Julia sets for holomorphic endomorphisms of $\mathbb{C}^n$

Stefan-M. Heinemann*
Institut für Mathematische Stochastik
Lotzestraße 13
D-37083 Göttingen

12th August 1995

Abstract

We give a definition for a Julia set $J(f)$ for generic classes of polynomial endomorphisms $f : \mathbb{C}^n \to \mathbb{C}^n$. For $n = 1$, our definition is equivalent to the usual one, which gives the points where the iterates of f do not form a normal family. Moreover, the Julia set $J(f_1 \times \ldots \times f_n) \subset \mathbb{C}^n$ for a product of one-dimensional polynomials $f_i : \mathbb{C} \to \mathbb{C}$ turns out to be the product $J(f_1) \times \ldots \times J(f_n)$ of the associated Julia sets $J(f_i) \subset \mathbb{C}$. For a special class of mappings $f : \mathbb{C}^2 \to \mathbb{C}^2$ which is not of this simple type, the so called Cantor skews, we investigate topological structure as well as measure theoretic aspects of the Julia sets obtained using our definition.

0 Introduction

The iteration of holomorphic functions of one variable has been a lively field of mathematical research for quite some time. The theory of iteration of rational functions on the Riemann sphere has become well developed during the last decades. It is now a natural question to ask for generalizations to the higher dimensional case.

The first thing to ask for is the class of mappings which we want to iterate. One possible choice are automorphisms, e.g. so called complex Hénon maps in $\mathbb{C}^2$. Another choice could be to consider holomorphic endomorphisms of (complex) projective spaces, hence homogeneous mappings. Our approach in this paper is different. We are interested in maps which show the same behaviour as polynomials in $\mathbb{C}$. We call these mappings which are defined in terms of their minimal and maximal growth rates (p, q)-*regular mappings*. They include a smaller class which we call *strict polynomials*. These classes of endomorphisms are closed under composition, hence also under iteration. Moreover, for (p, q)-regular maps $f : \mathbb{C}^n \to \mathbb{C}^n$, the set $K(f) \subset \mathbb{C}^n$ of points with bounded forward orbit turns out to be compact. Further, as (p, q)-regular maps are proper, they are compatible with the one-point-compactification of $\mathbb{C}^n$.

The second problem is to give a suitable definition for what is to become the substitute for "normal convergence" in one variable. Our approach, which we call *weakly normal convergence*, considers convergence on analytic sets instead of "full" open sets. The Julia set for a (p, q)-regular mapping f is then simply defined as the complement of the set of weakly normal points for the family $\{f^k\}$ of iterates of f. For $n = 1$, normal convergence and weakly normal convergence are equivalent, hence one obtains the usual Julia set. Moreover, for n-vectors of polynomials of one variable, the Julia set in $\mathbb{C}^n$ turns out to be exactly the product of the associated one-dimensional Julia sets.

In the third part we illustrate our ideas by looking at a class of examples $f : \mathbb{C}^2 \to \mathbb{C}^2$, the so called Cantor skews. We investigate the topological structure of their Julia sets, show that the dynamic gives a mixing repeller, and that the Julia set (in our definition) equals the support of the measure of maximal entropy. Finally we shall see that the Julia set also coincides with the support of the measure induced by the Green current, which is given by the Shilov boundary of $K(f)$.

*Research supported by SFB 170 "Geometrie und Analysis" and SPP "Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme" Georg-August-Universität Göttingen

1 Strict Polynomials and (p,q)-regular mappings

In dimension one the investigation of polynomials of one variable $f : \mathbb{C} \to \mathbb{C}$ has been an especially fruitful field of research. This is mainly due to several properties of polynomial mappings like finite mapping degree, surjectivity, the compactness of the set $K(f)$ of points with bounded orbit, the possibility to compute the Green function of the complement of $K(f)$ by a dynamical method. One might ask for one characteristic property of polynomials which distinguishes them from other entire mappings, and which implies all the qualities mentioned above. Then one can look for entire mappings $f : \mathbb{C}^n \to \mathbb{C}^n$ which satisfy a similar condition and investigate their dynamical behaviour.
A possible characterization is given by the following lemma.

Lemma 1.1
An entire mapping $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $p \in \mathbb{N}$ if only if one can find constants $k_1, k_2 > 0$, $r \in \mathbb{R}$ such that

$$k_1 \cdot |z|^p \leq |f(z)| \leq k_2 \cdot |z|^p$$

holds for all $|z| > r$ (see [11, p. 11]). □

By definition the constant mapping $f : z \mapsto 0$ has degree $-\infty$.
For $n > 1$ we replace the modulus $|\cdot|$ by a norm $\|\cdot\|$ which is compatible with the usual metric on $\mathbb{C}^n$. This then leads to the following definition.

Definition 1.2 (strict polynomial)
([9, ch. 1]) An entire mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a *strict polynomial* of degree $p \in \mathbb{N}$ if for some $k_1, k_2 > 0$, $r \in \mathbb{R}$,

$$k_1 \cdot \|z\|^p \leq \|f(z)\| \leq k_2 \cdot \|z\|^p \tag{1}$$

holds for $\|z\| > r$.

Let $\mathcal{S}$ denote the set of all such mappings. A strict polynomial of degree p is given by an n-vector of polynomials in n variables (*polynomial vector*) of degree less than or equal to p (cf. [4, p. 219]).
In order to weaken condition (1) a little, we introduce the concept of (p,q)-regularity.

Definition 1.3 ((p,q)-regular mapping)
[10, ch. 2] An entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called (p,q)-*regular* if for $p, q \in \mathbb{N}$, one can find constants $k_1, k_2 > 0$, $r \in \mathbb{R}$ such that

$$k_1 \cdot \|z\|^p \leq \|f(z)\| \leq k_2 \cdot \|z\|^q$$

holds for all $\|z\| > r$.

Evidently, the (p,p)-regular mappings are exactly the strict polynomials of degree p. (p,q)-regular mappings are also polynomial vectors of degree not exceeding q. We call a mapping *regular* if it is (p,q)-regular for some $p, q \in \mathbb{N}$. Let $\mathcal{R}$ denote the set of all such mappings.
From now on let us assume that $p \geq 2$.

Lemma 1.4
The composition of regular mappings is again regular.
Proof: Let $f, g : \mathbb{C}^n \to \mathbb{C}^n$ be (p', q')- respectively (p'', q'')-regular. Hence, for some strictly positive l_1, l_2, m_1, m_2, and $r', r'' \in \mathbb{R}$,

$$l_1 \cdot \|z\|^{p'} \leq \|f(z)\| \leq l_2 \cdot \|z\|^{q'}$$

for $\|z\| > r'$, and

$$m_1 \cdot \|z\|^{p''} \leq \|g(z)\| \leq m_2 \cdot \|z\|^{q''}$$

for $\|z\| > r''$.

If we define

$$\begin{aligned} p &:= p' \cdot p'', \\ q &:= q' \cdot q'', \\ k_1 &:= l_1 \cdot m_1^{p'}, \\ k_2 &:= l_2 \cdot m_2^{q'}, \\ r &:= \max\left\{r'', \sqrt[p'']{r'/m_1}\right\}, \end{aligned}$$

a simple calculation yields that, for $\|z\| > r$,

$$k_1 \cdot \|z\|^p \leq \|f(g(z))\| \leq k_2 \cdot \|z\|^q,$$

which shows that $f \circ g$ is (p,q)-regular. □

Concerning the iteration of strict polynomials we obtain the following result.

Corollary 1.5
The composition of strict polynomials is again a strict polynomial. For $f \in \mathcal{S}$, all iterates (for $k \in \mathbb{N}$)

$$f^k := \underbrace{f \circ \ldots \circ f}_{\text{k times}}$$

are also strict polynomials (of degree p^k). □

Let us now investigate some basic properties of (p,q)-regular mappings.

