cdot 3 \cdot 7$
$\{0^+, 0^-, 2\}, \{0^+, 4^+, 2\}, \{0^+, 4^-, 2\}, \{0^-, 4^+, 2\},$ $\{0^-, 4^-, 2\}, \{4^+, 4^-, 2\}$	$2^{14} \cdot 3^2$
$\{0^+, 0^-, 4^+, 4^-\}, \{0^+, 0^-, 4^+, 2\}, \{0^+, 0^-, 4^-, 2\},$ $\{0^+, 4^+, 4^-, 2\}, \{0^-, 4^+, 4^-, 2\}$	$2^{14} \cdot 3$
$\{0^+, 0^-, 4^+, 4^-, 2\}$	2^{14}

Therefore, if k is a field of characteristic different from 2, 3, 5 or 7, then the cohomology ring of the stabilizer Γ_σ of any simplex σ of X with constant coefficients in k is trivial and, as we have seen in the example after proposition 11, the

cohomology ring of Γ

$$H^*(\Gamma, k) \cong H^*(\Gamma \setminus X, k) \cong k$$

is also trivial, because $\Gamma \setminus X$ consists of a single simplex. In the following, we shall compute the ring $H^*(\Gamma, k)$ for the characteristics 7 and 5, which are the easiest of the non-trivial cases.

4.4 Cohomology of Γ at the prime 7

4.4.1 The cohomology rings of the stabilizers

Let k be a field of characteristic 7. In the following, k and $\mathbb{Z}$ are considered as trivial G-modules for any group G which appears, i.e. the action of G is given by

$$g.z = z \qquad \text{for } g \in G, z \in k \text{ or } z \in \mathbb{Z}.$$

In the following, I will compute the cohomology ring of the group $\Gamma = \mathrm{Spin}(L[\frac{1}{2}], q)$ with coefficients in the field k and the 7-part

$$H^q(\Gamma, \mathbb{Z})_{[7]} = \begin{cases} H^0(\Gamma, \mathbb{Z}) & \text{for } q = 0 \\ H^q(\Gamma, \mathbb{Z})_{(7)} & \text{for } q \neq 0 \end{cases}$$

of the integral cohomology ring of Γ.

By section 3.1.2 the cohomology ring $H^*(\Gamma, k)$ can be described in terms of the cohomology rings $H^*(\Gamma_\sigma, k)$ of the stabilizers of simplices σ in X and the restriction maps

$$H^*(\Gamma_\sigma, k) \to H^*(\Gamma_\tau, k),$$

if σ is a face of the simplex τ of X. The other ingredient for the calculation is the simplicial chain complex of the orbit space $\Gamma\backslash X$, which, in this case, can be identified with the chamber $\mathrm{Ch}_0 = \{\Lambda_{0^+}, \Lambda_{0^-}, \Lambda_2, \Lambda_{4^+}, \Lambda_{4^-}\}$. An analogous statement holds for the integral cohomology. Therefore consider first the cohomology rings of the stabilizers.

As above, let $(\overline{V}, \overline{q})$ be a hyperbolic quadratic space of dimension 8 over $\mathbb{F}_2$. Fix a canonical basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, e_{-1}, e_{-2}, e_{-3}, e_{-4}\}$ of $\overline{V}$, i.e.

$$\begin{aligned} q(e_i) &= \quad 0 \qquad \text{for } i \in I = \{\pm 1, \ldots, \pm 4\} \\ b(e_i, e_j) &= \begin{cases} 1 & \text{for } i = -j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Further, put $U_1 = \mathbb{F}_2 e_4$ and $U_4 = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2 \oplus \mathbb{F}_2 e_3 \oplus \mathbb{F}_2 e_4$.

The order of a stabilizer Γ_σ of a simplex σ of X is divisible by 7, if and only if $\dim \sigma \leq 2$ and σ contains no vertex of label 2. In these cases the 7-Sylow subgroup of Γ_σ is cyclic of order 7 and the quotient Γ_σ / U is isomorphic to one of the following groups:

1. $G := \mathrm{SO}(\overline{V}, \overline{q})$, if $\dim \sigma = 0$.
2. the stabilizer G_1 of the line $U_1 = \mathbb{F}_2 e_4$ in $\mathrm{SO}(\overline{V}, \overline{q})$ for $\dim \sigma = 1$.
3. the stabilizer of the flag $U_1 \subset U_4$ in $\mathrm{SO}(\overline{V}, \overline{q})$, if σ is of dimension 2.

First, I consider the groups $\mathrm{Gl}(n, \mathbb{F}_2)$ for $n = 3, 4, 5$. These groups have order $7 \cdot 6 \cdot 4$, $15 \cdot 14 \cdot 12 \cdot 8$ and $31 \cdot 30 \cdot 28 \cdot 24 \cdot 16$, respectively. Thus, their 7-Sylow subgroups are cyclic of order 7. Let C be such a group and fix embeddings

$$C \hookrightarrow \mathrm{Gl}(3, \mathbb{F}_2) \hookrightarrow \mathrm{Gl}(4, \mathbb{F}_2) \hookrightarrow \mathrm{Gl}(5, \mathbb{F}_2).$$

Then by proposition 6 we get injective homomorphisms

$$H^*(\mathrm{Gl}(5, \mathbb{F}_2), k) \hookrightarrow H^*(\mathrm{Gl}(4, \mathbb{F}_2), k) \hookrightarrow H^*(\mathrm{Gl}(3, \mathbb{F}_2), k) \hookrightarrow H^*(C, k)$$

and

$$H^*(\mathrm{Gl}(5, \mathbb{F}_2), \mathbb{Z})_{[7]} \hookrightarrow H^*(\mathrm{Gl}(4, \mathbb{F}_2), \mathbb{Z})_{[7]} \hookrightarrow H^*(\mathrm{Gl}(3, \mathbb{F}_2), \mathbb{Z})_{[7]} \hookrightarrow H^*(C, \mathbb{Z}).$$

With the notation $H^*(C, k) = k[x, y]$, where $\deg x = 2$, $\deg y = 1$ and $y^2 = 0$, (resp. $H^*(C, \mathbb{Z}) = \mathbb{Z}[\xi]$ with $\deg \xi = 2$ and $7\xi = 0$) of chapter 3 we can identify the rings $H^*(\mathrm{Gl}(n, \mathbb{F}_2), k)$ (resp. $H^*(\mathrm{Gl}(n, \mathbb{F}_2), \mathbb{Z})_{[7]}$) with their images in $k[x, y]$ (resp. $\mathbb{Z}[\xi]$).

Proposition 17. *For $n = 3, 4, 5$ one has*

$$H^*(\mathrm{Gl}(n, \mathbb{F}_2), k) = k[x^3, x^2 y]$$

and

$$H^*(\mathrm{Gl}(n, \mathbb{F}_2), \mathbb{Z})_{[7]} = \mathbb{Z}[\xi^3].$$

Proof. It follows from the lemma of Swan (lemma 15), that

$$H^*(\mathrm{Gl}(n, \mathbb{F}_2), k) = H^*(\mathcal{N}_{\mathrm{Gl}(n, \mathbb{F}_2)}(C), k) = H^*(C, k)^{\mathcal{N}_{\mathrm{Gl}(n, \mathbb{F}_2)}(C)}$$

and equally for the 7-part of integral cohomology. Here, $\mathcal{N}_{\mathrm{Gl}(n, \mathbb{F}_2)}(C)$ acts as usually by conjugation on C and this action induces a homomorphism $\mathcal{N}_{\mathrm{Gl}(n, \mathbb{F}_2)}(C) \to \mathrm{Aut}(C)$ of groups. By proposition 8.2., we have to show that the image of this map is the unique subgroup of order 3 in $\mathrm{Aut}(C) \cong \mathbb{F}_7^\times \cong \mathbb{Z}/6\mathbb{Z}$. In other words, if g is a generator of C, then g and g^{-1} are not conjugate in $\mathrm{Gl}(n, \mathbb{F}_2)$, but g and g^2 are.

First, I claim that g and g^{-1} are not conjugate in $\mathrm{Gl}(n, \mathbb{F}_2)$. Since g has order 7, the minimal polynomial m_g of g is a divisor of degree at most n of the polynomial

$$X^7 - 1 = (X - 1)(X^3 + X + 1)(X^3 + X^2 + 1) \in \mathbb{F}_2[X].$$

Since g is not the identity, it follows, that m_g is a multiple of $X^3 + X + 1$ or of $X^3 + X^2 + 1$. Therefore the characteristic polynomial of g is either $p_1 = (X - 1)^{n-3}(X^3+X+1)$ or $p_2 = (X-1)^{n-3}(X^3+X^2+1)$, because it has degree $n < 6$. The same holds for g^{-1}. But if p_1 is the characteristic polynomial of g, then $p_2(g^{-1}) = 0$, and it follows that p_2 is the characteristic polynomial of g^{-1}. Conversely, if p_2 is the characteristic polynomial of g, then p_2 is that of g^{-1}. In any case g and g^{-1} are not conjugate elements.

Now consider the natural action of the group $\mathrm{Gl}(3, \mathbb{F}_2)$ on the vector space ${\mathbb{F}_2}^3$. Note, that $\mathrm{Gl}(3, \mathbb{F}_2)$ acts on the seven elements of ${\mathbb{F}_2}^3 \setminus \{0\}$ and that this action induces an embedding of $\mathrm{Gl}(3, \mathbb{F}_2)$ into the symmetric group S_7 on 7 letters. Here the element g is mapped to a 7-cycle and it follows easily from this, that the centralizer of C in $\mathrm{Gl}(3, \mathbb{F}_2)$ is C itself. Therefore, the conjugacy class of g contains 24 elements. But if g would not be conjugate to g^2, then it would not be conjugate to g^i for $i = 1, 2, 3, 4, 5, 6$. Hence, there would be exactly 24 different 7-Sylow subgroups in $\mathrm{Gl}(3, \mathbb{F}_2)$, contradicting the Sylow theorems. Therefore g and g^2 are conjugate in $\mathrm{Gl}(3, \mathbb{F}_2)$, hence also in $\mathrm{Gl}(4, \mathbb{F}_2)$ and $\mathrm{Gl}(5, \mathbb{F}_2)$. □

Remark 22. The proof shows, that there are exactly two conjugacy classes of elements of order 7 in $\mathrm{Gl}(n, \mathbb{F}_2)$ for $n = 3, 4, 5$, and these can be distinguished by their characteristic polynomials. If $g \in \mathrm{Gl}(n, \mathbb{F}_2)$ is an element of order 7, then g and g^{-1} belong to different classes. An alternative proof of the proposition is provided by the Skolem-Noether theorem:

Any element g of order 7 in $\mathrm{Gl}(n, \mathbb{F}_2)$ generates a subfield $\mathbb{F}_8(g)$ of order 8 in the ring $\mathrm{Mat}(n, \mathbb{F}_2)$ of $n \times n$-matrices over $\mathbb{F}_2$, which contains the subfield $\mathbb{F}_2\mathrm{id} \cong \mathbb{F}_2$. Therefore any such g provides a homomorphism of $\mathbb{F}_2$-algebras

$$\mathbb{F}_8 \hookrightarrow \mathrm{Mat}(n, \mathbb{F}_2).$$

Since $\mathbb{F}_8$ is a simple ring and $\mathrm{Mat}(n, \mathbb{F}_2)$ is a central simple $\mathbb{F}_2$-algebra, it follows from the Skolem-Noether theorem ([Sch85] p. 291), that given two such homomorphisms, they differ only by an inner automorphism of $\mathrm{Mat}(n, \mathbb{F}_2)$. Therefore if two elements g, g' of order 7 in $\mathrm{Gl}(n, \mathbb{F}_2)$ are given, they are conjugate in $\mathrm{Gl}(n, \mathbb{F}_2)$, if and only if there is an isomorphism $\alpha : \mathbb{F}_8(g) \to \mathbb{F}_8(g')$ with $\alpha(g) = \alpha(g')$. But this is the case, if and only if g and g' have the same minimal polynomial over $\mathbb{F}_2$.

Corollary 11. *$H^*(\mathrm{SO}^+(6, \mathbb{F}_2), k) \cong k[x^3, x^2y]$ and $H^*(\mathrm{SO}^+(6, \mathbb{F}_2), \mathbb{Z})_{[7]} \cong \mathbb{Z}[\xi^3]$.*

Proof. This follows from the exceptional isomorphism

$$\mathrm{SO}^+(6, \mathbb{F}_2) \cong \mathrm{Gl}(4, \mathbb{F}_2),$$

which is provided by the fact, that both groups are finite simple groups of Lie type with Weyl group of type A_3. □

Now, we compute the cohomology rings of the stabilizers.

Proposition 18. *Let G_1 be the stabilizer of the subspace $U_1 = \mathbb{F}_2 e_4$ in $G_0 := \mathrm{SO}(\overline{V}, \overline{q})$ and let G_2 be the stabilizer of the flag $U_1 \subset U_4$, where $U_4 = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2 \oplus \mathbb{F}_2 e_3 \oplus \mathbb{F}_2 e_4$. Then*

1. *$H^*(G_0, k) \cong k[x^6, x^5 y]$ and $H^*(G_0, \mathbb{Z})_{[7]} \cong \mathbb{Z}[\xi^6]$.*
2. *$H^*(G_1, k) \cong H^*(G_2, k) \cong k[x^3, x^2 y]$ and $H^*(G_1, \mathbb{Z})_{[7]} \cong H^*(G_2, \mathbb{Z})_{[7]} \cong \mathbb{Z}[\xi^3]$.*

Proof. Again the computations for cohomology with coefficients in k resp. in $\mathbb{Z}$ go completely analogically, so I restrict myself to the first case.

The proof of 1. is almost identical to the computation of $H^*(\mathrm{O}^+(6, \mathbb{F}_2), \mathbb{F}_7)$ in [FP78] VIII 4.13. Consider the group $\mathrm{SO}(\overline{V}, \overline{q}) \cong \mathrm{SO}^+(8, \mathbb{F}_2)$. Let φ be the canonical embedding

$$\mathrm{Gl}(4, \mathbb{F}_2) \hookrightarrow \mathrm{SO}(\overline{V}, \overline{q}), \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix},$$

with respect to the basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, e_{-1}, e_{-2}, e_{-3}, e_{-4}\}$, defined in proposition 1 of section 1.13. Choose a 7-Sylow subgroup $C \subset \mathrm{Gl}(4, \mathbb{F}_2)$ and a generator $g \in C$. By the proof of proposition 17, the assertion follows, if we have shown, that $\varphi(g)$ and $\varphi(g^{-1})$ are conjugate in $\mathrm{SO}^+(8, \mathbb{F}_2)$.

Note, that the isometry τ of $\overline{V}$, which is defined by

$$\tau(e_i) = e_{-i} \qquad \text{for } i \in I$$

is a rotation (namely $\tau = \tau_{-e_1+e_{-1}} \circ \ldots \circ \tau_{-e_4+e_{-4}}$), hence contained in $\mathrm{SO}(\overline{V}, \overline{q})$. By construction,

$$\tau \varphi(g) \tau^{-1} = \varphi({}^t g^{-1}).$$

But g^{-1} and ${}^t g^{-1}$ have the same characteristic polynomial and it follows from remark 22, that there is a $h \in \mathrm{Gl}(4, \mathbb{F}_2)$ such, that $h({}^t g^{-1}) h^{-1} = g^{-1}$. Hence,

$$\varphi(g^{-1}) = \varphi(h({}^t g^{-1}) h^{-1}) = (\varphi(h)\tau)\varphi(g)(\varphi(h)\tau)^{-1}.$$

2. Let $\overline{W}$ denote the subspace of $\overline{V}$ generated by $\mathcal{B}_0 = \{e_1, e_2, e_3, e_{-1}, e_{-2}, e_{-3}\}$, which is hyperbolic of dimension 6, and let H be the hyperbolic plane $\overline{W}^\perp = \mathbb{F}_2 e_4 \oplus \mathbb{F}_2 e_{-4}$. Now, consider the canonical embedding

$$\mathrm{Gl}(3, \mathbb{F}_2) \hookrightarrow \mathrm{SO}(\overline{W}, \overline{q})$$

of section 1.13 which is associated to the basis $\mathcal{B}_0$ and let C be a 7-Sylow subgroup of $\mathrm{Gl}(3, \mathbb{F}_2)$, which is cyclic as we have seen above.

Then $\mathrm{SO}(\overline{W}, \overline{q})$ stabilizes U_1. Thus, there are embeddings

$$C \hookrightarrow \mathrm{SO}(\overline{W}, \overline{q}) \subset G_1.$$

I claim that $\mathcal{N}_{G_1}(C) = \mathcal{N}_{\mathrm{SO}(\overline{W},\overline{q})}(C)$.

Indeed, C acts on the subspaces $\overline{W}^+ = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2 \oplus \mathbb{F}_2 e_3$ and $\overline{W}^- = \mathbb{F}_2 e_{-1} \oplus \mathbb{F}_2 e_{-2} \oplus \mathbb{F}_2 e_{-3}$ via the inclusion

$$C \subset \mathrm{Gl}(3,\mathbb{F}_2) \hookrightarrow \mathrm{SO}(\overline{W},\overline{q})$$

and, as seen in the proof of proposition 17, C acts simply transitively on the sets $\overline{W}^+ \setminus \{0\}$ and $\overline{W}^- \setminus \{0\}$. It follows that

$$\overline{W}^C = (\overline{W}^+)^C \oplus (\overline{W}^-)^C = \{0\}$$

and

$$\overline{V}^C = \overline{W}^C \oplus H^C = H.$$

This implies for $g \in \mathcal{N}_{G_1}(C)$ that

$$gH = \overline{V}^{gCg^{-1}} = \overline{V}^C = H$$

and even $gx = x$ for any $x \in H$, since g is an isometry, which fixes e_4, by definition of G_1. Thus $\mathcal{N}_{G_1}(C) = \mathcal{N}_{\mathrm{SO}(\overline{W},\overline{q})}(C)$, as asserted. Now Swan's lemma implies that

$$\begin{aligned} H^*(G_1,k) &\cong H^*(C,k)^{\mathcal{N}_{G_1}(C)} \\ &\cong H^*(C,k)^{\mathcal{N}_{\mathrm{SO}(\overline{W},\overline{q})}(C)} \\ &\cong H^*(\mathrm{SO}(\overline{W},\overline{q}),k) \cong k[x^3,x^2y]. \end{aligned} \tag{4.2}$$

On the other hand, the image of $\mathrm{Gl}(3,\mathbb{F}_2)$ in $\mathrm{SO}(\overline{V},\overline{q})$ stabilizes the flag $U_1 \subset U_4 = U_1 \oplus \overline{W}^+$ and we get inclusions

$$C \subset \mathrm{Gl}(3,\mathbb{F}_2) \hookrightarrow G_2 \subset G_1 \subset \mathrm{SO}(\overline{V},\overline{q}),$$

which induce injective ring homomorphisms

$$H^*(G_1,k) \hookrightarrow H^*(G_2,k) \hookrightarrow H^*(\mathrm{Gl}(3,\mathbb{F}_2),k) \cong k[x^3,x^2y].$$

But, by (4.2) all these monomorphisms are isomorphisms. □

4.4.2 The computation of the cohomology groups

Let Σ_p denote the set of faces of dimension p of Ch_0 for $p = 0,\ldots,4$. To simplify notation put $P_0 := \Lambda_{0^+}$, $P_1 := \Lambda_{0^-}$, $P_2 := \Lambda_{4^+}$, $P_3 := \Lambda_{4^-}$, $P_4 := \Lambda_2$. For $\sigma = \{P_{i_0},\ldots,P_{i_p}\} \in \Sigma_p$ with $i_0 < \ldots < i_p$ let 1_σ be the canonical generator in the simplicial chain complex $C_p(\mathrm{Ch}_0)$ of Ch_0, which is associated to the simplex σ

equipped with the orientation, which is induced by the ordering $i_0 < \ldots < i_p$. Then we have the usual formula for the boundary operator

$$\delta(1_\sigma) = \sum_{j=0}^{p} (-1)^j 1_{\{P_{i_0},\ldots,\hat{P_{i_j}},\ldots,P_{i_p}\}}.$$

Let $\Sigma'_p \subset \Sigma_p$ denote the subset of those faces of Ch_0 of dimension p for $p = 0, 1, 2$, which do not contain the vertex $P_5 = \Lambda_2$ and put $\Sigma' = \Sigma'_0 \cup \Sigma'_1 \cup \Sigma'_2$. Further let K be the simplicial complex, which consists of the simplices in Σ'. This is just the boundary of the tetrahedron with vertices $\{P_0, P_1, P_2, P_3\}$. Then with the above notation $H^*(C, k) = k[x, y]$ and $H^*(C, \mathbb{Z}) = \mathbb{Z}[\xi]$ for a cyclic group C of order 7 we have shown

Theorem 13. *1. Let σ be a face of Ch_0. Then*

$$H^*(\Gamma_\sigma, k) = \begin{cases} k[x^6, x^5y] & \text{for } \sigma \in \Sigma'_0 \\ k[x^3, x^2y] & \text{for } \sigma \in \Sigma'_1 \cup \Sigma'_2 \\ k & \text{for } \sigma \notin \Sigma' \end{cases}$$

and

$$H^*(\Gamma_\sigma, \mathbb{Z})_{[7]} = \begin{cases} \mathbb{Z}[\xi^6] & \text{for } \sigma \in \Sigma'_0 \\ \mathbb{Z}[\xi^3] & \text{for } \sigma \in \Sigma'_1 \cup \Sigma'_2 \\ \mathbb{Z} & \text{for } \sigma \notin \Sigma' \end{cases}$$

2. If $\sigma \in \Sigma'$ and τ is a face of σ, then the restriction map

$$\mathrm{res}^{\Gamma_\tau}_{\Gamma_\sigma} : H^*(\Gamma_\tau, k) \to H^*(\Gamma_\sigma, k)$$

is injective. An analogous statement holds for $H^*(\quad, \mathbb{Z})_{[7]}$.

