Approximation of indifferent cycles

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Abstract

In this paper we deal with analytic families of polynomials or entire transcendental functions with one free singular value. The main topic is the approximation of non–hyperbolic funtions by hyperbolic functions and we discuss the (non–)convergence of the corresponding Julia sets. In particular we approximate indifferent cycles, more precisely, rationally indifferent cycles and linearizable irrationally indifferent cycles by attracting cycles and prove that the attracting basins converge (with respect to the Hausdorff metric) to the basin of the rationally indifferent cycle or to the Siegel disc attached to the non–linearizable irrationally indifferent cycle, respectively.

Key words: iteration, transcendental function, Fatou set, Julia set, parabolic cycle, polynomial, Siegel disc

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1 Introduction and motivation

The study of the dynamics of holomorphic functions naturally leads to the study of Julia sets: Attached to each holomorphic function f is a Julia set $J(f)$, where the dynamic is 'chaotic', and the Fatou set $F(f)$, where the dynamic is 'normal'. Several computer generated pictures show that the Julia sets may occur in different shape and topological type as the holomorphic function varies. This leads to the question: *How does the Julia set $J(f)$ depend on the function f?* An almost complete answer was given by Mañe, Sad and Sullivan in 1983, cf. [13]. They studied a family $\mathcal{R} : M \times \mathbb{P}_1 \to \mathbb{P}_1; (\lambda, z) \mapsto \mathcal{R}_\lambda(z)$ of rational functions of constant degree holomorphically depending on the parameter $\lambda \in M$ where the parameter space M is a domain in $\mathbb{C}$. They proved the existence of an open and dense subset $\widetilde{M}$ of M where the Julia set $J(\mathcal{R}_\lambda)$ depends continuously on λ (with respect to the Hausdorff distance).

Note that $\widetilde{M}$ is not the maximal set where the Julia set continuously depends on the parameter. In fact, if $\lambda_0 \in M \setminus \widetilde{M}$ and $F(\mathcal{R}_{\lambda_0})$ is the union of attracting basins then $\lim_{\lambda \to \lambda_0} J(\mathcal{R}_\lambda) = J(\mathcal{R}_{\lambda_0})$. On the other hand, by the result of Mañe, Sad and Sullivan there is a dense subset of $M \setminus \widetilde{M}$ where the Julia set does *not* continuoulsy depend on the parameter. In order to study the discontinuities, Baribeau and Ransford have introduced the so–called 'upper semicontinuous regularization' of a Julia set, see [2] and compare [17]. In the present paper we choose a different approach. It is motivated by the intensive study of the Mandelbrot family $\mathcal{M} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto z^2 + \lambda$ by Douady and Hubbard. We shall discuss this example in the next section in more detail, at this place we briefly summarize some facts.

1. If $\lambda \mapsto J(\mathcal{M}_\lambda)$ is discontinuous at some $\lambda_0 \in \mathbb{C}$ then $\mathcal{M}_{\lambda_0}$ has either a rationally indifferent cycle or a Siegel disc.

2. In both cases, $\lambda_0 \in \partial G$ for some hyperbolic component G of the Mandelbrot set M_2 and G is a component of the Mañe–Sad–Sullivan–set $\widetilde{M}$. In addition, ∂G is a piecewise analytic curve and $\lim J(\mathcal{M}_\lambda) = J(\mathcal{M}_{\lambda_0})$ as $\lambda \in G$ converges non–tangentially (with respect to ∂G) to λ_0.

3. Furthermore, $\lambda_0 \in \partial M_2$. Note that $\mathbb{C} \setminus M_2$ is also a hyperbolic component of the Mañe–Sad–Sullivan–set $\widetilde{M}$. But the boundary of this component is nowhere smooth and $\lim_{M_2 \not\ni \lambda \to \lambda_0} J(\mathcal{M}_\lambda) = J(\mathcal{M}_{\lambda_0})$ does not hold.

4. As is well known, in this example the critical value $v(\lambda) := \lambda$ plays a crucial role.

The purpose of the present paper is not only to prove the non–tangential limit property stated in item 2 of the above list but to generalize this result to arbitrary families of entire functions

(with one free singular value). In order to let the statement make sense it is necessary to show that the boundary of the component in question is smooth. As in the Mandelbrot example we expect that each hyperbolic component G of $\widetilde{M}$ belongs to exactly one of the following classes:

1. ∂G is a fractal and contains a dense set of parameters λ_0 such that the limit $\lim_{G \ni \lambda \to \lambda_0} J(\mathcal{R}_\lambda)$ does not exists.
2. G is a hyperbolic component, ∂G is a piecewise analytic curve and $J(\mathcal{M}_\lambda)$ converges to $J(\mathcal{M}_{\lambda_0})$ as $\lambda \in G$ converges to λ_0 non–tangentially with respect to ∂G.

The discussion of the Mandelbrot example in the subsequent section will give an idea how to distinguish between these two possibilities in terms of the dynamics of the free singular value; we shall generalize this idea to arbitrary families of entire functions with one free singular value. In the section 3 we recall some basic notations and definitions. In Section 4 we introduce families with one free singular value and state the Main Theorem. Then we reduce the proof to the discussion of two cases which are separately discussed in the the last two section.

2 The Mandelbrot example

In the Mandelbrot example

$$\mathcal{M} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto z^2 + \lambda$$

the Mañe–Sad–Sullivan set $\widetilde{M}$ is closely related to the so–called Mandelbrot set $M_2 := \{ \lambda \in \mathbb{C} \mid J(\mathcal{M}_\lambda) \text{ is connected } \}$:

$$\widetilde{M} = \mathbb{C} \setminus \partial M_2 \, .$$

The components of $\widetilde{M}$ split in two classes. The first type are the so–called hyperbolic components of $Int\widetilde{M}$; a component G of $\widetilde{M}$ belongs to this class if for each $\lambda \in G$ (or, equivalently, for at least one $\lambda \in G$) the polynomial $\mathcal{M}_\lambda$ has at least one finite attracting periodic point. We write H_2 for the union of all these components. If $\lambda \in \widetilde{M} \setminus H_2$ then $J(\mathcal{M}_\lambda)$ equals the filled–in Julia set $K(\mathcal{M}_\lambda)$ attached to $\mathcal{M}_\lambda$, that is to say, the Fatou set $F(\mathcal{M}_\lambda)$ equals the basin of attraction of ∞.

Now we fix $\lambda_0 \in \partial M_2$. If $\mathcal{M}_{\lambda_0}$ has neither a rationally indifferent cycle nor a linearizable irrationally indifferent cycle then $J(\mathcal{M}_{\lambda_0}) = K(\mathcal{M}_{\lambda_0})$. In this case, $\lim_{\lambda \to \lambda_0} J(\mathcal{M}_\lambda)$ exists and equals $J(\mathcal{M}_{\lambda_0})$, cf. [11, Proposition 1].
If $\mathcal{M}_{\lambda_0}$ has a linearizable irrationally indifferent cycle then we have $J(\mathcal{M}_{\lambda_0}) \neq K(\mathcal{M}_{\lambda_0})$ and

$$\lim_{\substack{\lambda \to \lambda_0 \\ \lambda \notin M_2}} J(\mathcal{M}_\lambda) = \lim_{\substack{\lambda \to \lambda_0 \\ \lambda \notin M_2}} K(\mathcal{M}_\lambda) = K(\mathcal{M}_{\lambda_0}) \neq J(\mathcal{M}_{\lambda_0}) \, ,$$

cf. [11, Proposition 1]. The case where $\mathcal{M}_{\lambda_0}$ has a rationally indifferent cycle is more complicated. For example let $\lambda_0 = 1/4$. Then $1/2$ is a rationally indifferent fixed point. Lavours has established the existence of a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset]1/4, \infty[\subset \mathbb{C} \setminus M_2$ converging to $1/4$ such that $\lim_{n\to\infty} J(\mathcal{M}_{\lambda_n}) = \lim_{n\to\infty} K(\mathcal{M}_{\lambda_n})$ exists but is neither $J(\mathcal{M}_{1/4})$ nor $K(\mathcal{M}_{1/4})$, cf. [12] and compare [14]. We summarize.