Theorem 1.6
(p,q)-regular mappings are proper.
Proof: In order to prove that a (p,q)-regular map $f : \mathbb{C}^n \to \mathbb{C}^n$ is proper we have to show that under f the inverse image $L := f^{-1}(K)$ of any compact set $K \Subset \mathbb{C}^n$ is again compact. Assume there was K such that L was not compact for a (p,q)-regular f (where k_1, k_2, r are defined as in definition **1.3**). In any case L is closed. If it were unbounded one could find $z^* \in L$ such that

$$\|z^*\| > \max_{z \in K}\left\{r, \sqrt[p]{(\|z\|+1)/k_1}\right\}.$$

But then

$$\|f(z^*)\| > \max_{z \in K}\{\|z\|+1\},$$

which gives a contradiction, since $f(z^*)$ lies in K. □

As an immediate consequence we derive the following corollary.

Corollary 1.7
A (p,q)-regular mapping admits a continuation to $\overline{\mathbb{C}^n} := \mathbb{C}^n \cup \{\infty\}$ by setting

$$f(\infty) := \infty.$$

□

That this compactification makes sense for dynamical purposes is seen by the next theorem.

Theorem 1.8
The attracting basin $F_\infty := F_\infty(f)$ for "infinity", i.e. the set of points whose forward orbits eventually leave any compact set in $\mathbb{C}^n$, is not empty.
Proof: If we define

$$R_f := \max\left\{r, \; 1 \Big/ \sqrt[p-1]{k_1}\right\},$$

we get, for $\|z\| \geq \delta \cdot R_f$ with $\delta > 1$,

$$\|f(z)\| \geq \delta \cdot \|z\|.$$

By induction it follows that

$$\left\|f^k(z)\right\| \geq \delta^k \cdot \|z\|,$$

hence $\left\|f^k(z)\right\|$ tends to infinity for $\|z\| > R_f$. □

Moreover, F_∞ is open, as it is the complement of the compact set

$$K = K(f) := \left\{z \in \mathbb{C}^n : \ \|f^k(z)\| \text{ stays bounded}\right\}.$$

This fact is easily seen, since with

$$B_{R_f} := \{z \in \mathbb{C}^n : \ \|z\| < R_f\}$$

we get

$$F_\infty = \bigcup_{k=0}^{\infty} f^{-k}\left(\mathbb{C}\overline{B_{R_f}}\right),$$

and

$$K = \bigcap_{k=0}^{\infty} f^{-k}\left(\overline{B_{R_f}}\right). \tag{2}$$

The fact that f is proper implies that f has constant rank n on a dense open subset of $\mathbb{C}^n$ (the complement of the critical locus $Crit$ where the Jacobi-determinant $\mathcal{J}_f$ of f vanishes) (cf. [14, p. 301]).
In the special case when f is a strict polynomial we prove the following result.

Theorem 1.9
A strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ is surjective and has mapping degree p^n.
Proof: Evidently only the terms with maximal degree are relevant in order to check if f is strict or not. Namely f is strict if and only if the homogeneous vector $[f]$ of terms of maximal degree $(= p)$ has no non-trivial zero. This follows from the fact that $f - [f]$ only contains monomials of degree less than or equal to $p - 1$. Hence

$$\|f(z) - [f](z)\| = O\left(\|z\|^{p-1}\right),$$

and these terms can be neglected in (1). This characterization of strictness allows us to extend a strict polynomial f from $\mathbb{C}^n$ to the n-dimensional complex projective space $\mathbb{P}^n$. All holomorphic endomorphisms of $\mathbb{P}^n$ can be represented as homogeneous polynomial vectors $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ without non-trivial zeros (cfer. [16, 1.§4]). Thus we add a variable z_0 to the n affine ones $z_1, \ldots, z_n$ and a $(n+1)$-st component z_0^p to the vector f, such that after multiplying each monomial in f with a suitable power of z_0 to get a homogeneous n-vector f^* of degree p, the resulting polynomial vector $\tilde{f} := (f^*, z_0^p)$ is also homogeneous of degree p. $\tilde{f}$ has no non-trivial zeros, since if $z_0 = 0$, then $[f]$ has to vanish, which implies that $z_1 = \ldots = z_n = 0$. If $z_0 \neq 0$, the last component does not vanish. In order to verify the theorem we have to show that each point $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ has p^n inverse images under f (counted with multiplicity), respectively, that $\tilde{c} = (c_1, \ldots, c_n, 1)$ does so for $\tilde{f}$. Let us check the zeros of the endomorphism represented by

$$\tilde{f}_{\tilde{c}} := \tilde{f} - \tilde{c} \cdot z_0^p,$$

where $\tilde{c} = (c_1, \ldots, c_n, c_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$. Bezout's theorem (see [16, p. 199]) gives the existence of exactly p^n zeros of $\tilde{f}_{\tilde{c}}$ (counted with multiplicity) as endomorphism of $\mathbb{P}^n$, provided one can show that $\tilde{f}_{\tilde{c}}$ has only isolated zeros. But if $c_{n+1} = 0$, then $z_0 = 0$ must hold, which implies like above that $[f]$ must also vanish, hence $z_0 = \ldots = z_{n+1} = 0$. For $c_{n+1} \neq 0$, without loss of generality $c_{n+1} = 1$, we get as first n components exactly $f - c$ which is still a strict polynomial of degree p. If there were a non-isolated zero of $\tilde{f}_{\tilde{c}}$ in $\mathbb{C}^{n+1}$ in this case, then the same would hold for $f - c$ in $\mathbb{C}^n$. But that implied the existence of an at least one-dimensional (unbounded) algebraic set were $f - c$ vanished in contradiction to (1). Hence each $c \in \mathbb{C}^n$ has p^n inverse images under f (counted with multiplicity) in $\mathbb{C}^n$. □

Corollary 1.10
A strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ has p^{nk} periodic points of order $k \in \mathbb{N}$ (counted with multiplicity). □

Corollary 1.11
For a strict polynomial f, the set $K(f)$ is not empty. □

Remark 1.12
Evidently strict polynomials are dense in the parameter space of polynomial vectors with maximal degree p, as one can see for a given polynomial vector by (if necessary) slightly varying the coefficients for the monomials with maximal degree in order to eliminate common zeros of these terms. As $\mathcal{R}$ contains $\mathcal{S}$ the same holds for (p, q)-regular mappings. □

2 Weakly normal convergence and Julia sets for regular maps

We will now define a suitable type of convergence for strict polynomials. We recall some properties of "one-dimensional Julia sets".
For the Julia set $J := J(f)$ of a polynomial $f : \mathbb{C} \to \mathbb{C}$ the following holds.

- J is a *compact completely invariant set* (i.e. $f(J) = J = f^{-1}(J)$), namely $\partial K(f)$; (see e.g. [3, sec. I.2 and cor. 11.1])
- f restricted to J is *topologically mixing* (i.e. for all open U with $U \cap J \neq \emptyset$ there exists $k_U \in \mathbb{N}$ such that $f^{k_U}(U) \supset J$); (see [3, th. 17.1])
- J is the support of the *measure of maximal entropy* $\log(\deg(f))$ for f; (see [13])
- J is the closure of the set of *repelling periodic points* of f. (see e.g. [3, th. 4.1])

Before we give our definition of a Julia set for the case $n > 1$, let us have a look at an example.

