A Note on the Dynamics of Homogeneous Maps

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Abstract

We investigate non-degenerate homogeneous maps $f : \mathbb{C}^n \to \mathbb{C}^n$ represented by n-vectors of complex polynomials in n variables. These maps admit two settings. Either one compactifies $\mathbb{C}^n$ by adding the point at infinity and regards f as endomorphism of $\overline{\mathbb{C}^n}$ (which is a strict polynomial), or one may remove 0 from $\mathbb{C}^n$ and regards f as inducing an endomorphism $\tilde{f}$ of the $(n-1)$-dimensional complex projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$ under the canonical projection $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}_{\mathbb{C}}$. In both cases it is possible to define JULIA sets $J(f)$, $J(\tilde{f})$, resp. We study their relation. It turns out that it may be the most convenient approach to work in a higher dimensional space with $J(f)$, which is obtained from merely polynomial dynamics, and derive the JULIA set $J(\tilde{f})$, which is contained in a lower dimensional manifold but comes from (in general more complicated) rational dynamics.

0 Introduction and Background

Iteration theory for holomorphic maps $\tilde{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, resp., is well developped. Its main 'basic ingredients' are normal convergence and MONTEL's normality criterion. We refer the reader to the books of BEARDON ([1]) and SCHIFF ([11]).

There has been some effort to extend the theory from $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 2$. It turned out that it is essential to find a suitable equivalent in $\mathbb{P}^n_{\mathbb{C}}$ for 'normal convergence' in $\mathbb{P}^1_{\mathbb{C}}$. SIBONY and FORNÆSS propose a sequence of JULIA sets $J_0 \supseteq \ldots J_k \ldots \supseteq J_{n-1}$, which are obtained by consideration of convergence on complex analytic sets of local codimension at most $0 \leq k \leq n-1$ rather than open sets. Of course, $k = 0$ just ressembles the usual normal convergence on open sets. We refer the reader to FORNÆSS's book [4].

Our approach to higher dimensional dynamics is rather 'affine' than 'projective'. We consider so called *strict polynomials*, i.e. holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^n$, which fulfill a growth condition of the following type: There exist $k_1, k_2 > 0$, $r \in \mathbb{R}$, $d \in \mathbb{N}$, such that

$$k_1 \cdot \|z\|^d \;\leq\; \|f(z)\| \;\leq\; k_2 \cdot \|z\|^d \tag{1}$$

for $\|z\| > r$.

Clearly, f must be given by a vector of polynomials (of algebraic degree d); for $n = 1$, the classes of polynomials and strict polynomials coincide. For a more extensive discussion see [7]. A suitable type of convergence is so called *weak normal convergence* on complex analytic sets $\mathcal{C}_x$ of dimension at least one:

We call a sequence of holomorphic maps (f_k) *weakly normal* in a point z if there exists an open set $U \ni z$ such that, for all $x \in U$, we can find open sets $V_x \ni x$, and (at least one dimensional) complex

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analytic sets $\mathcal{C}_x \ni x$, where each sequence $(f_k)|_{V_x \cap \mathcal{C}_x}$ is normal. In contrast to the projective setting we can include convergence to infinity. We refer to the $\mathcal{C}_x$ as *test varieties*.

The relationship to the theory of projective endomorphisms is easily obtained from the following observation. $f : \mathbb{C}^n \to \mathbb{C}^n$ is a strict polynomial if and only if $[f]$, the map obtained by taking only the terms of maximal algebraic degree of f, has no non-trivial zeros. This is equivalent to $[f]$ being a non-degenerate homogeneous map inducing a projective endomorphism. Of course, in our case, $f = [f]$ holds.

1 The JULIA set of a homogeneous map

We begin with an investigation of a family of mappings in dimension one. Let $d \geq 2$ be an integer, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers; we define

$$\underline{\Lambda} \;:=\; \liminf |\lambda_n|^{\frac{-1}{d^n}}$$

and

$$\overline{\Lambda} \;:=\; \limsup |\lambda_n|^{\frac{-1}{d^n}} \;.$$

For the sequence $(f_n)_{n\in\mathbb{N}}$ of polynomial maps

$$f_n : \; z \;\mapsto\; \lambda_n \cdot z^{d^n}$$

the following holds.