Theorem 1 *Let $\mathcal{M} : \mathbb{C} \times \mathbb{C}; (\lambda, z) \to \mathcal{R}_\lambda = z^2 + \lambda$ and $\lambda \in \mathbb{C}$. The mapping $\lambda \mapsto J(\mathcal{M}_\lambda)$ is continuous at λ_0 if and only if $\mathcal{M}_{\lambda_0}$ has neither a rationally indifferent cycle nor a linearizable irrationally indifferent cycle.*

Let be $\lambda_0 \in \partial M_2$ such that $\mathcal{M}_{\lambda_0}$ has a rationally indifferent cycle or a non–linearizable irrationally indifferent cycle. Then there exists at least one bounded hyperbolic component G of $Int(M_2)$ with $\lambda_0 \in \partial G$. G is a component of $\widetilde{M}$, too, and ∂G is a piecewise analytic curve. As a consequence of the Main Theorem, cf. section 4, we obtain

Theorem 2 *For each open triangle $\Delta \subset G$ satisfying $\partial\Delta \cap \partial G = \{\lambda_0\}$ the limit $\lim_{\substack{\lambda\to\lambda_0 \\ \lambda\in\Delta}} J(\mathcal{M}_\lambda)$ exists and is equal to $J(\mathcal{M}_{\lambda_0})$.*

Note that $\mathbb{C} \setminus M_2$ is another component of $\widetilde{M}$ carrying λ_0 on its boundary. But this time the boundary is nowhere smooth, hence it makes no sense to talk about non–tangential limits. Furthermore, for $\lambda \in \mathbb{C} \setminus M_2$ the polynomial $\mathcal{M}_\lambda$ has no finite attracting periodic points. The example $\lambda_0 = \frac{1}{4}$ and the case of linearizable irrationally indifferent cycles show that the limit property in the Theorem might become false in this cases. Om the other hand, for each $\lambda \in G$ the polynomial $\mathcal{M}_\lambda$ has a finite attracting periodic cycle which converges to an indifferent cycle of $\mathcal{M}_{\lambda_0}$ as $\lambda \in G$ tends to λ_0. This approximation of an indifferent cycle is the secret behind Theorem 2. Instead of Theorem 2 we shall establish an analogue for a more general setting, namely analytic families of entire (possibly transcendental) functions with one free singular value, see the Main Theorem in Section 4.

3 Notations

We begin by recalling some notations and basic facts from iteration theory. For more details we refer the reader to the surveys [1, 4, 7] and the monographs [3, 6, 16]. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function, that is to say, either a polynomial or an entire transcendental function. Throughout this paper, $f^{\circ n}$ denotes the n–th iterate of f. A point $\zeta \in \mathbb{C}$ is called

k–periodic if $f^{\circ k}(\zeta) = \zeta$ for some $k \in \mathbb{N}_+$, and k is called *minimal period* provided k is minimal with this property. The *cycle* $C := \{ f^{\circ n}(\zeta) \mid n = 1, \ldots, k \}$ attached to ζ is called attracting (or repelling) if the modulus of its multiplier $M(C) := \left(f^{\circ k}\right)'(\zeta)$ is smaller (respectively larger) than one. It is called indifferent if $|M(C)| = 1$. In the latter case we have $M(C) = e^{2\pi i t}$ for some $t \in \mathbb{R}$. If $t \in \mathbb{Q}$ then the cycle C is called a rationally indifferent cycle and an irrationally indifferent cycle otherwise. In case $t \in \mathbb{R} \setminus \mathbb{Q}$ the cycle is said to have a rotation number of *Diophantine Type* if t satisfies the following condition

(DC) There exist some constant $C > 0$ and some integer $n > 0$ such that $\left| t - \frac{p}{q} \right| > C \cdot q^{-n}$ holds for every number $\frac{p}{q} \in \mathbb{Q}$.

The *Julia set* $J(f)$ of f is defined as the closure (with respect to the complex sphere $\mathbb{P}_1$) of the set of all repelling periodic points of f. The *Fatou set* $F(f)$ of f is the set of all points $z \in \mathbb{P}_1$ such that there exists some open neighbourhood U of z where all iterates of f are defined, holomorphic and form a normal family. It is the complement (with respect to $\mathbb{P}_1$) of the Julia set. Both sets are completely invariant under f. In addition, the Julia set is always non–empty.

All important for the dynamics of f are the singular values of f. A point $v \in \mathbb{C}$ is called *singular value* of f if, for every neighbourhood U of v, there exists a branch of the inverse of f which is not holomorphic on U. The set $sing(f)$ of all singular values v of f consists of the critical values of f and also of the finite asymptotic values, if f is transcendental. Note that if f is transcendental then $sing(f) \subset \mathbb{C}$, while $\infty \in sing(f)$ if f is a polynomial of degree $\deg(f) \geq 2$. For $z \in \mathbb{C}$ the set $O^+(z) = \{ f^{\circ n}(z) \mid n \in \mathbb{N} \}$ is called the *orbit* of z (with respect to f). The *limit set* $\omega(z)$ of z (with respect to f) is the set of all accumulation points of the sequence $\{f^{\circ n}(z)\}_{n \in \mathbb{N}}$.

Recall that for two sets $A, B \subset \mathbb{C}$ the Hausdorff distance is defined as

$$h(A,B) := \inf \{ \varepsilon > 0 \mid A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A) \} \ ,$$

where $U_\varepsilon(X)$ denotes the ε–neighbourhood of a set $X \subset \mathbb{P}_1$ with respect to the chordal metric on $\mathbb{P}_1$. For a domain $G \in \mathbb{P}_1$ we shall write $\mathcal{O}(G)$ for the space of all functions holomorphic on G.

From now on we study families of entire functions depending holomorphically on a parameter $\lambda \in \mathbb{C}$ (or an arbitrary complex manifold of any finite dimension), that is to say a holomorphic function

$$\mathcal{F} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto \mathcal{F}_\lambda(z)$$

where $\mathcal{F}_\lambda$ is an entire function. In order to avoid pathological cases we assume either $\mathcal{F}_\lambda$ to be transcendental for each $\lambda \in \mathbb{C}$ or $\mathcal{F}_\lambda$ to be a polynomial with $\deg(\mathcal{F}_\lambda) \geq 2$ not depending on λ.

4 Main result

In this paper we are mainly interested in hyperbolic mappings.

Definition 3 *An entire mapping $f : \mathbb{C} \to \mathbb{C}$ is called* hyperbolic *if*

$$\cup_{v \in sing(f)} O^+(v) \subset\subset F(f)\,.$$

We shall restrict our attention to families with one free singular value.

Definition 4 *A holomorphic family $\mathcal{F} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of entire functions is said to have* one free singular value *if there exists a continuous function $v : \mathbb{C} \to \mathbb{C}$ such that for each $\lambda \in \mathbb{C}$:*

(i) $v(\lambda) \in sing(\mathcal{F}_\lambda)$,

(ii) $\cup_{v \in sing(\mathcal{F}_\lambda) \setminus \{v(\lambda)\}} \omega(v) \subset\subset F(\mathcal{F}_\lambda)$.