Since the set of faces of Ch_0 is a set of representatives for the Γ-orbits in $\Sigma(X)$, we get the E_1-term of the spectral sequence (3.5)

$$E_1^{p,q} = H^q(\Gamma, C^p(X, k)) \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, k) \tag{4.3}$$

and theorem 13 yields

Corollary 12. *The q-th row $(E_1^{*,q}, d_1^{*,q})$ of the spectral sequence (4.3) is of the form*

$p =$		0		1		2		3		4		5	
0	$\to$	k^4	$\to$	k^6	$\to$	k^4	$\to$	0	$\to$	0	$\to$	0	for $q > 0$ and $q \equiv 11, 12 (mod\ 12)$
0	$\to$	0	$\to$	k^6	$\to$	k^4	$\to$	0	$\to$	0	$\to$	0	for $q > 0$ and $q \equiv 5, 6 (mod\ 12)$
0	$\to$	0	$\to$	0	$\to$	0	$\to$	0	$\to$	0	$\to$	0	for $q < 0$ or $q \not\equiv 5, 6 (mod\ 6)$
0	$\to$	k^5	$\to$	k^{10}	$\to$	k^{10}	$\to$	k^5	$\to$	k	$\to$	0	for $q = 0$

It remains to compute the differential map $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ for all p and q. By proposition 11 we have

Proposition 19. *The complex $(E_1^{*,0}, d_1^{*,0})$ is isomorphic to the simplicial cochain complex of a 4-simplex with coefficients in k. In particular,*

$$E_2^{p,0} = \begin{cases} k & \text{for } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $q > 0$ we have $H^q(\Gamma_\sigma, k) \cong k$, if and only if $\dim \sigma = 0$ and $q \equiv 11, 12 (\mathrm{mod}\ 12)$ or if $\dim \sigma = 1, 2$ and $q \equiv 5, 6 (\mathrm{mod}\ 6)$. For all these σ and q, we choose now a generator $x_\sigma^{(q)}$ of $H^q(\Gamma_\sigma, k) \cong k$. Then via the isomorphisms

$$E_1^{p,q} \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, k) \cong \prod_{\sigma \in \Sigma'_p} H^q(\Gamma_\sigma, k)$$

we can consider the family $\{x_\sigma^{(q)}\}$ as a basis of $\bigoplus_{p \geq 0, q > 0} E_1^{p,q}$.

Let us begin with $q = 5, 6$.

Lemma 23. *For $q \in \{5, 6\}$, there is a set $\{x_\sigma^{(q)}\}_{\sigma \in \Sigma'_1 \cup \Sigma'_2}$, with the property, that $x_\sigma^{(q)}$ generates the vector space $H^q(\Gamma_\sigma, k)$ for any σ and that for $(\sigma, \tau) \in \Sigma'_2 \times \Sigma'_1$ with $\tau \leq \sigma$ we have*

$$\mathrm{res}^{\Gamma_\tau}_{\Gamma_\sigma} x_\tau^{(q)} = \begin{cases} -x_\sigma^{(q)} & \text{if } (\sigma, \tau) = (\{P_0, P_1, P_2\}, \{P_0, P_1\}) \\ & \text{or } (\sigma, \tau) = (\{P_1, P_2, P_3\}, \{P_2, P_3\}) \\ x_\sigma^{(q)} & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. First note, that we can consider the groups $G_\sigma := \Gamma_\sigma / U$ instead of Γ_σ for $\sigma \leq \mathrm{Ch}_0$, because by proposition 7, there are commutative diagrams

$$\begin{array}{ccc} H^*(G_\tau, k) & \xrightarrow[\pi^*]{\sim} & H^*(\Gamma_\tau, k) \\ \downarrow \mathrm{res}^{G_\tau}_{G_\sigma} & & \downarrow \mathrm{res}^{\Gamma_\tau}_{\Gamma_\sigma} \\ H^*(G_\sigma, k) & \xrightarrow[\pi^*]{\sim} & H^*(\Gamma_\sigma, k) \end{array}$$

for $\tau \leq \sigma$.

Now fix a vertex $P \in \Sigma'_0$ and let Q, R, S denote the other elements of Σ'_0 in any order. Consider the (non necessarily commutative) diagram

$$\begin{array}{ccccc} H^q(G_{\{P,Q\}}, k) & \xrightarrow[\mathrm{res}]{\sim} & H^q(G_{\{P,Q,R\}}, k) & \xleftarrow[\sim]{\mathrm{res}} & H^q(G_{\{P,R\}}, k) \\ \mathrm{res} \downarrow \sim & & & & \mathrm{res} \downarrow \sim \\ H^q(G_{\{P,Q,S\}}, k) & \xleftarrow[\sim]{\mathrm{res}} & H^q(G_{\{P,S\}}, k) & \xrightarrow[\mathrm{res}]{\sim} & H^q(G_{\{P,R,S\}}, k) \end{array} \tag{4.5}$$

Let $x^{(q)}_{\{P,Q,R\}}$ be an arbitrary generator of $H^q(G_{\{P,Q,R\}}, k) \cong k$. Then we can choose successively a basis $\{x^{(q)}_\sigma\}$ of $H^*(G_\sigma, k)$ for

$$\sigma = \{P,R\},\ \{P,R,S\},\ \{P,S\},\ \{P,Q,S\},\ \{P,Q\},$$

such that the restriction maps

$$\mathrm{res}^{G_{\{P,R\}}}_{G_{\{P,Q,R\}}},\ \mathrm{res}^{G_{\{P,R\}}}_{G_{\{P,R,S\}}},\ \mathrm{res}^{G_{\{P,S\}}}_{G_{\{P,R,S\}}},\ \mathrm{res}^{G_{\{P,S\}}}_{G_{\{P,Q,S\}}},\ \mathrm{res}^{G_{\{P,Q\}}}_{G_{\{P,Q,S\}}}$$

have the matrix (1) with respect to these bases. So, it remains to compute the matrix of $\mathrm{res}^{G_{\{P,Q\}}}_{G_{\{P,Q,R\}}}$ with respect to $x^{(q)}_{\{P,Q\}}$ and $x^{(q)}_{\{P,Q,R\}}$. This can be done in the following way.

Let C be a 7-Sylow subgroup of $G_{\{P,Q,R\}}$ and choose an element $g \in G_{\{P,R\}}$ such, that $C' := gCg^{-1}$ is a 7-Sylow subgroup contained in $G_{\{P,R,S\}}$. This is possible, because C and C' are both 7-Sylow subgroups of $G_{\{P,R\}}$. The map $\kappa^*_{g^{-1}}$, which is induced by

$$c_{g^{-1}} : gCg^{-1} \to C$$

on cohomology, is trivial on $H^q(G_{\{P,R\}}, k)$, because $g \in G_{\{P,R\}}$. Hence there is a commutative diagram

$$\begin{array}{ccc} H^q(G_{\{P,Q,R\}}, k) & \xrightarrow{\mathrm{res}} & H^q(C, k) \\ \uparrow{\scriptstyle \mathrm{res}} & & \\ H^q(G_{\{P,R\}}, k) & & \downarrow{\scriptstyle \kappa^*_{g^{-1}}} \\ \downarrow{\scriptstyle \mathrm{res}} & & \\ H^q(G_{\{P,R,S\}}, k) & \xrightarrow{\mathrm{res}} & H^q(C', k) \end{array}$$

We continue in this manner, choosing an element $g' \in G_{\{P,S\}}$, such that $C'' := g'C'(g')^{-1} \subset G_{\{P,Q,S\}}$, and a $g'' \in G_{\{P,Q\}}$, such that $C''' := g''C''(g'')^{-1} \subset G_{\{P,Q,R\}}$. Then, the map

$$\psi := (\mathrm{res}^{G_{\{P,Q,R\}}}_{C'''})^{-1} \circ \kappa^*_{(g'')^{-1}} \circ \kappa^*_{(g')^{-1}} \circ \kappa^*_{g^{-1}} \circ \mathrm{res}^{G_{\{P,Q,R\}}}_{C} \tag{4.6}$$

is the automorphism of $H^q(G_{\{P,Q,R\}}, k)$ obtained by going the diagram (4.5) around clockwise.

Now, let as above $(\overline{V}, \overline{q})$ denote a hyperbolic quadratic space of dimension 8 over $\mathbb{F}_2$ with canonical basis $\{e_i\}_{i=\pm1,\dots,\pm4}$. Then we consider the groups G_σ for

$$\sigma \in \{\{P,Q,R\}, \{P,R\}, \{P,R,S\}, \{P,S\}, \{P,Q,S\}, \{P,Q\}\}$$

as subgroups of $\mathrm{SO}(\overline{V}, \overline{q})$ via an isomorphism $\Gamma_P/U \cong \mathrm{SO}(\overline{V}, \overline{q})$. And by symmetry (cf. remark 21), we can choose this isomorphism in such a way, that the groups

$G_{\{P,R\}}$, $G_{\{P,S\}}, G_{\{P,Q\}}$ are the stabilizers of the totally isotropic subspaces $U_1 := \mathbb{F}_2 e_4$, $U_{4^-} := \mathbb{F}_2 e_{-1} \oplus \mathbb{F}_2 e_2 \oplus \mathbb{F}_2 e_3 \oplus \mathbb{F}_2 e_4$ and $U_{4^+} := \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2 \oplus \mathbb{F}_2 e_3 \oplus \mathbb{F}_2 e_4$ respectively. Hence $G_{\{P,Q,R\}}, G_{\{P,R,S\}}, G_{\{P,Q,S\}}$ are the stabilizers of the oriflamme flags $\{U_1, U_{4^+}\}, \{U_1, U_{4^-}\}, \{U_{4^+}, U_{4^-}\}$ respectively.

More concretely, consider the subspace $\overline{W} := \bigoplus_{i=\pm 1,\ldots,\pm 3} \mathbb{F}_2 e_i$ and the embedding

$$\iota : \mathrm{Gl}(3, \mathbb{F}_2) \hookrightarrow \mathrm{SO}(\overline{W}, \overline{q}) \to \mathrm{SO}(\overline{V}, \overline{q})$$

considered in the proof of proposition 18, where $a = (a_{ij}) \in \mathrm{Gl}(3, \mathbb{F}_2)$ is mapped to the isometry with matrix

$$\begin{pmatrix} a & & & \\ & 1 & & \\ & & {}^t a^{-1} & \\ & & & 1 \end{pmatrix}$$

with respect to the basis $e_1, \ldots, e_4, e_{-1}, \ldots, e_{-4}$. Then $\iota \mathrm{Gl}(3, \mathbb{F}_2)$ stabilizes $\{U_1, U_{4^+}\}$, thus it is contained in $G_{\{P,Q,R\}}$. Let C be a 7-Sylow subgroup of $\mathrm{Gl}(3, \mathbb{F}_2)$. Then we can choose g, g', g'' to be the maps

e_i	e_1	e_2	e_3	e_4	e_{-1}	e_{-2}	e_{-3}	e_{-4}
ge_i	e_1	e_{-2}	e_{-3}	e_4	e_{-1}	e_2	e_3	e_{-4}
$g'ge_i$	e_{-4}	e_{-2}	e_{-3}	e_{-1}	e_4	e_2	e_3	e_1
$g''g'ge_i$	e_{-1}	e_{-2}	e_{-3}	e_{-4}	e_1	e_2	e_3	e_4

which are easily seen to be rotations contained in $G_{\{P,R\}}, G_{\{P,S\}}, G_{\{P,Q\}}$ respectively, such that $gCg^{-1} \subset G_{\{P,R,S\}}$, $g'gCg^{-1}g'^{-1} \subset G_{\{P,Q,S\}}$ and $g''g'gCg^{-1}g'^{-1}g''^{-1} \subset G_{\{P,Q,R\}}$. By construction, if c is a generator of C, then $g''g'g\iota(c)g^{-1}g'^{-1}g''^{-1} = \iota({}^t c^{-1})$ and we know from the proof of proposition 17, that ${}^t c^{-1}$ is conjugate to c^{-1} in $\mathrm{Gl}(3, \mathbb{F}_2)$. So, proposition 8.2. shows that

$$\psi(x^{(q)}_{\{P,Q,R\}}) = (-1)^3 x^{(q)}_{\{P,Q,R\}} = -x^{(q)}_{\{P,Q,R\}},$$

hence

$$\mathrm{res}^{G_{\{P,Q\}}}_{G_{\{P,Q,R\}}} x^{(q)}_{\{P,Q\}} = \mathrm{res}^{G_{\{P,Q\}}}_{G_{\{P,Q,R\}}} \circ (\mathrm{res}^{G_{\{P,Q\}}}_{G_{\{P,Q,R\}}})^{-1} \circ \psi(x^{(q)}_{\{P,Q,R\}}) = -x^{(q)}_{\{P,Q,R\}}.$$

Now, apply this consideration first with

$$P = P_0,\ Q = P_1,\ R = P_2 \text{ and } S = P_3.$$

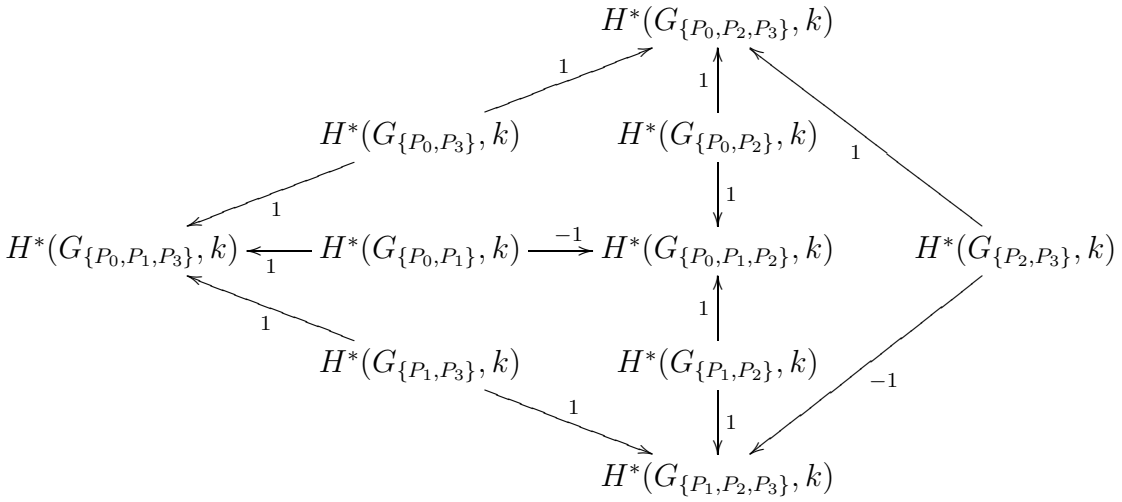
Then we have defined $x^{(q)}_\sigma$ for any $\sigma \in \Sigma'_1 \cup \Sigma'_2$ with vertex P_0 and have verified the assertion on the restriction maps $\mathrm{res}^{G_\tau}_{G_\sigma}$ for any $\tau \geq \{P_1\}$. After this continue with

$$P = P_1,\ Q = P_0,\ R = P_2 \text{ and } S = P_3$$

and with

$$P = P_2,\ Q = P_3,\ R = P_1 \text{ and } S = P_0,$$

where we can choose the element $x^{(q)}_{\{P,Q,R\}}$ to be the element $x^{(q)}_{\{P_0,P_1,P_2\}}$ in the second step and $x^{(q)}_{\{P_1,P_2,P_3\}}$ in the third step. This gives the diagram



Since, when you go any of the triangles around, the result is -1, it follows immediately, that the definitions of the $x^{(q)}_\sigma$ are compatible and that the diagram is commutative. This completes the proof. □

We extend the definition of $x^{(q)}_\sigma$ for $\sigma \in \Sigma'_1 \cup \Sigma'_2$ to an arbitrary positive $q \equiv 5, 6 (\text{mod } 6)$ by the formula

$$x^{(6r)}_\sigma = (x^{(6)}_\sigma)^r \text{ and } x^{(6r-1)}_\sigma = (x^{(6)}_\sigma)^{r-1} x^{(5)}_\sigma$$

for $r = 1, 2 \ldots$, where the products are cup products in the ring $H^*(\Gamma_\sigma, k) \cong H^*(G_\sigma, k)$. Then it follows by functoriality of the cup product (cf. [Bro82] V.3.7), that

$$\text{res}^{\Gamma_\tau}_{\Gamma_\sigma}(x^{(6r)}_\tau) = (\text{res}^{\Gamma_\tau}_{\Gamma_\sigma}(x^{(6)}_\tau))^r$$

and

$$\text{res}^{\Gamma_\tau}_{\Gamma_\sigma}(x^{(6r-1)}_\tau) = (\text{res}^{\Gamma_\tau}_{\Gamma_\sigma}(x^{(6)}_\tau))^{r-1} \text{res}^{\Gamma_\tau}_{\Gamma_\sigma}(x^{(5)}_\tau).$$

This yields the

Corollary 13. *Fix a positive number $q \equiv 5, 6 (mod\ 6)$ and a pair $(\sigma, \tau) \in \Sigma'_1 \times \Sigma'_2$ with $\tau \leq \sigma$.*

1. If $q \equiv 11, 12 (mod\ 12)$, then $\text{res}^{\Gamma_\tau}_{\Gamma_\sigma}(x^{(q)}_\tau) = x^{(q)}_\sigma$.

2. If $q \equiv 5, 6 (mod\ 12)$, then

$$\mathrm{res}^{\Gamma_\tau}_{\Gamma_\sigma} x^{(q)}_\tau = \begin{cases} -x^{(q)}_\sigma & \text{if } (\sigma,\tau) = (\{P_0,P_1,P_2\},\{P_0,P_1\}) \\ & \text{or } (\sigma,\tau) = (\{P_1,P_2,P_3\},\{P_2,P_3\}) \\ x^{(q)}_\sigma & \text{otherwise.} \end{cases} \tag{4.7}$$

It remains to define $x^{(q)}_\sigma$ for $\sigma \in \Sigma'_0$ and $q \equiv 11, 12 (\mathrm{mod}\ 12)$.