Lemma 1.1
$(f_n)_{n\in\mathbb{N}}$ is normal on $B_{\underline{\Lambda}}(0)$ (where it converges to 0) and on $\complement\overline{B_{\overline{\Lambda}}(0)}$ (where it converges to ∞). □

Corollary 1.2
If, with the above notation, the limit

$$\Lambda \;=\; \overline{\Lambda} \;=\; \underline{\Lambda}$$

exists, then the JULIA set $J\left((f_n)_{n\in\mathbb{N}}\right)$ of the sequence $(f_n)_{n\in\mathbb{N}}$ is given by

$$J\left((f_n)_{n\in\mathbb{N}}\right) \;=\; \partial B_\Lambda(0),$$

where we define $\partial B_0(0) = \{0\}$. □

Let us now investigate the dynamics of a (non-degenerate) homogeneous map

$$f : \; \mathbb{C}^2 \;\to\; \mathbb{C}^2$$

of (algebraic) degree $d \geq 2$.
For $z \in \mathbb{P}^1_{\mathbb{C}}$ denote with $\mathbb{C}_z$ the complex line

$$\left\{(x,y) \in \mathbb{C}^2 : \; (x:y) \;=\; z\right\}.$$

Clearly, f maps any line $\mathbb{C}_z$ to another line $\mathbb{C}_{\tilde{f}(z)}$, where $\tilde{f}$ is just the rational function

$$\tilde{f} : \; \mathbb{P}^1_{\mathbb{C}} \;\to\; \mathbb{P}^1_{\mathbb{C}}$$

of degree d which is induced by the restriction of f to $\mathbb{C}^2 \setminus \{0\}$ and the usual equivalence relation $\sim$ with

$$\left(\left(\mathbb{C}^2 \setminus \{0\}\right)/\sim\right) \;\cong\; \mathbb{P}^1_{\mathbb{C}}.$$

In order to describe the dynamics within the lines, i.e. the maps

$$f_z \;:=\; f|_{\mathbb{C}_z} : \; \mathbb{C}_z \;\to\; \mathbb{C}_{\tilde{f}(z)},$$

we will make use of suitable parametrizations

$$\varphi_z : \; \mathbb{C} \;\to\; \mathbb{C}_z.$$

Definition 1.3
A parametrization $\varphi_z : \mathbb{C} \to \mathbb{C}_z$ is called *admissable* if it is of the form

$$\varphi_z : \zeta \mapsto \zeta \cdot \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix},$$

where $\mu_z, \nu_z \in \mathbb{C}$,

$$(\mu_z : \nu_z) = z,$$

and

$$\left\| \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \right\| := \max\{|\mu_z|, |\nu_z|\} = 1.$$

This implies that φ_z is an isometry with respect to the metrics induced by the modulus on $\mathbb{C}$ and by the maximum norm on $\mathbb{C}^2$. Obviously, any admissable parametrization can be obtained from an arbitrary admissable parametrization by multiplication with some $s \in S^1$.

Propositon 1.4
In an admissable parametrization f_z takes the form

$$\zeta \mapsto \lambda_z \cdot \zeta^d,$$

where $|\lambda_z|$ does not depend on the choice of φ_z, $\varphi_{\tilde{f}(z)}$, resp.
Proof: Evidently,

$$\left| \varphi^{-1}_{\tilde{f}(z)} \circ f \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \right| = \left\| f \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \right\|$$

does not change if we multiply $\begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix}$ with some $s \in S^1$. □

We can be even more precise about the $|\lambda_z|$. As (since f is strict)

$$0 < k_1 = k_1 \cdot \left\| \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \right\|^d \leq \left\| f \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \right\| \leq k_2 \cdot \left\| \begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \right\|^d = k_2 < \infty,$$

we have

$$0 < k_1 \leq |\lambda_z| \leq k_2 < \infty,$$

moreover

$$\begin{aligned} \lambda : \mathbb{P}^1_{\mathbb{C}} &\to [k_1, k_2], \\ z &\mapsto |\lambda_z|, \end{aligned}$$

is continuous.
In order to check for normal convergence we might use the $\mathbb{C}_z$ as test varieties and investigate the dynamics of the sequences

$$\mathcal{F}_z := (f_z^1, f_z^2, f_z^3, \ldots) := \left(f_z, f_{\tilde{f}(z)} \circ f_z, f_{\tilde{f}^2(z)} \circ f_{\tilde{f}(z)} \circ f_z, \ldots\right).$$