This means that $\mathcal{F}_\lambda$ is hyperbolic if and only if $O^+(v(\lambda)) \subset\subset F(\mathcal{F}_\lambda)$. It is a well known fact that in the above setting the latter is the case if and only if $v(\lambda)$ is absorbed by some attracting cycle of $\mathcal{F}_\lambda$. In addition every component of $F(\mathcal{F}_\lambda)$ is preperiodic and Sullivan's classification of periodic components holds.

Since attracting cycles are persistent under changes of λ, the set

$$\mathcal{H}(\mathcal{F}) := \{\, \lambda \in \mathbb{C} \,|\, \mathcal{F}_\lambda \text{ is hyperbolic } \}$$

is an open set. A component G of $\mathcal{H}(\mathcal{F})$ is called a *hyperbolic component in the parameter space.* (Throughout this paper *component* means connected component.) Note that for $\lambda \in \mathcal{H}(\mathcal{F})$ each component of the Julia set $J(\mathcal{F}_\lambda)$ is a *hyperbolic component in the dynamical plane.* In the Mandelbrot example the interesting components turned out to be the components where the free singular value (the critical value λ) is absorbed by some attracting cycle $C(\lambda) \subset \mathbb{C}$, i. e. $C(\lambda) = \omega(v(\lambda))$ is an attracting cycle with $v(\lambda) \in A(\omega(v(\lambda)))$ and this basin of attraction does not contain any other singular value:

$$v(\lambda) = sing(\mathcal{F}_\lambda) \cap A(\omega(v(\lambda))) \text{ and } \omega(v(\lambda)) \subset \mathbb{C}\,.$$

The second inclusion is necessary in the polynomial case, only. We summarize some well known statements on hyperbolic components.

Proposition 5 *Let G be a component of $\mathcal{H}(\mathcal{F})$ such that $v(\lambda) = sing(\mathcal{F}_\lambda) \cap A(\omega(v(\lambda)))$ and $\omega(v(\lambda)) \subset \mathbb{C}$. Then*

(i) *G is simply connected.*

(ii) *For each $\lambda \in G$ the limit set $C(\lambda) := \omega(v(\lambda))$ is an attracting cycle of minimal period k with k only depending on G.*

(iii) *If $\mathcal{F}$ is a family of polynomials then the mapping $\Phi : G \to \mathbf{D}; \lambda \to M(C(\lambda))$ is a branched covering and ∂G is a piecewise analytic curve.*

Remark. In all know examples (iii) holds for families of entire transcendental functions, too, but a proof is not known to the author. The problem is the following: One can show that for $\lambda_0 \in \partial G$ the limit $\lim_{\substack{\lambda \to \lambda_0 \\ \lambda \in G}} C(\lambda)$ exists. In general it is an indifferent cycle of $\mathcal{F}_{\lambda_0}$. But the limit might become equal ∞, that is to say the attracting cycle $C(\lambda)$ escapes to ∞ as λ approaches the boundary point λ_0 of G.

Corollary 6 *For each $\lambda_0 \in \partial G \cap \mathbb{C}$ the limit*

$$C(\lambda_0) = \lim_{\substack{\lambda \to \lambda_0 \\ \lambda \in\ G}} C(\lambda)$$

exists and either it is an indifferent cycle with the multiplier

$$M(C(\lambda_0)) := \lim_{\substack{\lambda \to \lambda_0 \\ \lambda \in G}} M(C(\lambda))$$

or $\{\infty\} = \lim_{\substack{\lambda \to \lambda_0 \\ \lambda \in G}} M(C(\lambda))$.

In the latter case, $\mathcal{F}_{\lambda_0}$ has no indifferent cycle and hence the mapping $\lambda \to J(\mathcal{F}_\lambda)$ is continuous at λ_0, cf. [11, Main Theorem]. In the other case, $\mathcal{F}_{\lambda_0}$ has an indifferent cycle and ∂G is piecewise analytic near λ_0. We are interested in the convergence of the Julia sets as $\lambda \in G$ converges non–tangentially towards $\lambda_0 \in \partial G$. The main result of this paper is

Main Theorem. *Let G be a hyperbolic component in the parameter space of some holomorphic family $\mathcal{F}$ of entire functions with $v(\lambda) = sing(\mathcal{F}_\lambda) \cap A(\omega(v(\lambda)))$ and $\omega(v(\lambda)) \subset \mathbb{C}$, $\lambda_0 \in \partial G$, ∂G smooth near λ_0 and $\Delta \subset G$ a triangle such that $\partial\Delta \cap \partial G = \{\lambda_0\}$. In addition, if $\mathcal{F}_{\lambda_0}$ has a linearizable irrationally indifferent cycle then we assume the multiplier to satisfy the Diophantine Condition (DC). Then*

$$J(\mathcal{F}_{\lambda_0}) = \lim_{\substack{\lambda \to \lambda_0 \\ \lambda \in \Delta}} J(\mathcal{F}_\lambda)$$

(with respect to the Hausdorff distance).

Remarks.
1. If ∂G is piecewise analytic near λ_0 then λ_0 might be a cusp and no such triangle Δ needs to exists or, equivalently, it does not make sense to talk about the non–tangential limit $\lambda \to \lambda_0$. To exclude this possible case is the only purpose of the hypothesis on the smoothness of ∂G.
2. By suitable linear change of coordinates and reparameterization one can resolve the cusp of the boundary of hyperbolic components. For example, the unit disc $\mathbb{D}$ is a hyperbolic component of the family $\mathcal{P}_\lambda(z) = \lambda z + z^2$ and corresponds to the main cardiod of the Mandelbrot set. **3.** In the case where $\mathcal{F}$ is a family of rational functions, the hypothesis on the multiplier is not necessary, cf. [9, 10]. Unfortunately the proof involves quasiconformal surgery and does not carry over to transcendental functions.

5 Proof of the Main Theorem – preparations

Assume for a moment that an entire function f is transcendental and that

$$\cup_{\substack{v \in sing(f) \\ v \neq v_0}} O^+(v) \subset\subset F(f)$$

holds for a value $v_0 \in sing(f)$. Then f has neither wandering domains nor Baker domains. In addition, each component of $F(f)$ is simply connected, hence Herman rings can not occur. This proves

Lemma 7 **(i)** *Each component B of $F(f)$ is preperiodic and*

(ii) *if B is periodic then B is either a Siegel disc or a component of an immediate basin attached to an attracting or rationally indifferent cycle.*

This lemma holds in the case where f is a polynomial, too. Thus, in the situation of the Main Theorem we have to consider the following three cases.

Case 1. $\mathcal{F}_{\lambda_0}$ has a cycle of linearizable irrationally indifferent periodic points. Let S be a Siegel disc attached to one of these points and $B := \left\{ z \in \mathbb{C} \mid \mathcal{F}^n_{\lambda_0}(z) \in S \text{ for some } n \in \mathbb{N} \right\}$. Then $F(\mathcal{F}_{\lambda_0}) \setminus B$ is the union of attracting basins.

Case 2. $\mathcal{F}_{\lambda_0}$ has a rationally indifferent cycle. Then $C(\lambda_0) = \omega(v(\lambda_0))$. If B denotes the basin attached to $C(\lambda_0)$ then $F(\mathcal{F}_{\lambda_0}) \setminus B$ is the union of attracting basins.

Case 3. $\mathcal{F}_{\lambda_0}$ has a cycle of non–linearizable irrationally indifferent points or $C(\lambda_0) = \{\infty\}$. In this case $J(\mathcal{F}_{\lambda_0})$ is a repellor, that is to say, $F(\mathcal{F}_{\lambda_0})$ is the union of attracting basins.