Lemma 24. *Assume $q \equiv 11, 12 (mod\ 12)$. There is a family $\{x^{(q)}_\tau\}_{\tau\in\Sigma'_0}$, such that $x^{(q)}_\tau$ is a generator of $H^q(\Gamma_\tau, k)$ and such that*

$$\mathrm{res}^{\Gamma_\tau}_{\Gamma_\sigma} x^{(q)}_\tau = x^{(q)}_\sigma \tag{4.8}$$

holds for any pair $(\sigma,\tau) \in \Sigma' \times \Sigma'$ with $\tau \leq \sigma$.

Proof. By lemma 13, the formula (4.8) holds for $\dim \tau > 0$. But if $\tau \in \Sigma'_0$ and $\sigma \in \Sigma'_1 \cup \Sigma'_2$ with $\tau \leq \sigma$, then (4.8) forces the definition

$$x^{(q)}_\tau := (\mathrm{res}^{\Gamma_\tau}_{\Gamma_\sigma})^{-1}(x^{(q)}_\sigma).$$

This is well defined. Indeed, fix $P \in \Sigma'_0$ and let Q, R, S be the other vertices in Σ'_0. Then there are identities

$$\begin{aligned} (\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,Q\}}})^{-1} x^{(q)}_{\{P,Q\}} &= (\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,Q,R\}}})^{-1} x^{(q)}_{\{P,Q,R\}} = (\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,R\}}})^{-1} x^{(q)}_{\{P,R\}} \\ &= (\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,R,S\}}})^{-1} x^{(q)}_{\{P,R,S\}} = (\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,S\}}})^{-1} x^{(q)}_{\{P,S\}}. \end{aligned}$$

by the transitivity of the restriction maps. □

Note that if $q = 12r$ (resp. $q = 12r - 1$), then it follows that

$$x^{(q)}_P = (\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,Q\}}})^{-1} x^{(q)}_{\{P,Q\}} = ((\mathrm{res}^{\Gamma_P}_{\Gamma_{\{P,Q\}}})^{-1} x^{(12)}_{\{P,Q\}})^r = (x^{(12)}_P)^r$$

(resp. $x^{(q)}_P = (x^{(12)}_P)^{r-1} x^{(11)}_P$). Hence, if $\sigma \in \Sigma'$ and $q, q' \in \mathbb{Z}$ are chosen in such a way, that $x^{(q)}_\sigma$ and $x^{(q')}_\sigma$ are defined, then we get the formula

$$x^{(q)}_\sigma \cdot x^{(q')}_\sigma = \begin{cases} 0 & \text{if } q \text{ and } q' \text{ are odd} \\ x^{(q+q')}_\sigma & \text{otherwise.} \end{cases}$$

Now we are able to compute the differential maps $d_1^{p,q}$ for $q > 0$:

Theorem 14. *1. Let $q \equiv 11, 12 (mod\ 12)$ be a positive integer. Then the complex $(E_1^{*,q}, d_1^{*,q})$ is isomorphic to the simplicial cochain complex of K, which is the boundary of the 3-simplex with vertices $\{P_0, P_1, P_2, P_3\}$. In particular,*

$$E_2^{p,q} = \begin{cases} k & \text{for } p = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the kernel of the map $d_1^{0,q}$ is generated by the element

$$x^{(q)} := \sum_{P \in \Sigma_0'} x_P^{(q)}$$

and the image of $d_1^{1,q}$ is the subspace

$$\{\sum_{i=0}^{3} \lambda_i x_{\sigma_i}^{(q)} \mid \lambda_i \in k \text{ for } i = 0, 1, 2, 3, \ \sum_{i=0}^{3} (-1)^i \lambda_i = 0\} \subset E_1^{2,q},$$

where $\sigma_i := \{P_0, \ldots, P_3\} \setminus \{P_i\}$.

2. For $q \equiv 5, 6 (mod\ 12)$ the map

$$d_1^{1,q} : E_1^{1,q} \to E_1^{2,q}$$

is surjective and the kernel is generated by

$$a_1^{(q)} := x_{\{P_0,P_2\}}^{(q)} + x_{\{P_0,P_3\}}^{(q)} + x_{\{P_1,P_2\}}^{(q)} + x_{\{P_1,P_3\}}^{(q)}$$

and

$$a_2^{(q)} := x_{\{P_0,P_1\}}^{(q)} + x_{\{P_0,P_3\}}^{(q)} + x_{\{P_1,P_2\}}^{(q)} + x_{\{P_2,P_3\}}^{(q)}.$$

Hence

$$E_2^{p,q} \cong \begin{cases} k^2 & \text{for } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. 1. Let $\{f_\sigma\}$ be the basis of the simplicial cochain complex $C^*(K, k)$, which is dual to the basis 1_σ of $C_*(K)$, i.e.

$$f_\sigma(1_\tau) = \begin{cases} 1 & \text{for } \sigma = \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then, it follows directly from proposition 10 and lemma 24, that the map

$$E_1^{*,q} \to C^*(K, k), \ x_\sigma^{(q)} \mapsto f_\sigma$$

is an isomorphism of complexes. Therefore we can at once write down the matrices of the maps $d_1^{0,q}$ (resp. $d_1^{1,q}$) with respect to the bases

- $x^{(q)}_{\{P_0\}}, x^{(q)}_{\{P_1\}}, x^{(q)}_{\{P_2\}}, x^{(q)}_{\{P_3\}}$ of $E_1^{0,q}$
- $x^{(q)}_{\{P_0,P_1\}}, x^{(q)}_{\{P_0,P_2\}}, x^{(q)}_{\{P_0,P_3\}}, x^{(q)}_{\{P_1,P_2\}}, x^{(q)}_{\{P_1,P_3\}}, x^{(q)}_{\{P_2,P_3\}}$ of $E_1^{1,q}$
- $x^{(q)}_{\{P_0,P_1,P_2\}}, x^{(q)}_{\{P_0,P_1,P_3\}}, x^{(q)}_{\{P_0,P_2,P_3\}}, x^{(q)}_{\{P_1,P_2,P_3\}}$ of $E_1^{2,q}$,

for $q \equiv 11, 12 \pmod{12}$, namely

$$D^0 := \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad (\text{resp. } D^1 := \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}).$$

Now the other assertions of part 1. can be checked directly.

2. By corollary 13 we get the matrix D' of $d_1^{1,q}$ for $q \equiv 5, 6 (\text{mod } 12)$, if we change in D^1 the sign of the two coordinates which describe the components

$$H^q(\Gamma_\tau, k) \to H^q(\Gamma_\sigma, k)$$

of $d_1^{1,q}$ for $(\sigma, \tau) = (\{P_0, P_1, P_2\}, \{P_0, P_1\})$ or $(\sigma, \tau) = (\{P_1, P_2, P_3\}, \{P_2, P_3\})$. So, we get

$$D' := \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix})$$

and assertion 2. follows directly by calculation. □

Before we compute the differential $d_2^{*,*}$, let us consider the integral cohomology at the prime 7. Therefore, we have to compute the spectral sequence

$$E_{\mathbb{Z}}[7]_1^{p,q} = \begin{cases} H^q(\Gamma, C^p(X, \mathbb{Z})) & \text{for } q = 0 \\ H^q(\Gamma, C^p(X, \mathbb{Z}))_{(7)} & \text{for } q \neq 0 \end{cases}$$

defined in chapter 3.

Theorem 15. *1. $(E_{\mathbb{Z}}[7]_1^{*,0}, d_1^{*,0})$ is isomorphic to the simplicial cochain complex of $\Gamma \setminus X$.*

2. If q is odd or $q < 0$, then $E_{\mathbb{Z}}[7]_1^{p,q} = 0$.

3. Let $E_1^{p,q}$ denote the spectral sequence (4.3) with $k = \mathbb{Z}/7\mathbb{Z}$. Then for $q > 0$ even, the map $E_{\mathbb{Z}}[7]_1^{p,q} \to E_1^{p,q}$, which is induced by the canonical map $\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$, is an isomorphism of complexes of abelian groups.

Proof. 1. follows from proposition 11. And since $E_{\mathbb{Z}}[7]_1^{p,q} \cong \prod_{\sigma\in\Sigma_p} H^q(\Gamma_\sigma, \mathbb{Z})_{(7)}$ for $q \neq 0$, the fact, that

$$E_{\mathbb{Z}}[7]_1^{p,q} \cong \begin{cases} E_1^{p,q} & \text{for } q \neq 0 \text{ even} \\ 0 & \text{for } q \text{ odd} \end{cases}$$

is provided by theorem 13. That the map in 3. is an isomorphism of complexes follows easily from proposition 8 and from the functoriality of the construction with respect to modules of coefficients, cf. lemma 16 and proposition 10. □

Corollary 14.

$$E_{\mathbb{Z}}[7]_2^{p,q} \cong \begin{cases} \mathbb{Z} & \textit{for } p = q = 0 \\ \mathbb{Z}/7\mathbb{Z} & \textit{for } q > 0, q \equiv 0 (\textit{mod } 12) \textit{ and } p = 0, 2 \\ \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} & \textit{for } q > 0, q \equiv 6 (\textit{mod } 12) \textit{ and } p = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Obviously, if $E_{\mathbb{Z}}[7]_2^{p,q} \neq 0$, then $E_{\mathbb{Z}}[7]_2^{p+r,q-r+1} = 0$ for any $r \geq 2$. Therefore $E_{\mathbb{Z}}[7]_2^{p,q} = E_{\mathbb{Z}}[7]_\infty^{p,q}$ and it follows

Corollary 15. *For $q > 0$ we have*

$$H^q(\Gamma, \mathbb{Z})_{(7)} = \begin{cases} \mathbb{Z}/7\mathbb{Z} & \textit{for } q > 4,\ q \equiv 0, 2 (\textit{mod } 12) \\ \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} & \textit{for } q \equiv 7 (\textit{mod } 12) \\ 0 & \textit{otherwise.} \end{cases}$$

Now, an argument of Bockstein type provides

Corollary 16.

$$H^q(\Gamma, k) \cong k \qquad \textit{for positive } q \equiv 0 (\textit{mod } 12).$$

Proof. For positive $q \equiv 0 (\text{mod } 12)$, the 7-primary component of the long exact cohomology sequence, which is associated to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{7} \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/7\mathbb{Z} \to 0$$

is of the form

$$H^q(\Gamma, \mathbb{Z})_{(7)} \xrightarrow{0} H^q(\Gamma, \mathbb{Z})_{(7)} \xrightarrow{\alpha_*} H^q(\Gamma, \mathbb{Z}/7\mathbb{Z}) \xrightarrow{\delta} H^{q+1}(\Gamma, \mathbb{Z})_{(7)}.$$

Note, that this sequence is exact, since $H^n(\Gamma, \mathbb{Z})$ is a torsion module for $n > 0$. But $H^{q+1}(\Gamma, \mathbb{Z})_{(7)} = 0$ implies, that α_* is an isomorphism

$$H^q(\Gamma, \mathbb{Z}/7\mathbb{Z}) \xleftarrow{\sim} H^q(\Gamma, \mathbb{Z})_{(7)} \cong \mathbb{Z}/7\mathbb{Z}.$$

This implies the assertion for arbitrary k as follows. The group Γ is known to be of type WFL, in particular it contains a subgroup Γ' of finite index, such that $\mathbb{Z}$ possesses a finite free resolution over $\mathbb{Z}\Gamma'$, cf. section 3.2.4. But then we can choose a resolution F_* of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ such, that any module F_n is a free $\mathbb{Z}\Gamma$-module of finite rank, see [Bro82] VIII 4.3 and 5.1. In this case, there is a canonical isomorphism

$$\mathrm{Hom}_\Gamma(F_n, \mathbb{Z}/7\mathbb{Z}) \otimes k \to \mathrm{Hom}_\Gamma(F_n, k), \quad f \otimes x \mapsto x \cdot f$$

for any n, hence $H^q(\Gamma, k) \cong H^q(\Gamma, \mathbb{Z}/7\mathbb{Z}) \otimes k \cong k$. □

Theorem 16. *The spectral sequence 4.3 abuts in the E_2-term. In particular, if $q \geq 0$, then*

$$H^q(\Gamma, k) \cong \begin{cases} k & \text{for } q = 0 \text{ or } q \geq 4 \text{ with } q \equiv 0, 1, 2, 11 (\text{mod } 12) \\ k^2 & \text{for } q \equiv 6, 7 (\text{mod } 12) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $E_1^{p,q} = 0$ for $p < 0$ or $p > 2$ and any $q \neq 0$, the spectral sequence clearly abuts in the E_3-term. In order to compute the differential $d_2^{*,*}$, we have only to check the map

$$d_2^{0,q} : E_2^{0,q} \to E_2^{2,q-1} \qquad \text{for } q \equiv 0 (\text{mod } 12),$$

where $E_2^{0,q} \cong E_2^{2,q-1} \cong k$.

But if $d_2^{0,q} \neq 0$ would hold, then it would follow, that the spectral sequence would abut with $E_3^{0,q} = E_3^{1,q-1} = E_3^{2,q-2} = 0$, hence $H^q(\Gamma, k) = 0$, contradicting corollary 16. □

4.4.3 Products

Finally, it remains to compute the ring strucure on $H^q(\Gamma, k)$. The map

$$l' : \Sigma \to \{0, 1, 2, 3, 4\}, \quad P_i \mapsto i$$

extends uniquely to a labelling of X (cf. [Bro89] p. 30) and we have seen in chapter 3, that such a labelling induces a multiplicative structure on $E_1^{*,*}$, which is associative and for which we have computed a formula in proposition 13:

Let $\tau \in \Sigma_p$, $\tau' \in \Sigma_{p'}$ be some simplices and $\sigma = \{P_{i_0}, \ldots, P_{i_{p+p'}}\} \in \Sigma_{p+p'}$ with $i_0 < \ldots < i_{p+p'}$. The component

$$H^q(G_\tau, k) \otimes H^{q'}(G_{\tau'}, k) \to H^{q+q'}(G_\sigma, k)$$

of the multiplicative structure, is given by

$$\begin{array}{lll} H^q(G_\tau, k) \otimes H^{q'}(G_{\tau'}, k) & \xrightarrow{(-1)^{qp'} \mathrm{res}^{G_\tau}_{G_\sigma} \otimes \mathrm{res}^{G_{\tau'}}_{G_\sigma}} & H^q(G_\sigma, k) \otimes H^{q'}(G_\sigma, k) \\ & \xrightarrow{\cup} & H^{q+q'}(G_\sigma, k), \end{array} \tag{4.9}$$

if $\tau = \sigma_p = \{P_{i_0}, \ldots, P_{i_p}\}$ *and* $\tau' = \sigma^{p'} = \{P_{i_p}, \ldots, P_{i_{p+p'}}\}$*, and the trivial map otherwise.*

We have computed the restriction maps and the cup products occurring in this formula, already. It is well known, that this multiplicative structure induces another on $E_2^{*,*}$, which can be identified with the multiplication law in the ring $H^*(\Gamma, k)$.

First note, that there is a unit $1 \in E_1^{0,0}$ in $\bigoplus_{p,q} E_1^{p,q}$ given by $\sum_{P \in \Sigma_0} 1_P$, where 1_P denotes the unit in $H^*(\Gamma_P, k)$ for $P \in \Sigma_0$. Now let $x \in E_2^{0,12}, y \in E_2^{0,11}$ denote the elements, which are generated by the elements $x^{(12)} \in E_1^{0,12}$ and $x^{(11)} \in E_1^{0,11}$ defined in theorem 14, respectively.

Lemma 25. *If* $\sigma \in \Sigma'$ *and* q *is chosen such that* $x_\sigma^{(q)}$ *is defined, then*

$$x^{(12)} \cup x_\sigma^{(q)} = x_\sigma^{(q)} \cup x^{(12)} = x_\sigma^{(q+12)}.$$

In particular

$$x \cup __ = __ \cup x : \quad E_2^{p,q} \to E_2^{p,q+12}$$

is an isomorphism for any $q > 0$ *and any* p*.*

Proof. Assume that $\sigma = \{P_{i_0}, \ldots, P_{i_r}\}$ with $0 \leq i_0 < \ldots < i_r \leq 3$. Now, (4.9) provides

$$x^{(12)} \cup x_\sigma^{(q)} = \sum_{P \in \Sigma'} x_P^{(12)} \cup x_\sigma^{(q)} = x_{P_{i_0}}^{(12)} \cup x_\sigma^{(q)} = x_\sigma^{(12)} \circ x_\sigma^{(q)} = x_\sigma^{(q+12)},$$

where $\circ$ denotes the cup product in $H^*(\Gamma_\sigma, k)$. Analogously,

$$x_\sigma^{(q)} \cup x^{(12)} = x_\sigma^{(q)} \cup x_{P_{i_r}}^{(12)} = x_\sigma^{(q)} \circ x_\sigma^{(12)} = x_\sigma^{(q+12)}.$$

□

Remark 23. A similar calculation shows, that

$$x_\sigma^{(q)} \cup x^{(11)} = x_\sigma^{(q+11)}, \text{ but } x^{(11)} \cup x_\sigma^{(q)} = (-1)^{\dim \sigma} x_\sigma^{(q+11)},$$

since 11 is odd.

Therefore, by associativity and commutativity, the computation of the components

$$E_2^{p,q} \otimes E_2^{p',q'} \to E_2^{p+p',q+q'}$$

is reduced to the cases $0 < q \leq q' \leq 12$, $(p,q), (p',q') \neq (0,12)$.

Let $a_1, a_2 \in E_2^{1,6}$ and $b_1, b_2 \in E_2^{1,5}$ be the elements, which are induced by the elements $a_1^{(6)}, a_2^{(6)}, a_1^{(5)}, a_2^{(5)}$ of theorem 14, respectively.

Theorem 17. *The k-algebra $H^*(\Gamma, k)$, which is isomorphic to the algebra*

$$\bigoplus_{n\geq 0}\bigoplus_{p+q=n} E_2^{p,q},$$

that is provided by the spectral sequence and its multiplicative structure, is generated over $E_2^{0,0} \cong k$ by the elements x, y, a_1, a_2, b_1, b_2 of degree $12, 11, 7, 7, 6, 6$ respectively, and the multiplicative structure is determined by the fact, that it is associative and commutative in the graded sense, and by the further relations

- $b_1^2 = b_2^2 = 0$
- $b_1b_2 = yb_1 = yb_2 = a_1b_1 = a_2b_2 = 0$
- $a_1b_2 = -a_2b_1$
- $a_iy = xb_i, \quad i = 1, 2.$

Proof. That x, y, a_1, a_2, b_1, b_2 is a complete set of generators, follows at once by theorem 14 and lemma 25. Let us check the relations now.