Example 2.1

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.$$

Clearly $K := K(f)$ is the closed bidisk

$$K = \overline{\mathbb{B}} \times \overline{\mathbb{B}},$$

and the boundary ∂K splits into the three disjoint invariant sets $(S^1 \times S^1)$, $(S^1 \times \mathbb{B})$, and $(\mathbb{B} \times S^1)$. The two latter ones are not compact, but contain the compact invariant sets $S^1 \times \{0\}$ respectively $\{0\} \times S^1$. Evidently each of these three completely invariant compact sets

$$(S^1 \times S^1) \,\dot\cup\, (S^1 \times \{0\}) \,\dot\cup\, (\{0\} \times S^1)$$

gives rise to a topologically mixing system, however only $f|_{S^1 \times S^1}$ has topological entropy $\log(\deg(f)) = 2 \log 2$, the other two only $\log 2$ (one obtains these values by considering suitably normalized Lebesgue measure on the sets). Clearly $S^1 \times S^1$ is the set "to become J". The question is how to distinguish between ∂K and its proper subset J. Certainly – as our example shows – this is not possible by defining J to be the set of points where the family $\{f^k\}$ of iterates of f is not normal convergent, as this is the whole boundary ∂K. However, for (x^*, y^*) in $S^1 \times \mathbb{B}$ (in $\mathbb{B} \times S^1$), there is still some convergent behaviour. Namely $\{f^k\}$ restricted to $\{x^*\} \times \mathbb{B}$ (restricted to $\mathbb{B} \times \{y^*\}$) is a normal family. Moreover, this also holds for any (x', y') in the open neighbourhood $\mathbb{C} \times \mathbb{B} \cup \mathbb{B} \times \mathbb{C}$ of these (x^*, y^*).

The above kind of "partial convergence" motivates the following definition. Let $\{f_k\}$ be a family of holomorphic mappings $(f_k : U \to \mathbb{C}^n)$ on a domain $U \subseteq \mathbb{C}^n$.

Definition 2.2 (weakly normal)
$\{f_k\}$ is called *weakly normal* in a point $z \in U$ if there are

- an open neighbourhood V of z;
- a family C_x of at least one-dimensional (complex) analytic sets indexed by the points $x \in V$,

such that

- each x lies in the corresponding analytic set C_x;
- for each $x \in V$ the family $\{f_k\}$ restricted to $C_x \cap V$ is normal (including convergence to infinity).

Now we can state the definition for a the Julia set of a (p,q)-regular mapping.

Definition 2.3 (Julia set of a (p,q)-regular mapping)
The Julia set $J(f)$ for a (p,q)-regular mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of points where the family $\{f^k\}$ of iterates of f is not weakly normal.

Let us check if this gives $J = S^1 \times S^1$ for example **2.1**. Evidently, J is a closed set according to the definition of weakly normal. We already know that $J \subseteq S^1 \times S^1$. Clearly, one cannot obtain normal convergence on any (at least one-dimensional) analytic set containing one of the repelling periodic points $(\exp(2\pi ir), \exp(2\pi is))$, $r, s \in \mathbb{Q}$, of f. But these points are dense in $S^1 \times S^1$. We conclude that indeed $J = S^1 \times S^1$. □

Properties of $J(f)$:
J is closed and contained in ∂K, hence compact.
J is also forward invariant ($f(J) \subseteq J$), as (cf. definition **2.2**), for $z \in \complement J$, the inverse image $f^{-1}(V)$ of the open set V is again open and $f^{-1}(C_x)$ consists of analytic sets containing $f^{-1}(x)$, hence $f^{-1}(x) \subset \complement J$, and thus $f(J) \subseteq J$
To show that also $f^{-1}(J) \subseteq J$ we require the following two theorems.

Theorem 2.4 (Open Mapping Theorem)
([7, p. 108]) A holomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ with discrete fibers is open. □

and

Theorem 2.5 (Proper Mapping Theorem)
([7, p. 213]) For a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ which is proper, the image set $f(\mathcal{C})$ of an analytic set $\mathcal{C}$ is again an analytic set of the same dimension. □

Hence, for $z \in \complement J$, the image $f(V)$ of V is open, and the $f(\mathcal{C}_x)$ are again analytic sets which contain the $f(x)$. This implies $f(z) \in \complement J$, thus $f^{-1}(J) \subseteq J$. □

Remark 2.6
For $n = 1$, weakly normal and normal convergence give the same result as one-dimensional analytic sets are just open sets in $\mathbb{C}$. □

Let us have a look at the behaviour of products of one-dimensional maps. For n polynomials $f_i : \mathbb{C} \to \mathbb{C}$ in one variable, where $p_i := \deg(f_i) \geq 2$, we define a polynomial vector $f : \mathbb{C}^n \to \mathbb{C}^n$ by setting

$$f(z_1, \ldots, z_n) := (f_1(z_1), \ldots, f_n(z_n)) .$$

Evidently f is (p,q)-regular for $p := \min p_i$ and $q := \max p_i$. Let $J(f_i) \subset \mathbb{C}$ denote the Julia set of each $f_i : \mathbb{C} \to \mathbb{C}$. Then the following theorem holds.

Theorem 2.7
For $f : \mathbb{C}^n \to \mathbb{C}^n$ like above,

$$J := J(f) = \bigtimes_{i=1}^{n} J(f_i).$$

Proof: Clearly,

$$K := K(f) = \bigtimes_{i=1}^{n} K(f_i)$$

and

$$\partial K = \{(z_1, \ldots, z_n) \in K : \text{ there exists an } i \in \{1, \ldots, n\} \text{ such that } z_i \in J(f_i)\}.$$

We consider the subsets

$$\partial K_{(i)} := \left\{(z_i, \ldots, z_n) \in \partial K : \ z_i \in \overset{o}{K}(f_i)\right\}$$

(which might be empty for $\partial K(f_i) = K(f_i)$, e.g. for $J(f_i)$ a Cantor set).
For $z \in \partial K_{(i)}$, the connected component of $\overset{o}{K}(f_i)$ which contains z_i is denoted $\overset{o}{K}_z(f_i)$. If $z \in \partial K_{(i)}$ then we define

$$V := \left\{(x_1, \ldots, x_n) : \ x_i \in \overset{o}{K}_z(f_i), \text{ and } |x_j - z_j| < \varepsilon \text{ for all } j = 1, \ldots, n\right\}.$$

Clearly, for each $x \in V$, $\{f^k\}$ restricted to the intersection of V and

$$C_x := \{x_1\} \times \ldots \times \overset{o}{K}_z(f_i) \times \ldots \times \{x_n\}$$

is normal, hence z is a weakly normal point. This shows that

$$\partial K_{(i)} \cap J = \emptyset$$

for all $i = 1, \ldots, n$, and we conclude

$$J \subseteq \partial K \setminus \bigcup_{i=1}^{n} \partial K_{(i)} = \bigtimes_{i=1}^{n} J(f_i).$$

As, for each i, repelling periodic points of f_i are dense in $J(f_i)$, the same holds for the repelling periodic points of f in $\bigtimes_{i=1}^{n} J(f_i)$. By the same argument as in example **2.1**

$$J \supseteq \bigtimes_{i=1}^{n} J(f_i).$$

□

3 Cantor skews

The products we dealt with in theorem **2.7** are the simplest kind of polynomial vectors derived from one-dimensional mappings. Loosely speaking, each component is only influenced by itself, i.e. the i-th component f_i contains z_i as only variable. If we want to obtain a more complicated structure, we may consider skew products of polynomials.
For $n = 2$ and the two complex variables x, y, a skew product can be constructed from two ingredients. We choose a polynomial $q : \mathbb{C} \to \mathbb{C}$ in one variable, another polynomial $p : \mathbb{C}^2 \to \mathbb{C}$ in two variables. Instead of $p(x, y)$ we shall write $p_y(x)$ in order to indicate that we view p as a polynomial in x whose coefficients are dependent on y. We obtain a skew product $f : \mathbb{C}^2 \to \mathbb{C}^2$ by defining