For simplicity, in case 3. we define $B := \emptyset$. We now fix G, λ_0 and Δ as in the Main Theorem and B as above. By the preceding discussion and the definition of the Hausdorff distance the Main Theorem is a consequence of the following propositions.

Proposition 8 *For each $\delta > 0$ there exists some $\varepsilon > 0$ such that*

$$\lambda \in U_\varepsilon(\lambda_0) \Longrightarrow J(\mathcal{F}_{\lambda_0}) \subset U_\delta(J(\mathcal{F}_\lambda))\,.$$

Proposition 9 *Let $K \subset F(\mathcal{F}_{\lambda_0}) \setminus B$ be a compact set. Then there exists some $\varepsilon > 0$ such that*

$$\lambda \in U_\varepsilon(\lambda_0) \Longrightarrow K \subset F(\mathcal{F}_\lambda)\,.$$

Proposition 10 *Let B as above. In addition, if B is attached to a non–linearizable irrationally indifferent periodic point we assume the multiplier $M(C(\lambda_0))$ to satisfy the Diophantine condition (DC). For each compact set $K \subset B$ there exists some $\varepsilon > 0$ such that $\lambda \in \Delta$, $|\lambda - \lambda_0| < \varepsilon$ imply $K \subset F(\mathcal{F}_\lambda)$.*

Proposition 8 and Proposition 9 are folklore, for example cf. [8, 9, 11]. The next two sections are devoted to the proof of Proposition 10. B is either attached to a linearizable irrationally indifferent cycle (Case 1.) or is attached to a rationally indifferent cycle (Case 2.) or empty. In the latter case nothing is to do. We shall separately treat the first two possibilities.

We start with some preparations. Since the Julia set does not change if we replace the function in question by some iterate we may and will assume the periodic point to be a fixed point. The boundary of the hyperbolic component G is smooth near λ_0. Hence using the Implicit Function Theorem we are able to reparameterize the family such that we obtain a representation of the form

$$\mathcal{F}_\lambda(z) = \lambda m z + \sum_{\nu \geq 2} a_\nu(\lambda) z^\nu$$

where

1. $\lambda_0 = 1$,
2. $m = M(C(\lambda_0))$ is the multiplier of the indifferent fixed point 0 of $\mathcal{F}_{\lambda_0}$ and
3. the coefficient functions a_ν are holomorphic on some fixed neighbourhood of λ_0.

For simplicity we shall assume the coefficient functions to be holomorphic on the whole parameter space M and that $\mathbb{D} \subset M$. Now we choose an open triangle $\Delta \subset \mathbb{D}$ with $\partial\Delta \cap \partial\mathbb{D} = \{1\}$. Without loss of generality we may assume Δ to be symmetric with respect to the real axis.

6 Case 1. Siegel disc

This section bases on Siegel's proof of the existence of Siegel disc, cf. [15]. We start with some preparations. In this section we shall write $\lambda = 1+t$ and use both notations simultaneously. We assume $\Delta \subset \widetilde{\Delta} := \left\{ 0 < |t| < T \mid \frac{\pi}{2} + \delta < \arg t < \frac{3}{2}\pi - \delta \right\}$ for some small positive numbers δ and T. For simplicity we replace Δ by $\widetilde{\Delta}$. For every $t \in \Delta$ the origin $z_0 := 0$ is an attracting fixed point of $\mathcal{F}_t$. Let A_t denote its basin of attraction. Next we derive the statement of Proposition 10 from

Lemma 11 *For every $|t| \leq T$ let $f_t(z) = (1+t)mz + \sum_{n\geq 2} a_n(t)z^n$ be a function holomorphic on the closed disc $D_{1+R}(0)$ such that*

(i) $|z| = 1+R \Longrightarrow |f_t(z)| < 1+2R$ *and*

(ii) $a_n(0) = 0, a_n \in \mathcal{O}(D_T(0))$ *for every $n \geq 2$.*

Then for every $q \in]0,1[$ there exists some $\varepsilon \in]0,T[$ such that $t \in \Delta$, $|t| < \varepsilon$ implies $D_q(0)$ to be a subset of the basin of attraction A_t of z_0 with respect to f_t.

Proof of Proposition 10 – Case 1.
By assumption, for every number $\rho > 0$ there exists some biholomorphic mapping $\Phi : D_{1+2\rho}(0) \to G$ satisfying $\Phi(0) = z_0$. We have $(\mathcal{F}_{\lambda_0} \circ \Phi)(z) = \Phi(mz)$ on $D_{1+2\rho}(0)$ and we may choose ρ such that there exists some $q \in]0,1[$ satisfying $\Phi^{-1}(K) \subset D_q(0)$. Applying the lemma to the family $f_t := \Phi^{-1} \circ \mathcal{F}_\lambda \circ \Phi$ proves the proposition.

■

Proof of Lemma 11.

Step 1. Preparations.
The multiplier m has a representation of the form $m = \exp^{2\pi i\alpha}$ with some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfying (DC). For each integer $n \in \mathbb{N}$ we define $\widetilde{T}_n := \min_{p\in\mathbb{N}} \left|\alpha - \frac{p}{n}\right|$. Since α is irrational we obtain $\widetilde{T}_n > 0$. We introduce τ via $1+t = \exp(2\pi i\tau)$. Then $|\tau| < \frac{1}{2}\widetilde{T}_n$ yields $|(1+t)^n\alpha^n - 1| \geq \frac{1}{2}C_1\widetilde{T}_n$ for some constant $C_1 > 0$. By assumption of the proposition we have $\widetilde{T}_n > n^{-\nu}C_2$ for some constant $C_2 > 0$ and some integer $\nu \in \mathbb{N}$. For n sufficiently large we have $|t| \approx 2\pi|\tau|$. Thus $|t| \leq T_n := \frac{1}{7}\widetilde{T}_n$ implies $|\tau| < \frac{1}{2}\widetilde{T}_n$. Now for (almost) every $n \in \mathbb{N}$ we obtain $|(1+t)^n\alpha^n - 1| \geq T_n > C_3 n^{1-\nu}$ with some constant $C_3 > 0$. This yields

$$\lim_{n\to\infty} q^n T_n^{-2} = 0 \qquad \text{for } 0 < q < 1\,. \tag{1}$$

Step 2. Main step.

For $|t| \le T$ and T sufficiently small f_t maps $D_{1+\rho}(0)$ injectively into $D_{1+2\rho}(0)$. Using Cauchy's inequality we obtain an estimate for the coefficient functions a_n:

$$\begin{aligned} cl|a_n(t)| &\le (1+2\rho)(1+\rho)^{-n} \quad \text{for} \quad |t| \le T \\ |a_n'(t)| &\le 4(1+2\rho)(T(1+\rho))^{-n} \quad \text{for} \quad |t| \le \frac{1}{2}T\,. \end{aligned}$$

Now, $a_n(0) = 0$ implies

$$|a_n(t)| \le |t| \cdot 4(1+2\rho) \cdot (T(1+\rho))^{-n} \text{ for } |t| \le \frac{1}{2}T\,. \tag{2}$$

We shall partially linearize the function f_t. For every integer $N \ge 2$ there exists a holomorphic function $\Psi_{t,N}$ such that the power series of $g_{t,N} := \Psi_{t,N}^{-1} \circ f_t \circ \Psi_{t,N}$ has the representation

$$g_{t,N}(z) = (1+t)mz + \sum_{n\ge N} b_{N,n}(t) z^n \qquad \text{for } |t| \le T_N^*$$

where the coefficient function $b_{N,n}$ is holomorphic on $D_{T_N^*}(0)$ and $b_{N,n}(0) = 0$. For fixed N this is possible for small numbers T_N^*, only; we shall determine these in Step 4. At that place we shall also show that $2|t| \le T_N^*$ and $|z| = q$ imply

$$\left| \sum_{n\ge N} b_{N,n}(t) z^n \right| \le |t| C_4 q^n / T_N^*$$

for some constant $C_4 > 0$. By the maximum principle this remains true for $z \in D_q(0)$. By assumption there exists some number $\eta > 0$ such that

$$\frac{1}{2} \le |z| \le 1\ ,\ t \in \Delta \Longrightarrow |z| - |1+t| \cdot |z| \ge \eta |t|\,.$$

