A noncommutative index theorem for a manifold with boundary and cylindric ends

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultäten
der Georg-August-Universität zu Göttingen

vorgelegt
von Charlotte Wahl
aus Heidelberg

Göttingen 2003

D 7
Referent: Prof. Dr. Ulrich Bunke
Korreferent: Prof. Dr. Thomas Schick
Tag der mündlichen Prüfung: 4.6.2003

Contents

Introduction **6**

Summary 8

1 Preliminaries **10**

1.1 The geometric situation 10

1.2 The commutative index problem 12

1.3 The algebra of differential forms 13

1.3.1 The universal graded differential algebra 13

1.3.2 Supercalculus 16

1.3.3 The algebras $\mathcal{A}_\infty$ and $\hat{\Omega}_*\mathcal{A}_\infty$ 17

1.4 Lagrangian projections 20

1.4.1 Definition and properties 20

1.4.2 The Maslov index 22

2 The Fredholm operator and its index **24**

2.1 The operator D on M 24

2.1.1 Definition of D 24

2.1.2 Comparison with D_s 25

2.2 The operator D_I on $[0,1]$ 27

2.2.1 Definition and comparison with D_{I_s} 27

2.2.2 Generalized eigenspace decomposition 28

2.3 The operator D_Z on the cylinder 31

2.4 The index of D^+ 32

2.5 A perturbation with closed range 35

3 Heat semigroups and kernels **37**

3.1 Complex heat kernels 37

3.1.1 The heat kernel of $e^{-tD_{I_s}^2}$ 37

3.1.2 The heat kernel of $e^{-tD_s^2}$ 39

3.2 The heat semigroup on compact manifolds . . . 43
3.3 The heat semigroup on $[0,1]$ . . . 45
3.3.1 The semigroup . . . 45
3.3.2 The integral kernel . . . 47
3.4 The heat semigroup on the cylinder . . . 50
3.5 The heat semigroup on M . . . 56
3.5.1 Definitions . . . 56
3.5.2 The resolvents of $D(\rho)^2$ . . . 57
3.5.3 An approximation of the semigroup . . . 60
3.5.4 The semigroup $e^{-tD_s^2}$ . . . 61
3.5.5 The semigroup $e^{-tD(\rho)^2}$ . . . 64
3.5.6 The integral kernel . . . 65

4 The superconnection and the index theorem 69
4.1 The superconnection A_t^I associated to D_I . . . 70
4.1.1 The family $e^{-(A_t^I)^2}$ . . . 70
4.1.2 The integral kernel of $e^{-(A_t^I)^2}$ . . . 71
4.1.3 The η-form . . . 75
4.2 The superconnection associated to D_Z . . . 77
4.3 The superconnection $A(\rho)_t$ associated to $D(\rho)$ . . . 78
4.3.1 The family $e^{-A(\rho)_t^2}$ . . . 78
4.3.2 The integral kernel of $e^{-A(\rho)_t^2}$ . . . 80
4.4 The index theorem and its proof . . . 83
4.4.1 The supertrace . . . 83
4.4.2 The limit of $\mathrm{Tr}_s e^{-A(\rho)_t^2}$ for $t \to \infty$ . . . 84
4.4.3 The limit of $\mathrm{Tr}_s e^{-A_t^2}$ for $t \to 0$ . . . 91
4.4.4 $\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2}$ and $\frac{d}{d\rho}\mathrm{Tr}_s e^{-A(\rho)_t^2}$ . . . 92
4.4.5 The index theorem . . . 96

5 Definitions and Technics 99
5.1 Hilbert C^*-modules . . . 99
5.1.1 Bounded operators . . . 99
5.1.2 Fredholm operators . . . 100
5.1.3 Regular operators . . . 102
5.1.4 Decompositions of Hilbert C^*-modules . . . 106
5.2 Banach space valued functions . . . 107

5.2.1 Function spaces and tensor products 107
5.2.2 L^2-spaces and integral operators 108
5.2.3 Hilbert-Schmidt operators 109
5.2.4 Pseudodifferential operators 112
5.3 Projective systems and function spaces 114
5.3.1 Integral operators 114
5.3.2 Projections and the Chern character 116
5.4 Holomorphic semigroups 121
5.4.1 Generalities 121
5.4.2 Square roots of generators and perturbations 123

Introduction

Let $\mathbb{C}^{2n}$ be endowed with the skewhermitian form induced by $I_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and the standard hermitian product.

The Maslov index of a triple of Lagrangian subspaces of $\mathbb{C}^{2n}$ was introduced by Kashiwara [LV]. Bunke and Koch [BK] defined a generalized Maslov index for a family of triples of pairwise transverse Lagrangian subspaces. We generalize it to a noncommutative context:

For a C^*-algebra $\mathcal{A}$ we define the notion of a Lagrangian projection on $\mathcal{A}^{2n}$ and of transverse projections such that for $\mathcal{A} = \mathbb{C}$ a Lagrangian projection is a projection onto a Lagrangian submodule. We associate a class $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) \in K_0(\mathcal{A})$ to every triple $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ of pairwise transverse Lagrangian projections which we call the Maslov index of $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

Furthermore we prove an analogue of a formula relating the generalized Maslov index and η-forms [BK]:

Assume that there is a projective system $\{\mathcal{A}_i\}_{i\in\mathbb{N}_0}$ of involutive Banach algebras with $\mathcal{A}_0 = \mathcal{A}$, such that there are embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}$ and that $\mathcal{A}_i$ is closed with respect to the holomorphic functional calculus in $\mathcal{A}$ for all $i \in \mathbb{N}_0$. Assume furthermore that its projective limit $\mathcal{A}_\infty$ is dense in $\mathcal{A}$ [Lo].

We associate an η-form $\eta(\mathcal{P}_0, \mathcal{P}_1) \in \hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$ to a pair of transverse Lagrangian projections $(\mathcal{P}_0, \mathcal{P}_1)$ of $\mathcal{A}^{2n}$ with $\mathcal{P}_i \in M_{2n}(\mathcal{A}_\infty),\ i = 0, 1$. Here $\hat{\Omega}_*\mathcal{A}_\infty$ denotes the topological universal graded differential algebra introduced by Karoubi [Ka] and $[\ ,\ ]_s$ denotes the supercommutator. The η-form is defined via a superconnection associated to the operator $D_I := I_0 \frac{d}{dx}$ which acts on functions on $[0, 1]$ with values in $\mathcal{A}^{2n}$. The pair of Lagrangian projections defines the boundary conditions of D_I.

The Chern character homomorphism $\mathrm{ch} : K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\infty) \to H_{dR}(\mathcal{A}_\infty)$ [Ka] connects the K-theory of $\mathcal{A}$ with the de Rham homology $H_{dR}(\mathcal{A}_\infty)$ of $\mathcal{A}_\infty$.

Our main result is:

Theorem 0.0.1. *For a triple $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ of pairwise transverse Lagrangian projections with $\mathcal{P}_i \in M_{2n}(\mathcal{A}_\infty),\ i = 0, 1, 2$*

$$\mathrm{ch}\,\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [\eta(\mathcal{P}_0, \mathcal{P}_1) + \eta(\mathcal{P}_1, \mathcal{P}_2) + \eta(\mathcal{P}_2, \mathcal{P}_0)] \in H_{dR}(\mathcal{A}_\infty).$$

For $\mathcal{A} = \mathbb{C}$ the formula was obtained by Cappell, Lee and Miller [CLM]. For their result the transversality of the Lagrangian subspaces is not needed. Their proof relies on the axiomatic properties of the Maslov index whereas we follow the idea of Bunke and Koch who proved a similar result for the generalized Maslov index, and use index theory:

The Maslov index $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ is the index of a Dirac operator D^+ on a two dimensional spin manifold with six cylindric ends isometric to $[0, \infty) \times [0, 1]$. The projections $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ enter in the definition of boundary conditions for D^+. Then D^+ is a Fredholm operator between appropriate Hilbert $\mathcal{A}$-modules in the sense of Mishenko-Fomenko [MF]. Applying the superconnection formalism we prove a noncommutative index theorem for D^+. It turns out that the local term vanishes. The η-forms are the contributions from the cylindric ends.

We develop a noncommutative index theory for our situation: The Fréchet space $\hat{\Omega}_* \mathcal{A}_\infty$ is described as a projective limit of appropriate Banach spaces. Then we can deal with spaces of Banach space valued functions. We define appropriate function spaces and notions of trace class operators and Hilbert-Schmidt operators. In order to show that the left hand side of the equation is well-defined we find a compact surjective perturbation $D(\rho)^+$ of D^+ such that the index of $D(\rho)^+$ is its kernel. We show that the projection onto the kernel is Hilbert-Schmidt and that it is a projection onto a projective $\mathcal{A}_\infty$-module in the projective limit.

For the construction of the heat semigroups the theory of holomorphic semigroups is used. Applying a method of Lott [Lo] we gain knowledge about the resolvent sets of their generators. This leads to estimates of the semigroups for $t \to \infty$. The calculus of pseudodifferential operators helps to show that the resolvents are regularizing. It follows that the semigroups are smoothing. Using Duhamel's principle we prove the existence of the heat kernel and find estimates of it.

Since the proof of the existence of the semigroups and the heat kernels needs much effort, the method is described here for D_I in more detail:

On the Hilbert $\mathcal{A}$-module $L^2([0, 1], \mathcal{A}^{2d})$ the semigroup $e^{-tD_I^2}$ can easily be defined by the functional calculus of regular operators [La]. We show that it is a family of integral operators with integral kernels in $C^\infty([0, 1] \times [0, 1], M_{2d}(\mathcal{A}_\infty))$ and find estimates for the integral kernel, in particular for its behaviour at small times.

On the way we have to show that the semigroup $e^{-tD_I^2}$ restricts to a semigroup on the Banach space $L^2([0, 1], \mathcal{A}_i^{2n})$ for every $i \in \mathbb{N}$. If the boundary conditions are given by orthogonal submodules, the heat kernel is easily constructed and we show that it induces a holomorphic semigroup on $L^2([0, 1], \mathcal{A}_i^{2n})$ for every $i \in \mathbb{N}$. It turns out that D_I with general boundary conditions is unitarily equivalent to a bounded perturbation of the operator with orthogonal boundary conditions. We prove a perturbation lemma about holomorphic semigroups in order to show the existence of the semigroup $e^{-tD_I^2}$ on $L^2([0, 1], \mathcal{A}_i^{2n})$.

For getting estimates of the norm of the semigroup we study the resolvent set of the generator.

We prove that $e^{-tD_I^2}$ is smoothing by showing that the resolvent D_I^{-2} is regularizing.

Then we show that $e^{-tD_I^2}$ is a family of integral operators by comparing it with an approximation by a family of integral operators using Duhamel's principle. It will follow that the integral kernel of $e^{-tD_I^2}$ is in $C^\infty([0, 1] \times [0, 1], M_{2n}(\mathcal{A}_\infty))$. Duhamel's principle also allows to obtain estimates for the heat kernel.

Before the organisation of this thesis is described, I want to thank all people who supported it: in particular my supervisor Ulrich Bunke for drawing my attention to the problem and for discussions, Margit Rösler for the introduction into the theory of semigroups, and my friends and family for moral support.

Summary

The proof of the index theorem is organised in the following way:

In the first chapter the manifold M is described and we explain in more detail the family index theorem of Bunke and Koch [BK].

Then the differential algebra $\hat{\Omega}_*\mathcal{A}_\infty$, the de Rham homology $H_*^{dR}(\mathcal{A}_\infty)$ and the Chern character are introduced and investigated, Lagrangian projections and the Maslov index are defined and studied. Furthermore supercalculus and trace class operators are presented.

In the second chapter we introduce the operator $D = D^+ \oplus D^-$ on M whose boundary conditions are defined by a triple of Lagrangian projections $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

Furthermore the operator D_I is defined and its properties on the Hilbert $\mathcal{A}$-module $L^2([0,1], \mathcal{A}^{2n})$ are studied.

Then we show that D^+ is Fredholm when acting between appropriate Hilbert C^*-modules and that its index equals $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$. We define a compact perturbation $D(\rho)$ of D with closed range.

The third chapter is devoted to heat semigroups and their integral kernels, in particular those associated to D_I and $D(\rho)$.

In chapter four we introduce superconnections in order to define the η-form. Now finally the statement of the index theorem is well-defined. The remainder of the chapter is devoted to its proof. We have to introduce another family of operators $e^{-A(\rho)_t^2}$ where $A(\rho)_t$ is a rescaled superconnection associated to $D(\rho)$. As before we show that it is a family of integral operators with smooth integral kernels and obtain estimates for the integral kernels for small t.

As usual the proof of the index theorem compares the limit of the supertrace of $e^{-A(\rho)_t^2}$ for $t \to \infty$, which is the Chern character of the index of D^+, with the limit for $t \to 0$.

In chapter five many tools are presented: We introduce the function spaces we deal with, as for example the L^2- spaces, and study operators on them. In particular we recall the properties of Fredholm operators and regular operators on Hilbert C^*-modules, study Hilbert-Schmidt operators and pseudodifferential operators and develop the tools from holomorphic semigroup theory we need.

Notation and conventions: If not specified the vector spaces and algebras are complex, manifolds are smooth.

We often deal with $\mathbb{Z}/2$-graded spaces. Then $[\ ,\ ]_s$ denotes the supercommutator and tr_s the supertrace. In a graded context the tensor products are graded. For an ungraded space V, we denote by V^+ resp. V^- the same space endowed with a graduation: all elements are positive resp. negative.

Tensor products denoted by $\otimes$ are completed. The way of completion is indicated by a suffix in all but two cases: In the case of Hilbert C^*-modules $\otimes$ means the Hilbert C^*-module tensor product, and if one of the spaces is nuclear, then $\otimes$ means $\otimes_\pi$ or $\otimes_\varepsilon$. The algebraic tensor product is denoted by $\odot$.

By a differentiable function on an open subset of $[0,1]^n$ we understand a function that can be differentiably continued to an open subset of $\mathbb{R}^n$. This induces the notion of a differentiable function on a manifold with corners, in our case $M \times M$.

If S, X are sets with $S \subset X$, then the characteristic function of S is denoted by $1_S : X \to \{0,1\}$. If X is a metric space, $y \in X$ and $S \subset X$ then $d(y,S) := \inf_{x \in S} d(x,y)$. For $S_1, S_2 \subset X$ we set $d(S_1,S_2) := \inf_{x \in S_1} d(x,S_2)$

If E resp. F is a vector bundle on a space X resp. Y and $p_i(x_1,x_2) = x_i,\ i = 1,2$ for $(x_1,x_2) \in X \times Y$, then $E \boxtimes F = p_1^* E \otimes p_2^* F$ on $X \times Y$.

We use the notions from [BGV] in the context of spin geometry. In addition a Dirac bundle is a selfadjoint Clifford module endowed with a Clifford connection and we assume that the metric is parallel with respect to the connection.

The value of the constant C which we use in estimates may vary during a series of estimates without an explicit remark.

Chapter 1

Preliminaries

1.1 The geometric situation

In this section we define the spin manifold M and a Dirac bundle E on it. The Dirac operator associated to E will be the main object of our study. Another aim of this section is to fix the notation.

For $k \in \mathbb{Z}/6$ let Z_k be a copy of $\mathbb{R} \times [0,1]$. With the metric and orientation induced by the euclidian metric of $\mathbb{R}^2$ it is an oriented Riemannian manifold with boundary. Let (x_1^k, x_2^k) be the euclidian coordinates of Z_k.

For $r \geq -\frac{1}{2}$, $b \leq \frac{1}{3}$ and $k \in \mathbb{Z}/6$ let

$$F_k(r,b) := \{(x_1^k, x_2^k) \in \big(]r, \infty[\times[0,1]\big) \cup \big(]-1, r] \times ([0,b[\cup]1-b,1])\big)\} \subset Z_k \ .$$

We define

$$F(r,b) := \Big(\bigcup_{k \in \mathbb{Z}/6} F_k(r,b) \Big) / \sim$$

with $(x_1^k, x_2^k) \sim (-\frac{3}{2} - x_1^{k-1}, 1 - x_2^{k-1})$ for $(x_1^k, x_2^k) \in (]-1, -\frac{1}{2}[\times[0,b[)$ and $k \in \mathbb{Z}/6$.

Then $F(r,b)$ inherites the structure of an oriented Riemannian manifold from the sets $F_k(r,b)$.

The open set $F(-\frac{1}{2}, \frac{1}{3}) \setminus \overline{F(-\frac{1}{3}, \frac{1}{4})}$ is diffeomorphic to the open ring $B_1(0)^\circ \setminus B_{1/2}(0) \subset \mathbb{R}^2$ via an oriented diffeomorphism ϕ. We define the manifold with boundary

$$M := F(-\tfrac{1}{2}, \tfrac{1}{3}) \cup_\phi B_1(0)^\circ \ .$$

For $r > -\frac{1}{2}$, $b \leq \frac{1}{3}$ we identify $F(r,b)$ and $F_k(r,b)$ with the corresponding subsets in M. Note that with the coordinates above the sets $F_k(r,b)$ are coordinate patches of M.

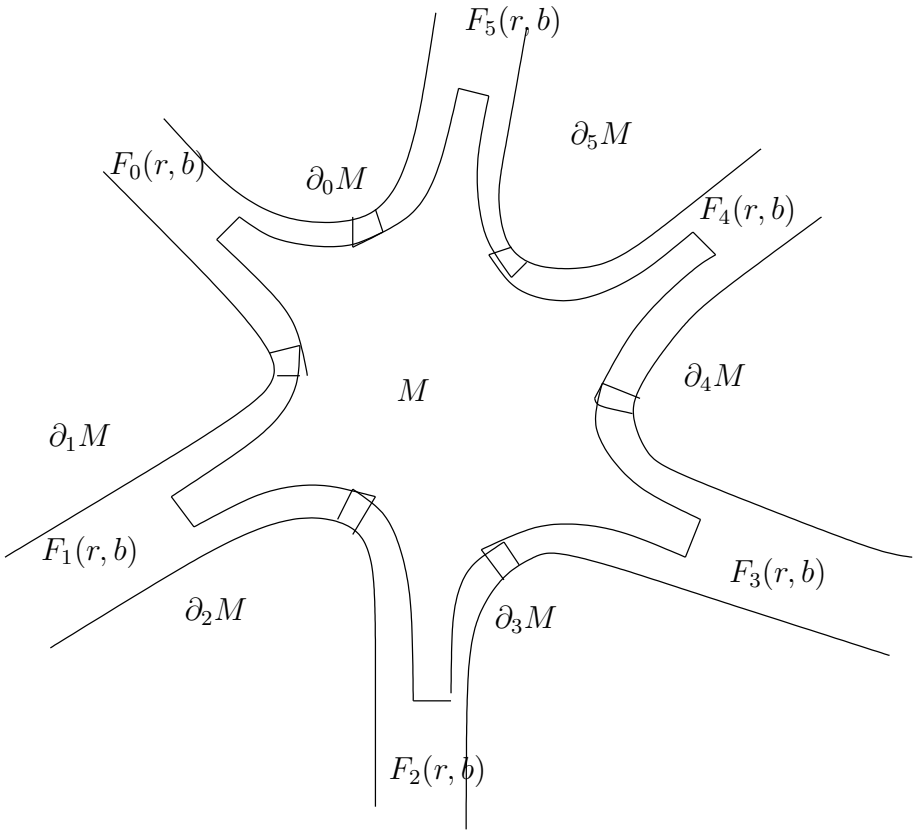
Extend the orientation and metric from $F := F(0, \frac{1}{4})$ to the whole of M such that M is an oriented Riemannian manifold, and endow TM with the Levi-Civita connection. In the following we identify TM and T^*M.

For $r \geq 0$ and $b \leq \frac{1}{4}$ we define an open covering $\mathcal{U}(r,b) = \{\mathcal{U}_k\}_{k \in J}$ of M as follows: Let J be the disjoint union of $\mathbb{Z}/6$ with a one-element set $\{\clubsuit\}$. For $k \in \mathbb{Z}/6$ let $\mathcal{U}_k := F_k(r,b)$ and let $\mathcal{U}_\clubsuit := M \setminus \overline{F(r+1, b/2)}$.

For $r \geq 0$ let $M_r := M \setminus \overline{F(r,0)}$.

The connected components of ∂M are labelled $\partial_k M,\ k \in \mathbb{Z}/6$, such that $\partial_k M \cap F_k(r,b) \subset \{x_2^k = 0\}$ and $\partial_{k+1} M \cap F_k(r,b) \subset \{x_2^k = 1\}$.

The manifold M can be embedded diffeomorphically into $\mathbb{R}^2$, even with a diffeomorphism that is an isometry outside M_r for some $r > 0$. The image of the embedding is illustrated by the following picture.



Now we define the Dirac operator: Choose a spin structure on M and fix $d \in \mathbb{N}$. Let S be the spinor bundle endowed with the Levi-Cività connection and with a hermitian metric such that it is a Dirac bundle. Let E be the graded vector bundle $((\mathbb{C}^+)^d \oplus (\mathbb{C}^-)^d) \otimes S$. The hermitian metric on S and the standard hermitian product on $\mathbb{C}^d$ induce a hermitian metric $< \cdot, \cdot >$ on E. Furthermore there are connections on E and its dual E^* induced by the Clifford connection on S. The Dirac operator associated to the Dirac bundle E is denoted by ∂_M.

The Clifford multiplication $c : TM \to \mathrm{End}E$ induces a skew-hermitian form on $E^{\pm}|_F$ that will be important for the definition of boundary conditions for ∂_M:

The oriented orthonormal frames $((-1)^k dx_1^k, (-1)^k dx_2^k)$ of $TM|_{F_k(0,\frac{1}{4})}$ patch together to an oriented orthonormal frame (e_1, e_2) of $TM|_F$. Hence we have an even parallel endomorphism

$$I = c(e_1)c(e_2) : E^{\pm}|_F \to E^{\pm}|_F$$

defining a skew-hermitian form on $E^{\pm}|_F$, namely

$$E^{\pm}|_F \otimes E^{\pm}|_F \to \mathbb{C},\ (x,y) \mapsto < x, Iy > \ .$$

Since the holonomy of $TM|_F$ is $4\pi\mathbb{Z}$ (measured with respect to any trivialisiation of TM on the whole of M), there are nonvanishing parallel sections of $S^+|_F$ and $S^-|_F$. We choose a parallel unit section s of $S^+|_F$ and fix once and for all the trivialisation of $E|_F$ defined for $x \in F$ by

$$E^+_x = ((\mathbb{C}^+)^d \otimes S^+_x) \oplus ((\mathbb{C}^-)^d \otimes S^-_x) \to (\mathbb{C}^+)^d \oplus (\mathbb{C}^+)^d \ ,$$

$$(v \otimes s(x)) \oplus (w \otimes ic(e_1)s(x)) \mapsto (v,w) \ ,$$

and

$$E^-_x = ((\mathbb{C}^+)^d \otimes S^-_x) \oplus ((\mathbb{C}^-)^d \otimes S^+_x) \to (\mathbb{C}^-)^d \oplus (\mathbb{C}^-)^d \ ,$$

$$(v \otimes ic(e_1)s(x)) \oplus (w \otimes s(x)) \mapsto (v,w) \ .$$

With respect to the trivialisation the endomorphism $I|_{E^+}$ corresponds to

$$I_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in M_{2d}(\mathbb{C})$$

and $I|_{E^-}$ corresponds to $-I_0$.

1.2 The commutative index problem

In this section we outline the family index theorem corresponding to our noncommutative problem. Our presentation follows [BK] with slight modifications.

Let $\mathbb{C}^{2d}$ be endowed with the standard hermitian product $<\ ,\ >$ and the skew-hermitian form induced by I_0 and $<\ ,\ >$.

Let B be a compact space and let (L_0, L_1, L_2) be a triple of pairwise transverse Lagrangian subbundles of the trivial bundle $B \times \mathbb{C}^{2d}$. For any $i = 0, 1, 2$ and $b \in B$ the Lagrangian subspace $L_i(b) \subset \mathbb{C}^{2d}$ defines a parallel Lagrangian subbundle of $E^+|_F$ via the trivialisation fixed in the previous section.

Let $D^+(b)$ be the Dirac operator associated to E with

$$\operatorname{dom} D^+(b) := \{s \in C^\infty_c(M, E^+) \mid s(x) \in L_i(b) \text{ for } x \in \partial_i M \cup \partial_{i+3} M,\ i = 0,1,2\} \ .$$

It turns out that for any $b \in B$ the kernel and cokernel of the closure of $D^+(b)$ are finite dimensional and that the K-theory class of the index bundle of the family $\{D^+(b)\}_{b\in B}$ equals the generalized Maslov index of (L_0, L_1, L_2) whose definition is as follows:

The triple (L_0, L_1, L_2) induces a nondegenerate hermitian form h on L_0; namely let $v = v_1 + v_2,\ w = w_1 + w_2 \in L_0(b)$ with $v_1, w_1 \in L_1(b), v_2, w_2 \in L_2(b)$, then

$$h_b(v,w) := < v_2, I_0 w_1 > \ .$$

The generalized Maslov index is the element $[L_0^+] - [L_0^-] \in K^0(B)$ where L_0^+ and L_0^- are subbundles of L_0 with $L_0^+ \oplus L_0^- = L_0$ and such that h is positive on L_0^+ and negative on L_0^-.

If B is a manifold and the bundles are smooth, then by a generalization of the Atiyah-Patodi index theorem the Chern character of the index bundle can be expressed in terms of η-forms. We outline their definition:

Let ∂ be the differentiation operator on $C^\infty([0,1],\mathbb{C}^{2d})$.

For $i \neq j$ and any $b \in B$ the operator $D_I(b) := I_0\partial$ with domain

$$\operatorname{dom} D_I(b) := \{s \in C^\infty([0,1],\mathbb{C}^{2d}) \mid s(0) \in L_i(b),\ s(1) \in L_j(b)\}$$

is essentially selfadjoint and its closure has a bounded inverse on $L^2([0,1],\mathbb{C}^{2d})$.

There is a family of rescaled superconnections A_t^I associated to σD_I on $C_1 \otimes \mathbb{C}^{2d}$ where C_1 is the Clifford algebra of the vector space $\mathbb{C}$: it is the graded complex unital algebra generated by the odd element σ with $\sigma^2 = 0$.

Let $\operatorname{Tr}_\sigma(a + \sigma b) := \operatorname{Tr}(a)$ (we do not want to specify here the space on which the trace Tr acts).