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix}.$$

The dynamics of f consists of two parts. The *base map* q acts on the *fibers* $\mathbb{C}_y := \mathbb{C} \times \{y\}$ by mapping $\mathbb{C}_y$ to $\mathbb{C}_{q(y)}$. Within each fiber we have the action of the *fiber maps* p_y. If we project to the first coordinate we obtain a family of holomorphic functions

$$\mathcal{P}_y := \{p_y, \ p_{q(y)} \circ p_y, \ p_{q^2(y)} \circ p_{q(y)} \circ p_y, \ \ldots\}$$

on each $\pi_1(\mathbb{C}_y)$. For this family we can compute the usual Julia set as subset of $\mathbb{C}_y$ (respectively $\pi_1(\mathbb{C}_y)$) where $\mathcal{P}_y$ is not normal. We call it J_y^*. On the other hand there is also the intersection of $J := J(f)$

with $\mathbb{C}_y$. We denote it J_y. Finally we define K_y as intersection of $K := K(f)$ and $\mathbb{C}_y$. Clearly, $\partial(\pi_1(K_y)) = \pi_1(J_y^*)$. We may compare two sets A, B of the above kinds, respectively their projections $\pi_1(A), \pi_1(B)$, by calculating their *Hausdorff distance* (see [4, p. 66]), which is defined on the space $\mathcal{P}_C$ of compact subsets of $\mathbb{C}$ by

$$d_{HAUSD}(\pi_1(A), \pi_1(B)) := \sup_{x \in \pi_1(A)} \left\{ \inf_{y \in \pi_1(B)} d(x, y) \right\},$$

where $d(\cdot, \cdot)$ denotes the usual metric on $\mathbb{C}$.
Before stating the definition of a Cantor skew we recall two facts from the theory of iteration of quadratic polynomials of one variable (cf. [1, §§9.7–9.10]). The Julia set for $x \mapsto x^2 + c$, where c is a complex constant, is a *Cantor set* if c is not in the *Mandelbrot set* . For $|c| < 1/4$ the Julia set is a *Jordan curve*. In both cases the mappings are *expanding* on J (also called *hyperbolic*). For a differentiable mapping f this means that there exist $C > 0$ and $\alpha > 1$ such that for all $z \in J$ and $k \in \mathbb{N}^*$

$$\left|(f^k)'(z)\right| > C \cdot \alpha^k.$$

Moreover, in the case of f being a quadratic polynomial this is also equivalent to the fact that the forward orbit of the finite critical point is bounded away from J.
A Cantor skew contains both of the above kinds of quadratic mappings. The base is given by a Cantor type mapping whereas the fiber maps are of Jordan type.

Definition 3.1 (Cantor skew)
A mapping $f : \mathbb{C}^2 \to \mathbb{C}^2$ is called a *Cantor skew* if it is a skew-product of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix},$$

where for

$$q(y) := y^2 + F$$

the complex constant F is not in the Mandelbrot set, hence $J(q)$ is a hyperbolic Cantor set. The fiber maps are given by

$$p_y(x) := x^2 + k(y) = x^2 + cy^2 + ey + f$$

with constants c, e, f such that for some $\varepsilon > 0$

$$\|k\|_{J(q)} := \sup_{y \in J(q)} |k(y)| \leq 1/4 - \varepsilon. \tag{3}$$

It will turn out that, for a Cantor skew, weakly normal convergence is equivalent to normal convergence on the fibers. We obtain the following theorem.

Theorem 3.2
For a Cantor skew and $y \in J(q)$, each J_y^* is a Jordan curve. Moreover, the mapping of metric spaces

$$\mathbb{J} : (J(q), d(\cdot, \cdot)) \to (\mathcal{P}_C(\mathbb{C}), d_{HAUSD}(\cdot, \cdot)),$$

defined by

$$y \mapsto \pi_1(J_y^*),$$

is continuous. This implies

$$J = \bigcup_{y \in J(q)} J_y^*,$$

hence, for all $y \in J(q)$,

$$J_y = J_y^*.$$

Proof: The set K must be contained in $B_{R_f} \times J(q)$, since on the complement of this set we obtain convergence to infinity. Hence, for the points $z = (x, y)$ in K, the only choice of $\mathcal{C}_z$ which makes sense in

order to get weakly normal convergence are the analytic sets of type $U \times \{y\}$, where $U \subset \pi_1(K_y)$ is an open set. For assume $\mathcal{C}_z$ was a connected analytic set whose projection to the second coordinate $\pi_2(\mathcal{C}_z)$ contained $y \in J(q)$ and another point $y' \neq y$. As $J(q)$ is totally disconnected, but $\pi_2(\mathcal{C}_z)$ is connected, in this case $\pi_2(\mathcal{C}_z)$ contains a point $y^* \in \complement J(q)$, which implies that $\mathcal{C}_z$ contains a point from $\complement K$. But that contradicts normal convergence of $\{f^k\}$ on $\mathcal{C}_z$. The same argument applied to the first coordinate shows that $U \subset \pi_1(K_y)$ must hold. Finally, this also implies that

$$J = \overline{\bigcup_{y \in J(q)} J_y^*}.$$

We define the closed annulus $\overline{A_{\alpha,\beta}}$ as $\overline{A_{\alpha,\beta}} := \{z \in \mathbb{C} : \ \alpha \leq |z| \leq \beta\}$. We use this notation to give a more precise estimate for the J_y^*.

Lemma 3.3
For a Cantor skew f and $y \in J(q)$, we have

$$\mathbb{J}(y) \subseteq \overline{A_{1/2+\sqrt{\varepsilon},\ 1/2+\sqrt{1/2-\varepsilon}}}.$$

Proof: The inner radius results from

$$\begin{aligned} |x^2 + k(y)| &\leq |x|^2 + |k(y)| \\ &< 1/4 + \sqrt{\varepsilon} + \varepsilon + 1/4 - \varepsilon \\ &= 1/2 + \sqrt{\varepsilon} \end{aligned}$$

for $|x| < 1/2 + \sqrt{\varepsilon}$ and $y \in J(q)$. This shows that $\{f^k\}\big|_{B_{1/2+\sqrt{\varepsilon}} \times \{y\}}$ is a normal family.
For the outer radius we observe that if $|x| > \delta \cdot \left(1/2 + \sqrt{1/2 - \varepsilon}\right)$ for some $\delta > 1$, and $y \in J(q)$, then

$$\begin{aligned} |x^2 + k(y)| &\geq |x|^2 - |k(y)| \\ &> \delta^2 \cdot \left(1/2 + \sqrt{1/2 - \varepsilon}\right)^2 - \delta^2 \cdot (1/4 - \varepsilon) \\ &= \delta^2 \cdot \left(1/2 + \sqrt{1/2 - \varepsilon}\right) \\ &> \delta \cdot |x|. \end{aligned}$$

Hence we obtain convergence to infinity on $\complement \overline{B_{1/2+\sqrt{1/2-\varepsilon}}} \times J(q)$. This gives the outer radius. □

Before we continue with the proof of **3.2**, we note that the forward orbits of the critical points in K are bounded away from the sets J_y^*. Here the critical locus is given by

$$Crit = \{0\} \times \mathbb{C} \cup \mathbb{C} \times \{0\}.$$

The orbit of the latter set is bounded away from K, as $J(q)$ is a hyperbolic Cantor set. For the first set we calculate, with $y \in J(q)$,

$$\begin{aligned} \limsup_{k \to \infty} |\pi_1 \circ f^k(0, y)| &\leq \limsup_{k \to \infty} \left(\left(\ldots \left((0 + 1/4 - \varepsilon)^2 + 1/4 - \varepsilon \right)^2 + \ldots \right)^2 + 1/4 - \varepsilon \right) \\ &= 1/2 - \sqrt{\varepsilon} \\ &< 1/2 + \sqrt{\varepsilon}. \end{aligned}$$

Now we can calculate the J_y^* by inverse iteration. For this we let $r > 1/2 + \sqrt{1/2 - \varepsilon}$ and define $\Gamma_{y,k}$ by induction.

$$\begin{aligned} \Gamma_{y,0} &:= \partial B_r\,; \\ \Gamma_{y,1} &:= p_y^{-1}\left(\Gamma_{q(y),0}\right); \\ &\vdots \\ \Gamma_{y,k+1} &:= p_y^{-1}\left(\Gamma_{q(y),k}\right). \end{aligned}$$

