Chapter 4

The superconnection and the index theorem

The notion of a superconnection on a free finitely generated $\mathbb{Z}/2$-graded module which we define now generalizes the notion of a connection on a free module [Kar].

In the family case superconnections usually act on infinite dimensional bundles ([BGV], ch. 9). In analogy the superconnections we consider act on modules with infinitely many generators. The following definition should be merely seen as a motivation for the definitions of the superconnections to come.

Definition 4.0.17. *Let $\mathcal{B}$ be a locally m-convex Fréchet algebra and let $p, q \in \mathbb{N}_0$.*

Let $V := (\mathbb{C}^+)^p \oplus (\mathbb{C}^-)^q$. Consider $V \otimes \hat{\Omega}_* \mathcal{B}$ as a $\mathbb{Z}/2$-graded space.

A SUPERCONNECTION ON $V \otimes \mathcal{B}$ *is an odd linear map*

$$A : V \otimes \hat{\Omega}_* \mathcal{B} \to V \otimes \hat{\Omega}_* \mathcal{B}$$

satisfying Leibniz rule:

For $\alpha \in V^{\pm} \otimes \hat{\Omega}_k \mathcal{B}$ and $\beta \in \hat{\Omega}_* \mathcal{B}$

$$A(\alpha\beta) = A(\alpha)\beta + (-1)^{\deg \alpha} \alpha \, \mathrm{d}\, \beta$$

where $\deg \alpha$ is the degree of α with respect to the $\mathbb{Z}/2$-graduation of $V \otimes \hat{\Omega}_* \mathcal{B}$.

The map A^2 is called the CURVATURE OF A.

As for a connection [Kar] it follows that the curvature of a superconnection is a right $\hat{\Omega}_* \mathcal{B}$-module map.

4.1 The superconnection A_t^I associated to D_I

4.1.1 The family $e^{-(A_t^I)^2}$

Let C_1 be the $\mathbb{Z}/2$-graded unital algebra generated by an odd element σ with $\sigma^2 = 1$. As a vector space it is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ via the map $C_1 \to \mathbb{C} \oplus \mathbb{C}, a + b\sigma \mapsto (a, b)$. We endow C_1 with the scalar product induced by the standard hermitian scalar product on $\mathbb{C} \oplus \mathbb{C}$.

We identify $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ with $C_1 \otimes L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

Let (P_0, P_1) be a pair of transverse Lagrangian projections with $P_i \in M_{2d}(\mathcal{A}_\infty),\ i = 0, 1$ and let D_I be the associated unbounded operator on $L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ from §3.3.1.

We define a superconnection A_I associated to the odd operator σD_I on the $\mathbb{Z}/2$-graded $\mathcal{A}_i$-module $L^2([0,1], C_1 \otimes \mathcal{A}_i^{2d})$:

Let $U \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ be as in Prop. 2.2.1 with $U(0)P_0U(0)^* = P_s$ and $U(1)P_1U(1)^* = 1 - P_s$. The differential $U^* \,\mathrm{d}\, U$ can be considered as a trivial superconnection on $L^2([0,1], C_1 \otimes \mathcal{A}_i^{2d})$.

We define

$$A_I := U^* \,\mathrm{d}\, U + \sigma D_I$$

and for $t \geq 0$

$$A_t^I := U^* \,\mathrm{d}\, U + \sqrt{t}\sigma D_I \ .$$

Since A_I is an odd map on $C_1 \otimes C_R^\infty([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ fulfilling Leibniz rule, it is called a superconnection, and A_t^I is called the corresponding rescaled superconnection.

The curvature of A_I is

$$A_I^2 = U^* \,\mathrm{d}^2\, U + U^* \,\mathrm{d}\, U\sigma D_I + \sigma D_I U^* \,\mathrm{d}\, U + D_I^2 = D_I^2 + \sigma[D_I, U^* \,\mathrm{d}\, U] \ .$$

From

$$\begin{aligned} [D_I, U^* \,\mathrm{d}\, U] &= -U^*[\mathrm{d}, UD_IU^*]U \\ &= -U^*([\mathrm{d}, D_{I_s}] + [\mathrm{d}, UI_0(\partial U^*)])U \\ &= -U^* \,\mathrm{d}(UI_0(\partial U^*))U =: R \end{aligned}$$

it follows that $A_I^2 = D_I^2 + \sigma R$ with $R \in C^\infty([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_\infty))$ vanishing near the boundary and fulfilling $R^* = R$.

The curvature of the rescaled superconnection A_t^I is

$$(A_t^I)^2 = tD_I^2 + \sqrt{t}\sigma R \ .$$

We see that the curvature and the rescaled curvature are right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-module maps.

Since A_I^2 is a bounded perturbation of D_I^2, it defines a holomorphic semigroup $e^{-tA_I^2}$ on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

In the following we restrict to $t \geq 0$:

By Volterra development

$$\begin{aligned} e^{-tA_I^2} &= \sum_{n=0}^{\infty}(-1)^n t^n \int_{\Delta^n} e^{-u_0 tD_I^2}\sigma R e^{-u_1 tD_I^2}\sigma R \dots e^{-u_n tD_I^2}\ du_0 \dots du_n \\ &= \sum_{n=0}^{\infty}\sigma^n(-1)^{\frac{(n+1)n}{2}} t^n I_n(t) \end{aligned}$$

with

$$I_n(t) := \int_{\Delta^n} e^{-u_0 tD_I^2} R e^{-u_1 tD_I^2} R \dots e^{-u_n tD_I^2}\ du_0 \dots du_n\ .$$

The series is finite on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

It follows that

$$e^{-(A_t^I)^2} = \sum_{n=0}^{\infty}\sigma^n(-1)^{\frac{(n+1)n}{2}} t^{n/2} I_n(t)\ .$$

The operators $I_n(t)$ obey the following recursive relation for $n \geq 1$:

$$\begin{aligned} I_n(t) &= \int_0^1 du_0\ e^{-u_0 tD_I^2} R \int_{(1-u_0)\Delta^{n-1}} e^{-u_1 tD_I^2} R \dots e^{-u_n tD_I^2}\ du_1 \dots du_n \\ &= \int_0^1 du_0\ (1-u_0)^{n-1} e^{-u_0 tD_I^2} R \int_{\Delta^{n-1}} e^{-(1-u_0)u_1 tD_I^2} R \dots e^{-(1-u_0)u_n tD_I^2}\ du_1 \dots du_n \\ &= \int_0^1 du_0\ (1-u_0)^{n-1} e^{-u_0 tD_I^2} R I_{n-1}((1-u_0)t)\ . \end{aligned}$$

Note that $e^{-(A_t^I)^2}$ and $I_n(t)$ are selfadjoint on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ in the sense of §5.2.3.

4.1.2 The integral kernel of $e^{-(A_t^I)^2}$

Since $e^{-tD_I^2}$ is a bounded semigroup on $C_R^m([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for every $m \in \mathbb{N}_0$ by Prop. 3.3.3, the family $I_n(t) : C_R^m([0,1], \mathcal{A}_i^{2d}) \to C_R^m([0,1], (\hat{\Omega}_n\mathcal{A}_i)^{2d})$ is uniformly bounded in $t \geq 0$.

In the following we write $|\cdot|$ for the norm on $M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i),\ \mu, i \in \mathbb{N}_0$.

Proposition 4.1.1. *For every $n \in \mathbb{N}_0$ and $t > 0$ the operator $I_n(t)$ is an integral operator. Let $p_t(x,y)^n$ be its integral kernel.*

1. *The map*

$$(0,\infty) \to C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_\infty))),\ t \mapsto \big(y \mapsto p_t(\cdot, y)^n\big)\ ,$$

   is smooth.

2. $p_t(x,y)^n = (p_t(y,x)^n)^*$.

3. *For every $l, m, n \in \mathbb{N}_0$ there is $C, \omega > 0$ such that for $t > 0$ and $x, y \in [0,1]$*

$$|\partial_x^l \partial_y^m p_t(x,y)^n| \leq C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}\ .$$

Proof. We prove the claim by induction on n. In degree $n = 0$ the assertions hold by Prop. 3.3.4 and Cor. 3.3.6.

Let $k_t(x, y)$ be the integral kernel of $e^{-tD_I^2}$.

For $f \in C_R^\infty([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$

$$(I_n(t)f)(x) = \int_0^1 \int_{[0,1]} \int_{[0,1]} (1-s)^{n-1} k_{st}(x,r) R(r) p_{(1-s)t}(r,y)^{n-1} f(y)\, dydrds$$

by induction and by the recursion formula above.

Since for fixed s the integrand is uniformly bounded in r and y, we can interchange the integration over r and y.

For the proof of the existence of the integral kernel and of 1) it suffices to show that the map

$$(s,t) \mapsto \big(y \mapsto \int_{[0,1]} (1-s)^{n-1} k_{st}(\cdot, r) R(r) p_{(1-s)t}(r,y)^{n-1} dr\big)$$

is a smooth map from $[0,1] \times (0,\infty)$ to $C^\infty([0,1], C_R^\infty[0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i)))$:

For $s \geq \frac{1}{2}$ this follows from the fact that by induction the map

$$(s,t) \mapsto \big(y \mapsto Rp_{(1-s)t}(\cdot, y)^{n-1}\big)$$

is a smooth map from $[\frac{1}{2}, 1] \times (0,\infty)$ to $C^\infty\big([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))\big)$. Furthermore the family $e^{-stD_I^2}$ is uniformly bounded on $C_R^l([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))$ for any $l \in \mathbb{N}_0$ by Prop. 3.3.3 and depends smoothly on s, t.

For $s \leq \frac{1}{2}$ the map is smooth since

$$(s,t) \mapsto \big(x \mapsto Rk_{st}(x,\cdot)^*\big)$$

is a smooth map with values in $C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_i)))$ and since the action of the family $I_{n-1}((1-s)t)$ on $C_R^m([0,1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)))$ depends smoothly on s, t and is uniformly bounded for any $m \in \mathbb{N}_0$.

Assertion 2) holds since $I_n(t)$ is selfadjoint in the sense of §5.2.3.

The preceding arguments and the following facts imply the estimate in 3):

By induction there is $C > 0$ such that the norm of $\big(y \mapsto Rp_{(1-s)t}(\cdot, y)^{n-1}\big)$ in $C^m\big([0,1], C_R^l([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))\big)$ is bounded by $C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}$ for $0 \leq s \leq \frac{1}{2},\ t > 0$. On the other hand by Cor. 3.3.6 the norm of $\big(x \mapsto Rk_{st}(x,\cdot)^*\big)$ is bounded in $C^l([0,1], C_R^m([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_i))$ by $C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}$ for $s > \frac{1}{2},\ t > 0$. □

The proof of the previous proposition did not use the fact that $D_I^2 + \sigma R$ is the curvature of a superconnection. Hence an analogous argument shows that $e^{-t(D_{I_s}^2 + \sigma R)}$ is an integral operator whose integral kernel can be written as

$$\sum_{n=0}^{\infty} (-1)^{\frac{(n+1)n}{2}} \sigma^n t^n p_t^s(x,y)^n$$

with $\big(y \mapsto p_t^s(\cdot, y)^n\big) \in C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))$ for $R = (P_s, 1 - P_s)$.

Lemma 4.1.2. *For every $l, m, n \in \mathbb{N}_0$ and $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$*

$$|\partial_x^l \partial_y^m p_t^s(x,y)^n| \leq C e^{-\frac{d(x,y)^2}{(4+\delta)t}}.$$

Proof. We prove the assertion by induction on n.

In degree $n = 0$ it holds by Cor. 3.1.2.

Let $\eta = \varepsilon/4$.

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function with $\chi(x) = 0$ for $x > \eta$ and $\chi(x) = 1$ for $x \leq \eta/2$.

Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$.

For $l, m \in \mathbb{N}_0$

$$\partial_x^l \partial_y^m p_t^s(x,y)^n = \int_0^1 \int_{[0,1]} (1-s)^{n-1} \partial_x^l k_{st}(x,r) R(r) \chi(d(x,r)) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1} dr ds$$

$$+ \int_0^1 \int_{[0,1]} (1-s)^{n-1} \partial_x^l k_{st}(x,r) R(r) \big(1 - \chi(d(x,r))\big) \chi(d(r,y)) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1} dr ds$$

$$+ \int_0^1 \int_{[0,1]} (1-s)^{n-1} \partial_x^l k_{st}(x,r) R(r) \big(1 - \chi(d(x,r))\big) \big(1 - \chi(d(r,y))\big) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1} dr ds \,.$$

We begin by estimating the first term on the right hand side: By induction there is $C > 0$ such that for $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and $0 < s < 1$ and $t > 0$

$$\| R \chi(d(\cdot, x)) \partial_y^m p_{(1-s)t}^s(\cdot, y)^{n-1} \|_{C_R^l} \leq C e^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}}$$

in $C_R^l([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$.

Since the operator $e^{-stD_{I_s}^2}$ is uniformly bounded on $C_R^l([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$, the first term is bounded by $C e^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}}$.

An analogous bound exists for the second term: By Cor. 3.1.2 there is $C > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and $0 < s < 1$ and $t > 0$ in $C_R^m([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$

$$\| \big(\partial_x^l k_{st}(x, \cdot) R\big)^* \big(1 - \chi(d(\cdot, x))\big) \chi(d(\cdot, y)) \|_{C_R^m} \leq C e^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}} \,.$$

Furthermore the integral kernel $(y, r) \mapsto (\partial_y^m p_{(1-s)t}^s(r,y)^{n-1})^*$ induces a uniformly bounded family of operators from $C_R^m([0,1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ to $C_R([0,1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$.

The third term is bounded by $C e^{-\frac{d(x,y)^2}{(4+\delta)t}}$ since by Cor. 3.1.2

$$|\partial_x^l k_{st}(x,r) R(r) \big(1 - \chi(d(x,r))\big)| \leq C e^{-\frac{d(x,r)^2}{(4+\delta)st}}$$

and by induction

$$|\big(1 - \chi(d(x,r))\big) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1}| \leq C e^{-\frac{d(r,y)^2}{(4+\delta)(1-s)t}}$$

for all $x, y, r \in [0,1]$, all $0 < s < 1$ and $t > 0$.

Hence there is $C > 0$ such that for all $x, y \in [0,1]$ and all $t > 0$

$$|\partial_x^l \partial_y^m p_t^s(x,y)^n| \leq C e^{-\frac{(d(x,y)-\varepsilon/2)^2}{(4+\delta)t}} \ .$$

The assertion follows now from Lemma 3.1.6.

□

As usual we apply Duhamel's principle in order to obtain an estimate for the kernel $p_t(x,y)^n$:

Recall the definitions of $\phi_k, \gamma_k,\ k = 0,1$, in §3.3.1.

Let $U_k \in M_{2d}(\mathcal{A}_\infty),\ k = 0,1$, be a unitary with $UI_0 = I_0 U$ and $U_k P_k U_k^* = P_s$ (for its existence see Lemma 1.4.3).

Let $W_n^k(t)$ be the integral operator with integral kernel

$$w_t^k(x,y)^n := U_k^* p_t^s(x,y)^n U_k$$

and denote by $W_n(t)$ the integral operator corresponding to

$$w_t(x,y)^n := \gamma_0(x) w_t^0(x,y)^n \phi_0(y) + \gamma_1(x) w_t^1(x,y)^n \phi_1(y) \ .$$

Let $W_0(0) := 1$ and $W_n(0) := 0$ for $n \geq 1$. Then $W_n(t)$ is a strongly continuous family of operators on $L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ for all $n \in \mathbb{N}_0$. For $f \in C_R^\infty([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ the map $[0,\infty) \to L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d}),\ t \mapsto W_n(t) f \in L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ is even smooth.

Furthermore for $t > 0$ the range of $W_n(t)$ is in $C_R^\infty([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$.

Hence Duhamel's principle yields for $f \in C_R^\infty([0,1], \mathcal{A}_i^{2d})$:

$$\begin{aligned}
&\big(e^{-tA_I^2} - \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} t^n W_n(t)\big) f \\
&= -\int_0^t e^{-sA_I^2} \big(\frac{d}{dt} + A_I^2\big) \sum_{n=0}^\infty (t-s)^n W_n(t-s) f \ ds \\
&= \int_0^t e^{-sA_I^2} \sum_{k=0,1} [\gamma_k, D_I^2]_s \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} (t-s)^n W_n^k(t-s) \phi_k f \ ds \\
&= \sum_{n=0}^\infty \int_0^t \sigma^n (-1)^{\frac{n(n+1)}{2}} \sum_{j=0}^n s^{n-j} (t-s)^j I_{n-j}(s) \sum_{k=0,1} (\gamma_k' \partial + \partial \gamma_k') W_j^k(t-s) \phi_k f \ ds \ .
\end{aligned}$$

It follows that

$$\big(I_n(t) - W_n(t)\big) f = t^{-n} \int_0^t \sum_{j=0}^n s^{n-j} (t-s)^j I_{n-j}(s) \sum_{k=0,1} (\gamma_k' \partial + \partial \gamma_k') W_j^k(t-s) \phi_k f \ ds \ .$$

Proposition 4.1.3. *For every $l, m, n \in \mathbb{N}_0$ and every $\delta > 0$ there is $C > 0$ such that*

$$|\partial_x^l \partial_y^m p_t(x,y)^n - \partial_x^l \partial_y^m w_t(x,y)^n| \leq Ct \sum_{k=0,1} e^{-\frac{d(y, \operatorname{supp} \gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp} \phi_k}(y)$$

for all $t > 0$ and all $x, y \in [0,1]$.

Proof. In $C^l_R([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))$

$$\begin{aligned}
&\|\partial_y^m p_t(\cdot,y)^n - \partial_y^m w_t(\cdot,y)^n\|_{C^l_R} \\
&= \|t^{-n}\int_0^t \sum_{j=0}^n s^{n-j}(t-s)^j I_{n-j}(s) \sum_{k=0,1}(\gamma_k'\partial + \partial\gamma_k')\partial_y^m\big(w^k_{t-s}(\cdot,y)^j\phi_k(y)\big)ds\|_{C^l_R} \\
&\leq \int_0^t \sum_{j=0}^n \|I_{n-j}(s) \sum_{k=0,1}(\gamma_k'\partial + \partial\gamma_k')\partial_y^m\big(w^k_{t-s}(\cdot,y)^j\phi_k(y)\big)\|_{C^l_R} ds\ .
\end{aligned}$$

From the previous lemma it follows that for every $j \in \mathbb{N}_0$ and $k = 0, 1$ there is $C > 0$ such that

$$\|(\gamma_k'\partial + \partial\gamma_k')\partial_y^m\big(w^k_{t-s}(\cdot,y)^j\phi_k(y)\big)\|_{C^l_R} \leq Ce^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $y \in [0,1]$, all $t > 0$ and $0 < s < t$.

The assertion follows from the fact that $I_{n-j}(s)$ is a uniformly bounded family of operators on $C^l_R([0,1], M_{2d}(\hat{\Omega}_{\leq n}\mathcal{A}_i))$.

□

Corollary 4.1.4. *For every $l, m, n \in \mathbb{N}_0$ and $\varepsilon > 0$ there are $c, C > 0$ such that*

$$|\partial_x^l\partial_y^m p_t(x,y)^n| \leq Ce^{-\frac{d(x,y)^2}{ct}}$$

for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$.

Proof. Note that the assertion is equivalent to the assertion that for every $l, m, n \in \mathbb{N}_0$ and $\varepsilon > 0$ there are $c, C > 0$ such that

$$|\partial_x^l\partial_y^m p_t(x,y)^n| \leq Ce^{-\frac{1}{ct}}$$

for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$. The estimate follows for $0 < t < 1$ from the previous proposition and Lemma 4.1.2 since $\operatorname{supp}\gamma_k' \cap \operatorname{supp}\phi_k = \emptyset$. For $t > 1$ it holds by the estimate in Prop. 4.1.1. □

4.1.3 The η-form

In the following the integral kernel of $D_I I_n(t)$ is denoted by $(D_I p_t)(x,y)^n$.

Lemma 4.1.5. *For every $n \in \mathbb{N}_0$ the integral*

$$\int_0^\infty t^{\frac{n-1}{2}} \int_{[0,1]} \operatorname{tr}(D_I p_t)(x,x)^n dxdt$$

is well-defined in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$.

Proof. The integral converges in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$ for all $n, \mu, i \in \mathbb{N}_0$: For $n = 0$ and $t \to 0$ the convergence follows from Cor. 3.3.8; for $n > 0$ and $t \to 0$ and for $n \in \mathbb{N}_0$ and $t \to \infty$ the convergence follows from Prop. 4.1.1. By Prop. 1.3.6 the integral converges in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$ as well. □

Let $\mathrm{tr}_\sigma(a+\sigma b) := \mathrm{tr}a$ for $a, b \in M_{2d}(\hat{\Omega}_*\mathcal{A}_i)$ and let Tr_σ be the corresponding trace on integral operators.

Definition 4.1.6. *The η-form of the superconnection A_I is*

$$\eta(A_I) := -\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \ \in \hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s} \ .$$

Since

$$\eta(A_I) = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty (-1)^n \int_0^\infty t^{n-\frac{1}{2}} \int_{[0,1]} \mathrm{tr}(D_I p_t)(x,x)^{2n} \ dxdt$$

the η-form is well-defined by the previous lemma.

For $\varepsilon \in]0, \frac{1}{2}]$ let $A_I(\varepsilon) := U_\varepsilon^* \,\mathrm{d}\, U_\varepsilon + \sigma D_I$ be a superconnection as above with

$$\mathrm{supp}(U_\varepsilon - 1) \subset [\varepsilon, 1-\varepsilon] \ .$$

Proposition 4.1.7. *The limit $\lim_{\varepsilon\to 0} \eta(A_I(\varepsilon))$ exists and does not depend on the choice of U_ε.*

Proof. Let $R_\varepsilon := [D_I, U_\varepsilon^* \,\mathrm{d}\, U_\varepsilon]$. It suffices to prove that for $y, z \in [0,1]$ and $s, t > 0$ the limit

$$\lim_{\varepsilon\to 0} \int_{[0,1]} k_s(y,x) R_\varepsilon(x) k_t(x,z) \ dx$$

exists and does not depend on the choice of U_ε.

Let $f(x) := k_s(y,x)$ and $g(x) := k_t(x,z)$.