Then the η-form

$$\eta(L_i, L_j) := -\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \in \Omega^*(B)$$

is well-defined.

The index theorem says

$$\operatorname{ch}(\operatorname{ind} D^+) = [\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)] \in H^*_{dR}(B)\ .$$

Before we formulate a noncommutative version of this problem we present what will play the role of the algebra $\Omega^*(B)$, the Chern character, the de Rham cohomology $H^*_{dR}(B)$ and the Lagrangian subbundles. This is done in the next sections.

1.3 The algebra of differential forms

1.3.1 The universal graded differential algebra

Let $\mathcal{B}$ be an involutive locally m-convex Fréchet algebra with unit. In particular, the group of invertible elements $\mathrm{Gl}(\mathcal{B})$ in $\mathcal{B}$ is open and the map $\mathrm{Gl}(\mathcal{B}) \to \mathrm{Gl}(\mathcal{B}),\ a \mapsto a^{-1}$ is continuous [Ma].

In this section we recall the definition of the topological universal graded differential algebra $\hat{\Omega}_*\mathcal{B}$ introduced by Karoubi [Kar] and collect its main properties. Most of the definitions of this section are either taken from [Kar] or are an immediate generalisation from the algebraic case described in [CQ].

We write $\otimes_\pi$ for the completed projective tensor product.

Let

$$\hat{\Omega}_k\mathcal{B} := \mathcal{B} \otimes_\pi (\otimes^k_\pi(\mathcal{B}/\mathbb{C}))$$

and

$$\hat{\Omega}_*\mathcal{B} := \prod_{k=0}^\infty \hat{\Omega}_k\mathcal{B}\ .$$

Consider $\hat{\Omega}_*\mathcal{B}$ as a $\mathbb{Z}$-graded vector space with the following additional structures:

Product: There is a graded continuous product on $\hat{\Omega}_*\mathcal{B}$ defined for elementary tensors by

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)(b_{k+1} \otimes b_{k+2} \otimes \ldots b_n) := \sum_{j=0}^{k} (-1)^{k-j} (b_0 \otimes b_1 \otimes \ldots \otimes b_j b_{j+1} \otimes b_{j+2} \otimes \ldots b_n) \ .$$

It is continuous, since the multiplication map $\mathcal{B} \otimes_\pi \mathcal{B} \to \mathcal{B}$ is continuous by the universal property of the projective tensor product. With the product $\hat{\Omega}_*\mathcal{B}$ is a locally m-convex Fréchet algebra.

Differential: There is a continuous differential d of degree one on the graded algebra $\hat{\Omega}_*\mathcal{B}$ defined on elementary tensors by

$$\mathrm{d}(b_0 \otimes b_1 \otimes \ldots \otimes b_k) := 1 \otimes b_0 \otimes b_1 \otimes \ldots \otimes b_k \ .$$

It satisfies the graded Leibniz rule: For $\alpha \in \hat{\Omega}_k\mathcal{B}$ and $\beta \in \hat{\Omega}_*\mathcal{B}$

$$\mathrm{d}(\alpha\beta) = (\mathrm{d}\,\alpha)\beta + (-1)^k \alpha(\mathrm{d}\,\beta) \ .$$

Furthermore

$$b_0 \otimes b_1 \otimes \ldots \otimes b_k = b_0\, \mathrm{d}\, b_1\, \mathrm{d}\, b_2 \ldots \mathrm{d}\, b_k \ .$$

If $\mathcal{B}$ is a Banach algebra, then d is a map of norm one.

Involution: We extend the $*$-operation on $\mathcal{B}$ to a continuous involution on $\hat{\Omega}_*\mathcal{B}$ by setting

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)^* := (1 \otimes b_k^* \otimes b_{k-1}^* \otimes \ldots \otimes b_1^*) b_0^*$$

or equivalently

$$(b_0\, \mathrm{d}\, b_1\, \mathrm{d}\, b_2 \ldots \mathrm{d}\, b_k)^* = (\mathrm{d}\, b_k^*\, \mathrm{d}\, b_{k-1}^* \ldots \mathrm{d}\, b_1^*) b_0^* \ .$$

For $\omega_1, \omega_2 \in \hat{\Omega}_*\mathcal{B}$

$$(\omega_1\omega_2)^* = \omega_2^*\omega_1^* \ ,$$

and for $\omega \in \hat{\Omega}_k\mathcal{B}$

$$(\mathrm{d}\,\omega)^* = (-1)^k\, \mathrm{d}(\omega^*) \ .$$

With these structures $\hat{\Omega}_*\mathcal{B}$ is an involutive graded differential Fréchet locally m-convex algebra. It behaves functorially in $\mathcal{B}$.

If $\omega \in \hat{\Omega}_*\mathcal{B}$, its part in degree n is denoted by ω^n .

Let

$$\hat{\Omega}_{\leq m}\mathcal{B} := \hat{\Omega}_*\mathcal{B} / \prod_{k=m+1}^{\infty} \hat{\Omega}_k\mathcal{B} \ .$$

We identify $\hat{\Omega}_{\leq m}\mathcal{B}$ as a graded vector space with the subspace of forms up to degree m in $\hat{\Omega}_*\mathcal{B}$.

Now we come to the definitions of the homology and the Chern character of $\mathcal{B}$:

The Fréchet space $\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$ generated by the supercommutators in $\hat{\Omega}_*\mathcal{B}$ is preserved by d by Leibniz rule. It follows that $(\hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}, \mathrm{d})$ is a complex.

Definition 1.3.1. *The* DE RHAM HOMOLOGY OF $\mathcal{B}$ *is*

$$H_*^{dR}(\mathcal{B}) := H_*(\hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}, \mathrm{d}) \ .$$

On the right hand side we take the topological homology, i.e. we quotient out the closure of the range of d *in order to obtain a Hausdorff space.*

We extend d to the a map $\mathrm{d} : M_n(\hat{\Omega}_*\mathcal{B}) \to M_n(\hat{\Omega}_*\mathcal{B}),\ A \mapsto \mathrm{d}(A)$ by applying d component by component. Note the difference between $\mathrm{d}\, A = \mathrm{d} \circ A$ and $\mathrm{d}(A)$. Sometimes we write $(\mathrm{d}\, A)$ for $\mathrm{d}(A)$.

For $A \in M_n(\hat{\Omega}_k\mathcal{B})$

$$\mathrm{d}\, A = (\mathrm{d}\, A) + (-1)^k A\, \mathrm{d} \ .$$

In the following a projection is not necessarily selfadjoint.

Definition 1.3.2. *Let $P \in M_n(\mathcal{B})$ be a projection. Then*

$$\mathrm{ch}(P) := \sum_{k=0}^{\infty} (-1)^k \mathrm{tr} P(\mathrm{d}(P))^{2k} \in \hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$$

is the CHERN CHARACTER FORM OF P.

Proposition 1.3.3. *1. If $P : [0,1] \to M_n(\mathcal{B})$ is a differentiable path of projections, then* $\mathrm{ch}(P)$ *is constant.*

2. The Chern character form is closed.

3. If $\mathcal{B}$ is a local Banach algebra [Bl], then the Chern character form induces a homomorphism

$$\mathrm{ch} : K_0(\mathcal{B}) \to H_*^{dR}(\mathcal{B}),\ \ \mathrm{ch}([P] - [Q]) := \mathrm{ch}(P) - \mathrm{ch}(Q) \ ,$$

called the CHERN CHARACTER.

Proof. First note

$$0 = \mathrm{d}((1-P)P) = (1-P)\,\mathrm{d}\, P - (\mathrm{d}\, P)P$$

for a projection $P \in M_n(\mathcal{B})$, hence $P(\mathrm{d}\, P)P = 0$ and $(1-P)\,\mathrm{d}\, P = (\mathrm{d}\, P)P$. From $0 = \mathrm{d}((1-P)P)$ it follows that $(\mathrm{d}\, P)P = (1-P)(\mathrm{d}\, P)$, therefore

$$P(\mathrm{d}\, P)^2 = (\mathrm{d}\, P)^2 P \ .$$

1) If P' be the derivative of P then

$$0 = ((1-P)P)' = (1-P)P' - P'P \ ,$$

hence $PP'P = 0$ and $(1-P)P' = P'P$.

Furthermore

$$(\mathrm{tr} P(\mathrm{d}(P))^{2k})' = \mathrm{tr} P'(\mathrm{d}(P))^{2k} + \mathrm{tr} P((\mathrm{d}(P))^{2k})' \ .$$

The first term on the right hand side vanishes by

$$\begin{aligned}\mathrm{tr}P'(\mathrm{d}(P))^{2k} &= \mathrm{tr}P'P(\mathrm{d}(P))^{2k} + \mathrm{tr}PP'(\mathrm{d}(P))^{2k} \\ &= \mathrm{tr}P'P(\mathrm{d}(P))^{2k}P + \mathrm{tr}PP'(\mathrm{d}(P))^{2k}P \\ &= 2\ \mathrm{tr}PP'P(\mathrm{d}(P))^{2k} \\ &= 0\ .\end{aligned}$$

The second term vanishes for $k > 0$ by

$$\begin{aligned}\mathrm{tr}P((\mathrm{d}(P))^{2k})' &= \sum_{i=0}^{2k-1} \mathrm{tr}P(\mathrm{d}\,P)^i(\mathrm{d}\,P)'(\mathrm{d}\,P)^{2k-i-1} \\ &= k\ \mathrm{tr}(\mathrm{d}\,P)^{2k-1}P(\mathrm{d}\,P)' + k\ \mathrm{tr}((\mathrm{d}\,P)^{2k-2}P(\mathrm{d}\,P)(\mathrm{d}\,P)' \\ &= k\ \mathrm{tr}(\mathrm{d}\,P)^{2k-1}(\mathrm{d}\,P)' \\ &= \tfrac{1}{2}\ (\mathrm{tr}(\mathrm{d}\,P)^{2k})' \\ &= \tfrac{1}{2}\ \mathrm{d}(\mathrm{tr}P(\mathrm{d}\,P)^{2k-1})' \\ &= 0\end{aligned}$$

2) follows from

$$\begin{aligned}\mathrm{d}\,\mathrm{tr}P(\mathrm{d}\,P)^{2k} &= \mathrm{tr}(\mathrm{d}\,P)^{2k+1} \\ &= \mathrm{tr}(1-P)(\mathrm{d}\,P)^{2k+1}(1-P) + \mathrm{tr}P(\mathrm{d}\,P)^{2k+1}P \\ &= 0\ .\end{aligned}$$

3) follows from 1) and 2). □

1.3.2 Supercalculus

Let $\mathcal{B}$ be as before.

In this section all spaces and tensor products are $\mathbb{Z}/2$-graded. If no graduation is specified we assume the graduation to be trivial.

Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}/2$-graded complex vector space with $\dim V^+ = m,\ \dim V^- = n$ and consider $\hat{\Omega}_*\mathcal{B}$ as a $\mathbb{Z}/2$-graded space with the even/odd-grading.

The space $V \otimes \hat{\Omega}_*\mathcal{B}$ is a free $\mathbb{Z}/2$-graded right $\hat{\Omega}_*\mathcal{B}$-module. It is furthermore a left supermodule of the superalgebra $\mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$. (Note that our setting differs from the corresponding one defined in [BGV], since we consider right $\hat{\Omega}_*\mathcal{B}$-modules. Sometimes this causes different signs.)

The supertrace $\mathrm{tr}_s : \mathrm{End}(V) \to \mathbb{C}$ extends to a supertrace

$$\mathrm{tr}_s : \mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B} \to \hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$$

$$\mathrm{tr}_s(T \otimes \omega) := \mathrm{tr}_s(T)\omega\ .$$

By quotienting out the supercommutator we ensure that $\mathrm{tr}_s([T_1, T_2]_s) = 0$ for $T_1, T_2 \in \mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$.

Note that the differential d acts on elements of $V \otimes \hat{\Omega}_* \mathcal{B}$ resp. $\mathrm{End}(V) \otimes \hat{\Omega}_* \mathcal{B}$ by

$$\mathrm{d}(A \otimes \omega) = (-1)^{\deg A} A \otimes \mathrm{d}\,\omega$$

for $A \in V^{\pm}$ resp. $A \in \mathrm{End}^{\pm}(V)$ and $\omega \in \hat{\Omega}_* \mathcal{B}$.

Since $\mathrm{tr}_s(A \otimes \omega) = 0$ for $A \in \mathrm{End}^{-}(V)$, it follows

$$\mathrm{tr}_s\, \mathrm{d}(T) = \mathrm{d}\, \mathrm{tr}_s T$$

for $T \in \mathrm{End}(V) \otimes \hat{\Omega}_* \mathcal{B}$.

Furthermore from Leipniz rule it follows for $A \otimes \omega \in \mathrm{End}^{\pm}(V) \otimes \hat{\Omega}_k \mathcal{B},\ v \in V \otimes \hat{\Omega}_* \mathcal{B}$:

$$\mathrm{d}((A \otimes \omega)v) = \mathrm{d}(A \otimes \omega)v + (-1)^{\deg A + k}(A \otimes \omega)\, \mathrm{d}\, v\ .$$

Now let M be a Riemannian manifold, possibly with boundary. In §5.2.3 we introduce the notion of a Hilbert-Schmidt operator. We use the notation of §5.2.2 and §5.2.3 in the following.

Definition 1.3.4. *A* TRACE CLASS OPERATOR ON $L^2(M, V \otimes \hat{\Omega}_{\leq \mu} \mathcal{B})$ *is an operator* $T = \sum A_i B_i$ *where* A_i, B_i *are Hilbert-Schmidt operators such that* $(x \mapsto k_{A_i}(x, \cdot)) \in C(M, L^2(M, \mathrm{End}V \otimes \hat{\Omega}_{\leq \mu} \mathcal{B}))$ *and* $(y \mapsto k_{A_i}(\cdot, y)) \in C(M, L^2(M, \mathrm{End}V \otimes \hat{\Omega}_{\leq \mu} \mathcal{B}))$*, and analogously for* B_i*.*

In particular a trace class operator T is a Hilbert-Schmidt operator with a uniquely defined continuous integral kernel k_T. We set

$$\mathrm{Tr}_s T := \int_M \mathrm{tr}_s k_T(x, x) \in \hat{\Omega}_{\leq \mu} \mathcal{B} / \overline{[\hat{\Omega}_{\leq \mu} \mathcal{B}, \hat{\Omega}_{\leq \mu} \mathcal{B}]}\ .$$

Note that

$$|\mathrm{Tr}_s A_i B_i| \leq \|A_i\|_{HS} \|B_i\|_{HS}$$

and

$$\mathrm{Tr}_s [A_i, B_i]_s = 0\ .$$

1.3.3 The algebras $\mathcal{A}_\infty$ and $\hat{\Omega}_* \mathcal{A}_\infty$

Let $(\mathcal{A}_j, \iota_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j)_{j \in \mathbb{N}_0}$ be a projective system of involutive Banach algebras with unit satisfying the following conditions (motivated by [Lo], §2.1):

- The algebra $\mathcal{A} := \mathcal{A}_0$ is a C^*-algebra.
- For any $j \in \mathbb{N}_0$ the map $\iota_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j$ is injective.
- For any $j \in \mathbb{N}_0$ the map $\iota_j : \mathcal{A}_\infty := \varprojlim_i \mathcal{A}_i \to \mathcal{A}_j$ has dense range.
- For any $j \in \mathbb{N}_0$ the algebra $\mathcal{A}_j$ is stable with respect to the holomorphic functional calculus in $\mathcal{A}$.

The projective limit $\mathcal{A}_\infty$ is an involutive locally m-convex Fréchet algebra with unit.

The motivating example is $\mathcal{A}_j = C^j(B)$ for a closed smooth manifold B.

Proposition 1.3.5. *For $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we have:*

1. *The map $\iota_j : \mathcal{A}_\infty \to \mathcal{A}_j$ is injective.*
2. *The algebras $M_n(\mathcal{A}_\infty)$ and $M_n(\mathcal{A}_j)$ are stable with respect to the holomorphic functional calculus in $M_n(\mathcal{A})$.*
3. *The map $\iota_{0*} : K_0(\mathcal{A}_\infty) \to K_0(\mathcal{A})$ is an isomorphism.*

Proof. 1) follows immediately.

2) follows from [Bo], Prop. A.2.2.

3) follows from [Bo], Th. A.2.1. □

The projective system $(\mathcal{A}_j, \iota_{j+1,j})_{j\in\mathbb{N}_0}$ induces two projective systems of involutive graded differential Fréchet algebras:

One of them is given by the maps

$$\iota_{j+1,j*} : \hat{\Omega}_*\mathcal{A}_{j+1} \to \hat{\Omega}_*\mathcal{A}_j \ .$$

Furthermore $(\hat{\Omega}_{\leq m}\mathcal{A}_j)_{m,j\in\mathbb{N}}$ is a projective system of involutive Banach graded differential algebras.

Their limits coincide:

The inclusion $\hat{\Omega}_{\leq m}\mathcal{A}_j \to \hat{\Omega}_*\mathcal{A}_j$ is left inverse to the projection $\hat{\Omega}_*\mathcal{A}_j \to \hat{\Omega}_{\leq m}\mathcal{A}_j$. The induced maps between the projective limits are inverse to each other. It follows that

$$\varprojlim_j \hat{\Omega}_*\mathcal{A}_j \cong \varprojlim_{j,m} \hat{\Omega}_{\leq m}\mathcal{A}_j \ .$$

Furthermore the inclusions $\iota_{j*} : \hat{\Omega}_*\mathcal{A}_\infty \to \hat{\Omega}_*\mathcal{A}_j$ induce a map

$$\iota_* : \hat{\Omega}_*\mathcal{A}_\infty \to \varprojlim_j \hat{\Omega}_*\mathcal{A}_j \ .$$

Proposition 1.3.6. *There are the following canonical isomorphisms of involutive Fréchet locally m-convex graded differential algebras:*

1. $\hat{\Omega}_*\mathcal{A}_\infty \cong \varprojlim_j \hat{\Omega}_*\mathcal{A}_j$,
2. $\hat{\Omega}_*\mathcal{A}_\infty \cong \varprojlim_{j,m} \hat{\Omega}_{\leq m}\mathcal{A}_j$.

There is the following canonical isomorphism of graded Fréchet spaces:

$$3. \quad \hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s} \cong \varprojlim_{j,m} \hat{\Omega}_{\leq m}\mathcal{A}_j / \overline{[\hat{\Omega}_{\leq m}\mathcal{A}_j, \hat{\Omega}_{\leq m}\mathcal{A}_j]_s} \ .$$

Proof. 1): It is enough to prove that the right hand side and the left hand side are isomorphic as topological vector spaces. This follows from the fact that projective limits and projective tensor products commute ([Kö] 41.6).

2) follows from the remarks preceding the proposition.

The following lemma implies 3). □

The importance of this proposition for our purposes is the following: Since it is easier to deal with Banach spaces than with Fréchet spaces we will prove our index theorem in $\hat{\Omega}_{\leq m}\mathcal{A}_j/\overline{[\hat{\Omega}_{\leq m}\mathcal{A}_j, \hat{\Omega}_{\leq m}\mathcal{A}_j]_s}$, making sure that the expressions in the index theorem behave well when taking the projective limit. By the proposition this will prove the index theorem in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$.

Lemma 1.3.7. *Let $(A_j, \iota_{j+1,j})_{j\in\mathbb{N}}$ be a projective system of Banach spaces and let $(B_j, \iota_{j+1,j}|_{B_j})_{j\in\mathbb{N}}$ be a projective system such that B_j is a closed subspace of A_j for all $j \in \mathbb{N}$.*

Let $A_\infty := \varprojlim_j A_j$ and $B_\infty := \varprojlim_j B_j$ and assume further that the image of A_∞ is dense in A_j and the image of B_∞ is dense in B_j for any $j \in \mathbb{N}$.

Then, canonically,

$$A_\infty/B_\infty \cong \varprojlim_j A_j/B_j \ .$$

Proof. For $j \in \mathbb{N}$ let $\iota_j : A_\infty \to A_j$ be the induced map. We prove that the map

$$\iota_\infty : A_\infty/B_\infty \to \varprojlim_j A_j/B_j$$

is an isomorphism:

It is well-defined since $\iota_j(B_\infty) \subset B_j$ for all $j \in \mathbb{N}$.

For injectivity let $a \in A_\infty/B_\infty$ with $\iota_\infty a = 0$. If $\tilde{a} \in A_\infty$ with $[\tilde{a}] = a$ then $\iota_j\tilde{a} \in B_j$ for all $j \in \mathbb{N}$, so $\tilde{a} \in B_\infty$, thus $a = 0$.

In order to prove surjectivity we assume that the norms of the maps $\iota_{j+1,j}$ are less than or equal to one. This can be obtained by rescaling the norms inductively.

Let $a \in \varprojlim_j A_j/B_j$ and choose $a_j \in A_j$ such that $[a_j] \in A_j/B_j$ is the image of a with respect to the map

$$\varprojlim_j A_j/B_j \to A_j/B_j \ .$$

As shown at the end of the proof we can assume

$$(*) \qquad |\iota_{j+1,j}a_{j+1} - a_j| \leq j^{-1} \ .$$

For $n \geq j$ let $\iota_{n,j} : A_n \to A_j$ be the injection. From $(*)$ it follows that for any $j \in \mathbb{N}$ the sequence $(\iota_{n,j}(a_n))_{n\geq j}$ is a Cauchy sequence in A_j. Let $\tilde{a}_j \in A_j$ be its limit.

Then $\iota_{j+1,j}\tilde{a}_{j+1} = \tilde{a}_j$ and $[\tilde{a}_j] = [a_j] \in A_j/B_j$. It follows that there is an element $\tilde{a} \in A_\infty$ with $\iota_j\tilde{a} = \tilde{a}_j$. Hence $\iota_\infty[\tilde{a}] = a$.

Now we show how to modify the sequence $(a_j)_{j\in\mathbb{N}}$ such that $(*)$ is fulfilled: We define $a'_1 := a_1$. Assume that a'_j is already defined. Then $\iota_{j+1,j}a_{j+1} - a'_j \in B_j$. Since the image of B_∞ is dense in B_j, we may choose $b_{j+1} \in B_\infty$ with

$$|\iota_j b_{j+1} - (\iota_{j+1,j}a_{j+1} - a'_j)| \leq j^{-1}$$

and define $a'_{j+1} := a_{j+1} + \iota_{j+1}b_{j+1}$. Now take $(a'_j)_{j\in\mathbb{N}}$. □

1.4 Lagrangian projections

Let $\mathcal{A}$ be as before.

In this section we define and study the analogue of Lagrangian subbundles and of the Maslov index bundle.

Definition 1.4.1. *Let $n \in \mathbb{N}$.*

Two selfadjoint projections $P_1, P_2 \in M_n(\mathcal{A})$ are called TRANSVERSE *if*

$$\operatorname{Ran} P_1 \oplus \operatorname{Ran} P_2 = \mathcal{A}^n \ .$$

We will often use the following transversality criterion: Two selfadjoint projections P_1, P_2 are transverse if and only there exist $a, b \in \mathrm{Gl}(\mathcal{A})$ such that $aP_1 + bP_2 \in M_n(\mathcal{A})$ is invertible, and this is equivalent to the invertibility of $aP_1 + bP_2$ for any $a, b \in \mathrm{Gl}(\mathcal{A})$.

1.4.1 Definition and properties

Definition 1.4.2. *For $n \in \mathbb{N}$ let $\mathcal{A}^{2n}$ be endowed with the standard $\mathcal{A}$-valued scalar product and let*

$$I_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : \mathcal{A}^n \oplus \mathcal{A}^n \to \mathcal{A}^n \oplus \mathcal{A}^n \ .$$

A LAGRANGIAN PROJECTION ON $\mathcal{A}^{2n}$ *is a selfadjoint projection $P \in M_{2n}(\mathcal{A})$ with*

$$PI_0 = I_0(1 - P) \ .$$

Note that if $\mathcal{A} = C(B)$ for some compact space B, then the Lagrangian projections are in one-to-one correspondence with the subbundles of $B \times \mathbb{C}^{2n}$ that are Lagrangian with respect to the skew-hermitian form induced by I_0. The transversality of two projections is equivalent to the transversality of the corresponding subbundles.

Note furthermore that any Lagrangian projection P on $\mathcal{A}^{2n}$ is a Lagrangian projection on $M_n(\mathcal{A})^2$ as well, where we consider $M_n(\mathcal{A})^2$ as a Hilbert $M_n(\mathcal{A})$-module.

Denote

$$P_s := \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ .$$

Lemma 1.4.3. *1. For every Lagrangian projection P of $\mathcal{A}^{2n}$ there is a unitary $p \in M_n(\mathcal{A})$ such that*

$$P = \tfrac{1}{2}\begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix} \ .$$

2. For every Lagrangian projection P on $\mathcal{A}^{2n}$ the unitary

$$U = \begin{pmatrix} 1 & 0 \\ 0 & p^* \end{pmatrix} \in M_{2n}(\mathcal{A})$$

with p as in 1) fulfills $UI_0 = I_0U$ and $UPU^* = P_s$.

Proof. 1) Since P is selfadjoint, there are $a, b, c \in M_n(\mathcal{A})$ with $a = a^*, c = c^*$ such that $P = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$. From $PI_0 = I_0(1-P)$ it follows that

$$\begin{pmatrix} ia & -ib \\ ib^* & -ic \end{pmatrix} = \begin{pmatrix} i(1-a) & -ib \\ ib^* & -i(1-c) \end{pmatrix} ,$$

thus $a = c = \frac{1}{2}$. Furthermore from $P^2 = P$ it follows that $2b$ is unitary.

2) is clear. □

Lemma 1.4.4. *Let $P \in M_{2n}(\mathcal{A}_\infty)$ be a Lagrangian projection of $\mathcal{A}^{2n}$ transverse to P_s. Let $\tilde{P} \in M_{2n}(\mathbb{C})$ be a complex Lagrangian projection. Then for every $0 < \varepsilon_1 < \varepsilon_2$ there is a smooth path of unitaries $U : [0, \varepsilon_2] \to M_{2n}(\mathcal{A}_\infty)$ such that*

1. *$U(0)PU(0)^* = \tilde{P}$,*
2. *U equals 1 on a neighbourhood of ε_2,*
3. *U is constant on $[0, \varepsilon_1]$,*
4. *U is diagonal with respect to the decomposition $\mathcal{A}^{2n} = \mathcal{A}^n \oplus \mathcal{A}^n$.*

Note that 4) implies taht $UI_0 = I_0U$.