By virtue of the choice of r we have that $\Gamma_{y,k+1}$ lies within the bounded component $\Delta_{y,k}$ of $\complement\Gamma_{y,k}$, hence $\Delta_{y,k+1} \Subset \Delta_{y,k}$, and we derive

$$K_y = \bigcap_{k=0}^{\infty} \overline{\Delta_{y,k}} \times \{y\} = \left(\lim_{k\to\infty} \overline{\Delta_{y,k}} \right) \times \{y\}.$$

Thus K_y is not empty and simply connected, since the critical point 0 (of p_y) lies in $\Delta_{y,k}$ for all $k \in \mathbb{N}$. Of course, $\Gamma_{y,0}$ is a Jordan curve for any $y \in J(q)$, in particular also for all $\Gamma_{q^k(y),0}$. These sets can be parametrized by the arguments of their points. We will "pull back" this parametrization to the $\Gamma_{y,k}$ and prove the existence of the limit set $\Gamma_{y,\infty}$, which is also a Jordan-curve.
We define an "extra layer" $\Gamma_{y,-1}$. Let W be a simply connected domain in $\mathbb{C}$ with $J(q) \Subset W$, such that the forward orbit of 0 under iteration of q does not meet $\overline{W}$. Then there are two inverse branches q_1^{-1}, q_2^{-1} of $q|_W$ which map W biholomorphically on two simply connected disjoint domains V_1 and V_2, respectively. Evidently $q(V_1 \cap J) = J$. We define

$$q^* := q|_{V_1}$$

and let

$$\Gamma_{y,-1} := p_{q^{*-1}(y)}(\Gamma_{q^{*-1}(y),0}).$$

Now let us take the canonical parametrization of $\Gamma_{y,0}$ by the argument. Each ray h_x from 0 through a point $x \in \Gamma_{y,0}$ has a unique intersection x' with $\Gamma_{y,-1}$ (which is a circle of radius r^2 with center $k(q^{*-1}(y))$). On the other hand each point of $\Gamma_{y,0}$ lies exactly on one h_x. Denote the directed segment of h_x from x' to x with $\gamma_{y,x,0}$. The length of the $\gamma_{y,x,0}$ is uniformly bounded by

$$\varrho := (r^2 + 1/4 - \varepsilon) - r.$$

Similar to the definition of $\Gamma_{y,k}$, we define $\gamma_{y,x,k}$ inductively as follows. Let

$$\gamma_{y,x,1} := \pi_1 \left(f_*^{-1} \left(\gamma_{p_y(x),q(y),0},\ q(y) \right) \right),$$

where the inverse branch f_*^{-1} is chosen such that $\gamma_{y,x,1}$ actually starts in x, and similarly for $\gamma_{y,x,k}$.
Since we have that

$$\left| p_y'(x) \right| \geq 2 \cdot (1/2 + \sqrt{\varepsilon}) = 1 + 2 \cdot \sqrt{\varepsilon} =: \lambda > 1$$

we conclude that

$$\text{length}(\gamma_{y,x,k}) \leq 1/\lambda^k \cdot \varrho.$$

Now, the concatenation

$$\gamma_{y,x} := (\gamma_{y,x,1} \circ \ldots \circ \gamma_{y,x,k} \circ \ldots)$$

has finite length which is bounded by

$$\varrho \cdot \sum_{k=1}^{\infty} \frac{1}{\lambda^k} = \frac{\varrho}{\lambda - 1} = \frac{\varrho}{2 \cdot \sqrt{\varepsilon}}.$$

From this we derive the existence of a well-defined endpoint which lies in $\partial\pi_1(K_y)$ $(= \pi_1(J_y^*))$. Proceeding as in the proof of [3, Th. 8.1] we see that each point of $\partial\pi_1(K_y)$ is endpoint of exactly one $\gamma_{y,x}$ since for each y the critical point 0 of p_y lies strictly inside $\pi_1(K_y)$. By mapping x to the endpoint of $\gamma_{y,x}$ we get a parametrization of $\pi_1(J_y^*)$ as Jordan curve.
For the proof of **3.2** let us assume that (3) is not only valid for $J(q)$, but even on an open neighbourhood $W \supset J(q)$, such that $q^{-1}(W) \subseteq W$. Hence we may extend the definition of $\Gamma_{y,0}$ from $y \in J(q)$ to all of W. We are interested in the behaviour of the inverse branches of f^n on the $\Gamma_{y,0} \times \{y\}$.
If we define

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} := f^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and

$$\begin{pmatrix} A_n & B_n \\ 0 & C_n \end{pmatrix} := Df^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and assume $y_0, \ldots, y_n \in W$, then we get

$$\begin{aligned} Df^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= Df \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \circ \ldots \circ Df \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} 2x_{n-1} & 2cy_{n-1}+e \\ 0 & 2y_{n-1} \end{pmatrix} \circ \ldots \circ \begin{pmatrix} 2x_0 & 2cy_0+e \\ 0 & 2y_0 \end{pmatrix} \\ &= \begin{pmatrix} 2x_{n-1} & 2y_{n-1}+e \\ 0 & 2y_{n-1} \end{pmatrix} \circ \begin{pmatrix} A_{n-1} & B_{n-1} \\ 0 & C_{n-1} \end{pmatrix}. \end{aligned}$$

Evidently

$$\begin{aligned} A_0 &= 1, \\ B_0 &= 0, \\ C_0 &= 1, \end{aligned}$$

and

$$\begin{aligned} A_n &= 2x_{n-1} \cdot A_{n-1}, \\ C_n &= 2y_{n-1} \cdot C_{n-1}, \\ B_n &= 2x_{n-1} \cdot B_{n-1} + 2y_{n-1} \cdot C_{n-1} + e \cdot C_{n-1} \\ &= 2x_{n-1} \cdot B_{n-1} + C_n + e \cdot C_{n-1}. \end{aligned}$$

This gives the following formulae:

$$\begin{aligned} A_n &= \prod_{i=0}^{n-1} 2x_i, \\ C_n &= \prod_{i=0}^{n-1} 2y_i, \\ B_n &= c \cdot \sum_{i=0}^{n-1} \left(\prod_{j=0}^{i} 2y_j \cdot \prod_{k=i+1}^{n-1} 2x_k \right) + e \cdot \sum_{i=-1}^{n-1} \left(\prod_{j=0}^{i} 2y_j \cdot \prod_{k=i+2}^{n-1} 2x_k \right). \end{aligned}$$

Furthermore, as $J(q)$ is a hyperbolic Cantor set, we know that for $y_0 \in J(q)$ and some $\alpha > 0$ and $\delta > 1$

$$\begin{aligned} |Dq^n(y_0)| &= \left| \prod_{i=0}^{n-1} 2y_i \right| \\ &= \prod_{i=0}^{n-1} |2y_i| \\ &\geq \alpha \cdot \delta^n. \end{aligned}$$

We can extend this inequality to $y_0, \ldots, y_n \in W$ (if necessary, by taking smaller α, δ).
For the inverse iteration we get

$$Df_*^{-n} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1/A_n & -B_n/(A_n \cdot C_n) \\ 0 & 1/C_n \end{pmatrix},$$

where f_*^{-l} maps $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ to $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. We obtain the following estimates.

$$\left| \frac{1}{A_n} \right| \leq \frac{1}{\lambda^n} \to 0, \tag{4}$$

and

$$\left|\frac{1}{C_n}\right| \leq \frac{1}{\alpha \cdot \delta^n} \to 0. \tag{5}$$

Furthermore,

$$\begin{aligned}\left|\frac{B_n}{A_n \cdot C_n}\right| &= \left| c \cdot \sum_{i=0}^{n-1} \frac{1}{\prod_{j=i+1}^{n-1} 2y_j \cdot \prod_{k=0}^{i} 2x_k} + e \cdot \sum_{i=-1}^{n-1} \frac{1}{\prod_{j=i+1}^{n-1} 2y_j \cdot \prod_{k=0}^{i+1} 2x_k} \right| \\ &\leq |c| \cdot \sum_{i=0}^{n-1} \frac{1}{\alpha \cdot \delta^{n-(i+1)} \cdot \lambda^{i+1}} + |e| \cdot \sum_{i=-1}^{n-1} \frac{1}{\alpha \cdot \delta^{n-(i+1)} \cdot \lambda^{i+2}}.\end{aligned}$$

