On the Convergence of Hyperbolic Components in Families of Finite Type

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Abstract. We study approximations of analytic families $G(\lambda, \cdot)$ of entire functions by analytic families $F_n(\lambda, \cdot)$, for example polynomials, where $\lambda \in \mathbb{C}$. In order to control the dynamics of these functions the families are assumed to be of constant finite type. In this setting we prove the convergence of the hyperbolic components as kernels in the sense of Carathéodory, which is a stronger notion than the usual pointwise convergence. We give an example that in general the hyperbolic components do not converge in the Hausdorff metric as the Julia sets do.

Key words: convergence, entire functions, Fatou sets, hyperbolic components, iteration, Julia sets, polynomials

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1 Introduction

Iteration theory, founded by Fatou and Julia [16, 19], has made a remarkable comeback during the last two decades. Publications on the iteration of polynomials and rational functions have received wide attention not only within mathematics — they have kindled interest in the concepts of iteration theory even among the general public. A fact that is not so well–known is that already Fatou studied the iteration of entire *transcendental* functions. The theory for the rational case has been well developped in the last years [12, 13, 25, 28, et al], however, there were substantially less publications on the transcendental case [1, 3, 9, et al]. Recently this particular field of iteration theory has received more attention and a number of expositions have appeared [5, 14, 15] as well as papers on specific problems [6, et al]. The theory of iteration for this class of functions develops along the lines of the rational case. However, often other and generally more complicated methods of proof need to be used and there is also a variety of phenomena that do not occur in the rational case. One of the questions is: which results carry over from the rational case and which do not?

An interesting approach to this question has been suggested in [11] and was illustrated in [10, 20]. On the whole complex plane $\mathbb{C}$ the family of polynomials $P_d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \lambda(1 + \frac{z}{d})^d$ converges uniformly on compact sets to the complex exponentials $E : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \lambda e^z$ as d tends to infinity. Due to this convergence certain dynamical properties of the polynomials $P_d(\lambda, \cdot)$ carry over to the exponentials $E(\lambda, \cdot)$. This makes it possible to study transcendental maps, the exponentials, by considering polynomials for which the theory is well–known. In [11] a combinatorial description of the parameter space of the exponentials $E(\lambda, \cdot)$ is given as a limit of structures of the polynomials $P_d(\lambda, \cdot)$. This is done by introducing symbolic dynamics in phase and parameter space. The pointwise convergence of hyperbolic components in parameter space is shown in [10, 11, 20]. The convergence of Julia sets w. r. t. the Hausdorff metric is shown in [20] for the above families. Similar convergence results for the class of rational functions can be found in [21] and in [22] for more general classes. In this context the set $SV(f)$ of the finite singular values of the function in question plays a crucial role. Note that in the above example all functions have 0 as their only finite singular value.

In this paper we extend the results in [10, 11, 20, 21] to the approximation in the class of entire transcendental functions of finite type. We deal with approximations of an analytic family $G(\lambda, \cdot)$ of entire functions by analytic families $F_n(\lambda, \cdot)$, often polynomials, where $\lambda \in \mathbb{C}$. In order to control the dynamics of these functions we assume that these families are of constant finite type, i. e. the sets of all finite singular values of these functions are finite and admit analytic parametrizations, cf. Section 4. We prove that the hyperbolic components convergence as kernels in the sense of Carathéodory. This is a stronger notion than the pointwise convergence that has been used up to now. Note that the hyperbolic components can not be expected to converge in the Hausdorff metric as the Julia sets do.

For any parameter value chosen from a hyperbolic component of the limit family, the Julia sets of the approximating functions converge w. r. t. the Hausdorff metric to the Julia set of the limit function. Proofs can be found in [20, 21] for special cases. However, since these proofs do not use specific properties of the considered function classes they apply in this case, too. For details we refer to [22].

Main Theorem: *Let $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $F_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be holomorphic families of functions holomorphic on the complex plane such that F_n converges to G uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$ as n tends to ∞. If these families are of constant finite type and $|SV(G(\lambda, \cdot))| = |SV(F_n(\lambda, \cdot))|$ then every hyperbolic component H of the family G is kernel of a sequence $\{H_n\}_{n \in \mathbb{N}}$ of hyperbolic components of the families F_n.*

Note that the methods in [9, 11, 20] can not be used in this general setting since they make use of explicit properties of the exponentials and the approximating polynomials. In this paper we give proofs of the stated convergence results that do not assume the functions to be in a specific form.

The paper is organized as follows. In Section 2 we recall basic facts from iteration theory as needed in later sections. The possible definitions of the hyperbolic components and the different notions of convergence are discussed in Section 3. Section 4 deals with convergence and approximation of functions of finite type. In particular, we give a necessary condition under which hyperbolicity is structurally stable property. In Section 5 the convergence results are stated and proven. Furthermore, we give examples illustrating the main theorem.