Proof. It is enough to prove the assertion for $\tilde{P} = P_s$.

Let p be as in the previous lemma applied to P.

Since P and P_s are transverse, $P - P_s$ is invertible. It follows that $p - 1$ and $p^* - 1$ are invertible, so $\log p$ and $\log p^*$ are well-defined if we choose the complement of $[0, \infty)$ in $\mathbb{C}$ as a domain for the logarithm.

Let $\chi : [0, 1] \to [0, 1]$ be a smooth function with $\chi|_{[0,\varepsilon_1]} = 0$ and $\chi(t) = 1$ for $t \in [\frac{\varepsilon_2 - \varepsilon_1}{2}, \varepsilon_2]$. Then the smooth path of unitaries

$$\gamma : [0, \varepsilon_2] \to M_n(\mathcal{A}),\ \gamma(t) := \exp(2\pi i \chi(t) + (1 - \chi(t)) \log p^*)$$

connects p^* with 1. Then $\gamma(t) \in M_n(\mathcal{A}_\infty)$ for all $t \in [0, \varepsilon_2]$ since $M_n(\mathcal{A}_\infty)$ is stable under holomorphic function calculus in $M_n(\mathcal{A})$ by Prop. 1.3.5.

Then

$$U := \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$$

satisfies the conditions. □

1.4.2 The Maslov index

Let (P_0, P_1, P_2) be a triple of pairwise transverse Lagrangian projections on $\mathcal{A}^{2n}$. Then we can write every $x \in \mathcal{A}^{2n}$ uniquely as $x = x_1 + x_2$ with $x_i \in \operatorname{Ran} P_i,\ i = 1, 2$, namely $x_1 = P_1(P_1 - P_2)^{-1}x$ and $x_2 = -P_2(P_1 - P_2)^{-1}x$.

The form

$$h : \operatorname{Ran} P_0 \times \operatorname{Ran} P_0 \to \mathcal{A},\ (x, y) \mapsto < x_2, I_0 y_1 >$$

is hermitian and its radical vanishes [Wa]. It can be represented by the matrix

$$A := -P_0(P_1 - P_2)^{-1}P_2 I_0 P_1(P_1 - P_2)^{-1}P_0 \in M_{2n}(\mathcal{A}) \ .$$

Since A is the composition of projections and invertibles, the range of A is closed, thus the restriction of A to the range of P_0 is invertible. Hence the hermitian form h is non-singular and defines an element in $K_0(\mathcal{A})$ [Ros].

Definition 1.4.5. *The* Maslov index $\tau(P_0, P_1, P_2) \in K_0(\mathcal{A})$ *of a triple of pairwise transverse* Lagrangian projections (P_0, P_1, P_2) *is the class of the hermitian form h in $K_0(\mathcal{A})$.*

We can express the Maslov index in terms of A as follows:

$$\tau(P_0, P_1, P_2) = [1_{\{x>0\}}(A)] - [1_{\{x<0\}}(A)] \in K_0(\mathcal{A}) \ .$$

Note that an even permutation of the projections leaves the Maslov index unchanged whereas an odd permutation turns it into its negative.

Proposition 1.4.6. *For $i = 0, 1, 2$ let $P_i : [0, 1] \to M_{2n}(\mathcal{A})$ be continuous paths of Lagrangian projections such that $P_i(t) - P_j(t)$ is invertible for all $i \neq j$ and all $t \in [0, 1]$.*

Then the Maslov index $\tau(P_0(t), P_1(t), P_2(t))$ does not depend on t.

Proof. The selfadjoint element $A(t) \in M_{2n}(\mathcal{A})$ defined by $(P_0(t), P_1(t), P_2(t))$ as above depends continuously on t for all $t \in [0, 1]$. It follows that the projections $1_{\{x>0\}}(A(t))$ and $1_{\{x<0\}}(A(t))$ also depend continuously on t, thus their K-theory classes are constant. □

Let B be a compact space and let (P_0, P_1, P_2) be a triple of pairwise transverse Lagrangian projections in $M_{2n}(C(B))$. Let (L_0, L_1, L_2) be the corresponding triple of Lagrangian subbundles of $B \times \mathbb{C}^{2n}$. Then the Maslov index bundle $[L_0^+] - [L_0^-]$ defined in §1.2 corresponds to $\tau(P_0, P_1, P_2)$ under the canonical isomorphism $K^0(B) \cong K_0(C(B))$.

Now we study in some detail the Maslov index of a pair (P_0, P_1, P_2) with $P_0 = P_s$. The general case can be reduced to that case by Lemma 1.4.3.

The Cayley transform yields a bijective map from the space of selfadjoint elements in $M_d(\mathcal{A})$ to space of projections transverse to P_s, namely

$$a \mapsto P(a) := \tfrac{1}{2} \begin{pmatrix} 1 & \frac{a-i}{a+i} \\ \frac{a+i}{a-i} & 1 \end{pmatrix} \ .$$

Lemma 1.4.7. *Let $a_1, a_2 \in M_n(\mathcal{A})$ be selfadjoint. Then $P(a_1)$ and $P(a_2)$ are transverse projections if and only if $a_1 - a_2$ is invertible.*

Proof. Let $U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

$$UP(a_j)U^* = (a_j^2+1)^{-1} \begin{pmatrix} a_j^2 & -ia_j \\ ia_j & 1 \end{pmatrix} .$$

Now $P(a_1)$ and $P(a_2)$ are transverse if and only if

$$(a_1^2+1)UP(a_1)U^* - (a_2^2+1)UP(a_2)U^*$$

is invertible and this is the case if and only if $a_1 - a_2$ is invertible. □

Lemma 1.4.8. *Let (P_s, P_1, P_2) be a triple of pairwise transverse projections and let $a_1, a_2 \in M_n(\mathcal{A})$ be such that $P_i = P(a_i),\ i = 1, 2$. Let $p^+ := 1_{\{x>0\}}(a_1 - a_2)$.*

1. *There are continuous paths $P_1, P_2 : [0, 2] \to M_{2n}(\mathcal{A})$ such that $P_s, P_1(t), P_2(t)$ are pairwise transverse for all $t \in [0, 2]$ and such that $P_1(2) = P(2p^+ - 1)$ and $P_2(2) = P(1 - 2p^+)$.*

2. $$\tau(P_s, P_1, P_2) = [p^+] - [1 - p^+] .$$

Proof. 1) For $t \in [0, 1]$ define $a_1(t) := \frac{1}{2}(t(a_1 - a_2) + (1 - t)a_1)$ and $a_2(t) := \frac{1}{2}(t(a_2 - a_1) + (1 - t)a_2)$. Then $a_1(t) - a_2(t) = a_1 - a_2$ is invertible, thus the projections $P(a_1(t))$ and $P(a_2(t))$ are transverse. Furthermore $a_1(0) = a_1$ and $a_1(1) = a_1 - a_2$ whereas $a_2(0) = a_2$ and $a_2(1) = a_2 - a_1$.

For $t \in [1, 2]$ let $a_1(t)$ be a path of invertible selfadjoint elements with $a_1(1) = a_1 - a_2$ and $a_1(2) = p^+ - (1 - p^+)$ and let $a_2(t) = -a_1(t)$.

Then the paths $P(a_1(t))$ and $P(a_2(t))$ satisfy the conditions.

2) From 1) and Prop. 1.4.6 it follows that $\tau(P_s, P_1, P_2) = \tau(P_s, P(2p^+ - 1), P(1 - 2p^+))$.

Note that the Cayley transform of $2p^+ - 1$ is $i(2p^+ - 1)$. By computing the matrix A we see that $1_{\{x>0\}}(A) = p^+$ and $1_{\{x<0\}}(A) = (1 - p^+)$. □

Chapter 2

The Fredholm operator and its index

2.1 The operator D on M

2.1.1 Definition of D

Now we come back to the geometric situation described in §1.1.

We tensorize the bundle E with the C^*-algebra $\mathcal{A}$ in order to get an $\mathcal{A}$-vector bundle [MF]. Furthermore for $i, \mu \in \mathbb{N}_0$ we consider the bundle $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ of right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-modules. Keep in mind that E can be trivialized on M via a global orthonormal frame. Thus we do not need a theory of Banach space bundles in this context.

The hermitian metrics on E extends to an $\mathcal{A}$-valued scalar product $< \cdot, \cdot >$ on $E \otimes \mathcal{A}$ and to an $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-valued non-degenerated product on $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ (see §5.2.3 for this notion). Furthermore parallel transport is defined on $E \otimes \mathcal{A}$ resp. $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$.

By the trivialisation of $E|_F$ fixed in §1.1 we identify $(E|_F \otimes \mathcal{A}, I)_x,\ x \in F$ with $(\mathcal{A}^{4d}, I_0 \oplus (-I_0))$ as a Hilbert $\mathcal{A}$-module with a skew-hermitian structure, and $(E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)_x,\ x \in F$ with $(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}$ as a right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-module with an $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-valued non-degenerated product.

Recall that the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$ can be defined as the completion of $C_c^\infty(M, E \otimes \mathcal{A})$ with respect to the norm induced by the $\mathcal{A}$-valued scalar product

$$< f, g > := \int_M < f(x), g(x) > dx \ .$$

(For fixing notation a short summary about Hilbert $\mathcal{A}$-modules can be found in §5.1.1.) By Lemma 5.1.20 any orthonormal basis of the Hilbert space $L^2(M, E)$ is an orthonormal basis of $L^2(M, E \otimes \mathcal{A})$ and hence $L^2(M, E \otimes \mathcal{A})$ is isomorphic to $l^2(\mathcal{A})$.

We introduce Schwartz spaces of sections of $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$:

First for $k \in \mathbb{Z}/6$ we define the Schwartz space

$$\mathcal{S}(Z_k, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) := \mathcal{S}(\mathbb{R}) \otimes_\pi C^\infty([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \ .$$

For $r \geq 0,\ 0 \leq b \leq \frac{1}{4}$ choose a partition of unity $\{\phi_k\}_{k\in J}$ subordinate to the covering $\mathcal{U}(r,b)$.

The embedding $F_k(r,b) \hookrightarrow Z_k,\ k \in \mathbb{Z}/6$ and the trivialisation of $E|_F$ induce a map

$$C^\infty(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^\infty(Z_k, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}),\ f \mapsto \phi_k f\ .$$

As a vector space let $\mathcal{S}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ be the largest subspace of $C^\infty(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ such that for all $k \in \mathbb{Z}/6$ the maps

$$\mathcal{S}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to \mathcal{S}(Z_k, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}),\ f \mapsto \phi_k f$$

are well-defined, and let the topology on $\mathcal{S}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ be the weakest topology such that these maps and the embedding into $C^\infty(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ are continuous. Then $\mathcal{S}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is a Fréchet space. It does not depend on the choice of r and b, neither on the choice of the partition of unity.

Now we come to the definition of spaces of sections with boundary conditions induced by a triple $R = (P_0, P_1, P_2)$ of Lagrangian projections of $\mathcal{A}^{2d}$ with $P_i \in M_{2d}(\mathcal{A}_\infty),\ i = 0,1,2$.

For $k \in \mathbb{N}_0 \cup \{\infty\}$ let

$$\begin{aligned} C^k_R(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i) \ :=\ & \{f \in C^k(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i) \mid (P_i \oplus P_i)\partial_M^l f(x) = \partial_M^l f(x) \\ & \text{for } x \in \partial_i M \cup \partial_{i+3} M,\ i = 0,1,2;\ l \in \mathbb{N}_0, l \leq k\} \end{aligned}$$

endowed with the subspace topology.

Analogously we define $C^k_{R0}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $C^k_{Rc}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Furthermore we write $\mathcal{S}_R(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for the space $\mathcal{S}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i) \cap C^\infty_R(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with the topology induced by $\mathcal{S}(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Now we are ready to introduce the operator D:

Fix a triple of pairwise transverse Lagrangian projections $R = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ of $\mathcal{A}^{2d}$ with $\mathcal{P}_i \in M_{2d}(\mathcal{A}_\infty),\ i = 0,1,2$. We define $D : \operatorname{dom} D \to L^2(M, E\otimes\mathcal{A})$ to be the closure of the Dirac operator ∂_M with domain $C^\infty_{Rc}(M, E\otimes\mathcal{A})$ and let D^+ resp. D^- be the restriction of D to the sections of E^+ resp. E^-.

Note that D is a symmetric unbounded operator on $L^2(M, E\otimes\mathcal{A})$.

2.1.2 Comparison with D_s

Fix a triple of pairwise transverse Lagrangian projections $(\mathcal{P}^s_0 := P_s, \mathcal{P}^s_1, \mathcal{P}^s_2)$ with $\mathcal{P}^s_1, \mathcal{P}^s_2 \in M_{2d}(\mathbb{C})$. Let $D_s : \operatorname{dom} D_s \to L^2(M, E\otimes\mathcal{A})$ to be the closure of the Dirac operator ∂_M with domain $C^\infty_{Rc}(M, E\otimes\mathcal{A})$ for $R = (\mathcal{P}^s_0, \mathcal{P}^s_1, \mathcal{P}^s_2)$. By [BK] the operator D_s restricts to a selfadjoint operator on the Hilbert space $L^2(M, E)$.

Let $W \in C^\infty(M, \operatorname{End}^+ E\otimes\mathcal{A}_\infty)$ be such that

- $WW^* = 1$,
- $W(x)(\mathcal{P}_i \oplus \mathcal{P}_i)W(x)^* = (\mathcal{P}^s_i \oplus \mathcal{P}^s_i)$ for all $x \in \partial_i M \cup \partial_{i+3} M$ with $i = 0,1,2$,

- W is parallel on $M \setminus F$ and on a neighbourhood of ∂M,
- for all $k \in \mathbb{Z}/6$ the restriction of W to $F_k(0, \frac{1}{4})$ depends only on the coordinate x_2^k,
- W commutes with the Clifford multiplication.

Proposition 2.1.1. *1. We have $WDW^* = D_s + Wc(dW^*)$ with $c(dW^*)|_F := c(e_2)\partial_{e_2}W^*$ and $c(dW^*)|_{M\setminus F} := 0$. In particular, $Wc(dW^*) \in C^\infty(M, \mathrm{End}^- E \otimes \mathcal{A}_\infty)$.*

2. The operator D is regular and selfadjoint.

Proof. 1) For $R = (\mathcal{P}_0^s, \mathcal{P}_1^s, \mathcal{P}_2^s)$ and $f \in C^\infty_{cR}(M, E \otimes \mathcal{A})$ we have $(WDW^*f)|_{M\setminus F} = (D_s f)|_{M\setminus F}$ and

$$\begin{aligned}(WDW^*f)|_F &= (D_s f)|_F + W[c(e_2)\partial_{e_2}, W^*]_s(f|_F)\\ &= (D_s f)|_F + Wc(e_2)(\partial_{e_2}W^*)(f|_F)\ .\end{aligned}$$

2) The restriction of D_s to the Hilbert space $L^2(M, E)$ is selfadjoint. Hence $(1 + D_s^2)$ has a bounded inverse on $L^2(M, E)$. It follows that the range of $(1 + D_s^2)$ on $L^2(M, E \otimes \mathcal{A})$ is dense. Thus D_s is regular. By an analogous argument the operators $D_s \pm i$ have dense range. From Lemma 5.1.17 it follows that D_s is selfadjoint.

By 1) the operator D is a bounded perturbation of D_s. It follows that D is selfadjoint. Prop. 5.1.13 implies that D is regular as well. □

The existence of a section W fulfilling the properties above is proved in the following lemma and proposition:

Lemma 2.1.2. *There is a parallel unitary section $U \in C^\infty(M, \mathrm{End}E^+ \otimes \mathcal{A}_\infty)$ such that $U\mathcal{P}_0U^* = \mathcal{P}_0^s = P_s$.*

Proof. By the isomorphism $E^+ \otimes \mathcal{A} \cong (\mathcal{A}^+)^d \otimes S^+ \oplus (\mathcal{A}^-)^d \otimes S^-$ any $A \in M_{2d}(\mathcal{A})$ of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in M_d(\mathcal{A})$ defines a parallel section of $\mathrm{End}E^+ \otimes \mathcal{A}$.

By Lemma 1.4.3 there is a unitary $U \in M_{2d}(\mathcal{A}_\infty)$ of that form such that $U\mathcal{P}_0U^* = \mathcal{P}_0^s = P_s$. □

Proposition 2.1.3. *For any $0 < b < \frac{1}{4}$ there is a section $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$ satisfying the properties above and such that W is parallel on $M \setminus F(r, b)$ for all $r \geq 0$.*

Proof. By the previous lemma we may assume $\mathcal{P}_0 = P_s$.

In the following we identify $\partial M \times [0, b]$ with $\{x \in M \mid d(x, \partial M) \leq b\}$.

Since for $i = 1, 2$ the projection $\mathcal{P}_i$ is transverse to P_s, by Lemma 1.4.4 there are smooth paths $W_i : [0, b] \to M_{2d}(\mathcal{A}_\infty)$ of unitaries with $[W_i, I_0] = 0$ such that $W_i(0)\mathcal{P}_i W_i^*(0) = \mathcal{P}_i^s$ and such that W_i is equal to the identity in a neighbourhood of b and constant on $[0, \frac{b}{2}]$. They induce maps $\tilde{W}_i := \mathrm{id} \times W_i : (\partial_i M \cup \partial_{i+3} M) \times [0, b] \to M_{2d}(\mathcal{A}_\infty)$. The map

$$\cup_i(\tilde{W}_i \oplus \tilde{W}_i) : \partial M \times [0, b] \to M_{4d}(\mathcal{A}_\infty)$$

can be extended by 1 to a smooth section $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$. By construction W has the right properties. □

2.2 The operator D_I on $[0,1]$

2.2.1 Definition and comparison with D_{I_s}

Now we switch our attention to the unit interval.

Let ∂ be the differentiation operator.

For a pair $R=(P_0,P_1)$ of transverse Lagrangian projections with $P_j \in M_{2d}(\mathcal{A}_\infty),\ j=0,1$, and $k \in \mathbb{N}_0 \cup \{\infty\}$ we define the function space

$$\begin{aligned} &C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \\ &:= \{f \in C^k([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \mid P_x(I_0\partial)^l f(x) = (I_0\partial)^l f(x) \text{ for } x=0,1;\ l \in \mathbb{N}_0,\ l \leq k\} \end{aligned}$$

and endow it with the induced topology.

Given a pair (P_0,P_1) of transverse Lagrangian projections on $\mathcal{A}^{2d}$ we write D_I for the closure of the operator $I_0\partial$ as an unbounded operator on the Hilbert $\mathcal{A}$-module $L^2([0,1],\mathcal{A}^{2d})$ with domain $C^\infty_R([0,1],\mathcal{A}^{2d})$.

If $(P_0,P_1)=(P_s,1-P_s)$, then we write D_{I_s}.

Proposition 2.2.1. *Let $P_0,P_1 \in M_{2d}(\mathcal{A}_\infty)$ be transverse Lagrangian projections of $\mathcal{A}^{2d}$. Then for any $0<x_1<x_2<1$ there is $U \in C^\infty([0,1],M_{2d}(\mathcal{A}_\infty))$ such that*

1. *$UU^*=1$,*
2. *$UI_0=I_0U$,*
3. *$U(0)P_0U(0)^*=P_s$,*
4. *$U(1)P_1U(1)^*=1-P_s$,*
5. *U is constant on $[0,x_1]$ and on $[x_2,1]$.*

Proof. By Lemma 1.4.3 there is a unitary $U_0 \in M_{2d}(\mathcal{A}_\infty)$ with $U_0I_0=I_0U_0$ and $U_0P_0U_0^*=P_s$. Since $U_0P_1U_0^*$ is transverse to P_s, we can apply Lemma 1.4.4 to $P=P_1$ and $\tilde{P}=1-P_s$ in order to get a smooth path of unitaries $U_1:[0,1]\to M_{2d}(\mathcal{A}_\infty)$ such that $U(t):=U_1(t)U_0$ has the right properties. □

Proposition 2.2.2. *Let (P_0,P_1) a pair of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0,P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_I be the associated operator.*

1. *Let U be as in the previous proposition with $U(0)P_0U(0)^*=P_s$ and $U(1)P_1U(1)^*=1-P_s$. Then $UD_IU^*=D_{I_s}+UI_0(\partial U^*)$ and $UI_0(\partial U^*) \in C^\infty([0,1],M_{2d}(\mathcal{A}_\infty))$.*
2. *The operator D_I is regular and selfadjoint.*

Proof. 1) For $f \in C^\infty_R([0,1],\mathcal{A}^{2d}_\infty)$ with $R=(P_s,1-P_s)$ we have:

$$\begin{aligned} UD_IU^*f &= UI_0\partial U^*f \\ &= D_{I_s}f + U[I_0\partial,U^*]f \\ &= D_{I_s}f + UI_0(\partial U^*)f\ . \end{aligned}$$

2) follows as in Prop. 2.1.1. □

2.2.2 Generalized eigenspace decomposition

Now we show that we can find a decomposition of $L^2([0,1],\mathcal{A}^{2d})$ in free finitely generated $\mathcal{A}$-modules preserved by D_I. For $d=1$ these can be understood as analogues of eigenspaces of D_I.

Assume that the boundary conditions of D_I are given by a pair (P_0,P_1) of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0=P_s$.

By Lemma 1.4.3 the general case can be reduced to that case.

Let $p\in M_d(\mathcal{A})$ be such that $P_1=\frac{1}{2}\begin{pmatrix}1 & p^*\\ p & 1\end{pmatrix}$.

The transversality of P_0 and P_1 implies that $1-p$ is invertible. It follows that $\log p$ is defined for $\log:\mathbb{C}\setminus[0,\infty)\to\mathbb{C}$.

Lemma 2.2.3. *Assume $d=1$.*

Then for $k\in\mathbb{Z}$ the right $\mathcal{A}$-module spanned by the function

$$f_k(x)=\frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\exp[(-\tfrac{1}{2}\log p+\pi k i)x]+\begin{pmatrix}0\\1\end{pmatrix}\exp[(\tfrac{1}{2}\log p-\pi k i)x]\right)$$

is preserved by D_I and $D_I f_k=\lambda_k f_k$ with $\lambda_k:=-\frac{1}{2}i\log p-\pi k$.

The system $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of the Hilbert $\mathcal{A}$-module $L^2([0,1],\mathcal{A}^2)$.

Note that $\lambda_k f_k=f_k\lambda_k$ and $\sigma(\lambda_k)\subset]-\pi k,-\pi(k-1)[$.

Proof. In the following we call a system $\{a_k\}_{k\in\mathbb{Z}}\subset l^2(\mathcal{A})$ square-summable.

Clearly the system $\{f_k\}_{k\in\mathbb{Z}}$ is orthonormal in the sense of def. 5.1.19.

In order to prove that these functions form a basis we first show that there is an orthogonal projection onto the closure of their span. This implies that the span is orthogonally complemented. The second step will be to see that the complement is trivial. Then the claim follows from Prop. 5.1.20.

The system

$$\{e_l^{\pm}(x):=v_{\pm}e^{2\pi i l x}\}_{l\in\mathbb{Z},\pm}$$

with $v_+:=\begin{pmatrix}1\\0\end{pmatrix}$, $v_-:=\begin{pmatrix}0\\1\end{pmatrix}$ is an orthonormal basis of $L^2([0,1],\mathcal{A}^2)$ by Prop. 5.1.20.

We have

$$\begin{aligned}<f_k,e_l^{\pm}> &= \tfrac{1}{\sqrt{2}}\int_0^1\exp((\mp\tfrac{1}{2}\log p^*\mp\pi k i)x)\exp(2\pi i l x)\,dx\\ &= \tfrac{1}{\sqrt{2}}\left(\frac{(-1)^k\exp(\mp\frac{1}{2}\log p^*)-1}{\mp\frac{1}{2}\log p^*+\pi i(2l\mp k)}\right),\end{aligned}$$

hence for any $l\in\mathbb{Z}$ and for $\pm$ the system $\{<f_k,e_l^{\pm}>\}_{k\in\mathbb{Z}}$ is square-summable. Then

$$Pe_l^{\pm}:=\sum_{k\in\mathbb{Z}}f_k<f_k,e_l^{\pm}>,$$

is well-defined.

The linear extension of P to the algebraic span of the functions $e_l^{\pm}$ has norm one. It follows that its closure is an orthogonal projection

$$P : L^2([0,1], \mathcal{A}^2) \to \overline{\text{span}\{f_k \mid k \in \mathbb{Z}\}} \ .$$

It remains to show that the kernel of the projection P is trivial.

Let $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L^2([0,1], \mathcal{A}^2)$ with $< f_k, g >= 0$ for all $k \in \mathbb{Z}$.

Hence for all $k \in \mathbb{Z}$

$$(*) \qquad \int_0^1 \exp((-\tfrac{1}{2}\log p^* - \pi k i)x) g_1(x) + \exp((\tfrac{1}{2}\log p^* + \pi k i)x) g_2(x) \ dx = 0 \ .$$

Since $\exp(-\frac{1}{2}(\log p^*)\ x) g_1(x)$ is in $L^2([0,1], \mathcal{A})$, there is a unique square-summable system $\{\lambda_l\}_{l\in\mathbb{Z}}$ in $\mathcal{A}$ such that

$$\sum_{l\in\mathbb{Z}} \lambda_l e^{2\pi i l x} = \exp(-\tfrac{1}{2}(\log p^*)\ x) g_1(x) \ .$$

Inserting this in $(*)$ and evaluating the integral for k even leads to

$$\lambda_{k/2} - \int_0^1 \exp((\tfrac{1}{2}\log p^* + \pi k i)x) g_2(x) dx = 0 \ .$$

It follows that

$$\exp(\tfrac{1}{2}(\log p^*)\ x) g_2(x) = \sum_{l\in\mathbb{Z}} (-\lambda_l) e^{-2\pi i l x} \ .$$

Substituting again and evaluating $(*)$ for $k = 2\nu + 1$ with $\nu \in \mathbb{Z}$, we obtain

$$\begin{aligned} 0 &= \int_0^1 \sum_{l\in\mathbb{Z}} \lambda_l (e^{\pi i (2(l-\nu)-1)x} - e^{-\pi i (2(l-\nu)-1)x}) dx \\ &= -\sum_{l\in\mathbb{Z}} \lambda_l \frac{4}{\pi i (2(l-\nu)-1)} \ . \end{aligned}$$

Note that for every $l \in \mathbb{Z}$ the function

$$a_l : \mathbb{Z} \to \mathbb{C}, \ \nu \mapsto \frac{2}{\pi i (2(l-\nu)-1)}$$

is in $l^2(\mathbb{C})$. We claim that $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathbb{C})$. Then $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathcal{A})$ as well, thus $\lambda_l = 0$ for all $l \in \mathbb{Z}$ and hence $g_1 = g_2 = 0$.