Then

$$\begin{aligned}
&\int_{[0,1]} f(x) R_\varepsilon(x) g(x) \ dx \\
&= \int_{[0,1]} f(x) [I_0 \partial_x, U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x)] g(x) \ dx \\
&= \int_{[0,1]} \partial_x \big(f(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g(x) \big) \ dx \\
&\quad - \int_{[0,1]} f'(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g(x) \ dx - \int_{[0,1]} f(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g'(x) \ dx \\
&= f(1) I_0 U_\varepsilon^*(1) \,\mathrm{d}\, U_\varepsilon(1) g(1) - f(0) I_0 U_\varepsilon^*(0) \,\mathrm{d}\, U_\varepsilon(0) g(0) \\
&\quad - \int_{[0,1]} f'(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g(x) \ dx - \int_{[0,1]} f(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g'(x) \ dx \\
&= - \int_{[0,1]} f'(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g(x) \ dx - \int_{[0,1]} f(x) I_0 U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g'(x) \ dx \ .
\end{aligned}$$

Since for $\varepsilon \to 0$ the term $U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g(x)$ resp. $U_\varepsilon^*(x) \,\mathrm{d}\, U_\varepsilon(x) g'(x)$ converges to $\mathrm{d}\, g(x)$ resp. $\mathrm{d}\, g'(x)$, it follows that

$$\lim_{\varepsilon\to 0} \int_{[0,1]} f(x) R_\varepsilon(x) g(x) \ dx = f(0) I_0 \,\mathrm{d}\, g(0) - f(1) I_0 \,\mathrm{d}\, g(1) \ .$$

□

Definition 4.1.8. *We call*

$$\eta(P_0, P_1) := \lim_{\varepsilon \to 0} \eta(A_I(\varepsilon))$$

the η-FORM ASSOCIATED TO THE PAIR (P_0, P_1).

4.2 The superconnection associated to D_Z

Let (P_0, P_1) be a pair of transverse Lagrangian projections with $P_i \in M_{2d}(\mathcal{A}_\infty)$, $i = 1, 2$ and let D_Z be the asssociated operator on $L^2(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$ that was defined in §3.4.

In this section we define a superconnection associated to D_Z.

Let $U \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ be as in Prop. 2.2.1 with $U(0)P_0U(0)^* = P_s$ and $U(1)P_1U(1)^* = 1 - P_s$. We consider U as a function on Z depending only on the variable x_2 and set

$$\tilde{W} := U \oplus U \in C^\infty(Z, M_{2d}(\mathcal{A}_\infty)) \ .$$

We call $A_Z := \tilde{W}^* \, \mathrm{d} \, \tilde{W} + D_Z$ a superconnection associated to D_Z and $A_t^Z := \tilde{W}^* \, \mathrm{d} \, \tilde{W} + \sqrt{t} D_Z$ the corresponding rescaled superconnection.

Its curvature is

$$\begin{aligned} A_Z^2 &= D_Z^2 + [\tilde{W}^* \, \mathrm{d} \, \tilde{W}, D_Z]_s \\ &= D_Z^2 + \tilde{W}^* [\mathrm{d}, \tilde{W} c(dx_2) \partial_{x_2} \tilde{W}^*]_s \tilde{W} \\ &= D_Z^2 - c(dx_1) \tilde{W}^* [\mathrm{d}, \tilde{W} I \partial_{x_2} \tilde{W}^*]_s \tilde{W} \\ &= D_Z^2 - c(dx_1) \tilde{W}^* \, \mathrm{d}(\tilde{W} I \partial_{x_2} \tilde{W}^*) \tilde{W} \\ &= D_Z^2 + c(dx_1)(R \oplus (-R)) \end{aligned}$$

with $R = -U^* \, \mathrm{d}(U I_0(\partial U^*)) U$.

Let $\tilde{R} = c(dx_1)(R \oplus (-R))$. Then $\tilde{R} \in C^\infty(Z, M_{4d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ and $\tilde{R}^* = -\tilde{R}$.

The negative of the curvature generates a holomorphic semigroup $e^{-tA_Z^2}$ on $L^2(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$.

Let D_I be the closed operator on $L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ associated to (P_0, P_1). From §3.4 recall that $e^{-tD_Z^2} = e^{-t\tilde{D}_I^2} e^{-t\Delta_{\mathbb{R}}}$.

By Prop. 4.1.1 the operator

$$I_n(t) = \int_{\Delta^n} e^{-u_0 t D_I^2} R e^{-u_1 t D_I^2} R \ldots e^{-u_n t D_I^2} \, du_0 \ldots du_n$$

is an integral operator. Its integral kernel is denoted by $p_t^I(x, y)^n$.

Since $e^{-t\Delta_{\mathbb{R}}}$ commutes with $\tilde{R}$ and $c(dx_1)$ commutes with $e^{-tD_Z^2}$ and $R \oplus (-R)$, we obtain from Volterra development:

$$\begin{aligned} e^{-tA_Z^2} &= \sum_{n=0}^{\infty} (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D_Z^2} \tilde{R} e^{-u_1 t D_Z^2} \tilde{R} \ldots e^{-u_n t D_Z^2} \, du_0 \ldots du_n \\ &= \sum_{n=0}^{\infty} (-1)^n t^n e^{-t\Delta_{\mathbb{R}}} \int_{\Delta^n} e^{-u_0 t \tilde{D}_I^2} \tilde{R} e^{-u_1 t \tilde{D}_I^2} \tilde{R} \ldots e^{-u_n t \tilde{D}_I^2} \, du_0 \ldots du_n \\ &= \sum_{n=0}^{\infty} (-1)^n t^n c(dx_1)^n e^{-t\Delta_{\mathbb{R}}} \big(I_n(t) \oplus (-1)^n I_n(t) \big) \ . \end{aligned}$$

We define

$$p_t^Z(x,y)^n := c(dx_1)^n \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} \left(p_t^I(x_2,y_2)^n \oplus (-1)^n p_t^I(x_2,y_2)^n\right) .$$

It follows that the integral kernel of $e^{-tA_Z^2}$ is

$$\sum_{n=0}^{\infty} (-1)^n t^n p_t^Z(x,y)^n$$

and the integral kernel of $e^{-(A_t^Z)^2}$ is

$$\sum_{n=0}^{\infty} (-1)^n t^{\frac{n}{2}} p_t^Z(x,y)^n .$$

Note that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^2$

$$\mathrm{tr}_s \partial_x^\alpha \partial_y^\beta p_t^Z(x,y)^n = 0 .$$

Furthermore

$$(D_Z)_x p_t^Z(x,y)^n = c(dx_1)(\partial_{x_1} - I\partial_{x_2}) p_t^Z(x,y)^n .$$

Since $\mathrm{tr}_s c(dx_1)\partial_{x_1} p_t^Z(x,y)^n$ vanishes for $x_1 = y_1$ and $\mathrm{tr}_s c(dx_1) I \partial_{x_2} p_t^Z(x,y)^n$ vanishes for all $x, y \in Z$,

$$\mathrm{tr}_s (D_Z)_x p_t^Z(x,y)^n = 0$$

if $x = y$.

Furthermore the integral kernel $p_t^Z(x,y)^n$ satisfies the following Gaussian estimate:

Lemma 4.2.1. *Let $\alpha, \beta \in \mathbb{N}_0^2$. For every $\varepsilon > 0$ there are $c, C > 0$ such that for all $x, y \in Z$ with $d(x,y) > \varepsilon$ and all $t > 0$*

$$|\partial_x^\alpha \partial_y^\beta p_t^Z(x,y)^n| \leq C e^{-\frac{d(x,y)^2}{ct}} .$$

Proof. The assertion follows from Prop. 4.1.1 and Cor. 4.1.4. The arguments are as in the proof of Lemma 3.4.4. □

4.3 The superconnection $A(\rho)_t$ associated to $D(\rho)$

4.3.1 The family $e^{-A(\rho)_t^2}$

In §3.5.1 we fixed $r_0, b_0 > 0$ such that

$$\mathrm{supp}\, k_K \cap \left((F(r_0,b_0) \times M) \cup (M \times F(r_0,b_0))\right) = \emptyset .$$

Let $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$ be as in §2.1.2 and assume that W is parallel on $M \backslash F(r, b_0)$ for any $r \geq 0$. By Prop. 2.1.3 this is possible. Then $[W, K]_s = 0$.

We define a superconnection associated to $D(\rho)$ on the $\mathbb{Z}/2$-graded $\mathcal{A}_i$-module $L^2(M, E \otimes \mathcal{A}_i)$ by

$$A(\rho) := W^* \,\mathrm{d}\, W + D(\rho) .$$

The corresponding rescaled superconnection is

$$A(\rho)_t := W^* \,\mathrm{d}\, W + \sqrt{t} D(\rho) \ .$$

The curvature of $A(\rho)$ is

$$\begin{aligned} A(\rho)^2 &= W^* \,\mathrm{d}^2\, W + [D(\rho), W^* \,\mathrm{d}\, W]_s + D(\rho)^2 \\ &= D(\rho)^2 + W^* [\mathrm{d}, W c(dW^*)]_s W \\ &=: D(\rho)^2 + \mathcal{R} \ . \end{aligned}$$

We used Prop. 2.1.1 and the fact that $[W, K]_s = 0$.

Then $\mathcal{R} \in C^\infty(M, \mathrm{End}E \otimes \hat{\Omega}_{\leq 1}\mathcal{A}_i)$ with $\mathcal{R}|_{M \setminus F(r,b_0)} = 0$ for all $r \geq 0$ and

$$\begin{aligned} \mathcal{R}|_F &= W^* [\mathrm{d}, W c(e_2) \partial_{e_2} W^*)]_s W \\ &= -c(e_1) W^* \,\mathrm{d}(W I \partial_{e_2} W^*) W \ . \end{aligned}$$

Furthermore for every $k \in \mathbb{Z}/6$ the restriction of $\mathcal{R}$ to $\mathcal{U}_k$ is of the form $c(e_1)(R \oplus (-R))$ with $R \in C^\infty(\mathcal{U}_k, M_{2d}(\hat{\Omega}_{\leq 1}\mathcal{A}_i))$ and R is independent of the variable x_2^k.

The rescaled curvature is

$$A(\rho)_t^2 = tD(\rho)^2 + \sqrt{t}\mathcal{R} \ .$$

Both, $A(\rho)^2$ and $A(\rho)_t^2$, are right $\hat{\Omega}_{\leq \mu}\mathcal{A}_i$-module homomorphisms.

Since $A(\rho)^2$ is a bounded perturbation of $D(\rho)^2$, it generates a holomorphic semigroup $e^{-tA(\rho)^2}$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$.

In the following we only consider $t \geq 0$.

The Volterra development of $e^{-tA(\rho)^2}$ is

$$\begin{aligned} e^{-tA(\rho)^2} &= \sum_{n=0}^{\infty} (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D(\rho)^2} \mathcal{R} e^{-u_1 t D(\rho)^2} \mathcal{R} \ldots e^{-u_n t D(\rho)^2} \ du_0 \ldots du_n \\ &=: \sum_{n=0}^{\infty} (-1)^n t^n I_n(\rho, t) \ . \end{aligned}$$

It follows that

$$e^{-A(\rho)_t^2} = \sum_{n=0}^{\infty} (-1)^n t^{n/2} I_n(\rho, t) \ .$$

For $n \in \mathbb{N}$ there is the following recursive relation:

$$I_n(\rho, t) = \int_0^1 du_0 \ (1 - u_0)^{n-1} e^{-u_0 t D(\rho)^2} \mathcal{R} I_{n-1}(\rho, (1 - u_0)t) \ .$$

For $\rho \neq 0$ the family $t \mapsto I_n(\rho, t)$ is uniformly bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$. By Cor. 3.5.14 it acts as a strongly continuous family of operators on $C_R^m(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ and there are $C, l > 0$ such that the action is bounded by $C(1 + t)^l$.

Note that $I_n(\rho, t)^* = (-1)^n I_n(\rho, t)$.

4.3.2 The integral kernel of $e^{-A(\rho)_t^2}$

In this section we prove that $I_n(\rho,t)$ is an integral operator for $t>0$ by our standard method: We construct an approximation of the family $I_n(\rho,t)$ by a family of integral operators and compare it with $I_n(\rho,t)$ by Duhamel's principle.

Let $\{\phi_k\}_{k\in J}$ and $\{\gamma_k\}_{k\in J}$ be as defined in §3.5.1.

For $k\in\mathbb{Z}/6$ the function $W|_{\mathcal{U}_k}:\mathcal{U}_k\to M_{4d}(\mathcal{A}_\infty)$ does not depend on the coordinate x_1^k. We extend it to a section $\tilde{W}_k:Z_k\to M_{4d}(\mathcal{A}_\infty)$ which is independent of x_1^k as well. Then the superconnection $A_{Z_k}:=\tilde{W}_k^*\,\mathrm{d}\,\tilde{W}_k+D_{Z_k}$ coincides on $\mathcal{U}_k$ with the superconnection $A(\rho)$.

For $k\in\mathbb{Z}/6$ and $n\in\mathbb{N}_0$ let $w(\rho)_t^k(x,y)^n$ be the restriction of $p_t^{Z_k}(x,y)^n$ to $\mathcal{U}_k\times\mathcal{U}_k$.

Let furthermore $w(\rho)_t^{\clubsuit}(x,y)^0$ be the restriction of the integral kernel of $e^{-tD_N(\rho)^2}$ to $\mathcal{U}_{\clubsuit}\times\mathcal{U}_{\clubsuit}$ for $t>0$ and set $w(\rho)_t^{\clubsuit}(x,y)^n=0$ for $n>0$.

This is reasonable since $A(\rho)^2$ equals $D(\rho)^2$ on $\mathcal{U}_{\clubsuit}$.

We extend $w(\rho)_t^k(x,y)^n$ by zero to $M\times M$ and set

$$w(\rho)_t(x,y)^n:=\sum_{k\in J}\gamma_k(x)w(\rho)_t^k(x,y)^n\phi_k(y)\ .$$

Write $W_n(\rho,t)$ for the corresponding integral operator. It is a bounded operator from $L^2(M,E\otimes\mathcal{A}_i)$ to $L^2(M,E\otimes\hat{\Omega}_{\le n}\mathcal{A}_i)$ as well as from $C_R^m(M,E\otimes\mathcal{A}_i)$ to $C_R^m(M,E\otimes\hat{\Omega}_{\le n}\mathcal{A}_i)$.

Set $W_0(\rho,0)=1$ and $W_n(\rho,0)=0$ for $n>0$.

Then for $f\in L^2(M,E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ the family $W_n(\rho,t)f\in L^2(M,E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ depends continuously on t for all $t\in[0,\infty)$, and for $f\in C_{Rc}^\infty(M,E\otimes\mathcal{A}_i)$ even smoothly.

By Duhamel's principle it follows that for $f\in C_{Rc}^\infty(M,E\otimes\mathcal{A}_i)$

$$\begin{aligned}
&\big(e^{-tA(\rho)^2}-\sum_{n=0}^\infty(-1)^nt^nW_n(\rho,t)\big)f\\
&=\ -\int_0^t e^{-sA(\rho)^2}(\frac{d}{dt}+A(\rho)^2)\sum_{k\in J}\sum_{n=0}^\infty(-1)^n(t-s)^n\gamma_kW_n^k(\rho,t-s)\phi_kf\ ds\\
&=\ \int_0^t e^{-sA(\rho)^2}\sum_{k\in J}[\gamma_k,D(\rho)^2]_s\sum_{n=0}^\infty(-1)^n(t-s)^nW_n^k(\rho,t-s)\phi_kf\ ds\\
&=\ \sum_{n=0}^\infty(-1)^n\int_0^t\sum_{m=0}^n s^{n-m}(t-s)^mI_{n-m}(\rho,s)\sum_{k\in J}[\gamma_k,D(\rho)^2]_sW_m^k(\rho,t-s)\phi_kf\ ds\ .
\end{aligned}$$

Hence

$$\big(I_n(\rho,t)-W_n(\rho,t)\big)f=t^{-n}\int_0^t\sum_{m=0}^n s^{n-m}(t-s)^mI_{n-m}(\rho,s)\sum_{k\in J}[\gamma_k,D(\rho)^2]_sW_m^k(\rho,t-s)\phi_kf\ ds\ .$$

In the following $|\cdot|$ denotes the fibrewise norm of $(E\boxtimes E^*)\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i$.

Proposition 4.3.1. *The operator $I_n(\rho,t)$ is an integral operator for $t>0$. Let $p(\rho)_t(x,y)^n$ be its integral kernel.*

1. *The map $(0,\infty)\to C^\infty(M\times M,(E\boxtimes E^*)\otimes\hat\Omega_n\mathcal{A}_\infty), t\mapsto p(\rho)_t^n$ is smooth.*
2. *$p(\rho)_t(x,y)^n=(-1)^n(p(\rho)_t(y,x)^n)^*$.*
3. *For every $T>0$ there are $C,c>0$ such that*

$$|p(\rho)_t(x,y)^n-w(\rho)_t(x,y)^n|\le Ct\left(\sum_{k\in J}e^{-\frac{d(y,\operatorname{supp}d\gamma_k)^2}{ct}}1_{\operatorname{supp}\phi_k}(y)+\rho 1_{\mathcal{U}_\clubsuit}(y)\right)$$

   for all $0<t<T$, $\rho\in[-1,1]$ and all $x,y\in M$.

4. *Let $\rho\neq 0$. Then there are $C,c>0$ and $j\in\mathbb{N}$ such that*

$$|p(\rho)_t(x,y)^n-w(\rho)_t(x,y)^n|\le Ct(1+t)^j e^{-\frac{d(y,\mathcal{U}_\clubsuit)^2}{ct}}$$

   for all $t>0$ and all $x,y\in M$.

   Statements analogous to 3), 4) hold for the partial derivatives of $p(\rho)_t(x,y)^n$ in x of y with respect to unit vector fields on M.

Proof. The proof follows the lines of the proof of Prop. 3.5.12.

In order to show the existence of the integral kernel and 1) we need only investigate $I_n(\rho,t)-W_n(\rho,t)$.

For $f\in C^\infty_{Rc}(M,E\otimes\mathcal{A}_i)$

$$\begin{aligned}&\big(I_n(\rho,t)-W_n(\rho,t)\big)f\\ &= t^{-n}\int_0^t\int_M\sum_{m=0}^n s^{n-m}(t-s)^m I_{n-m}(\rho,s)\sum_{k\in J}[\gamma_k,A(\rho)^2]_s w(\rho)^k_{t-s}(\cdot,y)^m\phi_k(y)f(y)\,dyds\ .\end{aligned}$$

Note that $[\gamma_k,A(\rho)^2]=[c(d\gamma_k),D]_s$ for $k\in\mathbb{Z}/6$.

For $\rho\in\mathbb{R}$ and $t>0$ the family $I_{n-m}(\rho,s):C^\nu_R(M,E\otimes\hat\Omega_{\le\mu}\mathcal{A}_i)\to C^\nu_R(M,E\otimes\hat\Omega_{\le\mu+n-m}\mathcal{A}_i)$ is uniformly bounded in $s<t$.

The function

$$\tau\mapsto\big(y\mapsto\sum_{k\in J}[\gamma_k,A(\rho)^2]_s w(\rho)^k_\tau(\cdot,y)^m\phi_k(y)\big)$$

is smooth from $(0,\infty)$ to $C^l(M,C^\nu_{Rc}(M,E\otimes\hat\Omega_{\le m+1}\mathcal{A}_i)\otimes E^*)$ for any $l,\nu\in\mathbb{N}_0$.

If $k\in\mathbb{Z}/6$, then by Lemma 4.2.1 there are $c,C>0$ such that for all $y\in M$ and $0<\tau$

$$\|[\gamma_k,A(\rho)^2]_s w^k_\tau(\cdot,y)^m\phi_k(y)\|_{C^\nu}\le Ce^{-\frac{d(y,\operatorname{supp}d\gamma_k)^2}{c\tau}}1_{\operatorname{supp}\phi_k}(y)\ .$$

Furthermore $w(\rho)^\clubsuit_\tau(x,y)^m=0$ for $m>0$. The kernel $w(\rho)^\clubsuit_\tau(x,y)^0$ is equal to the kernel $e(\rho)^\clubsuit_\tau(x,y)$ in the proof of Prop. 3.5.12 and was estimated there. It follows that for $T,\delta>0$ there is $C>0$ such that

$$\|[\gamma_\clubsuit,A(\rho)^2]_s w(\rho)^\clubsuit_\tau(\cdot,y)^m\phi_\clubsuit(y)f(y)\|_{C^\nu}\le C\big(e^{-\frac{d(y,\operatorname{supp}d\gamma_\clubsuit)^2}{(4+\delta)\tau}}+|\rho|\big)1_{\operatorname{supp}\phi_\clubsuit}(y)$$

for $y \in M$, $\rho \in [-1,1]$ and $0 < \tau < T$.

Analogous estimates hold for the derivatives in y and also in τ by the heat equation.

The existence and 1) follow now by standard arguments.

Assertion 2) follows from $I_n(\rho,t)^* = (-1)^n I_n(\rho,t)$.

In order to prove 3) we combine the estimates above with the fact that for any $T > 0$ the operator $I_n(\rho,t) : C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is uniformly bounded in $\rho \in [-1,1]$ and $0 \leq t < T$.

4) follows from the estimates by taking into account that for every $\rho \neq 0$ there is $C > 0$ and $j \in \mathbb{N}$ such that the norm of $I_n(\rho,t) : C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is bounded by $C(1+t)^j$ for all $t > 0$. □

Corollary 4.3.2. *Let $\rho \neq 0$. For every $\varepsilon > 0$ and $m, n \in \mathbb{N}_0$ there are $c, C > 0$ and $j \in \mathbb{N}$ such that for $t > 0$ and $x, y \in M$ with $d(x,y) > \varepsilon$*

$$|D(\rho)^m_x p(\rho)_t(x,y)^n| \leq C(1+t)^j \big(e^{-\frac{d(x,y)^2}{ct}} + e^{-\frac{d(y,\mathcal{U}_\clubsuit)^2}{ct}}\big).$$

Proof. This follows from Lemma 4.2.1. □

Corollary 4.3.3. *Let $k \in \mathbb{Z}/6$.*

1. *For every $T > 0$ and $m, n \in \mathbb{N}_0$ there are $c, C > 0$ such that for all $x, y \in \mathcal{U}_k$, for $0 < t < T$ and $\rho \in [-1,1]$*

$$|D(\rho)^m_x p(\rho)_t(x,y)^n - (D_{Z_k})^m_x p^{Z_k}_t(x,y)^n| \leq C e^{-\frac{d(y,\mathcal{U}_\clubsuit)}{ct}} .$$

2. *For every $\rho \neq 0$ and $n \in \mathbb{N}_0$ there is $c, C > 0$ and $j \in \mathbb{N}$ such that for all $x, y \in \mathcal{U}_k$ and $t > 0$*

$$|D(\rho)^m_x p(\rho)_t(x,y)^n - (D_{Z_k})^m_x p^{Z_k}_t(x,y)^n| \leq C(1+t)^j e^{-\frac{d(y,\mathcal{U}_\clubsuit)}{ct}} .$$

4.4 The index theorem and its proof

4.4.1 The supertrace

In the following tr_s denotes the supertrace on the fibres of $(E \otimes E^*) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i = \mathrm{End}E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ and Tr_s the corresponding trace for trace class operators as defined in §1.3.2.

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function with $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Let $\phi_r : M \to [0,1],\ \phi_r(x) := \chi(d(M_r, x))$ for $r > 0$.