If $\lim_{n\to\infty} q^n / T_N^* = 0$ for every $q \in]0,1[$ then there exists some integer N_0 depending on q such that $C_4 q^{N_0} < \eta \cdot T_{N_0}^*$. This yields

$$t \in \Delta\ ,\ |t| \le \frac{1}{2} T_{N_0}^*\ ,\ |z| = q\ ,\ N \ge N_0 \Longrightarrow |g_{t,N}(z)| < |z|\,.$$

Thus for $|t| \le \frac{1}{2} T_{N_0}^*$ and every $N \ge N_0$ the disc $D_q(0)$ is invariant with respect to $g_{t,N}$, in other words, $D_q(0)$ is a subset of the basin of attraction of z_0 with respect to $g_{t,N}$. This implies the

lemma, since for every N and for t tending to 0 the mappings $\Psi_{t,N}$ converge to the identity uniformly on $\mathbb{D}$.

Step 3. Estimates.
In this paragraph we study the conjugacies $\Psi_{t,N}$ and derive some estimates required in the next step. We write

$$\Psi_{t,N} = z + \sum_{n=2}^{N} c_n(t) z^n$$

where c_n is recursively defined by $c_1 \equiv 1$ and

$$\sum_{n=2}^{\infty} \left[m c_n(t) \cdot (1+t) \left((1+t)^{n-1} \cdot m^{n-1} - 1 \right) z^n \right] = \sum_{n=2}^{\infty} a_n(t) \left(z + \sum_{\nu=2}^{\infty} c_\nu(t) z^\nu \right) .$$

This yields

$$m \cdot c_n(t) \cdot (1+t) \left((1+t)^{n-1} \cdot m^{n-1} - 1 \right) = \sum_{\nu=2}^{\infty} a_\nu(t) \cdot \sum_{\substack{l_1+\ldots+l_\nu \\ 1 \le l_1 \cdots \le l_\nu}} c_{l_1}(t) \cdots c_{l_\nu}(t) .$$

Due to the first step in this proof there exists some integer N such that $2T_N \le T$. Thus for $|t| \le T_n^2$ and $n = 2, \ldots, N$ we obtain

$$\sqrt{|t|} \le \left| (1+t)^{n-1} \cdot m^{n-1} - 1 \right| .$$

Inserting this and equation (2) into the equation above yields

$$|c_n(t)| \le \sqrt{|t|} \cdot \frac{4(1+2R)}{|1+t|T} \cdot \sum_{\nu=2}^{\infty} \left[(1+R)^\nu \sum |c_{l_1}(t) \cdots c_{l_\nu}(t)| \right] .$$

Now we define functions γ_n recursively by $\gamma_1 \equiv 1$ and

$$\gamma_n(t) := \sqrt{|t|} \cdot \frac{4(1+2R)}{|1+t|T} \cdot \sum_{\nu=2}^{\infty} \left[(1+r)^\nu \sum \gamma_{l_1}(t) \cdots \gamma_{l_\nu}(t) \right] .$$

This gives $|c_n(t)| \le \gamma_n(t)$ for $|t| \le T_N^2$ and $n = 1, \ldots, N$. We want to estimate γ_n. For this purpose we write

$$k_t := \sqrt{|t|} \cdot \frac{4(1+2R)}{|1+t|T(1+R)}$$

and

$$\psi_t(z) := z + \sum_{n=2}^{\infty} \gamma_n(t) z^n .$$

Then we obtain

$$\psi_t(z) - z = \frac{k_t (\psi_t(z))^2}{1 + R - \psi_t(z)}$$

hence $\psi(t)$ is given by

$$(1+k_t)\psi_t^2(z) - \psi_t(1+R+z) + z(1+R) = 0\,. \tag{3}$$

Thus ψ_t is holomorphic provided that the discriminant d_t of this polynomial equation does *not* vanish. One readily computes $d_t = (1+R-z)^2 - 4k_t z$. We have $k_0 = 0$ and hence $d_0(z) = (1+R-z)^2$. Because k_t depends continuously on t there exists some positive number $\widetilde{T}$ such that

$$|t| \le \widetilde{T}\,,\ |z| \le 1 + \frac{R}{2} \Longrightarrow D_t(z) \neq 0\,.$$

This yields

$$\psi_t(z) = \frac{1}{2(1+k_t)}\left[(1+R) + z - \sqrt{(1+R-z)^2 - 4k_t z}\right]$$

and

$$\psi_t(0) = 0\,.$$

Because of $\lim_{t\to 0} k_t = 0$ and after decreasing $\widetilde{T}$ we obtain for given τ:

$$|t| \le \widetilde{T}\,,\ |z| \le 1 + \frac{R}{2} \Longrightarrow |\psi_t(z) - z| < \tau\,.$$

Step 4. The end.

By construction we have $\psi_t(z) - z = \sum_{n\in\mathbb{N}} \gamma_n(t) z^n$ with $\gamma_n(t) > 0$. Hence we obtain

$$|z| \le 1 + \frac{R}{2} \Longrightarrow |\psi_t(z) - z| \le \psi_t\left(1+\frac{R}{2}\right) - \left(1+\frac{R}{2}\right)\,.$$

On the other hand we have

$$|\Psi_{t,N}(z) - z| \le \sum_{n=0}^{N} \gamma_n(t)|z|^n \le \sum_{n\in\mathbb{N}} \gamma_n(t)|z|^n\,.$$

This proves

$$|t| \le \min\{\widetilde{T}, T_N^2\}\,,\ |z| \le 1 + \frac{R}{2} \Longrightarrow |\Psi_{t,N}(z) - z| \le \tau\,.$$

After choosing τ sufficiently small and N_0 sufficiently large we obtain for $N > N_0$:

1. $\Psi_{t,N}$ maps $\mathbb{D}$ into the disc $\{|z| < 1 + \frac{R}{8}\}$
2. f_t maps $\{|z| < 1 + \frac{R}{8}\}$ into $\{|z| < 1 + \frac{R}{4}\}$.
3. $\{|z| < 1 + \frac{R}{4}\} \subset \Psi_{t,N}\left(\{|z| < 1 + \frac{R}{2}\}\right)$
4. $\psi_{t,N}$ is holomorphic and injective on $\{|z| < 1 + \frac{R}{2}\}$.

These four statements prove

Claim 1. $g_{t,n} = \Psi_{t,N}^{-1} \circ f_t \circ \Psi_{t,N} \in \mathcal{O}(\mathbb{D})$ and $g_{t,N}(\mathbb{D}) \subset \{|z| < 1 + \frac{R}{2}\}$.

The power series of $g_{t,N}$ has the representation

$$g_{t,N}(z) = (1+t)mz + \sum_{n>N} b_{N,n}(t) z^n$$

where $b_{N,n}(0) = 0$ and $|b_{N,n}(t)| \leq 1 + \frac{R}{2}$. This implies

$$|b_{N,n}(t)| \leq 4|t| \cdot \left(1 + \frac{R}{2}\right) \cdot T_N^{-2}$$

for $|t| \leq \frac{1}{2} T_N^2$ and $N \geq N_0$. Finally we obtain

$$|z| \leq q \Longrightarrow \left| \sum_{n>N} b_{N,n}(t) z^n \right| \leq \frac{1}{2} C |t| q^n \cdot T_N^{-2}$$

for some positive constant C. By definition of T_N and assumption we have $\lim_{N\to\infty} q^N T_N^{-2} = 0$.