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2 Notation and Background

In this section we set up notations and recall the basic properties of Fatou and Julia sets. The reader interested in further details and proofs is referred to the expositions [5] and [7] or to the monographs [4, 8, 27]. For a domain $M \subset \mathbb{C}$ let $\mathcal{O}(M)$ denote the set of *holomorphic* functions $f : M \to \mathbb{C}$. A function f is called *entire* if it is holomorphic on $\mathbb{C}$, i. e. it is an entire transcendental function or a polynomial.

Consider an entire function f on $\mathbb{C}$ and let $f^{\circ n}$ denote the n-th iterate of f. If f is transcendental then f and all $f^{\circ n}$ have an essential singularity at ∞. We denote the Riemann sphere by $\mathbb{P}_1$ and call $\zeta \in \mathbb{P}_1$ a *periodic point of period n of f* if for some integer $n > 0$ the following holds:

(i) $f^{\circ n}$ is complex analytic on some neighbourhood $U \subset \mathbb{P}_1$ of ζ
(ii) $f^{\circ n}(\zeta) = \zeta$.

A periodic point ζ of f is called attracting or repelling if $|(f^{\circ n})'(\zeta)|$ is smaller resp. larger than one. If $(f^{\circ n})'(\zeta) = 0$ then ζ is called a super–attracting periodic point. We define the *Julia set $\mathcal{J}(f)$ of f* as the closure (w. r. t. the Riemann sphere) of the set of all repelling periodic points of f, and the *Fatou set $\mathcal{F}(f)$ of f* as the complement (w. r. t. the Riemann sphere) of $\mathcal{J}(f)$. These definitions immediately imply

Lemma 2.1 *The Julia set is completely invariant, i. e.* $z \in \mathcal{J}(f)$ *implies* $f(z) \in \mathcal{J}(f)$ *unless* f *is transcendental and* $z = \infty$*, and* $f(z) = w \in \mathcal{J}(f)$ *implies* $z \in \mathcal{J}(f)$*.*

The Fatou set and Julia set are viewed as subsets of the Riemann sphere $\mathbb{P}_1$. We keep in mind that polynomials are complex analytic self mappings of $\mathbb{P}_1$. Throughout this paper we write $f(S) := \{f(z) \mid z \in S \text{ and } f \text{ is defined in } z\}$, where f is an entire function and $S \subset \mathbb{P}_1$. We call a point $v \in \mathbb{P}_1$ a *singular value of* f if for every neighborhood U of v there exists a branch of the inverse of f which is not holomorphic on U. Let $SV(f)$ denote the set of all *finite* singular values of f. For transcendental functions the set $SV(f)$ consists of all finite *critical values of* f and of all finite *asymptotic values of* f. For polynomials $SV(f)$ consists only of critical values. If ζ is not an essential singularity of f, i. e. $\zeta \neq \infty$ if f is an transcendental function, then $f^{\circ n}(\zeta)$ is defined for all $n \in \mathbb{N}$ and we call $O^+(\zeta) := \{f^{\circ n}(\zeta) \mid n \in \mathbb{N}\}$ the (forward) orbit of ζ. To every attracting periodic point ζ there belongs the *attracting cycle* $Z := O^+(\zeta)$. The *basin of attraction of* Z is defined to be the set of all points $z \in \mathbb{C}$ such that every accumulation point of $O^+(z)$ belongs to Z. Each component of a basin of attraction is a component of the Fatou set $\mathcal{F}(f)$ and that each basin of attraction contains at least one singular value.

An important role in the iteration theory of transcendental functions is played by wandering domains and Baker domains. A component U of the Fatou set of a transcendental function f is called a *wandering domain* if $f^{\circ m}(U) \cap f^{\circ n}(U) = \emptyset$ for all integers $n, m \in \mathbb{N}$ satisfying $0 \leq n < m$. A Baker domain B is a periodic component of the Fatou set of a transcendental function f such that $\lim_{k\to\infty} f^{\circ(kp+m)} \equiv \infty$ uniformly on compact subsets of B for some integers $m, p \in \mathbb{N}$.

3 Hyperbolicity

In iteration theory it is standard to use

Definition 3.1 *A polynomial or rational function* f *is called* hyperbolic *if the forward orbits of its singular values are relatively compact in the Fatou set F(f), i. e.*

$$O^+(SV(f)) \subset\subset \mathcal{F}(f)\,.$$

In the case that f is a polynomial (or a rational function for that matte) the set of singular values $SV(f)$ consists only of critical values of f and we have

Proposition 3.1 *Let f be a polynomial or a rational function. Then the following statements are equivalent:*

(H1) *f is hyperbolic.*

(H2) *$O^+(SV(f)) \subset\subset \mathcal{F}(f)$.*

(H3) *$SV(f) \subset \cup A(Z_\nu)$ where the union is taken over all attracting cycles of f.*

(H4) *$O^+(SV(f)) \subset\subset \cup A(Z_\nu)$ where the union is taken over all attracting cycles of f.*

(H5) *There exists some integer n such that $|(f^{\circ n})'(z)| > c$ for every $z \in \mathcal{J}(f)$ and some constant $c > 1$.*

The proof can be found in [7].