Since $b_ib_j \in E_2^{2,10} = 0$ for $i, j \in \{1, 2\}$ and $yb_i \in E_2^{1,16} = 0$, we have only to compute the following products:

$a_iy = xb_i$ follows from lemma 25 and the remark after it,

$$\begin{aligned} a_1^{(6)} \cup a_1^{(5)} &= (x^{(6)}_{\{P_0,P_2\}} + x^{(6)}_{\{P_0,P_3\}} + x^{(6)}_{\{P_1,P_2\}} + x^{(6)}_{\{P_1,P_3\}}) \\ &\quad \cup (x^{(5)}_{\{P_0,P_2\}} + x^{(5)}_{\{P_0,P_3\}} + x^{(5)}_{\{P_1,P_2\}} + x^{(5)}_{\{P_1,P_3\}}) \\ &= 0, \end{aligned}$$

$$\begin{aligned} a_2^{(6)} \cup a_2^{(5)} &= (x^{(6)}_{\{P_0,P_1\}} + x^{(6)}_{\{P_0,P_3\}} + x^{(6)}_{\{P_1,P_2\}} + x^{(6)}_{\{P_2,P_3\}}) \\ &\quad \cup (x^{(5)}_{\{P_0,P_1\}} + x^{(5)}_{\{P_0,P_3\}} + x^{(5)}_{\{P_1,P_2\}} + x^{(5)}_{\{P_2,P_3\}}) \\ &= x^{(6)}_{\{P_0,P_1\}} \cup x^{(5)}_{\{P_1,P_2\}} + x^{(6)}_{\{P_1,P_2\}} \cup x^{(5)}_{\{P_2,P_3\}} \\ &= -x^{(6)}_{\{P_0,P_1,P_2\}} \circ x^{(5)}_{\{P_0,P_1,P_2\}} - x^{(6)}_{\{P_1,P_2,P_3\}} \circ x^{(5)}_{\{P_1,P_2,P_3\}} \\ &= -x^{(11)}_{\{P_0,P_1,P_2\}} - x^{(11)}_{\{P_1,P_2,P_3\}} \in \mathrm{im}(d_1^{1,11}), \end{aligned}$$

where again $\circ$ denotes the cup product in $H^*(\Gamma_\sigma, k)$ for any σ. Hence we have $a_1b_1 = a_2b_2 = 0$.

For $q = 5, 6$, it follows from

$$\begin{aligned} a_1^{(6)} \cup a_2^{(q)} &= (x^{(6)}_{\{P_0,P_2\}} + x^{(6)}_{\{P_0,P_3\}} + x^{(6)}_{\{P_1,P_2\}} + x^{(6)}_{\{P_1,P_3\}}) \\ &\quad \cup (x^{(q)}_{\{P_0,P_1\}} + x^{(q)}_{\{P_0,P_3\}} + x^{(q)}_{\{P_1,P_2\}} + x^{(q)}_{\{P_2,P_3\}}) \\ &= x^{(6)}_{\{P_0,P_2\}} \cup x^{(q)}_{\{P_2,P_3\}} + x^{(6)}_{\{P_1,P_2\}} \cup x^{(q)}_{\{P_2,P_3\}} \\ &= x^{(q+6)}_{\{P_0,P_2,P_3\}} - x^{(q+6)}_{\{P_1,P_2,P_3\}} \notin \mathrm{im}(d_1^{1,q+6}), \end{aligned}$$

that $c := a_1 a_2 \in E_2^{2,11}$ and $d := a_1 b_2 \in E_2^{2,12}$ are not zero. Further, since

$$\begin{aligned} a_2^{(6)} \cup a_1^{(5)} &= (x^{(6)}_{\{P_0,P_1\}} + x^{(6)}_{\{P_0,P_3\}} + x^{(6)}_{\{P_1,P_2\}} + x^{(6)}_{\{P_2,P_3\}}) \\ &\quad \cup (x^{(5)}_{\{P_0,P_2\}} + x^{(5)}_{\{P_0,P_3\}} + x^{(5)}_{\{P_1,P_2\}} + x^{(5)}_{\{P_1,P_3\}}) \\ &= x^{(6)}_{\{P_0,P_1\}} \cup x^{(5)}_{\{P_1,P_2\}} + x^{(6)}_{\{P_0,P_1\}} \cup x^{(5)}_{\{P_1,P_3\}} \\ &= -x^{(11)}_{\{P_0,P_1,P_2\}} + x^{(11)}_{\{P_0,P_1,P_3\}}, \end{aligned}$$

we see, that $a_1^{(6)} \cup a_2^{(5)} + a_2^{(6)} \cup a_1^{(5)} \in \mathrm{im}(d_1^{1,11})$, thus $a_1 b_2 + a_2 b_1 = 0 \in E_2^{2,11}$.

It follows from graded commutativity, that there are further relations

$$a_1^2 = a_2^2 = y^2 = 0.$$

But the elements $x^n, a_1 x^n, a_2 x^n, b_1 x^n, b_2 x^n, c x^n, d x^n$, where n runs through the set of non negative integers, form a basis of the k-vector space $H^*(\Gamma, k)$ and by the relations, any product of the generators can be transformed into an element of this form, up to sign. Therefore the relations determine the ring structure. □

Corollary 17. *The associative and graded commutative $\mathbb{Z}$-algebra $H^*(\Gamma, \mathbb{Z})_{[7]}$ is generated over $H^0(\Gamma, \mathbb{Z})_{[7]} \cong \mathbb{Z}$ by generators ξ, α_1, α_2 of degree $12, 7, 7$ respectively together with the relations $7\xi = 7\alpha_1 = 7\alpha_2 = 0$.*

Proof. Consider the homomorphism of rings

$$\alpha_* : H^*(\Gamma, \mathbb{Z})_{[7]} \to H^*(\Gamma, \mathbb{Z}/7\mathbb{Z}),$$

which is induced by $\alpha : \mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$. Now, the assertion follows, since α_* is injective in positive degrees, which can be derived from the Bockstein long exact sequence as in the proof of corollary 16. □

Remark 24. Again, commutativity in the graded sense implies the further relations $\alpha_1^2 = \alpha_2^2 = (\alpha_1 + \alpha_2)^2 = 0$.

4.4.4 An outlook onto Farrell-Tate Cohomology

These results become more elegant, if we use *Farrell-Tate cohomology* instead of the usual one. On the other hand we loose some information about the cohomology groups of low degree then. Since the computation of the Farrell-Tate cohomology goes step by step in parallel to the computations, we have done, I will give only a short sketch of the changes. For definitions and properties I refer the reader to chapter X of [Bro82].

The Farrell-Tate cohomology is defined for groups G of finite virtuell cohomological dimension and denoted by $\hat{H}^*(G, M)$, where M is a G-module of coefficients.

But since Γ contains subgroups of finite index, which act freely on the finite dimensional contractible space X, it follows that Γ has virtuell cohomological dimension $4 = \dim X$.

Exactly as in the case of ordinary cohomology, it is possible to compute $\hat{H}^*(\Gamma, k)$ by considering equivariant cohomology

$$\hat{H}^*(\Gamma, C^*(X, k))$$

and there is a spectral sequence $(\hat{E}_r^{*,*}, d_r^{*,*})_{r\geq 0}$ with

$$\hat{E}_1^{p,q} = \hat{H}^q(\Gamma, C^p(X, k)) \cong \prod_{\sigma\in\Sigma_p} \hat{H}^q(\Gamma_\sigma, k),$$

which converges to $\hat{H}^*(\Gamma, k)$. Note that, although $\hat{E}_r^{*,*}$ is no first quadrant spectral sequence, there are no difficulties about convergence, because $\hat{E}_1^{p,q} \neq 0$ only for $0 \leq p \leq \dim X = 4$. The groups $\hat{H}^q(\Gamma_\sigma, k)$ result to be just the Tate cohomology groups of Γ_σ, because the stabilizers Γ_σ have finite order. And it is clear from our calculations how they look like.

With a similar notation as above, the Tate cohomology groups of a cyclic group C of order 7 are of the form

$$\hat{H}^*(C, k) \cong k[x, x^{-1}, y] \quad \text{with } \deg x = 2, \deg y = 1 \text{ and } y^2 = 0.$$

Now in analogy to theorem 13, we find

$$\hat{H}^*(\Gamma_\sigma, k) = \begin{cases} k[x^6, x^{-6}, x^5 y] & \text{for } \sigma \in \Sigma_0' \\ k[x^3, x^{-3}, x^2 y] & \text{for } \sigma \in \Sigma_1' \cup \Sigma_2' \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\hat{E}_1^{p,q}$ for $p = \ldots, -1, 0, 1, 2, 3, \ldots$ is of the form

$$\begin{aligned} 0 \to k^4 \to k^6 \to k^4 \to 0 \quad & \text{for } q \equiv 11, 12 (\text{mod } 12) \\ 0 \to 0 \to k^6 \to k^4 \to 0 \quad & \text{for } q \equiv 5, 6 (\text{mod } 12). \end{aligned}$$

Since Tate cohomology and ordinary group cohomology for finite groups coincide in positive degrees, we can identify $\hat{E}_1^{p,q}$ with $E_1^{p,q}$ for $q > 0$. Then we extend our basis $x_\sigma^{(q)}$ of $\bigoplus_{p,q} E_1^{p,q}$ to a basis of $\bigoplus_{p,q} \hat{E}_1^{p,q}$ by the following formulas. For $\sigma \in \Sigma_1' \cup \Sigma_2'$ we put

$$\begin{aligned} x_\sigma^{(-q)} &:= (x_\sigma^{(q)})^{-1} \qquad \text{for } q > 0,\ q \equiv 0 (\text{mod } 6), \\ x_\sigma^{(0)} &:= x_\sigma^{(6)} \circ x_\sigma^{(-6)} \qquad \text{and} \\ x_\sigma^{(q+5)} &:= x_\sigma^{(q)} x_\sigma^{(5)} \qquad \text{for any } q \equiv 0 (\text{mod } 6), \end{aligned}$$

where the cup products in $\hat{H}^*(\Gamma_\sigma, k)$ are used. Analogously, we define

$$\begin{array}{rcll} x_\sigma^{(-q)} & := & (x_\sigma^{(q)})^{-1} & \text{for } q > 0,\ q \equiv 0(\text{mod } 12), \\ x_\sigma^{(0)} & := & x_\sigma^{(12)} \circ x_\sigma^{(-12)} & \text{and} \\ x_\sigma^{(q+11)} & := & x_\sigma^{(q)} x_\sigma^{(11)} & \text{for any } q \equiv 0(\text{mod } 12) \end{array}$$

for $\sigma \in \Sigma_0'$. Then it is easy to check, that the restriction maps $\text{res}_{\Gamma_\sigma}^{\Gamma_\tau} : \hat{H}^*(\Gamma_\tau, k) \to \hat{H}^*(\Gamma_\sigma, k)$ for $\tau \leq \sigma$ can be computed for $q \leq 0$ as for positive degrees. Hence $(\hat{E}_1^{*,q}, d_1^{*,q})$ is isomorphic as a complex of vector spaces over k to $E_1^{*,12}$, if $q \equiv 11, 12(\text{mod } 12)$, and to $E_1^{*,6}$, if $q \equiv 5, 6(\text{mod } 12)$. This yields

$$\hat{E}_2^{p,q} \cong \begin{cases} k & \text{for } q \equiv 11, 12(\text{mod } 12) \text{ and } p = 0, 2 \\ k^2 & \text{for } q \equiv 5, 6(\text{mod } 12) \text{ and } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The formulas for the multiplicative structure stay also valid in the present context and it follows that we can consider the k-algebra $\bigoplus_{p,q} E_2^{p,q}$ with generators x, y, a_1, a_2, b_1, b_2 as a subring of $\bigoplus_{p,q} \hat{E}_2^{p,q}$. Now put

$$x^{(-12)} := \sum_{P \in \Sigma_0'} x_P^{(-12)}.$$

Then it can be shown as in the proof of lemma 25, that for any $x_\sigma^{(q)}$ defined in the present context

$$x^{(12)} \cup x_\sigma^{(q)} = x_\sigma^{(q)} \cup x^{(12)} = x_\sigma^{(q+12)}$$

and

$$x^{(-12)} \cup x_\sigma^{(q)} = x_\sigma^{(q)} \cup x^{(-12)} = x_\sigma^{(q-12)}.$$

In particular, $x^{(-12)}$ is an inverse element of $x^{(12)}$. Now it follows as in the proof of theorem 17, that the elements

$$x^n, yx^n, a_1x^n, a_2x^n, b_1x^n, b_2x^n, cx^n, dx^n$$

with $n \in \mathbb{Z}$ form a basis of $\bigoplus_{p,q} \hat{E}_2^{p,q}$. But since $c = a_1a_2$, $b_ix = a_iy$ $(i = 1, 2)$ and $dx = a_1b_2x = a_1a_2y$, the relations in theorem 17 yield

$$\bigoplus_{p,q} \hat{E}_2^{p,q} \cong k[x, x^{-1}] \otimes_k \bigwedge[a_1, a_2, y],$$

where $\bigwedge[a_1, a_2, y]$ denotes the exterior algebra associated with the vector space, which is generated by a_1, a_2 and y over k.

Theorem 18.

$$\hat{H}^*(\Gamma, k) \cong k[x, x^{-1}] \otimes_k \bigwedge[a_1, a_2, y] \tag{4.10}$$

with $\deg x = 12$, $\deg y = 11$ *and* $\deg a_1 = \deg a_2 = 7$.

Proof. As above we have to show

$$d_2^{0,q} = 0 \quad \text{for any } q \equiv 0 (\text{mod } 12). \tag{4.11}$$

This can be done exactly as above, using integral Farrell-Tate cohomology and the long exact cohomology sequence associated to $0 \to \mathbb{Z} \xrightarrow{7} \mathbb{Z} \to \mathbb{Z}/7\mathbb{Z} \to 0$. But there is another argument.

Farrell-Tate cohomology always coincides with usual group cohomology for degrees greater than the virtuell cohomological dimension of the group. Thus we have $\hat{H}^q(\Gamma, k) \cong H^q(\Gamma, k)$ for $q > 4$. This implies (4.11) for positive q. If $q \leq 0$, then write $q = -7r + s$ with $r, s > 0$. Since $d_2^{*,*}$ is a derivation on the k-algebra $\bigoplus_{p,q} \hat{E}_2^{p,q}$, [Bro82] X 4.5, it follows that

$$d_2(x^{-7r+s}) = 7x^{-6r} d_2(x^{-r}) x^s + x^{7r} d_2(x^s) = 0,$$

because 7 is the characteristic of k. □

The discussion of integral Farrell-Tate cohomology differs from ordinary cohomology with coefficients in $\mathbb{Z}$ only by the fact, that $\hat{H}^n(\Gamma, \mathbb{Z})$ is a torsion module for any q (hence also for $q = 0$). Therefore it is reasonable to consider for any prime l the l-primary component $\hat{H}^*(\Gamma, \mathbb{Z})_{(l)}$, which is a subring, clearly.

As in chapter 3, it is easy to prove, that, if we take the l-primary component $(\hat{E}_1^{p,q})_{(l)}$ of $\hat{E}_1^{p,q} = \hat{H}^q(\Gamma, C^p(X, \mathbb{Z}))$ for any q, p, we get a spectral sequence converging to $\hat{H}^*(\Gamma, \mathbb{Z})_{(l)}$.

A cyclic group C of order 7 has

$$\hat{H}^*(C, \mathbb{Z}) \cong (\mathbb{Z}/7\mathbb{Z})[\xi, \xi^{-1}] \quad \text{with } \deg \xi = 2$$

and by Swan's lemma we have

$$\hat{H}^*(\Gamma_\sigma, \mathbb{Z})_{(7)} = \begin{cases} (\mathbb{Z}/7\mathbb{Z})[\xi^6, \xi^{-6}] & \text{for } \sigma \in \Sigma_0' \\ (\mathbb{Z}/7\mathbb{Z})[\xi^3, \xi^{-3}] & \text{for } \sigma \in \Sigma_1' \cup \Sigma_2' \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as for ordinary cohomology theory, it follows, that the homomorphism, which is induced by the canonical surjection $\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$, maps $\hat{H}^*(\Gamma, \mathbb{Z})_{(7)}$ isomorphically onto the subring of $\hat{H}^*(\Gamma, \mathbb{Z}/7\mathbb{Z})$, generated by x, x^{-1}, a_1, a_2. Thus, we have shown

Corollary 18.

$$\hat{H}^*(\Gamma, \mathbb{Z})_{(7)} \cong (\mathbb{Z}/7\mathbb{Z})[x, x^{-1}] \otimes \bigwedge[a_1, a_2],$$

where $\deg x = 12$ *and* $\deg a_1 = \deg a_2 = 7$.

4.5 The cohomology in characteristic 5

In this section k denotes a field of characteristic 5, on which Γ acts trivially, and $\mathbb{F}_5 \subset k$ its prime field. Now we shall compute the cohomology rings $H^*(\Gamma, k)$ and $H^*(\Gamma, \mathbb{Z})_{[5]}$. As in characteristic 7, it follows from the Lyndon-Hochschild-Serre spectral sequence of the group extension

$$1 \to U \to \Gamma \to \Gamma/U \to 1$$

that there are canonical isomorphisms $H^*(\Gamma, k) \cong H^*(\Gamma/U, k)$ and $H^*(\Gamma, \mathbb{Z})_{[5]} \cong H^*(\Gamma/U, \mathbb{Z})_{[5]}$. Therefore, we will consider in the following always the group $G := \Gamma/U$ instead of Γ. The computation will be simplified by the fact, that there can be found generators of the cohomology rings of the stabilizers, which are essentially independent from special choices. This is provided mainly by the following simple observation.

Remark 25. Let p be a prime number and k' be a field of characteristic p. Then the cohomology ring $H^*(C, k')$ of a cyclic group C of order p is of the form $k'[x, y]$ with $\deg x = 2$, $\deg y = 1$ and $y^2 = 0$. The generators x, y of $H^*(C, k')$ can be chosen most canonically as follows. Note first, that the map $H^*(C, \mathbb{F}_p) \to H^*(C, k')$, which is induced by the inclusion $\mathbb{F}_p \subset k'$ of the prime field, is injective and induces an isomorphism $k' \otimes_{\mathbb{F}_p} H^*(C, \mathbb{F}_p) \cong H^*(C, k')$. Therefore assume, that

a) x and y are elements of $H^*(C, \mathbb{F}_p)$, i.e. classes, which are represented by $\mathbb{F}_p$-valued cocycles.

Furthermore, the long exact cohomology sequence, which is associated to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{\alpha} \mathbb{F}_p \to 0$$

implies, that the induced maps

$$H^1(C, \mathbb{F}_p) \xrightarrow{\delta} H^2(C, \mathbb{Z})$$

and

$$H^2(C, \mathbb{Z}) \xrightarrow{\alpha_*} H^2(C, \mathbb{F}_p)$$

are bijective. Therefore, to any generator $x \in H^2(C, \mathbb{F}_p)$ it is associated canonically a generator $y \in H^1(C, \mathbb{F}_p)$, such that

b) $x = (\alpha_* \circ \delta)y$.

Note, that x and y are unique with these properties up to an isomorphism of $H^*(C, k')$, which is induced by an automorphism of C. Indeed, if x' and y' are other generators satisfying **a)** and **b)**, then $x' = \lambda x$ with λ in $\mathbb{F}_p^\times$ by **a)** and **b)**

implies, that $y' = \lambda y$ holds, too. Now, part 2. of proposition 8 asserts, that there is indeed an automorphism of C, which acts by multiplication with λ on $H^1(C, k')$ and $H^2(C, k')$. A pair $(x, y) \in H^2(C, k') \times H^1(C, k')$, which satisfies **a)** and **b)**, is called in the following a pair of canonical generators for $H^*(C, k')$.