■

7 Case 2. Rationally indifferent cycles

In the first step we reduce $\mathcal{F}$ to a certain normal form. Then we show that we may assume K to be a subset of some small neighbourhood of z_0. In the third step, for every $\lambda \in \Delta$ sufficiently close to λ_0 we shall construct a neighbourhood $U_\lambda \subset A_\lambda^*(z_0)$ of z_0 containing K.

Step 1. Normal form.
z_0 is a rationally indifferent fixed point of $\mathcal{F}_1$ thus $m^n = 1$ for some integer $n > 0$. Since the Julia set does not change if we replace the function in question by some iterate we may and will assume $\mathcal{F}$ to have a power series of the form

$$(\lambda, z) \mapsto \lambda z + b_2(\lambda) z^2 + \dots$$

with holomorphic functions $b_\nu \in \mathcal{O}(M)$. Furthermore without loss of generality we may and will assume the triangle Δ to be symmetric with respect to the real axis.

Step 2. Small neighbourhood.
Firstly, we show that we may assume $K \subset U \cap A^*$ where U is an arbitrary small (but fixed)

neighbourhood of $z_0 = 0$. On every compact subset of A^* the iterates $\mathcal{F}_{\lambda_0}^{\circ n}$ converge uniformly to the constant function $g \equiv z_0$. Thus for every $z \in A^*$ there exists an open neighbourhood $W \subset A^*$ and some integer $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{\lambda_0}^{\circ n}(W) \subset\subset U \cap A^*$ for every $n \geq n_0$. Now we choose $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{\lambda_0}^{\circ n}(K) \subset U \cap A^*$ for every $n \geq n_0$. Using the continuity of $\mathcal{F}$ we obtain the existence of some $\delta_1 > 0$ and some connected neighbourhood $W \subset\subset A^* \cap U$ of $\mathcal{F}_{\lambda_0}^{\circ n}(K)$ with $\mathcal{F}_{\lambda}^{\circ n_0}(K) \subset W$ for $|\lambda - \lambda_0| < \delta_1$. Let us assume that the proposition is already proved for the compact set $\overline{W}$. We may assume that the number δ obtained by applying the proposition on $\overline{W}$ is smaller than δ_1. The invariance of $F(\mathcal{F}_\lambda)$ yields $K \subset F(\mathcal{F}_\lambda)$ for $\lambda \subset \Delta$ and $|\lambda - \lambda_0| < \delta$. Hence $\mathcal{F}_{\lambda_0}^{\circ n}(K) \subset U \cap A^*$ (for every $n \in \mathbb{N}$) with some small $\delta > 0$.

The immediate basin $A^*(z_0, \mathcal{F}_{\lambda_0})$ of the rational indifferent fixed point z_0 splits into a finite number of components, cf. [5], we denote this number by k. For a sufficiently small neighbourhood U of z_0 there exists an injective holomorphic function ψ_λ such that the power series of $\psi_{\lambda_0} \circ \mathcal{F}_{\lambda_0} \circ \psi_{\lambda_0}^{-1}$ in z_0 has the representation

$$z \mapsto z + z^{m+1} + b_{m+2} z^{m+2} + \dots$$

Elementary calculations with power series yield ψ_{λ_0} to be a polynomial. Using a suitable neighbourhood V of λ_0 and suitable distortions ψ_λ of ψ_{λ_0} we obtain a holomorphic family of polynomials ψ_λ such that for every $\lambda \in V$ the function $\psi_\lambda \circ \mathcal{F}_\lambda \circ \psi_\lambda^{-1}$ has the power series

$$(\lambda, z) \mapsto \lambda z + b_{m+1}(z) z^{m+1} + b_{m+2}(\lambda) z^{m+2} + \dots$$

where $z \in U$ and $b_\mu \in \mathcal{O}(V)$. As it is done in [5] we now transfer $\mathcal{F}_\lambda$ into a translation. By $\mathbb{C}^{*[m]}$ we denote the Riemannian surface of $z \mapsto z^{1/m}$ and by $w : \mathbb{C}^{*[m]} \to \mathbb{C}^*$ the canonical coordinates on $\mathbb{C}^{*[m]}$. The lift Φ of $\phi : z \mapsto z^{-m}$ with respect to w then Φ is a biholomorphic mapping from $\mathbb{C}^*$ onto $\mathbb{C}^{*[m]}$. For $0 < R < r < \infty$ we write $\mathbb{C}_r^{*[m]} := \left\{ P \in \mathbb{C}^{*[m]} \;\middle|\; |w(P)| > r \right\}$, $U = \Phi^{-1}(\mathbb{C}_r^{*[m]})$ and $\mathbb{C}_R^{*[m]} := \left\{ P \in \mathbb{C}^{*[m]} \;\middle|\; |w(P)| > R \right\}$. If R is sufficiently large then the function $f_\lambda := \Phi \circ \mathcal{F}_\lambda|_U \circ \Phi^{-1}$ is a holomorphic mapping from $\mathbb{C}_r^{*[m]}$ into $\mathbb{C}_R^{*[m]}$. One calculates

$$f_\lambda(w) = \lambda^{-m} w + b^*(\lambda) + O(w^{-1/w})$$

where $b^* \in \mathcal{O}(V)$ and $b^*(\lambda_0) \neq 0$. Since we are interested in parameters λ near $\lambda_0 = 1$ we may assume f_λ to have the form

$$w \mapsto \lambda^{-m} w - 1 + O(w^{-1/m})\,. \tag{4}$$

It is possible to maximize the modulus of the higher terms independently on λ by $C \cdot |w|^{-1/m}$ (with a suitable constant $C > 0$).

We have already proved that we may assume $\mathcal{F}_{\lambda_0}^{\circ n}(K) \subset U \cap A^*(z_0, \mathcal{F}_{\lambda_0})$ for every $n \in \mathbb{N}$. This yields $f_{\lambda_0}^{\circ n}(\Phi(K)) \subset \mathbb{C}_r^{*[m]}$ for every $n \in \mathbb{N}$ and a suitable number $r > 0$. Since we may choose U arbitrarily large we may choose r arbitrarily small. Then $P \in \mathbb{C}_r^{*[m]}$ yields

$$\mathrm{Re}(w(f_{\lambda_0}(P))) < \mathrm{Re}(w(P)) - \frac{1}{2}\,.$$

For $n \in \mathbb{N}$ and $P \in \Phi(K)$ we obtain

$$\mathrm{Re}(w(f_{\lambda_0}^{\circ n}(P))) < \mathrm{Re}(w(P)) - \frac{n}{2}\,. \tag{5}$$

Thus for $P \in f_{\lambda_0}^{\circ n}(\Phi(K))$ and $n \in \mathbb{N}$ we may assume $\mathrm{Re}(w(P)) < -r$. The triangle Δ is symmetric with respect to the real axis, let 2β be the angle of Δ at λ_0. By construction we have $\beta \in]0, \frac{\pi}{2}[$ and define $\eta := \min\{\frac{\beta}{2}, \frac{1}{2}(\frac{\pi}{2} - \beta)\}$. We may choose r such that $|\mathrm{Im}(w(f_{\lambda_0}(P))) - \mathrm{Im}(w(P))| < \frac{1}{2}\tan(\frac{\eta}{2})$ for every $P \in \mathbb{C}_r^{*[m]}$. Then $n \in \mathbb{N}$ and $P \in \Phi(K)$ yields

$$\pi - \frac{\eta}{2} < \arg(w(f_{\lambda_0}^{\circ n}(P)) < \pi + \frac{\eta}{2}\,. \tag{6}$$

Together with equation (5) this proves the existence of some number $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $P \in \Phi(K)$ the inequality $\pi - \eta < \arg(w(f_{\lambda_0}^{\circ n}(P)) < \pi + \eta$ holds. Thus for proving the proposition we may and shall assume K to satisfy

$$\Phi(K) \subset \mathcal{B}'(r,\beta) := \left\{ P \in \mathbb{C}_r^{*[m]} \;\middle|\; |\pi - \arg(w(P))| < \eta \text{ and } \mathrm{Re}(w(P)) < -r \right\}\,.$$

We write $\mathcal{B}(r,\beta) := w(\mathcal{B}'(r,\beta))$.