Remark. If f is hyperbolic then $\mathcal{F}(f) = \cup A(Z_\nu)$, but the reverse is not true. Consider the quadratic polynomial $f(z) = z^2 - 2$. Then $\mathcal{J}(f) = [-2, 2]$, $\mathcal{F}(f) = A(\infty)$ but $\mathcal{J}(f)$ contains the critical value -2. Hence (H3) is not satisfied and f is not hyperbolic.

In the case that f is an entire transcendental function the set $SV(f)$ of finite singular values consists of the finite critical values and of the finite asymptotic values of f. Furthermore, in general the number of singular values is not finite. This causes some difficulties concerning the equivalences in Proposition 3.1. (H2) and (H4) are still equivalent, but the proof is harder. Certainly, (H4) implies (H3). The reverse is not true any longer. In order to illustrate the difficulties we give

We remark that property (H5) does not make sense for transcendental functions since f is required to be holomorphic on some open neighbourhood of $\mathcal{J}(f)$. If f is for example an entire transcendental functions then $\infty \in \mathcal{J}(f)$. Since ∞ is an essential singularity f it is not complex analytic on any open neighbourhood of $\mathcal{J}(f)$. Nevertheless, in the case where f is an entire transcendental function similar results as in the rational case have been proven [24, 26].

We can summarize the above discussion by the

Question. *Which of the properties (H2) – (H4) is appropriate to define hyperbolicity for transcendental functions?*

The standard way to avoid the above difficulties is to consider the class of functions of finite type.

Definition 3.2

a.) A function f holomorphic on $\mathbb{C}$ is of finite type *if its set $SV(f)$ of finite singular values is finite.*

b.) A function f of finite type and holomorphic on $\mathbb{C}$ is called hyperbolic *if $O^+(SV(f)) \subset\subset \mathcal{F}(f)$.*

Remark. Polynomials are always of finite type. Another example is $f(z) = \exp(p(z))$, where p is a polynomial.

For functions of finite type Sullivan's classification of the components of the Fatou set holds, in particular, there are no wandering or Baker domains, see [2, 15, 17]. In the sequel we will only consider functions of finite type. For this class of functions we have

Proposition 3.2 *Let f be an entire function on $\mathbb{C}$ of finite type. Then (H2) – (H4) are equivalent.*

Proof: Certainly, (H4) implies (H2) and (H3). f is of finite type, i. e. $SV(f)$ is finite. Consequently, (H3) implies (H4). It remains to prove that (H2) implies (H4).
As in the rational case one shows that (H2) implies that f does not have any Siegel discs, Herman rings or rationally indifferent cycles. Since f is of finite type it does not have any wandering domains or Baker domains. Hence, each component of the Fatou set is subset of some attracting basin.

□

From now on we study families of entire maps depending on a complex parameter. Consider a holomorphic function $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C} : (\lambda, z) \mapsto G(\lambda, \cdot)$ where $|SV(G(\lambda, \cdot))| = N(G)$ for some integer $N(G)$ and every $\lambda \in \mathbb{C}$. Here and in the sequel the singular values are *counted without multiplicity.*

Definition 3.3 *Consider $M_H := \{\lambda \in \mathbb{C} \mid G(\lambda, \cdot)$ is hyperbolic $\}$. A connected component of M_H is called a* hyperbolic component *of G.*

Remarks.
1. We note that due to Proposition 4.3 the set $M_H(G) := \{\lambda \in \mathbb{C} \mid G(\lambda, \cdot) \text{ is hyperbolic}\}$ is open.
2. The parameter space $\mathbb{C}$ may be replaced by an arbitrary domain $M \subset \mathbb{C}$.

Example 1. The classical example is the quadratic family

$$G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto z^2 + \lambda \, .$$

The well–known Mandelbrot set is defined as

$$M_2 := \{\lambda \in \mathbb{C} \mid \{G(\lambda, \cdot)^{\circ k}(\lambda)\}_{k \in \mathbb{N}} \text{ is bounded}\}.$$

For every $\lambda \in \mathbb{C}$ the polynomial $G(\lambda, \cdot)$ has two critical values, namely λ and ∞. Since ∞ is a super–attracting fixed point for any λ the critical value λ determines the dynamics.

For $\lambda \notin M_2$ the critical value λ lies in the basin of attraction of ∞. Hence $H_{\text{ext}} := \mathbb{C} \setminus M_2$ is a hyperbolic component of the quadratic family. If $\lambda \in \partial(H_{\text{ext}}) = \partial M_2$ then λ can not be in the basin of ∞. Consequently, the boundary of M_2 can be described as follows: ∂M_2 is the set of those parameter values where the critical value λ, the so–called *free* critical value, leaves the basin of ∞.

We now turn our attention to the bounded hyperbolic components of the quadratic family. Obviously, these components are in the interior of M_2. Note that it is an open question whether the union of all bounded hyperbolic components is dense in M_2. If H is a bounded hyperbolic component and $\lambda \in H$ then $G(\lambda, \cdot)$ has a bounded attracting cycle A_λ containing the critical value λ in its basin of attraction. The boundary of a bounded hyperbolic component can be described as the set of parameters such that the cycle A_λ becomes indifferent. Although it seems to be well–known, [23] seems to contain the first direct proof that the Julia sets depend analytically on the parameter on the components of the interior of M_2.