The latter term is bounded by

$$\frac{|c|}{\alpha} \cdot \frac{n}{\sigma^n} + \frac{|e|}{\alpha} \cdot \frac{n+1}{\sigma^{n+1}} \to 0, \tag{6}$$

where

$$\sigma := \min\{\lambda, \delta\} > 1.$$

Theorem 3.2 now follows from an application of the Weierstraß criterion. Choose for $\eta > 0$ some $\ell \in \mathbb{N}$ such that

$$\ell > \log\left(\frac{3 \cdot \varrho}{2 \cdot \sqrt{\varepsilon} \cdot \eta}\right) \Big/ \log(\lambda).$$

This implies

$$\begin{aligned}\varrho \cdot \sum_{k=\ell}^{\infty} \frac{1}{\lambda^k} &= \varrho \cdot \frac{1}{\lambda^\ell} \cdot \frac{1}{1 - 1/\lambda} \\ &= \frac{\varrho}{2 \cdot \sqrt{\varepsilon}} \cdot \frac{1}{\lambda^\ell} \\ &< \eta/3.\end{aligned}$$

Hence we have for all $y' \in J(q)$ that

$$d_{HAUSD}(\mathbb{J}(y'), \Gamma_{y',\ell}) < \eta/3.$$

For arbitrary $(x, y_0) \in J^*_{y_0}$, we choose a point $x_0 \in \Gamma_{y_0,\ell}$ such that

$$|x_0 - x| < \eta/3.$$

The ℓ-th image (x_ℓ, y_ℓ) of (x_0, y_0) lies in $\Gamma_{y_\ell,0} \times \{y_\ell\}$. By compactness of $\Gamma_{y_\ell,0}$ there exists $\vartheta > 0$ such that for each $x^* \in \Gamma_{y_\ell,0}$, all inverse branches of f^ℓ which map J_{y_ℓ} to J_{y_0}, are defined on some open connected neighbourhood of $\{x^*\} \times B_\vartheta(y_\ell)$. Let τ be such that

$$\bigcup_{y' \in B_\tau(y)} \Gamma_{y',\ell} \times \{y'\} \subseteq f^{-\ell}\left(\bigcup_{y^* \in B_\vartheta(y_\ell)} \Gamma_{y^*,0}\right).$$

Clearly, for each inverse branch $f_*^{-\ell}$ of f^ℓ, we have that

$$\left| f_*^{-\ell}\begin{pmatrix} x_\ell \\ y^* \end{pmatrix} - f_*^{-\ell}\begin{pmatrix} x_\ell \\ y_\ell \end{pmatrix} \right| \leq \left(\frac{|c|}{\alpha} \cdot \frac{\ell}{\sigma^\ell} + \frac{|e|}{\alpha} \cdot \frac{\ell+1}{\sigma^{\ell+1}} + \frac{1}{\alpha \cdot \delta^l}\right) \cdot |y^* - y_\ell|.$$

If we choose y' close to y such that

$$|q^\ell(y') - y_\ell| < \frac{\eta}{3} \Big/ \left(\frac{|c|}{\alpha} \cdot \frac{\ell}{\sigma^\ell} + \frac{|e|}{\alpha} \cdot \frac{\ell+1}{\sigma^{\ell+1}} + \frac{1}{\alpha \cdot \delta^\ell}\right), \tag{7}$$

then by taking

$$\begin{pmatrix} x^* \\ y' \end{pmatrix} := f_*^{-\ell} \begin{pmatrix} x_\ell \\ q^\ell(y') \end{pmatrix},$$

where $f_*^{-\ell}$ maps (x_ℓ, y_ℓ) to (x_0, y_0), and $(x', y') \in J^*_{y'}$ is chosen such that $|x^* - x'| < \eta/3$, we see that

$$d_{HAUSD}(\mathbb{J}(y), \mathbb{J}(y')) < \eta/3 + \eta/3 + \eta/3 = \eta.$$

Thus $\bigcup_{y' \in J(q)} J^*_{y'}$ is already closed in $\mathbb{C}^2$ as the graph of $\mathbb{J}$. Hence this set is equal to J. □

Theorem 3.4
Repelling periodic points of f are dense in J.
Proof: Repelling periodic points y (under q) are dense in $J(q)$. For each of these y, repelling periodic points of f are dense in J_y. For given $\eta > 0$ and $(x, y) \in J$, fix $\delta < \eta/3$ such that, for $|y - y'| < \delta$,

$$d_{HAUSD}(\mathbb{J}(y), \mathbb{J}(y')) < \eta/3.$$

Now choose $y' \in J(q) \cap B_{\delta(y)}$ which is periodic under iteration of q. We find a point $x' \in \mathbb{J}(y')$ such that $|x - x'| < \eta/3$, and then a periodic point (x^*, y') such that

$$d((x', y'), (x^*, y')) < \eta/3.$$

Hence it follows that

$$d((x, y), (x^*, y')) < \eta.$$

□

Theorem 3.5
$f|_J$ is topologically mixing.
Proof: We know that the assertion of the theorem holds for $q|_{J(q)}$ and all $f|J_y$ for y periodic under q. Let U be an open set with $U \cap J \neq \emptyset$. We can assume that U is equal to $B_\eta(x) \times B_\eta(y)$, for some periodic point (x, y) of f and $\eta > 0$. Now we find k' such that, for $V := f^{k'}(U)$,

$$J_y \subset V.$$

For some $\vartheta > 0$, we even get that

$$\bigcup_{y' \in B_\vartheta(y)} J_{y'} \subset V.$$

But for $B_\vartheta(y)$, we can find $k'' \in \mathbb{N}$ such that

$$q^{k''}(B_\vartheta(y)) \supset J(q).$$

We define $k := k' + k''$, and then conclude that

$$f^k(U) \supset J.$$

□

Theorem 3.6
Cantor skews are mixing repellers (for the definition cf. [15]).
This means that the following three conditions are satisfied:

- $\exists C > 0, \alpha > 1$, such that $\left\| (T_z f^k(z)) \cdot u \right\|_* \geq C \cdot \alpha^k \cdot \|u\|_*$ for all $z \in J,\ u \in T_z(\mathbb{C}^2)$ (tangent space of $\mathbb{C}^2$), and $\|\cdot\|$ a Riemannian metric on $T(\mathbb{C}^2)$;
- $J = \left\{ z \in \mathbb{C}^2 :\ f^k(z) \in V := A_{1/2+\sqrt{\varepsilon/2},\ 1/2+\sqrt{1/2-\varepsilon/2}} \times B_{R_f} \text{ for all } k \in \mathbb{N} \right\}$;
- f is topologically mixing.

Proof: The third condition was proved above. The first property is easily derived from (4), (5), (6). For the second statement, the factor B_{R_f} is evident; for the factor $A_{1/2+\sqrt{\varepsilon/2},\ 1/2+\sqrt{1/2-\varepsilon/2}}$, we note that

$$\begin{aligned} J &\subset \overline{A_{1/2+\sqrt{\varepsilon},\ 1/2+\sqrt{1/2-\varepsilon}}} \times B_{R_f} \\ &\subset A_{1/2+\sqrt{\varepsilon/2},\ 1/2+\sqrt{1/2-\varepsilon/2}} \times B_{R_f}. \end{aligned}$$

If the orbit of x leaves $A_{1/2+\sqrt{\varepsilon/2},\ 1/2+\sqrt{1/2-\varepsilon/2}}$ then either it stays in $B_{1/2+\sqrt{\varepsilon}} \supset \overline{B_{1/2+\sqrt{\varepsilon/2}}}$, which implies weakly normal convergence, or it leaves $B_{1/2+\sqrt{1/2-\varepsilon}} \subset B_{1/2+\sqrt{1/2-\varepsilon/2}}$, hence converges to infinity. In either case (x,y) cannot be in J. If the orbit stays in V, then the modulus of the derivative

$$\left| \frac{\partial}{\partial x} \left(\pi_1 \circ f^k(x,y) \right) \right| = \left| \prod_{i=0}^{k-1} 2 \cdot x_i \right|,$$

where $x_i := \pi_1(f^i(x,y))$, is bigger than $\left| 2 \cdot (1/2 + \sqrt{\varepsilon/2}) \right|^k$. Hence the modulus of the derivative turns to infinity. (x,y) must be in J as there cannot be normal convergence on any set $U \times \{y\}$, for an open set U containing x. □

For mixing repellers Ruelle ([15]) showed the following.