Definition 4.4.1. *Let K be a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ such that $\phi_r K \phi_r$ is a trace class operator for all $r \geq 0$. Then we define*

$$\mathrm{Tr}_s K := \lim_{r\to\infty} \mathrm{Tr}_s\big(\phi_r K \phi_r\big)$$

if the limit exists.

On trace class operators this definition coincides with the supertrace Tr_s defined in §1.3.2.

Proposition 4.4.2. *For $\rho \in \mathbb{R}$ and $t > 0$ the supertraces*

$$\mathrm{Tr}_s e^{-A(\rho)_t^2}$$

and

$$\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}$$

exist in the sense of the previous definition.

Proof. We show the assertion for $e^{-A(\rho)_t^2}$, the proof for $D(\rho)e^{-A(\rho)_t^2}$ is analogous.

The operator $\phi_r e^{-A(\rho)_t^2}\phi_r$ is trace class for $t > 0$ since

$$\phi_r e^{-A(\rho)_t^2}\phi_r = (\phi_r e^{-A(\rho)_t^2/2})(e^{-A(\rho)_t^2/2}\phi_r)$$

is a decomposition in Hilbert-Schmidt operators. Note that it follows that the operators $I_n(\rho, t)$ are also trace class for $t > 0$.

We show that $\mathrm{tr}_s p(\rho)_t(x,x)^n$ is in $L^1(M, \hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s})$:

By Cor. 4.3.3 there are $c, C > 0$ such that

$$|p(\rho)_t(x,x)^n - p_t^{Z_k}(x,x)^n| \leq Ce^{-cd(x,\mathcal{U}_\clubsuit)^2}$$

for all $x \in \mathcal{U}_k$.

Since $\mathrm{tr}_s p_t^{Z_k}(x,x)^n = 0$ by §4.2,

$$|\mathrm{tr}_s p(\rho)_t(x,x)^n| \leq Ce^{-cd(x,\mathcal{U}_\clubsuit)^2}\ .$$

Now the assertion follows. □

4.4.2 The limit of $\mathrm{Tr}_s e^{-A(\rho)_t^2}$ for $t \to \infty$

This section is devoted to the proof of

Theorem 4.4.3. *Let $\rho \neq 0$.*

Let P_0 be the projection onto the kernel of $D(\rho)$. Then for $T > 0$ there is $C > 0$ such that for all $t > T$

$$|\mathrm{Tr}_s e^{-A(\rho)_t^2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \mathrm{Tr}_s (P_0 W \,\mathrm{d}\, W^* P_0)^{2n}| \leq C t^{-\frac{1}{2}}$$

and

$$|\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}| \leq C t^{-1}$$

in $\hat{\Omega}_{\leq \mu} \mathcal{A}_i / \overline{[\hat{\Omega}_{\leq \mu} \mathcal{A}_i, \hat{\Omega}_{\leq \mu} \mathcal{A}_i]_s}$.

Note that $(P_0 W \,\mathrm{d}\, W^* P_0)^{2n} = W (W^* P_0 W)(\mathrm{d}\, W^* P_0 W)^{2n} W^*$, hence by Lemma 5.3.5 it follows that $(P_0 W \,\mathrm{d}\, W^* P_0)^{2n}$ is a trace class operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

The proof is subdivided in some lemmata. It relays on the tools developped in §5.2.3.

Throughout the section $\rho \neq 0$ is fixed.

In the following $|\cdot|$ denotes the norm on $\hat{\Omega}_{\leq \mu} \mathcal{A}_i / \overline{[\hat{\Omega}_{\leq \mu} \mathcal{A}_i, \hat{\Omega}_{\leq \mu} \mathcal{A}_i]_s}$ resp. the fibrewise norm of $(E \boxtimes E^*) \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$ (depending on the context), and $\|\cdot\|$ denotes the operator norm on $B(L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i))$.

Lemma 4.4.4. *Let $\nu = 0, 1$. For any $T > 0$ there are $\varepsilon, C > 0$ such that for all $t > T$*

$$|\mathrm{Tr}_s D(\rho)^\nu e^{-A(\rho)_t^2} - \mathrm{Tr}_s \phi_t D(\rho)^\nu e^{-A(\rho)_t^2}| \leq C e^{-\varepsilon t}.$$

Proof. We prove the case $\nu = 0$, the case $\nu = 1$ can be proved analogously.

By Cor. 4.3.3 for any $n \in \mathbb{N}_0$ there are $c, C, r > 0$ such that

$$|p(\rho)_t(x,x)^n - p_t^{Z_k}(x,x)^n| \leq C(1+t)^j e^{-\frac{d(x, M_r)^2}{ct}}$$

for $t > 0$ and $x \in \mathcal{U}_k$ with $k \in \mathbb{Z}/6$.

Hence there are $c, C > 0$ such that for all $x \in M$ and $t > T$

$$|\mathrm{tr}_s p(\rho)_t(x,x)| \leq C e^{-\frac{d(x, M_r)^2}{ct}}$$

and by that there are $C, \varepsilon > 0$ such that for all $t > r$

$$\begin{aligned} |\mathrm{Tr}_s e^{-A(\rho)_t^2} - \mathrm{Tr}_s \phi_t e^{-A(\rho)_t^2}| &= |\mathrm{Tr}_s (1 - \phi_t) e^{-A(\rho)_t^2}| \\ &\leq C \int_t^\infty e^{-\frac{(r'-r)^2}{ct}} dr' \\ &\leq C e^{-\varepsilon t} . \end{aligned}$$

□

Let $P_1 := 1 - P_0$.

Lemma 4.4.5. *Let $\nu = 0, 1$ and $k \in \mathbb{N}_0$. Then*

$$\mathrm{Tr}_s \int_{\Delta^k} \phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t \; du_0 \dots du_k$$

$$= \int_{\Delta^k} \mathrm{Tr}_s\big(\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t\big) \; du_0 \dots du_k \; .$$

For $T > 0$ there are $C, \varepsilon > 0$ such that for all $t > T$ and

$$\Big| \int_{\Delta^k} \mathrm{Tr}_s\big(\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t\big) \; du_0 \dots du_k \Big| \leq C e^{-\varepsilon t} \; .$$

Proof. Note that for any $(u_0, \dots u_k) \in \Delta_k$ we can find $i \in \{0, 1, \dots, k\}$ such that $u_i \geq \frac{1}{k+1}$. We begin by showing that for any $T > 0$ there are $C, \varepsilon > 0$ such that for $1 > u_i \geq \frac{1}{k+1}$, for $1 > u_0, \dots, u_{i-1} \geq 0$ and for $t > T$ the family

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-\frac{u_i}{2} tD(\rho)^2}$$

is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$. If not specified the estimates in the following hold for $1 > u_i > \frac{1}{k+1}$, for $1 > u_0, \dots, u_{i-1} \geq 0$ and for $t > T$.

We have that

$$\begin{aligned}
&\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-\frac{u_i}{2} tD(\rho)^2} \\
&\quad = \phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-\frac{u_i}{2} tD(\rho)^2} \phi_{(t+6)} \\
&\qquad + \phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} \dots P_1 e^{-u_{i-1} tD(\rho)^2} \mathcal{R} \phi_{(t+6)} P_1 e^{-\frac{u_i}{2} tD(\rho)^2} (1 - \phi_{t+6}) \\
&\qquad + \phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} \dots P_1 e^{-u_{i-1} tD(\rho)^2} \mathcal{R} (1 - \phi_{(t+6)}) P_1 e^{-\frac{u_i}{2} tD(\rho)^2} (1 - \phi_{(t+6)}) \; .
\end{aligned}$$

Consider the first term on the right hand side: By Cor. 3.5.16 there is $C > 0$ such that the family $e^{-\frac{u_i}{4} tD(\rho)^2} \phi_{(t+6)}$ is family of Hilbert-Schmidt operators with norm bounded by Ct.

Furthermore by Prop. 3.5.10 and Cor. 3.5.13 there are $\varepsilon, C > 0$ such that the operator norm of

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-\frac{u_i}{4} tD(\rho)^2}$$

is bounded by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$.

By Prop. 5.2.12 it follows that the first term is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$ for some $C, \varepsilon > 0$.

In the second term the factor

$$\phi_{(t+6)} P_1 e^{-\frac{u_i}{2} tD(\rho)^2} (1 - \phi_{(t+6)}) = (\phi_{(t+6)} e^{-\frac{u_i}{4} tD(\rho)^2}) P_1 e^{-\frac{u_i}{4} tD(\rho)^2} (1 - \phi_{(t+6)})$$

is a Hilbert-Schmidt operator bounded by some $C e^{-\varepsilon t}$ with $C, \varepsilon > 0$. Hence the second term is bounded in the Hilbert-Schmidt norm by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$ for some $C, \varepsilon > 0$.

The estimate of the third term requires more effort:

We prove by induction on $j \in \mathbb{N}$ that there is $C > 0$ such that

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \dots P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)})$$

is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by $C u_0^{-\frac{\nu}{2}} (1+t)$ for $t > T$ and $0 \leq u_0, \dots, u_j \leq 1$. Then it follows that the third term is uniformly bounded by some $Ce^{-\varepsilon t}$ for some $C, \varepsilon > 0$ since $P_1 e^{-\frac{u_i}{2} t D(\rho)^2}$ is exponentially decaying for $t \to \infty$ by Prop. 3.5.10.

For $j = 0$ the assertion follows from Cor. 3.5.16 by

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} (1 - \phi_{(t+6)}) = \phi_t D(\rho)^\nu e^{-u_0 t D(\rho)^2} (1 - \phi_{(t+6)}) - \phi_t D(\rho)^\nu P_0 (1 - \phi_{(t+6)}) \; .$$

Now assume the assertion is true for $j - 1$. We have

$$\begin{aligned} &\phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 \dots \mathcal{R} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)}) \\ &= \phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} P_1 \mathcal{R} P_1 \dots P_1 e^{-u_{j-1} t D(\rho)^2} P_1 \mathcal{R} \phi_{(t+3)} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)}) \\ &\quad + \phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} P_1 \dots P_1 e^{-u_{j-1} t D(\rho)^2} P_1 (1 - \phi_{(t+3)}) \mathcal{R} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)}) \; . \end{aligned}$$

Both terms on the right hand side are bounded in the Hilbert-Schmidt norm by $C u_0^{-\nu/2} (1+t)$ for some $C > 0$ and all $t > T$ and $0 \leq u_0, \dots, u_j \leq 1$: the first term since $\phi_{(t+3)} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)})$ is bounded by $C(1+t)$ by Cor. 3.5.16, the second term by induction.

Now we have finished the claim the proof began with.

An analogous proof shows that for any $T > 0$ there are $C, \varepsilon > 0$ such that the family

$$P_1 e^{-\frac{u_i}{2} t D(\rho)^2} \mathcal{R} P_1 e^{-u_{i+1} t D(\rho)^2} \mathcal{R} \dots P_1 e^{-u_k t D(\rho)^2} \phi_t$$

is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $Ce^{-\varepsilon t}$ for all $u_i > \frac{1}{k+1}$, $0 \leq u_{i+1}, \dots, u_k \leq 1$ and $t > T$.

It follows that we can interchange the integration and the supertrace ant that

$$\int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-u_k t D(\rho)^2} \phi_t \big) du_0 \dots du_k$$

is bounded by $Ce^{-\varepsilon t}$ for some $C, \varepsilon > 0$ and all $t > T$.

This implies the assertion. □

We have

$$\mathrm{Tr}_s \phi_t D(\rho)^\nu e^{-A(\rho)_t^2} = \sum_{k=0}^{\infty} (-1)^k t^{k/2} \; \mathrm{Tr}_s \phi_t D(\rho)^\nu I_k(\rho, t) \; .$$

From arguments analogous to those in the proof of the previous lemma it follows that

$$\mathrm{Tr}_s \phi_t D(\rho)^\nu I_k(\rho, t) = \int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t D(\rho)^\nu e^{-u_0 t D(\rho)^2} \mathcal{R} e^{-u_1 t D(\rho)^2} \mathcal{R} \dots e^{-u_k t D(\rho)^2} \phi_t \big) \; du_0 \dots du_k \; .$$

Note that on the right hand side of the last equation the supertrace is applied to trace class operators.

Hence the decomposition $e^{-u_i tD(\rho)^2} = P_0 + P_1 e^{-u_i tD(\rho)^2}$ induces a decomposition of $\mathrm{Tr}_s \phi_t D(\rho)^\nu I_k(\rho, t)$ in a sum of 2^{k+1} terms. Let $P^\nu_{jk}(t)$ be the sum of all those terms containing j factors of the form $P_1 e^{-u_i tD(\rho)^2}$.

Thus

$$\mathrm{Tr}_s \phi_t D(\rho)^\nu e^{-A(\rho)_t^2} = \sum_{k=0}^{\infty} (-1)^k t^{\frac{k}{2}} \sum_{j=0}^{k+1} P^\nu_{jk}(t) \ .$$

From $P_0 \mathcal{R} P_0 = P_0 [W^* \, \mathrm{d}\, W, D(\rho)]_s P_0 = 0$ it follows that $P^\nu_{jk}(t) = 0$ for $j < \frac{k}{2}$.

Furthermore for k even

$$\begin{aligned} P^\nu_{\frac{k}{2}k}(t) \ &= \ \int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t D(\rho)^\nu P_0 \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_3 tD(\rho)^2} \mathcal{R} P_0 \dots \\ &\qquad \dots P_0 \mathcal{R} P_1 e^{-u_{k-1} tD(\rho)^2} \mathcal{R} P_0 \phi_t \big) \ du_0 \dots du_k \ . \end{aligned}$$

Since $D(\rho) P_0 = 0$, it follows that $P^1_{\frac{k}{2}k}(t) = 0$.

Furthermore by the previous lemma

$$|P^\nu_{(k+1)k}(t)| \leq C e^{-\varepsilon t}$$

for t large.

Now we study the behaviour of the remaining cases for large t:

Lemma 4.4.6. *Let $\nu = 0, 1$ and $j, k \in \mathbb{N}_0$ with $\frac{k}{2} \leq j \leq k$. For every $T > 0$ there is $C > 0$ such that for $t > T$*

$$|P^\nu_{jk}(t)| \leq C t^{-j} \ .$$

Furthermore for k odd, for every $n \in \mathbb{N}$ and $T > 0$ there is $C > 0$ such that for $t > T$

$$|P^\nu_{\frac{k+1}{2}k}(t)| \leq C t^{-n} \ .$$

Proof. For $j \leq k$ the operator $P^\nu_{jk}(t)$ is a sum of terms of the form

$$\int_{\Delta^k} \mathrm{Tr}_s (A(u_0, \dots u_i, t) P_0)(P_0 B(u_{i+1}, \dots u_k, t)) \ du_0 \dots du_k \ ,$$

where A and B are continuous families of bounded operators on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ for $u_0 \neq 0$.

Since P_0 is a Hilbert-Schmidt operator by Prop. 5.3.7, Prop. 5.2.12 implies that

$$\begin{aligned} &| \int_{\Delta^k} \mathrm{Tr}_s \big(A(u_0, \dots u_i, t) P_0 \big) \big(P_0 B(u_{i+1}, \dots u_k, t) \big) \ du_0 \dots du_k | \\ &\leq \|P_0\|^2_{HS} \int_{\Delta^k} \|A(u_0, \dots u_i, t)\| \|B(u_{i+1}, \dots u_k, t)\| \ du_0 \dots du_k \ . \end{aligned}$$

Let $\omega > 0$ be such that there is $C > 0$ with $\|P_1 e^{-tD(\rho)^2}\| \leq Ce^{-\omega t}$ for all $t > 0$. Then

$$\begin{aligned}
|P^\nu_{jk}(t)| &\leq C \sum_{\lambda=0,1} \int_0^1 du_0 \ (u_0 t)^{-\lambda/2} e^{-\omega u_0 t} \int_{(1-u_0)\Delta^{k-1}} \exp(-\sum_{i=1}^{j-1} \omega u_i t) \ du_1 \dots du_k \\
&= C \sum_{\lambda=0,1} \int_0^1 du_0 \ (u_0 t)^{-\lambda/2} e^{-\omega u_0 t} \int_0^{(1-u_0)} e^{-\omega s t} \ \mathrm{vol}(s\Delta^{j-2}) \ \mathrm{vol}\big((1-u_0-s)\Delta^{k-j}\big) \ ds \\
&= C \sum_{\lambda=0,1} \int_0^1 du_0 \ (u_0 t)^{-\lambda/2} e^{-\omega u_0 t} \int_0^{(1-u_0)} e^{-\omega s t} \ \frac{s^{j-2}(1-u_0-s)^{k-j}}{(j-2)!(k-j)!} ds \\
&= Ct^{-j} \sum_{\lambda=0,1} \int_0^t dy \ y^{-\lambda/2} e^{-\omega y} \int_0^{(t-y)} e^{-\omega x} \ \frac{x^{j-2}(1-y/t-x/t)^{k-j}}{(j-2)!(k-j)!} \ dx \\
&\leq Ct^{-j} \sum_{\lambda=0,1} \Big(\int_0^\infty y^{-\lambda/2} e^{-\omega y} \ dy \Big) \Big(\int_0^\infty e^{-\omega x} \ \frac{x^{j-2}}{(j-2)!(k-j)!} \ dx \Big) \\
&\leq Ct^{-j} \ .
\end{aligned}$$

This shows the first statement.

For k odd

$$\begin{aligned}
&P^\nu_{\frac{k+1}{2}k}(t) \\
&= \int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t D(\rho)^\nu P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_3 t D(\rho)^2} \dots P_0 \mathcal{R} P_1 e^{-u_k t D(\rho)^2} \big) \ du_0 \dots du_k \\
&\quad + \int_{\Delta^k} \mathrm{Tr}_s \big(D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_2 t D(\rho)^2} \dots P_0 \mathcal{R} P_1 e^{-u_{k-1} t D(\rho)^2} \mathcal{R} P_0 \phi_t \big) \ du_0 \dots du_k \ .
\end{aligned}$$

From Prop. 3.5.6 it follows that for any $n \in \mathbb{N}$ there is $C > 0$ such that

$$\|(1-\phi_t)P_0\|_{HS} + \|P_0(1-\phi_t)\|_{HS} \leq Ct^{-n} \ .$$

The second estimate follows then from the cyclicity of the sypertrace since $P_0 P_1 = P_1 P_0 = 0$. □

The results so far obtained yield that the supertrace $\mathrm{Tr}_s D(\rho) e^{-A(\rho)^2_t}$ can be estimated by Ct^{-1} for $t > T$. This shows the second estimate in the theorem.

Furthermore the previous lemmata imply that for $t > T$ there is $C > 0$ such that

$$|\mathrm{Tr}_s \phi_t e^{-A(\rho)^2_t} - \sum_{n=0}^\infty t^n \ P^0_{n(2n)}(t)| \leq Ct^{-\frac{1}{2}} \ .$$

Hence it remains to study the behaviour of $t^n P_{n(2n)}(t)$ for $t \to \infty$.

Define

$$\begin{aligned}
\mathcal{P}_n(t) \ := \ &\int_{\Delta^{2n}} P_0 \mathcal{R} P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \dots \\
&\dots P_0 \mathcal{R} P_1 e^{-u_{n-1} t D(\rho)^2} \mathcal{R} P_0 \ du_0 \dots du_{2n} \ .
\end{aligned}$$

By $\mathcal{P}_n(t) = P_0\mathcal{P}_n(t)P_0$ this is a trace class operator.

Furthermore $P^0_{n(2n)}(t) = \mathrm{Tr}_s\phi_t\mathcal{P}_n(t)$.

By Prop. 3.5.6 for any $j \in \mathbb{N}$ there is $C > 0$ such that

$$\|P_0 - \phi_t P_0\|_{HS} \leq Ct^{-j} \ ,$$

hence

$$|\mathrm{Tr}_s\mathcal{P}_n(t) - P^0_{n(2n)}(t)| \leq Ct^{-j} \ .$$

In the next lemma we show that for $T > 0$ there is $C > 0$ such that

$$\|t^n\mathcal{P}_n(t) - (-1)^n\frac{1}{n!}(P_0WdW^*P_0)^{2n}\| \leq Ct^{-1}$$

in $B(L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ for $t > T$. The first estimate of the theorem follows then from the fact that for every $j > 0$ there is $C > 0$ such that

$$\begin{aligned}
&|t^n P^0_{n(2n)}(t) - (-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0WdW^*P_0)^{2n}| \\
&\leq \ Ct^{n-j} + |t^n \ \mathrm{Tr}_s[P_0^2\big(\mathcal{P}_n(t) - (-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0WdW^*P_0)^{2n}\big)]| \\
&\leq \ Ct^{n-j} + \|P_0\|^2_{HS}\|\mathcal{P}_n(t) - (-1)^n\frac{1}{n!}(P_0WdW^*P_0)^{2n}\| \ .
\end{aligned}$$

Lemma 4.4.7. *Let $k, n \in \mathbb{N}_0$ with $n \leq k$. For $t \to \infty$ the term*

$$t^n\int_{\Delta^k} P_0\mathcal{R}P_1e^{-u_0tD(\rho)^2}P_1\mathcal{R}P_0\mathcal{R}P_1e^{-u_1tD(\rho)^2}\mathcal{R}P_0\ldots P_0\mathcal{R}P_1e^{-u_{n-1}tD(\rho)^2}\mathcal{R}P_0 \ du_0\ldots du_k$$

converges in $B(L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ to

$$(-1)^n\frac{1}{(k-n)!} \ (P_0W\,\mathrm{d}\,W^*P_0)^{2n}$$

with $O(t^{-1})$.

Proof. For $i, j \in \{0, 1\}$ let ${}_id_j := P_iW\,\mathrm{d}\,W^*P_j$.

By $\mathcal{R} = [W^*\,\mathrm{d}\,W, D(\rho)]_s$

$$\begin{aligned}
P_1\mathcal{R}P_0 &= P_1D(\rho)W^*\,\mathrm{d}\,WP_0 \\
P_0\mathcal{R}P_1 &= P_0W^*\,\mathrm{d}\,WD(\rho)P_1
\end{aligned}$$

and thus

$$P_0\mathcal{R}P_1e^{-tD(\rho)^2}\mathcal{R}P_0 = {}_0d_1D(\rho)^2e^{-tD(\rho)^2}{}_1d_0.$$

This term is uniformly bounded for $t \to 0$.

Hence the integral

$$\int_0^t {}_0d_1D(\rho)^2e^{-sD(\rho)^2}{}_1d_0 \ ds$$

converges and equals ${}_0d_1(e^{-tD(\rho)^2} - 1) \ {}_1d_0$.