We finish this section by recalling different notions of convergence of domains. Let $\{H_n\}_{n \in \mathbb{N}}$ be a sequence of domains in $\mathbb{C}$ and $H \subset \mathbb{C}$ a domain. There are three different notions of convergence of domains that we use in the sequel.

(P) H is the *pointwise limit* of the sequence if every point $z \in H$ is contained in almost every H_n and every point $w \notin H$ is contained in only finitely many H_n or in ∂H.

(K) H is the *kernel* of the sequence if every compact subset of H is contained in almost every H_n and this is not true for any domain $\widetilde{H}$ satisfying $H \subsetneq \widetilde{H}$.

(H) H is the limit of the sequence *w. r. t. the Hausdorff distance* if for every $\varepsilon > 0$ there exists some integer n_0 such that $\partial H \subset U_\varepsilon(\partial H_n)$ and $\partial H_n \subset U_\varepsilon(\partial H)$ holds for every $n \geq n_0$.

Certainly, (H) implies (K) which in turn implies (P). It turns out that the hyperbolic components of families of finite type are kernels of the approximating families, but in general can not be expected to converge with respect to the Hausdorff distance.

4 The Robustness of Hyperbolicity

Our main goal is to investigate the convergence of hyperbolic components of families of functions depending on a complex parameter. We are concerned with an entire family

$$G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto G(\lambda, z)$$

and entire families

$$F_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto F_n(\lambda, z)\,, n \in \mathbb{N},$$

such that the functions F_n converge to G uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$.

In [10, 11, 20] this is studied for the case that $G = \lambda e^z$ is the standard exponential family and $F_n = \lambda(1 + \frac{z}{d})^d$. In particular, there it is proved that for every hyperbolic component H of G there exist hyperbolic components H_n of F_n for almost every integer $n \in \mathbb{N}$ such that the H_n tend to H pointwise. Here we generalize this convergence result to the class of functions of finite type and also use the stronger notion (K) of kernel convergence.

General Assumption (Constant finite type).
In the sequel we assume that the families G and F_n are of constant finite type, i. e. there are integers $N(G), N(F_n)$ such that

$$|SV(G(\lambda,\cdot))| = N(G) < \infty \text{ and } |SV(F_n(\lambda,\cdot))| = N(F_n) < \infty$$

(counted without multiplicity) independantly of $\lambda \in \mathbb{C}$ and that the sets $SV(G(\lambda,\cdot))$ and $SV(F_n(\lambda,\cdot))$ have holomorphic parametrizations on $\mathbb{C}$. This means that there are analytic functions $s_\nu : \mathbb{C} \to \mathbb{C}$, $\nu = 1,\ldots,N(G)$, such that $SV(G(\lambda,\cdot)) = \{s_1(\lambda),\ldots,s_{N(G)}(\lambda)\}$ and the analogue holds for the $SV(F_n(\lambda,\cdot))$.

For some statements we need the additional

Condition (N). $G(\lambda,\cdot)$ and $F_n(\lambda,\cdot)$ are of the same constant finite type, i. e. $N(G) = N(F_n)$ for (almost) all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

We begin by considering some consequences of uniform convergence.

Lemma 4.1 *Let g be an entire function of finite type. Furthermore let a sequence $\{f_n\}_{n\in\mathbb{N}}$ of entire functions converge uniformly to g on compact subsets of $\mathbb{C}$. Let $w \in \mathbb{C}$ be a singular value of g. Then there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} w_n = w$ and w_n is a singular value of f_n for almost every $n \in \mathbb{N}$.*

Proof: Suppose w is a critical value. Then there exists some $c \in \mathbb{C}$ such that $g(c) = w$ and $g'(c) = 0$. For simplicity we assume c to be a simple root of g'. By Rouché's Theorem there is a sequence $\{c_n\}_{n\in\mathbb{N}}$ satisfying $\lim\limits_{n\to\infty} c_n = c$, $\lim\limits_{n\to\infty} f_n(c_n) = w$ and $f_n'(c_n) = 0$ for almost every $n \in \mathbb{N}$. We choose $w_n := f_n(c_n)$.