Theorem 3.7
For a mixing repeller and a real-analytic function

$$\varphi : V \to \mathbb{R},$$

the series

$$\zeta^*(u) := \exp\left(\sum_{k=1}^{\infty} \frac{u^k}{k} \sum_{x \in Fix(f^k)} \sum_{l=0}^{k-1} \varphi(f^l(x)) \right)$$

has positive radius of convergence. ζ^* can be continued to a meromorphic function $\zeta(u)$ which has a simple pole at $\exp(P(\varphi)) > 0$. All other zeros or poles are of bigger modulus. Furthermore,

$$P : \varphi \mapsto P(\varphi)$$

is convex and there exists a unique Radon-measure μ on J such that

$$P(\varphi + \psi) - P(\varphi) \geq \mu(\psi),$$

for all real analytic $\varphi, \psi : V \to \mathbb{C}$. In particular, μ is an f-invariant Gibbs measure. □

The proof uses the existence of Markov-partitions for expanding maps. In our case we can construct these partitions in a geometric way, which also leads to the construction of the measure of maximal entropy.

Lemma 3.8
There exists a continuous function

$$g : J(q) \to \mathbb{C},$$

such that the graph

$$\{(g(y), y)\}_{y \in J(q)} \subset J$$

is forward invariant under f.
Proof: Fix $x^* := r \in \Gamma_{y,0}$, and define

$$\begin{aligned} g_0 : J(q) &\to \mathbb{C} \\ y &\mapsto x^*. \end{aligned}$$

g_k is determined by continuation of that branch of p_y^{-1} which maps $g_{k-1}(q(y))$ to a complex number with positive real part. The sequence is uniformly convergent, as

$$\|g_k(y) - g_{k+1}(y)\|_{J(q)} \leq \frac{\kappa}{\lambda^k},$$

where for

$$\kappa := \max_{y \in J(q)} \operatorname{dist}\left(x^*, g_1(y)\right),$$

and $\operatorname{dist}(\cdot,\cdot)$ is measured along rectifiable paths in $\mathbb{C}\overline{B_{1/2+\sqrt{\varepsilon}}}$.
Moreover, for $y' = q(y)$ and any $k \in \mathbb{N}^*$

$$\begin{aligned} f(g_k(y), y) &= (g_{k-1}(q(y)), q(y)) \\ &= (q_{k-1}(y'), y'). \end{aligned}$$

Hence the graph of the continuous limit function g is forward invariant. □

A Markov-partition is now given by the following construction.
We establish a conjugation

$$\begin{array}{ccc} S^1 \times J(q) & \xrightarrow{(x \mapsto x^2, q)} & S^1 \times J(q) \\ \downarrow \pi & & \downarrow \pi \\ J & \xrightarrow{\quad f \quad} & J \end{array} \tag{8}$$

by setting

$$\begin{aligned} \pi(1, y) &:= (g(y), y), \\ \pi(-1, y) &:= (-g(y), y), \end{aligned}$$

and continue in the obvious way such that (8) commutes. Continuity is again clear by means of (4), (5), (6). In "y-direction" we may now "cut" according to the Cantor set-structure of $J(q)$. In "x-direction" the images of

$$\left(\left\{ \left(\exp\left(\frac{j}{2^k} 2\pi i \right) \right) \right\} \times J(q) \right)_{i=0,\ldots,2^k-1}$$

under π give the boundaries of the sets in the partition. Furthermore, with π we can transport the product of the measure of maximal entropy on $J(q)$ (cfer. [13]) and the Lebesgue measure on S^1 (which gives maximal entropy for $x \mapsto x^2$) to J and obtain an invariant measure of entropy $2\log 2$, hence maximal entropy for f (see [8]).
Another way to obtain a measure of maximal entropy is the following ([2], [12, ch. 5]):
The family of plurisubharmonic (psh) functions on $\mathbb{C}^n$ with *minimal growth* is defined as

$$\mathcal{G} := \{u \text{ psh on } \mathbb{C}^n : \ u(z) \leq \log(1 + \|z\|) + C_u\}$$

where C_u is a constant. We define for a compact $K \Subset \mathbb{C}^n$

$$G_K^*(z) := \sup\{u \in \mathcal{G} : \ u \leq 0 \text{ on } K\}.$$

The *generalized Green function* for K in $\mathbb{C}^n$ is then given by

$$G_K(z) := \limsup_{\zeta \to z} G_K^*(\zeta).$$

G_K is uniquely determined and by a result of Siciak (see [2]) one can compute G_K by

$$G_K(z) = \sup_{P \in \mathcal{P}} \left\{ \frac{1}{\deg(P)} \log(|P(z)|) \right\}, \tag{9}$$

where $\mathcal{P}$ is a certain class of polynomials. By applying dd^C to G_K one obtains a current λ_K whose n-th product induces a measure μ_K with support exactly the Shilov boundary $\partial_{SH}K$ of K.
In our situation ($K = K(f)$ for a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ of degree $p \geq 2$) we can take

$$\mathcal{P} := \{\pi_i \circ f^k\}_{i=1,\ldots,n,\ k\in\mathbb{N}}$$

in (9), since we deduce from (1) that for $\|z\| > R_f$

$$k_1^{p^k-1} \cdot \|z\|^{p^k} \leq \left\|f^k(z)\right\| \leq k_2^{p^k-1} \cdot \|z\|^{p^k},$$

and

$$\frac{1}{p^k} \cdot \log\left(k_1^{p^k-1} \cdot \|z\|^{p^k}\right) \leq \frac{1}{p^k} \cdot \log\left\|f^k(z)\right\| \leq \frac{1}{p^k} \cdot \log\left(k_2^{q^k-1} \cdot \|z\|^{p^k}\right).$$

From

$$\log\|z\| + \frac{p^k-1}{p^k} \cdot \log(k_1) \leq \frac{1}{p^k} \cdot \log\left\|f^k(z)\right\| \leq \log\|z\| + \frac{p^k-1}{p^k} \cdot \log(k_2)$$

we derive the existence of the limit

$$G'_K(z) := \lim_{k\to\infty} \frac{1}{p^k} \cdot \log\left\|f^k(z)\right\|$$

on $\complement K$. Minimal growth, continuity, and $G'_K|_K \equiv 0$ are evident. Hence $G'_K = G_K$. As by construction

$$G_K(f(z)) = d \cdot G_K(z),$$

we see that

$$\mu_K \circ f = d^n \cdot \mu_K,$$

which implies that μ_K has maximal entropy.
Since $\mathrm{supp}(\mu_K) = \partial_{SH}K$ it is natural to ask for the relationship of $\partial_{SH}K$ and J. This question is answered by the following theorem.

Theorem 3.9
For a Cantor skew $f : \mathbb{C}^n \to \mathbb{C}^n$,

$$J(f) = \partial_{SH}(K(f)).$$

Before proving the theorem we give some necessary definitions (see [6]):
Let $\mathbb{A}_0$ denote the algebra of functions which are holomorphic on some neighbourhood of K. Let

$$\mathbb{A} := \overline{\mathbb{A}_0}$$

be its closure (in the algebra $\mathcal{C}(K)$ of continuous functions with the topology of uniform convergence). The space of maximal ideals $\mathcal{A}$ of $\mathbb{A}$ is in this case (note that by (2) K is polynomially convex) isomorphic to K. Each of those ideals consists of all functions which vanish at a point $z \in K$. Hence in the following we will define the terms *determining set* and *boundary* for K though they are usually defined for $\mathcal{A}$.