For $n \in \mathbb{N}$ and $k \geq n$

$$t^n \int_{\Delta^k} P_0 \mathcal{R} P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \dots P_0 \mathcal{R} P_1 e^{-u_{n-1} t D(\rho)^2} \mathcal{R} P_0 \; du_0 \dots du_k$$

$$= t^n \int_{\Delta^k} {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1d_0 {}_0d_1 D(\rho)^2 e^{-u_1 t D(\rho)^2} {}_1d_0 \dots {}_0d_1 D(\rho)^2 e^{-u_{n-1} t D(\rho)^2} {}_1d_0 \; du_0 \dots du_k \; .$$

For $n \in \mathbb{N}$ set

$$D_n := \{ \sum_{i=0}^{n-1} u_i \leq 1; \; 0 \leq u_i \leq 1, \; i = 0, \dots, n-1 \} \; .$$

By integration on $u_n, \dots, u_k$ the previous term equals

$$t^n \int_{D_n} {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1d_0 \; \dots {}_0d_1 D(\rho)^2 e^{-u_{n-1} t D(\rho)^2} {}_1d_0 \; \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i) \Delta^{k-n}) \; du_0 \dots du_{n-1} \; .$$

We prove by induction that this term converges to $\frac{1}{(k-n)!} ({}_0d_1 1 d_0)^n$ with $O(t^{-1})$.

For $n = 1$ and $k = 1$ the term equals 1.

For $n = 1$ and $k > 1$ the claim follows by induction on k: By partial integration

$$\begin{aligned} & \int_0^1 {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1d_0 \, \mathrm{vol}((1 - u_0) \Delta^{k-1}) \; du_0 \\ & = \; {}_0d_1 e^{-t D(\rho)^2} {}_1d_0 + \int_0^1 {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1d_0 \, \mathrm{vol}((1 - u_0) \Delta^{k-2}) \; du_0 \end{aligned}$$

since

$$\partial_{u_0} \, \mathrm{vol}((1 - u_0) \Delta^{k-1}) = - \, \mathrm{vol}((1 - u_0) \Delta^{k-2}) \; .$$

Furthermore ${}_0d_1 e^{-t D(\rho)^2} {}_1d_0$ decays exponentially for $t \to \infty$.

Assume now that the claim holds for $n - 1$ and any $k \geq n - 1$.

Then by partial integration on u_{n-1} the term equals

$$t^n \int_{\Delta^{n-1}} {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1d_0 \; \dots {}_0d_1 \Big[-t^{-1} e^{-x t D(\rho)^2} \, \mathrm{vol}((1 - \sum_{i=0}^{n-2} u_i - x) \Delta^{k-n}) \Big]_0^{u_{n-1}} {}_1d_0 \; du_0 \dots du_{n-1}$$

$$+ \quad t^{n-1} \int_{D_n} {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1d_0 \; \dots {}_0d_1 e^{-u_{n-1} t D(\rho)^2} {}_1d_0 \Big(\partial_{u_{n-1}} \, \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i) \Delta^{k-n}) \Big) \; du_0 \dots du_{n-1}$$

Note that the first integral vanishes for $x = u_{n-1}$.

We obtain

$$t^{n-1} \int_{D_{n-1}} {}_0d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} \dots {}_0d_1 D(\rho)^2 e^{-u_{n-2} t D(\rho)^2} {}_1d_0 {}_0d_1 1 d_0 \, \mathrm{vol}((1 - \sum_{i=0}^{n-2} u_i) \Delta^{k-n}) \; du_0 \dots du_{n-2}$$

$$-t^{n-1}\int_{D_{n-1}} {}_0d_1D(\rho)^2e^{-u_0tD(\rho)^2}{}_1d_0\dots{}_0d_1e^{-u_{n-1}tD(\rho)^2}{}_1d_0\,\mathrm{vol}((1-\sum_{i=0}^{n-1}u_i)\Delta^{k-n-1})\,du_0\dots du_{n-1}\,.$$

There are $C,\omega>0$ such that the last term is bounded by

$$Ct^{n-1}\int_0^1 e^{-s\omega t}\,\mathrm{vol}(s\Delta^{n-1})\,\mathrm{vol}((1-s)\Delta^{k-n-1})\,ds\,,$$

hence by an estimate analogous to the one in the proof of the previous lemma it follows that the last term vanishes with $O(t^{-1})$ for $t\to\infty$. Thus it will not contribute to the limit.

By induction the first term converges with $O(t^{-1})$ to

$$\frac{1}{(k-1-(n-1))!}({}_0d_{11}d_0)^n=\frac{1}{(k-n)!}({}_0d_{11}d_0)^n\,.$$

From $(W^*\,\mathrm{d}\,W)^2=0$ it follows that ${}_0d_{11}d_0=-{}_0d_0^2$. This shows the assertion of the lemma.

□

4.4.3 The limit of $\mathrm{Tr}_s e^{-A_t^2}$ for $t\to 0$

Recall the definition of $\mathcal{N}$ from §2.5.

Theorem 4.4.8. *1.* $\lim_{t\to0}\mathrm{Tr}_s e^{-A_t^2}=\mathcal{N}$.

2. $\lim_{t\to0}\mathrm{Tr}_s De^{-A_t^2}=0$.

Proof. 1) From Prop. 4.3.1 it follows that

$$\lim_{t\to0}\mathrm{Tr}_s e^{-A_t^2}=\sum_{n=0}^{\infty}(-1)^n\lim_{t\to0}\,t^{\frac{n}{2}}\,\mathrm{Tr}_s W_n(0,t)\,.$$

Furthermore

$$\mathrm{Tr}_s W_n(0,t)=\sum_{k\in J}\int_{\mathcal{U}_k}\gamma_k(x)\,\mathrm{tr}_s w(0)_t^k(x,x)^n\phi_k(x)\,dx\,.$$

Now $\mathrm{tr}_s w(\rho)_t^k(x,x)^n=0$ for $k\in\mathbb{Z}/6$ and for all $n\in\mathbb{N}_0$ and $t>0$ by §4.2.

Furthermore $w(0)_t^{\clubsuit}(x,y)^n$ vanishes for $n>0$.

Recall that $\mathcal{U}_{\clubsuit}$ contains the isolated point $*$. Since $w(0)_t^{\clubsuit}(x,y)^0$ is the integral kernel of $e^{-tD_N^2}$

$$\lim_{t\to0}\mathrm{tr}_s w(0)_t^{\clubsuit}(x,x)^0=\mathcal{N}1_*(x)$$

in $C(\mathcal{U}_{\clubsuit})$ by the local index theorem ([BGV], Th.4.2) and by $\mathrm{ch}(E/S)=\mathrm{ch}((\mathbb{C}^+)^d)-\mathrm{ch}((\mathbb{C}^-)^d)=0$.

2) Prop. 4.3.1 also implies that

$$\lim_{t\to0}\mathrm{Tr}_s De^{-A_t^2}=\sum_{n=0}^{\infty}(-1)^n\lim_{t\to0}\,t^{\frac{n}{2}}\,\mathrm{Tr}_s DW_n(0,t)\,.$$

Since $DW_0(0,t)$ is odd, its supertrace vanishes. Furthermore $\mathrm{Tr}_s DW_n(0,t)=0$ by §4.2.

□

4.4.4 $\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2}$ and $\frac{d}{d\rho}\mathrm{Tr}_s e^{-A(\rho)_t^2}$

Lemma 4.4.9. *1.*

$$\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2} = -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{dt} e^{-A(\rho)_t^2} \ .$$

2.

$$\frac{d}{d\rho}\mathrm{Tr}_s e^{-A(\rho)_t^2} = -\,\mathrm{d}\,\mathrm{Tr}_s \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} \ .$$

Proof. 1) First we calculate $\frac{d}{dt}e^{-A(\rho)_t^2}$:

Consider the holomorphic semigroup $e^{-t'(D(\rho)^2+z\mathcal{R})}$ depending on the parameter z. By the semigroup law

$$\frac{d}{dt'}e^{-t'(D(\rho)^2+z\mathcal{R})} = -(D(\rho)^2+z\mathcal{R})e^{-t'(D(\rho)^2+z\mathcal{R})}$$

and by Duhamel's formula (Prop. 5.4.4)

$$\frac{d}{dz}e^{-t'(D(\rho)^2+z\mathcal{R})} = -\int_0^{t'} e^{-(t'-s)(D(\rho)^2+z\mathcal{R})}\mathcal{R}e^{-s(D(\rho)^2+z\mathcal{R})}ds \ .$$

It follows that

$$\begin{aligned}
\frac{d}{dt}e^{-A(\rho)_t^2} &= \frac{d}{dt}e^{-t(D(\rho)^2+t^{-1/2}\mathcal{R})} \\
&= \frac{d}{dt'}e^{-t'(D(\rho)^2+t^{-1/2}\mathcal{R})}|_{(t'=t)} - \tfrac{1}{2}t^{-3/2}\frac{d}{dz}e^{-t(D(\rho)^2+z\mathcal{R})}|_{(z=t^{-1/2})} \\
&= -(D(\rho)^2+t^{-1/2}\mathcal{R})e^{-t(D(\rho)^2+t^{-1/2}\mathcal{R})} \\
&\quad + \tfrac{1}{2}t^{-3/2}\int_0^t e^{-(t-s)(D(\rho)^2+t^{-1/2}\mathcal{R})}\mathcal{R}e^{-s(D(\rho)^2+t^{-1/2}\mathcal{R})}ds \\
&= -t^{-1}A(\rho)_t^2 e^{-A(\rho)_t^2} + \tfrac{1}{2}t^{-1/2}\int_0^1 e^{-(1-s)A(\rho)_t^2}\mathcal{R}e^{-sA(\rho)_t^2}ds \ .
\end{aligned}$$

Now we prove that for any $n \in \mathbb{N}_0$ and $t_1, t_2 > 0$ there are $r, c, C > 0$ such that

$$|\frac{d}{dt}\mathrm{tr}_s p(\rho)_t(x,x)^n| \leq Ce^{-cd(x,M_r)^2}$$

for $t_1 < t < t_2$. This yields that

$$\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2} = \mathrm{Tr}_s \frac{d}{dt}e^{-A(\rho)_t^2} \ .$$

By Cor. 4.3.3 and the fact that the pointwise supertrace of the integral kernel $(A_t^{Z_k})^2 e^{-(A_t^{Z_k})^2}$ vanishes for $k \in \mathbb{Z}/6$ there are $r, c, C > 0$ such that the pointwise supertrace of the integral kernel of $A(\rho)_t^2 e^{-A(\rho)_t^2}$ can be estimated by $Ce^{-cd(x,M_r)^2}$ for $t_1 < t < t_2$. The integral kernel of $\int_0^1 e^{-(1-s)A(\rho)_t^2}\mathcal{R}e^{-sA(\rho)_t^2}ds$ is the sum over $m, n \in \mathbb{N}$ of the terms

$$(-1)^{m+n}t^{\frac{m+n}{2}}\int_0^1 (1-s)^{m/2}s^{n/2}\int_M p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)p(\rho)_{st}(y,x)^n \ dyds \ .$$

We have that

$$\begin{aligned} & p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \\ = \;& p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y)\big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n\big) \\ & + \big(p(\rho)_{(1-s)t}(x,y)^m - w(\rho)_{(1-s)t}(x,y)^m\big)\mathcal{R}(y) w(\rho)_{st}(y,x)^n \\ & + w(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) w(\rho)_{st}(y,x)^n \ . \end{aligned}$$

By Prop. 4.3.1 and the fact that the operator $I_m(\rho,(1-s)t)$ is uniformly bounded on $C_R(M, E\otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ for $0 \leq (1-s)t \leq t_2$ there are $C,c,r>0$ such that for all $x\in M$ and $0\leq s\leq 1$ and $t_1 \leq t \leq t_2$

$$|\int_M p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y)\big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n\big)dy| \leq Ce^{-cd(x,M_r)^2} \ .$$

An analogous estimate holds for the second term.

By the results in §4.2 it follows that

$$\mathrm{tr}_s w(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) w(\rho)_{st}(y,x)^n = 0$$

since $\mathcal{R}|_{\mathcal{U}_\clubsuit} = 0$ and since for $k\in \mathbb{Z}/6$ the section $\mathcal{R}|_{\mathcal{U}_k}$ is of the form $c(e_1)(R\oplus(-R))$.

Hence there are $r,c,C>0$ such that for $x\in M_r$, $0\leq s\leq 1$ and $t_1\leq t\leq t_2$

$$(*) \qquad |\mathrm{tr}_s \int_M p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \ dy| \leq Ce^{-cd(x,M_r)^2} \ .$$

Now the claim from the beginning of the proof follows.

Next we show that

$$\mathrm{Tr}_s \int_0^1 e^{-(1-s)A(\rho)_t^2} \mathcal{R} e^{-sA(\rho)_t^2} ds = \mathrm{Tr}_s \mathcal{R} e^{-A(\rho)_t^2}$$

or equivalently

$$\begin{aligned} & \int_M \int_0^1 (1-s)^{m/2} s^{n/2} \int_M \mathrm{tr}_s p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \ dydsdx \\ = & \int_0^1 (1-s)^{m/2} s^{n/2} \int_M \int_M \mathrm{tr}_s p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \ dxdyds \ . \end{aligned}$$

We can interchange the integration over s and x by the estimate $(*)$.

Fix s and t. Consider once more the decomposition of $p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n$ from above:

From Cor. 4.3.2 it follows that for $\varepsilon>0$ there are $r,c,C>0$ such that for all $x,y\in M$

$$\begin{aligned} & |p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y)\big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n\big)| \\ & \leq \ C\big(1_{[0,\varepsilon]}(d(x,y)) + e^{-cd(x,y)^2} + e^{-cd(x,M_r)^2}\big)e^{-cd(y,M_r)^2} \ . \end{aligned}$$

The second term can be estimated in an analogous manner and the supertrace of the third term vanishes as we saw.

Hence we can interchange dx and dy.

Finally 1) follows from

$$\begin{aligned}\mathrm{Tr}_s \frac{d}{dt} e^{-A(\rho)_t^2} &= \mathrm{Tr}_s(-t^{-1}A(\rho)_t^2 + \tfrac{1}{2}t^{-1/2}\mathcal{R})e^{-A(\rho)_t^2} \\ &= -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{dt} e^{-A(\rho)_t^2} \ .\end{aligned}$$

2) Since $A(\rho)_t^2 = tD^2 + t\rho[D,K] + t\rho^2K^2 + \sqrt{t}\mathcal{R}$, Duhamel's formula (Prop. 5.4.4) and the chain rule imply that

$$\frac{d}{d\rho} e^{-A(\rho)_t^2} = -\int_0^1 e^{-(1-s)A(\rho)_t^2} \frac{dA(\rho)_t^2}{d\rho} e^{-sA(\rho)_t^2} ds \ .$$

Note that the operator $\frac{dA(\rho)_t^2}{d\rho}$ is a finite integral operator whose integral kernel is compactly supported.

By arguments similar to those in the proof of 1) we can prove that

$$\frac{d}{d\rho}\mathrm{Tr}_s e^{-A(\rho)_t^2} = \mathrm{Tr}_s \frac{d}{d\rho} e^{-A(\rho)_t^2} = -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{d\rho} e^{-A(\rho)_t^2} \ .$$

Since $A(\rho)_t$ is a bounded perturbation of $-\sqrt{t}WD_sW^*$, Lemma 3.5.9 and Prop. 5.4.10 show that $A(\rho)_t$ and $e^{-A(\rho)_t^2}$ commute.

It follows that

$$\begin{aligned}\frac{dA(\rho)_t^2}{d\rho} e^{-A(\rho)_t^2} &= [A(\rho)_t, \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s \\ &= [W^* \,\mathrm{d}\, W, \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s + \sqrt{t}[D(\rho), \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s \ .\end{aligned}$$

The first term of the last line is an integral operator with integral kernel $W^*d(k)W$ if k is the integral kernel of $W\frac{dA(\rho)_t}{d\rho}e^{-A(\rho)_t^2}W^*$. Hence the supertrace of the first term equals

$$\mathrm{d}\,\mathrm{Tr}_s \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} \ .$$

Now consider the second term. Let P be the orthogonal projection onto the range of K. It is a Hilbert-Schmidt operator. Since $\frac{dA(\rho)_t}{d\rho} = \sqrt{t}K$,

$$\mathrm{Tr}_s[D(\rho), \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s = \mathrm{Tr}_s[D(\rho)P, Ke^{-A(\rho)_t^2}]_s \ .$$

Since $D(\rho)P$ and $Ke^{-A(\rho)_t^2}$ are Hilbert-Schmidt operators, the supertrace vanishes by def. 1.3.2. □

Let D_{I_k} be the operator D_I from §3.3.1 with boundary conditions given by the pair $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{k+1 \bmod 3})$.

In §4.3.2 we defined $A_t^{Z_k} = \tilde{W}_k^* \,\mathrm{d}\, \tilde{W}_k + \sqrt{t}D_{Z_k}$ such that $A_t^{Z_k}$ coincides with $A(\rho)_t$ on $\mathcal{U}_k$. There is $U_k \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ such that $\tilde{W}_k(x_1,x_2) = U_k(x_2) \oplus U_k(x_2)$. Let $A_t^{I_k} = U_k^* \,\mathrm{d}\, U_k + \sqrt{t}\sigma D_{I_k}$.

Lemma 4.4.10.

$$\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2} = -\frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}/6} \mathrm{Tr}_\sigma D_{I_k} e^{-(A_t^{I_k})^2} - \frac{1}{2\sqrt{t}}\,\mathrm{d}\,\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2} \ .$$

Proof. Since $\frac{dA(\rho)_t^2}{dt} = [\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s$, it follows from the previous lemma that

$$\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2} = -\mathrm{Tr}_s[\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s e^{-A(\rho)_t^2} \ .$$

Furthermore

$$\begin{aligned} -[\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s e^{-A(\rho)_t^2} &= -[A(\rho)_t, \frac{dA(\rho)_t}{dt} e^{-A(\rho)_t^2}]_s \\ &= -\frac{1}{2\sqrt{t}}[W^*\,\mathrm{d}\,W, D(\rho)e^{-A(\rho)_t^2}]_s - \frac{1}{2}[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s \ . \end{aligned}$$

The supertrace of the first supercommutator in the last line equals

$$-\frac{1}{2\sqrt{t}}\,\mathrm{d}\,\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2} \ .$$

Now consider the second term:

As in the beginning of §4.4.1 let $\chi : \mathbb{R} \to [0,1]$ be a smooth function with $\chi(x) = 1$ for $x \le 0$ and $\chi(x) = 0$ for $x \ge 1$. Let $\phi_r : M \to [0,1],\ \phi_r(x) = \chi(d(M_r, x))$ for $r > 0$.

We have that

$$\begin{aligned} &-\tfrac{1}{2}\mathrm{Tr}_s[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s \\ &= -\tfrac{1}{2}\lim_{r\to\infty}\left(\mathrm{Tr}_s\sqrt{\phi_r}D(\rho)^2 e^{-A(\rho)_t^2}\sqrt{\phi_r} + \mathrm{Tr}_s\sqrt{\phi_r}D(\rho)e^{-A(\rho)_t^2}D(\rho)\sqrt{\phi_r}\right) \ . \end{aligned}$$

Since for $\nu \in \mathbb{N}_0$ the operators $\sqrt{\phi_r}D(\rho)^\nu e^{-A(\rho)_t^2/2}$ and $e^{-A(\rho)_t^2/2}D(\rho)^\nu\sqrt{\phi_r}$ are odd Hilbert-Schmidt operators, it follows that

$$\begin{aligned} &-\tfrac{1}{2}\mathrm{Tr}_s[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s \\ &= -\tfrac{1}{2}\lim_{r\to\infty}\left(\mathrm{Tr}_s e^{-A(\rho)_t^2/2}\phi_r D(\rho)^2 e^{-A(\rho)_t^2/2} - \mathrm{Tr}_s e^{-A(\rho)_t^2/2}D(\rho)\phi_r D(\rho)e^{-A(\rho)_t^2/2}\right) \\ &= \tfrac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s e^{-A(\rho)_t^2/2}c(d\phi_r)D(\rho)e^{-A(\rho)_t^2/2} \\ &= \tfrac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s c(d\phi_r)D(\rho)e^{-A(\rho)_t^2} \ . \end{aligned}$$

For $r > 0$ we define the function $\chi_r : Z \to \mathbb{R},\ \chi_r(x) := \chi(x_1 - r)$. Cor. 4.3.3 implies

$$\tfrac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s c(d\phi_r)D(\rho)e^{-A(\rho)_t^2} = \tfrac{1}{2}\lim_{r\to\infty}\sum_{k\in\mathbb{Z}/6}\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2} \ .$$

Recall from §4.2 that the integral kernel of $e^{-(A_t^{Z_k})^2}$ is $\sum_{n=0}^{\infty}(-1)^n t^{\frac{n}{2}} p_t^{Z_k}(x,y)^n$ with

$$p_t^{Z_k}(x,y)^n = c(dx_1)^n \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}}\left(p_t^{I_k}(x_2,y_2)^n \oplus (-1)^n p_t^{I_k}(x_2,y_2)^n\right) \ .$$

Hence the integral kernel of $c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2}$ is

$$\sum_{n=0}^{\infty}(-1)^n t^{\frac{n}{2}}\chi'(x_1-r)(I\partial_{x_2}-\partial_{x_1})p_t^{Z_k}(x,y)^n \ .$$

An easy computation shows that

$$\begin{aligned}&\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2}\\ &= -\frac{1}{\sqrt{4\pi t}}\sum_{n=0}^{\infty}t^{\frac{n}{2}}\int_0^1 \mathrm{tr}_s c(dx_1)^n\big((D_{I_k}p_t^{I_k})(x_2,x_2)^n\oplus(-1)^{n+1}(D_{I_k}p_t^{I_k})(x_2,x_2)^n\big)dx_2 \ .\end{aligned}$$

Comparison with def. 4.1.6 and the subsequent remark yields that

$$\mathrm{tr}_s c(dx_1)^n\big((D_{I_k}p_t^{I_k})(x_2,x_2)^n\oplus(-1)^{n+1}(D_{I_k}p_t^{I_k})(x_2,x_2)^n\big)=2i^n\mathrm{tr}_\sigma\sigma^n(D_{I_k}p_t^{I_k})(x_2,x_2)^n \ .$$

It follows that

$$\begin{aligned}\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2} &= -\frac{2}{\sqrt{4\pi t}}\sum_{n=0}^{\infty}(-1)^n t^n\int_0^1\mathrm{tr}(D_{I_k}p_t^{I_k})(x_2,x_2)^{2n}dx_2\\ &= -\frac{1}{\sqrt{\pi t}}\mathrm{Tr}_\sigma D_{I_k}e^{-(A_t^{I_k})^2} \ ,\end{aligned}$$

hence

$$-\tfrac{1}{2}\mathrm{Tr}_s[D(\rho),D(\rho)e^{-A(\rho)_t^2}]_s=-\sum_{k\in\mathbb{Z}/6}\frac{1}{\sqrt{4\pi t}}\mathrm{Tr}_\sigma D_{I_k}e^{-(A_t^{I_k})^2} \ .$$

□

4.4.5 The index theorem

Now we turn back to the notation used in Prop. 2.5.1.