We denote their JULIA sets in the $\mathbb{C}_z$ with J_z.
Let us fix admissable parametrizations of the lines $\mathbb{C}_z, \mathbb{C}_{\tilde{f}(z)}, \mathbb{C}_{\tilde{f}^2(z)}, \ldots$ Then the maps in $\mathcal{F}_z$ take the form

$$\begin{aligned} f_z^1(\zeta) &= \lambda_z^{d^0} \cdot \zeta^{d^1}, \\ f_z^2(\zeta) &= \left(\lambda_{\tilde{f}(z)}^{d^0} \cdot \lambda_z^{d^1}\right) \cdot \zeta^{d^2}, \\ f_z^3(\zeta) &= \left(\lambda_{\tilde{f}^2(z)}^{d^0} \cdot \lambda_{\tilde{f}(z)}^{d^1} \cdot \lambda_z^{d^2}\right) \cdot \zeta^{d^3}, \\ &\ldots \end{aligned}$$

In view of **1.2** we have to check the existence of the limit

$$|\Lambda_z| \;:=\; \lim_{n\to\infty}\left|\prod_{k=0}^{n-1}\lambda_{\tilde{f}^{n-1-k}(z)}^{d^k}\right|^{\frac{-1}{d^n}} \;=\; \lim_{n\to\infty}\prod_{k=0}^{n-1}\left|\lambda_{\tilde{f}^k(z)}^{\frac{-1}{d^{k+1}}}\right|, \tag{2}$$

which we call the *escape radius* for $\mathbb{C}_z$. But since we know that, for all k,

$$0 \;<\; k_1 \;\leq\; \left|\lambda_{\tilde{f}^k(z)}\right| \;\leq\; k_2 \;<\; \infty,$$

we see that, for all n,

$$k_2^{-\frac{d^n-1}{d^{n+1}-d^n}} \;\leq\; \prod_{k=0}^{n-1}\left|\lambda_{\tilde{f}^k(z)}^{\frac{-1}{d^{k+1}}}\right| \;\leq\; k_1^{-\frac{d^n-1}{d^{n+1}-d^n}},$$

further

$$1 \;\leftarrow\; k_2^{\frac{-1}{d^n(d-1)}} \;\leq\; \prod_{k=n}^{\infty}\left|\lambda_{\tilde{f}^k(z)}^{\frac{-1}{d^{k+1}}}\right| \;\leq\; k_1^{\frac{-1}{d^n(d-1)}} \;\to\; 1$$

□

Corollary 1.5

$$J_z \;:=\; J\left(\mathcal{F}_z\right) \;=\; \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}_z,\; \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| = |\Lambda_z|\right\}.$$

□

Let d_{SPH} be the spherical metric on $P^1_{\mathbb{C}}$ and denote with d_{HAUSD} the HAUSDORFF metric on the space $P_C\left(\mathbb{C}^2\right)$ of compact subsets of $\mathbb{C}^2$.

Propositon 1.6
The map

$$\begin{aligned} j:\; \left(P^1_{\mathbb{C}}, d_{SPH}\right) &\;\to\; \left(\mathbb{C}^2, d_{HAUSD}\right), \\ z &\;\mapsto\; J_z, \end{aligned}$$

is continuous.
Proof: It suffices to show that the map

$$z \;\mapsto\; |\Lambda_z|$$

is continuous. □

If $z \in F\left(\tilde{f}\right)$, the dependence of J_z on z is even stronger.

Propositon 1.7
The union of the J_zs for $z \in F\left(\tilde{f}\right)$ entirely consists of (complex) analytic sets.

Proof: We can assume that $z \in F\left(\tilde{f}\right)$ is contained in a small ball $U \subseteq F\left(\tilde{f}\right)$ such that, for all $k \in \mathbb{N}$, $\tilde{f}^k(U)$ is simply connected and contained in a ball of the form $B_\theta\left(1-\theta\right)$, for some fixed $0 < \theta < 1/2$. This enables us to use the following admissable parametrization for the $\mathbb{C}_z$, $z \in \bigcup_{k=0}^{\infty}\tilde{f}\left(U\right) \subseteq \mathbb{B}$:

$$\begin{pmatrix} \mu_z \\ \nu_z \end{pmatrix} \;:=\; \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

With this parametrization, we obtain

$$\begin{aligned} \lambda_z \;&:=\; \varphi^{-1}_{\tilde{f}(z)} \circ f\begin{pmatrix} z \\ 1 \end{pmatrix} \\ &=\; \pi_1 \circ f\begin{pmatrix} z \\ 1 \end{pmatrix}\Big/\,\tilde{f}(z). \end{aligned}$$

