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In this booklet a simple system of axioms for the Euclidean geometry will be given. The Pseudo-Euclidean geometry which is important for the theory of relativity will be derived as a limit of the discrete geometry. This is a geometry wich has only finitely many points and it gives a method to understand the well known laws of quantum physics more simply. To be able to deal with the general relativity the space may also have a curvature. Then it is no longer Euclidean.

Axioms for the Elementary Geometry and the Ancient Greek Theorems

Introduction

The aim of this paper is to prove that in 2 dimensions the synthetic geometry is isomorphic with the Euclidean analytic geometry. For doing this we need axioms which are completely evident. The evidence comes from the geometric visualization.

1. The historical development of our conception of geometry is very important for our way of thinking in general. The opinion of *Kant* was that the Eucdlidean geometry is demanded by our reason. Later on mathematicians like Gauss, Lobatchevsky, Bolyai discovered that there are other geometries which are as consistent as the Euclidean geometry is. However, it has to be stressed that Goedel has shown that the consistency of all these geometries cannot be proved.

The construction of the non-Euclidean geometry is done by mathematical methods. These all are based on formal logic. We have to distinguish between a logical theory and a *mental picture.* Our reason is not a unity without inner tension. It consists of different organs which may be in conflict with each other. There is harmony in those parts only, which were subject to biological evolution. It may be that nowadays we have to look from this point of view at the Kant antinomies.

It is quite certain: We have both the possibility for spatial visualization and the possibility for logical construction. And they are different. In planar geometry there is the theorem of Jordan on simply closed curves. In visualization its statement is only accessible for piecewise smooth curves. But in this case, already it is tiring to prove it

logically. However, in the geometric view it is completely trivial: The plane is decomposed by every simply connected closed curve into 2 disjoint domains.
On the other hand already the ancient Greeks saw that it is impossible to obtain deeper theorems on polygons by visualization alone. It is necessary to use logic. In this way the method of mathematical proof was found. It has not been forgotten to the present day.
Finally, Minkowski and Einstein saw that space and time are the same thing. When we change the (inertial) coordinate system, space and time quantities will be mixed. In mental picture time and space are essentially differrent entities.

The geometric visualization is very much practical. Many things become simpler. Already, Goethe's Faust longed for an all–embracing view: "Dass ich nicht mehr mit saurem Schweiss zu sagen brauche, was ich nicht weiss. ... Schau alle Wirkenskraft und Samen und tu nicht mehr in Worten kramen."

Certainly, many people cannot distinguish between the visual geometry and the logical geometry. This seems to be similar to the situation during the time of Goethe, when nobody saw the difference of light and colour, although this is a very simple thing: The same colour can be produced by a lot of different mixtures of light. Here the proof of difference uses the fact that deductions in visual geometry are simpler than those in any logical theory. – If you wish to draw something in mathematical TEX, you have to compute all objects. That is very tiring. Please, imagine that Rembrandt had had to paint his pictures in that way!

2. But what is visualization? Many people say, it is just a sight produced by recognizing things, frequently. But that is not a description. If you see a spatial picture, then the view is towards a point, all the times. But the whole figure is there during the whole time, it is present. We can do experiments with the figure. The visualization is like the product of a parallel computer or that on an analogue computer. – In formal logic only one result is there, always. So the logic is like a simple digital computer.

There are important cases, where the visualization is not spatial. The positive integers

$\mathbb{N}$ are obtained by a construction without end. If $M \subset \mathbb{N}$ is any subset, then we see that there is a smallest number in M. This is evident since we see that any numbers is produced by the preceeding number. Here we replaced space by time.

Now, we shall make a list of the axioms of geometry. They all are statements, which cannot be obtained by logical conclusion. But they are "evident". That means that there is a visualization in which the axiom is visuable. The reason for the axiom is not of mathematical nature but it comes fron psychology.

3. We shall condider the plane in our geomeric visualization. We shall take all arrows which begin in a base point O and go to another point. We shall establish a list of axioms. They are evident in our geometric mental picture (but not logically necessary). But we have to know that at first the mental vision produces only a very bounded domain around the origin. Then we have to make the domain larger and larger and have to glue together in order to obtain the infinite space. This is a typical mathematical construction. The full space of our mental picture can be only obtained in such a way.

The geometric vizualization has dimension 3. Therefore, we should do the glueing in the 3-dimensional case. Assume that we constructed a bounded part of the infinite space already such that it is isomorphic with a small ball (isomorphic concernig the line segments and the angles) and then take a large ball; we glue the small one identically into the large one and so we get the enlargement of our space. Finally by a homothety in O the large ball can be mapped isomorphically onto the small one. We repeat our procedure. One thing is evident: we cannot enlarge the small ball such that we obtain curvature and still stay in the region of visuability.