Let A_{I_k} be as in the previous section.

Theorem 4.4.11.

$$\mathrm{ch}(\mathrm{ind}\,D^+)=\tfrac{1}{2}[\sum_{k\in\mathbb{Z}/6}\eta(A_{I_k})]\in H_{ev}^{dR}(\mathcal{A}_\infty) \ .$$

Here we understand $\mathrm{ind}\,D^+$ as a class in $K_0(\mathcal{A}_\infty)$ via the isomorphism $K_0(\mathcal{A})\cong K_0(\mathcal{A}_\infty)$ induced by the injection $\mathcal{A}_\infty\hookrightarrow\mathcal{A}$.

Proof. Let $\rho\neq 0$.

By §2.5

$$\mathrm{ind}\,D^+=[\mathrm{Ker}\,D(\rho)^2]-[\mathcal{A}^{\mathcal{N}}]$$

in $K_0(\mathcal{A})$. Let P_0 be the projection onto the kernel of $D(\rho)^2$.

From Prop. 3.5.6 and Prop. 5.3.7 it follows that

$$\mathrm{ind}\,D^+=[\mathrm{Ran}_\infty P_0]-[\mathcal{A}_\infty^{\mathcal{N}}]=[\mathrm{Ran}_\infty WP_0W^*]-[\mathcal{A}_\infty^{\mathcal{N}}]$$

in $K_0(\mathcal{A}_\infty)$, hence in $H^{dR}(\mathcal{A}_\infty)$

$$\mathrm{ch}(\mathrm{ind}\, D^+) = \mathrm{ch}[\mathrm{Ran}_\infty\, WP_0W^*] - \mathcal{N}.$$

By Prop. 5.3.7 in $H^{dR}(\mathcal{A}_\infty)$

$$\begin{aligned}\mathrm{ch}[\mathrm{Ran}_\infty\, WP_0W^*] &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(WP_0W^*\,\mathrm{d}\,WP_0W^*)^{2n}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0W^*\,\mathrm{d}\,WP_0)^{2n}\ .\end{aligned}$$

Let $\rho, T > 0$. By Th. 4.4.3 and Th. 4.4.8 in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0W^*\,\mathrm{d}\,WP_0)^{2n} - \mathcal{N} &= \lim_{t\to\infty}\mathrm{Tr}_s e^{-A(\rho)_t^2} - \lim_{t\to 0}\mathrm{Tr}_s e^{-A_t^2}\\ &= \int_T^\infty \frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2}dt\\ &\quad + \int_0^\rho \frac{d}{d\rho'}\mathrm{Tr}_s e^{-A(\rho')_T^2}d\rho'\\ &\quad + \int_0^T \frac{d}{dt}\mathrm{Tr}_s e^{-A_t^2}dt\ .\end{aligned}$$

The integrals converge by the results of §4.4.4, the estimate of Th. 4.4.3 and Th. 4.4.8 and

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0W^*\,\mathrm{d}\,WP_0)^{2n} - \mathcal{N} &= -\sum_{k\in\mathbb{Z}/6}\int_0^\infty \frac{1}{\sqrt{4\pi t}}\mathrm{Tr}_s D_{I_k}e^{-(A_t^{I_k})^2}dt\\ &\quad - \mathrm{d}\int_T^\infty \frac{1}{2\sqrt{t}}\mathrm{Tr}_s D(\rho)e^{-A(\rho)_t^2}dt\\ &\quad - \mathrm{d}\int_0^\rho \mathrm{Tr}_s\frac{dA(\rho')_t}{d\rho'}e^{-A(\rho')_t^2}d\rho'\\ &\quad - \mathrm{d}\int_0^T \frac{1}{2\sqrt{t}}\mathrm{Tr}_s D(0)e^{-A_t^2}dt\ .\end{aligned}$$

The assertion follows from the definition of the η-form, Def. 4.1.6. □

Corollary 4.4.12. *In* $H^{dR}(\mathcal{A}_\infty)$

$$\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [\eta(\mathcal{P}_0, \mathcal{P}_1) + \eta(\mathcal{P}_1, \mathcal{P}_2) + \eta(\mathcal{P}_2, \mathcal{P}_0)]\ .$$

Proof. For $\rho > 0$ we define a family $A(\rho, b)$ of superconnections associated to $D(\rho)$: For $0 < b < b_0$ let $W(b) \in C^\infty(M, \mathrm{End}^+E \otimes \mathcal{A}_\infty)$ be as in §4.3.1 such that $W(b)$ is parallel on $M \setminus F(r, b)$ for any $r \geq 0$. As in §4.3.1 we set

$$A(\rho, b) := W(b)^*\,\mathrm{d}\,W(b) + D(\rho)\ .$$

For every $k \in \mathbb{Z}/6$ this induces a family of superconnections $A_{I_k}(b)$. By Prop. 4.1.7 and the subsequent definition

$$\lim_{b \to 0} \eta(A_{I_k}(b)) = \eta(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{k+1 \bmod 3}) \ .$$

The assertion follows now by the previous theorem and Prop. 2.4.4. □

Chapter 5

Definitions and Technics

5.1 Hilbert C^*-modules

5.1.1 Bounded operators

Let $\mathcal{A}$ be a unital C^*-algebra with norm $|\cdot|$.

In order to fix notation we recall some facts about Hilbert $\mathcal{A}$-modules:

Definition 5.1.1. *A* PRE-HILBERT $\mathcal{A}$-MODULE *is a right $\mathcal{A}$-module H with an $\mathcal{A}$-valued scalar product $< \ , \ >: H \times H \to \mathcal{A}$, that is*

1. $< \ , \ >$ *is $\mathcal{A}$-linear in the second variable,*
2. $< x, y >=< y, x >^*$,
3. $< x, x >\geq 0$ *for all* $x \in H$,
4. *if* $< x, x >= 0$, *then* $x = 0$.

If H is complete with respect to the norm $\|v\| := |< v, v >|$, then H is called a HILBERT $\mathcal{A}$-MODULE.

The right $\mathcal{A}$-module $\{(a_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} a_n^* a_n \text{ converges}\}$ endowed with the $\mathcal{A}$-valued scalar product

$$< (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} >:= \sum_{n=1}^{\infty} a_n^* b_n$$

is denoted by $l^2(\mathcal{A})$. It can also be defined with index set $\mathbb{Z}$.

Let M be a measure space and let $< \ , \ >$ be the standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^n$. Then the Hilbert $\mathcal{A}$-module $L^2(M, \mathcal{A}^n)$ is defined in the following way: By

$$< f, g >_{L^2} = \int_M < f(x)g(x) > dx$$

an $\mathcal{A}$-valued scalar product is defined on the quotient of the space of simple functions on M with values in $\mathcal{A}^n$ by those simple functions that vanish on the complement of some

set of measure zero. Hence the quotient is a pre-Hilbert $\mathcal{A}$-module. The corresponding Hilbert $\mathcal{A}$-module is $L^2(M, \mathcal{A}^n)$.

Let H be a Hilbert $\mathcal{A}$-module.

A submodule $U \subset H$ is called complemented if $U^\perp = \{x \in H \mid \ < x, u >= 0 \ \forall u \in U\}$ satisfies $U \oplus U^\perp = H$.

For example any projective submodule in H is complemented.

Let H_1, H_2 be Hilbert $\mathcal{A}$-modules. The elements of

$$\begin{aligned} B(H_1, H_2) \ &= \ \{T : H_1 \to H_2 \mid T\text{is continuous and } \exists T^* : H_2 \to H_1 \text{ with} \\ &\quad < Tv, w >_{H_2} =< v, T^* w >_{H_1} \ \ \forall v \in H_1, \ w \in H_2\} \end{aligned}$$

are called bounded operators from H_1 to H_2. They form a Banach space with respect to the operator norm. With the composition as a product $B(H_1) := B(H_1, H_1)$ is a C^*-algebra. Note that the existence of an adjoint must be required.

A continuous $\mathcal{A}$-module map $K : H_1 \to H_2$ is called compact if it can be approximated in the operator norm topology by finite rank operators.

If K is compact, then K is adjointable, thus $K \in B(H_1, H_2)$.

A projection in $B(H)$ is compact if and only if it is a projection onto a projective submodule of H.

If the range of $T \in B(H_1, H_2)$ is complemented, we call its complement the cokernel $\operatorname{Coker} T$. Clearly a necessary condition is that the range of T is closed. The following proposition shows that it is sufficient:

Proposition 5.1.2. *Suppose that $T \in B(H_1, H_2)$ has closed range.*

1. $\operatorname{Ker} T$ *is a complemented submodule of* H_1,
2. $\operatorname{Ran} T$ *is a complemented submodule of* H_2,
3. $T^* : H_2 \to H_1$ *also has closed range.*

Proof. [La], Th. 3.2 □

5.1.2 Fredholm operators

Let H_1, H_2 be Hilbert $\mathcal{A}$-modules which are isomorphic to $l^2(\mathcal{A})$.

There are two different notions of Fredholm operators from H_1 to H_2 which we present now. In the theory of Fredholm operators developped by Miščenko and Fomenko [MF] it is not required that the operators are adjointable:

Definition 5.1.3. *A* Fredholm operator in the sense of Miščenko/Fomenko *is an $\mathcal{A}$-linear continuous map $F : H_1 \to H_2$, not necessarily with adjoint, such that there are decompositions $H_1 = M_1 \oplus N_1$ and $H_2 = M_2 \oplus N_2$ with the following properties:*

1. *N_1, N_2 are projective $\mathcal{A}$-modules and M_1, M_2 are closed $\mathcal{A}$-modules.*

2. *The operator F is diagonal: $F = F_M \oplus F_N$ with $F_M : M_1 \to M_2$ and $F_N : N_1 \to N_2$.*

3. *The component $F_M : M_1 \to M_2$ is an isomorphism.*

The INDEX OF F *is defined as*

$$\operatorname{ind} F := [N_1] - [N_2] \in K_0(\mathcal{A}) \ .$$

Proposition 5.1.4. *An $\mathcal{A}$-linear continuous map $F : H_1 \to H_2$ is Fredholm in the sense of Miščenko/Fomenko if and only if there exists an $\mathcal{A}$-linear continuous map $G : H_2 \to H_1$ such that $FG - 1$ and $GF - 1$ are compact.*

Proof. [MF]. □

From the proposition it follows that if F is Fredholm in the sense of Miščenko/Fomenko, then for any compact operator K the operator $F + K$ is Fredholm in the sense of Miščenko/Fomenko.

Proposition 5.1.5. *If $F : H_1 \to H_2$ is a Fredholm operator in the sense of Miščenko/Fomenko and $K : H_1 \to H_2$ is a compact operator, then*

$$\operatorname{ind} F = \operatorname{ind}(F + K) \ .$$

Proof. [MF], Lemma 2.3. □

In our context the following notion, discussed in [WO], is more appropriate:

Definition 5.1.6. *An operator $F \in B(H_1, H_2)$ is* FREDHOLM *if there is a compact operator $K : H_1 \to H_2$ such that* $\operatorname{Ran}(F + K)$ *is closed and such that* $\operatorname{Ker}(F + K)$ *and* $\operatorname{Coker}(F + K)$ *are projective $\mathcal{A}$-modules.*

The INDEX OF F *is*

$$\operatorname{ind} F := [\operatorname{Ker}(F + K)] - [\operatorname{Coker}(F + K)] \ .$$

Fortunately for adjointable operators both notions are equivalent:

Lemma 5.1.7. *An operator $F \in B(H_1, H_2)$ is Fredholm if and only if it is Fredholm in the sense of Miščenko/Fomenko.*

Proof. If F is Fredholm, let $K \in B(H_1, H_2)$ be such that $\operatorname{Ran}(F + K)$ is closed and $\operatorname{Ker}(F + K)$ and $\operatorname{Coker}(F + K)$ are projective $\mathcal{A}$-modules. By Prop. 5.1.2 the kernel of $F + K$ is complemented. Define $N_1 := \operatorname{Ker}(F + K)$, let M_1 be its orthogonal complement, let $N_2 := \operatorname{Coker}(F + K)$ and $M_2 := \operatorname{Ran}(F + K)$. These decompositions show that $F + K$ is Fredholm in the sense of Miščenko/Fomenko.

Assume now that F is Fredholm in the sense of Miščenko/Fomenko and let $H_1 = M_1 \oplus N_1$, $H_2 = M_2 \oplus N_2$ be the corresponding decompositions. Let P be the orthogonal projection onto N_1 and let P_{N_1} be the projection onto N_1 along M_1. Since N_1 is projective, P is compact, hence $P_{N_1} = PP_{N_1}$ is compact and thus FP_{N_1} is compact as well. Then $F - FP_{N_1}$ is an adjointable operator with closed range M_2, with kernel N_1 and cokernel N_2. It follows that F is Fredholm. □

Another important property of Fredholm operators is the following:

Proposition 5.1.8. *If $F \in B(H_1, H_2)$ is Fredholm and* $\operatorname{Ran} F$ *is closed, then* $\operatorname{Ker} F$ *and* $\operatorname{Coker} F$ *are projective modules.*

Hence

$$\operatorname{Ind} F = [\operatorname{Coker} F] - [\operatorname{Ker} F] \ .$$

Proof. Let $P_{\operatorname{Ker} F} \in B(H_1)$ resp. $P_{\operatorname{Coker} F} \in B(H_2)$ be the orthogonal projection onto the kernel resp. cokernel of F. It exist by Prop. 5.1.2. We have to prove that it is compact.

There is $E \in B(H_2, H_1)$ such that $EF = 1 - P_{\operatorname{Ker} F}$ and $FE = 1 - P_{\operatorname{Coker} F}$.

Let G be a parametrix of F. It follows that

$$P_{\operatorname{Ker} F} = 1 - EF = (1 - GF)(1 - EF)$$

and

$$P_{\operatorname{Coker} F} = 1 - FE = (1 - FE)(1 - FG)$$

are compact operators. □

Proposition 5.1.9. *If $F : [0, 1] \to B(H_1, H_2)$ is a continuous path of Fredholm operators, then the map $[0, 1] \to K_0(\mathcal{A}),\ t \mapsto \operatorname{ind} F(t)$ is constant.*

Proof. see [WO], Prop. 17.3.4. □

5.1.3 Regular operators

In this section we study unbounded operators on Hilbert $\mathcal{A}$-modules.

Let H be a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued scalar product $< \ , \ >$. Let $D : \operatorname{dom} D \to H$ be a densily defined operator on H.

Lemma 5.1.10. *If the adjoint D^* of D is densily defined, then D is closable.*

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\operatorname{dom} D$ such that (f_n, Df_n) converges to $(0, f)$ in $H \oplus H$ for $n \to \infty$. Then for every $g \in \operatorname{dom} D^*$

$$< f, g >= \lim_{n \to \infty} < Df_n, g >= \lim_{n \to \infty} < f_n, Dg >= 0.$$

Since $\operatorname{dom} D^*$ is dense in H, it follows that $f = 0$. □

If D is closed, then

$$< f, g >_D := < f, g > + < Df, Dg >$$

is an $\mathcal{A}$-valued scalar product on $\operatorname{dom} D$ with respect to which $\operatorname{dom} D$ is a Hilbert $\mathcal{A}$-module. It is denoted by $H(D)$.

Lemma 5.1.11. *Assume that D is closed.*

1. *Suppose that D has a densily defined adjoint D^*. Then* $\operatorname{Ker} D^* = (\operatorname{Ran} D)^{\perp}$.
2. $\operatorname{Ker} D$ *is complemented in $H(D)$ if and only if* $\operatorname{Ker} D$ *is complemented in H.*

Proof. 1) Since for $f \in \operatorname{Ker} D^*$ and $h \in \operatorname{dom} D$

$$< f, Dh >=< D^*f, h >= 0 \ ,$$

the $\mathcal{A}$-module $\operatorname{Ker} D^*$ is a submodule of $(\operatorname{Ran} D)^\perp$.

On the other hand the linear functional

$$\operatorname{dom} D \to \mathcal{A}, \ f \mapsto < Df, g >$$

vanishes for $g \in (\operatorname{Ran} D)^\perp$. Thus $g \in \operatorname{dom} D^*$ and $D^*g = 0$.

2) Note first that for $g \in \operatorname{Ker} D$ and $f \in \operatorname{dom} D$ the conditions $< f, g >_D = 0$ and $< f, g >= 0$ are equivalent.

If $\operatorname{Ker} D$ is complemented in $H(D)$, then $\operatorname{Ker} D$ is complemented in H since

$$\begin{aligned} H = \overline{H(D)} &= \overline{(\operatorname{Ker} D)^{\perp_{H(D)}} \oplus \operatorname{Ker} D} \\ &= (\operatorname{Ker} D)^\perp \oplus \operatorname{Ker} D \ . \end{aligned}$$

On the other hand if $\operatorname{Ker} D$ is complemented in H, we can decompose $g \in H(D)$ in a sum $g = g_1 + g_2$ with $g_1 \in \operatorname{Ker} D$ and $g_2 \in (\operatorname{Ker} D)^\perp$. From $\operatorname{Ker} D \subset H(D)$ it follows that $g_2 = g - g_1 \in H(D)$, hence $g_2 \in (\operatorname{Ker} D)^{\perp_{H(D)}}$. □

Recall that D is called regular if it is closed with densily defined adjoint D^* and if $1+D^*D$ has dense range, or equivalently if it is closed with densily defined adjoint and if its graph is complemented in $H \times H$.

If D is regular, then $1 + D^*D$ has a bounded inverse.

In the following we denote the adjoint of an operator $A \in B(H(D), H)$ by $A^T \in B(H, H(D))$ in order to distinguish it from the adjoint A^* of A as an unbounded operator on H.

Lemma 5.1.12. *Assume that D is closed.*

1. *The operator D is regular if and only if the inclusion $\iota : H(D) \to H$ is in $B(H(D), H)$ and $(1 + D^*D)$ is selfadjoint.*

   Then $\iota^T = (1 + D^*D)^{-1} \in B(H, H(D))$ and $(1 + D^*D)^{-\frac{1}{2}} : H \to H(D)$ is an isometry.

2. *Assume that D is regular and selfadjoint. Then $D \in B(H(D), H)$ and $D^T = D(1 + D^2)^{-1}$.*

Proof. If D is regular, then for $v \in H(D)$ and $w \in H$

$$\begin{aligned} < \iota v, w > &= < v, w > \\ &= < v, (1 + D^*D)(1 + D^*D)^{-1}w > \\ &= < v, (1 + D^*D)^{-1}w > + < Dv, D(1 + D^*D)^{-1}w > \\ &= < v, (1 + D^*D)^{-1}w >_D \ . \end{aligned}$$

This shows $\iota^T = (1 + D^*D)^{-1}$.

Now the converse direction:

Let $v \in H$. Then for any $w \in \mathrm{dom}(1 + D^*D)$

$$< v, w >=< \iota^T v, w >_D =< \iota^T v, (1 + D^*D)w > \ .$$

Since $(1+D^*D)$ is selfadjoint, it follows that $\iota^T v \in \mathrm{dom}(1+D^*D)$ and $(1+D^*D)\iota^T v = v$. This shows that $(1 + D^*D)$ is surjective and that ι^T is a right inverse of $(1 + D^*D)$. Since $(1 + D^*D)$ is bounded below, it is injective as well. It follows that $(1 + D^*D)$ is invertible and ι^T is its inverse.

The remaining parts are immediate.

□

Proposition 5.1.13. *Let D_0 be a regular selfadjoint operator and assume $D = D_0 + V$ with $V \in B(H)$.*

1. *Then D is closed.*
2. *The identity map induces a continuous isomorphism from $H(D_0)$ to $H(D)$.*
3. *$D \in B(H(D_0), H)$.*
4. *Suppose that D is selfadjoint. Then D is regular.*

Proof. 1) From $\mathrm{dom}\, D^{**} = \mathrm{dom}\, D_0^{**} = \mathrm{dom}\, D_0 = \mathrm{dom}\, D$ it follows that $D = D^{**}$. Thus D is closed.

Assertion 2) follows from the fact that there is $C > 0$ such that for all $f \in H(D_0)$

$$\begin{aligned} \|f\|_D^2 &\leq \|f\|^2 + \|D_0 f\|^2 + \| < Vf, D_0 f > \| + \| < D_0 f, Vf > \| + \|Vf\|^2 \\ &\leq C(\|f\|^2 + \|D_0 f\|^2) + 2\|Vf\|\|D_0 f\| \\ &\leq C\|f\|_{D_0}^2 \ . \end{aligned}$$

We applied Cauchy-Schwarz inequality.

3) By 2) the operator $D : H(D_0) \to H$ is continuous. By the previous lemma the adjoint of $D = D_0 + V\iota : H(D_0) \to H$ is

$$D^T = D_0(1 + D_0^2)^{-1} + (1 + D_0^2)^{-1}V^* : H \to H(D_0) \ .$$

4) By 3) the operator $D + i : H(D_0) \to H$ is an adjointable bounded operator. By [La], Lemma 9.7, the range of $D + i$ is closed. Thus it is complemented by Prop. 5.1.2. From Lemma 5.1.11 it follows that the cokernel of $D + i$ agrees with the kernel of $D - i$. By [La], Lemma 9.7, the operator $D - i$ is injective. It follows that $\mathrm{Coker}(D + i) = \{0\}$. By [La], Lemma 9.8, this shows that D is regular. □

Proposition 5.1.14. *Let D be a regular and selfadjoint operator on H with closed range.*

1. *The cokernel of D exists and* $\mathrm{Ker}\, D = \mathrm{Coker}\, D$. *In particular* $\mathrm{Ker}\, D$ *is complemented.*

2. *The $\mathcal{A}$-module* $\operatorname{dom} D \cap \operatorname{Ran} D$ *is dense in* $\operatorname{Ran} D$ *and* $\operatorname{dom} D = \operatorname{Ker} D \oplus (\operatorname{dom} D \cap \operatorname{Ran} D)$, *thus* $D = 0 \oplus D|_{\operatorname{Ran} D}$ *and* $D|_{\operatorname{Ran} D}$ *is invertible.*

Proof. 1) Since $D \in B(H(D), H)$ by Lemma 5.1.12 and since the range of D is closed, it is complemented by prop 5.1.2. Its complement is $\operatorname{Coker} D$. Since D is selfadjoint, $\operatorname{Ker} D = \operatorname{Coker} D$ by Lemma 5.1.11.

2) Let $P : H \to \operatorname{Ran} D$ be the orthogonal projection. From $(1 - P)(\operatorname{dom} D) \subset \operatorname{Ker} D \subset \operatorname{dom} D$ we conclude that $P(\operatorname{dom} D) \subset \operatorname{dom} D$. The assertion follows because $P(\operatorname{dom} D)$ is dense in $P(H) = \operatorname{Ran} D$. □

We will need the following $\mathbb{Z}/2$-version of the previous proposition:

If $H = H^+ \oplus H^-$ is $\mathbb{Z}/2$-graded, then we call a closed operator D on H even resp. odd if $\operatorname{dom} D$ decomposes in $(\operatorname{dom} D)^+ \oplus (\operatorname{dom} D)^-$ and if the action of D is even resp. odd.