The Fourier transform of

$$a_0 : \nu \mapsto \frac{-2}{\pi i (2\nu + 1)}$$

is

$$h(x) = -i e^{-\pi i x} (1_{[0,1/2]}(x) - 1_{]1/2,1]}(x)) \in L^2(\mathbb{R}/\mathbb{Z}) \ .$$

Since $a_l(\nu) = a_0(\nu - l)$, the Fourier transform of a_l is $h(x)e^{-2\pi i l x}$.

By $hh^* = 1$ the system $\{h(x)e^{-2\pi i l x}\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$. This implies that $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathbb{C})$. □

For general d there is a decomposition of $L^2([0,1]\mathcal{A}^{2d})$ in $\mathcal{A}$-modules of rank d:

Proposition 2.2.4. *For $k \in \mathbb{Z}$ let $U_k \subset L^2([0,1], \mathcal{A}^{2d})$ be the right $\mathcal{A}$-module spanned by the column vectors of*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \exp[(-\frac{1}{2}\log p + \pi k i)x] \\ \exp[(\frac{1}{2}\log p - \pi k i)x] \end{pmatrix} \in C^\infty([0,1], M_{2d\times d}(\mathcal{A}_\infty)) \ .$$

Each U_k is a free right $\mathcal{A}$-module of rank d and

$$L^2([0,1], \mathcal{A}^{2d}) = \hat{\bigoplus_{k\in\mathbb{Z}}} U_k \ .$$

The sum is orthogonal.

Furthermore $U_k \subset \operatorname{dom} D_I$ and for $f \in U_k$

$$D_I f = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} f \in U_k$$

with $\lambda_k = -\frac{1}{2} i \log p - \pi k$.

Proof. The projections $P_0, P_1 \in M_{2d}(\mathcal{A})$ are Lagrangian projections on $M_d(\mathcal{A})^2$ as we remarked after def 1.4.1.

Let $\tilde{D}_I$ be the closure of $I_0\partial$ on $L^2([0,1], M_d(\mathcal{A})^2)$ with domain $C^\infty_R([0,1], M_{2d}(\mathcal{A}))$ with $R = (P_0, P_1)$. Then $\tilde{D}_I = \oplus^d D_I$ with respect to the decomposition

$$L^2([0,1], M_d(\mathcal{A})^2) = L^2([0,1], \mathcal{A}^{2d})^d$$

that is induced by the decomposition of a matrix into its column vectors.

By the previous proposition the Hilbert $M_d(\mathcal{A})$-module $L^2([0,1], M_d(\mathcal{A})^2)$ has an orthonormal basis $\{f_k\}_{k\in\mathbb{Z}}$ such that

$$\tilde{D}_I f_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} f_k \ .$$

For $k \in \mathbb{Z}$ let P_k be the orthogonal projection onto the span of f_k in $L^2([0,1], M_d(\mathcal{A})^2)$. It is diagonal with respect to the decomposition $L^2([0,1], M_d(\mathcal{A})^2) = L^2([0,1], \mathcal{A}^{2d})^d$.

Hence

$$L^2([0,1], \mathcal{A}^{2d}) = \hat{\bigoplus_{k\in\mathbb{Z}}} P_k L^2([0,1], \mathcal{A}^{2d}) \ .$$

The assertion follows now since $P_k L^2([0,1], \mathcal{A}^{2d}) = U_k$. The module U_k is free since $\exp[(-\frac{1}{2}\log p + \pi k i)x]$ is invertible in $M_d(\mathcal{A})$ for all $x \in [0,1]$. □

Corollary 2.2.5. *Let $\lambda \in \mathbb{C}$. The operator $D_I - \lambda$ has a bounded inverse on $L^2([0,1], \mathcal{A}^{2d})$ if and only if $\exp(-2i\lambda) \notin \sigma(p)$.*

Proof. By the previous proposition $(D_I - \lambda)|_{U_k}$ is invertible if and only if $e^{-2i\lambda} \notin \sigma(p)$. Furthermore for λ with $e^{-2i\lambda} \notin \sigma(p)$ the inverse of $(D_I - \lambda)|_{U_k}$ is uniformly bounded in k. Hence the closure of $\oplus_k (D_I - \lambda)|_{U_k}$ has a bounded inverse by Cor. 5.1.22. In particular $\oplus_k D_I|_{U_k}$ is selfadjoint. Since D_I is a selfadjoint extension of $\oplus_k D_I|_{U_k}$, it follows that D_I is the closure of $\oplus_k D_I|_{U_k}$. Hence $D_I - \lambda$ has a bounded inverse if $\exp(-2i\lambda) \notin \sigma(p)$. □

2.3 The operator D_Z on the cylinder

Let $X = \mathbb{R}, \mathbb{R}/\mathbb{Z}$. Endow $X \times [0,1]$ with the euclidean metric and a spin structure and let (x_1, x_2) be the euclidian coordinates of $X \times [0,1]$. Let $S = S^+ \oplus S^-$ be the spinor bundle on $X \times [0,1]$ endowed with the Levi-Cività connection and with a hermitian metric such that it is a Dirac bundle. Then $((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d) \otimes S$ is graded $\mathcal{A}$-vector bundle on $X \times [0,1]$ with an $\mathcal{A}$-valued scalar product induced by the hermitian metric on S and the standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^d$. Let ∂_Z be the Dirac operator associated to the Dirac bundle $\big((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d\big) \otimes S$.

We choose a parallel unit section s of S^+ and identify $((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d) \otimes S$ with the trivial bundle $(X \times [0,1]) \times \big((\mathcal{A}^+)^{2d} \oplus (\mathcal{A}^-)^{2d}\big)$ by the isomorphisms

$$((\mathcal{A}^+)^d \otimes S_x^+) \oplus ((\mathcal{A}^-)^d \otimes S_x^-) \to (\mathcal{A}^+)^d \oplus (\mathcal{A}^+)^d \ ,$$

$$(v \otimes s(x)) \oplus (w \otimes ic(dx_1)s(x)) \mapsto (v, w) \ ,$$

and

$$((\mathcal{A}^+)^d \otimes S_x^-) \oplus ((\mathcal{A}^-)^d \otimes S_x^+) \to (\mathcal{A}^-)^d \oplus (\mathcal{A}^-)^d \ ,$$

$$(v \otimes ic(dx_1)s(x)) \oplus (w \otimes s(x)) \mapsto (v, w)$$

for $x \in X \times [0,1]$.

Let $I := c(dx_1)c(dx_2) = I_0 \oplus (-I_0)$. Then

$$\partial_Z = c(dx_1)(\partial_{x_1} - I\partial_{x_2}) \ .$$

Given a pair (P_0, P_1) of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_i \in M_{2d}(\mathcal{A}_\infty),\ i = 0,1$ we define D_Z to be the closure of ∂_Z with domain

$$\{f \in C_c^\infty(X \times [0,1], (\mathcal{A}_\infty)^{4d}) \mid (P_i \oplus P_i)f(x,i) = f(x,i) \text{ for all } x \in X,\ i = 0,1\}$$

on the Hilbert $\mathcal{A}$-module $L^2(X \times [0,1], \mathcal{A}^{4d})$.

We write $H(D_Z)$ for the Hilbert $\mathcal{A}$-module whose underlying right $\mathcal{A}$-module is $\operatorname{dom} D_Z$ and whose $\mathcal{A}$-valued scalar product is $< f, g >_{D_Z} = < f, g > + < D_Z f, D_Z g >$ (see §5.1.3).

Proposition 2.3.1. *1. The operator D_Z is selfadjoint on $L^2(X \times [0,1], \mathcal{A}^{4d})$ and has a bounded inverse.*

2. If $X = \mathbb{R}/\mathbb{Z}$, then the inclusion $\iota : H(D_Z) \to L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d})$ is compact.

Proof. The proof is as in the family case [BK]:

1) By Lemma 1.4.3 we may assume $P_0 = P_s$. Recall from Prop. 2.2.4 that the operator D_I with boundary conditions (P_0, P_1) induces a decomposition

$$L^2([0,1], (\mathcal{A}^+)^{2d}) = \hat{\bigoplus_{l \in \mathbb{Z}}} U_l$$

such that for each $l \in \mathbb{Z}$ there is $\lambda_l \in M_d(\mathcal{A})$ with $D_I f = \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l \end{pmatrix} f$ for $f \in U_l$.

We define submodules $U_{l,+} := U_l$ and $U_{l,-} := ic(dx_1)U_l$ of $L^2([0,1], \mathcal{A}^{4d})$. Then

$$\bigoplus_{l\in\mathbb{Z}} (U_{l,+} \oplus U_{l,-}) L^2(X)$$

is dense in $L^2(X \times [0,1], \mathcal{A}^{4d})$.

Now consider the case $X = \mathbb{R}$:

For $l \in \mathbb{Z}$ let $\partial_{l,\pm}$ be the closure of the unbounded operator $(\partial_{x_1} \pm \lambda_l)$ on $U_{l,\pm}L^2(\mathbb{R})$ with domain $U_{l,\pm}\mathcal{S}(\mathbb{R})$ and let ∂_e be the closure of $c(dx_1)\big(\bigoplus_{l\in\mathbb{Z},\pm} \partial_{l,\pm}\big)$.

The operator D_Z is an extension of ∂_e.

We claim that ∂_e has a bounded inverse on $L^2(\mathbb{R} \times [0,1], \mathcal{A}^{4d})$.

The Fourier transform on $L^2(\mathbb{R})$ induces an automorphism on $U_{l,\pm}L^2(\mathbb{R})$. Conjugation by it transforms $\partial_{l,\pm}$ into multiplication by

$$ix_1 \pm \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l \end{pmatrix} .$$

Since $\sigma(\lambda_l) \subset \mathbb{R}^*$, we see that the operator $\partial_{l,\pm}$ has a bounded inverse and the norm of the inverse tends to zero for $l \to \pm\infty$. By Cor. 5.1.22 it follows that the closure of $\oplus_{l,\pm}\partial_{l,\pm}$ has a bounded inverse.

Hence the operator ∂_e has a bounded inverse as well. In particular it is selfadjoint, thus $D_Z = \partial_e$. For $X = \mathbb{R}$ the assertion follows.

If $X = \mathbb{R}/\mathbb{Z}$, then the spaces $U_{l,\pm}L^2(\mathbb{R}/\mathbb{Z})$ decompose further into a direct sum

$$U_{l,\pm}L^2(\mathbb{R}/\mathbb{Z}) = \hat{\bigoplus_{k\in\mathbb{Z}}} V_{kl,\pm}$$

with $V_{kl,\pm} := e^{2\pi ikx_1}U_{l,\pm}$. Note that $V_{kl,\pm}$ is isomorphic to $\mathcal{A}^d$ as a Hilbert $\mathcal{A}$-module.

Let $\partial_{kl,\pm} \in B(V_{kl,\pm})$ be defined by

$$\partial_{kl,\pm} f := (\partial_{x_1} \pm \lambda_l)f = \begin{pmatrix} 2\pi ik \pm \lambda_l & 0 \\ 0 & 2\pi ik \pm \lambda_l \end{pmatrix} f$$

and let ∂_e be the closure of $c(dx_1)\big(\bigoplus_{k,l,\pm} \partial_{kl,\pm}\big)$.

The operator D_Z is an extension of ∂_e.

Since $|(2\pi ik \pm \lambda_l)^{-1}|$ tends to zero for $k, l \to \pm\infty$, the closure of $\oplus_{kl,\pm}\partial_{kl,\pm}$ has a bounded – even compact – inverse by Cor. 5.1.22 and hence ∂_e has a compact inverse as well.

Now 1) follows as above.

Furthermore for $X = \mathbb{R}/\mathbb{Z}$ the operator $D_Z^{-1} \in B(L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d}))$ is compact. This implies 2) since $\iota = D_Z^{-1}D_Z : H(D_Z) \to L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d})$. □

2.4 The index of D^+

For an open precompact subset U of M we define $H^1_{R0}(U, E \otimes \mathcal{A})$ to be the closure of $C^\infty_{Rc}(U, E \otimes \mathcal{A})$ in $H(D)$ (see §5.1.3 for the definition of $H(D)$).

Note that $H(D)$ is isomorphic to $l^2(\mathcal{A})$ as a Hilbert $\mathcal{A}$-module, since $L^2(M, E \otimes \mathcal{A})$ is isomorphic to $l^2(\mathcal{A})$ and since $L^2(M, E \otimes \mathcal{A})$ and $H(D)$ are isomorphic by Lemma 5.1.12. Recall that the boundary conditions of D are given by $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

Lemma 2.4.1. *For any $r \geq 0$ the inclusion $\iota : H^1_{R0}(M_r, E \otimes \mathcal{A}) \to L^2(M, E \otimes \mathcal{A})$ is compact.*

Proof. Let $\mathcal{V} := \{V_k\}_{k \in L}$ be an open covering of M_r such that the index set L is a finite subset of $\mathbb{N}$ with $1 \in L$ and assume that $\mathcal{V}$ satisfies the following conditions:

- $V_1 = M \setminus \overline{F(r, 1/6)}$;
- for $k > 1$ there is an isometry $V_k \cong]0, \frac{1}{2}[\times [0, \frac{1}{5}[$. In particular V_k is in the flat region and has exactly one boundary component.

First we prove that the maps $\iota_k : H^1_{R0}(V_k, E \otimes \mathcal{A}) \to L^2(V_k, E \otimes \mathcal{A})$ are compact for all $k \in L$.

The compactness of ι_1 follows from the Sobolev embedding theorem ([MF], Lemma 3.3).

For $k \neq 1$ let $i \in \{0, 1, 2\}$ be such that $\partial V_k \subset (\partial_i M \cup \partial_{i+3} M)$ and set $P_k := \mathcal{P}_i$. Let D_k be the operator D_Z on the bundle $(\mathbb{R}/\mathbb{Z} \times [0, 1]) \times \mathcal{A}^{4d}$ with boundary conditions given by the pair $(P_k, 1 - P_k)$.

The map ι_k is compact since it factorizes through the map $H(D_k) \to L^2(\mathbb{R}/\mathbb{Z} \times [0, 1], \mathcal{A}^{4d})$ which is compact by Prop. 2.3.1.

Let $\{\phi_k\}_{k \in L}$ be a smooth partition of unity subordinate to the covering $\mathcal{V}$ such that $\partial_{e_2} \phi_k(x) = 0$ for all $x \in \partial M$ and $k \in L$. Multiplication with ϕ_k is a bounded map from $H^1_{R0}(M_r, E \otimes \mathcal{A})$ to $H^1_{R0}(V_k, E \otimes \mathcal{A})$, hence by $\iota = \sum_{k \in L} \iota_k \phi_k$, the inclusion ι is compact. □

Let $H(D)^+$ be the subspace of $H(D)$ containing all even elements.

Proposition 2.4.2. *The operator*

$$D^+ : H(D)^+ \to L^2(M, E^- \otimes \mathcal{A})$$

is a Fredholm operator.

Proof. By constructing a parametrix for D^+ we show that D^+ is Fredholm in the sense of Miščenko/Fomenko (see Prop. 5.1.4). Then from Lemma 5.1.12 it follows that $D^+ \in B(H(D)^+, L^2(M, E^- \otimes \mathcal{A}))$, hence Lemma 5.1.7 implies that D^+ is Fredholm.

The construction of the parametrix is analogous to the construction in the family case [BK]:

Choose a smooth partition of unity $\{\phi_k\}_{k \in J}$ subordinate to the covering $\mathcal{U}(0, \frac{1}{4})$ defined in §1.1 and a system of smooth functions $\{\gamma_k\}_{k \in J}$ on M such that for all $k \in J$

- $\operatorname{supp} \gamma_k \subset \mathcal{U}_k$ and $\gamma_k \phi_k = \phi_k$,
- $\partial_{e_2} \gamma_k(x) = \partial_{e_2} \phi_k(x) = 0$ for all $x \in \partial M$.

Now we define a local parametrix Q_k of D on $\mathcal{U}_k,\ k \in J$:

By §1.1 the set $\mathcal{U}_k$ is a subset of Z_k for $k \in \mathbb{Z}/6$. Let D_{Z_k} be the operator on $L^2(Z_k, \mathcal{A}^{4d})$ with boundary conditions given by $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$. It was defined in §2.3. By Prop. 2.3.1 it is invertible, hence we can set

$$Q_k := D_{Z_k}^{-1} : L^2(Z_k, \mathcal{A}^{4d}) \to H(D_{Z_k}) \ .$$

Since the symbol of D is elliptic and since $\mathcal{U}_\clubsuit$ is precompact, there is a parametrix $Q_\clubsuit : L^2(\mathcal{U}_\clubsuit, E\otimes\mathcal{A}) \to H^1_{R0}(\mathcal{U}_\clubsuit, E\otimes\mathcal{A})$ such that $DQ_\clubsuit - 1$ resp. $Q_\clubsuit D - 1$ is compact on $L^2(\mathcal{U}_\clubsuit, E\otimes\mathcal{A})$ resp. $H^1_{R0}(\mathcal{U}_\clubsuit, E\otimes\mathcal{A})$ [MF]. Furthermore $Q_\clubsuit$ can be chosen to be an odd operator on $L^2(\mathcal{U}_\clubsuit, E\otimes\mathcal{A})$.

We claim that

$$Q := \sum_{k\in J} \gamma_k Q_k \phi_k : L^2(M, E\otimes\mathcal{A}) \to H(D)$$

is a parametrix of D. Since Q is odd, it follows that $Q^- : L^2(M, E^-\otimes\mathcal{A}) \to H(D)^+$ is a parametrix of D^+.

In the following calculations the operators D_{Z_k} and the restriction of D to $\mathcal{U}_k$ are denoted by D as well. Let $\sim$ denote equality up to compact operators.

On $L^2(M, E\otimes\mathcal{A})$

$$\begin{aligned} DQ - 1 &= \sum_{k\in J} [D,\gamma_k] Q_k\phi_k + \sum_{k\in J} \gamma_k D Q_k \phi_k - 1 \\ &\sim \sum_{k\in J} c(d\gamma_k) Q_k \phi_k \ . \end{aligned}$$

Since $c(d\gamma_k)Q_k\phi_k$ is bounded from $L^2(M, E\otimes\mathcal{A})$ to $H^1_{R0}(M_r, E\otimes\mathcal{A})$ for any $k\in J$ and $r>0$ big enough, $c(d\gamma_k)Q_k\phi_k$ is a compact operator on $L^2(M, E\otimes\mathcal{A})$ for each $k\in J$ by the previous lemma. Hence $DQ-1$ is compact.

On $H(D)$

$$\begin{aligned} QD - 1 &= \sum_{k\in J} \gamma_k Q_k [D,\phi_k] + \sum_{k\in J} \gamma_k Q_k D \phi_k - 1 \\ &\sim \sum_{k\in J} \gamma_k Q_k c(d\phi_k) \ . \end{aligned}$$

Here $c(d\phi_k) : H(D) \to L^2(\mathcal{U}_k, E\otimes\mathcal{A})$ is a compact operator by the previous lemma since $\operatorname{supp} c(d\phi_k)$ is precompact for all $k\in J$. Since $\gamma_k Q_k : L^2(\mathcal{U}_k, E\otimes\mathcal{A}) \to H(D)$ is bounded, the operator $QD-1 : H(D)\to H(D)$ is compact. □

Lemma 2.4.3. *Let $P_0, P_1, P_2 : [0,1] \to M_{2d}(\mathcal{A})$ be continuous paths of Lagrangian projections and assume that $P_0(t), P_1(t), P_2(t)$ are pairwise transverse for each $t\in[0,1]$ Let $R(t) = (P_0(t), P_1(t), P_2(t))$. We define $D(t)$ to be the closure of ∂_M on $L^2(M, E\otimes\mathcal{A})$ with domain $C^\infty_{R(t)c}(M, E\otimes\mathcal{A})$.*

Let $\operatorname{ind} D(t)^+$ be the index of the Fredholm operator $D(t)^+ : H(D(t))^+ \to L^2(M, E\otimes\mathcal{A})$. Then

$$\operatorname{ind} D(0)^+ = \operatorname{ind} D(1)^+ \in K_0(\mathcal{A}) \ .$$

Proof. There is a continuous path of unitaries $[0,1] \to C^\infty(M, \operatorname{End} E^+ \otimes \mathcal{A}_\infty),\ t\mapsto W_t$, such that $W_t D(t)^+ W_t^* = D_s^+ + W_t c(dW_t^*)$. By Prop. 5.1.13

$$D_s^+ + W_t c(dW_t^*) \in B(H(D_s)^+, L^2(M, E^-\otimes\mathcal{A})) \ .$$

Hence the family

$$D_s^+ + W_t c(dW_t^*) : H(D_s)^+ \to L^2(M, E^-\otimes\mathcal{A})$$

is a continuous path of Fredholm operators, Prop. 5.1.9 implies $\operatorname{ind} W_0 D(0)^+ W_0^* = \operatorname{ind} W_1 D(1)^+ W_1^*$. The assertion follows. □

Proposition 2.4.4. *The index of* $D^+ : H(D)^+ \to L^2(M, E^- \otimes \mathcal{A})$ *is*

$$\operatorname{ind} D^+ = \tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) \in K_0(\mathcal{A}) \ .$$

Proof. The argument is analogous to the one in [BK].

By Lemma 2.1.2 we may assume $\mathcal{P}_0 = P_s$.

Let $a_j := i\frac{p_j+1}{p_j-1} \in M_{2d}(\mathcal{A})$. Then $\mathcal{P}_j = P(a_j),\ j = 1, 2$ in the notation of §1.4.2 .

Let $p^+ := 1_{\{x>0\}}(a_1 - a_2)$.

From Lemma 1.4.8 it follows that

$$\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [p^+] - [1 - p^+] \in K_0(\mathcal{A}) \ .$$

Furthermore by Lemma 1.4.8 there are continuous paths $P_1, P_2 : [0, 2] \to M_{2d}(\mathcal{A})$ of Lagrangian projections with $P_j(0) = \mathcal{P}_j,\ j = 1, 2$, further with $P_1(2) = P(2p^+ - 1)$ and $P_2(2) = P(1 - 2p^+)$ and such that $P_s, P_1(t), P_2(t)$ are pairwise transverse for all $t \in [0, 2]$.

For $t \in [0, 2]$ let $D(t)$ be the Dirac operator on $L^2(M, E \otimes \mathcal{A})$ whose boundary conditions are given by the triple $(P_s, P_1(t), P_2(t))$. The previous lemma implies $\operatorname{ind} D(0)^+ = \operatorname{ind} D(2)^+$.

We show that the index of $D(2)^+$ equals $[p^+] - [1 - p^+]$:

Let $Q_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{C})$, $Q_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $Q_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Then

$$P_1(2) = (Q_1 \otimes p^+) \oplus (Q_2 \otimes (1 - p^+))$$

and

$$P_2(2) = (Q_2 \otimes p^+) \oplus (Q_1 \otimes (1 - p^+))$$

with respect to the decomposition

$$E^+ \otimes \mathcal{A} = (S \otimes p^+\mathcal{A}^n) \oplus (S \otimes (1 - p^+)\mathcal{A}^n) \ .$$

The Dirac operator ∂_M respects the decomposition. By [BK] the Dirac operator associated to the bundle $(\mathbb{C}^+ \oplus \mathbb{C}^-) \otimes S$ has index 1 if the boundary conditions are given by the triple (Q_0, Q_1, Q_2), and index -1 if the boundary conditions are given by (Q_0, Q_2, Q_1). It follows

$$\operatorname{ind} D(2)^+ = [p^+] - [1 - p^+] \ .$$

□

2.5 A perturbation with closed range

Imitating the construction in [BGV] §9.5 we will define a compact perturbation of D with closed range. Since D is Fredholm, the kernel and cokernel of the perturbation will be projective $\mathcal{A}$-modules.

Choose an orthonormal basis $\{\psi_i\}_{i \in \mathbb{N}} \subset L^2(M, E^-)$ such that $\psi_i \in C_c^\infty(M, E^-)$ and $\operatorname{supp} \psi_i \in M \setminus \partial M$ for all $i \in \mathbb{N}$. By Lemma 5.1.20 this is an orthonormal basis of $L^2(M, E^- \otimes \mathcal{A})$ as well.

Since D^+ is a Fredholm operator, there is a projective $\mathcal{A}$-module $P \subset L^2(M, E^- \otimes \mathcal{A})$ and a closed $\mathcal{A}$-module $Q \subset \operatorname{Ran} D^+$ such that $P \oplus Q = L^2(M, E^- \otimes \mathcal{A})$. By Prop. 5.3.3 there is $\mathcal{N} \in \mathbb{N}$ such that $L_{\mathcal{N}} := \operatorname{span}\{\psi_i \mid i = 1, \ldots, \mathcal{N}\}$ fulfills $L_{\mathcal{N}} + P = L^2(M, E^- \otimes \mathcal{A})$. In particular it follows that

$$L_{\mathcal{N}} + \operatorname{Ran} D^+ = L^2(M, E^- \otimes \mathcal{A}) \ .$$

Let $M' := M \cup *$ be the disjoint union of M and one isolated point. Let E'^+ be the hermitian vector bundle $E^+ \cup (* \times \mathbb{C}^{\mathcal{N}})$ on M', where we endow $\mathbb{C}^{\mathcal{N}}$ with the standard hermitian product, and let E'^- be the hermitian bundle $E^- \cup (* \times \{0\})$. Let $E' = E'^+ \oplus E'^-$. Extend D by zero to a selfadjoint odd operator D' on $L^2(M', E' \otimes \mathcal{A})$.

As D is regular, D' is regular as well.

Furthermore $D' : H(D') \to L^2(M', E' \otimes \mathcal{A})$ is a Fredholm operator and

$$\operatorname{ind} D'^+ = \operatorname{ind} D^+ + [\mathcal{A}^{\mathcal{N}}] \ .$$

We extend the standard basis of $\mathbb{C}^{\mathcal{N}}$ by zero to a system $\{e_k\}_{k=1,\ldots,\mathcal{N}}$ of sections of E'^+ and define a finite integral operator K on $L^2(M', E' \otimes \mathcal{A})$ by

$$Kf := \sum_{k=1}^{\mathcal{N}} e_k < \psi_k, f > + \psi_k < e_k, f > \ .$$

Set $D(\rho) := D' + \rho K$ for $\rho \in \mathbb{R}$.