We already know that the limit $|\Lambda_z|$ exists and is continuous in z. We have to check the possibility to find proper arguments such that Λ_z becomes holomorphic in z. We directly see that, for $z \in \bigcup_{k=0}^{\infty} \tilde{f}^k(U)$, we have

$$0 \;<\; \omega \;:=\; \frac{\min\limits_{z \in \overline{B_\theta(1-\theta)}} \left| \pi_1 \circ f \begin{pmatrix} z \\ 1 \end{pmatrix} \right|}{1} \;<\; |\lambda_z| \;<\; \frac{\max\limits_{z \in \overline{B_\theta(1-\theta)}} \left| \pi_1 \circ f \begin{pmatrix} z \\ 1 \end{pmatrix} \right|}{1-2\cdot\theta} \;=:\; \Omega \;<\; \infty.$$

Also, the arguments of the λ_z (fix the argument of λ_z for one $z \in B_\theta(1-\theta)$ and use analytic continuation for the other $\zeta \in B_\theta(1-\theta)$) are bounded, say, for $z \in B_\theta(1-\theta)$,

$$-\infty \;<\; \alpha \;<\; \arg(\lambda_z) \;<\; A \;<\; \infty.$$

We define

$$\lambda_{(k)}(\zeta) \;:=\; \lambda_{\tilde{f}^k(\zeta)}^{\frac{-1}{d^{k+1}}}.$$

If we take, for each $\tilde{f}^k(U)$ separately, the branch of $\sqrt[d^k]{\cdot}$ which sends the λ_z with $\arg(\lambda_z) = 0$ to their (positive) real roots, then

$$\alpha/d^k \;<\; \arg\left(\lambda_{(k)}(z)\right) \;<\; A/d^k,$$

hence the limit

$$\Lambda_z \;:=\; \prod_{k=0}^{\infty} \lambda_{(k)}(z)$$

exists for all $z \in \bigcup_{k=0}^{\infty} \tilde{f}(U)$, and is holomorphic in z. The J_z are obtained as union of the graph of the analytic function (defined on U)

$$\Lambda : \; z \;\mapsto\; \Lambda_z$$

multiplied by arbitrary $s \in S^1$.

$$\bigcup_{z \in U} J_z \;=\; \bigcup_{\substack{s \in S^1 \\ z \in U}} s \cdot \Lambda_z \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

□

Corollary 1.8

$$J \;\subseteq\; \bigcup_{z \in J(\tilde{f})} J_z.$$

□

We know from the continuity of j that the $\mathbb{C}_z$, where z is a (repelling) periodic point of $\tilde{f}$, are dense in $\bigcup_{z \in J(\tilde{f})} J_z$. On such a line $\mathbb{C}_z$ with $f^n(\mathbb{C}_z) = \mathbb{C}_z$ the map f^n takes the form

$$\zeta \;\mapsto\; \left(\prod_{k=0}^{n-1} \lambda_{\tilde{f}^k(z)}^{d^{n-1-k}} \right) \cdot \zeta^{d^n}$$

In its JULIA set J_z repelling periodic points of f^k are dense. We conclude that the repelling periodic points of f are dense in $\bigcup_{z \in J(\tilde{f})} J_z$. As they are in J, we have shown

$$J \;\supseteq\; \bigcup_{z \in J(\tilde{f})} J_z,$$

hence

$$J \;=\; \bigcup_{z \in J(\tilde{f})} J_z \;=\; \bigcup_{z \in J(\tilde{f})} |\Lambda_z| \cdot S^1. \tag{3}$$

Corollary 1.9
J is the closure of repelling periodic points of f. □

With the above result it is easy to see that f operates on J in a chaotic way.

Theorem 1.10
The action of f on J is topologically transitive.

Proof: Let $U \cap J \neq \emptyset$ be open. Hence there exists a periodic point $z \in J\left(\tilde{f}\right)$ such that $U \cap \mathbb{C}_z$ contains a point $\mathfrak{z}^* \in J_z$ and a neighbourhood $\mathfrak{z}^* \in V \subset \mathbb{C}_z$. Since we know that the theorem holds for JULIA-sets of polynomials $p: \mathbb{C} \to \mathbb{C}$, we can find an $\ell_1 \in \mathbb{N}$ such that