Proposition 5.1.15. *Let H be a $\mathbb{Z}/2$-graded Hilbert $\mathcal{A}$-module and let D be an odd regular selfadjoint operator on H.*

Suppose that $D^+ : (\operatorname{dom} D)^+ \to H^-$ is surjective.

Then the range of D is closed and complemented and $\operatorname{Ker} D^+ = \operatorname{Ker} D = \operatorname{Coker} D = \operatorname{Coker} D^-$ and this module is complemented.

Proof. Since D^+ is surjective, D^- is injective and so $\operatorname{Ker} D^+ = \operatorname{Ker} D$.

Let P_+ be the orthogonal projection onto H^+. Since D is odd, $DP_+ = P_+ D$.

By Lemma 5.1.12 the operator $DP_+ : H(D) \to H$ is adjointable with adjoint $P_+ D(D^2 + 1)^{-1}$. From $P^+ D|_{H(D)^\pm} = D^\pm$ it follows that $D^-(D^2+1)^{-1} : H^- \to H(D)^+$ is the adjoint of $D^+ : H(D)^+ \to H^-$.

Since D^+ is surjective, $\operatorname{Ker} D^+$ is complemented in $H(D)^+$ and the range of the adjoint $D^-(D^2 + 1)^{-1} : H^- \to H(D)^+$ is closed. Furthermore

$$D^-(D^2 + 1)^{-1} = (D^2 + 1)^{-\frac{1}{2}} D^-(D^2 + 1)^{-\frac{1}{2}} ,$$

and $(D^2 + 1)^{-1/2} : H^\pm \to H(D)^\pm$ is an isomorphism by Lemma 5.1.12, hence $\operatorname{Ran} D^-$ is closed, too. □

Proposition 5.1.16. *Let D be a regular selfadjoint operator on H.*

1. *For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the operator $D - \lambda$ is invertible.*
2. *Assume that the range of D is closed and let P be the projection onto the kernel of D. Then there is $c > 0$ such that the spectrum of $(D + P)$ is contained in $\mathbb{R}\setminus]-c, c[$ and the spectrum of D is contained in $(\mathbb{R}\setminus]-c, c[) \cup \{0\}$.*

Proof. This follows from the functional calculus for regular operators ([La], Th. 10.9) and from the decomposition in Prop. 5.1.14. □

The following criteria for selfadjointness and regularity will be useful:

Lemma 5.1.17. *Let D be a symmetric regular operator such that the ranges of $D + i$ and of $D - i$ are dense in H. Then D is selfadjoint.*

Proof. By [La], Lemma 9.7 the operators $D+i$ and of $D-i$ have closed ranges. It follows that they have a bounded inverse on H. Then they are adjoint to each other, thus D is selfadjoint. □

Lemma 5.1.18. *Assume that D is symmetric and has an inverse $D^{-1} \in B(H)$. Then D is regular.*

Proof. From $D^{-1} \in B(H)$ it follows that the graph of D^{-1} is complemented, hence the graph of D is complemented as well. Since D is symmetric, the adjoint is densily defined. Hence D is regular. □

5.1.4 Decompositions of Hilbert C^*-modules

Let H be a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued scalar product $< \ , \ >$. Let $J = \{1, \dots, m\} \subset \mathbb{N}$ resp. $J = \mathbb{N}$. If $J = \mathbb{N}$, then set $m = \infty$.

Definition 5.1.19. *A system $\{f_k\}_{k\in J} \subset H$ is called* ORTHONORMAL *if for all $k, l \in J$*

$$< f_k, f_l >= \delta_{kl} \ .$$

It is called an ORTHONORMAL BASIS OF *H if for all $f \in H$ there is $(a_n)_{n\in J} \subset \mathcal{A}$ such that $f = \sum_{n=1}^{m} f_n a_n$.*

Since $a_n =< f_n, f >$, the coefficients are uniquely defined by the system.

Lemma 5.1.20. *Let $\{f_k\}_{k\in J}$ be an orthonormal system in H whose span is dense in H. Then it is an orthonormal basis of H and H is isomorphic as a Hilbert $\mathcal{A}$-module to $\mathcal{A}^m$ if $m < \infty$ and to $l^2(\mathcal{A})$ else.*

Proof. Let P_n be the orthogonal projection onto the span of the first n vectors of the system $\{f_k\}_{k\in J}$. On the algebraic span of $\{f_k\}_{k\in J}$ the projection P_n converges strongly to the identity for $n \to \infty$. Since $\|P_n\| = 1$ for all $n \in \mathbb{N}$, it follows that P_n converges strongly to the identity on H.

The isomorphism is given by $f \mapsto (< f_n, f >)_{n\in J}$. □

Lemma 5.1.21. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of pairwise orthogonal closed subspaces of H such that $\oplus_{i\in\mathbb{N}} U_i$ is dense in H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of operators with $T_i \in B(U_i)$ and assume that there is $c \in \mathbb{R}$ such that $\|T_i\| \le c$ for all $i \in \mathbb{N}$.*

Then the closure T of the operator $\oplus_{i\in\mathbb{N}} T_i$ is in $B(H)$ and $\|T\| \le c$.

Proof. The spectral radius of an operator $A \in B(H)$ is denoted by $r(A)$.

Write $T(n)$ for the restriction of T on $\oplus_{i=1}^{n} U_i$. Then for all $n \in \mathbb{N}$

$$\|T(n)\|^2 = r(T(n)^*T(n)) = \max_{1\le i\le n} r(T_i^*T_i) = \max_{1\le i\le n} \|T_i^*T_i\| \le c^2 \ .$$

For $v \in \operatorname{dom} T$ there is $n \in \mathbb{N}$ such that $v \in \oplus_{i=1}^{n} U_i$. Then $Tv = T(n)v$ and thus

$$\|Tv\| = \|T(n)v\| \le c\|v\| \ .$$

It follows that $\oplus_{i\in\mathbb{N}} T_i$ is bounded. Since it is densily defined, its closure is a bounded operator on H. The adjoint is given by the closure of $\oplus_{i\in\mathbb{N}} T_i^*$. □

Corollary 5.1.22. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of pairwise orthogonal closed subspaces of H such that $\oplus_{i\in\mathbb{N}}U_i$ is dense in H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of operators such that $T_i^{-1}\in B(U_i)$ and assume that there is $c\in\mathbb{R}$ such that $\|T_i^{-1}\|\leq c$ for all $i\in\mathbb{N}$. Then the closure T of the map $\oplus_{i\in\mathbb{N}}T_i$ is invertible with inverse in $B(H)$.*

Proof. The operator $\oplus_{i\in\mathbb{N}}T_i^{-1}$ is inverse to $\oplus_{i\in\mathbb{N}}T_i$. It fulfills the conditions of the previous lemma and hence its closure is a bounded operator on H. It is the inverse of the closure of T. □

5.2 Banach space valued functions

5.2.1 Function spaces and tensor products

Let V be a Fréchet space.

If a space of V-valued functions can be described as an ε- or π-tensor product of V with a complex function space, then every bounded operator on the complex function space extends to a bounded operator on the corresponding space of V-valued functions.

In the following we list some important examples. The proofs can be found in [Tr] or they are easy generalisations. If not specified the functions are assumed complex valued.

- Let M be a compact topological space.

  Then the inclusion $C(M)\odot V\to C(M,V)$ extends to an isomorphism
  $$C(M)\otimes_\varepsilon V\cong C(M,V)\ .$$

  For compact spaces M,N
  $$C(M\times N)\cong C(M)\otimes_\varepsilon C(N)\ .$$

- Let $U\subset\mathbb{R}^n$ be open and precompact. For all $m\in\mathbb{N}_0$ there is a canonical isomorphism
  $$C_0^m(U)\otimes_\varepsilon V\cong C_0^m(U,V)\ .$$

- Let M be a closed smooth manifold. For all $m\in\mathbb{N}_0$ there is a canonical isomorphism
  $$C^m(M)\otimes_\varepsilon V\cong C^m(M,V)\ .$$

- Let $U\subset\mathbb{R}^n$ be open and precompact. Then $C_0^\infty(U)$ is nuclear, in particular there are canonical isomorphisms
  $$C_0^\infty(U)\otimes_\pi V\cong C_0^\infty(U)\otimes_\varepsilon V\cong C_0^\infty(U,V)\ .$$

- Let M be a closed smooth manifold. Then $C^\infty(M)$ is nuclear, in particular there are canonical isomorphisms
  $$C^\infty(M)\otimes_\pi V\cong C^\infty(M)\otimes_\varepsilon V\cong C^\infty(M,V)\ .$$

  For closed smooth manifolds M,N
  $$C^\infty(M\times N)\cong C^\infty(M)\otimes C^\infty(N)\ .$$

- The space of Schwartz functions $\mathcal{S}(\mathbb{R})$ is nuclear, in particular
  $$\mathcal{S}(\mathbb{R})\otimes_\pi V\cong\mathcal{S}(\mathbb{R})\otimes_\varepsilon V\cong\mathcal{S}(\mathbb{R},V)\ .$$

5.2.2 L^2-spaces and integral operators

Let E be Banach space with norm $|\cdot|$. Let End$E$ be the Banach algebra of bounded operators on E. We denote the operator norm on End$E$ by $|\cdot|$ as well.

Definition 5.2.1. *Let M be a measure space and $p \in \mathbb{N}$.*

The Banach space $L^p(M,E)$ is defined to be the completion of the quotient of the space of simple E-valued functions on M by the subspace of functions vanishing outside a set of measure zero with respect to the norm

$$\|f\| := \left(\int_M |f(x)|^p dx\right)^{\frac{1}{p}} .$$

In order to avoid confusion we make the following convention: If $E = \mathcal{A}^n$ for a C^*-algebra $\mathcal{A}$, then $L^2(M,E)$ denotes the Hilbert $\mathcal{A}$-module defined in §5.1.1 and not the space just defined. In general they do not coincide.

Lemma 5.2.2. *Let M_1, M_2 be σ-finite measure spaces. Then the map*

$$L^2(M_1 \times M_2, E) \cong L^2(M_1, L^2(M_2, E)),\ \ f \mapsto (x \mapsto f(x,\cdot))$$

is an isometry.

Proof. The lemma follows from Fubini. □

Proposition 5.2.3. *Let M be a measure space.*

Let $k : M \times M \to \mathrm{End}E$ be such that the integral kernel $|k(x,y)|$ defines a bounded operator $|K|$ on $L^2(M)$. Then k defines a bounded operator on $L^2(M,E)$ with norm less than or equal to $|||K|||$.

Proof. For a simple function $f : M \to E$

$$\| \int_M k(x,y) f(y) dy \|_{L^2} \leq \| \int_M |k(x,y)| |f(y)| dy \|_{L^2} \leq |||K|||\ \|f\|_{L^2}.$$

□

Corollary 5.2.4. *Let M be a measure space.*

There is a norm-decreasing map

$$L^2(M \times M, \mathrm{End}E) \to B(L^2(M,E), L^2(M,E))$$

$$k \mapsto \left(f \mapsto Kf := \int_M k(\ .\ , y) f(y) dy\right) .$$

Corollary 5.2.5. *The convolution induces a continuous map*

$$L^1(\mathbb{R}^n, \mathrm{End}E) \to B(L^2(\mathbb{R}^n, E)),\ \ f \mapsto (g \mapsto f * g) .$$

Proof. The convolution with $f \in L^1(\mathbb{R}^n, \mathrm{End}E)$ is an integral operator with integral kernel $f(x-y)$. Since $|f| \in L^1(\mathbb{R}^n)$ for $f \in L^1(\mathbb{R}^n, \mathrm{End}E)$, the convolution with $|f|$ is bounded on $L^2(\mathbb{R}^n)$. Thus the assertion follows from the previous proposition. □

Lemma 5.2.6. *For any $f \in L^p(\mathbb{R}^n, E)$ the translation map*

$$\tau f : \mathbb{R}^n \to L^p(\mathbb{R}^n, E),\ y \mapsto \tau_y f\ ,$$

defined by

$$\tau_y f(x) := f(x-y)\ ,$$

is continuous.

Proof. The proof is analogous to the case $E = \mathbb{C}$, see [Co], ch. VII, Prop. 9.2. □

5.2.3 Hilbert-Schmidt operators

Let $\mathcal{B}$ be an involutive Banach algebra with unit, and let M be a σ-finite measure space. In this section all operators are assumed to be right $\mathcal{B}$-module maps. In particular we identify $\mathrm{End}(\mathcal{B}^n)$ with $M_n(\mathcal{B})$.

Lemma 5.2.7. *Let $k \in L^2(M \times M, M_n(\mathcal{B}))$ and let K be the corresponding integral operator. Then k is uniquely defined by K.*

Proof. It is enough to show that k vanishes in $L^2(M \times M, M_n(\mathcal{B}))$ if $K = 0$.

Let $\mathcal{B}'$ be the topological dual of $\mathcal{B}$. Applying $\lambda \in \mathcal{B}'$ componentwise yields maps $\lambda : M_n(\mathcal{B}) \to M_n(\mathbb{C})$ and $\lambda : \mathcal{B}^n \to \mathbb{C}^n$.

For $f \in L^2(M, \mathbb{C}^n)$ almost everywhere

$$\lambda(\int_M k(x,y)f(y)dy) = \int_M \lambda(k(x,y))f(y)dy = 0\ .$$

It follows $\lambda(k(x,y)) = 0$ in $L^2(M \times M, M_n(\mathbb{C}))$. Since $\mathcal{B}'$ is separable, the set containing all $(x,y) \in M \times M$ such that there is $\lambda \in \mathcal{B}'$ with $\lambda(k(x,y)) \neq 0$ has measure zero. Outside this set k vanishes. □

Definition 5.2.8. *A* Hilbert-Schmidt operator on $L^2(M, \mathcal{B}^n)$ *is an integral operator with an integral kernel in $L^2(M \times M, M_n(\mathcal{B}))$. Let A be a Hilbert-Schmidt operator on $L^2(M, \mathcal{B}^n)$. Its integral kernel in $L^2(M \times M, M_n(\mathcal{B}))$ is denoted by k_A. We define*

$$\|A\|_{HS} := \|k_A\|\ ,$$

where the norm on the right hand side is taken in $L^2(M \times M, M_n(\mathcal{B}))$.

The normed space of Hilbert-Schmidt operators on $L^2(M, \mathcal{B}^n)$ is denoted by $HS(L^2(M, \mathcal{B}^n))$.

Note that $HS(L^2(M, \mathcal{B}^n))$ is a Banach algebra and that the inclusion $HS(L^2(M, \mathcal{B}^n)) \to B(L^2(M, \mathcal{B}^n))$ is bounded. Prop. 5.2.12 below shows that $HS(L^2(M, \mathcal{B}^n))$ is a left $B(L^2(M, \mathcal{B}^n))$-module.

Definition 5.2.9. *Let E be a Banach right $\mathcal{B}$-module with norm $|\cdot|$.*

A $\mathcal{B}$-valued non-degenerated product on E *is a sesquilinear map $< \ , \ >: E \times E \to \mathcal{B}$ with the following properties:*

1. $< v, wb >=< v, w > b$ *and* $< vb, w >= b^* < v, w >$ *for all* $v, w \in E,\ b \in \mathcal{B}$,

2. *if* $< v, w >= 0$ *for all* $w \in E$, *then* $v = 0$,

3. *there is* $C > 0$ *such that* $| < v, w > | \leq C|v|_E\ |w|_E$ *for all* $v, w \in E$.

Definition 5.2.10. *Let* $<\ ,\ >$ *be a* $\mathcal{B}$*-valued non-degenerated product on* E. *A bounded operator* $T : E \to E$ *is called* ADJOINTABLE *if there is a map* $T^* : E \to E$ *satisfying*

$$< v, Tw >=< T^*v, w >$$

for all $v, w \in E$.

Lemma 5.2.11. *Let* $T : E \to E$ *be adjointable.*

1. *The adjoint* T^* *is unique.*

2. T^* *is a right* $\mathcal{B}$*-module map.*

3. T^* *is bounded.*

4. $T^{**} = T$.

5. $(ST)^* = T^*S^*$.

Proof. Let T^* be an adjoint of T and $\Gamma(T^*)$ its graph. Then

$$\Gamma(T^*) \subset G := \{(x, y) \in E \times E \mid\ < y, w > + < -x, Tw >= 0\ \forall w \in E\}\ .$$

Let $v \in E$. There is a unique $v_1 \in E$ with $(v, v_1) \in G$ since from

$$< v_1, w >=< v, Tw >=< v_2, w >\ \ \forall w \in E$$

it follows $< v_1 - v_2, w >= 0$ for all $w \in E$ and therefore $v_1 = v_2$. This shows $\Gamma(T^*) = G$.

2) If $(x, y), (v, w) \in \Gamma(T^*)$ and $b \in \mathcal{B}$, then $(xb + v, yb + w) \in \Gamma(T^*)$ by the proof of 1).

3) Since $\Gamma(T^*)$ is closed, the operator T^* is bounded.

4) From 1) it follows $\Gamma(T^{**}) = \Gamma(T)$.

5) $< (ST)^*v, w >=< v, STw >=< S^*v, Tw >=< T^*S^*v, w >$. □

On $\mathcal{B}^n$ there is a standard $\mathcal{B}$-valued non-degenerated product, namely

$$< v, w >:= \sum_{i=1}^{n} v_i^* w_i\ .$$

All elements of $M_n(\mathcal{B})$ are adjointable and taking the adjoint is a bounded linear map.

Furthermore there is a standard $\mathcal{B}$-valued non-degenerated product on $L^2(M, \mathcal{B}^n)$ defined by

$$< f, g >_{L^2}:= \int_M < f(x), g(x) > dx\ .$$

We check condition 2) of Def. 5.2.9:

We use that every $\lambda \in \mathcal{B}'$ induces a map $\lambda : \mathcal{B}^n \to \mathbb{C}^n$ by componentwise application.

If $f \in L^2(M, \mathcal{B}^n)$ with $< f, g >_{L^2} = 0$ for all $g \in L^2(M, \mathcal{B}^n)$, then in particular for all $g \in L^2(M, \mathbb{C}^n)$ and $\lambda \in \mathcal{B}'$

$$\int_M \lambda(f(x)^*)g(x)dx = 0 \ ,$$

hence $\lambda(f(x))$ equals zero outside a set of measure zero. Since $\mathcal{B}'$ is separable, it follows $f = 0$ in $L^2(M, \mathcal{B}^n)$.

All operators in $HS(L^2(M, \mathcal{B}^n))$ are adjointable: If $A \in HS(L^2(M, \mathcal{B}^n))$, then A^* is an integral operator whose integral kernel is $k_{A^*}(x, y) = k_A(y, x)^*$. In particular it is in $L^2(M \times M, M_n(\mathcal{B}))$ and taking the adjoint is a bounded map on $HS(L^2(M, \mathcal{B}^n))$.

Proposition 5.2.12. *1. Let $A \in B(L^2(M, \mathcal{B}^n))$, $K \in HS(L^2(M, \mathcal{B}^n))$. Then $AK \in HS(L^2(M, \mathcal{B}^n))$.*

Furthermore there is $C > 0$ such that

$$\|AK\|_{HS} \leq C\|A\|\|K\|_{HS}$$

for all $A \in B(L^2(M, \mathcal{B}^n))$, $K \in HS(L^2(M, \mathcal{B}^n))$.

2. If A is an adjointable bounded operator on $L^2(M, \mathcal{B}^n)$ and $K \in HS(L^2(M, \mathcal{B}^n))$ then $KA \in HS(L^2(M, \mathcal{B}^n))$.

Furthermore there is $C > 0$ with

$$\|KA\|_{HS} \leq C\|A^*\|\|K\|_{HS}$$

for every such A and K.

Proof. 1) There is an isomorphism

$$L^2(M \times M, M_n(\mathcal{B})) \cong L^2(M \times M, \mathcal{B}^n)^n$$

that is equivariant with respect to the left $M_n(\mathcal{B})$-action on both spaces. Furthermore the map

$$L^2(M \times M, \mathcal{B}^n) \to L^2(M, L^2(M, \mathcal{B}^n)), \ k \mapsto (y \mapsto k(\cdot, y))$$

is an isomorphism by Lemma 5.2.2. The operator A induces a bounded map on $L^2(M, L^2(M, \mathcal{B}^n))$, namely $k \mapsto (y \mapsto Ak(\cdot, y))$, clearly its norm is less than or equal to the norm of A on $L^2(M, \mathcal{B}^n)$.

2) The map $K \mapsto KA$ is a composition of the following maps on $HS(L^2(M, \mathcal{B}^n))$:

$$K \overset{*}{\mapsto} K^* \overset{A^*}{\mapsto} A^*K^* = (KA)^* \overset{*}{\mapsto} KA \ .$$

By 1) and the fact that taking the adjoint is bounded on $HS(L^2(M, \mathcal{B}^n))$ these maps are bounded. □

Note that the integral kernel of $(TA)^*$ is $A_x^* k_{T^*}(x, y)$. By the suffix we indicate on which variable A^* acts.

Definition 5.2.13. *1. Let*

$$< \ , \ >: \mathcal{B}^n \times \mathcal{B}^n \to \mathcal{B}$$

be as above. For $e \in \mathcal{B}^n$ define the map

$$e^* : \mathcal{B}^n \to \mathcal{B}, \ v \mapsto < e, v > \ .$$

2. An integral operator A on $L^2(M, \mathcal{B}^n)$ is called FINITE *if there is $k \in \mathbb{N}$ and there are functions $f_j, h_j \in L^2(M, \mathcal{B}^n)$, $j = 1 \dots k$, such that*

$$k_A(x, y) = \sum_{j=1}^{k} f_j(x) h_j(y)^* \ .$$

5.2.4 Pseudodifferential operators

Let E be a Banach space.

Let U be an open precompact subset of $\mathbb{R}^n$. Recall the notion of a symbol of order m on U:

Definition 5.2.14. *A function $a \in C^\infty(U \times \mathbb{R}^n, M_l(\mathbb{C}))$ is called a* SYMBOL OF ORDER *$m \in \mathbb{R}$, if it is compactly supported in the first variable and if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ the expressions*

$$\sup_{x \in U, \ \xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+|\beta|} |\partial_{x^\alpha} \partial_{\xi^\beta} a(x, \xi)|$$

are finite.

These are seminorms on the space $S^m(U, M_l(\mathbb{C}))$ of symbols of order m on U that turn $S^m(U, M_l(\mathbb{C}))$ into a Fréchet space.

In order to simplify formula involving Fourier transform we rescale the Lebesgue measure on $\mathbb{R}^n$ by setting $d'x := (2\pi)^{\frac{n}{2}} dx$.