Since K is an odd selfadjoint compact operator, the operator $D(\rho)$ is selfadjoint and odd and

$$\operatorname{ind} D(\rho)^+ = \operatorname{ind} D'^+$$

for all $\rho \in \mathbb{R}$.

Furthermore $D(\rho)$ is regular by Prop. 5.1.13.

By construction $D(\rho)^+$ is surjective for $\rho \neq 0$. Hence by Prop. 5.1.15 and Prop. 5.1.8 its kernel is a projective submodule of $H(D')^+$, and the kernel of $D(\rho)^+$ is trivial.

This shows the following proposition:

Proposition 2.5.1. *For $\rho \in \mathbb{R}$ the operator $D(\rho)^+$ is a Fredholm operator. For $\rho \neq 0$*

$$\operatorname{ind} D^+ = \operatorname{ind} D(\rho)^+ - [\mathcal{A}^{\mathcal{N}}] = [\operatorname{Ker} D(\rho)] - [\mathcal{A}^{\mathcal{N}}] \ .$$

From now on we write D, E, M for D', E', M' and we extend the operator D_s by zero to our new manifold M. Furthermore we redefine the open covering $\mathcal{U}(r, b)$ from §1.1 by adding the isolated point to the set $\mathcal{U}_{\clubsuit}$.

Chapter 3

Heat semigroups and kernels

3.1 Complex heat kernels

In this section we recall some properties of heat kernels associated to Dirac operators on complex vector bundles. In particular we study the heat semigroups of D_{I_s} and D_s.

3.1.1 The heat kernel of $e^{-tD_{I_s}^2}$

Since D_{I_s} is selfadjoint on $L^2([0,1],\mathbb{C}^{2d})$, the operator $-D_{I_s}^2$ generates a semigroup $e^{-tD_{I_s}^2}$ on $L^2([0,1],\mathbb{C}^{2d})$. In this section we determine the corresponding family of integral kernels by using the method of images (see [Ta], 3.7), and study its properties.

The space $L^2([0,1],\mathbb{C}^{2d})$ decomposes into an orthogonal sum

$$L^2([0,1],P_s\mathbb{C}^{2d}) \oplus L^2([0,1],(1-P_s)\mathbb{C}^{2d})$$

and the semigroup $e^{-tD_{I_s}^2}$ is diagonal with respect to this decomposition.

We define an embedding

$$\tilde{} : L^2([0,1],\mathbb{C}^{2d}) \to L^2(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d})$$

by requiring that $\tilde{}$ is a right inverse of the map

$$L^2(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d}) \to L^2([0,1],\mathbb{C}^{2d}),\ f \mapsto f|_{[0,1]}$$

and that the image of $L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ resp. $L^2([0,1],P_s\mathbb{C}^{2d})$ consists of functions that are odd resp. even with respect to $y=0$ and $y=2$ and even resp. odd with respect to $y=1$ and $y=3$.

Let $R=(P_s,1-P_s)$. Since

$$C_R^\infty([0,1],\mathbb{C}^{2d}) = C_l^\infty([0,1],P_s\mathbb{C}^{2d}) \oplus C_r^\infty([0,1],(1-P_s)\mathbb{C}^{2d})$$

with

$$\begin{aligned} & C_l^\infty([0,1],P_s\mathbb{C}^{2d}) \\ & \quad := \ \{f \in C^\infty([0,1],P_s\mathbb{C}^{2d}) \mid (i\partial)^{2k} f(1) = 0,\ (i\partial)^{2k+1} f(0) = 0\ \forall k \in \mathbb{N}_0\} \end{aligned}$$

and

$$\begin{aligned} &C_r^\infty([0,1],(1-P_s)\mathbb{C}^{2d}) \\ &\quad := \quad \{f\in C^\infty([0,1],(1-P_s)\mathbb{C}^{2d}) \mid (i\partial)^{2k} f(0)=0,\ (i\partial)^{2k+1} f(1)=0\ \forall k\in \mathbb{N}_0\}\ , \end{aligned}$$

the embedding $\tilde{}$ maps the space $C_R^\infty([0,1],\mathbb{C}^{2d})$ into $C^\infty(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d})$.
Furthermore it intertwines the operators $D_{I_s}^2$ on $C_R^\infty([0,1],\mathbb{C}^{2d})$ and $-\partial^2$ on $C^\infty(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d})$.
It is well-known (see [Ta], 3.7) that the scalar heat kernel associated to ∂^2 on $\mathbb{R}/4\mathbb{Z}$ is

$$H(t,x,y)=(4\pi t)^{-\frac{1}{2}}\sum_{k\in\mathbb{Z}} e^{-\frac{(x-y+4k)^2}{4t}}\ .$$

Hence for $f\in L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ and $x\in[0,1]$

$$\begin{aligned} (e^{-tD_{I_s}^2}f)(x) &= (e^{t\partial^2}\tilde{f})(x) \\ &= \int_0^1 H(t,x,y)f(y)dy+\int_1^2 H(t,x,y)f(2-y)dy \\ &\quad +\int_2^3 H(t,x,y)(-f(y-2))dy+\int_3^4 H(t,x,y)(-f(4-y))dy\ . \end{aligned}$$

It follows that the action of $e^{-tD_{I_s}^2}$ on the space $L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ is given by the scalar integral kernel

$$(x,y)\mapsto H(t,x,y)+H(t,x,2-y)-H(t,x,y+2)-H(t,x,4-y)\ .$$

Analogously we conclude that the action of $e^{-tD_{I_s}^2}$ restricted to $L^2([0,1],P_s\mathbb{C}^{2d})$ is given by the integral kernel

$$(x,y)\mapsto H(t,x,y)-H(t,x,2-y)-H(t,x,y+2)+H(t,x,4-y)\ .$$

This yields the integral kernel k_t of $e^{-tD_{I_s}^2}$.
In the following we write $C_R^\infty([0,1],M_{2d}(\mathbb{C}))$ for the subspace of those functions in $C^\infty([0,1],M_{2d}(\mathbb{C}))$ whose column vectors are in $C_R^\infty([0,1],\mathbb{C}^{2d})$.

Lemma 3.1.1. *The map*

$$(0,\infty)\to C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\mathbb{C}))),\ \ t\mapsto \big(y\mapsto k_t(\cdot,y)\big)\ ,$$

is smooth.
For $\phi,\psi\in C^\infty([0,1])$ with $\operatorname{supp}\phi\cap\operatorname{supp}\psi=\emptyset$ the map $t\mapsto\big(y\mapsto\phi(\cdot)k_t(\cdot,y)\psi(y)\big)$ can be continued by zero to a smooth map from $[0,\infty)$ to $C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\mathbb{C})))$.

Proof. This follows from the corresponding well-known properties of H. □

Lemma 3.1.2. *Let $m,n\in\mathbb{N}_0$. Then there is $C>0$ such that for all $x,y\in[0,1]$ and all $t>0$*

$$|\partial_x^m\partial_y^n k_t(x,y)|\le C(1+t^{-\frac{m+n+1}{2}})(e^{-\frac{d(x,y)^2}{4t}})\ .$$

Proof. The assertion follows from the explicit formula of H above. When estimating the derivatives we take into account that for all $m\in\mathbb{N}$ the function $(x,y,t)\mapsto\frac{(x-y)^{2m}}{t^m}e^{-\frac{(x-y)^2}{4t}}$ can be continuously extended by zero to $t=0$. □

3.1.2 The heat kernel of $e^{-tD_s^2}$

The operator D_s that was defined in §2.1.2 acts on the Hilbert space $L^2(M,E)$ as a self-adjoint operator. Hence $-D_s^2$ generates a strongly continuous semigroup on $L^2(M,E)$. In this section we prove the existence of the integral kernel of $e^{-tD_s^2}$ and study its properties.

Let $Z = [0,1] \times \mathbb{R}$ and let D_Z be the operator defined in §2.3 with boundary conditions given by a pair (P_0, P_1) with $P_0, P_1 \in M_{2d}(\mathbb{C})$. Then D_Z is selfadjoint on $L^2(Z, \mathbb{C}^{4d})$. We study the semigroup $e^{-tD_Z^2}$ on $L^2(Z,\mathbb{C}^{4d})$ as well, since later we will compare $e^{-tD_s^2}$ on the cylindric ends with $e^{-tD_Z^2}$ for appropriate boundary conditions.

Since the proofs are standard, they are only sketched here.

First note that a solution $u : \mathbb{R} \to \operatorname{dom} D_s$ of the initial-value problem

$$\frac{d}{dt}u(t) = iD_s u(t),\ u(0) = f$$

with $f \in C^\infty_{Rc}(M,E)$ is unique by an energy estimate. An analogous statement holds for a solution of the corresponding problem for D_Z.

Lemma 3.1.3. *If $f \in C^\infty_{Rc}(M,E)$, then $d(x, \operatorname{supp} f) \le |t|$ for any $x \in \operatorname{supp}(e^{itD_s} f)$.*
An analogous result holds for D_Z on $L^2(Z,\mathbb{C}^{4d})$.

This property is called "finite propagation speed property".

Proof. The proof relies on a cutting-and-pasting argument.

For $j \in \mathbb{Z}/6$ let $V_j := \{x \in M \mid d(x, \partial_j M) < \frac{1}{4}\}$ and let $W := M \setminus \partial M$. These sets cover M.

By an oriented isometric embedding $V_j \hookrightarrow \mathbb{R}^2$ we may identify V_j with $\{x_2 \ge 0\}$ and ∂V_j with $\{x_2 = 0\}$. Then the restriction of ∂_M to V_j extends to a translation invariant differential operator $\partial_{\mathbb{R}^2} : C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d}) \to L^2(\mathbb{R}^2, \mathbb{C}^{4d})$. Let $D_{\mathbb{R}^2}$ be the closure of $\partial_{\mathbb{R}^2}$ as an unbounded operator on $L^2(\mathbb{R}^2, \mathbb{C}^{4d})$.

We apply the method of images in order to obtain an embedding

$$\tilde{} : C^\infty_{Rc}(V_j, E) \hookrightarrow C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d})$$

intertwining the operators D_s and $D_{\mathbb{R}^2}$:

Let

$$\begin{aligned} &C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \\ &\quad := \quad \{f \in C^\infty_c(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \mid (\partial_{e_2}^{2k+1} f)(x) = 0,\ \forall k \in \mathbb{N}_0,\ \forall x \in \partial_j M\} \end{aligned}$$

and

$$\begin{aligned} &C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+) \\ &\quad := \quad \{f \in C^\infty_c(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+) \mid (\partial_{e_2}^{2k} f)(x) = 0,\ \forall k \in \mathbb{N}_0,\ \forall x \in \partial_j M\}\ . \end{aligned}$$

Then

$$C^\infty_{Rc}(V_j, E^+) = C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \oplus C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+)$$

and

$$C^\infty_{Rc}(V_j, E^-) = c(e_1)C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3}E^+) \oplus c(e_1)C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3})E^+) \ .$$

For $f \in C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3}E^+)$ resp. $f \in C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3})E^+)$ we define $\tilde{f}$ by first extending f by zero to the half plane $\{x_2 \geq 0\}$ and then reflecting such that $\tilde{f}$ is even resp. odd with respect to $\{x_2 = 0\}$.

For $f \in C^\infty_{Rc}(V_j, E^-)$ we set $\tilde{f} := ic(e_1)\tilde{}(ic(e_1)f)$.

For $D_{\mathbb{R}^2}$ the finite propagation speed property holds. Hence the assertion of the lemma holds for all $f \in C^\infty_{Rc}(V_j, E)$ with $\operatorname{supp} f \subset \{x \in M \mid d(x, \partial_j M) < \frac{3}{16}\}$ and for $|t| < \frac{1}{16}$.

For $f \in C^\infty_c(W, E)$ with $\operatorname{supp} f \subset \{x \in M \mid d(x, \partial M) > \frac{1}{16}\}$ and for $|t| < \frac{1}{16}$ the assertion holds by the standard theory of hyperbolic equations on open subsets of $\mathbb{R}^2$.

Since every $f \in C^\infty_c(M, E)$ can be written as $f = f_W + f_0 + \ldots f_5$ with $f_W \in C^\infty_c(W, E)$ and $f_j \in C^\infty_{Rc}(V_j, E)$, the assertion holds for every $f \in C^\infty_c(M, E)$ and for $|t| < \frac{1}{16}$, and by the group property of e^{itD_s} it follows for all $t \in \mathbb{R}$.

The proof for D_Z is analogous. □

For $k \in \mathbb{N}_0$ let $H^k(\mathbb{C}, D_s)$ be the Hilbert space whose underlying vector space is $\operatorname{dom} D^k_s$ and whose scalar product is given by

$$< f, g >_{H^k} := < (1 + D^2_s)^{\frac{k}{2}} f, (1 + D^2_s)^{\frac{k}{2}} g > \ .$$

Define $H^k(\mathbb{C}, D_Z)$ analogously.

Lemma 3.1.4. *Let $k \in \mathbb{N},\ k \geq 2$.*

1. *There is an embedding $H^k(\mathbb{C}, D_Z) \to C^{k-2}(Z, \mathbb{C}^{4d})$.*
2. *There is an embedding $H^k(\mathbb{C}, D_s) \to C^{k-2}(M, E)$.*

Proof. We sketch the proof of 2), the proof of 1) is analogous.

Let $D_{\mathbb{R}^2}$ be as in the previous proof.

For fixed $r > 0$ the constants in the Gårding inequality for the elliptic operator $(1 + D^2_{\mathbb{R}^2})^k$ on balls $B_r(x),\ x \in \mathbb{R}^2$, can be chosen independent of x.

For $j \in \mathbb{Z}/6$ the embedding $C^\infty_{Rc}(V_j, E) \hookrightarrow C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d})$ that was defined in the previous proof intertwines the Dirac operators D_s on $C^\infty_{Rc}(V_j, E)$ and $D_{\mathbb{R}^2}$ on $C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d})$. Hence the Gårding inequality for the operator $(1 + D^2_s)^k$ on balls $B_{1/8}(x) \subset M$ with $x \in \partial M$ holds with constants independent of x as well.

For $r > 0$ fixed and small enough we can also find global constants for the Gårding inequality for $(1 + D^2_s)^k$ on balls $B_r(x) \subset M$ with $B_r(x) \cap \partial M = \emptyset$. This follows from the fact that M is of bounded geometry.

Then the assertion follows from the Sobolev embedding theorem on balls by standard arguments. □

Corollary 3.1.5. *The operators $e^{-tD^2_Z}$ on $L^2(Z, \mathbb{C}^{4d})$ and $e^{-tD^2_s}$ on $L^2(M, E)$ are integral operators with smooth integral kernels.*

Proof. This follows from the previous two lemmata (see [Ro], Lemma 5.6). □

Lemma 3.1.6. *Let $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a function and assume that for every $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $r > \varepsilon$*

$$f(r,t) \leq Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}} .$$

Then for all $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $r > \varepsilon$

$$f(r,t) \leq Ce^{-\frac{r^2}{(4+\delta)t}} .$$

Proof. Choose $0 < a < 1$ with $\frac{1-a}{4+\delta/2} > \frac{1}{4+\delta}$ and let $m > \frac{2}{a}$.
Then there is $C > 0$ such that for all $r > \varepsilon$

$$f(r,t) \leq Ce^{-\frac{(r-\varepsilon/m)^2}{(4+\delta/2)t}} .$$

It follows that

$$\begin{aligned} f(r,t) &\leq Ce^{-\frac{(1-a)r^2}{(4+\delta/2)t}} e^{\frac{r}{(4+\delta/2)t}(-ar+\frac{2\varepsilon}{m})} e^{-\frac{(\varepsilon/m)^2}{(4+\delta/2)t}} \\ &\leq Ce^{-\frac{r^2}{(4+\delta)t}} . \end{aligned}$$

In the last step we used the fact that $\frac{r}{(4+\delta/2)t}(-ar + \frac{2\varepsilon}{m}) < 0$ for $r > \varepsilon$. □

Lemma 3.1.7. *Let N be closed spin manifold resp. let $N = M, Z$. If N is closed, let E_N be a Dirac bundle on N and let D_N be the associated Dirac operator. If $N = M$ resp. $N = Z$, then let $E_N = E$ resp. $E_N = Z \times \mathbb{C}^{4d}$ and let $D_N = D_s$ resp. $D_N = D_Z$. Let k_t be the integral kernel of $e^{-tD_N^2}$.*

For every $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $t > 0,\ r > \varepsilon$ and $x \in N$

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy \leq Ce^{-\frac{r^2}{(4+\delta)t}} .$$

Analogous estimates hold for the partial derivatives in x and y with respect to unit vector fields on N.

Proof. Let $k \in 2\mathbb{N}$ with $k > \frac{\dim N}{2}$.

Let $S(x,\varepsilon) := \{u \in C_c^\infty(B_\varepsilon(x), E_N) \mid \|(1 + D_N^2)^{-\frac{k}{2}} u\| \leq 1\}$.

Then by the Sobolev embedding theorem resp. by Lemma 3.1.4 there is $C > 0$ such that for all $x \in N$

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy \leq C \sup_{u \in S(x,\varepsilon/2)} \|e^{-tD_N^2} u\|^2_{N \setminus B_r(x)} .$$

By a standard argument (see the proof of [CGT], Prop. 1.1), it follows that

$$\begin{aligned} \int_{N \setminus B_r(x)} |k_t(x,y)|^2 \, dy &\leq Ct^{-1/2} \int_{r-\varepsilon/2}^{\infty} |(1 + (\frac{d}{ids})^2)^{k/2} e^{-s^2/4t}| \, ds \\ &= C \int_{\frac{r-\varepsilon/2}{\sqrt{t}}}^{\infty} |(1 + t^{-1}(\frac{d}{ids'})^2)^{k/2} e^{-s'^2/4}| \, ds' . \end{aligned}$$

Thus there is $l \in \mathbb{N}$ such that

$$\begin{aligned}\int_{N\setminus B_r(x)} |k_t(x,y)|^2\, dy &\leq C(1+t^{-l}) \int\limits_{\frac{r-\varepsilon/2}{\sqrt{t}}}^{\infty} (1+s'^l)e^{-s'^2/4}\, ds' \\ &\leq C(1+t^{-l})e^{-\frac{(r-\varepsilon/2)^2}{(4+\delta/2)t}} \\ &\leq Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}}\ .\end{aligned}$$

Then the assertion follows by applying the previous lemma to $f(r,t) := \sup_{x\in N} \int_{N\setminus B_r(x)} |k_t(x,y)|^2\, dy$. □

Lemma 3.1.8. *Let k_t be as in the previous lemma.*

For any $\varepsilon, \delta > 0$ there is $C < \infty$ such that for all $x, y \in N$ with $d(x,y) > \varepsilon$ and all $t > 0$

$$|k_t(x,y)| \leq Ce^{-\frac{d(x,y)^2}{(4+\delta)t}}\ .$$

Analogous estimates hold for the partial derivatives in x and y with respect to unit vector fields on N.

Proof. Let $S(y,\varepsilon)$ and $k \in 2\mathbb{N}$ be as in the proof of the previous lemma. By the Sobolev embedding theorem resp. Lemma 3.1.4 there is $C > 0$ such that for all $r > \varepsilon$ and all $x, y \in N$ with $d(x,y) \geq r$

$$|k_t(x,y)| \leq C \sup_{u\in S(y,\varepsilon/4)} \|(1+D_N^2)^{\frac{k}{2}} e^{-tD_N^2} u\|^2_{N\setminus B_{r-\varepsilon/4}(y)}\ .$$

As in the proof of the previous lemma this implies

$$|k_t(x,y)| \leq Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}}\ .$$

Then the result follows from Lemma 3.1.6 with $f(r,t) := \sup_{x,y\in N:\ d(x,y)=r} |k_t(x,y)|$. □

For the next lemma assume that $U \subset M$ is an open set for which at least one of the following properties holds:

1. U is precompact and $\overline{U} \cap \partial M = \emptyset$,
2. there is $k \in \mathbb{Z}/6$ such that $U \subset F_k(0, \frac{1}{4})$.

In the first case there is a compact manifold N and a Dirac bundle E_N on N such that there is an Dirac bundle isomorphism $E|_U \to E_N$, whose base map is an isometry. We identify $E|_U$ with its image in E_N. Then D_s coincides with D_N on U.

In the second case U is a subset of Z_k by §1.1. Let D_{Z_k} be the operator D_Z on $Z_k \times \mathbb{C}^{4d}$ with boundary conditions given by $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$. Then D_s coincides with D_{Z_k} on U.

Lemma 3.1.9. *Let U be as in 1) resp. 2). Let k_t be the integral kernel of the heat semigroup of D_s on M and let k'_t be the integral kernel of the heat semigroup of D_N resp. D_{Z_k}.*

For every $T > 0$ and $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $0 < t < T$, $r > \varepsilon$ and $x, y \in U$ with $B_r(x) \subset U$ and $B_r(y) \subset U$

$$|k_t(x,y) - k'_t(x,y)| \leq e^{-\frac{r^2}{(4+\delta)t}} .$$

Analogous estimates hold for the partial derivatives with respect to unit vector fields on U.

Proof. The notation is as in the proof of Lemma 3.1.7.

The estimate follows from

$$\begin{aligned} |k_t(x,y) - k'_t(x,y)| &= \sup_{\phi \in S(x,\varepsilon)} \sup_{\psi \in S(y,\varepsilon/2)} | < \phi, e^{-tD_s^2}\psi > - < \phi, e^{-tD_N^2}\psi > | \\ &\leq Ct^{-\frac{1}{2}} \sup_{\phi \in S(x,\varepsilon)} \sup_{\psi \in S(y,\varepsilon/2)} | < \phi, \int_{\mathbb{R}} e^{-s^2/4t}(e^{isD_s} - e^{isD_N})\psi > | \\ &\leq Ct^{-\frac{1}{2}} \int_{r-\varepsilon/2}^{\infty} |(1 + (\frac{d}{ids})^2)^k e^{-s^2/4t}| . \end{aligned}$$

Here we used that $(e^{isD_s} - e^{isD_N})\psi = 0$ for $|s| \leq r - \varepsilon/2$ by the finite propagation speed property (Lemma 3.1.3) and the uniqueness of solutions of hyperbolic equations. □

3.2 The heat semigroup on compact manifolds

Let $\mathcal{B}$ be a Banach algebra with unit.

Let N be a closed spin manifold of dimension n. Let E_N be a Dirac bundle on N that is trivial as a vector bundle and let D_N be the associated Dirac operator.

Let $k_t \in C^\infty(N \times N, E_N \boxtimes E_N)$ be the induced heat kernel.

According to Cor. 5.2.4 the heat kernel defines a family of bounded operators on $L^2(N, E_N \otimes \mathcal{B})$. The operators are smoothing, thus they restrict to a family of bounded operators on $C^m(N, E_N \otimes \mathcal{B})$ for any $m \in \mathbb{N}_0$. In order to show that the family extends to a holomorphic semigroup we have to study its behaviour for small t.

We define $-D_N^2$ as a closed operator on $L^2(N, E_N \otimes \mathcal{B})$ by requiring that $C^\infty(N, E_N \otimes \mathcal{B})$ is a core of $-D_N^2$.

For $t \to 0$ the heat kernel can be estimated as follows:

Lemma 3.2.1. *Let $\varepsilon > 0$ be smaller than the injectivity radius of N and let $\chi : [0, \infty) \to [0, 1]$ be a smooth monotonously decreasing function such that $\chi(r) = 1$ for $r \leq \varepsilon/2$ and $\chi(r) = 0$ for $r \geq \varepsilon$.*

Let A be a differential operator of order m on $C^\infty(N, E_N)$.

Then there is $C > 0$ such that for all $x, y \in N$ and for all $t > 0$

$$|A_x k_t(x,y)| \leq C + Ct^{-(n+m)/2} e^{-d(x,y)^2/4t} \chi(d(x,y)) \sum_{i=0}^{m} d(x,y)^i t^{\frac{i}{2}} .$$

Proof. This follows from [BGV], Prop. 2.46, and its proof. □

Proposition 3.2.2. *Let A be a differential operator of order m on $C^\infty(N, E_N \otimes \mathcal{B})$. Then there is $C > 0$ such that the action of the integral kernel $A_x k_t(x,y)$ on $L^2(N, E_N \otimes \mathcal{B})$ is bounded by $C(1 + t^{-m/2})$ for all $t > 0$.*

Proof. Choose a finite open covering $\{U_\nu\}_{\nu \in I}$ of N of normal coordinate patches and assume that for every $x, y \in U_\nu$ the shortest geodesics connecting x and y is in U_ν.

Then there are $c_1, c_2 > 0$ such that for all $\nu \in I$ and all $x, y \in U_\nu$

$$c_1 |x - y|_\nu \leq d(x,y) \leq c_2 |x - y|_\nu \ ,$$

where $|\cdot|_\nu$ denotes the euclidian distance on U_ν defined by the coordinates.

Let $\{\phi_\nu\}_{\nu \in I}$ be a partition of unity subordinate to the covering $\{U_\nu\}_{\nu \in I}$.

Let $\varepsilon > 0$ be smaller than the injectivity radius of N and so small that $\{x \in N \mid d(x, \operatorname{supp} \phi_\nu) \leq \varepsilon\} \subset U_\nu$ for every $\nu \in I$. Let χ be as in the previous lemma. Then it follows that

$$\phi_\nu(x) \chi(d(x,y)) \leq \phi_\nu(x) \chi(c_1 |x - y|_\nu) 1_{U_\nu}(y) \ .$$

By the previous lemma there is $C > 0$ such that for all $x, y \in N$ and $t > 0$

$$|A_x k_t(x,y)| \leq C + C t^{-(n+m)/2} \sum_{\nu \in J} \phi_\nu(x) \Big(e^{-c_1^2 |x-y|_\nu^2 / 4t} \chi(c_1 |x-y|_\nu) \sum_{i=0}^{m} c_2^i |x-y|_\nu^i t^{\frac{i}{2}} \Big) 1_{U_\nu}(y) \ .$$

The ν-th term of the outer sum is supported on $U_\nu \times U_\nu$. It is of the form $\phi_\nu(x) f(x - y) 1_{U_\nu}(y)$ in the coordinates of $U_\nu \times U_\nu$ with $f \in L^1(\mathbb{R}^n)$ and there is $C > 0$ such that

$$\|f\|_{L^1} \leq C t^{n/2} (1 + t^{m/2})$$

for all $t > 0$.