Assume now that w is an asymptotic value and that the statement is false. This means that g is a transcendental function and consequently $\infty \in \mathcal{J}(g)$. Then there exists some simply connected neighbourhood $W \subset \mathbb{C}$ of w such that W does not contain any singular value of f_n for any $n \in \mathbb{N}$. This implies the existence of an unbounded domain U such that $w \in \partial(g(U))$ but $w \notin g(\partial U \cap \mathbb{C})$. We fix $\zeta_0 \in U$ such that $g(\zeta_0) \in W$. We may assume $\widetilde{U} := g(U) \subset\subset W$. Then $\widetilde{U}$ is kernel w.r.t. $g(\zeta_0)$ of the sequence $\{f_n(U)\}_{n\in\mathbb{N}}$. We write $\widetilde{U}_n := f_n(U)$ and $z_n := f_n(\zeta_0)$. Now, for every $n \in \mathbb{N}$ there exist

functions $h_n \in \mathcal{O}(\widetilde{U})$ satisfying $h_n \circ f_n = id|_U$ and $h_n(z_n) = \zeta_0$. Clearly $\left(\lim_{n\to\infty} h_n\right) \circ g = id|_U$, hence, $h = \lim_{n\to\infty} h_n \in \mathcal{O}(U)$ is not constant. In particular, h satisfies $h(g(\zeta_0)) = \zeta_0$. W does not contain any singularity of f_n^{-1}, thus, h_n extends holomorphically to W.

Since $W \subset\subset \mathbb{C}$ we may choose two points in $\mathbb{C}$, say ζ_1 and ζ_2, such that $\zeta_1 \neq \zeta_2$, $\{\zeta_1, \zeta_2\} \subset \mathbb{C}$ and $\{g(\zeta_1), g(\zeta_2)\} \cap W = \emptyset$. We may also assume $\{f_n(\zeta_1), f_n(\zeta_2)\} \cap W = \emptyset$ for every $n \in \mathbb{N}$.
By construction, we have $h_n(W) \cap \{\zeta_1, \zeta_2, \infty\} = \emptyset$. This implies that $\{h_n\}_{n\in\mathbb{N}}$ is a normal family on W. On $\widetilde{U} \subset W$ we have $h := \lim_{n\to\infty} h_n$, which yields $h = \lim_{n\to\infty} h_n$ uniformly on compact subsets of W and $h \in \mathcal{O}(W)$. The function $h|_{\widetilde{U}}$ is not constant, hence, h is a non–constant holomorphic function on W. The property $h_n \circ f_n = id$ carries over to h: $g \circ h = id|_W$. Now $\widetilde{U} \subset\subset W$ implies $U = h(\widetilde{U}) \subset\subset \mathbb{C}$. But U was supposed to be unbounded, a contradiction.

□

The following lemma is well–known, cf. [18].

Lemma 4.2 *Let g be a hyperbolic entire function. Then for every sequence $\{f_n\}_{n\in\mathbb{N}}$ of entire functions converging to g uniformly on compact subsets of $\mathbb{C}$ and every set $K \subset\subset \mathcal{F}(g)$ there exists some number $n_0 \in \mathbb{N}$ such that $K \subset\subset \mathcal{F}(f_n)$ for every $n \geq n_0$.*

Combining these two lemmas we obtain

Proposition 4.3 *Hyperbolicity is a structurally stable property for families of constant finite type, i. e. the set of hyperbolic functions is open in parameterspace.*

5 Main Results

It is our aim to prove the kernel convergence of hyperbolic components, the main theorem. One can not expect a hyperbolic component of the limit family to be a limit w. r. t. the Hausdorff distance of hyperbolic components of the approximating families.

Example 2. Consider the families

$$G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \left(1 + \lambda^2\right) z + z^2$$

and

$$F_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \left(1 + \lambda^2 - \frac{1}{n}\right) z + z^2 \, .$$

For every $n \in \mathbb{N}$ the polynomial $F_n(0, \cdot)$ is hyperbolic, but $G(0, \cdot)$ is not. We look for those parameter values where these polynomials have $z_0 = 0$ as an attracting fixed point. This set forms a hyperbolic component for the family in question. The parameter values where z_0 is an indifferent fixed point form the boundary of the associated hyperbolic components. In fact, for every $n \in \mathbb{N}$ these boundaries are Cassini–type curves and in the case of the limit family G we obtain a lemniscate, cf. Figure 1. Hence, in this case there are hyperbolic components H^1, H^2 of G which are not a limit w. r. t. the Hausdorff distance of hyperbolic components H_n of the families F_n.

As an immediate consequence of the robustness of the hyperbolicity, cf. prop. 4.3, we get for families.

Lemma 5.1 *Let $\lambda_0 \in M$ be such that $G(\lambda_0, \cdot)$ is hyperbolic.*
(a) There exists a neighbourhood $U \subset M$ of λ_0 such that $G(\lambda, \cdot)$ is hyperbolic for every $\lambda \in U$.
(b) Suppose that condition (N) holds. Then there exists some neighbourhood $U \subset M$ of λ_0 such that $F_n(\lambda, \cdot)$ is hyperbolic for all $\lambda \subset U$ and almost every $n \in \mathbb{N}$.