$$f^{\ell_1}(V) \supset J_z.$$

By continuity of j we even get, for some $\epsilon > 0$,

$$f^{\ell_1}(U) \supset \bigcup_{\zeta \in B_\epsilon(z)} J_\zeta.$$

Now we find ℓ_2 such that

$$\tilde{f}^{\ell_2}(B_\epsilon(z)) \supset J\left(\tilde{f}\right);$$

evidently

$$f^{\ell_1+\ell_2}(U) \supset J.$$

□

In addition to its geometric interpretation in $\mathbb{C}_z$, $|\Lambda_z|$ plays an important role in the potential theory of $K(f)$. With $\mathfrak{z} := (x, y) \in \mathbb{C}^2 \setminus \{0\}$, $(x : y) = z$, we can rewrite (2) as

$$\begin{aligned} |\Lambda_z| &= \lim_{n\to\infty} \prod_{k=0}^{n-1} \left|\lambda_{(k)}\right| \\ &= \lim_{n\to\infty} \prod_{k=0}^{n-1} \left(\frac{\left\|f^{k+1}(\mathfrak{z})\right\|}{\|f^k(\mathfrak{z})\|^d}\right)^{-\frac{1}{d^{k+1}}} \\ &= \frac{1}{\|\mathfrak{z}\|} \cdot \lim_{n\to\infty} \|f^n(\mathfrak{z})\|^{-\frac{1}{d^n}}. \end{aligned} \tag{4}$$

It is easy to see that the *generalized Green function* G_K (cf. [2]) for K in $\mathbb{C}^2$ can be obtained as

$$G_K(\mathfrak{z}) = \lim_{n\to\infty} \frac{1}{d^n} \cdot \log \|f^n(\mathfrak{z})\| \tag{5}$$

Combining (4) and (5) we obtain the following relation for $\mathfrak{z} \in \mathbb{C}^2 \setminus K$.

Corollary 1.11

$$\begin{aligned} |\Lambda_z| &= \frac{1}{\|\mathfrak{z}\|} \cdot \exp\left(G(\mathfrak{z})\right), \\ \log|\Lambda_z| + \log\|\mathfrak{z}\| &= G(\mathfrak{z}), \end{aligned}$$

resp. □

G_K shows the following behaviour under iteration.

$$G_K \circ f = d \cdot G_K, \tag{6}$$

moreover, since

$$\mu_K := \left(dd^C G_K\right)^2$$

induces a finite measure supported on the *Shilov boundary* $\partial_{SH}K$ of K (cf. [5]), we have found a measure of maximal entropy $2\log d$ (cf. [6]) as

$$\mu_K \circ f = d^2 \cdot \mu_K.$$

In dimension 1 this unique measure is supported on the Julia set (see [10] — note that the convergence of the measure in $\overline{\mathbb{C}}$ can be shown by consideration of currents in $\mathbb{C}^2$ (see [8], together with an investigation of general maps with superattractive fixedpoints)). Hence we have to prove

Theorem 1.12
J equals the SHILOV-boundary $\partial_{SH}K$ of K.
Proof: $\partial_{SH}K \neq \emptyset$, and, for any $\mathfrak{z}^* \in J$,

$$\overline{\bigcup_{k=0}^{\infty} f^{-k}(\mathfrak{z}^*)} \;=\; J.$$

Hence, it is sufficient to prove the implications

$$\mathfrak{z}^* \in \partial_{SH}K \wedge f(\mathfrak{z}) = \mathfrak{z}^* \implies \mathfrak{z} \in \partial_{SH}K,$$

and

$$\mathfrak{z} \in \partial K \setminus J \implies \mathfrak{z} \notin \partial_{SH}K.$$

We recall that $\mathfrak{z} \in \partial_{SH}K$ if and only if there exists, for each neighbourhood $U \ni \mathfrak{z}$, a function φ_U holomorphic on some neighbourhood V of K, such that $|\varphi_U|$ has its maximum on K in $U \cap V$ but has only smaller values in $(K \cap V) \setminus U$. This already proves the second statement. We simply apply the maximum modulus principle to $\mathcal{C}_{\mathfrak{z}}$.