4.5.1 The 5-Sylow subgroups of the stabilizers and its normalizers

Fix again a chamber $\mathrm{Ch}_0 = \{\Lambda_{0^+}, \Lambda_{0^-}, \Lambda_2, \Lambda_{4^+}, \Lambda_{4^-}\}$ of X, whose faces provide a set of representatives for the G-orbits in the set of cells of X, and let Σ'_0 and Σ'_1 be the set of faces of Ch_0, which have the dimension 0 and 1 respectively and which do not contain the vertex Λ_2 of label 2. From the orders of the stabilizers G_σ for $\sigma \leq \mathrm{Ch}_0$, which are computed in section 4.3, we see that G_σ contains a non-trivial 5-Sylow subgroup, if and only if σ is an element in $\Sigma'_0 \cup \Sigma'_1$. More concretely, for $\tau \in \Sigma'_0$ the stabilizer G_τ is isomorphic to the special orthogonal group $\mathrm{SO}(\overline{V}, \overline{q})$ of a hyperbolic quadratic space $(\overline{V}, \overline{q})$ of dimension 8 over $\mathbb{F}_2$, and if $\sigma \in \Sigma'_1$ contains τ as a face, then the stabilizer of τ corresponds to the stabilizer of an isotropic line or of a maximal totally isotropic subspace of $(\overline{V}, \overline{q})$. But such stabilizers are isomorphic by remark 21. Therefore, when we calculate the cohomology of the stabilizers, we can restrict ourself to the group $\mathrm{SO}(\overline{V}, \overline{q})$ and the stabilizer of an isotropic line in it.

The special orthogonal group $\mathrm{SO}(\overline{V}, \overline{q})$ has the order $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$, hence it contains an abelian 5-Sylow subgroup of order 5^2. This can be described as follows.

The hyperbolic space $\overline{V}$ can be decomposed into an orthogonal sum

$$\overline{V} = \overline{V}_1 \perp \overline{V}_2$$

of two regular quadratic spaces $(\overline{V}_1, \overline{q})$ and $(\overline{V}_2, \overline{q})$ of dimension 4 and Witt index 1, cf. remark 17. Recall that there is only one isometry class of such spaces, which contain exactly 5 non-zero isotropic and 10 anisotropic vectors. Then for $i = 1, 2$ the action of $\mathrm{O}(\overline{V}_i, \overline{q})$ on the set of non-zero isotropic vectors of $\overline{V}_i$ induces isomorphisms

$$\mathrm{O}(\overline{V}_i, \overline{q}) \cong S_5 \text{ and } \mathrm{SO}(\overline{V}_i, \overline{q}) \cong A_5 \tag{4.12}$$

with the symmetric and the alternating group on 5 letters. More precisely, the reflections τ_x defined by the anisotropic vectors $x \in \overline{V}_i$ correspond to the 10 transpositions of S_5. Now let $C_i \subset \mathrm{SO}(\overline{V}_i, \overline{q}) \subset \mathrm{O}(\overline{V}_i, \overline{q})$ denote a cyclic subgroup of order 5 with generator g_i for $i = 1, 2$. Then $P := C_1 \times C_2$ is a 5-Sylow subgroup of $\mathrm{SO}(\overline{V}, \overline{q})$.

It follows at once from the isomorphisms (4.12), that the normalizer $\mathcal{N}_{\mathrm{O}(\overline{V}_i, \overline{q})}(C_i)$ is generated by g_i and an element $\tau_i \in \mathrm{O}(\overline{V}_i, \overline{q})$ of order 4 with $\tau_i g_i \tau_i^{-1} = g_i^2$ for

$i = 1, 2$ and that the normalizer of C_i in $\mathrm{SO}(\overline{V}_i, \overline{q})$ is the subgroup generated by g_i and τ_i^2. Note, that $\tau_i^2 g_i \tau_i^{-2} = g_i^{-1}$. Now, the normalizer of P in $\mathrm{SO}(\overline{V}, \overline{q})$ can be obtained in the following way. For $i = 1, 2$ the element g_i acts by cyclic permutation on the five non-zero isotropic vectors $u_1^{(i)}, \ldots, u_5^{(i)}$ of $\overline{V}_i$ and these generate the subspace $\overline{V}_i$. It follows

$$\overline{V}_i^{g_i^{\alpha_i}} = \mathbb{F}_2(u_1^{(i)} + \ldots + u_5^{(i)})$$

for $1 \leq \alpha_1, \alpha_2 \leq 4$. But by symmetry, any four of the elements $u_1^{(i)}, \ldots, u_5^{(i)}$ are a basis of $\overline{V}_i$. Therefore, since $u_1^{(i)}, \ldots, u_5^{(i)}$ are linear dependent, it follows $u_1^{(i)} + \ldots + u_5^{(i)} = 0$, because this is the only non-trivial linear combination, which includes any of the five vectors. Hence

$$\overline{V}^{g_1^{\alpha_1}} = \overline{V}_2,\ \overline{V}^{g_2^{\alpha_2}} = \overline{V}_1 \text{ and } \overline{V}^{g_1^{\alpha_1} g_2^{\alpha_2}} = (0).$$

Therefore the set $\{\overline{V}_1, \overline{V}_2\}$ of subspaces of $\overline{V}$ is determined as the set of subspaces, which are fixed pointwise by some non trivial element of P. Hence $\{u\overline{V}_1, u\overline{V}_2\} = \{\overline{V}_1, \overline{V}_2\}$ for any $u \in \mathcal{N}_{\mathrm{SO}(\overline{V}, \overline{q})}(P)$. The subgroup of all $u \in \mathrm{SO}(\overline{V}, \overline{q})$ with $u\overline{V}_1 = \overline{V}_1$ and $u\overline{V}_2 = \overline{V}_2$ is the subgroup

$$(\mathrm{O}(\overline{V}_1, \overline{q}) \times \mathrm{O}(\overline{V}_2, \overline{q})) \cap \mathrm{SO}(\overline{V}, \overline{q}) = (\mathrm{SO}(\overline{V}_1, \overline{q}) \times \mathrm{SO}(\overline{V}_2, \overline{q})) \cdot \langle \tau_1 \tau_2 \rangle.$$

But by Witt's theorem, every isometry $\overline{V}_1 \xrightarrow{\sim} \overline{V}_2$ can be extended to an isometry of $(\overline{V}, \overline{q})$, which can clearly be assumed to be an element of $\mathrm{SO}(\overline{V}, \overline{q})$. Furthermore since all Sylow subgroups are conjugate to each other, we can choose such an isometry σ satisfying $C_2 = \sigma C_1 \sigma^{-1}$ and $g_2 = \sigma g_1 \sigma^{-1}$. Therefore, the full normalizer of P in $\mathrm{SO}(\overline{V}, \overline{q})$ is easily seen to be of the form

$$(P \rtimes \langle \tau_1^2, \tau_2^2 \rangle) \rtimes \langle \tau_1 \cdot \tau_2, \sigma \rangle.$$

Let $(x_i, y_i) \in H^2(C_i, k) \times H^1(C_i, k)$ be a pair of canonical generators for $H^*(C_i, k)$ for $i = 1, 2$, as defined in remark 25. Now recall, that the Künneth formula provides an isomorphism

$$H^*(P, k) \cong H^*(C_1, k) \hat{\otimes} H^*(C_2, k) = k[x_1, x_2] \otimes \bigwedge [y_1, y_2],$$

which is a tensor product of a polynomial algebra on two generators of degree 2 and an exterior algebra on two generators of degree 1. By Swan's lemma $H^*(\mathrm{SO}(\overline{V}, \overline{q}), k)$ is the subring of invariants of $H^*(P, k)$ with respect to the action of the elements τ_1^2, τ_2^2, $\tau_1 \cdot \tau_2$ and σ. Clearly, we have

$$\begin{aligned} &\tau_1^2(x_1) = -x_1,\ \tau_1^2(y_1) = -y_1,\ \tau_1^2(x_2) = x_2,\ \tau_1^2(y_2) = y_2 \\ &\tau_2^2(x_2) = -x_2,\ \tau_2^2(y_2) = -y_2,\ \tau_2^2(x_1) = x_1,\ \tau_2^2(y_1) = y_1. \end{aligned}$$

Thus,

$$H^*(P,k)^{\langle \tau_1^2, \tau_2^2\rangle} = k[x_1^2, x_2^2] \otimes \bigwedge [x_1y_1, x_2y_2]. \tag{4.13}$$

Now, σ acts on this ring by permutation of the indices, i.e. $\sigma(x_1^2) = x_2^2$, $\sigma(x_2^2) = x_1^2$, $\sigma(x_1y_1) = x_2y_2$ and $\sigma(x_2y_2) = x_1y_1$. It is well known and not very difficult to check, that

$$(k[x_1^2, x_2^2] \otimes \bigwedge [x_1y_1, x_2y_2])^\sigma = k[x_1^2 + x_2^2, x_1^2x_2^2] \otimes \bigwedge [x_1y_1 + x_2y_2, x_1^2x_2y_2 + x_2^2x_1y_1], \tag{4.14}$$

cf. [AM94] ch. III thm. 4.2, where the notation in (4.13) and (4.14) means, that the rings described are in fact isomorphic to a tensor product of a polynomial algebra and an exterior algebra on generators of the respective degree. By the identities

$$\begin{aligned} \tau_1 \cdot \tau_2(x_1^2 + x_2^2) &= -(x_1^2 + x_2^2) \\ \tau_1 \cdot \tau_2(x_1^2x_2^2) &= x_1^2x_2^2 \\ \tau_1 \cdot \tau_2(x_1y_1 + x_2y_2) &= -(x_1y_1 + x_2y_2) \\ \tau_1 \cdot \tau_2(x_1^2x_2y_2 + x_2^2x_1y_1) &= x_1^2x_2y_2 + x_2^2x_1y_1, \end{aligned}$$

it follows

Proposition 20.

$$H^*(\mathrm{SO}(\overline{V}, \overline{q}), k) \cong H^*(P,k)^{\langle \tau_1^2, \tau_2^2, \tau_1\cdot\tau_2, \sigma\rangle} = k[a,b] \otimes \bigwedge [c,d]$$

is the tensor product of a polynomial algebra with two generators $a = x_1^4 + x_2^4$ and $b = x_1^2x_2^2$ of degree 8 and an exterior algebra on two generators $c = x_1^3y_1 + x_2^3y_2$ and $d = x_1^2x_2y_2 + x_2^2x_1y_1$ of degree 7.

The 5-part of integral cohomology can be computed quite analogously.

Lemma 26.

$$H^*(P, \mathbb{Z}) = \mathbb{Z}[\xi_1, \xi_2, \eta]$$

with $5\xi_1 = 5\xi_2 = 5\eta = \eta^2 = 0$, $\deg \xi_1 = \deg \xi_2 = 2$ and $\deg \eta = 3$. Here η can be chosen to be the image of $y_1 \cdot y_2$ under the Bockstein homomorphism $H^2(P, \mathbb{Z}/5\mathbb{Z}) \to H^3(P, \mathbb{Z})$.

I sketch a proof of this well known fact, because it is instructive for the following.

Proof. Consider the field $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$. It follows from the Künneth formula in the form (3.3)in section 3.1.1, that $H^n(P, \mathbb{Z})$ is a $\mathbb{F}_5$-vector space for $n > 0$ of dimension $\frac{n}{2}$, if n is even, and of dimension $\frac{n-1}{2}$, if n is odd. Further, the long exact cohomology sequence associated to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to \mathbb{F}_5 \to 0$$

of P-modules shows, that the ring homomorphism

$$\iota : H^*(P, \mathbb{Z}) \to H^*(P, \mathbb{F}_5)$$

is injective in positive degrees, and that the Bockstein map

$$\delta : H^n(P, \mathbb{F}_5) \to H^{n+1}(P, \mathbb{Z})$$

is surjective for any $n \geq 0$. Therefore $H^2(P, \mathbb{Z})$ is generated by $\xi_1 = \delta(y_1)$ and $\xi_2 = \delta(y_2)$. Recall from remark 25, that since (x_i, y_i) is a canonical pair of generators for $H^i(C, k)$, we have $\iota(\xi_i) = \iota \circ \delta(y_i) = x_i$ for $i = 1, 2$. By exactness, the kernel of

$$\delta : H^2(P, \mathbb{F}_5) \to H^3(P, \mathbb{Z})$$

is generated by $x_1 = \iota(\xi_1)$ and $x_2 = \iota(\xi_2)$. Therefore $\eta := \delta(y_1 \cdot y_2)$ is a non-trivial element of $H^3(P, \mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z}$. □

In the rest of this paragraph, we assume that y_1, y_2 are contained in $H^1(P, \mathbb{F}_5) \subset H^1(P, k)$, $\xi_1 = \delta(y_1), \xi_2 = \delta(y_2) \in H^2(P, \mathbb{Z})$, $x_1 = \iota(\xi_1), x_2 = \iota(\xi_2) \in H^2(P, k)$ and $\eta = \delta(y_1 \cdot y_2) \in H^3(P, \mathbb{Z})$.

It follows that τ_1^2 fixes ξ_2 and acts on ξ_1 and η by multiplication with -1 and similar for τ_2^2. Therefore $H^*(P, \mathbb{Z})^{\langle \tau_1^2, \tau_2^2 \rangle}$ is generated over $\mathbb{Z}$ by

$$\xi_1^2,\ \xi_2^2,\ \xi_1\eta,\ \xi_2\eta.$$

But $\tau_1 \cdot \tau_2(\xi_i) = 2\xi_i$ for $i = 1, 2$ and $\tau_1 \cdot \tau_2(\eta) = -\eta$, hence $\tau_1 \cdot \tau_2(\xi_i^2) = -\xi_i$ and $\tau_1 \cdot \tau_2(\xi_i\eta) = -2\xi_i\eta$. This provides the fact, that $H^*(P, \mathbb{Z})^{\langle \tau_1^2, \tau_2^2, \tau_1 \cdot \tau_2 \rangle}$ is generated by $\xi_1^4, \xi_1^2\xi_2^2, \xi_2^4$. An easy argument shows

Proposition 21.

$$H^*(\mathrm{SO}(\overline{V}, \overline{q}), \mathbb{Z})_{[5]} = H^*(P, \mathbb{Z})^{\langle \tau_1^2, \tau_2^2, \tau_1 \cdot \tau_2, \sigma \rangle} = \mathbb{Z}[\alpha, \beta]$$

is a polynomial algebra with two generators $\alpha := \xi_1^4 + \xi_2^4$ *and* $\beta := \xi_1^2 \cdot \xi_2^2$ *of degree 8.*

Now, let $G_0 \subset \mathrm{SO}(\overline{V}, \overline{q})$ be the stabilizer of an isotropic line λ contained in $\overline{V}_2$. By (4.12) we find a reflection $\rho \in \mathrm{O}(\overline{V}_2, \overline{q}_2)$, which fixes λ. Thus, there is an injective homomorphism

$$\alpha : \mathrm{O}(\overline{V}_1, \overline{q}_1) \hookrightarrow G_0,\ u \mapsto u \cdot \rho^{D(u)},$$

where $D(u) \in \mathbb{Z}/2\mathbb{Z}$ denotes the Dickson invariant of u. The image $\alpha(\mathrm{O}(\overline{V}_1, \overline{q}_1)) \subset G_0$ contains the cyclic subgroup C_1 and this is a 5-Sylow subgroup of G_0, because

$$|G_0| = 2^{12} \cdot 3^2 \cdot 5 \cdot 7, \quad \text{cf. section 4.1.1.}$$

Therefore, it follows from Swan's lemma, that

$$H^*(G_0,k) \cong H^*(S_5,k) \text{ and } H^*(G_0,\mathbb{Z})_{[5]} \cong H^*(S_5,\mathbb{Z})_{[5]}.$$

In order to distinguish the elements of $H^*(C_1,k)$ and $H^*(C_1,\mathbb{Z})$ from the elements of $H^*(P,k)$ and $H^*(P,\mathbb{Z})$, I write in the following x,y (resp. ξ) instead of x_1, y_1 (resp. ξ_1), when they are considered as elements of $H^*(C_1,k)$ and $H^*(C_1,\mathbb{Z})$. Hence we have $H^*(C_1,k) = k[x,y]$ with $\deg x = 2$, $\deg y = 1$ and $y^2 = 0$ and $H^*(C_1,\mathbb{Z}) = \mathbb{Z}[\xi]$ with $\deg \xi = 2$ and $5\xi = 0$. It follows immediatly from Swan's lemma

Proposition 22.

$$H^*(G_0,k) = k[x^4, x^3y] \textit{ and } H^*(G_0,\mathbb{Z})_{[5]} = \mathbb{Z}[\xi^4].$$

Note, that the generators $v := x^4$ and $w := x^3y$ of $H^*(G_0,k)$ are independent from the canonical pair (x,y), which is chosen above, since two canonical pairs differ only by a common factor $\lambda \in \mathbb{F}_5^\times$.

4.5.2 The restriction maps

Recall from the description of the Tits building in chapter 2, that there are two conjugacy classes of stabilizers of maximal totally isotropic subspaces in $\mathrm{SO}(\overline{V},\overline{q})$, and let the subgroups $G_1, G_2 \subset \mathrm{SO}(\overline{V},\overline{q})$ be representatives of these classes. By remark 21 the groups G_1 and G_2 are isomorphic to G_0 and it follows from the discussion above, that there are unique isomorphisms

$$H^*(G_i,k) \to H^*(G_0,k) \qquad \text{for } i = 1,2,$$

which are induced by isomorphisms $G_0 \cong G_i$. Therefore their cohomology rings can be identified naturally in a unique way:

$$H^*(G_1,k) = H^*(G_2,k) = H^*(G_0,k) = k[v,w].$$

Now, for $i = 0,1,2$ the restriction maps

$$\mathrm{res}_{G_i}^{\mathrm{SO}(\overline{V},\overline{q})} : H^*(\mathrm{SO}(\overline{V},\overline{q}),k) \to H^*(G_i,k)$$

will be computed. It will turn out, that the special set of generators a,b,c,d of $H^*(\mathrm{SO}(\overline{V},\overline{q}),k)$ can be described completely in terms of these restriction maps.

By construction, the restriction map $\mathrm{res}_{C_1}^P : H^*(P,k) \to H^*(C_1,k)$ is given by

$$\begin{aligned} x_1 &\mapsto x \quad y_1 \mapsto y \\ x_2 &\mapsto 0 \quad y_2 \mapsto 0. \end{aligned}$$

Now, the diagram of inclusions

$$\begin{array}{ccc} P & \hookleftarrow & C_1 \\ \downarrow & & \downarrow \\ \mathrm{SO}(\overline{V}, \overline{q}) & \hookleftarrow & G_0 \end{array}$$

induces in cohomology

$$\begin{array}{ccc} H^*(P,k) & \longrightarrow & H^*(C_1,k) \\ \uparrow & & \uparrow \\ H^*(\mathrm{SO}(\overline{V}, \overline{q}),k) & \longrightarrow & H^*(G_0,k) \end{array}$$

This gives

$$\begin{array}{rcl} \mathrm{res}_{G_0}^{\mathrm{SO}(\overline{V},\overline{q})}(a) & = & \mathrm{res}_{C_1}^{P}(x_1^4 + x_2^4) = x^4 = v \\ \mathrm{res}_{G_0}^{\mathrm{SO}(\overline{V},\overline{q})}(b) & = & \mathrm{res}_{C_1}^{P}(x_1^2 x_2^2) = 0 \\ \mathrm{res}_{G_0}^{\mathrm{SO}(\overline{V},\overline{q})}(c) & = & \mathrm{res}_{C_1}^{P}(x_1^3 y_1 + x_2^3 y_2) = x^3 y = w \\ \mathrm{res}_{G_0}^{\mathrm{SO}(\overline{V},\overline{q})}(d) & = & \mathrm{res}_{C_1}^{P}(x_1^2 x_2 y_2 + x_2^2 x_1 y_1) = 0. \end{array} \tag{4.15}$$

We have also to compute the restriction maps $\mathrm{res}_{G_1}^{\mathrm{SO}(\overline{V},\overline{q})}$ and $\mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})}$, where G_1 and G_2 are the stabilizers of maximal totally isotropic subspaces W_1, W_2 in $(\overline{V}, \overline{q})$, which represent vertices of different type in the Tits building associated with $(\overline{V}, \overline{q})$, cf. section 2.2. Therefore, I first construct two such spaces. As above fix a generator $g_1 \in C_1$. Let $\rho \in \mathcal{N}_{\mathrm{O}(\overline{V},\overline{q})}(P)$ be a map with $\rho g_1 \rho^{-1} \in C_2$ (thus $\rho \overline{V}_1 = \overline{V}_2$) and put $h := g_1 \rho g_1 \rho^{-1} \in P$. Further let f_1 be a non-zero isotropic vector in $\overline{V}_1$ and put $f = f_1 + \rho f_1$.