We consider $L^2(U, E^l)$ as a subspace of $L^2(\mathbb{R}^n, E^l)$.

Note that the Fourier transform is bounded from $L^1(\mathbb{R}^n, E)$ to $C_0(\mathbb{R}^n, E)$.

A symbol $a \in S^m(U, M_l(\mathbb{C}))$ defines a continuous operator

$$\mathrm{Op}(a) : C_c^\infty(U, E^l) \to C_c^\infty(U, E^l), \ (\mathrm{Op}(a) f)(x) = \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi) \ d'\xi \ .$$

An operator defined by a symbol of order m is called a pseudodifferential operator of order m.

Lemma 5.2.15. *1. For $m < -\frac{n}{2}$ and $a \in S^m(U, M_l(\mathbb{C}))$ the operator $\mathrm{Op}(a)$ extends to a bounded operator on $L^2(U, E^l)$ and the map*

$$\mathrm{Op} : S^m(U, M_l(\mathbb{C})) \to B(L^2(U, E^l))$$

is continuous.

2. *Let $m < -\frac{n}{2}$ and $\nu, k \in \mathbb{N}_0$ with $k < -\frac{n}{2} - m$. Then for $a \in S^m(U, M_l(\mathbb{C}))$ the operators*

$$\mathrm{Op}(a) : C_0^\nu(U, E^l) \to C_0^{\nu+k}(U, E^l)$$

and

$$\mathrm{Op}(a) : L^2(U, E^l) \to C_0^k(U, E^l)$$

are continuous.

Proof. 1) Let $m < -\frac{n}{2}$.

The Fourier transform induces a bounded map

$$S^m(U, M_l(\mathbb{C})) \to C_c^\infty(U, L^2(\mathbb{R}^n, M_l(\mathbb{C}))),\ a \mapsto (x \mapsto \hat{a}(x, \cdot)) \ .$$

For $a \in S^m(U, M_l(\mathbb{C}))$ and $f \in L^2(U, E^l)$ we define

$$(\mathrm{Op}(a)f)(x) := \int_{\mathbb{R}^n} \hat{a}(x, z) f(-x - z) d'z \ .$$

The map

$$\mathbb{R}^n \to L^2(\mathbb{R}^n, E^l),\ x \mapsto (z \mapsto f(-x - z))$$

is continuous by Lemma 5.2.6, hence $\mathrm{Op}(a)f \in C_0(U, E^l)$ and

$$\| \mathrm{Op}(a)f \|_{C_0} \leq \sup_{x \in U} \|\hat{a}(x, \cdot)\|_{L^2} \|f\|_{L^2} \ .$$

Since U is precompact, the inclusion $C_0(U, E^l) \hookrightarrow L^2(U, E^l)$ is well-defined and continuous. Hence $\mathrm{Op}(a)$ is a bounded operator on $L^2(U, E^l)$. Furthermore the estimate shows that the map

$$S^m(U, M_l(\mathbb{C})) \to B(L^2(U, E^l)),\ a \mapsto \mathrm{Op}(a)$$

is continuous.

2) First let $m < -\frac{n}{2}$ and $k = 0$.

If $f \in C_0^\nu(U, E^l)$, then $x \mapsto (z \mapsto f(-x - z))$ is in $C_0^\nu(\mathbb{R}^n, L^2(\mathbb{R}^n, E^l))$.

It follows as above that $\mathrm{Op}(a)f \in C_0^\nu(U, E^l)$ and

$$\| \mathrm{Op}(a)f \|_{C^\nu} \leq C \sup_{|\alpha| \leq \nu} \sup_{x \in U} \|\partial_x^\alpha \hat{a}(x, \cdot)\|_{L^2} \|f\|_{C^\nu} \ .$$

We go on by induction on k: Assume that the assertion holds for $k - 1$ and all $a \in S^m(U, M_l(\mathbb{C}))$ with $m < -\frac{n}{2} - k + 1$.

We prove the assertion for k and $a \in S^m(U, M_l(\mathbb{C}))$ with $m < -\frac{n}{2} - k$:

If $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$, then for $f \in C_0^\infty(U, E^l)$

$$\begin{aligned}
&\partial_x^\alpha \int_{\mathbb{R}^n} \hat{a}(x, z) f(-x - z) d'z \\
&= \int_{\mathbb{R}^n} \partial_x^\alpha \hat{a}(x, z) f(-x - z) d'z + \int_{\mathbb{R}^n} \hat{a}(x, z) \partial_x^\alpha f(-x - z) d'z \\
&= \int_{\mathbb{R}^n} \partial_x^\alpha \hat{a}(x, z) f(-x - z) d'z + \int_{\mathbb{R}^n} \hat{a}(x, z) \partial_z^\alpha f(-x - z) d'z \\
&= \int_{\mathbb{R}^n} \partial_x^\alpha \hat{a}(x, z) f(-x - z) d'z - \int_{\mathbb{R}^n} \partial_z^\alpha \hat{a}(x, z) f(-x - z) d'z \ .
\end{aligned}$$

The last step is justified since the symbols $\xi^\alpha a(x,\xi)$ and $\partial_x^\alpha a(x,\xi)$ are in $S^{m+1}(U, M_l(\mathbb{C}))$, thus by $m+1 < -\frac{n}{2}$ the functions

$$x \mapsto (z \mapsto \partial_z^\alpha \hat{a}(x,z))$$

and

$$x \mapsto (z \mapsto \partial_x^\alpha \hat{a}(x,z))$$

are both in $C_c^\infty(U, L^2(\mathbb{R}^n, M_l(\mathbb{C})))$.

It follows that the map

$$f \mapsto \partial^\alpha(\mathrm{Op}(a)f)$$

is a pseudodifferential operator of degree $m+|\alpha| = m+1$. By induction it is a bounded operator from $C^\nu(U, E^l)$ to $C^{\nu+k-1}(U, E^l)$. Since this holds for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$, it follows that $\mathrm{Op}(a)$ is continuous from $C^\nu(U, E^l)$ to $C^{\nu+k}(U, E^l)$.

From an analogous induction argument it follows that $\mathrm{Op}(a)$ is continuous from $L^2(U, E^l)$ to $C_0^k(U, E^l)$. For $k = 0$ this was proven in 1). □

5.3 Projective systems and function spaces

The projective systems $(\mathcal{A}_i)_{i\in\mathbb{N}_0}$ and $(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)_{i,\mu\in\mathbb{N}_0}$ from §1.3.3 induce projective systems of spaces $\left(L^2(M, \mathcal{A}_i^l)\right)_{i\in\mathbb{N}_0}$ and $\left(L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)\right)_{i,\mu\in\mathbb{N}_0}$.

Recall the convention fixed in §5.2.2: The space $L^2(M, \mathcal{A}^l)$ is the Hilbert $\mathcal{A}$-module defined in §5.1.1. For $\mu \in \mathbb{N}_0$ and $i \in \mathbb{N}$ the space $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ was defined in §5.2.2.

In the following we investigate the behaviour some particular classes of operators on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ under the projective limit.

5.3.1 Integral operators

Hilbert-Schmidt operators on $L^2(M, \mathcal{A}_i^l)$ have the property that they extend to bounded operators on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_j)^l)$ for all $j \in \mathbb{N}_0$ with $j \leq i$ and all $\mu \in \mathbb{N}_0$. We investigate how the spectrum depends on j, μ.

A result in this direction was proved by Lott ([Lo], §6.1.) for the case that M is a closed manifold. We extend Lott's method in order to obtain an analogous result for certain non-compact manifolds M with boundary (in particular for the manifold defined in §1.1).

Let $[0,1]^n$ be endowed with a measure of the form hdx where h is a positive continuous function on $[0,1]^n$ and dx is the Lebesgue measure.

In the proof we use that there exists a Schauder basis of $C([0,1]^n)$ which is orthonormal in $L^2([0,1]^n)$ (here and in the following $L^2([0,1]^n)$ is defined with respect to hdx). For $h = 1$ a Franklin system [Se] yields such a basis $\{f_n\}_{n\in\mathbb{N}}$, then for general h the system $\{h^{-\frac{1}{2}} f_n\}_{n\in\mathbb{N}}$ is one.

Proposition 5.3.1. *1. Let hdx be a measure on $[0,1]^n$ as above.*

Let $k \in C([0,1]^n \times [0,1]^n, M_l(\mathcal{A}_i))$ and let K be the corresponding integral operator. Assume that $1-K$ is invertible in $B(L^2([0,1]^n, \mathcal{A}^l))$. Then $1-K$: $L^2([0,1]^n, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) \to L^2([0,1]^n, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ is invertible.

2. *Let M be a Riemannian manifold of dimension n, possibly with boundary. Suppose that there is an exaustion $\{K_m\}_{m\in\mathbb{N}}$ of M such that K_m is diffeomorphic to $[0,1]^n$ for every $m \in \mathbb{N}$. Let $k \in L^2(M\times M, M_l(\mathcal{A}_i)) \cap C(M\times M, M_l(\mathcal{A}_i))$ and assume furthermore that $x \mapsto k(x,\cdot)$ and $y \mapsto k(\cdot, y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$.*

   If $1-K$ is invertible in $B(L^2(M,\mathcal{A}^l))$, then $1-K$ is invertible in $B(L^2(M,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l))$.

Proof. 1) Choose a basis of $C([0,1]^n)$ which is orthonormal with respect to hdx and let P_N denote the projection onto the first N basis vectors. It is an integral operator with integral kernel in $L^2([0,1]^n\times[0,1]^n)$, thus it acts continuously on $L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$.

We decompose $L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ into the direct sum

$$P_N L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) \oplus (1-P_N)L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$$

and write

$$1-K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition.

If we find N such that d is invertible on $(1-P_N)L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ and prove that then $a-bd^{-1}c$ is invertible on $P_N L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$, we can conclude that $(1-K)$ is invertible by the equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} .$$

First we show that

$$d = (1-P_N)(1-K)(1-P_N)$$

is invertible for N big enough. By Prop. 5.2.12 the operator $(1-P_N)K(1-P_N)$ is a Hilbert-Schmidt operator. Its integral kernel is continuous.

For $N\to\infty$ the projections P_N converge strongly to the identity on $C([0,1]^n)$. Since

$$C([0,1]^n\times[0,1]^n, M_l(\mathcal{A}_i)) = C([0,1]^n)\otimes_\varepsilon C([0,1]^n)\otimes_\varepsilon M_l(\mathcal{A}_i) ,$$

there is N such that the norm of the integral kernel of $(1-P_N)K(1-P_N)$ is smaller than $\frac{1}{2}$ in $C([0,1]^n\times[0,1]^n, M_l(\mathcal{A}_i))$.

For that N the series

$$(1-P_N) + \sum_{\nu=1}^{\infty} ((1-P_N)K(1-P_N))^\nu$$

converges as a bounded operator on $(1-P_N)L^2([0,1]^n,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ and inverts d. Hence $a-bd^{-1}c$ is well-defined.

Via the basis we identify $a-bd^{-1}c$ with an element of $M_{Nl}(\mathcal{A}_i)$. Since $1-K$ is invertible on $L^2(M,\mathcal{A}^l)$, the matrix $a-bd^{-1}c$ is invertible in $M_{Nl}(\mathcal{A})$. By Prop. 1.3.5 it follows that $a-bd^{-1}c$ is invertible in $M_{Nl}(\mathcal{A}_i)$ as well.

2): Let $m \in \mathbb{N}$ be such that

$$\|(1 - 1_{K_m}(x))k(x,y)(1 - 1_{K_m}(y))\|_{L^2(M\times M, M_l(\mathcal{A}_i))} \leq \frac{1}{2} \ .$$

Write

$$(1-K) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition

$$L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) = 1_{K_m}L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) \oplus (1 - 1_{K_m})L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) \ .$$

By the choice of $m \in \mathbb{N}$ the entry $d = 1 - (1 - 1_{K_m})K(1 - 1_{K_m})$ is invertible on $(1 - 1_{K_m})L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$.

We prove that $a - bd^{-1}c$ is invertible on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ and then the assertion follows as in the proof of 1).

On $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$

$$a - bd^{-1}c = 1_{K_m} - (1_{K_m}K1_{K_m} + bd^{-1}c) \ .$$

The operator $1_{K_m}K1_{K_m} + bd^{-1}c$ is an integral operator on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ with continuous integral kernel: The integral kernel of b is $1_{K_m}(x)k(x,y)(1 - 1_{K_m}(y))$ and $x \mapsto 1_{K_m}(x)k(x,\cdot)(1 - 1_{K_m})$ is in $C(K_m, L^2(M, M_l(\mathcal{A}_i)))$. For the integral kernel $(1 - 1_{K_m}(x))k(x,y)1_{K_m}(y)$ of c we have $y \mapsto (1-1_{K_m})k(\cdot,y)1_{K_m}(y) \in C(K_m, L^2(M, M_l(\mathcal{A}_i)))$. It follows that $bd^{-1}c$ is an integral operator with continuous kernel on $K_m \times K_m$. Clearly the integral kernel of $1_{K_m}K1_{K_m}$ is continuous as well.

Since $a - bd^{-1}c$ is invertible on $L^2(K_m, \mathcal{A}^l)$ and since the measure on K_m pulled back by an orientation preserving diffeomorphism $[0,1]^n \to K_m$ is of the form hdx, we conclude by 1) that $a - bd^{-1}c$ is invertible on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ as well. □

Corollary 5.3.2. *Let k be an integral kernel as in part 2) of the proposition and let K be the corresponding integral operator. Then for $\lambda \in \mathbb{C}^*$ the operator $K - \lambda$ is invertible on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ if and only if $K - \lambda$ is invertible on $L^2(M, \mathcal{A}^l)$.*

Proof. For $\lambda \in \mathbb{C} \setminus \{0\}$ the integral kernel k/λ fulfills the conditions of the lemma. Thus if $\lambda - K = \lambda(1 - K/\lambda)$ is invertible on $L^2(M, \mathcal{A}^l)$, then $\lambda - K$ is invertible on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ as well. □

5.3.2 Projections and the Chern character

Proposition 5.3.3. *Let $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of $l^2(\mathcal{A})$. Let M be a closed and N a projective submodule of $l^2(\mathcal{A})$ such that $l^2(\mathcal{A}) = M \oplus N$. Let P be the projection onto N along M and let P_n be the orthogonal projection onto $L_n := \operatorname{span}\{e_i \mid i = 1, \ldots, n\}$.*

If $n \in \mathbb{N}$ such that

$$\|P(1 - P_n)\| \leq \frac{1}{2}$$

then:

(i) The $\mathcal{A}$-module $N' := P_n(N)$ is projective and the maps

$$P_n : N \to N' \text{ and } P : N' \to N$$

are isomorphisms.

(ii) $l^2(\mathcal{A}) = M \oplus N'$.

Note that for $n \in \mathbb{N}$ large enough $\|P(1-P_n)\| \leq \frac{1}{2}$ since P is a compact operator.

Proof. From $\|P(1-P_n)\| \leq \frac{1}{2}$ it follows that

$$\|1_N - (PP_n)|_N\| \leq \frac{1}{2} .$$

Hence $(PP_n)|_N : N \to N$ is invertible and thus for any finite set G of generators of N the set $PP_n(G)$ generates N as well.

The module $N' := P_n(N)$ is projective. Furthermore, since $(PP_n)|_N$ is an isomorphism, the maps

$$P_n : N \to N' \text{ and } P : N' \to N$$

are isomorphisms as well.

It remains to show $N' \oplus M = l^2(\mathcal{A})$.

The intersection $N' \cap M$ is trivial: If $x \in N' \cap M$, then $Px = 0$, hence, as $P : N' \to N$ is an isomorphism, $x = 0$.

Let now $x \in l^2(\mathcal{A})$. Since $PP_n : N \to N$ is invertible, there is $y \in N$ such that $Px = PP_ny$.

Then

$$\begin{aligned} x &= (1-P)x + Px \\ &= (1-P)x + PP_ny \\ &= (1-P)x + P_ny - (1-P)P_ny \\ &= (1-P)(x - P_ny) + P_ny . \end{aligned}$$

Since $(1-P)(x-P_ny) \in M$ and $P_ny \in N'$, it follows that $x \in M \oplus N'$. □

Definition 5.3.4. *Let V be a $\mathbb{Z}/2$-graded finite dimensional vector space. Let K be an integral operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with integral kernel $k : M \times M \to \mathrm{End}(V) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$. Then we define $\mathrm{d}(K)$ to be the integral operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with integral kernel $\mathrm{d}(k(x,y))$. (The action of d on $\mathrm{End}(V) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ was described in §1.3.2.)*

Note that if K is of degree n with respect to the $\mathbb{Z}/2$-grading on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, then

$$\mathrm{d}(Kf) = \mathrm{d}(K)f + (-1)^n K\,\mathrm{d}\,f .$$

Lemma 5.3.5. *Let M be a σ-finite measure space.*

Let P be a Hilbert-Schmidt operator on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ with integral kernel in $L^2(M \times M, M_l(\mathcal{A}_i))$ and assume that $P^2 = P$. Then

$$P\,\mathrm{d}\,P\,\mathrm{d}\,P = P(\mathrm{d}(P))^2 .$$

Proof. As for matrices (see the beginning of the proof of Prop. 1.3.3) $(\mathrm{d}\,P) = P(\mathrm{d}\,P) + (\mathrm{d}\,P)P$ and $P(\mathrm{d}\,P)P = 0$.

It follows that

$$\begin{aligned} P\,\mathrm{d}\,P\,\mathrm{d}\,P &= P(\mathrm{d}\,P)\,\mathrm{d}\,P = P(\mathrm{d}\,P)^2P \\ &= P(\mathrm{d}\,P)(\mathrm{d}\,P) - P(\mathrm{d}\,P)P(\mathrm{d}\,P) \\ &= P(\mathrm{d}(P))^2. \end{aligned}$$

□

We introduced the notion of trace class operators in §1.3.2. For the proof of the next proposition we have to define a generalized trace:

If A, B are Hilbert-Schmidt operators on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ where M is a σ-finite measure space, then we define

$$\mathrm{Tr}(A,B) := \int_M \mathrm{tr} k_A(x,y)k_B(y,x)\ dydx \in \hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]}\ .$$

We have

$$|\mathrm{Tr}(A,B)| \leq \|A\|_{HS}\|B\|_{HS}$$

and

$$\mathrm{Tr}(A,B) = \mathrm{Tr}(B,A)\ .$$

If A, B are as in §1.3.2 such that in particular AB is trace class, then $\mathrm{Tr}(A,B) = \mathrm{Tr}AB$.

Lemma 5.3.6. *Let $P : [0,1] \to HS(L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l))$ be a differentiable path of Hilbert-Schmidt operators with integral kernels in $L^2(M \times M, M_l(\mathcal{A}_i))$, and assume that $P(t)^2 = P(t)$ for every $t \in [0,1]$.*

Then for $k \in \mathbb{N}_0$ in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]}$

$$\mathrm{Tr}\big(P(1), (\mathrm{d}\,P(1))^{2k}\big) - \mathrm{Tr}\big(P(0), (\mathrm{d}\,P(0))^{2k}\big) = 0\ .$$

Proof. As for matrices (see Prop. 1.3.3). □

If S, T are Hilbert-Schmidt operators on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ that define Hilbert-Schmidt operators $S_{\nu,j}, T_{\nu,j}$ on $L^2(M, (\hat{\Omega}_{\leq\nu}\mathcal{A}_j)^l)$ for all $j, \nu \in \mathbb{N}$ with $j \geq i$ and $\nu \geq \mu$ then we set

$$\mathrm{Tr}(S,T) := \varprojlim_{\nu,j} \mathrm{Tr}(S_{\nu,j}, T_{\nu,j}) \in \hat{\Omega}_*(\mathcal{A}_\infty)/\overline{[\hat{\Omega}_*(\mathcal{A}_\infty), \hat{\Omega}_*(\mathcal{A}_\infty)]_s}\ .$$

This notation has still another meaning: namely the trace of (S,T) in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$. The context will decide what is meant. An analogous convention is made for the trace of trace class operators.

Proposition 5.3.7. *Let M be a Riemannian manifold of dimension d, possibly with boundary. Suppose that there is an exaustion $\{K_m\}_{m\in\mathbb{N}}$ of M such that K_m is diffeomorphic to $[0,1]^d$ for all $m \in \mathbb{N}$.*

Let $P \in B(L^2(M, \mathcal{A}^l))$ be a projection onto a projective submodule of $L^2(M, \mathcal{A}^l)$. Assume further that for any $i \in \mathbb{N}$ it restricts to a bounded projection on $L^2(M, \mathcal{A}_i^l)$ and that $P(L^2(M, \mathcal{A}_i^l)) \subset C(M, \mathcal{A}_i^l)$. Let

$$\mathrm{Ran}_\infty P := \bigcap_{i\in\mathbb{N}} P(L^2(M, \mathcal{A}_i^l))\ .$$

(i) *The projection P is a Hilbert-Schmidt operator of the form $\sum_{j=1}^{m} f_j(x)h_j(x)^*$ with $f_j, h_j \in \mathrm{Ran}_\infty P$.*

(ii) *The intersection $\mathrm{Ran}_\infty P$ is a projective $\mathcal{A}_\infty$-module. The classes $[\mathrm{Ran}\, P] \in K_0(\mathcal{A})$ and $[\mathrm{Ran}_\infty P] \in K_0(\mathcal{A}_\infty)$ correspond to each other under the canonical isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\infty)$.*

(iii) *In $H_*^{dR}(\mathcal{A}_\infty)$*

$$\mathrm{ch}[\mathrm{Ran}_\infty P] = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \mathrm{Tr}(P\,\mathrm{d}\,P)^{2n} \ .$$

Proof. (i): Let $\{e_n\}_{n\in\mathbb{N}} \subset C_0^\infty(M, \mathbb{C}^l)$ be an orthonormal basis of $L^2(M, \mathbb{C}^l)$.

Let P_n be the projection onto the span of the first n basis vectors. It is a Hilbert-Schmidt operator with integral kernel in $C_0^\infty(M \times M, M_l(\mathbb{C}))$.

In particular $P_n \in B(L^2(M, \mathcal{A}_i^l))$ for any $i \in \mathbb{N}$.

First we consider the situation on $L^2(M, \mathcal{A}^l)$:

Since P is compact, there is $n \in \mathbb{N}$ such that on $L^2(M, \mathcal{A}^l)$

$$\|P(P_n - 1)\| \leq \frac{1}{2} \ .$$

Hence by Prop. 5.3.3 the map $PP_nP : \mathrm{Ran}\, P \to \mathrm{Ran}\, P$ is an isomorphism.

It follows that $\mathrm{Ker}\, PP_nP = (\mathrm{Ran}\, P)^\perp = \mathrm{Ker}\, P$ and therefore

$$P = 1 - P_{\mathrm{Ker}\, P} = 1 - P_{\mathrm{Ker}\, PP_nP} \ .$$

Here $P_{\mathrm{Ker}\, P}$ resp. $P_{\mathrm{Ker}\, PP_nP}$ denotes the orthogonal projection onto $\mathrm{Ker}\, P$ resp. $\mathrm{Ker}\, PP_nP$.