The assertion follows now from Prop. 5.2.3 and Cor. 5.2.5. □

Proposition 3.2.3. *1. The family of integral kernels $k_t(x,y)$ defines a bounded strongly continuous semigroup on $L^2(N, E_N \otimes \mathcal{B})$ that may be extended to a bounded holomorphic semigroup. Its generator is $-D_N^2$.*

2. The family of integral kernels $k_t(x,y)$ defines a bounded strongly continuous semigroup on $C^m(N, E_N \otimes \mathcal{B})$ for every $m \in \mathbb{N}_0$.

Proof. 1) By the previous proposition the action of the integral kernel $k_t(x,y)$ on $L^2(N, E_N \otimes \mathcal{B})$ is uniformly bounded for $t > 0$. On $L^2(N, E_N) \odot \mathcal{B}$ it converges strongly to the identity. Thus $k_t(x,y)$ induces a bounded strongly continuous semigroup on $L^2(N, E_N \otimes \mathcal{B})$.

On $C^\infty(N, E_N \otimes \mathcal{B})$ the action of the generator coincides with the action of $-D_N^2$. Since $C^\infty(N, E_N \otimes \mathcal{B})$ is invariant under the semigroup and dense in $L^2(N, E_N \otimes \mathcal{B})$, it is a core for the generator. Hence the generator is $-D_N^2$.

By the previous proposition there is $C > 0$ such that on $L^2(N, E_N \otimes \mathcal{B})$ for all $0 < t < 1$

$$\|D_N^2 e^{-tD_N^2}\| < Ct^{-1} \ .$$

Since $\operatorname{Ran} e^{-tD_N^2} \subset C^\infty(N, E_N \otimes \mathcal{B}) \subset \operatorname{dom} D_N^2$ for $t > 0$, it follows, by Prop. 5.4.3, that $e^{-tD_N^2}$ extends to a holomorphic semigroup.

The integral kernel of $D_N^2 e^{-tD_N^2}$ is exponentially decaying in the supremumsnorm for $t \to \infty$, hence $D_N^2 e^{-tD_N^2}$ is exponentially decaying as an operator on $L^2(N, E_N \otimes \mathcal{B})$.

By Prop. 5.4.3 this shows that the holomorphic extension is bounded.

2) follows from the fact that $k_t(x, y)$ defines a strongly continuous bounded semigroup on $C^m(N, E_N)$ by [BGV], th. 2.30, and that $C^m(N, E_N \otimes \mathcal{B}) \cong C^m(N, E_N) \otimes_\varepsilon \mathcal{B}$. □

3.3 The heat semigroup on $[0, 1]$

3.3.1 The semigroup

Let (P_0, P_1) be a pair of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_I be the associated operator on $L^2([0, 1], \mathcal{A}^{2d})$ defined in §2.2.1.

Now we define an action of D_I on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$: Let D_I, as an unbounded operator on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$, be the closure of $I_0\partial$ with domain $C_R^\infty([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

Let $U \in M_{2d}(\mathcal{A}_\infty)$ be a unitary with $UI_0 = I_0U$ and $UP_0U^* = P_s$ and let $p \in M_d(\mathcal{A}_\infty)$ be such that

$$UP_1U^* = \tfrac{1}{2}\begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix} \ .$$

The unitaries U and p exist by Lemma 1.4.3.

Proposition 3.3.1. *Let $\lambda \in \mathbb{C}$ with $\exp(-2i\lambda) \notin \sigma(p)$.*

1. *The operator $D_I - \lambda$ has a bounded inverse on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.*
2. *The inverse $(D_I - \lambda)^{-1}$ maps $C_R^l([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ isomorphically onto $C_R^{l+1}([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for any $l \in \mathbb{N}_0$.*
3. *The inverse $(D_I - \lambda)^{-1}$ maps $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ continuously to $C([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.*

Proof. The inverse of $D_I - \lambda$ is given by

$$((D_I - \lambda)^{-1} f)(x) = \int_0^x I_0 e^{I_0\lambda(x-y)} f(y)dy + \int_0^1 e^{I_0\lambda(x-y)} A(y)f(y)dy$$

with

$$A(y) = U^* \frac{i}{p - e^{-2i\lambda}} \begin{pmatrix} p & e^{-2i\lambda(1-y)} \\ pe^{-2i\lambda y} & e^{-2i\lambda} \end{pmatrix} U \ .$$

It is easily seen that this map fulfills 1),2) and 3). □

Furthermore we define D_I as an unbounded operator on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$, $l \in \mathbb{N}_0$, by setting $\operatorname{dom} D_I := C^{l+1}_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. By the previous proposition D_I is a closed operator on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

As before D_{I_s} denotes D_I with $R = (P_s, 1-P_s)$.

Now we show that $-D_I^2$ generates a holomorphic semigroup on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. This will be done by first proving that $-D_{I_s}^2$ generates a holomorphic semigroup and by then applying Prop. 5.4.10.

Furthermore the knowledge of the resolvent set of $-D_I^2$ yields a norm estimate of the semigroup $e^{-tD_I^2}$ for large t.

Lemma 3.3.2. *Assume that $R = (P_s, 1-P_s)$.*

The operator $-D_{I_s}^2$ is the generator of a bounded holomorphic semigroup $e^{-tD_{I_s}^2}$ on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$and on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for any $l \in \mathbb{N}_0$.

Proof. Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$ (see §3.1.1) and let $S(t)$ be the induced integral operator.

By Lemma 3.1.2 and Prop. 5.2.3 the family $S(t)$ is uniformly bounded on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for $t > 0$ and the family $D_{I_s}^2 S(t)$ is bounded on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ by $C(1+t^{-1})e^{-\omega t}$ for some $C, \omega > 0$ and all $t > 0$.

Since $S(t)$ converges strongly to the identity on $L^2([0,1],\mathbb{C}^{2d}) \odot \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ for $t \to 0$, it is a strongly continuous semigroup.

By Prop. 5.4.3 it extends to a bounded holomorphic semigroup.

The kernel H from §3.1.1 defines a bounded holomorphic semigroup on $C^k(\mathbb{R}/4\mathbb{Z},(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well by Prop. 3.2.3. This implies that $S(t)$ restricts to a bounded holomorphic semigroup on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. In particular it follows that $C^\infty_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is a core of the generator of $S(t)$. Thus the generator is $-D_{I_s}^2$. □

Proposition 3.3.3. *The operator $-D_I^2$ generates a holomorphic semigroup $e^{-tD_I^2}$ on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for all $k \in \mathbb{N}_0$, and there are $C, \omega > 0$ such that for all $t \geq 0$*

$$\|e^{-tD_I^2}\| \leq Ce^{-\omega t}$$

on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

Proof. The following arguments hold on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$:

The operator $D_I - U^*D_{I_s}U$ is bounded by Prop. 2.2.2. Since $U^*D_{I_s}U$ has a bounded inverse by Prop. 3.3.1 and $-U^*D_{I_s}^2U$ generates a bounded holomorphic semigroup by the previous lemma, we can apply Prop. 5.4.10: It follows that $-D_I^2$ generates a holomorphic semigroup.

By Prop. 3.3.1 there is $\omega > 0$ such that the spectrum of D_I^2 on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is in $]\omega,\infty[$. Hence by Prop. 5.4.2 there is $C > 0$ with $\|e^{-tD_I^2}\| \leq Ce^{-\omega t}$. □

3.3.2 The integral kernel

In this section we use a method that will be our standard method for proving the existence of an integral kernel of a semigroup: We construct an approximation of the semigroup $e^{-tD_I^2}$ by integral operators by cutting and pasting. Using Duhamel's principle we prove that the error term is an integral operator as well. At the same time we obtain estimates for the integral kernel of $e^{-tD_I^2}$.

Let $R = (P_0, P_1)$ be the boundary conditions of D_I.

Let D_{I_0} resp. D_{I_1} be defined as D_I with boundary conditions given by $(P_0, 1 - P_0)$ resp. $(1 - P_1, P_1)$.

From Lemma 1.4.3 and §3.1.1 it follows that $e^{-tD_{I_k}^2},\ k = 0, 1$ is an integral operator for $t > 0$. Its integral kernel is denoted by $e_t^k(x, y)$.

Let $\phi_0 : [0, 1] \to [0, 1]$ be a smooth function with $\operatorname{supp}\phi_0 \subset [0, \frac{2}{3}[$ and $\operatorname{supp}(1 - \phi_0) \subset]\frac{1}{3}, 1]$ and let $\phi_1 := (1 - \phi_0)$. Furthermore choose smooth functions $\gamma_0, \gamma_1 : [0, 1] \to [0, 1]$ with

- $\gamma_k|_{\operatorname{supp}\phi_k} = 1,\ k = 0, 1$,
- $\operatorname{supp}\gamma_k' \cap \operatorname{supp}\phi_k = \emptyset,\ k = 0, 1$,
- $\operatorname{supp}\gamma_0 \subset [0, \frac{5}{6}]$ and $\operatorname{supp}\gamma_1 \subset [\frac{1}{6}, 1]$.

Write E_t for the integral operator corresponding to the integral kernel

$$e_t(x, y) := \gamma_0(x)e_t^0(x, y)\phi_0(y) + \gamma_1(x)e_t^1(x, y)\phi_1(y) \ .$$

Set $E_0 := 1$.

Then E_t is strongly continuous on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and on $C_R^l([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ at any $t \geq 0$.

For $f \in C_R^\infty([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ the map $[0, \infty) \to L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}),\ t \mapsto E_t f$ is even differentiable. Hence by Duhamel's principle (Prop. 5.4.5) we have for $f \in C_R^\infty([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ in $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$:

$$(*) \qquad e^{-tD_I^2}f - E_t f = -\int_0^t e^{-(t-s)D_I^2}(\frac{d}{ds} + D_I^2)E_t f \ ds \ .$$

In the next proposition we apply this equation: We prove that $e^{-tD_I^2}$ is an integral operator with smooth integral kernel for $t > 0$ by showing that the right hand side is an integral operator with smooth integral kernel.

First we fix some notation:

In the following the norm on $M_{2d}(\mathcal{A}_i)$ is denoted by $|\cdot|$. Furthermore $C_R^k([0, 1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)),\ k \in \mathbb{N}_0 \cup \{\infty\}$ means the subspace of $C^k([0, 1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ containing those functions whose column vectors are elements of $C_R^k([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. Then any bounded operator on $C_R^k([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ acts as a bounded operator on $C_R^k([0, 1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ in an obvious way.

Proposition 3.3.4. *For $t > 0$ the operator $e^{-tD_I^2}$ is an integral operator. Let k_t be its integral kernel.*

1. *The map*

$$(0,\infty) \to C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\mathcal{A}_\infty))),\ t \mapsto \big(y \mapsto k_t(\cdot, y)\big)$$

is well-defined and smooth.

2. $k_t(x,y) = k_t(y,x)^*$.

3. *For every $m, n \in \mathbb{N}_0$ and every $\delta > 0$ there is $C > 0$ such that*

$$|\partial_x^m \partial_y^n k_t(x,y) - \partial_x^m \partial_y^n e_t(x,y)| \leq Ct \sum_{k=0,1} e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $t > 0$ and all $x, y \in [0,1]$.

Proof. Let $f \in C_R^\infty([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. From $(*)$ it follows that

$$e^{-tD_I^2} f - E_t f = \sum_{k=0,1} \int_0^t \int_{[0,1]} e^{-sD_I^2} (\gamma_k' \partial + \partial \gamma_k') e_{t-s}^k(\cdot, y) \phi_k(y) f(y)\ dyds\ .$$

By Lemma 3.1.1 the map

$$t \mapsto \big(y \mapsto (\gamma_k' \partial + \partial \gamma_k') e_t^k(\cdot, y) \phi_k(y)\big)$$

can be extended by zero to a smooth map from $[0,\infty)$ to $C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\mathcal{A}_i)))$.

Since $e^{-sD_I^2}$ acts as a uniformly bounded operator on $C_R^\infty([0,1], M_{2d}(\mathcal{A}_i))$ by Lemma 3.1.1, it follows that the operator on the right hand side is an integral operator with smooth integral kernel.

Hence also $e^{-tD_I^2}$ is an integral operator with smooth integral kernel. Its integral kernel satisfies 1) by the preceding arguments.

The selfadjointness of $e^{-tD_I^2}$ implies 2).

Since $d(\operatorname{supp}\phi_k, \operatorname{supp}\gamma_k') > \varepsilon$ for some $\varepsilon > 0$, there is $C > 0$ by Lemma 3.1.2 such that for all $x, y \in [0,1]$ and $t > 0$

$$\begin{aligned}
&|\partial_x^m \partial_y^n \big(k_t(x,y) - e_t(x,y)\big)| \\
&\leq\ C \sum_{k=0,1} \int_0^t \|e^{-sD_I^2} (\gamma_k' \partial + \partial \gamma_k') \partial_y^n \big(e_{t-s}^k(\cdot, y)\phi_k(y)\big)\|_{C^m}\ ds\ 1_{\operatorname{supp}\phi_k}(y) \\
&\leq\ C \sum_{k=0,1} \int_0^t e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)(t-s)}} ds\ 1_{\operatorname{supp}\phi_k}(y) \\
&\leq\ C \sum_{k=0,1} t e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}}\ 1_{\operatorname{supp}\phi_k}(y)\ .
\end{aligned}$$

This shows statement 3). □

Corollary 3.3.5. *Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.*
For every $m,n \in \mathbb{N}_0$ and $\delta,\varepsilon > 0$ we find $C > 0$ such that for all $x,y \in [0,1]$ with $d(x,y) > \varepsilon$ and $t > 0$

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq Ce^{-\frac{d(x,y)^2}{(4+\delta)t}} + Ct \sum_{k=0,1} e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y).$$

Proof. This follows from the previous proposition and Lemma. 3.1.2. □

Corollary 3.3.6. *Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.*
Let ω be as in Prop. 3.3.2. For every $m,n \in \mathbb{N}_0$ there is $C > 0$ such that for any $t > 0$ and any $x,y \in [0,1]$

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}})e^{-\omega t} \ .$$

Proof. There is $C > 0$ such that for all $x,y \in [0,1]$ and all $0 < t < 1$

$$|\partial_x^m \partial_y^n k_t(x,y) - \partial_x^m \partial_y^n e_t(x,y)| \leq Ct \sum_{k=0,1} e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{5t}} \ 1_{\operatorname{supp}\phi_k}(y) \ ,$$

hence by Lemma 3.1.2

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}}) \ .$$

For all $t > 1$ and $y \in [0,1]$

$$k_t(\cdot,y) = e^{-(t-1)D_I^2} k_1(\cdot,y) \ .$$

The assertion follows now since $(y \mapsto k_1(\cdot,y)) \in C^n([0,1], C_R^m([0,1], M_{2d}(\mathcal{A}_i)))$ and since by Prop. 3.3.3 the action of $e^{-(t-1)D_I^2}$ on $C_R^m([0,1], M_{2d}(\mathcal{A}_i))$ is bounded by $Ce^{-\omega t}$ for some $C,\omega > 0$ and every $t > 1$. □

The following facts will be needed when we define the η-form.

Lemma 3.3.7. *Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$.*
Then for all $x,y \in [0,1]$ and $t > 0$

$$\operatorname{tr}(D_{I_s})_x k_t(x,y) = 0.$$

Proof. Let $S := 2P_s - 1 \in \mathrm{Gl}_{2d}(\mathbb{C})$. Then $S^2 = 1$, $SI + IS = 0$, $SP_s = P_s$ and $S(1-P_s) = -(1-P_s)$.
This implies $SD_{I_s}e^{-tD_{I_s}^2} + D_{I_s}e^{-tD_{I_s}^2}S = 0$. Therefore

$$S(D_{I_s})_x k_t(x,y) + (D_{I_s})_x k_t(x,y) S = 0 \ ,$$

hence

$$\operatorname{tr}(D_{I_s})_x k_t(x,y) = \operatorname{tr}(-S(D_{I_s})_x k_t(x,y) S) = -\operatorname{tr}(D_{I_s})_x k_t(x,y) \ .$$

It follows that $\operatorname{tr}(D_{I_s})_x k_t(x,y) = 0$. □

Corollary 3.3.8. *Let now $(D_I k_t)$ be the integral kernel of $D_I e^{-tD_I^2}$. We have uniformly on $[0,1]$:*

$$\lim_{t \to 0} \operatorname{tr}(D_I k_t)(x,x) = 0 \ .$$

Proof. By the previous lemma $\operatorname{tr}(D_I)_x e_t(x,y) = 0$ for all $x,y \in [0,1]$. Then the assertion follows from the estimate in Prop. 3.3.4. □

3.4 The heat semigroup on the cylinder

Let $Z = \mathbb{R} \times [0,1]$.

Let $R = (P_0, P_1)$ be a pair of pairwise transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_Z be the associated operator on $L^2(Z, \mathcal{A}^{4d})$ defined in §2.3.

In this section we define an action of D_Z as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ and study its properties. If not specified the notation is as in §2.3.

First we define the following function spaces and operators:

For $k \in \mathbb{N}_0 \cup \{\infty\}$ let

$$C^k_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) := \{f \in C^k(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \mid (P_i \oplus P_i)(\partial^l_Z f)(x,i) = f(x,i) \text{ for } x \in \mathbb{R};\ i = 0,1; l \in \mathbb{N}_0, l \leq k\} .$$

Further suffixes, like c or $0 \ldots$, have their usual meaning.

These spaces are endowed with the subspace topologies.

For a Fréchet space V we define the Schwarz spaces

$$\mathcal{S}(Z, V) := \mathcal{S}(\mathbb{R}) \otimes C^\infty([0,1], V) .$$

Moreover let $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ be $\mathcal{S}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \cap C^\infty_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as a vector space with the topology induced by $\mathcal{S}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let D_Z as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ be the closure of ∂_Z with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Note that at the moment it is not clear whether D^2_Z is closed on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. We define Δ to the closure of $\partial^2_Z = -\partial^2_{x_1} - \partial^2_{x_2}$ on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let $\Delta_{\mathbb{R}}$ be the closure of $-\partial^2_{x_1}$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let $\tilde{D}_I$ be the closure of $I\partial_{x_2}$ as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let D_I be the operator on $L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ from §3.3.1 with boundary conditions defined by (P_0, P_1).

By Prop. 3.3.1 the operator D_I has a bounded inverse on $L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. Since the space $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ can be identified with $L^2(\mathbb{R}, L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$ by Lemma 5.2.2, it follows that $\tilde{D}_I$ has a bounded inverse on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence $-\tilde{D}^2_I$ is closed.

By an analogous argument it follows that $-\tilde{D}^2_I$ generates a bounded holomorphic semigroup on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with integral kernel $k^I_t(x_2, y_2) \oplus k^I_t(x_2, y_2)$ for $t > 0$, where k^I_t is the integral kernel of $e^{-tD^2_I}$. It exists by Prop. 3.3.4.

Since $\Delta = \Delta_{\mathbb{R}} + \tilde{D}_I^2$, we have a natural candidate for the integral kernel of a semigroup generated by $-\Delta$, namely

$$k_t^Z(x,y) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} (k_t^I(x_2,y_2) \oplus k_t^I(x_2,y_2)) \ .$$

In the next proposition the corresponding family of integral operator is studied.

In the following let $\omega > 0$ be such that

$$\|e^{-t\tilde{D}_I^2}\| \leq Ce^{-\omega t}$$

on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ for all $t \geq 0$. The existence of such an ω follows from Prop. 3.3.3.

Proposition 3.4.1. *1. The integral kernel $k_t^Z(x,y)$ defines a holomorphic semigroup on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ whose generator is $-\Delta$.*

2. For every $m \in \mathbb{N}_0$ the integral kernel $k_t^Z(x,y)$ defines a holomorphic semigroup on $C_R^m(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. It is denoted by $e^{-t\Delta}$ as well.

3. Let A be a differential operator of order m on $C^\infty(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Then for the operator $Ae^{-t\Delta}$ on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well as for $Ae^{-t\Delta} : C_R^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \to C^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, $n \in \mathbb{N}_0$ we have:

There is $C > 0$ such that

$$\|Ae^{-t\Delta}\| \leq C(1 + t^{-m/2})e^{-\omega t}$$

for all $t > 0$.

Proof. 1) By Lemma 5.2.3 the kernel $\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}}$ defines a uniformly bounded family of operators on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ for $t > 0$. For $t \to 0$ it converges strongly to the identity on $L^2(Z) \odot (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}$, thus it is a strongly continuous semigroup. The space $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is invariant under the action of the semigroup and the action of the generator on that space is given by $\partial_{x_2}^2$. Hence the generator is $-\Delta_{\mathbb{R}}$.

By checking the assumptions of Prop. 5.4.3 we show that the semigroup $e^{-t\Delta_{\mathbb{R}}}$ extends to a holomorphic one:

The operator $(i\partial_{x_1})e^{-t\Delta_{\mathbb{R}}}$ equals the convolution with the function

$$g(x_1) := -c(dx_1) \frac{1}{\sqrt{4\pi t}} \left(\frac{x_1}{2t}\right) e^{-x_1^2/4t} \ .$$

Since there is $C > 0$ such that for $0 < t < 1$

$$\|g\|_{L^1} \leq Ct^{-1/2} \ ,$$

it follows that

$$\|(i\partial_{x_1})e^{-t\Delta_{\mathbb{R}}}\| \leq Ct^{-1/2} \ ,$$

on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, thus for $0 < t < 1$

$$\|\Delta_{\mathbb{R}} e^{-t\Delta_{\mathbb{R}}}\| \leq \|(i\partial_{x_1})e^{-(t/2)\Delta_{\mathbb{R}}}\|^2 \leq Ct^{-1} \ .$$

Hence the semigroup $e^{-t\Delta_{\mathbb{R}}}$ is holomorphic on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Note that this estimate also holds on $C_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ showing that $e^{-t\Delta_{\mathbb{R}}}$ is a holomorphic semigroup on $C_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well.

Since the semigroups $e^{-t\Delta_{\mathbb{R}}}$ and $e^{-t\tilde{D}_I^2}$ commute with each other, their composition is a holomorphic semigroup. The space $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is invariant under the action of the semigroup and the generator acts on it as $\partial_{x_1}^2+\partial_{x_2}^2$. Thus the generator is $-\Delta$.

2) Since $(i\partial_{x_j})e^{-t\Delta}f = e^{-t\Delta}(i\partial_{x_j})f,\ j=0,1$, for $f\in C_R^n(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}), n\in\mathbb{N}$, the assertion can be reduced to the case $n=0$.

From Prop. 3.3.3 it follows that the action of the integral kernel $k_t^I(x_2,y_2)\oplus k_t^I(x_2,y_2)$ on $C_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ extends to a holomorphic semigroup. Furthermore in 1) we showed that the integral kernel $\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}$ defines a holomorphic semigroup on $C_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence the kernel $k_t^Z(x,y)$ defines a semigroup on $C_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ that extends to a holomorphic one.

3) We can restrict to the case that A has constant coefficients and furthermore to the case $n=0$ by the argument in the proof of 2).

In the following the operator norms can be understood with respect to the action on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well as with respect to the action on $C_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

The differential operator A is a sum of operators $a_{hk}\tilde{D}_I^h(i\partial_{x_1})^k$ with $a_{hk}\in M_{4d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $h+k\leq m$. We have

$$\tilde{D}_I^h(i\partial_{x_1})^k e^{-t\Delta} = \tilde{D}_I^h e^{-t\tilde{D}_I^2}(i\partial_{x_1})^k e^{-t\Delta_{\mathbb{R}}}\ .$$

By Cor. 5.4.9 there is $C>0$ such that for $0<t$

$$\|D_I^h e^{-tD_I^2}\| \leq Ct^{-h/2}e^{-\omega t}\ .$$

It follows that

$$\|\tilde{D}_I^h e^{-t\tilde{D}_I^2}\| \leq Ct^{-h/2}e^{-\omega t}\ .$$

By the estimate in the proof of 1)

$$\|(i\partial_{x_1})^k e^{-t\Delta_{\mathbb{R}}}\| \leq \|(i\partial_{x_1})e^{-(t/k)\Delta_{\mathbb{R}}}\|^k \leq Ct^{-k/2}$$

for $0<t<1$.

Now the assertion follows from the fact that $e^{-t\Delta_{\mathbb{R}}}$ is uniformly bounded. □

Corollary 3.4.2. *Let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda^2<\omega$.*

1. *The operator $D_Z-\lambda$ is invertible on $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.*
2. *The operator $D_Z-\lambda$ is invertible on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.*
3. *$\Delta = D_Z^2$.*

Proof. 1) For any seminorm $\|\cdot\|$ of $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ there is $C>0$ such that $\|e^{-t\Delta_{\mathbb{R}}}f\|\leq C\|f\|$ and $\|e^{-t\tilde{D}_I^2}f\|\leq Ce^{-\omega t}\|f\|$ for all $f\in\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence $e^{-t\Delta}$ restricts to a bounded operator on $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ and the integral

$$G(\lambda) := \int_0^\infty (D_Z+\lambda)e^{-t(\Delta-\lambda^2)}dt$$

converges as a bounded operator on $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. It inverts $D_Z-\lambda$ on $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

2) The operator $G(\lambda)$ extends to a bounded operator on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ since by the previous proposition there is $C>0$ such that on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$

$$\|(D_Z+\lambda)e^{-t(\Delta-\lambda^2)}\|\leq C(1+t^{-\frac{1}{2}})e^{-(\omega-\operatorname{Re}\lambda^2)t}$$

for all $t>0$.

From 1) it follows that $G(\lambda)$ inverts $D_Z-\lambda$ on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well.

3) From 1) it follows that $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is a core for the closure of D_Z^2, hence the closure of D_Z^2 equals Δ. From 2) it follows that the operator D_Z^2 is closed. □

Proposition 3.4.3. *Let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda<\omega$.*

1. *The operator $(D_Z^2-\lambda)^{-1}$ maps $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ continuously to $C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.*
2. *Let $n\in\mathbb{N},n\geq 2$. The operator $(D_Z^2-\lambda)^{-n}$ maps $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ continuously to $C^{2n-3}(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.*

Proof. 1) For $\operatorname{Re}\lambda<\omega$ we have on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

$$\begin{aligned}(D_Z^2-\lambda)^{-1} &= \int_0^\infty e^{-t(D_Z^2-\lambda)}dt\\ &= \int_0^\infty e^{\lambda t}e^{-t\Delta_{\mathbb{R}}^2}e^{-t\tilde{D}_I^2}dt\ .\end{aligned}$$

By Prop. 3.3.1 the operator $D_I^{-1}:L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})\to C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is bounded. Thus the family of operators

$$e^{-t\tilde{D}_I^2}=\tilde{D}_I^{-1}\tilde{D}_Ie^{-t\tilde{D}_I^2}:L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})\to L^2(\mathbb{R},C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$$

is bounded by $C(1+t^{-\frac{1}{2}})e^{-\omega t}$ for all $t>0$.