Rephrasing this in terms of the convergence of hyperbolic components gives

Corollary 5.2 *Under condition (N) every hyperbolic component H of the family G is contained in a kernel of a sequence $\{H_n\}_{n \in \mathbb{N}}$ of hyperbolic components H_n of the families F_n.*

The necessicity of condition (N) is illustrated by

Example 3. Consider the families

$$F_n : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \to z^2 \left(\lambda + \left(\frac{3 - 2\lambda n}{n^2}\right) z + \left(\frac{\lambda n - 2}{n^3}\right) z^2\right)$$

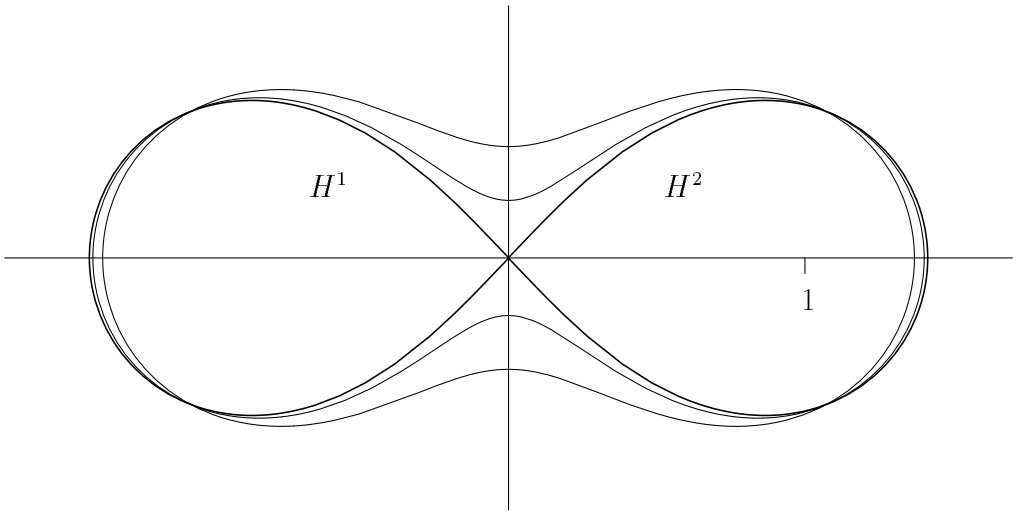

Figure 1: In general the hyperbolic components do not converge w. r. t. Hausdorff metric. In Example 2 the limit of the hyperbolic components of the attracting fixed point 0 are the two kernels H_1 and H_2.

and

$$G : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \to \lambda z^2 \ ,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. As usual the functions F_n converge to G uniformly on compact subsets of $\mathbb{C}^* \times \mathbb{C}$ as n tends to infinity. $F_n(\lambda, \cdot)$ is a polynomial of degree 4, hence it has exactly 3 (finite) critical values. On the other hand, $G(\lambda, \cdot)$ is a quadratic polynomial having $w = 0$ as an super–attracting fixed point (and no other finite critical values). This proves condition (N) is not satisfied in this example.
By construction, for every $\lambda \in \mathbb{C}^*$ the function $G(\lambda, \cdot)$ is hyperbolic. One calculates that

$$F_n(\lambda, n) = n \ \text{ and } \ F_n'(\lambda, n) = 1$$

for every $\lambda \in \mathbb{C}^*$ and every $n \in \mathbb{N}$. This proves that $F_n(\lambda, \cdot)$ has a rationally indifferent fixed point. But every rationally indifferent fixed point is an element of the Julia set and attracts a critical value. Thus $F_n(\lambda, \cdot)$ cannot be

hyperbolic. In other words, the families F_n do not have any hyperbolic components. Consequently the hyperbolic component $\mathbb{C}^*$ of the family G can not be the kernel of a sequence of hyperbolic components of the approximating families.

It is our aim to prove the reverse statement of corollary 5.2. The key ingredient is

Theorem 5.3 *Let $H \subset \mathbb{C}$ be kernel of a sequence $\{H_n\}_{n\in\mathbb{N}}$ of hyperbolic components of the families F_n. Then either*

(i) $G(\lambda,\cdot)$ is not hyperbolic for any $\lambda \in H$ or

(ii) $H \subset \widetilde{H}$ for some hyperbolic component $\widetilde{H}$ of the family G.

Proof: The idea of the proof is to show that case (ii) holds if case (i) does not. In the sequel we assume that case (i) does not hold. That means that there is a hyperbolic component $\widetilde{H}$ of G with $\widetilde{H} \cap H \neq \emptyset$. We take a domain $B \subset\subset H \cap H_n$ for (almost) all $n \in \mathbb{N}$. The idea is now to show that $B \subset \widetilde{H}$. This is done by repeatedly using the Theorem of Montel for the parametrizations of the singular values.

Without loss of generality we may assume $\{0,1\} \subset J(G(\lambda,\cdot)), J(F_n(\lambda,\cdot))$ for all $\lambda \in \mathbb{C}$ and every $n \in \mathbb{N}$. By definition, $\infty \notin SV(\cdot)$ holds. Hence, we have $SV(G(\lambda,\cdot)), SV(F_n(\lambda,\cdot)) \subset \mathbb{C} \setminus \{0,1\}$.