Definition 3.10 (determining set)
A closed subset $Q \Subset K$ is a *determining set* if for each $\varphi \in \mathbb{A}$ there exists a $z^* \in Q$ such that

$$|\varphi(z^*)| = \|\varphi\|_K.$$

For example, K itself is a determining set.

Definition 3.11 (boundary)
A minimal (i.e. no proper subset is also determining) determining set Q is called a *boundary* of K.

Theorem 3.12
For K, a uniquely determined boundary exists. It is called the *Shilov boundary* $\partial_{SH}K$ of K. □

Theorem 3.13
A point $z \in K$ lies in $\partial_{SH}K$ if and only if for each neighbourhood $U \ni z$ there exists a function $\varphi_U \in \mathbb{A}$ such that $|\varphi_U|$ has its maximum in U but has only smaller values on $\complement U$. □

Now we are able to prove theorem 3.9.
Proof: The inclusion $\partial_{SH}K \subseteq J$ is obvious. Fix a holomorphic function $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ on some neighbourhood of K. Then for $y^* \in J(q)$, the function of one variable, defined by

$$\varphi_{y^*} : x \mapsto \varphi(x, y^*)$$

is holomorphic on $\pi_1(K_y)$. The uniform limit of such φ_{y^*} is again holomorphic in $\pi_1(\overset{o}{K}_{y^*})$ and continuous on $\partial\pi_1(K_{y^*}) = \pi_1(J_{y^*})$. By the maximum modulus principle it takes its maximal modulus in J_{y^*}. For $(x, y) \in J$, we shall now construct functions like those in theorem 3.13 in order to prove the inclusion $J \supseteq \partial_{SH}K$.
We need the following lemma.

Lemma 3.14
For a Jordan-domain $G \subset \mathbb{C}$,

$$\partial_{SH}G = \partial G.$$

Proof: The inclusion $\partial_{SH}G \subseteq \partial G$ is clear from the above; for each $z^* \in \partial G$ and $\delta > 0$, we can choose $z' \in \complement\overline{G}$ such that

$$|z^* - z'| < \delta/2.$$

Then

$$\varphi_\delta(z) := \frac{1}{z - z'}$$

is an extremal function for $B_\delta(z^*)$. □

We shall also apply Mergelyan's theorem.

Theorem 3.15 (Mergelyan)
(see [5, III, §2]) For a compact set $K \subset \mathbb{C}$ whose complement $\complement K$ is connected, φ holomorphic on $\overset{o}{K}$ and continuous on ∂K, $\varepsilon > 0$, we can always find a polynomial P such that

$$\|f(z) - P(z)\|_K < \varepsilon,$$

hence P approximates φ uniformly. □

We proceed with the proof of theorem 3.9:
For $(x, y) \in J$ and $\eta > 0$, we find an extremal function

$$\varphi : \pi_1(K_y) \to \mathbb{C}$$

which takes its maximal modulus 1 in $B_{\eta/2}(x)$. Moreover, for some $1/2 > \delta > 0$,

$$\|\varphi(x)\|_{\pi_1(K_y)\setminus B_{\eta/2}(x)} < \delta/4.$$

We can approximate φ by a polynomial P such that

$$\|P - \varphi\|_{\pi_1(K_y)} < \delta/4.$$

P still is an extremal function, as it takes its maximal modulus which is at least $1 - \delta/4$ in $B_{\eta/2}(x)$. On the complement of $B_{\eta/2}(x) \cap \pi_1(K_y)$ its modulus is smaller than $\delta/2$.
On a ι-neighbourhood $(0 < \iota < \infty)$ $B_\iota(\pi_1(K_y))$, the derivative P' is bounded, i.e.

$$\|P'\|_{B_\iota(\pi_1(K_y))} < \omega < \infty.$$

If we choose $\kappa > 0$ such that

$$\begin{aligned} \omega \cdot \kappa &< \delta/4 \\ \kappa &< \iota,\ \eta/2, \end{aligned}$$

then, for $B_\kappa(\pi_1(K_y))$,

$$\begin{aligned} \|P\|_{B_\kappa(\pi_1(K_y))\setminus B_\eta(x)} &< \delta/4 + \delta/4 + \delta/4 = 3\delta/4 \\ \|P\|_{B_\kappa(\pi_1(K_y))\cap B_\eta(x)} &> 1 - \delta/4 - \delta/4 = 1 - \delta/2 \end{aligned}$$

By continuity of $\mathbb{J}$ we can find a neighbourhood of y (e.g. $B_\zeta(y)$ for some $\zeta > 0$, $\zeta < \eta$) such that

$$\mathbb{J}(B_\zeta(y)) \subset B_\kappa(J_y).$$

For $B_\zeta(y)$, we define $Q \in \mathbb{A}$ by

$$Q(y) := \begin{cases} 1 \text{ for } y \in J(q) \cap B_\zeta(y); \\ 0 \text{ else.} \end{cases}$$

Then

$$\Phi(x, y) := P(x) \cdot Q(y)$$

is an extremal function for (x, y), as

$$\begin{aligned} \|\Phi\|_{(B_\eta(x)\times B_\zeta(y))\cap K} &> 1 - \delta/2 > 1/2 \\ \|\Phi\|_{\left(\complement B_\eta(x)\times B_\zeta(y)\right)\cap K} &< 3\delta/4 < 1/2 \\ \|\Phi\|_{\left(\mathbb{C}\times\complement B_\zeta(y)\right)\cap K} &= 0. \end{aligned}$$

□

References

[1] A. F. Beardon. *Iteration of Rational Functions*, volume 132 of *Graduate Texts in Mathematics*. Springer, 1991.

[2] E. Bedford and B. A. Taylor. Fine topology, Šilov boundary and dd^C. *Journal of Functional Analysis*, 72:225–251, 1987.

[3] H. Brolin. Invariant sets under iteration of rational functions. *Arkiv för Matematik*, 6:103–144, 1965.

[4] J. Dieudonné. *Grundzüge der modernen Analysis 1*, volume 8 of *Logik und Grundlagen der Mathematik*. Vieweg, 1985.

[5] D. Gaier. *Vorlesungen über Approximation im Komplexen*. Birkhäuser, 1980.

[6] I. R. Gelfand, D. A. Raikow, and G. E. Schilow. *Kommutative normierte Algebren*. VEB Verlag der Wissenschaften, Berlin, 1964.

[7] H. Grauert and R. Remmert. *Coherent analytic sheaves*. Springer, 1984.

[8] M. Gromov. On the entropy of holomorphic mappings. *Preprint d'Institut des Hautes Etudes scientifiques*.

[9] S. M. Heinemann. Iteration holomorpher Abbildungen in $\mathbb{C}^n$. *Diplomarbeit Universität Göttingen*, 1993.

[10] S. M. Heinemann. Dynamische Aspekte holomorpher Abbildungen in $\mathbb{C}^n$. *Dissertation Universität Göttingen*, 1994.

[11] G. Jank and L. Volkmann. *Meromorphe Funktionen und Differentialgleichungen.* UTB. Birkhäuser, 1985.

[12] M. Klimek. *Pluripotential theory*, volume 6 of *London Mathematical Society Monographs.* Oxford University Press, 1991.

[13] M. Yu. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory and Dynamical Systems*, 3:351–385, 1983.

[14] W. Rudin. *Function theory in the unit ball of* $\mathbb{C}^n$, volume 241 of *Grundlehren der mathematischen Wissenschaften.* Springer, 1980.

[15] D. Ruelle. Repellers for real analytic maps. *Ergodic Theory and Dynamical Systems*, 2:99–108, 1982.

[16] I. R. Shafarevich. *Basic Algebraic Geometry.* Springer study edition. Springer, 1974.