Furthermore we can find $r > 0$ such that $B_r(0) \setminus \{0\}$ is in the resolvent set of PP_nP.

Then

$$\begin{aligned} P &= 1 - P_{\mathrm{Ker}\, PP_nP} \\ &= 1 - \frac{1}{2\pi i} \int_{|\lambda|=r} (\lambda - PP_nP)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=r} \left(\lambda^{-1} - (\lambda - PP_nP)^{-1}\right) d\lambda \\ &= -\frac{1}{2\pi i} PP_nP \int_{|\lambda|=r} \lambda^{-1} (\lambda - PP_nP)^{-1} d\lambda \ . \end{aligned}$$

Since P_n is a Hilbert-Schmidt operator on $L^2(M, \mathcal{A}^l)$, the operator P is one as well.

In order to show that the equation holds on $L^2(M, \mathcal{A}_i^l)$ as well for all $i \in \mathbb{N}$, we prove that

$$R := \int_{|\lambda|=r} \lambda^{-1} (\lambda - PP_nP)^{-1} d\lambda$$

is well-defined in $B(L^2(M, \mathcal{A}_i^l))$ or equivalently that $(\lambda - PP_nP)$ is invertible in $B(L^2(M, \mathcal{A}_i^l))$ for $|\lambda| = r$.

The integral kernel of PP_nP is $k(x,y) := \sum_{j=1}^{n} Pe_j(x)(Pe_j(y))^*$. It is in $L^2(M\times M, M_l(\mathcal{A}_i))\cap C(M\times M, M_l(\mathcal{A}_i))$ for all $i \in \mathbb{N}_0$ and the maps $x \mapsto k(x,\cdot)$ and $y \mapsto k(\cdot,y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$. From Cor. 5.3.2 we conclude that the spectrum of PP_nP on $B(L^2(M,\mathcal{A}_i^l))$ is independent of i, thus R is a bounded operator on $L^2(M,\mathcal{A}_i^l)$ for all $i \in \mathbb{N}_0$.

This and the equation $PR = PRP$ show that P is an integral operator with integral kernel

$$k_P(x,y) = -\frac{1}{2\pi i}\sum_{j=1}^{n} Pe_j(x)(PR^*Pe_j(y))^* \ .$$

The integral kernel is in $L^2(M\times M, M_l(\mathcal{A}_i))$ for all $i \in \mathbb{N}_0$ and is of the form we asserted.

(ii): Since $\|(P_n-1)P\| \le \frac{1}{2}$ in $B(L^2(M,\mathcal{A}^l))$, the operator $(1+(P_n-1)P)$ is invertible in $B(L^2(M,\mathcal{A}^l))$. The integral kernel κ of $(P_n-1)P$ is continuous and the maps $x \mapsto \kappa(x,\cdot)$ and $y \mapsto \kappa(\cdot,y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$. Hence by Prop. 5.3.1 the operator $(1+(P_n-1)P)$ is invertible in $B(L^2(M,\mathcal{A}_i^l))$ for any $i \in \mathbb{N}_0$. From $P_nP = (1+(P_n-1)P)P$ it follows that

$$P_nP(L^2(M,\mathcal{A}_i^l)) \cong \operatorname{Ran} P(L^2(M,\mathcal{A}_i^l)) \ .$$

Furthermore $Q := (1+(P_n-1)P)P(1+(P_n-1)P)^{-1} \in B(L^2(M,\mathcal{A}_i^l))$ is a projection onto $P_nP(L^2(M,\mathcal{A}_i^l))$. Hence $Q(L_n(\mathcal{A}_i)) = P_nP(L^2(M,\mathcal{A}_i^l))$. Identify $L_n(\mathcal{A}_i)$ with $\mathcal{A}_i^n$ via the basis. Then Q restricted to $L_n(\mathcal{A}_i)$ is given by a projection $Q' \in M_n(\mathcal{A}_\infty)$. From $\operatorname{Ran}_\infty P \cong Q'(\mathcal{A}_\infty^n)$ it follows that $[\operatorname{Ran} P] = [Q']$ in $K_0(\mathcal{A})$ and $[\operatorname{Ran}_\infty P] = [Q']$ in $K_0(\mathcal{A}_\infty)$. This shows the assertion.

(iii): Let P_n, Q and Q' be as in the proof of (ii). In the following computations we use the fact that $P_nQ = Q$, hence Q is a finite Hilbert-Schmidt operator. In $H^{dR}(\mathcal{A}_\infty)$

$$\begin{aligned}
\operatorname{ch}[\operatorname{Ran}_\infty P] &= \operatorname{ch}(Q') \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}\operatorname{tr}(Q'\,\mathrm{d}\,Q')^{2k} \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}\operatorname{Tr}((P_nQP_n)\,\mathrm{d}(P_nQP_n))^{2k} \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}\operatorname{Tr}\big((P_nQ),(\mathrm{d}\,P_nQ)^{2k}\big) \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}\operatorname{Tr}(Q,(\mathrm{d}\,Q)^{2k}) \ .
\end{aligned}$$

Since

$$H : [0,1] \to B(L^2(M,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)) \ ,$$
$$H(t) = (1+t(P_n-1)P)P(1+t(P_n-1)P)^{-1}$$

is a differentiable path of finite projections with $H(0) = P$ and $H(1) = Q$ for any $i,\mu \in \mathbb{N}$, the difference $\operatorname{Tr}\big(Q,(\mathrm{d}\,Q)^{2k}\big) - \operatorname{Tr}(P\,\mathrm{d}\,P)^{2k}$ is exact in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty,\hat{\Omega}_*\mathcal{A}_\infty]_s}$ by the previous lemma.

This shows the assertion. □

5.4 Holomorphic semigroups

5.4.1 Generalities

Let X be a Banach space.

We collect some well-known facts about holomorphic semigroups on X and prove a technical lemma (Lemma 5.4.1) which we need in the next section.

Recall that a densily defined operator Z on X generates a holomorphic semigroup e^{tZ} if and only if there is $\omega \geq 0$ such that $Z - \omega$ is δ-sectorial for some $\delta \in]0, \pi/2]$, i.e. if and only if

$$\Sigma_{\delta+\pi/2} = \{\lambda \in \mathbb{C}^* \mid |\arg \lambda| < \pi/2 + \delta\}$$

is a subset of $\rho(Z - \omega)$ and for any ε with $0 < \varepsilon < \delta$ there is $C > 0$ such that for all $\lambda \in \Sigma_{\pi/2+\varepsilon}$

$$\|(Z - \omega - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} .$$

The semigroup is bounded holomorphic if we can choose $\omega = 0$. Then in particular e^{tZ} is uniformly bounded for $t \geq 0$.

Lemma 5.4.1. *Let Z be a densily defined operator on X and let $\omega \in \mathbb{R}$ and $\delta > 0$ be such that*

1. $\Sigma_{\delta+\pi/2} \cup \{0\} \subset \rho(Z + \omega)$,
2. *for every $\alpha < \delta$ there are $R, C > 0$ such that*

$$\|(Z - \lambda)^{-1}\| \leq \frac{C}{|\lambda|}$$

for $\lambda \in \Sigma_{\alpha+\pi/2} \setminus B_R(0)$.

Then $Z + \omega$ is δ-sectorial.

Proof. Let $\varepsilon < \delta$ and choose $\alpha \in]\varepsilon, \delta[$. We have to show that there is $C > 0$ such that

$$\|(Z + \omega - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} .$$

Let $r > \max(R, |\omega|)$ be such that $\overline{\Sigma_{\varepsilon+\pi/2}} \subset \big((\overline{\Sigma_{\alpha+\pi/2}} \cup B_r(0)) + \omega\big)$. (If $\omega \leq 0$, this is fulfilled for all $r > \max(R, |\omega|)$.)

The compact set $K := \overline{\Sigma_{\varepsilon+\pi/2}} \cap (B_r(0) + \omega)$ is a subset of $\rho(Z + \omega)$. The resolvent $(Z + \omega - \lambda)^{-1}$ is uniformly bounded on K, in particular there is $C > 0$ such that for $\lambda \in K \setminus \{0\}$

$$\|(Z + \omega - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} .$$

If $\lambda \in \Sigma_{\varepsilon+\pi/2} \setminus K$, then $\lambda - \omega \in \Sigma_{\alpha+\pi/2} \setminus B_r(0)$ thus by assumption there is $C > 0$ such that for all $\lambda \in \Sigma_{\varepsilon+\pi/2} \setminus K$

$$\|(Z + \omega - \lambda)^{-1}\| \leq \frac{C}{|\lambda - \omega|} .$$

It follows that there is $C > 0$ such that for all $\lambda \in \Sigma_{\varepsilon+\pi/2}$

$$\begin{aligned}\|(Z+\omega-\lambda)^{-1}\| &\leq \left(\frac{C}{|\lambda|}\right)\left(\frac{|\lambda-\omega|+|\omega|}{|\lambda-\omega|}\right)\\ &\leq \left(\frac{C}{|\lambda|}\right)\left(1+\frac{|\omega|}{|\lambda-\omega|}\right)\\ &\leq \left(\frac{C}{|\lambda|}\right)\left(1+\frac{|\omega|}{(r-|\omega|)}\right)\\ &\leq \frac{C}{|\lambda|}\ .\end{aligned}$$

This proves the assertion. □

The lemma yields the following connection between the spectrum of a δ-sectorial operator Z and the behaviour of the holomorphic semigroup e^{tZ} for $t \to \infty$.

Proposition 5.4.2. *If Z is a δ-sectorial operator and there is $\omega > 0$ such that*

$$\{\operatorname{Re}\lambda \geq 0\} \subset \rho(Z+\omega)\ ,$$

then for any $\omega' < \omega$ there is $C > 0$ such that for all $t \geq 0$

$$\|e^{tZ}\| \leq Ce^{-\omega' t}\ .$$

Proof. For all $\omega' < \omega$ there is $0 < \delta' \leq \delta$ such that $\Sigma_{\delta'+\pi/2} \cup \{0\} \subset \rho(Z+\omega')$ and thus by the previous lemma $Z+\omega'$ is δ'-sectorial. □

Proposition 5.4.3. *If e^{tZ} is a strongly continuous semigroup with generator Z such that $\operatorname{Ran} e^{tZ} \subset \operatorname{dom} Z$ for all $t \in (0,\infty)$ and if there is $T > 0$ and $C > 0$ such that for $t \leq T$*

$$\|Ze^{tZ}\| \leq Ct^{-1}\ ,$$

then e^{tZ} extends to a holomorphic semigroup.

If the estimate holds for all $t > 0$, then the extension is bounded holomorphic.

Proof. The assertion follows immediately from [Da], Th. 2.39. □

Part 1) of the next proposition is known under the name Volterra development and the formula in part 2) is called Duhamel's formula:

Proposition 5.4.4. *Let Z be the generator of a strongly continuous semigroup and let $M, \omega \in \mathbb{R}$ be such that $\|e^{tZ}\| \leq Me^{\omega t}$ for all $t \geq 0$.*

1. *Let $R \in B(X)$. Then $Z+R$ is the generator of a strongly continuous semigroup and for all $t \geq 0$*

$$e^{t(Z+R)} = \sum_{n=0}^{\infty}(-1)^n t^n \int_{\Delta^n} e^{u_0 tZ} R e^{u_1 tZ} R \ldots e^{u_n tZ}\ du_0 \ldots du_n$$

with $\Delta^n = \{u_0 + \cdots + u_n = 1; u_i \geq 0,\ i = 0,\ldots,n\}$. Furthermore

$$\|e^{t(Z+R)}\| \leq Me^{(\omega + M\|R\|)t}\ .$$

2. Let $R_1, \dots, R_n \in B(X)$. For $t \geq 0$ the map

$$\mathbb{C}^n \to B(X),\ (z_1, \dots, z_n) \mapsto e^{t(Z+z_1R_1+\dots+z_nR_n)}$$

is analytical and for $i = 0, \dots n$

$$\frac{d}{dz_i} e^{t(Z+z_1R_1+\dots+z_nR_n)} = \int_0^t e^{(t-s)(Z+z_1R_1+\dots+z_nR_n)} R_i e^{s(Z+z_1R_1+\dots+z_nR_n)}\ ds\ .$$

Proof. 1) follows from [Da], Th. 3.1 and the proof of it.

2) The analyticity follows from 1).

For the formula it is enough to consider $n = 1$. Let $R := R_1$.

For $z_0 \in \mathbb{C}$ by 1)

$$e^{(Z+zR)t} = \sum_{n=0}^{\infty} (z - z_0)^n (-1)^n t^n \int_{\Delta^n} e^{u_0 t(Z+z_0R)} R e^{u_1 t(Z+z_0R)} R \dots e^{u_n t(Z+z_0R)}\ du_0 \dots du_n$$

and the series converges absolutely for all $z \in \mathbb{C}$. This implies that

$$\frac{d}{dz} e^{(Z+zR)t}|_{z_0} = -t \int_0^1 e^{u_0 t(Z+z_0R)} R e^{(1-u_0)t(Z+z_0R)} du_0\ .$$

The assertion follows now by the change of variables $s := (1 - u_0)t$. □

The following proposition is known in the literature as Duhamel's principle ([Ta], §9):

Proposition 5.4.5. *Let Z be the generator of a strongly continuous semigroup on X. Let $u \in C^1([0, \infty), X)$ such that $\frac{d}{dt}u(t) - Zu(t) \in \operatorname{dom} Z$ for all $t \in [0, \infty)$. Then*

$$e^{tZ}u(0) - u(t) = -\int_0^t e^{sZ} \big(\frac{d}{dt} - Z\big) u(t - s) ds\ .$$

5.4.2 Square roots of generators and perturbations

Assume that D is a densily defined closed operator on a Banach space X with bounded inverse and such that $-D^2$ is δ-sectorial.

There are well-defined fractional powers $(D^2)^\alpha$ for $\alpha \in \mathbb{R}$ ([RR], 11.4.2). These are densily defined closed operators that coincide for $\alpha \in \mathbb{Z}$ with the usual powers and satisfy

$$(D^2)^{\alpha+\beta} f = (D^2)^\alpha (D^2)^\beta f$$

for all $\alpha, \beta \in \mathbb{R}$ and $f \in \operatorname{dom}(D^2)^\gamma$ with $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$. For $\alpha \leq 0$ the operator $(D^2)^\alpha$ is bounded and depends in a strongly continuous way of α.

It follows in particular that for $\alpha \geq 0$ the operator $(D^2)^{-\alpha}$ is a bounded inverse of $(D^2)^\alpha$.

Define

$$|D| := (D^2)^{\frac{1}{2}}.$$

By [Ka], Th. 2, the operator $-|D|$ is $\delta + (\pi/2 - \delta)/2$-sectorial and can be expressed in terms of the resolvents of D^2.

Note that for every $n \in \mathbb{N}$ the domain of $|D|^n$ is a core of $|D|$ and $\operatorname{dom} D^n$ is a core of D.

Lemma 5.4.6. *1. Let P be a projection on X, i.e. we demand P^2 to be densily defined and $P^2|_M = P|_M$ for some set $M \subset \operatorname{dom} P^2$ dense in X.*

If P is closed, then P is bounded.

2. Let I be a involution on X, i.e. assume that I^2 is densily defined and that there is a set $M \subset \operatorname{dom} I^2$ dense in X with $I^2|_M = 1|_M$.

If I is closed, then I is bounded.

Proof. 1) If $f \in M$ then $f = (1-P)f + Pf$ and $(1-P)Pf = P(1-P)f = 0$. We conclude $M \subset \operatorname{Ker} P + \operatorname{Ker}(1-P) \subset \operatorname{dom} P$. If P is closed, then $\operatorname{Ker} P + \operatorname{Ker}(1-P)$ is closed, hence $\operatorname{Ker} P + \operatorname{Ker}(1-P) = X$, thus $\operatorname{dom} P = X$.

2) The operator $P := \frac{1}{2}(1-I)$ is a closed projection on X in the sense of 1). Thus P is bounded. It follows that I is bounded as well. □

Proposition 5.4.7. *1. The operator $|D|^{-1}D : \operatorname{dom} D \to X$ extends to a bounded involution I on X.*

2. $\operatorname{dom} D = \operatorname{dom} |D|$ and $I(\operatorname{dom} D) \subset \operatorname{dom} D$.

3. $|D| = ID = DI$ and $D = I|D| = |D|I$.

Proof. 1) The operator D^{-1} commutes with the resolvents of D^2. It follows that $|D|^{-1}D^{-1} = D^{-1}|D|^{-1}$. Hence $|D|^{-1}(\operatorname{dom} D) \subset \operatorname{dom} D$ because of $\operatorname{dom} D = D^{-1}X$, so $\operatorname{dom} D^2 \subset \operatorname{dom}(|D|^{-1}D)^2$.

If $f \in \operatorname{dom} D$ then $|D|^{-1}Df = D|D|^{-1}f$. For $f \in \operatorname{dom} D^2$ it follows that $(|D|^{-1}D)^2 f = f$.

We prove that $|D|^{-1}D$ is closable, then the assertion follows from the previous lemma:

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom} D$ converging to zero. Then $|D|^{-1}f_n \in \operatorname{dom} D$ converges to zero for $n \to \infty$. But then if $|D|^{-1}Df_n = D|D|^{-1}f_n$ converges, the limit is zero since D is closed and injective. Thus $|D|^{-1}D$ is closable.

2) The composition $IDI : \operatorname{dom} |D| \to X$ is well defined and closed. It coincides with D on $\operatorname{dom} D^2$, hence it is a closed extension of D. It follows that $\operatorname{dom} D \subset \operatorname{dom} |D|$. The inclusion $\operatorname{dom} |D| \subset \operatorname{dom} D$ is shown analogously. □

Let $P := \frac{1}{2}(1+I)$ with I as in the previous proposition. It is a bounded projection on X. By the proposition $P \operatorname{dom} D \subset \operatorname{dom} D$, and P commutes with D and $|D|$.

From $ID = |D|$ and $I|D| = D$ it follows $PD = -P|D|$. Thus with respect to the decomposition $X = PX \oplus (1-P)X$

$$D = \begin{pmatrix} PDP & 0 \\ 0 & -(1-P)D(1-P) \end{pmatrix}$$

and

$$|D| = \begin{pmatrix} PDP & 0 \\ 0 & (1-P)D(1-P) \end{pmatrix} .$$

Taking into account that $-|D|$ is $\delta + (\pi/2 - \delta)/2$-sectorial it follows for the resolvent set of D:

Proposition 5.4.8. *For $\lambda \in \mathbb{C}$*

$$\{\lambda, -\lambda\} \subset \rho(D) \Leftrightarrow \{\lambda, -\lambda\} \subset \rho(|D|) \ .$$

Thus if $\lambda \in \mathbb{C}$ with $-\lambda^2 \in \Sigma_{\pi/2+\delta}$, then $\lambda \in \rho(D)$.
Furthermore for every $\delta' < \delta$ there is $C > 0$ such that for all λ with $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$

$$\|(D-\lambda)^{-1}\| \leq \frac{C}{|\lambda|} \ .$$

Corollary 5.4.9. *Let $\omega \geq 0$ be such that there is $C > 0$ with $\|e^{-tD^2}\| \leq Ce^{-\omega t}$ for all $t \geq 0$.*

For every $\alpha \in \mathbb{R}$ and $\omega' < \omega$ there is $C > 0$ such that for all $t > 0$

$$\||D|^\alpha e^{-tD^2}\| \leq Ct^{-\alpha/2}e^{-\omega' t} \ .$$

For every $n \in \mathbb{N}$ and $\omega' < \omega$ there is $C > 0$ such that for all $t > 0$

$$\|D^n e^{-tD^2}\| \leq Ct^{-n/2}e^{-\omega' t} \ .$$

Proof. The first assertion is [RR], Lemma 11.36, and the second one follows from the first one by $D = I|D|$ and $DI = ID$. □

Proposition 5.4.10. *Let A be a bounded operator and let $\delta' < \delta$.*

1. *There is $R > 0$ such that $D + A - \lambda$ has a bounded inverse if $|\lambda| > R$ and $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$.*
2. *There is $\omega > 0$ such that $-(D+A)^2 + \omega$ is δ'-sectorial.*
3. *$D + A$ commutes with $e^{-t(D+A)^2}$.*

Proof. By Prop. 5.4.8 there is $M > 0$ such that for all λ with $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$

$$\|(D-\lambda)^{-1}\| \leq \frac{M}{|\lambda|}.$$

Hence the Neumann series

$$(D + A - \lambda)^{-1} = (D-\lambda)^{-1} \sum_{n=0}^{\infty} (A(D-\lambda)^{-1})^n$$

converges for $|\lambda| > M\|A\|$ and $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$.

This shows 1).

Now we show that there is $\omega > 0$ such that $-(D+A)^2 + \omega$ and δ' satisfy the assumptions of Lemma 5.4.1:

If $|\lambda| > 2M\|A\|$ and $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$, then

$$\begin{aligned}
\|(D+A-\lambda)^{-1}\| &= \|\sum_{n=0}^{\infty}(D-\lambda)^{-1}\left(A(D-\lambda)^{-1}\right)^n\| \\
&\leq \sum_{n=0}^{\infty}\|A\|^n\|(D-\lambda)^{-1}\|^{n+1} \\
&\leq \sum_{n=0}^{\infty}\frac{M^{n+1}\|A\|^n}{|\lambda|^{n+1}} \\
&= \frac{M}{|\lambda|}(1-\frac{M\|A\|}{|\lambda|})^{-1} \\
&\leq \frac{2M}{|\lambda|}\ .
\end{aligned}$$

Let $\mu \in \{|\mu| > 4M^2\|A\|^2\} \cap \Sigma_{\delta'+\pi/2}$. If $\lambda \in \mathbb{C}$ with $-\lambda^2 = \mu$ then $\lambda \in \rho(D+A)$, hence the resolvent

$$(-(D+A)^2-\mu)^{-1} = -(D+A-\lambda)^{-1}(D+A+\lambda)^{-1}$$

exists. By the calculations it is bounded by

$$\|(-(D+A)^2-\mu)^{-1}\| \leq \frac{4M^2\|A\|^2}{|\mu|}\ .$$

Furthermore there is $\omega > 4M^2\|A\|^2$ such that

$$\Sigma_{\delta'+\pi/2} \subset \left(\{|\mu| > 4M^2\|A\|^2\} \cap \Sigma_{\delta'+\pi/2} - \omega\right)$$

and thus

$$\Sigma_{\delta'+\pi/2} \subset \rho\big(-(D+A)^2+\omega\big)\ .$$

Assertion 2) follows now from Lemma 5.4.1.

3) follows from the fact that $e^{-t(D+A)^2}$ can be expressed in terms of the resolvents of $(D+A)$ by Cauchy's theorem. □

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