Let $s_1, \ldots, s_m \in \mathcal{O}(\mathbb{C})$ be the parametrizations of the $N(G) := m$ singular values of $G(\lambda,\cdot)$. Let $s_{n,1}, \ldots, s_{n,m}, \ldots \in \mathcal{O}(\mathbb{C})$ be the parametrizations of the $N(F_n) \geq m$ singular values of $F_n(\lambda,\cdot)$. ¿From now on we fix $\mu \in \{1,\ldots,m\}$. Note that due to Lemma 4.1 and Montel's Theorem we have $s_{n,\mu} \to s_\mu$ uniformly on compact sets since we have $s_{n,\mu} \subset \mathbb{C}\setminus\{0,1\}$ (after renumbering the $s_{\nu,\mu}$ if necessary).

For $\lambda \in B$ we have $s_{n,\mu} \notin \mathcal{J}(F_n(\lambda,\cdot))$ and the invariance of Julia sets yields $F_n^{\circ\nu}(\lambda, s_{n,\mu}) \notin \mathcal{J}(F_n(\lambda,\cdot))$ and, in particular, $F_n^{\circ\nu}(\lambda, s_{n,\mu}) \in \mathbb{C} \setminus \{0,1\}$ for all $\lambda \in B$. In other words $\{F_n^{\circ\nu}(\lambda, s_{n,\mu}) | n \in \mathbb{N}\}$ is a normal family for any fixed ν. We call the limit

$$S_{\nu,\mu} : B \to \mathbb{P}_1; \lambda \mapsto S_{\nu,\mu}(\lambda) := \lim_{n\to\infty} F_n^{\circ\nu}(\lambda, s_{n,\mu}(\lambda))$$

and conclude that $S_{\nu,\mu}(\lambda) = G^{\circ\nu}(\lambda, s_\mu(\lambda))$ due to the uniform convergence on compact sets of the $F_n(\lambda,\cdot)$ to $G(\lambda,\cdot)$ and the $s_{n,\mu}$ to s_μ. If $G^{\circ\nu}(\lambda, s_\mu(\lambda)) \equiv 0,1$ for some smallest ν then G would not be hyperbolic on $B \cap \widetilde{H}$, a contradiction. We conclude that indeed

$$S_{\nu,\mu}(\lambda) = G^{\circ\nu}(\lambda, s_\mu(\lambda)) \in \mathbb{C} \setminus \{0,1\}$$

for all $\lambda \in B$ and $\nu \in \mathbb{N}$. In other words $\{S_{\nu,\mu}(\lambda) | \nu \in \mathbb{N}\}$ is a normal family. Let S denote a limit function. For simplicity we assume $S := \lim_{n\to\infty} S_{\nu,\mu}(\lambda)$.

For the limit function we have either $S \equiv 0, 1, \infty$ or $S(B) \in \mathbb{C} \setminus \{0, 1\}$. Since $\widetilde{H}$ is a hyperbolic component of $G(\lambda, \cdot)$ we have on $B \cap \widetilde{H}$ for the limit $S(\lambda) = a(\lambda)$, where $a(\lambda)$ is an attracting periodic point of, say, period p of $G(\lambda, \cdot)$. Consequently, $S \not\equiv 0, 1$.
In case $S \equiv \infty$ the Identity Theorem yields that ∞ is an attracting periodic point of $G(\lambda, \cdot)$ for every $\lambda \in B$. In particular, $G(\lambda, \cdot)$ is an polynomial, and thus ∞ is a super–attracting fixed point and the basin of ∞ is containing the singular value $s_\mu(\lambda)$ in its basin.

We now assume $S \not\equiv \infty$. Then we have $a_n(\lambda) \to a(\lambda)$ for all $\lambda \in B \cap \widetilde{H}$, where $a_n(\lambda)$ is an attracting periodic point of period p of $F_n(\lambda, \cdot)$. But since $B \subset\subset H_n$ we conclude that $a_n(\lambda)$ is an attracting periodic point of period p of $F_n(\lambda, \cdot)$ for all $\lambda \in B$. In particular, $a_n(B) \in \mathbb{C} \setminus \{0, 1\}$ and $\{a_n | n \in \mathbb{N}\}$ is a normal family, which converges on $B \cap \widetilde{H}$ to the well–defined function a. Hence, we can extend a to the whole of B. Due to the uniform convergence of the $F_n(\lambda, \cdot)$ to $G(\lambda, \cdot)$ the point $a(\lambda)$ is a non–repelling point of $G(\lambda, \cdot)$ for all $\lambda \in B$. Since $a(\lambda)$ is attracting on $B \cap \widetilde{H}$ we conclude from the Maximum Modulus Principle that it is even attracting on the whole of $B \subset\subset H_n$. Finally we conclude

$$S_\mu(\lambda) \in \mathcal{F}(G(\lambda, \cdot))$$

for all $\lambda \in B$. Since μ was arbitrary we have shown that according to Proposition 3.1 (H3) B is in a hyperbolic component of G and, hence, $B \subset \widetilde{H}$. Since $B \subset\subset H$ was arbitrary we conclude $H \subset \widetilde{H}$ and the proof is completed.