Lemma 27. *The elements $h^\nu f$ for $\nu = 0, \ldots, 4$ generate a totally isotropic and $\langle h \rangle$-invariant subspace W_ρ of dimension 4 in $\overline{V}$.*

Proof. That W_ρ is $\langle h \rangle$-invariant is clear. Note that

$$h^\nu f = g_1^\nu f_1 + (\rho g_1 \rho^{-1})^\nu \rho f_1 = g_1^\nu f_1 + \rho g_1^\nu f_1,$$

because $g_1 \in C_1$ and $\rho g_1 \rho^{-1} \in C_2$. Therefore, any element of W_ρ is of the form $w = z + \rho z$ with $z \in \overline{V}_1$, which proves that $\overline{q}(w) = \overline{q}(z) + \overline{q}(\rho z) = 2\overline{q}(z) = 0$, i.e. W_ρ is totally isotropic. Further by construction, the elements $g_1^\nu f_1$ for $\nu = 0, \ldots, 4$ are the five non-zero isotropic vectors of $\overline{V}_1$, which generate $\overline{V}_1$(cf. the last section). Therefore the linear projection $p : \overline{V} \to \overline{V}_1$ with kernel $\overline{V}_2$ maps W_ρ surjectively onto $\overline{V}_1$. But p has a splitting

$$\mathrm{id}_{\overline{V}_1} + \rho : \overline{V}_1 \to W_\rho,$$

hence $\dim W_\rho = \dim \overline{V}_1 = 4$. □

Now apply this construction, where ρ is the element σ (resp. $\tau_2\sigma$) defined in the computation of $H^*(\mathrm{SO}(\overline{V},\overline{q}),k)$. Then we have

$$h = g_1\sigma g_1\sigma^{-1} = g_1g_2 \qquad (\text{resp. } h = g_1\tau_2 g_2\tau_2^{-1} = g_1g_2^2).$$

If G_1 (resp. G_2) is the stabilizer of $W_\sigma := (\mathrm{id}_{\overline{V}_1}+\sigma)\overline{V}_1$ (resp. $W_\rho := (\mathrm{id}_{\overline{V}_1}+\tau_2\sigma)\overline{V}_1$) in $\mathrm{SO}(\overline{V},\overline{q})$, then $C_\sigma := \langle g_1g_2\rangle$ (resp. $C_\rho := \langle g_1g_2^2\rangle$) is a 5-Sylow subgroup of G_1 (resp. G_2). Identifying C_σ and C_ρ with C_1 via some isomorphism we can write

$$H^*(C_\sigma,k) = H^*(C_\rho,k) = H^*(C_1,k) = k[x,y].$$

Lemma 28. *1. The restriction map $H^*(P,k)\to H^*(C_\sigma,k)$ is of the form*

$$x_1\mapsto x,\ x_2\mapsto x,\ y_1\mapsto y,\ y_2\mapsto y$$

up to an automorphism of C_σ.
2. On the other hand $\mathrm{res}^P_{C_\rho}: H^*(P,k)\to H^*(C_\rho,k)$ is given by

$$x_1\mapsto x,\ x_2\mapsto 3x,\ y_1\mapsto y,\ y_2\mapsto 3y$$

up to an automorphism of C_ρ.

Proof. The assertions 1. and 2. in the lemma can be verified by the same arguments, therefore I prove here only 2. The generator $g_1g_2^2$ of C_ρ is fixed by the automorphism κ of the elementary abelian group P, which is defined by $\kappa(g_1)=g_2^2$ and $\kappa(g_2)=g_1^3$. This map induces an automorphism κ^* of $H^*(P,k)$ with

$$\kappa^*(x_1)=2x_2,\ \kappa^*(x_2)=3x_1,\ \kappa^*(y_1)=2y_2 \text{ and } \kappa^*(y_2)=3y_1.$$

Therefore the eigenspace with eigenvalue -1 of the endomorphism $\kappa^*: H^i(P,k)\to H^i(P,k)$ for $i=2$ (resp. $i=1$) is generated by x_1+3x_2 (resp. y_1+3y_2). Since $\mathrm{res}^P_{C_\rho}\circ\kappa^* = \mathrm{res}^P_{C_\rho}$, it follows, that the kernel of the restriction map is generated in degree 2 and 1 by the elements x_1+3x_2 and y_1+3y_2. Therefore, x_1 is mapped to a non trivial element of $H^2(C_\rho,\mathbb{F}_5)\subset H^2(C_\rho,k)$, say x. Then $\mathrm{res}^P_{C_\rho}x_2 = 3x$ follows from the identity

$$0 = \mathrm{res}^P_{C_\rho}(x_1+3x_2) = \mathrm{res}^P_{C_\rho}x_1 + 3\mathrm{res}^P_{C_\rho}x_2.$$

The analogous identities hold for y_1 and y_2, since the pair (x,y) is chosen to be a canonical set of generators. This proves 2. □

This provides the values

$$\begin{array}{rcl}
\mathrm{res}^{\mathrm{SO}(\overline{V},\overline{q})}_{G_1}(a) &=& \mathrm{res}^P_{C_\sigma}(x_1^4+x_2^4) = 2x^4 = 2v\\
\mathrm{res}^{\mathrm{SO}(\overline{V},\overline{q})}_{G_1}(b) &=& \mathrm{res}^P_{C_\sigma}(x_1^2x_2^2) = v\\
\mathrm{res}^{\mathrm{SO}(\overline{V},\overline{q})}_{G_1}(c) &=& \mathrm{res}^P_{C_\sigma}(x_1^3y_1+x_2^3y_2) = 2x^3y = 2w\\
\mathrm{res}^{\mathrm{SO}(\overline{V},\overline{q})}_{G_1}(d) &=& \mathrm{res}^P_{C_\sigma}(x_1^2x_2y_2+x_2^2x_1y_1) = 2x^3y = 2w.
\end{array} \tag{4.16}$$

and

$$\begin{array}{rcl} \mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})}(a) & = & \mathrm{res}_{C_\rho}^{P}(x_1^4 + x_2^4) = 2x^4 = 2v \\ \mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})}(b) & = & \mathrm{res}_{C_\rho}^{P}(x_1^2 x_2^2) = -x^4 = -v \\ \mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})}(c) & = & \mathrm{res}_{C_\rho}^{P}(x_1^3 y_1 + x_2^3 y_2) = 2x^3 y = 2w \\ \mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})}(d) & = & \mathrm{res}_{C_\rho}^{P}(x_1^2 x_2 y_2 + x_2^2 x_1 y_1) = -2x^3 y = -2w. \end{array} \tag{4.17}$$

Summing up, we have shown

Lemma 29. *The bases $\{a, b\}$ of $H^8(\mathrm{SO}(\overline{V}, \overline{q}), k)$ and $\{c, d\}$ of $H^7(\mathrm{SO}(\overline{V}, \overline{q}), k)$ are determined uniquely by the properties (4.15), (4.16) and (4.17).*

Proof. If $\{a', b'\}$ is given as in the lemma, then b' is contained in the kernel of $\mathrm{res}_{G_0}^{\mathrm{SO}(\overline{V},\overline{q})}$, hence, it is of the form λb with $\lambda \in k^\times$. Then (4.16) implies that $b' = b$. Since $\ker(\mathrm{res}_{G_1}^{\mathrm{SO}(\overline{V},\overline{q})}) \cap \ker(\mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})}) = \{0\}$, the set

$$(\mathrm{res}_{G_1}^{\mathrm{SO}(\overline{V},\overline{q})})^{-1}(2 \cdot \mathrm{res}_{G_1}^{\mathrm{SO}(\overline{V},\overline{q})} b') \cap (\mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})})^{-1}(-2 \cdot \mathrm{res}_{G_2}^{\mathrm{SO}(\overline{V},\overline{q})} b'),$$

which contains a', consists also of only one element, which is a, clearly. For the generators of degree 7 the assertion is proved in the same manner. □

Remark 26. The group $\mathrm{Out}(\mathrm{SO}(\overline{V}, \overline{q})) = \mathrm{Aut}(\mathrm{SO}(\overline{V}, \overline{q}))/\mathrm{Inn}(\mathrm{SO}(\overline{V}, \overline{q}))$ of automorphisms modulo inner automorphisms of $\mathrm{SO}(\overline{V}, \overline{q})$ acts by permutation on the set of conjugacy classes $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ of the parabolic subgroups G_0, G_1 and G_2 of $\mathrm{SO}(\overline{V}, \overline{q})$, cf. remark 21. From the last lemma, it can be read off, how $\mathrm{Out}(\mathrm{SO}(\overline{V}, \overline{q}))$ acts on $H^*(\mathrm{SO}(\overline{V}, \overline{q}), k)$. Let ψ_1 be for example an automorphism of $\mathrm{SO}(\overline{V}, \overline{q})$, which fixes $\mathcal{C}_0$ and permutes $\mathcal{C}_1$ with $\mathcal{C}_2$, then the map

$$\psi_1^* : H^*(\mathrm{SO}(\overline{V}, \overline{q}), k) \to H^*(\mathrm{SO}(\overline{V}, \overline{q}), k),$$

is necessarily of the form

$$\begin{array}{rcl} \psi_1^*(a) & = & a \\ \psi_1^*(b) & = & -b \\ \psi_1^*(c) & = & c \\ \psi_1^*(d) & = & -d, \end{array}$$

such that the lemma is satisfied. A "triality" automorphism ψ_2, which maps $\mathcal{C}_0$ to $\mathcal{C}_1$, $\mathcal{C}_1$ to $\mathcal{C}_2$ and $\mathcal{C}_2$ to $\mathcal{C}_0$, induces on cohomology a map ψ_2^* given by

$$\begin{array}{rcl} \psi_2^*(a) & = & 2(a - b) \\ \psi_2^*(b) & = & a + 2b \\ \psi_2^*(c) & = & 2c - d \\ \psi_2^*(d) & = & 2(c + d). \end{array}$$

4.5.3 The computation of the cohomology ring

We shall now compute the spectral sequence

$$E_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} H^q(G_\sigma, k).$$

By the results of the last section, the group $H^q(G_\sigma, k)$ is trivial for $q > 0$ unless $\sigma \in \Sigma_0' \cup \Sigma_1'$ and $q \equiv 0, 7 (\mathrm{mod}\ 8)$ or $\sigma \in \Sigma_0'$, $q \geq 14$ and $q \equiv 6 (\mathrm{mod}\ 8)$. Now, we use bases as in lemma 29 to describe the differential maps

$$d_1^{0,q} : \bigoplus_{\sigma \in \Sigma_0} H^q(G_\sigma, k) \to \bigoplus_{\sigma \in \Sigma_1} H^q(G_\sigma, k)$$

for $q = 8, 7$.

Let G_{0^+}, G_{0^-}, G_{4^+}, G_{4^-} denote the stabilizers of the vertices $\Lambda_{0^+}, \Lambda_{0^-}, \Lambda_{4^+}, \Lambda_{4^-}$ and $G_{\{0^+,0^-\}}$, $G_{\{0^+,4^+\}}$, $G_{\{0^+,4^-\}}$, $G_{\{0^-,4^+\}}$, $G_{\{0^-,4^-\}}$, $G_{\{4^+,4^-\}}$ be the stabilizers of the 1-simplices $\{\Lambda_{0^+}, \Lambda_{0^-}\}$, $\{\Lambda_{0^+}, \Lambda_{4^+}\}$, $\{\Lambda_{0^+}, \Lambda_{4^-}\}$, $\{\Lambda_{0^-}, \Lambda_{4^+}\}$, $\{\Lambda_{0^-}, \Lambda_{4^-}\}$, $\{\Lambda_{4^+}, \Lambda_{4^-}\}$ in $G = \Gamma/U$ respectively. For

$$\alpha \in \{\{0^+, 0^-\}, \{0^+, 4^+\}, \{0^+, 4^-\}\{0^-, 4^+\}, \{0^-, 4^-\}, \{4^+, 4^-\}\}$$

there are unique canonical generators v_α and w_α of $H^8(G_\alpha, k)$ and $H^7(G_\alpha, k)$. Furthermore, if we identify G_{0^+} with $\mathrm{SO}(\overline{V}_{\Lambda_{0^+}}, \overline{q})$, G_{0^-} with $\mathrm{SO}(\overline{V}_{\Lambda_{0^-}}, \overline{q})$, G_{4^+} with $\mathrm{SO}(\overline{V}^{\Lambda_{4^+}}, \overline{q})$ and G_{4^-} with $\mathrm{SO}(\overline{V}^{\Lambda_{4^-}}, \overline{q})$, then we find for any $\beta \in \{0^+, 0^-, 4^+, 4^-\}$ bases a_β, b_β of $H^8(G_\beta, k)$ and c_β, d_β of $H^7(G_\beta, k)$ as described in lemma 29 and these bases are unique up to a common factor ± 1 of b_β and d_β. Here we make the following choices

$$\begin{aligned}
\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(b_{0^+}) &= v_{\{0^+,4^+\}} \\
\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(d_{0^+}) &= 2w_{\{0^+,4^+\}} \\
\mathrm{res}^{G_{0^-}}_{G_{\{0^-,4^-\}}}(b_{0^-}) &= v_{\{0^-,4^-\}} \\
\mathrm{res}^{G_{0^-}}_{G_{\{0^-,4^-\}}}(d_{0^-}) &= 2w_{\{0^-,4^-\}} \\
\mathrm{res}^{G_{4^+}}_{G_{\{0^+,4^+\}}}(b_{4^+}) &= v_{\{0^+,4^+\}} \\
\mathrm{res}^{G_{4^+}}_{G_{\{0^+,4^+\}}}(d_{0^+}) &= 2w_{\{0^+,4^+\}} \\
\mathrm{res}^{G_{4^-}}_{G_{\{0^-,4^-\}}}(b_{4^-}) &= v_{\{0^-,4^-\}} \\
\mathrm{res}^{G_{4^-}}_{G_{\{0^-,4^-\}}}(d_{0^-}) &= 2w_{\{0^-,4^-\}}
\end{aligned}$$

Summing up, we have fixed a basis $\mathfrak{A} := \{a_{0^+}, b_{0^+}, a_{0^-}, b_{0^-}, a_{4^+}, b_{4^+}, a_{4^-}, b_{4^-}\}$ of

$$E_1^{0,8} = \bigoplus_{P \in \Sigma_0'} H^8(G_P, k)$$

and a basis $\mathfrak{C} := \{c_{0^+}, d_{0^+}, c_{0^-}, d_{0^-}, c_{4^+}, d_{4^+}, c_{4^-}, d_{4^-}\}$ of $E_1^{0,7}$, furthermore bases

$$\mathfrak{V} := \{v_{\{0^+,0^-\}}, v_{\{0^+,4^+\}}, v_{\{0^+,4^-\}}, v_{\{0^-,4^+\}}, v_{\{0^-,4^-\}}, v_{\{4^+,4^-\}}\}$$

and

$$\mathfrak{W} := \{w_{\{0^+,0^-\}}, w_{\{0^+,4^+\}}, w_{\{0^+,4^-\}}, w_{\{0^-,4^+\}}, w_{\{0^-,4^-\}}, w_{\{4^+,4^-\}}\}$$

of $E_1^{1,8}$ and $E_1^{1,7}$, respectively.

Theorem 19. *1. The map $d_1^{0,8}$ (resp. $d_1^{0,7}$) is surjective and its kernel is generated by $a := a_{0^+} + a_{0^-} + a_{4^+} + a_{4^-}$ and $b := b_{0^+} + b_{0^-} + b_{4^+} + b_{4^-}$ (resp. $c := c_{0^+} + c_{0^-} + c_{4^+} + c_{4^-}$ and $d := d_{0^+} + d_{0^-} + d_{4^+} + d_{4^-}$).*
2. The differential map $d_1^{0,q} : E_1^{0,q} \to E_1^{1,q}$ is surjective for any positive q.
3. The cohomology ring $H^*(\Gamma, k) \cong H^*(G, k)$ is isomorphic as a ring to the kernel of the differential

$$d_1^{0,*} : \bigoplus_{\tau \in \Sigma_0'} H^*(G_\tau, k) \to \bigoplus_{\sigma \in \Sigma_1'} H^*(G_\sigma, k),$$

in particular

$$\dim_k H^q(\Gamma, k) := \begin{cases} 1 & \text{for } q = 0 \\ \frac{q}{2} - 2 & \text{for } q > 0,\ q \equiv 0 \ (mod\ 8) \\ q - 5 & \text{for } q > 0,\ q \equiv 7 \ (mod\ 8) \\ \frac{q}{2} - 3 & \text{for } q \geq 14,\ q \equiv 6 \ (mod\ 8) \\ 0 & \text{otherwise} \end{cases}$$

Proof. 1. Using the formula for $d_1^{*,*}$ in proposition 10, it follows immediately from lemma 29 above, that $d_1^{0,8}$ (resp. $d_1^{0,7}$) has the matrices

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -2 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 2 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 & 2 & -2 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the bases $\mathfrak{A}$ and $\mathfrak{V}$ (resp. $\mathfrak{C}$ and $\mathfrak{W}$) and the assertions in 1. can be verified immediatly.

Part 2. has to be checked for $q \equiv 7, 8 (\text{mod } 8)$ only, where $E_1^{1,q}$ is not the trivial module. For $q = 7$ and $q = 8$ it is checked already in 1. Now, we can use the multiplicative structure of the spectral sequence (4.5.3) defined in section 3.1.3. Let the products of this multiplicative structure be denoted by the symbol $\cup$ and the products in the rings $H^*(G_\sigma, k)$ for $\sigma \in \Sigma_0' \cup \Sigma_1'$ by $\circ$. Then it follows from proposition 13, that for any $z = \sum_{\sigma \in \Sigma_1'} z_\sigma \in \bigoplus_{\sigma \in \Sigma_1'} H^*(G_\sigma, k)$ we have

$$\begin{aligned} z \cup a &= z_{\{0^+,0^-\}} \cup a_{0^-} + z_{\{0^+,4^+\}} \cup a_{4^+} + z_{\{0^+,4^-\}} \cup a_{4^-} \\ &\quad + z_{\{0^-,4^+\}} \cup a_{4^+} + z_{\{0^-,4^-\}} \cup a_{4^-} + z_{\{4^+,4^-\}} \cup a_{4^-} \\ &= z_{\{0^+,0^-\}} \circ v_{\{0^+,0^-\}} + z_{\{0^+,4^+\}} \circ 2v_{\{0^+,4^+\}} + z_{\{0^+,4^-\}} \circ 2v_{\{0^+,4^-\}} \\ &\quad + z_{\{0^-,4^+\}} \circ 2v_{\{0^-,4^+\}} + z_{\{0^-,4^-\}} \circ 2v_{0^-,4^-} + z_{\{4^+,4^-\}} \circ v_{\{4^+,4^-\}}. \end{aligned}$$

From this it follows easily, that multiplication with a provides an isomorphism $E_1^{1,q} \to E_1^{1,q+8}$ for any $q > 0$, because the cup product with v_σ gives an isomorphism $H^q(G_\sigma, k) \to H^{q+8}(G_\sigma, k)$ for any $\sigma \in \Sigma_1'$. Therefore we can write any element $z \in E_1^{1,q}$ with $q \equiv 7, 8 (\text{mod } 8)$ as a product $z = z_0 \cup a^n$ with $z_0 \in E_1^{1,q_0}$ for $q_0 = 7$ or $q_0 = 8$ and $n \in \mathbb{N}_0$. By 1. there is an element $x \in E_1^{0,q_0}$, such that $d_1^{0,q_0}(x) = z_0$. Then it follows with $d_1 = d_1^{*,*}$

$$d_1(x \cup a^n) = d_1(x) \cup a^n + (-1)^{deg(x)} x \cup n a^{n-1} d_1(a) = z_0 \cup a^n = z,$$

since d_1 is a derivation of the ring $\bigoplus_{p,q} E_1^{p,q}$ (cf. [Bro89] ch. X 4.5) and a is an element in the kernel of d_1 by 1. This proves 2.