Step 3. Final step.

For every $\lambda \in \Delta$ and λ close to λ_0 we shall construct a punctured neighbourhood $W_\lambda \subset \mathbb{C}^*$ of ∞ which is invariant with respect to $g_\lambda : w \mapsto \lambda^{-m}w - 1$ and contains $\mathcal{B}(r,\beta)$. Then we prove W_λ to be invariant under small perturbations of g_λ. Using w-coordinates we show f_λ to be such a small perturbation. Thus $w^{-1}(W_\lambda)$ is invariant with respect to f_λ and for $\lambda \in \Delta$ and λ sufficiently close to λ_0 the set $U_\lambda := \Phi^{-1}(w^{-1}(W_\lambda)) \cup \{0\}$ is a neighbourhood of 0 invariant with respect to $\mathcal{F}_\lambda$. This implies $K \subset U_\lambda \subset F(\mathcal{F}_\lambda)$ and completes the proof.

By the maximum principle, for the invariance of W_λ with respect to g_λ it is sufficient to show $g_\lambda(\partial W_\lambda) \subset W_\lambda$. We fix $r > 0$ and write $\mathcal{B}_0 := \partial\mathcal{B}(r,\beta) \cap \{\mathrm{Re}w = -r\}$. Then $g_{\lambda_0}(\mathcal{B}_0) \subset \mathcal{B}(r,\beta)$ and the distance $d_{\mathrm{eucl}}\,(g_{\lambda_0}(\mathcal{B}_0), \partial\mathcal{B}(r,\beta))$ does not depend on r. This yields $g_\lambda(\mathcal{B}_0) \subset \mathcal{B}(r,\beta)$. There exists a positive constant C_1 not depending on r and a positive constant δ_1 depending on r such that $|\lambda - \lambda_0| < \delta_1$ implies $d_{\mathrm{eucl}}\,(g_\lambda(\mathcal{B}_0), \partial\mathcal{B}(r,\beta)) > C_1$.

We write $\lambda = 1 - a + ib$. In particular we have $a \in]0,1[$ and $|b| \leq a\tan(\lambda)$. For symmetry we may and for simplicity we shall assume $b \geq 0$. There exists some number $\delta_2 \in]0, \min\{\tan\eta, \delta_1\}[$ such that $0 \leq \arg(\lambda) < \eta$ if $|\lambda - \lambda_0| \leq \delta_2$. This implies $0 \geq \arg(\lambda^{-1}) > -\eta$. For every $w \in \mathcal{B}_1 := \partial\mathcal{B}(r,\beta) \cap \{\arg w = \pi + \eta\}$ we obtain $\pi < \arg(\lambda^{-1}w) \leq \pi + \eta$ and $\mathrm{Re}(\lambda^{-1}w) < \mathrm{Re}(w)$. This yields $g_\lambda(w) \in \mathcal{B}(r,\beta)$ where $\mathrm{Im}(w)$ and $\mathrm{Re}(g_\lambda(w)) \leq \mathrm{Re}(w) - 1$. Thus there exists some constant $C_2 > 0$ independent on r with $d_{\mathrm{eucl}}\,(g_\lambda(\mathcal{B}_1), \partial\mathcal{B}(r,\beta)) > C_2$. We are searching for $w \in \mathcal{B}_2 := \partial\mathcal{B}_2 \cap \{\arg(w) = \pi - \eta\}$ satisfying $g_\lambda(w) \in \mathcal{B}(r,\beta)$. For this the condition $\pi - \eta < \arg(g_\lambda(w) - w) < \pi$ is sufficient. We write $w = -x + iy \in \mathcal{B}_2$ where the numbers x and y are positive by construction. We have

$$-(1-a)x + by \leq -((1-a)^2 + b^2)x \tag{7}$$

or, equivalently, $by \leq x(a(1-a)-b^2)$. This yields $\mathrm{Re}(\lambda^{-1}w) \leq \mathrm{Re}(w)$. By construction we have $b < \tan\beta$ and $y = x\tan\eta$. Hence (7) is a consequence of the inequality $\tan\eta \leq (1-a)\cot\beta - b$. We have assumed $\eta \leq \frac{1}{2}(\frac{\pi}{2}-\beta) < \frac{\pi}{2}-\beta$ which implies $\tan\eta < \cot\beta$. For λ near λ_0 the numbers a and b are small, thus there exists a number $\delta_3 \in]0,\delta_2[$ such that $\tan\eta < (1-a)\cot\beta - b$ provided $\lambda \in \Delta$ and $|\lambda-\lambda_0| < \delta_3$. For these λ

$$w \in \mathcal{B}_2 \Longrightarrow \mathrm{Re}(\lambda^{-1}w) \leq \mathrm{Re}(w) \text{ and } \mathrm{Re}(g_\lambda(w)-w) \leq -1 \tag{8}$$

holds. Because of $b \geq 0$ we have $\mathrm{Im}(\lambda^{-1}w) \geq \mathrm{Im}(w)$ and $\pi > \arg(g_\lambda(w)-w)$. This yields

$$0 < \tan(\pi - \arg(g_\lambda(w)-w)) = -\frac{\mathrm{Im}(g_\lambda(w)-w)}{\mathrm{Re}(g_\lambda(w)-w)} \leq \mathrm{Im}(g_\lambda(w)-w)$$

for every $w \in \mathcal{B}_2$. Now we look for $w \in \mathcal{B}_2$ satisfying $\mathrm{Im}(g_\lambda(w)-w) \leq \tan\frac{\eta}{2}$. The monotonicity of tangent gives $\pi > \arg(g_\lambda(w)-w) > \pi-\eta$ and $g_\lambda(w) \in \mathcal{B}(r,\beta)$. We want to show

$$\mathrm{Im}(g_\lambda(w)-w) = \big(xb + (1-a)y - y((1-a)^2+b^2)\big)\cdot|\lambda|^{-2} \leq \tan\frac{\eta}{2}\,.$$

By $y = \tan(x)$ and $|\lambda-1| \leq \delta_3$ we obtain the condition

$$x\left(b + a(1-a)\tan\eta - b^2\tan\eta\right) \leq (1-\delta_3)^2\tan\frac{\eta}{2}\,.$$

By $b \leq a\tan\beta$ and $0 \leq a \leq 1$ the inequality

$$xa\left(\tan\beta + \tan\eta + a\tan^2\beta\tan\eta\right)\cdot(1-\delta_3)^{-2} \leq \tan\frac{\eta}{2}$$

turns out to be a sufficient condition. It is possible to estimate the term in the bracket by some constant C^*. For a sufficiently small there exists a point $\zeta_0(\lambda) \in \mathcal{B}_2$ such that in

$$xaC^* \leq \tan\frac{\eta}{2} \tag{9}$$

equality holds. We choose δ_3 such that for every $|\lambda-\lambda_0| < \delta_3$ and $\lambda \in \Delta$ such a point $w = x+iy$ exists. Let $\mathcal{B}_{2,\lambda}$ denote the set of the points $w \in \mathcal{B}_2$ such satisfying (9):

$$\mathcal{B}_{2,\lambda} := \{\, w \in \mathcal{B}_2 \mid \mathrm{Re}(w) \geq \mathrm{Re}(\zeta_0(\lambda))\} \ .$$