□

Proof of the Main Theorem: Due to Corollary 5.2 every compact set in H is contained in some hyperbolic components H_n of the F_n. Let $\widetilde{H}$ be the kernel of the sequence $\{H_n\}_{n\in\mathbb{N}}$, in particular, $H \subset \widetilde{H}$. We have to show $H = \widetilde{H}$ for which it suffices to show $\widetilde{H} \subset H$. According to Theorem 5.3 this is the case or $G(\lambda, \cdot)$ is not hyperbolic for all $\lambda \in \widetilde{H}$. This contradicts the assumption that the hyperbolic component H is a subset of $\widetilde{H}$.

□

Remark. One might expect $H = \widetilde{H}$. We now give an example that this is not generally the case.

Example 4. Consider the families

$$F_n : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \to \lambda z^2 \left(1 + \frac{4 - \lambda - 2n}{n^2} z + \frac{n + \lambda - 3}{n^3} z^2\right)$$

and

$$G : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \to \lambda z^2 \ .$$

The functions F_n converge to G uniformly on compact subsets of $\mathbb{C}^* \times \mathbb{C}$ as n tends to infinity. $F_n(\lambda, \cdot)$ is a polynomial of degree 4, hence, it has exactly three (finite) critical values. On the other hand, $G(\lambda, \cdot)$ is a quadratic polynomial having $\hat{\zeta} = 0$ as a super–attracting fixed point (and no further finite critical values).
By construction the function $G(\lambda, \cdot)$ is hyperbolic for every $\lambda \in \mathbb{C}^*$. One calculates, that

$$F_n(\lambda, n) = n \quad \text{and} \quad F_n'(\lambda, n) = \lambda$$

for every $\lambda \in \mathbb{C}^*$ and every $n \in \mathbb{N}$. For $\lambda \in \mathbb{D}^* := \{z \in \mathbb{C}^* \,|\, |z| < 1\}$ the point $\zeta_n := n$ is an attracting but not super–attracting fixed point of $F_n(\lambda, \cdot)$, and in the basin of this fixed point there is at least one non (pre–)periodic critical value ζ_n of $F_n(\lambda, \cdot)$. Since $\hat{\zeta}$ is an attracting fixed point we obtain $\zeta_n \neq \hat{\zeta}$. But there must be a third critical value. One can show, that for large $n \in \mathbb{N}$ there is a critical point $c_n \approx \frac{n}{2}$. This implies

$$v_n := F_n(\lambda, c_n) \approx \left(\frac{n}{2}\right)^2 \ .$$

Now, for sufficiently large n one can prove the critical value v_n to be an element of the basin of infinity which is a super–attracting fixed point. This yields $\mathbb{D}^*$ to be an hyperbolic component of F_n for almost every $n \in \mathbb{N}$. The kernel H of this constant sequence of hyperbolic components is $\mathbb{D}^*$ again. But the limit family G has only one hyperbolic component, namely $\widetilde{H} := \mathbb{C}^*$. This proves $H \subsetneq \widetilde{H}$. On the other hand we have shown that for $\lambda \in \mathbb{D}^*$ the function $F_n(\lambda, \cdot)$ has three distinct finite critical values, but the limit function has only one. Hence, condition (N) is not satisfied in this example.

We finish this section by giving a large class of examples for the application of the main theorem.

Example 5. Take polynomials p and q and consider the family

$$G : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \lambda \cdot p(z) \cdot e^{q(z)} \ .$$

Recall that $P_n(z) = (1 + \frac{z}{n})^n$ converges to e^z uniformly on compact subsets of $\mathbb{C}$ as n tends to infinity. We define

$$F_n : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \lambda \cdot p(z) \cdot P_n(q(z)) .$$

Then for every $\lambda \in \mathbb{C}^*$ we have

$$\lim_{n\to\infty} F_n = G$$

w. r. t. uniform convergence on compact subsets of $\mathbb{C}^* \times \mathbb{C}$.

For $\lambda \in \mathbb{C}^*$ and $n \in \mathbb{N}$ the function $F_n(\lambda, \cdot)$ is in fact a polynomial and does not have any asymptotic value. We now look for critical values and critical points. The critical points turn out to be the roots of $q(z) + n$ and $Q_n := p' + \frac{1}{n}p' \cdot q + p \cdot q'$. To every root of $q(z) + n$ there belongs the critical value $w_0 = 0$. The polynomials Q_n converge to $Q := p' + p \cdot q'$. Both, Q_n and Q are of degree $d := \deg(p) + \deg(q) - 1$, hence Q_n produces d critical values $w_1(n), \ldots, w_d(n)$. These converge to critical values $w_1, \ldots, w_d$ of the function $G(\lambda, \cdot)$. $G(\lambda, \cdot)$ does not have any further critical values, since the critical points are roots of Q which is of degree d. But there is an asymptotic value, namely $w_0 = 0$.
This proves that in this example condition (N) is satisfied. By the main theorem every hyperbolic component of the family G is kernel of a sequence $\{H_n\}_{n\in\mathbb{N}}$ of hyperbolic components H_n of the families F_n.

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