3. As in characteristic 7 it follows from proposition 11, that $(E_1^{p,0}, d_1^{p,0})$ is isomorphic to the simplicial cochain complex of the chamber Ch_0. Hence $E_2^{p,0} = 0$ for any positive p and it follows from part 2., that the spectral sequence $E_r^{p,q}$ is concentrated in the row $E_2^{0,q}$ for $r = 2$. This provides the first assertion, because $E_r^{p,q}$ converges to $H^*(\Gamma, k)$ and the ring structure of $H^*(\Gamma, k)$ is determined by the multiplicative structure on the spectral sequence (cf. [Bro89] ch. X 4.5). Therefore, it remains to check the formulas for the dimensions of the cohomology spaces. Recall, that for $\sigma \in \Sigma_1'$ we have

$$\dim_k H^q(G_\sigma, k) = \begin{cases} 1 & \text{for } q \geq 0,\ q \equiv 7, 8 \text{ (mod 8)} \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, for $x \in \Sigma_0'$ proposition 20 provides

$$\dim_k H^q(G_x, k) = \begin{cases} \frac{q}{8} + 1 & \text{for } q \geq 0,\ q \equiv 0 \text{ (mod 8)} \\ 2 \cdot (\frac{q-7}{8} + 1) = \frac{q+1}{4} & \text{for } q \geq 0,\ q \equiv 7 \text{ (mod 8)} \\ \frac{q-14}{8} + 1 = \frac{q-6}{8} & \text{for } q \geq 14,\ q \equiv 6 \text{ (mod 8)} \\ 0 & \textit{otherwise.} \end{cases}$$

On the other hand, by part 2. above, we have the following formula for the dimensions of the cohomology spaces $H^q(G,k)$ for $q>0$:

$$\dim_k H^q(G,k) = 4\ \dim_k H^q(G_x,k) - 6\, H^q(G_\sigma,k).$$

This completes the proof. □

We have seen in the last section, that for any simplex $\sigma \in \Sigma_0' \cup \Sigma_1'$ the 5-part of the integral cohomology ring is of the form

$$H^q(G_\sigma,\mathbb{Z})_{[5]} \cong \begin{cases} \mathbb{Z} & \text{for } q=0 \\ H^q(G_\sigma,\mathbb{Z}/5\mathbb{Z}) & \text{for } q>0,\ q\equiv 0 \pmod 8 \\ 0 & \text{otherwise} \end{cases}$$

and that the isomorphism in positive degrees $q \equiv 0 \pmod 8$ is provided by the natural map, which is induced by $\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$. Now, exactly as in characteristic 7, it follows from the functoriality of the spectral sequence with respect to the module of coefficients (cf. lemma 16):

Corollary 19. *The 5-part of the integral cohomology of* Γ *is of the form*

$$H^q(\Gamma,\mathbb{Z})_{[5]} \cong H^q(G,\mathbb{Z})_{[5]} \cong \begin{cases} \mathbb{Z} & \textit{for } q=0 \\ H^q(\Gamma,\mathbb{Z}/5\mathbb{Z}) & \textit{for } q>0,\ q\equiv 0\ (\textit{mod } 8) \\ 0 & \textit{otherwise} \end{cases}$$

and the natural map

$$H^q(\Gamma,\mathbb{Z})_{[5]} \to H^q(\Gamma,\mathbb{Z}/5\mathbb{Z})$$

is injective for $q>0$.

4.5.4 Generators and relations

It remains to describe the ring $H^*(\Gamma,k)$ by a minimal set of generators and relations. Note first, that the subring, which is generated by $1, a, b, c$ and d is of the form

$$k[a,b] \otimes_k \bigwedge[c,d],$$

i.e. a tensor product of a polynomial with an exterior algebra on two generators of degree 8 and 7 respectively, isomorphic to the cohomology rings $H^*(G_x,k)$ for a vertex $x \in \Sigma_0'$. Comparing dimensions, it follows at once, that one has to choose further generators of degree 14, 15 and 16. We shall do this and prove, that $H^*(G,k)$ is generated by the elements of degree ≤ 16.

Let $1_{0^+}, 1_{0^-}, 1_2, 1_{4^+}, 1_{4^-}$ be the units of the rings $H^*(G_x, k)$ for $x \in \Sigma_0$. Then, considered as elements of the ring

$$E_1^{0,*} = \bigoplus_{x \in \Sigma_0} H^*(G_x, k)$$

the set $D = \{1_{0^+}, 1_{0^-}, 1_2, 1_{4^+}, 1_{4^-}\}$ provides a system of orthogonal idempotents in the ring $\bigoplus_{x\in\Sigma_0} H^*(G_x, k)$, i.e. any two elements $e, e' \in D$ satisfy

$$e \cdot e' = \begin{cases} e & \text{for } e = e' \\ 0 & \text{for } e \neq e' \end{cases}$$

and $1_{0^+} + 1_{0^-} + 1_2 + 1_{4^+} + 1_{4^-} = 1$. It follows from theorem 19, that we can consider $H^*(G, k)$ as subring of $E_1^{0,*}$. Therefore we can write $a_{0^+} = a \cdot 1_{0^+}$, $b_{0^+} = b \cdot 1_{0^+}$, $a_{0^-} = a \cdot 1_{0^-}$ and so on.

We begin the computation in degree 16. The formula in proposition 10 together with the funcoriality of the cup-product with respect to group homomorphisms yields

$$\begin{aligned}
d_1(a_{0^+}^2) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(a_{0^+})^2 - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(a_{0^+})^2 - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(a_{0^+})^2 \\
&= -v^2_{\{0^+,0^-\}} + v^2_{\{0^+,4^+\}} + v^2_{\{0^+,4^-\}}, \\
d_1(a_{0^+}b_{0^+}) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(b_{0^+}) - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(b_{0^+}) \\
&\quad -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(b_{0^+}) \\
&= 0 - 2v^2_{\{0^+,4^+\}} + 2v^2_{\{0^+,4^-\}}, \\
d_1(b_{0^+}^2) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(b_{0^+})^2 - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(b_{0^+})^2 - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(b_{0^+})^2 \\
&= 0 - v^2_{\{0^+,4^+\}} - v^2_{\{0^+,4^-\}}
\end{aligned}$$

and similar for the other indices. Therefore, with respect to the bases

$$a_{0^+}^2, a_{0^+}b_{0^+}, b_{0^+}^2, a_{0^-}^2, a_{0^-}b_{0^-}, b_{0^-}^2, a_{4^+}^2, a_{4^+}b_{4^+}, b_{4^+}^2, a_{4^-}^2, a_{4^-}b_{4^-}, b_{4^-}^2 \tag{4.18}$$

and

$$v^2_{\{0^+,0^-\}}, v^2_{\{0^+,4^+\}}, v^2_{\{0^+,4^-\}}, v^2_{\{0^-,4^+\}}, v^2_{\{0^-,4^-\}}, v^2_{\{4^+,4^-\}}$$

the map $d_1^{0,16}$ has the matrix

$$\begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 0 & 0 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\
1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 1 \\
0 & 0 & 0 & 1 & 2 & -1 & -1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

and the kernel is generated by

$$a^2 = \sum_{i\in\{0^+,0^-,4^+,4^-\}} a_i^2,\ ab = \sum_{i\in\{0^+,0^-,4^+,4^-\}} a_i b_i,\ b^2 = \sum_{i\in\{0^+,0^-,4^+,4^-\}} b_i^2$$

and the additional generators

$$\begin{aligned} u_1 &:= a_{0^+}^2 + b_{0^+}^2 + a_{0^-}^2 + b_{0^-}^2 = (b+2a)(b-2a)(1_{0^+} + 1_{0^-}) \\ u_2 &:= 2a_{0^+}b_{0^+} + b_{0^+}^2 + 2a_{4^-}b_{4^-} + b_{4^-}^2 = b(b+2a)(1_{0^+} + 1_{4^-}) \\ u_3 &:= -2a_{0^-}b_{0^-} + b_{0^-}^2 - 2a_{4^-}b_{4^-} + b_{4^-}^2 = b(b-2a)(1_{0^-} + 1_{4^-}). \end{aligned}$$

Lemma 30. *The subring of $H^*(G,k)$, which is generated by those homogeneous elements of $H^*(G,k)$, whose degree is divisible by 8, is of the form*

$$k[a,b] \oplus k[a,b]u_1 \oplus k[a,b]u_2 \oplus k[a,b]u_3.$$

Proof. Assume, that $f_1, f_2, f_3 \in k[a,b]$ are polynomials satisfying

$$f_1u_1 + f_2u_2 + f_3u_3 \in k[a,b]. \tag{4.19}$$

Since the element (4.19), written in terms of the basis (4.18), contains no summand with index 4^+, it follows that

$$f_1u_1 + f_2u_2 + f_3u_3 = 0.$$

Comparing the coefficients of 1_{0^+}, 1_{0^-} and 1_{4^-}, it follows

$$\begin{aligned} f_1(b+2a)(b-2a) + f_2(b+2a)b = 0 \ &\Rightarrow\ f_2 = -\frac{b-2a}{b} f_1 \\ f_1(b+2a)(b-2a) + f_3(b-2a)b = 0 \ &\Rightarrow\ f_3 = -\frac{b+2a}{b} f_1 \end{aligned}$$

and therefore

$$0 = f_2(b+2a)b + f_3(b-2a)b = -2(b+2a)(b-2a)f_1,$$

which implies $f_1 = f_2 = f_3 = 0$. Thus we have proved, that for any integer $q \geq 16$, which is divisible by 8, the subspace of homogeneous elements of degree q in $k[a,b,u_1,u_2,u_3]$ has dimension

$$\dim_k k[a,b]_q + 3 \cdot k[a,b]_{q-16} = \frac{q}{8} + 1 + 3(\frac{q-16}{8} + 1) = \frac{q}{2} - 2,$$

where $k[a,b]_l$ denotes the subspace of homogeneous elements of degree l of $k[a,b]$ for $l \in \mathbb{Z}$. This is the whole subspace of homogeneous elements of degree q in $H^*(G,k)$ by theorem 19. □

The generators of degree 15 can be treated exactly in the same manner. Since

$$\begin{aligned}
d_1(a_{0^+}c_{0^+}) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(c_{0^+}) - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(c_{0^+})\\
&\quad -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(c_{0^+})\\
&= -v_{\{0^+,0^-\}}w_{\{0^+,0^-\}} + v_{\{0^+,4^+\}}w_{\{0^+,4^+\}} + v_{\{0^+,4^-\}}w_{\{0^+,4^-\}},\\
d_1(b_{0^+}c_{0^+}) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(b_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(c_{0^+}) - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(b_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(c_{0^+})\\
&\quad -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(b_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(c_{0^+})\\
&= 0 - 2v_{\{0^+,4^+\}}w_{\{0^+,4^+\}} + 2v_{\{0^+,4^-\}}w_{\{0^+,4^-\}},\\
d_1(a_{0^+}d_{0^+}) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(d_{0^+}) - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(d_{0^+})\\
&\quad -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(a_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(d_{0^+})\\
&= 0 + v_{\{0^+,4^+\}}w_{\{0^+,4^+\}} - v_{\{0^+,4^-\}}w_{\{0^+,4^-\}}\\
d_1(b_{0^+}d_{0^+}) &= -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(b_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,0^-\}}}(d_{0^+}) - \mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(b_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^+\}}}(d_{0^+})\\
&\quad -\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(b_{0^+})\mathrm{res}^{G_{0^+}}_{G_{\{0^+,4^-\}}}(d_{0^+})\\
&= 0 - 2v_{\{0^+,4^+\}}w_{\{0^+,4^+\}} - 2v_{\{0^+,4^-\}}w_{\{0^+,4^-\}}
\end{aligned}$$

and similar for the other indices, we get for d_1^{15} the matrix

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 2 & 0 & 0 & 0 & 0\\
1 & 2 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 1 & 2\\
0 & 0 & 0 & 0 & 1 & 2 & -1 & -2 & -1 & -2 & 1 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -2 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

with respect to the bases

$$\begin{aligned}
&a_{0^+}c_{0^+},\ b_{0^+}c_{0^+},\ a_{0^+}d_{0^+},\ b_{0^+}d_{0^+},\ a_{0^-}c_{0^-},\ b_{0^-}c_{0^-},\ a_{0^-}d_{0^-},\ b_{0^-}d_{0^-},\\
&a_{4^+}c_{4^+},\ b_{4^+}c_{4^+},\ a_{4^+}d_{4^+},\ b_{4^+}d_{4^+},\ a_{4^-}c_{4^-},\ b_{4^-}c_{4^-},\ a_{4^-}d_{4^-},\ b_{4^-}d_{4^-}
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&v_{\{0^+,0^-\}}w_{\{0^+,0^-\}},\ v_{\{0^+,4^+\}}w_{\{0^+,4^+\}},\ v_{\{0^+,4^-\}}w_{\{0^+,4^-\}},\\
&v_{\{0^-,4^+\}}w_{\{0^-,4^+\}},\ v_{\{0^-,4^-\}}w_{\{0^-,4^-\}},\ v_{\{4^+,4^-\}}w_{\{4^+,4^-\}}.
\end{aligned}$$

The kernel of d_1^{15} has the basis, which consists of the elements ac, bc, ad, bd and the

further generators

$$\begin{aligned} v_1 &:= (2ac+bd)(1_{0^+}+1_{0^-}) \\ v_2 &:= b(c+d)(1_{0^+}+1_{4^+}) \\ v_3 &:= (2a+b)d(1_{0^+}+1_{4^-}) \\ v_4 &:= b(c-d)(1_{0^-}+1_{4^+}) \\ v_5 &:= (2a-b)d(1_{0^-}+1_{4^-}) \\ v_6 &:= (2a+b)(c+d)(1_{4^+}+1_{4^-}). \end{aligned}$$

Lemma 31. *For $q \equiv 7 \pmod 8$ any homogeneous element of degree q in $H^*(G,k)$ is contained in*

$$k[a,b]c + k[a,b]d + \sum_{i=1}^{6} k[a,b]v_i.$$

Proof. Let $f_1, \ldots, f_6 \in k[a,b]$ be polynomials, such that

$$f_1v_1 + \ldots + f_6v_6 \in k[a,b]c + k[a,b]d. \tag{4.21}$$

Then in (4.21) the coefficients of $c\,1_{0^+}$, $c\,1_{0^-}$, $c\,1_{4^+}$ and $c\,1_{4^-}$ are necessarily equal, hence

$$2af_1 + bf_2 = 2af_1 + bf_4 = bf_2 + bf_4 + (2a+b)f_6 = (2a+b)f_6,$$

which provides $f_2 = f_4 = 0$ and

$$f_6 = \frac{2a}{2a+b}f_1. \tag{4.22}$$

Similarly, comparing the coefficients of $d\,1_{0^+}$, $d\,1_{0^-}$, $d\,1_{4^+}$ and $d\,1_{4^-}$, it follows, that

$$bf_1 + (2a+b)f_3 = bf_1 + (2a-b)f_5 = (2a+b)f_6 = (2a+b)f_3 + (2a+b)f_6,$$

hence $f_3 = f_5 = 0$ and by (4.22) we get

$$\frac{2a}{2a+b}f_1 = f_6 = \frac{b}{2a+b}f_1,$$

thus $f_1 = \ldots = f_6 = 0$. Therefore for $q \equiv 7 \pmod 8$ the subspace of homogeneous elements of degree q in $k[a,b]c + k[a,b]d + \sum_{i=1}^{6} k[a,b]v_i$ has exactly the dimension

$$\begin{aligned} 2\dim_k k[a,b]_{q-7} + 6\dim_k k[a,b]_{q-14} &= 2(\frac{q-7}{8}+1) + 6(\frac{q-15}{8}+1) \\ &= q-5 = \dim_k H^q(G,k), \end{aligned}$$

where again $k[a,k]_l$ denotes the subspace of homogeneous elements of degree l of $k[a,b]$ for $l \in \mathbb{Z}$. This finishes the proof. □

Since $d_1^{0,14} = 0$, we can choose as generators in degree 14 beside cd the elements

$$\begin{aligned} w_1 &:= cd\,1_{0^-} \\ w_2 &:= cd\,1_{4^+} \\ w_3 &:= cd\,1_{4^-}. \end{aligned}$$

Here it is obvious, that for $f_1, f_2, f_3 \in k[a,b]$ with $f_1w_1 + f_2w_2 + f_3w_3 \in k[a,b]cd$ we have $f_1 = f_2 = f_3 = 0$, and if $q \geq 14$ is an integer with $q \equiv 6 \pmod 8$, then the subspace of homogeneous elements of degree q in

$$k[a,b]cd + \sum_{i=1}^{3} k[a,b]w_i$$

has the dimension

$$4(\dim_k k[a,b]_{q-14} + 1) = 4(\frac{q-14}{8} + 1) = \frac{q}{2} - 3 = \dim_k H^q(G,k).$$

Therefore we have proven, that the ring $H^*(G,k) = H^*(\Gamma,k)$ has the basis

$$\begin{aligned} \mathfrak{T} \;=\; & \{f, fc, fd, fcd, fu_1, fu_2, fu_3, fv_1, fv_2, fv_3, fv_4, fv_5, fv_6, fw_1, fw_2, fw_3 \mid \\ & f \text{ is a monomial in } k[a,b]\} \end{aligned}$$

over k and it follows

Theorem 20. *The ring $H^*(\Gamma,k)$ is generated over $H^0(\Gamma,k) \cong k$ by two generators c, d of degree 7, two generators a, b of degree 8, three generators w_1, w_2, w_3 of degree 14, six generators $v_1, \ldots, v_6$ of degree 15 and three generators u_1, u_2, u_3 of degree 16. The ring structure is determined by the fact, that $H^*(\Gamma,k)$ is graded commutative, and by the further relations*

$$\begin{aligned} & c^2 = d^2 = (c+d)^2 = 0 \\ & w_iw_j = 0 \qquad \text{for } 1 \leq i,j \leq 3 \\ & w_iv_j = 0 \qquad \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq 6 \end{aligned}$$

and some other relations, expressing the products u_iu_j $(1 \leq i,j \leq 3)$, u_iv_j $(1 \leq i \leq 3,\ 1 \leq j \leq 6)$, v_iv_j $(1 \leq i,j \leq 6)$ and w_iu_j $(1 \leq i,j \leq 3)$ as linear combinations with respect to the basis $\mathfrak{T}$ over k.