Furthermore the family

$$e^{-t\Delta_{\mathbb{R}}}:L^2(\mathbb{R},C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))\to C(\mathbb{R},C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$$

is bounded by $\sup\limits_{x_1\in\mathbb{R}}\|\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}\|_{L^2_{y_1}}$, hence by some $Ct^{-1/4}$.

It follows that the integral converges in the bounded operators from $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ to $C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

2) We show that for any $k\in\mathbb{N}_0,\ k\leq 2n-3$ the map

$$\tilde{D}_I^k(i\partial_{x_1})^{2n-3-k}(D_Z^2-\lambda)^{-n}:\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})\to C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$$

extends to a bounded operator from $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ to $C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

On $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$

$$(i\partial_{x_1})^2=\Delta_{\mathbb{R}}=D_Z^2-\tilde{D}_I^2$$

and

$$i\partial_{x_1} = -ic(dx_1)D_Z + i\tilde{D}_I \ .$$

Note that $c(dx_1)$ anticommutes with $\tilde{D}_I$.

By these facts and by 1) it is enough to show that the operator

$$(D_Z^2 - \lambda)\tilde{D}_I^k D_Z^{2n-3-k}(D_Z^2 - \lambda)^{-n} = \tilde{D}_I^k D_Z^{2n-3-k}(D_Z^2 - \lambda)^{-n+1}$$

extends to a bounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Since D_Z and $\tilde{D}_I$ are invertible, Prop. 5.4.7 implies that there are bounded involutions I_1, I_2 on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ satisfying $\tilde{D}_I = I_1|\tilde{D}_I|$ and $D_Z = I_2|D_Z|$. Furthermore $[I_1, \tilde{D}_I] = 0,\ [I_2, D_Z] = 0$ and $[I_j, D_Z^2] = 0,\ j = 1, 2$. Thus we only need to consider the operator

$$\begin{aligned} &|\tilde{D}_I|^k |D_Z|^{2n-3-k}(D_Z^2 - \lambda)^{-n+1} \\ &= \left(|\tilde{D}_I|^{\frac{k}{n-1}} |D_Z|^{\frac{(2n-3-k)}{n-1}}(D_Z^2 - \lambda)^{-1}\right)^{n-1} \ . \end{aligned}$$

On $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ the term in brackets can be re-written as

$$\int_0^\infty |\tilde{D}_I|^{\frac{k}{n-1}} e^{-\frac{t}{2}\tilde{D}_I^2} e^{-\frac{t}{2}D_{\mathbb{R}}^2} |D_Z|^{\frac{(2n-3-k)}{n-1}} e^{-\frac{t}{2}D_Z^2} dt \ .$$

The integral converges as a bounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ since by Cor. 5.4.9 the integrand can be estimated for small t by

$$C\big(\frac{t}{2}\big)^{-(\frac{k}{2n-2})}\big(\frac{t}{2}\big)^{-(\frac{2n-3-k}{2n-2})} = C\big(\frac{t}{2}\big)^{-(\frac{2n-3}{2n-2})}$$

and since for large t it is exponentially decaying. □

In the following $|\cdot|$ denotes the norm on $M_{4d}(\mathcal{A}_i)$.

Lemma 3.4.4. *For every $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{N}_0^2$ there are $c, C > 0$ such that for all $x, y \in Z$ with $d(x, y) > \varepsilon$ and all $t > 0$*

$$|\partial_x^\alpha \partial_y^\beta k_t^Z(x, y)| \leq Ce^{-\frac{d(x,y)^2}{ct}} \ .$$

Proof. For $m, n \in \mathbb{N}_0$ there are $C, c > 0$ such that

$$|\partial_{x_2}^m \partial_{y_2}^n k_t^I(x_2, y_2)| \leq Ce^{-\frac{(x_2-y_2)^2}{ct}}$$

for $|x_2 - y_2| \geq \varepsilon/2$ and $t > 0$. This follows from Cor. 3.3.5 for $t < 1$ and from Cor. 3.3.6 for $t \geq 1$. For $|x_2 - y_2| \leq \varepsilon/2$ the left hand side it bounded by $C(1 + t^{-\frac{m+n+1}{2}})$ by Cor. 3.3.6.

Hence there are $c_1, c_2, C > 0$ such that for $d(x, y) > \varepsilon$ and $t > 0$

$$\begin{aligned} &|\partial_x^\alpha \partial_y^\beta \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} \big(k_t^I(x_2, y_2) \oplus k_t^I(x_2, y_2)\big)\right)| \\ &\leq\ C\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} \Big(e^{-\frac{(x_2-y_2)^2}{c_1 t}} 1_{\mathbb{R}\setminus[0,\varepsilon/2]}(|x_2 - y_2|) \\ &\qquad +(1 + t^{-\frac{|\alpha|+|\beta|+2}{2}}) 1_{[0,\varepsilon/2]}(|x_2 - y_2|)\Big) \\ &\leq\ Ce^{-\frac{d(x,y)^2}{c_2 t}} \ . \end{aligned}$$

□

In the next lemma we write $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ for the subspace of $\mathcal{S}(Z, M_{4d}(\mathcal{A}_i))$ containing those functions whose columns are in $\mathcal{S}_R(Z, \mathcal{A}_i^{4d})$. Then operators on $\mathcal{S}_R(Z, \mathcal{A}_i^{4d})$ act on $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ as well – column by column. The space $C^\infty_{Rc}(Z, M_{4d}(\mathcal{A}_i))$ is analogously defined.

Lemma 3.4.5. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$. Let $\xi_1, \xi_2 \in C^\infty(Z)$ with $\operatorname{supp}\xi_1 \cap \operatorname{supp}\xi_2 = \emptyset$ and assume that $\operatorname{supp}\xi_2$ is compact.*

Then for any $n \in \mathbb{N}$ the operator $\xi_1(D_Z^2 - \lambda)^{-n}\xi_2$ is an integral operator.

Let κ be its integral kernel. Then $(y \mapsto \kappa(\cdot, y)) \in C_c^\infty(Z, \mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i)))$ and $(x \mapsto \kappa(x, \cdot)^*) \in \mathcal{S}(Z, C^\infty_{Rc}(Z, M_{4d}(\mathcal{A}_i)))$.

In particular $\xi_1(D^2 - \lambda)^{-n}\xi_2$ maps $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ continuously to $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Proof. We prove the claim first for $n = 1$:

Let $f \in C^\infty_{Rc}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

In $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$

$$\begin{aligned}\xi_1(D_Z^2 - \lambda)^{-1}\xi_2 f &= \int_0^\infty \xi_1 e^{-t(D_Z^2-\lambda)}\xi_2 f \; dt \\ &= \int_0^\infty \int_Z \xi_1 k_t^Z(\cdot, y) e^{\lambda t}\xi_2(y) f(y) \; dy dt \; .\end{aligned}$$

Let $\varepsilon > 0$ be such that $d(\operatorname{supp}\xi_1, \operatorname{supp}\xi_2) > \varepsilon$.

By the previous lemma it follows that there are $c, C > 0$ such that for all $x, y \in Z$ and all $t > 0$

$$\begin{aligned}|\xi_1(x)e^{\lambda t}k_t^Z(x, y)\xi_2(y)| &\leq C1_{\{t>1\}}(t)|\xi_1(x)|e^{(\operatorname{Re}\lambda-\omega)t}e^{-\frac{(x_1-y_1)^2}{4t}}|\xi_2(y)| \\ &\quad + C1_{\{t\leq 1\}}(t)|\xi_1(x)|e^{-\frac{d(x,y)^2}{ct}}|\xi_2(y)| \; .\end{aligned}$$

Analogous estimates hold for the partial derivatives. Hence we can interchange the order of integration. It follows that $\xi_1(D_Z^2 - \lambda)^{-1}\xi_2$ is an integral operator with integral kernel

$$\kappa(x, y) := \int_0^\infty \xi_1(x)e^{-\lambda t}k_t^Z(x, y)\xi_2(y)dt \; .$$

The other statements of the lemma also follow from the estimates for $n = 1$.

For $n > 1$ choose a smooth compactly supported function $\psi : Z \to [0, 1]$ such that $\operatorname{supp}\psi \cap \operatorname{supp}\xi_1 = \emptyset$ and $\operatorname{supp}(1 - \psi) \cap \operatorname{supp}\xi_2 = \emptyset$. Then

$$\begin{aligned}\xi_1(D_Z^2 - \lambda)^{-n}\xi_2 &= \xi_1(D_Z^2 - \lambda)^{-1}\psi(D_Z^2 - \lambda)^{-n+1}\xi_2 \\ &\quad + \xi_1(D_Z^2 - \lambda)^{-1}(1 - \psi)(D_Z^2 - \lambda)^{-n+1}\xi_2 \; .\end{aligned}$$

By induction the lemma can be applied to $\xi_1(D_Z^2-\lambda)^{-1}\psi$ and $(1-\psi)(D_Z^2-\lambda)^{-n+1}\xi_2$. The statement of the lemma follows for $\xi_1(D_Z^2 - \lambda)^{-n}\xi_2$ from this and the fact that by Cor. 3.4.2 the operator $(D_Z^2 - \lambda)^{-m}$ acts continuously on $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ for all $m \in \mathbb{N}$. □

3.5 The heat semigroup on M

3.5.1 Definitions

Recall that we defined $D(\rho)^2$ as an unbounded operator on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$ in §2.1.1 and §2.5. Now we define an action of it on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$:

Let $D(\rho)_i^2$ as an unbounded operator on $L^2(M, E \otimes \mathcal{A}_i)$ be the restriction of $D(\rho)^2$ to

$$\operatorname{dom} D(\rho)_i^2 := \{f \in \operatorname{dom} D(\rho)^2 \cap L^2(M, E \otimes \mathcal{A}_i) \mid D(\rho)^2 f \in L^2(M, E \otimes \mathcal{A}_i)\} \ .$$

It is closed.

Let $D(\rho)_{\mu i}^2$ be the closure of the unbounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ whose domain is given by the right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-submodule generated by $\operatorname{dom} D(\rho)_i^2$ and whose action is defined by $D(\rho)_{\mu i}^2(f\omega) := (D(\rho)_i^2 f)\omega$ for $f \in \operatorname{dom} D(\rho)_i^2,\ \omega \in \hat{\Omega}_{\leq\mu}\mathcal{A}_i$.

Note that the notation is misleading: It suggests that $D(\rho)_{\mu i}^2$ is the square of some unbounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, but this is not clear for the moment.

In the following we suppress the indices.

Define D_s^2 as a closed operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ in an analogous way.

We will often make use of cutting and pasting arguments. Here we fix the setting:

Let $0 < b_0 \leq \frac{1}{4}$ be small enough and $r_0 > 0$ large enough such that

$$\operatorname{supp} k_K \cap \big((F(r_0, b_0) \times M) \cup (M \times F(r_0, b_0))\big) = \emptyset \ ,$$

with $F(r_0, b_0)$ as in §1.1. Here k_K denotes the integral kernel of K.

Let $\mathcal{U}(r_0, b_0)$ be the open covering that was defined in §1.1 and modified in §2.5.

Choose a smooth partition of unity $\{\phi_k\}_{k\in J}$ subordinate to $\mathcal{U}(r_0, b_0)$ and smooth functions $\{\gamma_k\}_{k\in J}$ on M such that for all $k \in J$

- $\operatorname{supp} \gamma_k \subset \mathcal{U}_k$,
- $\operatorname{supp}(1 - \gamma_k) \cap \operatorname{supp} \phi_k = \emptyset$,
- the derivatives $\partial_{e_2}(\phi_k|_F)$ and $\partial_{e_2}(\gamma_k|_F)$ vanish in a neighbourhood of ∂M.

Let E_N be a Dirac bundle on a compact spin manifold N that is trivial as a vector bundle and assume that there is a Dirac bundle isomorphism $E|_{\mathcal{U}_\clubsuit} \to E_N$, whose base map is an isometric embedding. Let D_N be the associated Dirac operator. We identify $\mathcal{U}_\clubsuit$ with its image in N and $E|_{\mathcal{U}_\clubsuit}$ with its image in E_N.

Since the support of k_K is in $\mathcal{U}_\clubsuit \times \mathcal{U}_\clubsuit$, the restriction of $D(\rho) = D + \rho K$ to $\mathcal{U}_\clubsuit$ extends to an operator $D_N + \rho K$ on the sections of E_N.

For $k \in \mathbb{Z}/6$ let D_{Z_k} be the operator D_Z on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ from §3.4 with boundary conditions given by the pair $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$.

3.5.2 The resolvents of $D(\rho)^2$

This section has three different aims:

Applying a method of Lott ([Lo], §6.1.) we investigate the resolvent set of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Furthermore we prove a kind of Sobolev embedding theorem – more precisely an analogue of Lemma 3.1.4 for the operator $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Third we obtain more information about the kernel of $D(\rho)^2$ on $L^2(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, namely that there is a projection on it and that this projection is a Hilbert-Schmidt operator with a smooth integral kernel.

Let $\omega > 0$ be such that there is $C > 0$ with

$$\|e^{-tD^2_{Z_k}}\| \leq Ce^{-\omega t}$$

for all $t \geq 0$ and all $k \in \mathbb{Z}/6$.

Let $\nu \in \mathbb{N}$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ we define a parametrix of $(D(\rho)^2 - \lambda)^\nu$:

For $k \in \mathbb{Z}/6$ let $Q_k(\lambda) = (D^2_{Z_k} - \lambda)^{-\nu}$. It is well-defined by Cor. 3.4.2.

Let $Q_\clubsuit(\lambda)$ be a local parametrix of $(D^2 - \lambda)^\nu$ on $\mathcal{U}_\clubsuit$ defined by the symbol of $(D^2 - \lambda)^\nu$.

The operator

$$Q(\lambda) := \sum_{k\in J} \phi_k Q_k(\lambda)\gamma_k$$

acts as a bounded operator on the spaces $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by §5.2.4 and by Cor. 3.4.2.

Lemma 3.5.1. *For any $\rho \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ the closure of $Q(\lambda)(D(\rho)^2-\lambda)^\nu-1$ is an integral operator $\mathcal{K}$ with smooth integral kernel $\kappa \in L^2(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$.*

Furthermore $(x \mapsto \kappa(x,\cdot)^*) \in \mathcal{S}(M, C^\infty_{cR}(M, E \otimes \mathcal{A}_i) \otimes E^*)$ and $(y \mapsto \kappa(\cdot, y)) \in C^\infty_c(M, \mathcal{S}_R(M, E \otimes \mathcal{A}_i) \otimes E^*)$.

In particular $\mathcal{K}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Proof. The difference $(D(\rho)^2-\lambda)^\nu-(D^2-\lambda)^\nu$ is an integral operator with smooth integral kernel whose support is contained in $\operatorname{supp} k_K$. Hence we need only investigate $Q(\lambda)(D^2 - \lambda)^\nu - 1$.

For any $k \in J$ choose a function $\xi_k \in C^\infty_c(M)$ with values in $[0, 1]$ and such that $\operatorname{supp}\xi_k \subset \mathcal{U}_k$, $\xi_k|_{\operatorname{supp} d\gamma_k} = 1$ and $\operatorname{supp}\phi_k \cap \operatorname{supp}\xi_k = \emptyset$. Furthermore assume that $\partial_{e_2}(\xi_k|_F)$ vanishes in a neighbourhood of ∂M.

By induction on ν we show that

$$[\gamma_k, (D^2 - \lambda)^\nu] = \xi_k[\gamma_k, (D^2 - \lambda)^\nu] \ :$$

For $\nu = 0$ the claim is trivial and for general ν

$$\begin{aligned}[\gamma_k, (D^2 - \lambda)^\nu] &= (D^2 - \lambda)^{\nu-1}\big(c(d\gamma_k)D + Dc(d\gamma_k)\big) \\ &\quad +[\gamma_k, (D^2 - \lambda)^{\nu-1}](D^2 - \lambda) \\ &= \xi_k(D^2 - \lambda)^{\nu-1}\big(c(d\gamma_k)D + Dc(d\gamma_k)\big) + [\gamma_k, (D^2 - \lambda)^{\nu-1}](D^2 - \lambda) \ .\end{aligned}$$

In the following for simplicity the operators D_{Z_k}, $k \in \mathbb{Z}/6$ are denoted by D as well. Furthermore $\sim$ means equality up to integral operators with smooth compactly supported integral kernels.

Then on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$

$$\begin{aligned} Q(\lambda)(D^2-\lambda)^\nu - 1 &= \sum_{k\in J} \phi_k Q_k(\lambda)[\gamma_k, (D^2-\lambda)^\nu] + \sum_{k\in J} \phi_k Q_k(\lambda)(D^2-\lambda)^\nu \gamma_k - 1 \\ &\sim -\sum_{k\in J} \phi_k Q_k(\lambda)\xi_k[\gamma_k, (D^2-\lambda)^\nu] \ . \end{aligned}$$

For all $k \in J$ the operator $\phi_k Q_k(\lambda)\xi_k$ is an integral operator whose integral kernel has the properties stated in the lemma. This follows for $k \in \mathbb{Z}/6$ from Lemma 3.4.5 and for $k = \clubsuit$ from the properties of pseudodifferential operators. Now the assertion is shown. □

Proposition 3.5.2. *Let $\rho \in \mathbb{R}$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ such that $D(\rho)^2 - \lambda$ has a bounded inverse on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$.*

Then $D(\rho)^2 - \lambda$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

The inverse $(D(\rho)^2 - \lambda)^{-1}$ acts as a bounded operator on the space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Note that in particular $D(\rho)^2 - \lambda$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for all $\lambda < 0$.

Proof. The operator $Q(\lambda)$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and satisfies $Q(\lambda)(D(\rho)^2 - \lambda) = 1 - \mathcal{K}$ by the previous lemma. Here $\mathcal{K}$ is an integral operator with smooth integral kernel $\kappa \in L^2(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$ and $(x \mapsto \kappa(x, \cdot)) \in C(M, L^2(M, E \otimes \mathcal{A}_i) \otimes E^*)$ and $(y \mapsto \kappa(\cdot, y)) \in C(M, L^2(M, E \otimes \mathcal{A}_i) \otimes E^*)$.

We want to apply Lemma 5.3.1. Since in general $1 - \mathcal{K}$ is not invertible on $L^2(M, E \otimes \mathcal{A})$, we modify the parametrix:

Choose an integral kernel $s \in C_c^\infty(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$ vanishing near $(\partial M \times M) \cup (M \times \partial M)$ such that in $B(L^2(M, E \otimes \mathcal{A}))$

$$\|\mathcal{K} - S(D(\rho)^2 - \lambda)\| \leq \tfrac{1}{2}.$$

This choice is possible since by assumption $(D(\rho)^2 - \lambda)$ has a bounded inverse, hence also $(D(\rho)^2 - \overline{\lambda})$ has a bounded inverse on $L^2(M, E \otimes \mathcal{A})$.

It follows that

$$(Q(\lambda) + S)(D(\rho)^2 - \lambda) = 1 - \big(\mathcal{K} - S(D(\rho)^2 - \lambda)\big)$$

has a bounded inverse on $L^2(M, E \otimes \mathcal{A})$.

Lemma 5.3.1 implies that $1 - \mathcal{K} - S(D(\rho)^2 - \lambda)$ is invertible on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Thus

$$\big(1 - (\mathcal{K} - S(D(\rho)^2 - \lambda))\big)^{-1}(Q(\lambda) + S)$$

is a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. On $L^2(M, E \otimes \mathcal{A})$ it inverts $D(\rho)^2 - \lambda$, hence it is the inverse of $D(\rho)^2 - \lambda$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as well.

Since $Q(\lambda)$ acts continuously on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $\mathcal{K}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by the previous lemma, the operator $(D(\rho)^2 - \lambda)^{-1}$ acts continously on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by

$$\begin{aligned}(D(\rho)^2 - \lambda)^{-1} &= (1 - \mathcal{K})(D(\rho)^2 - \lambda)^{-1} + \mathcal{K}(D(\rho)^2 - \lambda)^{-1} \\ &= Q(\lambda) + \mathcal{K}(D(\rho)^2 - \lambda)^{-1} \ .\end{aligned}$$

□

Corollary 3.5.3. *The space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is a core of $D(\rho)^2$.*

Proposition 3.5.4. *Let $\rho \in \mathbb{R}$ and let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ be such that $(D(\rho)^2 - \lambda)$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.*

Then for any $\nu \in \mathbb{N}$, $\nu \geq 2$, the operator $(D(\rho)^2 - \lambda)^{-\nu}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C_R^{2\nu-3}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Proof. Let $Q(\lambda)(D(\rho)^2 - \lambda)^\nu = 1 - \mathcal{K}$ as before, thus

$$(D(\rho)^2 - \lambda)^{-\nu} = Q(\lambda) + \mathcal{K}(D(\rho)^2 - \lambda)^{-\nu} \ .$$

By Prop. 3.4.3 and §5.2.4 the operator $Q(\lambda)$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C_R^{2\nu-3}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Furthermore $\mathcal{K}$ is smoothing. □

Corollary 3.5.5. *The kernel of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is a subspace of $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.*

Proof. Let $\lambda \in \mathbb{C}$ be as in the previous proposition and assume furthermore $\lambda \neq 0$.

Then $(D(\rho)^2 - \lambda)^{-\nu} f = (-\lambda)^{-\nu} f$ for $f \in \operatorname{Ker} D(\rho)^2$ and every $\nu \in \mathbb{N}$. By the previous proposition it follows that the elements of $\operatorname{Ker} D(\rho)^2$ are smooth.

For $k \in \mathbb{Z}/6$ and $f \in \operatorname{Ker} D(\rho)^2$

$$D_{Z_k}^2 \phi_k f \in C_{cR}^\infty(Z_k, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \ .$$

From Cor. 3.4.2 it follows that

$$\phi_k f = D_{Z_k}^{-2}(D_{Z_k}^2 \phi_k f) \in \mathcal{S}_R(Z_k, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \ .$$

Hence $f \in \mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. □

Proposition 3.5.6. *Let $\rho \neq 0$.*

Let P be the projection onto the kernel of $D(\rho)$ on $L^2(M, E \otimes \mathcal{A})$.

Then P is a finite Hilbert-Schmidt operator whose integral kernel is of the form $\sum_{j=1}^m f_j(x)h_j(y)^*$ with $f_j, h_j \in \bigcap_i \big(\operatorname{Ker} D(\rho) \cap L^2(M, E \otimes \mathcal{A}_i)\big) \subset \mathcal{S}_R(M, E \otimes \mathcal{A}_\infty)$.

Furthermore $\operatorname{Ker} D(\rho)^2 = PL^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $\operatorname{Ran} D(\rho)^2 = (1 - P)L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Hence there is a decomposition

$$L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) = \operatorname{Ker} D(\rho)^2 \oplus \operatorname{Ran} D(\rho)^2$$

and

$$D(\rho)^2 = D(\rho)^2|_{\operatorname{Ker} D(\rho)^2} \oplus D(\rho)^2|_{\operatorname{Ran} D(\rho)^2} \ .$$

Proof. First consider the situation on $L^2(M, E \otimes \mathcal{A})$: Since the range of $D(\rho)$ is closed, there is an orthogonal projection P onto the kernel of $D(\rho)$ by Prop. 5.1.14. Furthermore $D(\rho)$ is selfadjoint, hence $\operatorname{Ker} D(\rho) = \operatorname{Ker} D(\rho)^2$ by Prop. 2.1.1. The range of $D(\rho)^2$ is closed, thus zero is an isolated point in the spectrum of $D(\rho)^2$ on $L^2(M, E \otimes \mathcal{A})$.

Hence, for r small enough,

$$P = \frac{1}{2\pi i} \int_{|\lambda|=r} (D(\rho)^2 - \lambda)^{-1} d\lambda \ .$$

From Prop. 3.5.2 it follows that zero is an isolated point in the spectrum of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ as well. Thus P is well-defined as a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. By Prop. 5.3.7 it is a Hilbert-Schmidt operator whose integral kernel is as asserted.

The remaining parts follow from the spectral theory for closed operators on Banach spaces ([Da], th. 2.14). □

Corollary 3.5.7. *Let* $\operatorname{Re} \lambda < \omega$.

1. *Let* $\rho \neq 0$ *and let* P *be the orthogonal projection onto the kernel of* $D(\rho)^2$.

   If $D(\rho)^2 + P - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \mathcal{A})$, *then* $D(\rho)^2 + P - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *and the inverse acts as a bounded operator on the space* $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *as well.*

2. *Let* P_0 *be the orthogonal projection onto* $\operatorname{Ker} D_s^2$. *If* $D_s^2 + P_0 - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \mathcal{A})$, *then* $D_s^2 + P_0 - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *and the inverse acts as a bounded operator on the space* $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *as well.*

In particular there is $c > 0$ *such that* $\{\operatorname{Re} \lambda < c\}$ *is in the resolvent set of* $D(\rho)^2 + P$ *resp.* $D_s^2 + P_0$.

Proof. From the previous proposition it follows that $P(1 - \lambda)^{-1} + (1 - P)(D(\rho)^2 - \lambda)^{-1}$ inverts $D(\rho)^2 + P - \lambda$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Since P acts as a bounded operator on the space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by Cor. 3.5.5 and $(D(\rho)^2 - \lambda)^{-1}$ is bounded on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by Prop. 3.5.2, the operator $(D(\rho)^2 + P - \lambda)^{-1}$ is bounded on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ as well.

2) follows analogously. □

3.5.3 An approximation of the semigroup

By cutting and pasting we construct an integral operator that for small t behaves similar to a semigroup generated by $-D(\rho)^2$:

Recall the definitions of §3.5.1.

Let $e(\rho)_t^{\clubsuit}(x, y)$ be the restriction of the integral kernel of $e^{-t(D_N + \rho K)^2}$ to $\mathcal{U}_{\clubsuit} \times \mathcal{U}_{\clubsuit}$ and for $k \in \mathbb{Z}/6$ let $e(\rho)_t^k(x, y)$ be the restriction of the integral kernel of $e^{-tD_{Z_k}^2}$ to $\mathcal{U}_k \times \mathcal{U}_k$. Extend these functions by zero to $M \times M$. Clearly for $k \in \mathbb{Z}/6$ we have $e(\rho)_t^k(x, y) = e(0)_t^k(x, y)$.