For $w \in \mathcal{B}_{2,\lambda}$ we estimate $d_{\mathrm{eucl}}(g_\lambda(w),\mathcal{B}_1) > r\tan\eta$, $d_{\mathrm{eucl}}(g_\lambda(w),\mathcal{B}_2) > \tan\frac{\eta}{2}$ and $d_{\mathrm{eucl}}(g_\lambda(w),\mathcal{B}_0) > 1$. Without loss of generality we may assume $r \geq 1$ and obtain

Claim 2. $\lambda \in \Delta$, $|\lambda-\lambda_0| < \delta_3 \implies d_{\mathrm{eucl}}\left(g_\lambda(\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_{2,\lambda}), \partial\mathcal{B}(r,\beta)\right) > \min\{C_1, C_2, C_3\}\,.$

The constant $C_3 := \min\{1, \tan\frac{\eta}{2}\}$ does not depend on r. Now we connect $\zeta_0(\lambda)$ with the point $\zeta_1(\lambda) \in \mathcal{B}_1$ by a curve Γ_λ running in the complement of $\mathcal{B}(r,\beta)$, and we write $\mathcal{B}_{1,\lambda} := \{w \in \mathcal{B}_1 \mid \mathrm{Re}(w) \geq \mathrm{Re}(\zeta_1(\lambda)\}$. We want to proof the set $\mathcal{B}_\lambda := \Gamma_\lambda \cup \mathcal{B}_{1,\lambda} \cup \mathcal{B}_0 \cup \mathcal{B}_{2,\lambda}$ to be the boundary of a simply connected neighbourhood W_λ of infinity and invariant with respect to

g_λ. We construct Γ_λ by choosing a curve γ_λ^* such that at every point $w \in \gamma_\lambda^*$ the tangent vector equals $g_\lambda(w) - w$. To this end we solve the ordinary differential equation

$$\gamma_\lambda^*(t) = g_\lambda(\gamma_\lambda^*(t)) - \gamma_\lambda^*(t)$$

under the initial condition $\gamma_\lambda^*(0) = w^* \in \mathbb{C}$. The solution is a logarithmic spiral emanating in $w_0(\lambda)$ and running to infinity as t tends to infinity:

$$\gamma_\lambda^*(t) = w_0(\lambda) + (w^* - w_0(\lambda)) \exp(w_0^{-1} t)\,,$$

where $w_0(\lambda) := \lambda/(1-\lambda)$ is a repelling fixed point for g_λ. $\gamma_\lambda := \gamma_\lambda^*$ is one possibility. But it is more comfortable to choose for γ_λ the circle centered at $w_0(\lambda)$ with radius $\rho(\lambda) := |w_0(\lambda) - \zeta_0(\lambda)|$. Let V_λ be the unbounded component of $\mathbb{C} \setminus \gamma_\lambda$. We prove

Claim 3. V_λ is invariant with respect to g_λ.

Proof. It is sufficient to show $g_\lambda(\partial V_\lambda) \subset V_\lambda$. Let $w^* \in \partial V_\lambda = \gamma_\lambda$. Then $g_\lambda(w^*) - w^*$ is the tangent vector of the spiral γ_λ^* at the point w^* and we obtain $g_\lambda(\zeta^*) \in V_\lambda$. Now we estimate $\rho(\lambda)$. We have $\rho(\lambda) = |w_0(\lambda) - \zeta_0(\lambda)|$ and $\zeta_0(\lambda) = a^{-1}C'(-1 + i\tan\eta)$ where C' is a positive constant. This yields $|\zeta(\lambda)| = a^{-1}C''$ where C'' is a positive constant independent on r and λ. Then by $0 \le \arg((1-\lambda)^{-1}) \le \beta$ and $0 \le \arg(\lambda) \le \frac{\eta}{2}$ for each λ satisfying $|\lambda - \lambda_0| < \delta_3$ we obtain $0 \le \arg(w_0(\lambda)) \le \pi - \eta$. Now $a \le |1-\lambda| \le a\cos^{-1}\beta$ guarantees the existence of constants $c_1, c_2 \in \mathcal{F}$ such that $0 < c_1 < h(\lambda) < c_2$ where $h : \lambda \mapsto a|w_0(\lambda)|$ and $\lambda \in \Delta$. This implies $\rho(\lambda) \ge a^{-1}\sqrt{C''^2 + h^2(\lambda)}$. The estimate of the function h guarantees the existence of some constant $C > 0$ with $\rho(\lambda) \ge a^{-1}C$. Thus we obtain

Claim 4. For every r there exists some number $\delta_4 \in]0, \delta_3[$ such that $\lambda \in \Delta$ and $|\lambda - \lambda_0| < \delta_4$ imply $\gamma_\lambda \cap D_r(0) = \emptyset$.

In order to estimate $d_{\text{eucl}}(\gamma_\lambda, g_\lambda(\gamma_\lambda))$ we fix $w = w_0(\lambda) + \rho(\lambda)e^{it} \in \gamma_\lambda$ and consider $|g_\lambda(w) - w_0(\lambda)| - \rho(\lambda)$. A short calculation yields $|g_\lambda(w) - w_0(\lambda)| = \rho(\lambda)/\lambda$. This gives

$$|g_\lambda(w) - w_0(\lambda)| - \rho(\lambda) = \left(\rho(\lambda)(1 - |\lambda|)\right)/|\lambda| \ge (1-\delta_4)C/(\delta_4|\lambda|)\,.$$

After assuming $0 < \delta_4 < \frac{1}{2}$ we obtain

Claim 5. There exists some constant $C_4 > 0$ independent on r such that

$$d_{\text{eucl}}\left(g_\lambda(\gamma_\lambda), \gamma_\lambda\right) \ge d_{\text{eucl}}\left(g_\lambda(\partial V_\lambda), \partial\mathcal{B}(r,\beta) \setminus V_\lambda\right) > C_4$$

provided $|\lambda - \lambda_0| < \delta_4$ and $\lambda \in \Delta$.

We write $W_\lambda = V_\lambda \cup \mathcal{B}(r,\beta)$. Claim 2 and Claim 5 yield

Claim 6. For every $\lambda \in \Delta$ satisfying $|\lambda - \lambda_0| < \delta_4$ we have $g_\lambda(W_\lambda) \subset W_\lambda$ and $d_{\text{eucl}}(g_\lambda(W_\lambda), \partial W_\lambda) > \min\{C_1, C_2, C_3, C_4\}$ where the constants C_1, C_2, C_3, C_4 do *not* depend on r.

The representation of f_λ with respect to w–coordinates and of g_λ differ by terms of order $|w|^{\frac{-1}{m}}$, only, thus there exists $r > 0$ such that

$$|w(f_\lambda(P)) - g_\lambda(w(P))| < \min\{C_1, C_2, C_3, C_4\}$$

for every $P \in \mathbb{C}_r^{*[m]}$. Now Claim 6 proves $U_\lambda := \Phi^{-1}(\{P \mid w(P) \in W_\lambda\})$ to be invariant with respect to f_λ for every $\lambda \in \Delta$ satisfying $|\lambda - \lambda_0| < \delta_4(r)$.

■

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