It is easy to write down the relations explicitely, so I give only some examples:

$$\begin{aligned}
u_1^2 &= (a^2+b^2)(1_{0^+}+1_{0^-})u_1 = (a^2+b^2)u_1 \\
u_2^2 &= (2ab+b^2)(1_{0^+}+1_{4^-})u_2 = (2ab+b^2)u_2 \\
u_3^2 &= (-2ab+b^2)(1_{0^-}+1_{4^-})u_3 = (-2ab+b^2)u_3 \\
u_1u_2 &= b(b+2a)^2(b-2a)1_{0^+} \\
&= 3b(b+2a)^2(b-2a)((1_{0^+}+1_{0^-})+(1_{0^+}+1_{4^-})-(1_{0^-}+1_{4^-})) \\
&= 3(b^2+2ab)u_1+3(a^2+b^2)u_2-3(b^2-ab-a^2)u_3 \\
u_1u_3 &= b(b+2a)(b-2a)^2 1_{0^-} = 3(b^2-2ab)u_1-3(b^2+ab-a^2)u_2+3(a^2+b^2)u_3 \\
u_2u_3 &= b^2(b+2a)(b-2a)1_{4^-} = -3b^2u_1+3(b^2-2ab)u_2+3(b^2+2ab)u_3
\end{aligned}$$

By corollary 19, this determines the 5-part of integral cohomology, too.

Corollary 20. *The 5-part of integral cohomology $H^*(\Gamma,\mathbb{Z})_{[5]}$ is generated over*

$$H^0(\Gamma,\mathbb{Z})_{[5]} \cong \mathbb{Z}$$

by two generators a, b of degree 8 and three generators u_1, u_2, u_3 of degree 16 with

$$5a = 5b = 5u_1 = 5u_2 = 5u_3 = 0.$$

This ring is commutative and its ring structure is determined by the six relations following theorem 20.

To get more insight into the structure of the ring $H^*(\Gamma, k)$, we take the point of view of [Qui71] and consider the ring

$$A := H^*(\Gamma,k)/J$$

where J is the ideal of nilpotent elements of $H^*(\Gamma, k)$.

Lemma 32. $A \cong k[a, b, u_1, u_2, u_3]$.

Proof. Clearly, the generators $c, d, w_1, w_2, w_3, v_1, \ldots, v_6$ of degree 7, 14 and 15 are nilpotent elements. On the other hand, the algebra $k[a, b, u_1, u_2, u_3]$ is a subring in the direct sum

$$k[a_{0^+}, b_{0^+}] \oplus k[a_{0^-}, b_{0^-}] \oplus k[a_{4^+}, b_{4^+}] \oplus k[a_{4^-}, b_{4^-}]$$

of polynomial algebras, hence it is reduced. □

By the general theory, the maximal finite subgroups of Γ are in bijection with the vertices of the Bruhat-Tits building X. Since Γ acts transitively on the chambers of X, there are exactly five conjugacy classes of such subgroups and four of them contain non-trivial 5-Sylow subgroups. These are elementary abelian groups of order 25. Since no stabilizer of a simplex σ with $\dim \sigma \geq 1$ contains a subgroup of order 25, it follows from the geometry of X, that any subgroup $P \subset \Gamma$ of order 25 is contained in an unique maximal finite subgroup of Γ. Indeed, assume that P fixes two different vertices x, y of X. Then P fixes also the unique geodesic $[x, y]$ between x and y (cf. [Bro82] ch. VI.3) and thus all simplices of X, which are crossed by $[x, y]$, contradiction. This proves, that there are exactly four conjugacy classes of subgroups of order 25, which are maximal among the elementary abelian subgroups related to the prime number 5. It follows by a fundamental result of [Qui71], that the affine algebra A is of Krull dimension 2 and contains exactly four minimal prime ideals, which are in canonical bijection with these conjugacy classes of elementary abelian subgroups of Γ.

We shall now determine the four minimal prime ideals of A and describe the scheme Proj$A$, which is defined by the homogeneous spectrum of the graded ring A. From the relations

$$\begin{array}{rcl} u_1(a^2+b^2-u_1) & = & 0 \\ u_2(b^2+2ab-u_2) & = & 0 \\ u_3(b^2-2ab-u_3) & = & 0 \end{array} \tag{4.23}$$

it follows, that any prime ideal of A contains one of the two factors of each equation in (4.23).

Let $\mathfrak{P}_{4^+}$ be the ideal in A, which is generated by u_1, u_2 and u_3. Then by lemma 30 the quotient $A/\mathfrak{P}_{4^+}$ is isomorphic to the polynomial ring $k[a, b]$, in particular it is an integral domain. Therefore $\mathfrak{P}_{4^+}$ is a minimal prime ideal of A.

Lemma 33. *There are three automorphisms ψ_{0^+}, ψ_{0^-}, ψ_{4^-} of A, which fix a and b and satisfy*

$$\begin{array}{lll} \psi_{0^+}(u_1) = a^2+b^2-u_1 & \psi_{0^-}(u_1) = a^2+b^2-u_1 & \psi_{4^-}(u_1) = u_1 \\ \psi_{0^+}(u_2) = b^2+2ab-u_2 & \psi_{0^-}(u_2) = u_2 & \psi_{4^-}(u_2) = b^2+2ab-u_2 \\ \psi_{0^+}(u_3) = u_3 & \psi_{0^-}(u_3) = b^2-2ab-u_3 & \psi_{4^-}(u_3) = b^2-2ab-u_3. \end{array}$$

Proof. It is only necessary to check, that for $i = 0^+, 0^-, 4^-$ the elements $\psi_i(u_1)$, $\psi_i(u_2)$, $\psi_i(u_3)$ satisfy the same relations as u_1, u_2, u_3. This is straight forward. For example, we have

$$\begin{array}{rcl} \psi_{0^+}(u_1)^2 & = & (a^2+b^2)^2 - 2(a^2+b^2)u_1 + u_1^2 \\ & = & (a^2+b^2)^2 - (a^2+b^2)u_1 \\ & = & (a^2+b^2)(a^2+b^2-u_1) \\ & = & (a^2+b^2)\,\psi_{0^+}(u_1), \end{array}$$

$$\begin{aligned}\psi_{0^+}(u_2)^2 &= (b^2+2ab)^2 - 2(b^2+2ab)u_2 + u_2^2\\ &= (b^2+2ab)^2 - (b^2+2ab)u_2\\ &= (b^2+2ab)(b^2+2ab-u_2)\\ &= (b^2+2ab)\,\psi_{0^+}(u_2),\end{aligned}$$

$$\begin{aligned}\psi_{0^+}(u_1)\quad \psi_{0^+}(u_2) &= (a^2+b^2)(b^2+2ab) - (b^2+2ab)u_1 - (a^2+b^2)u_2 + u_1u_2\\ = \quad & (a^2+b^2)(b^2+2ab) + (3(b^2+2ab) - (b^2+2ab))u_1\\ & +(3(a^2+b^2) - (a^2+b^2))u_2 - 3(b^2-ab-a^2)u_3\\ = \quad & 6(a^2+b^2)(b^2+2ab) - 3(b^2+2ab)u_1 - 3(a^2+b^2)u_2 - 3(b^2-ab-a^2)u_3\\ = \quad & 3(b^2+2ab)(a^2+b^2-u_1) + 3(a^2+b^2)(b^2+2ab-u_2) - 3(b^2-ab-a^2)u_3\\ = \quad & 3(b^2+2ab)\,\psi_{0^+}(u_1) + 3(a^2+b^2)\,\psi_{0^+}(u_2) - 3(b^2-ab-a^2)\psi_{0^+}(u_3),\end{aligned}$$

$$\begin{aligned}\psi_{0^+}(u_1)\quad \psi_{0^+}(u_3) &= (a^2+b^2)u_3 - 3(b^2-2ab)u_1 + 3(b^2+ab-a^2)u_2 - 3(a^2+b^2)u_3\\ = \quad & 3(b^2-2ab)(a^2+b^2-u_1) - 3(b^2+ab-a^2)(b^2+2ab-u_2) + 3(a^2+b^2)u_3\\ = \quad & 3(b^2-2ab)\psi_{0^+}(u_1) - 3(b^2+ab-a^2)\psi_{0^+}(u_2) + 3(a^2+b^2)\psi_{0^+}(u_3),\end{aligned}$$

$$\begin{aligned}\psi_{0^+}(u_2)\quad \psi_{0^+}(u_3) &= (b^2+2ab)u_3 + 3b^2u_1 - 3(b^2-2ab)u_2 - 3(b^2+2ab)u_3\\ = \quad & -3b^2(a^2+b^2-u_1) + 3(b^2-2ab)(b^2+2ab-u_2) + 3(b^2+2ab)u_3\\ = \quad & -3b^2\psi_{0^+}(u_1) + 3(b^2-2ab)\psi_{0^+}(u_2) + 3(b^2+2ab)\psi_{0^+}(u_3).\end{aligned}$$

This proves, that ψ_{0^+} is an automorphism. □

Therefore, the ideals $\mathfrak{P}_i = \psi_i(\mathfrak{P}_{4^+})$ are also minimal prime ideals of A and the rings $A/\mathfrak{P}_i$ are polynomial rings on two generators of degree 8 for $i = 0^+, 0^-, 4^-$. As remarked above, these are all minimal prime ideals.

Remark 27. Note, that for example the ideal I with generators $a^2+b^2-u_1$, u_2, u_3 is no prime ideal in A. Indeed, the equation

$$\begin{aligned}(a^2+b^2-u_1)u_3 &= (a^2+b^2)u_3 - 3(b^2-2ab)u_1 + 3(b^2+ab-a^2)u_2 - 3(a^2+b^2)u_3\\ = \; & 3(b^2-2ab)(a^2+b^2-u_1) + 3(b^2+ab-a^2)u_2 + 3(a^2+b^2)u_3 - 3(b^2-2ab)(a^2+b^2)\end{aligned}$$

implies, that the image of the product $(b^2-2ab)(a^2+b^2)$ is zero in A/I. Therefore, if $\mathfrak{P}$ is a prime ideal with $\mathfrak{P} \supset I$, then $\mathfrak{P}$ contains (b^2-2ab) or (a^2+b^2), hence $\mathfrak{P} \supset \mathfrak{P}_{0^-}$ or $\mathfrak{P} \supset \mathfrak{P}_{4^+}$. Arguing in this manner one can show directly, that $\mathfrak{P}_{0^+}$, $\mathfrak{P}_{0^-}$, $\mathfrak{P}_{4^+}$ and $\mathfrak{P}_{4^-}$ are the only minimal prime ideals of A.

These ideals are related to the vertices Λ_{0^+}, Λ_{0^-}, Λ_{4^+} and Λ_{4^-} in the following way. Since in this special situation we have

$$E_\infty^{0,*} = E_2^{0,*} = H^*(G,k)$$

(cf. theorem 19), it follows from lemma 17, that the restriction maps

$$\mathrm{res}^G_{G_i} : H^*(G,k) \to H^*(G_i,k)$$

are given by $x \mapsto x1_i$ for $i \in \{0^+, 0^-, 4^+, 4^-\}$ and $x \in H^*(G,k)$. From this formula it follows, that $\mathfrak{P}_i$ is the unique minimal prime ideal of A, which is contained in the kernel of $\mathrm{res}^G_{G_i}$ for $i \in \{0^+, 0^-, 4^+, 4^-\}$.

Note, that the prime ideals $\mathfrak{P}_{0^+}$, $\mathfrak{P}_{0^-}$, $\mathfrak{P}_{4^+}$ and $\mathfrak{P}_{4^-}$ are homogeneous ideals of A. Therefore, they define the irreducible components of the scheme $\mathrm{Proj}(A)$. If we divide the grading by 8, i.e. if we consider a and b as generator of degree 1 and u_1, u_2, u_3 as elements of degree 2, then we get four projective lines Z_{0^+}, Z_{0^-}, Z_{4^+} and Z_{4^-}. The intersections $Z_{4^+} \cap Z_i$ for $i = 0^+, 0^-, 4^-$ are defined as subschemes of Z_{4^+} by the equations

$$\begin{aligned} Z_{4^+} \cap Z_{0^+} &: b(b+2a) = (b+2a)(b-2a) = 0 \\ Z_{4^+} \cap Z_{0^-} &: b(b-2a) = (b+2a)(b-2a) = 0 \\ Z_{4^+} \cap Z_{4^-} &: b(b+2a) = b(b-2a) = 0. \end{aligned}$$

By the symmetries, which are given by the automorphisms ψ_{0^+}, ψ_{0^-} and ψ_{4^-} we can conclude, that $\mathrm{Proj}(A)$ consists of four lines and any two of them intersect in a unique point of multiplicity 2.

We finish the section with the remark, that the knowledge about the 7-part and the 5-part of the integral cohomology ring is equivalent to the computation of the cohomology ring $H^*(\Gamma, \mathbb{Z}[\frac{1}{6}])$ with constant coefficients in $\mathbb{Z}[\frac{1}{6}]$. This can be seen as follows.

Since all stabilizers Γ_σ of simplices σ of X are finite,

$$E_1^{p,q} = \sum_{\sigma \in \Sigma_p} H^*(\Gamma_\sigma, M)$$

is a torsion group for any Γ-module M and any $q > 0$. Furthermore, since $\Gamma \setminus X$ is contractible, $E_2^{p,0} = 0$ for $p > 0$. This implies, that $H^q(\Gamma, M)$ is a torsion module for any $q > 0$. But the only primes which divide the order of some stabilizer Γ_σ are 2, 3, 5 and 7. Therefore, we get a decomposition

$$H^q(\Gamma, M) = H^q(\Gamma, M)_{(2)} \oplus H^q(\Gamma, M)_{(3)} \oplus H^q(\Gamma, M)_{(5)} \oplus H^q(\Gamma, M)_{(7)}$$

for $q > 0$. On the other hand, if multiplication with a prime p defines an automorphism of M, then we have $H^q(G, M)_{(p)} = 0$ for any finite group G and $q > 0$ and consequently $H^q(\Gamma, M)_{(p)} = 0$. Applying these considerations to the ring $M = \mathbb{Z}[\frac{1}{6}]$ we get

$$H^q(\Gamma, \mathbb{Z}[\frac{1}{6}]) = H^q(\Gamma, \mathbb{Z}[\frac{1}{6}])_{(5)} \oplus H^q(\Gamma, \mathbb{Z}[\frac{1}{6}])_{(7)}.$$

Now, consider the group $\mathbb{Z}[\frac{1}{6}]/\mathbb{Z}$.

Lemma 34. *For any prime $p \neq 2, 3$, multiplication with p is an automorphism of $\mathbb{Z}[\frac{1}{6}]/\mathbb{Z}$.*

Proof. The map $m_p : \mathbb{Z}[\frac{1}{6}]/\mathbb{Z} \to \mathbb{Z}[\frac{1}{6}]/\mathbb{Z},\ x \mapsto px$ is obviously injective. Conversely, consider $z \in \mathbb{Z}[\frac{1}{6}]$. Then we can write z in the form $\frac{m}{6^{(p-1)n}}$ with $m, n \in \mathbb{Z}$ and $n \geq 0$. Since $(6^{(p-1)n} - 1)$ is divisible by p, say $(6^{(p-1)n} - 1) = lp$ with $l \in \mathbb{Z}$, we get

$$z = m - \frac{m(6^{(p-1)n} - 1)}{6^{(p-1)n}} = m - p\frac{ml}{6^{(p-1)n}}.$$

Thus $m_p(-\frac{ml}{6^{(p-1)n}} + \mathbb{Z}) = z + \mathbb{Z}$. □

The lemma implies

$$H^q(\Gamma, \mathbb{Z}[\frac{1}{6}]/\mathbb{Z}) = H^q(\Gamma, \mathbb{Z}[\frac{1}{6}]/\mathbb{Z})_{(2)} \oplus H^q(\Gamma, \mathbb{Z}[\frac{1}{6}]/\mathbb{Z})_{(3)}$$

for $q > 0$. Now, the long exact cohomology sequence which is associated to

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}[\frac{1}{6}] \to \mathbb{Z}[\frac{1}{6}]/\mathbb{Z} \to 0$$

shows, that for $p = 5$ and $p = 7$ the induced map $\iota^* : H^*(\Gamma, \mathbb{Z}) \to H^*(\Gamma, \mathbb{Z}[\frac{1}{6}])$ is bijective on the p-torsion. Therefore the corollaries 20 and 17 yield

Theorem 21. *The cohomology ring $H^*(\Gamma, \mathbb{Z}[\frac{1}{6}])$ is defined over $H^*(\Gamma, \mathbb{Z}[\frac{1}{6}]) \cong \mathbb{Z}[\frac{1}{6}]$ by generators $\alpha_1, \alpha_2, a, b, \xi, u_1, u_2, u_3$ of degree 7, 7, 8, 8, 12, 16, 16 and 16 and the relations*

$$7\alpha_1 = 7\alpha_2 = 7\xi = 0 \quad \text{and} \quad 5a = 5b = 5u_1 = 5u_2 = 5u_3 = 0$$
$$\alpha_1^2 = \alpha_2^2 = (\alpha_1 + \alpha_2)^2 = 0$$

together with the six relations following theorem 20.

4.6 Remarks on the remaining primes 3 and 2

For a complete calculation of the cohomology of Γ with constant coefficients, the discussion of the characteristics $p = 2$ and $p = 3$ is left. This should not be done here. I give only some hints about problems which may appear in characteristic 3. The case $p = 2$ requires some different methods and cannot be discussed here. But there is announced a very similar result by H.-W. Henn and J. Lannes [HL], which states, that the reduction map modulo 3 $\mathrm{O}(L[\frac{1}{2}], q) \to \mathrm{O}(L[\frac{1}{2}]/3L[\frac{1}{2}], q)$ induces an isomorphism in homology

$$H_*(\mathrm{O}(L[\frac{1}{2}], q), \mathbb{F}_2) \to H_*(\mathrm{O}(L[\frac{1}{2}]/3L[\frac{1}{2}], q), \mathbb{F}_2)$$

and the latter group is the orthogonal group of a hyperbolic quadratic space of dimensien 8 over the prime field $\mathbb{F}_3$. This homology is computed in [FP78], III 5.12.

In characteristic 3 the situation is similar in many respects to the cases $p = 5$ and $p = 7$. From the description in (4.1) it can be seen, that the 3-Sylow subgroups of the group Γ_{Λ_2} are elementary abelian subgroups of order 3^4. Also the other stabilizers have elementary abelian 3-Sylow subgroups except the stabilizers of the special vertices Λ_{0^+}, Λ_{0^-}, Λ_{4^+} and Λ_{4^-}, whose 3-Sylow subgroups are of the form

$$(\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}/3\mathbb{Z}.$$

This can be derived from a decomposition of the hyperbolic quadratic space $(\overline{V}, \overline{q})$ of dimension 8 over $\mathbb{F}_2$ into an orthogonal sum of four anisotropic subspaces of dimension 2. Therefore the cohomology of such a stabilizer Γ_x cannot be computed with Swan's lemma, directly. Nevertheless it is still true, that the cohomology of Γ_x is detected by an elementary abelian subgroup of order 3^4. We can again consider the group $\mathrm{SO}(\overline{V}, \overline{q})$ instead of Γ_x. The assertion is proved in [FP78] III 5.4-5.8 and VIII 4.3-4.7 for orthogonal groups and this proof remains also valid for our group $\mathrm{SO}(\overline{V}, \overline{q})$.[2] As the crucial point here, it remains the question, wether the spectral sequence abutes at the E_2-term or not.

[2]One has only to note, that the unitary group $U(n, \mathbb{F}_4)$ injects as a connected group over $\mathbb{F}_2$ into $\mathrm{SO}(2n, \mathbb{F}_2)$.

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