For simplicity, let us assume that $K \subset \mathbb{B} \times \mathbb{B}$. We note that $f^{-1}(\mathfrak{z}^*)$ consists of $k \leq d^2$ points $\mathfrak{z}_1, \ldots, \mathfrak{z}_k$. For given $U \ni \mathfrak{z}_1$ we fix a radius ϱ so small that

$$B_\varrho(\mathfrak{z}_1) \;\subset\; U,$$

and the $B_i := B_\varrho(\mathfrak{z}_i)$ are pairwise disjoint. For $\mathfrak{z}^*$ we find a neighbourhood $W \ni \mathfrak{z}^*$ such that

$$f^{-1}(W) \;\subset\; \bigcup_{i=1}^{k} B_i.$$

From the assumption it follows that a function φ_W exists. Furthermore, for $0 < \sigma < 1/2$, we can construct a holomorphic map (see [7])

$$\psi_\sigma : \; \mathbb{B} \times \mathbb{B} \;\to\; \mathbb{B}$$

such that

$$\inf_{\mathfrak{z} \in B_1} |\psi_\sigma| \;\geq\; \sigma,$$

$$\sup_{\substack{\mathfrak{z} \in B_i \\ i>1}} |\psi_\sigma| \;\leq\; \sigma.$$

A suitable φ_U can be defined by setting

$$\varphi_U(\mathfrak{z}) \;:=\; \psi_\sigma(\mathfrak{z}) \cdot \varphi_W \circ f(\mathfrak{z}),$$

which proves the first implication. □

We have seen that

$$\pi \circ J(f) \;=\; J\left(\tilde{f}\right).$$

What happens if we also project μ_K to $\mathbb{P}^1_{\mathbb{C}}$? In $\mathbb{C}^2$, $\mathbb{C}^2 \setminus \{0\}$, resp., μ_K is the (pluri-)harmonic (dd^C as higher dimensional extension to Δ) measure of K w.r.t. ∞ as well as the measure of maximal entropy for f. f has mapping degree d^2, $\tilde{f}$ and $\zeta \mapsto \lambda_z \cdot \zeta^d$ have degree d, hence the measure $\pi \circ \mu_K$ which is supported on $J\left(\tilde{f}\right)$ fulfills the equality

$$(\pi \circ \mu_K) \circ \tilde{f} \;=\; d \cdot \pi \circ \mu_K;$$

the projection of the measure of maximal entropy for f in $\mathbb{C}^2$ to $\overline{\mathbb{C}}$ gives the measure of maximal entropy for $\tilde{f}$ in $\overline{\mathbb{C}}$.

Whereas for polynomials $\tilde{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ measure of maximal entropy and harmonic measure w.r.t. ∞ coincide (see [3]), this is, in general, not the case. The following theorem, due to LOPES ([9]), holds.

Theorem 1.13
If, for a rational map $\tilde{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\tilde{f}(\infty) = \infty$, the measure of maximal entropy for $\tilde{f}$ equals the harmonic measure of $K\left(\tilde{f}\right)$ w.r.t. ∞, then $\tilde{f}$ is conjugate to a polynomial. □

It is easy to see that $\tilde{f}$ is a polynomial (in z) if and only if f has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} g(x,y) \\ s \cdot y^d \end{pmatrix} \tag{7}$$

with g a homogeneous polynomial of degree $d \in \mathbb{N}$, $s \in \mathbb{C}^*$ (After a suitable conjugation, we may assume $s = 1$). We deduce

Corollary 1.14
For a homogeneous strict polynomial $f : \mathbb{C}^2 \to \mathbb{C}^2$ which fixes the (complex) line $\mathbb{C}_\infty$, the following holds: If the projection $\pi \circ \mu_K$ equals the harmonic measure of $K\left(\tilde{f}\right)$ w.r.t. ∞, then f is given as a skew product of the form (7) in $\mathbb{C}^2$. □

In this case we can be more precise about the shape of $J(f)$ than in (3). Evidently

$$f^n \begin{pmatrix} x \\ y \end{pmatrix} = y^{(d^n)} \cdot \begin{pmatrix} \tilde{f}^n(z) \\ 1 \end{pmatrix}, \tag{8}$$

where $z = (x : y)$ (we can neglect the cases $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $z = \infty$). From (8) we deduce that the repelling periodic points of period k are given by

$$\left\{(x,y) \in \mathbb{C}^2 : \; y^{(d^k)} = y \neq 0, \; x = z \cdot y, \; z \text{ repelling periodic point of } \tilde{f}\right\},$$

namely the moduli of the eigenvalues of their differentials are $\left|\left(\tilde{f}^k\right)'(z)\right|$, d^k, resp. Finally, we get

Corollary 1.15
For a homogeneous strict polynomial f represented as a skew product of the form (7) with $s = 1$, the JULIA set takes the form

$$J(f) = \bigcup_{\substack{z \in J(\tilde{f}) \\ y \in S^1}} y \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

□

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