Denote the family of integral operators corresponding to the integral kernel

$$e(\rho)_t(x, y) := \sum_{k \in J} \gamma_k(x) e(\rho)_t^k(x, y) \phi_k(y)$$

on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by $E(\rho)_t$.

By the results in §3.2 and §3.4 it is a strongly continuous family of bounded operators and has the following properties:

1. For $t \to 0$ the family $E(\rho)_t$ converges strongly to $E(\rho)_0 := 1$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.
2. If $f \in C^\infty_{cR}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, then the map $[0,\infty) \to L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i),\ t \mapsto E(\rho)_t f$ is differentiable.
3. $\operatorname{Ran} E(\rho)_t \subset \mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t > 0$.
4. Let A be a differential operator on $C^\infty(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ of order m with bounded coefficients. By Prop. 3.2.2 and Prop. 3.4.1 the operator $AE(\rho)_t$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t > 0$. Moreover, for $T > 0$ there is $C > 0$ such that for all $T > t > 0$:
$$\|AE(\rho)_t\| \leq Ct^{-\frac{m}{2}} .$$
5. For all $m \in \mathbb{N}_0$ and $T > 0$ there is $C > 0$ such that for all $y \in \mathcal{U}_\clubsuit$, $\rho \in [-1,1]$ and $0 < t < T$ in $C^m_R(\mathcal{U}_\clubsuit, (E \otimes E_y) \otimes \mathcal{A}_i)$
$$\|e(\rho)^\clubsuit_t(\cdot, y) - e(0)^\clubsuit_t(\cdot, y)\|_{C^m_R} \leq C|\rho| .$$
An analogous estimate holds for the partial derivatives in y with respect to unit vector fields on $\mathcal{U}_\clubsuit$.

The last statement follows by Volterra development (Prop. 5.4.4): On N

$$\begin{aligned} & e^{-tD_N(\rho)^2} - e^{-tD_N^2} \\ &= \rho \sum_{n=1}^{\infty} (-1)^n \rho^{n-1} t^n \int\!\!\int_{\Delta^n} e^{u_0 t D_N^2} \big([D_N, K]_s + \rho K^2\big) e^{u_1 t D_N^2} \dots e^{u_n t D_N^2}\ du_0 \dots du_n , \end{aligned}$$

and the sum is an integral operator whose integral kernel is uniformly bounded in $0 < t < T$ and $\rho \in [-1,1]$.

3.5.4 The semigroup $e^{-tD_s^2}$

By Cor. 3.1.5 the operator $e^{-tD_s^2}$ on the Hilbert space $L^2(M, E)$ is an integral operator with smooth integral kernel k_t for $t > 0$. In this section we show that k_t defines a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and that this semigroup extends to a holomorphic one.

For D_s let $e^k_t(x,y)$ be an integral kernel that is defined analogously to $e(\rho)_t(x,y)$ for $D(\rho)$ and let
$$e_t(x,y) := \sum_{k \in J} \gamma_k(x) e^k_t(x,y) \phi_k(y) .$$

The corresponding family of operators is denoted by E_t. We set $E_0 := 1$. The properties of E_t are analogous to those of $E(\rho)_t$ from the previous section.

For $f \in C^\infty_{cR}(M, E)$
$$e^{-tD_s^2} f - E_t f = \int_0^t e^{-sD_s^2} \left(\frac{d}{dt} + D_s^2\right) E_{t-s} f\ ds$$
by Duhamel's principle.

Proposition 3.5.8. *The heat kernel k_t associated to D_s defines a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with generator $-D_s^2$ that extends to a bounded holomorphic semigroup.*

Let A be a differential operator of order $m \in \mathbb{N}_0$ with bounded smooth coefficients. Then for any $t > 0$ the operator $Ae^{-tD_s^2}$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and for any $T > 0$ there is $C > 0$ such that for $0 < t < T$

$$\|Ae^{-tD_s^2}\| \leq Ct^{-\frac{m}{2}} \ .$$

Proof. First we show that for $T > 0$ there is $C > 0$ such that for $0 < t < T$ the difference $A_x k_t(x, y) - A_x e_t(x, y)$ is bounded by Ct in $L^2(M \times M, E \boxtimes E^*)$.

For $k \in J$ let $\chi_k \in C_c^\infty(M)$ be a function with values in $[0, 1]$, with compact support in $\mathcal{U}_k$ and equal to one on a neighbourhood of $\operatorname{supp} d\gamma_k$.

From the equation above it follows that

$$\begin{aligned} & A_x k_t(x, y) - A_x e_t(x, y) \\ &= -\sum_{k\in J} \int_0^t \int_M A_x k_s(x, r)[\partial_M, c(d\gamma_k(r))]_s e^k_{t-s}(r, y)\phi_k(y) \ drds \ . \end{aligned}$$

This can be re-written as

$$\begin{aligned} &-\sum_{k\in J} \int_0^t \int_M (1 - \chi_k(x))A_x k_s(x, r)[(\partial_M)_r, c(d\gamma_k(r))]_s e^k_{t-s}(r, y)\phi_k(y) \ drds \\ &-\sum_{k\in J} \int_0^t \int_M \chi_k(x)A_x(k_s(x, r) - e^k_s(x, r))[(\partial_M)_r, c(d\gamma_k(r))]_s e^k_{t-s}(r, y)\phi_k(y) \ drds \\ &-\sum_{k\in J} \int_0^t \int_M \chi_k(x)A_x e^k_s(x, r)[(\partial_M)_r, c(d\gamma_k(r))]_s e^k_{t-s}(r, y)\phi_k(y) \ drds \ . \end{aligned}$$

Using Lemma 3.1.8 and Lemma 3.1.9 we estimate the norms of the three terms in $E_x \otimes E_y$:

Since $\operatorname{supp} d\gamma_k \cap \operatorname{supp} \phi_k = \emptyset$ and $\operatorname{supp} d\gamma_k \cap \operatorname{supp}(1 - \chi_k) = \emptyset$, there is $C > 0$ such that for $x, y \in M$ and $t > 0$ the norm of the first term is bounded by

$$C \sum_{k\in J} t(1 - \chi_k(x))e^{-\frac{d(x,\operatorname{supp} d\gamma_k)^2}{(4+\delta)t}} e^{-\frac{d(y,\operatorname{supp} d\gamma_k)^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y) \ ,$$

and such that for $x, y \in M$ and $t > 0$ the norms of the second and third term are bounded by

$$Ct\chi_k(x)e^{-\frac{d(y,\operatorname{supp} d\gamma_k)^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y) \ .$$

When estimating the third term we used that the action of the integral kernel $e^k_s(x, r)\chi_k(r)$ is uniformly bounded for $k = \clubsuit$ on $C^n(\mathcal{U}_\clubsuit, E \otimes E_y)$ by Prop. 3.2.2 and for $k \in \mathbb{Z}/6$ on $C^n_R(\mathcal{U}_k, E \otimes E_y)$ by Prop. 3.4.1.

Analogous estimates hold for the derivatives in y with respect to unit vector fields on M.

Hence $A_x k_t(x, y) - A_x e_t(x, y)$ is bounded by Ct in $L^2(M \times M, E \boxtimes E^*)$ for $0 < t < T$ and some $C > 0$. By Cor. 5.2.4 the corresponding family of operators on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$

is bounded by Ct for $0 < t < T$, hence $A_x k_t(x,y)$ defines a family of bounded operators on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Write $S(t)$ for the integral operator induced by the integral kernel $k_t(x,y)$.

By property 4) in §3.5.3 there is $C > 0$ such that $\|AE_t\| \leq Ct^{-\frac{m}{2}}$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $0 < t < T$ and some $C > 0$, hence

$$(*) \qquad \|AS(t)\| \leq Ct^{-\frac{m}{2}} \ .$$

The fact that $S(t)$ extends to a bounded holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is an almost immediate consequence of $(*)$:

Since E_t converges strongly to the identity on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t \to 0$, the operator $S(t)$ also does. Furthermore the kernels k_t obey the semigroup law, hence $S(t)$ is a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Note that the range of $S(t) - E_t$ is a subset of $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Hence $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is invariant under the action of $S(t)$. It follows that $-D_s^2$ is the generator of $S(t)$.

From Prop. 5.4.3 and the estimate $(*)$ applied to $A = D_s^2$ it follows that the semigroup $S(t) = e^{-tD_s^2}$ extends to a holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

It remains to show that the holomorphic semigroup is bounded:

Let P_0 be the projection onto the kernel of D_s^2.

By Cor. 3.5.7 there is $c > 0$ such that $\{\operatorname{Re}\lambda \leq c\}$ is in the resolvent set of $D_s^2 + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Hence by Prop. 5.4.2 the holomorphic semigroup $e^{-t(D_s^2+P_0)}$ is bounded. Thus

$$e^{-tD_s^2} = e^{-t(D_s^2+P_0)}(1 - P_0) + P_0$$

is bounded as well. □

Recall that in §3.5.1 we fixed the domain of D_s^2 as an unbounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, but not the domain of D_s. This is done now:

Let D_s be the closure on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ of the Dirac operator ∂_M with domain $\mathcal{S}_R(M, E) \odot \hat{\Omega}_{\leq\mu}\mathcal{A}_i$.

Corollary 3.5.9. *Let P_0 be the projection onto the kernel of D_s^2.*

Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda^2 < 0$.

Then the operators $D_s + P_0$ and $D_s - \lambda$ have a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Proof. By Cor. 3.5.7 there is $c > 0$ such that $\{\operatorname{Re}\lambda \leq 2c\}$ is in the resolvent set of $D_s^2 + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

By the previous proposition and Prop. 5.4.2 it follows that there is $C > 0$ such that for all $t > 0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$

$$\|(D_s + P_0)e^{-t(D_s^2+P_0)}\| \leq C(1 + t^{-\frac{1}{2}})e^{-ct} \ .$$

Thus

$$G := \int_0^\infty (D_s + P_0)e^{-t(D_s^2+P_0)}dt$$

is a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Furthermore $G = (D_s + P_0)^{-1}$ on the Hilbert space $L^2(M, E)$. From Cor. 3.5.7 it follows that G acts as a bounded operator on $\mathcal{S}_R(M, E)$. Hence G inverts $D_s + P_0$ on $\mathcal{S}_R(M, E) \odot \hat{\Omega}_{\leq\mu}\mathcal{A}_i$, thus G inverts $D_s + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

The proof of the fact that

$$\int_0^\infty (D_s + \lambda)e^{-t(D_s^2-\lambda^2)}$$

inverts $(D_s - \lambda)$ for $\operatorname{Re}\lambda^2 < 0$ is analogous. □

From this corollary and from the analogue of Cor. 3.5.3 for D_s^2 we conclude that D_s^2 as it was defined in the beginning of §3.5.1 is the square of D_s.

3.5.5 The semigroup $e^{-tD(\rho)^2}$

This section is devoted to the study of the holomorphic semigroup generated by $-D(\rho)^2$.

First we prove its existence:

Let $D(\rho)$ be defined on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as the closure of the operator $\partial_M + \rho K$ with domain $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Let P_0 be the orthogonal projection onto the kernel of D_s^2. Then

$$D(\rho) = W^*(D_s + P_0)W + \left(c(dW^*)W + \rho K - W^*P_0W\right) .$$

The family $e^{-t(D_s+P_0)^2} = e^{-tD_s^2}(1 - P_0) + P_0$ is a holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by the previous section and $D_s + P_0$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by Cor. 3.5.9.

From Prop. 5.4.10 it follows that $D(\rho)^2$ as defined in §3.5.1 is the square of $D(\rho)$ and furthermore that $-D(\rho)^2$ generates a holomorphic semigroup which commutes with $D(\rho)$. This shows part of the following proposition.

For $\rho \neq 0$ let P be the orthogonal projection onto the kernel of $D(\rho)^2$.

Proposition 3.5.10. *1. Let $\rho \in \mathbb{R}$. The operator $-D(\rho)^2$ generates a holomorphic semigroup $e^{-tD(\rho)^2}$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. For every $t \geq 0$ the operator $e^{-tD(\rho)^2}$ depends analytically on ρ. For $\rho \neq 0$ the semigroup is bounded holomorphic.*

2. For every $T > 0$ there is $C > 0$ such that

$$\|e^{-tD(\rho)^2}\| \leq C$$

for all $\rho \in [-1, 1]$ and $0 \leq t \leq T$.

3. For $\rho \neq 0$ there are $C, \omega > 0$ such that for all $t \geq 0$

$$\|(1 - P)e^{-tD(\rho)^2}\| \leq Ce^{-\omega t} .$$

Proof. 1) The operator $e^{-tD(\rho)}$ depends analytically on ρ by Prop. 5.4.4.

Now let $\rho \neq 0$.

By Cor. 3.5.7 there is $c > 0$ such that $\{\operatorname{Re}\lambda \leq 2c\}$ is in the resolvent set of $D(\rho)^2 + P$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, hence by Lemma 5.4.2 there is $C > 0$ such that the holomorphic semigroup $e^{-t(D(\rho)^2+P)}$ is bounded by Ce^{-ct} for all $t \geq 0$, thus for $T > 0$ there is $C > 0$ such that for $t > T$

$$\|D(\rho)^2 e^{-tD(\rho)^2}\| = \|D(\rho)^2 e^{-t(D(\rho)^2+P)}\| \leq Ce^{-ct} .$$

Prop. 5.4.3 implies that the semigroup $e^{-tD(\rho)^2}$ is bounded holomorphic.

The equality $(1-P)e^{-tD(\rho)^2} = (1-P)e^{-t(D(\rho)^2+P)}$ implies 3).

2) follows from Prop. 5.4.4. □

The following proposition shows that $e^{-tD(\rho)^2}$ is smoothing:

Proposition 3.5.11. *1. Let $n \in \mathbb{N}_0$. For every $\rho \in \mathbb{R}$ and every $t > 0$ the operator $e^{-tD(\rho)^2}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C^n_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.*

2. Let $n \in 2\mathbb{N}$, $n \geq 4$. For every $\rho \neq 0$ the family $e^{-tD(\rho)^2} : C^n_{cR}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^{n-3}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is uniformly bounded.

3. Let $n \in 2\mathbb{N}$, $n \geq 4$. For every $T > 0$ the family $e^{-tD(\rho)^2} : C^n_{cR}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^{n-3}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is uniformly bounded in $0 \leq t < T$ and in $\rho \in [-1, 1]$.

Proof. 1) follows from

$$e^{-tD(\rho)^2} = (D(\rho)^2 + 1)^{-n}(D(\rho)^2 + 1)^n e^{-tD(\rho)^2}$$

since $(D(\rho)^2 + 1)^n e^{-tD(\rho)^2}$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t > 0$ by Cor. 5.4.9 and since $(D(\rho)^2 + 1)^{-n}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C^{2n-3}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $n \in \mathbb{N}$, $n \geq 2$ by Prop. 3.5.4.

2) and 3) follow by Prop. 3.5.4 from

$$e^{-tD(\rho)^2} = (D(\rho)^2 + 1)^{-n} e^{-tD(\rho)^2} (D(\rho)^2 + 1)^n .$$

□

3.5.6 The integral kernel

In order to prove that the operator $e^{-tD(\rho)^2}$ is an integral operator we use the method from §3.3.2: By Duhamel's principle we compare $e^{-tD(\rho)^2}$ and $E(\rho)_t$.

In the following $|\cdot|$ denotes the fibrewise norm of $(E \boxtimes E^*) \otimes \mathcal{A}_i$.

Proposition 3.5.12. *For every $\rho \in \mathbb{R}$ and every $t > 0$ the operator $e^{-tD(\rho)^2}$ is an integral operator with smooth integral kernel. Let $k(\rho)_t(x, y)$ be its integral kernel.*

1. The map $(0, \infty) \to C^\infty(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_\infty)$, $t \mapsto k(\rho)_t$ is smooth.

2. $k(\rho)_t(x, y) = k(\rho)_t(y, x)^*$.

3. For every $T > 0$ there are $c, C > 0$ such that

$$|k(\rho)_t(x,y) - e(\rho)_t(x,y)| \leq Ct \left(\sum_{k \in J} e^{-\frac{d(y, \operatorname{supp} d\gamma_k)^2}{ct}} 1_{\operatorname{supp} \phi_k}(y) + |\rho| 1_{\mathcal{U}_\clubsuit}(y) \right)$$

for all $0 < t < T$, $\rho \in [-1,1]$ and $x, y \in M$.

4. Let $\rho \neq 0$. There are $c, C > 0$ such that

$$|k(\rho)_t(x,y) - e(\rho)_t(x,y)| \leq Cte^{-\frac{d(y,\mathcal{U}_\clubsuit)^2}{ct}}$$

for all $t > 0$ and all $x, y \in M$.

Statements analogous to 3) and 4) hold for the partial derivatives in x and y with respect to unit vector fields on M.

Proof. In order to prove 1) it is enough to consider the operator $e^{-tD(\rho)^2} - E(\rho)_t$.
Let $f \in C^\infty_{cR}(M, E \otimes \mathcal{A}_i)$. Then by Duhamel's principle

$$\begin{aligned} & e^{-tD(\rho)^2} f - E(\rho)_t f \\ = \; & -\sum_{k \in J} \int_0^t \int_M e^{-sD(\rho)^2} \big(\frac{d}{dt} + D(\rho)^2\big) \gamma_k e(\rho)^k_{t-s}(\cdot, y) \phi_k(y) f(y) \; dyds \; . \end{aligned}$$

The map

$$\begin{aligned} & (0, \infty) \to C^\infty(M, C^\infty_{cR}(M, E \otimes \mathcal{A}_\infty) \otimes E) \; , \\ & \tau \mapsto \big(y \mapsto \big(\frac{d}{dt} + D(\rho)^2\big) e(\rho)^k_\tau(\cdot, y) \phi_k(y)\big) \end{aligned}$$

is smooth.

We can write

$$\begin{aligned} & \big(\frac{d}{dt} + D(\rho)^2\big) \gamma_k e(\rho)^k_\tau(\cdot, y) \phi_k(y) \\ = \; & -\sum_{k \in J} [\partial_M, c(d\gamma_k)]_s e(0)^k_\tau(\cdot, y) \phi_k(y) \\ & -\rho\big(\big[[\partial_M, K]_s + \rho K^2, \gamma_\clubsuit\big]_s e(\rho)^\clubsuit_\tau(\cdot, y) \phi_\clubsuit(y) \\ & -[\partial_M, c(d\gamma_\clubsuit)]_s \big(e(\rho)^\clubsuit_\tau(\cdot, y) - e(0)^\clubsuit_{t-s}(\cdot, y)\big) \phi_\clubsuit(y) \; . \end{aligned}$$

We estimate the three terms on the right hand side:

From the estimates in Lemma 3.1.8 and Lemma 3.4.4 it follows that for any $m \in \mathbb{N}_0$ there are $C, c > 0$ such that

$$\|[\partial_M, c(d\gamma_k)]_s e(0)^k_\tau(\cdot, y) \phi_k(y)\|_{C^m_R} \leq Ce^{-\frac{d(y, \operatorname{supp} d\gamma_k)^2}{c\tau}} 1_{\operatorname{supp} \phi_k}(y)$$

in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E^*_y$ for all $k \in J$, $0 < \tau < T$, $\rho \in [-1,1]$ and $y \in M$.

Since $\big[[\partial_M, K]_s + \rho K^2, \gamma_\clubsuit\big]_s$ is a finite Hilbert-Schmidt operator with smooth compactly supported integral kernel, for any $m \in \mathbb{N}_0$ there is $C > 0$ such that

$$\|\rho\big[[\partial_M, K]_s + \rho K^2, \gamma_\clubsuit\big]_s e(\rho)^\clubsuit_\tau(\cdot, y) \phi_\clubsuit(y) \; \|_{C^m_R} \leq C|\rho| 1_{\mathcal{U}_\clubsuit}(y)$$

for $0 < \tau < T,\ \rho \in [-1,1]$ and $y \in M$ in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E^*_y$.

Furthermore by §3.5.3, property 5), there is $C > 0$ such that

$$\|[\partial_M, c(d\gamma_{\clubsuit})]_s \big(e(\rho)^{\clubsuit}_{\tau}(\cdot, y) - e(0)^{\clubsuit}_{\tau}(\cdot, y)\big)\phi_{\clubsuit}(y)\|_{C^m_R} \leq C|\rho| 1_{\mathcal{U}_{\clubsuit}}(y)$$

in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E_y$ for $0 < \tau < T,\ \rho \in [-1,1]$ and $y \in M$.

Analogous estimates hold for the derivatives in y and also in τ since by the heat equation the derivatives with respect to τ can be expressed in terms of the derivatives with respect to x.

From these estimates and from Prop. 3.5.11 it follows that we can interchange the order of integration. Hence $e^{-tD(\rho)^2} - E(\rho)_t$ is an integral operator. Furthermore it follows that for its integral kernel a statement analogous to 1) holds. This shows 1).

The property $k(\rho)_t(x,y) = k(\rho)_t(y,x)^*$ follows from the selfadjointness of $e^{-tD(\rho)^2}$.

Statement 2) and 3) follow from the estimates and from Prop. 3.5.11. For the proof of 3) we also take the fact into account that for fixed $\rho \neq 0$ the kernels $e(\rho)^{\clubsuit}_t$ and $e^{\clubsuit}_t$ and their derivatives are uniformly bounded in t with $t > T$. □

Corollary 3.5.13. *For every $\nu \in \mathbb{N}_0$ and $T > 0$ there is $C > 0$ such that*

$$\|D(\rho)^{\nu} e^{-tD(\rho)^2}\| \leq C t^{-\frac{\nu}{2}}$$

for $0 < t < T$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$.

Proof. By Duhamel's principle, for $f \in C^{\infty}_{Rc}(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$,

$$\begin{aligned} & D(\rho)^{\nu} e^{-tD(\rho)^2} f - D(\rho)^{\nu} E(\rho)_t f \\ = \ & -\sum_{k \in J} \int_0^t \int_M e^{-sD(\rho)^2} D(\rho)^{\nu} [D(\rho)^2, \gamma_k]_s e(\rho)^k_{t-s}(\cdot, y)\phi_k(y) f(y) dy ds \ . \end{aligned}$$

There is $C > 0$ such that this term is bounded in $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ by some Ct for $0 < t < T$.

By Prop. 3.2.2 and Prop. 3.4.1 there is $C > 0$ such that on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$

$$\|D(\rho)^{\nu} E(\rho)_t\| \leq C t^{-\frac{\nu}{2}} \ .$$

The assertion follows. □

Corollary 3.5.14. *For every $\rho \neq 0$ and $m \in \mathbb{N}$ the family of integral kernels $k(\rho)_t$ defines a strongly continuous semigroup on $C^m_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ bounded by $C(1+t)^{\frac{3}{2}}$ for some $C > 0$ and all $t > 0$.*

It is denoted by $e^{-tD(\rho)^2}$ as well.

Proof. By the estimates in the proposition the integral kernel $k(\rho)_t - e(\rho)_t$ defines an operator on $C^m_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ bounded by $Ct^{3/2}$ for every $t > 0$ and some $C > 0$.

For $k \in \mathbb{Z}/6$ the action of the integral kernel $e(\rho)^k_t$ on $C^m_R(\mathcal{U}_k, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ is strongly continuous and is uniformly bounded by Prop. 3.4.1.

By Cor. 3.2.3 and Prop. 5.4.4 the family $e^{-t(D_N+\rho K)^2}$ is a strongly continuous semigroup on $C^m(N, E_N \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. It is bounded since its integral kernel is uniformly bounded for $t > 1$. Hence the action of $e(\rho)_t^{\clubsuit}$ on $C^m(\mathcal{U}_{\clubsuit}, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is strongly continuous and is uniformly bounded

Since furthermore the operator induced by $e(\rho)_t$ converges strongly to the identity for $t \to 0$, the operator induced by $k(\rho)_t$ also does. It satisfies the semigroup property, hence it is a strongly continuous semigroup. □

Corollary 3.5.15. *Let $\rho \neq 0$ and $n \in \mathbb{N}$.*

For every $\varepsilon > 0$ we can find $C, c > 0$ such that for $x, y \in M$ with $d(x, y) > \varepsilon$ and $t > 0$

$$|k(\rho)_t(x, y)| \leq C\big(e^{-\frac{d(x,y)^2}{ct}} + te^{-\frac{d(y,\mathcal{U}_{\clubsuit})^2}{ct}}\big) .$$

An analogous statement holds for the derivatives in x and y with respect to unit vector fields on M.

In the next corollary we use the notion of a Hilbert-Schmidt operator and the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ that are defined in §5.2.3.

Corollary 3.5.16. *Let $\rho \neq 0$.*

For every $r > 0$ and $\nu \in \mathbb{N}_0$ the operators $1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}$ and $D(\rho)^\nu e^{-tD(\rho)^2}1_{M_r}$ are Hilbert-Schmidt operators.

1. *For every $n \in \mathbb{N}_0$ and $T > 0$ there is $C > 0$ such that for any $r > 0$ and $t > T$*

$$\|1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}\|_{HS} \leq C(1 + r)$$

and

$$\|D(\rho)^\nu e^{-tD(\rho)^2}1_{M_r}\|_{HS} \leq C(1 + r) .$$

2. *For every $n \in \mathbb{N}_0$ and $\varepsilon > 0$ there is $C > 0$ such that for every $r, t > 0$*

$$\|1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}(1 - 1_{M_{r+\varepsilon}})\|_{HS} \leq C(1 + r)t^{1/2}$$

and

$$\|(1 - 1_{M_{r+\varepsilon}})D(\rho)^\nu e^{-tD(\rho)^2}1_{M_r}\|_{HS} \leq C(1 + r)t^{1/2} .$$

Proof. 1) Since by Prop. 3.5.10 the semigroup $e^{-tD(\rho)^2}$ is bounded, it follows from Prop. 5.2.12 that there is $C > 0$ such that for all $r > 0$ and $t > T$

$$\|1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}\|_{HS} \leq C\|1_{M_r}D(\rho)^\nu e^{-TD(\rho)^2}\|_{HS} .$$

By the previous corollary there are $C, c > 0$ such that

$$|1_{M_r}(x)D(\rho)^\nu_x k(\rho)_T(x, y)| \leq C1_{M_r}(x)(e^{-cd(x,y)^2} + e^{-cd(y,M_r)^2})$$

for all $r > 0$ and $x, y \in M$. This yields the asserted estimate. The second estimate in 1) is proven analogously.

2) follows from the previous corollary and 1). □