The full space is the projective limit of our sequence of constructions, finally. By definition it has the following property: it has the same structure in the large as in an arbitrary small neighborhood around the origin. Its curvature vanishes everywhere. So its geometry is not elliptic or hyperbolic but Euclidean.

In the classical Greek period (i.e. in the time of Perikles) the outer space was constructed in such a way. There was the space of places nearer than the fixed stars. It

was an open (near) ball. The fixed stars were on the boundary of this ball, the so called fixed star sphere. But there was space beyond the closed near ball. It was a large ball. So the near ball was a subball of the large ball, as it always happens in our construction. The Greeks did the first step of it!

We should see that also the other mathematical objects like curve, surface etc. are not in the naive vision. The philosopher Zeno from Elea found that the point did (ontologically) not exist. But we can get it by a mathematical construction. We also can say: a point is a spot together with the knowledge that it should be arbitrary small. We may make it so small as we like it.

In the axioms for geometry given by Euclides the full constructed infinite space was used. But the region in it near to infinity is not visuable. Therefore the axiom of parallels became doubtful. In our axiomatic system only arbitrary large bounded parts of the full space and bounded line segments are used. Here we have the full visuability. We take always the model of a bounded ball. Therefore, there are psychlogical reasons that by our system nobody would have been lead to the doubts: The Euclidean geometry would have been considered as the only really existing one.

By our axioms we shall immediately obtain a 2–dimensional real vector space. Thereafter, the group of rotations will be introduced. This will give the notion of angle. Practically, this is the Euclidean structure, already.

4. For the ancient Greeks the area of a figure was given by nature. No mathematical definition was necessary (not that by the integral of Lebesgue). They should have seen the relation to the area of squares. But for them the area was just a magnitude and congruent figures had the same magnitude. So it was clear that the areas of congruent figures were the same. Of course, their conclusion was wrong. They had made an incomplete analysis! The kind of thinking of the ancient Greeks still regarded content, very much, but it was not formal.

But the classical theorems like that of Pythagoras and that of Thales are trivial for Euclidean vectorspaces. We shall prove that the visual plane is an Euclidean vectorspace.

So we proved these thorems. It is clear that the theorem of Pythagoras has actually nothing to do with the area of a square. So, moreover, the old proofs do not fit into the structure of modern mathematics.

5. Let us go a little bit more into the visual plane! Our reason consists of several domains, which work with completely different methods. First, there is the

1) Logic. It is marked by simple rules on modifications of expressions. There is something before the logic, which is felt by us directly. But here everything is still very much indistinct. But then concepts and predicates are formed. These are given into a language, which has to be constructed, and it contains terms for concepts, predicates and for propositions. Now the conclusions will be done by formal modifications in the language. And this is the beginnig of the logic method. With the help of the language also new concepts can be constructed out of certain more simple basic ones.

In mathematics this procedure is especially distinct. Here, only a very small number of basic concepts is necessary. There are the basic rules for conclusion, there are the axioms, which contain the basis of the desired theory. There is practically no content. Only, formal rules are desired. The language describes it, precisely.

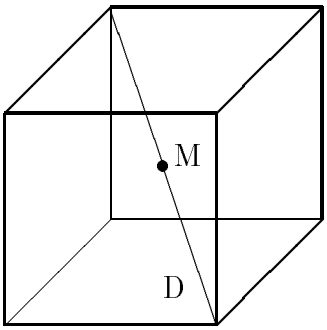
2) The situation is different, if we use instead of logic the mental picture. We call this process the geometric **visualization**. The visualization will be created by a special activity of our neuro physiological system. We can see this e.g. at the following occurrences: Sometimes it can happen that we look at a black–white map in a newspaper in order to see the countries. But we see the complement, the oceans, as if they were the countries. But this only for a moment. Then, suddenly the sight is reversed. There are even better examples for this phenomenon. So there are drawings, on which at first you see 6 cubes, and after reversion only 5 (or vice–versa). – The reversion produces a psychological sensation. There were some experiments in neuro psychology (using the spin of atomic kernels). These proved that there is an anatomic correlate. Both sights correspond with states of the brain, where two sets of billions on neurons oscillate, synchronously. In the reversion the sets are exchanged by others. See e.g. the talk

given by Wolf Singer (Frankfurt) at the German academy Leopoldina (Halle) in April 1996 (published in the book on that conference).

The n-dimensional real number space $\mathbb{R}^n$ is basic for mathematics. Here, n can be arbitrary large. But we can only picture the conditions in $\mathbb{R}^3$. It is impossible to see more than 3 lines, which are orthogonal to each other. In the visible space we have points, curves, surfaces. Then, the next object of higher dimension is a domain, which locally covers the whole space, already. Nothing more is possible!

In geometric visualization space and time are completely different things. There is an absolute time, which is independent of place and motion. But at the beginning of this century Einstein saw that there is a principle of relativity in physics that covers mechanics and optics. That lead to the Lorentz transformations and to the Minkowski geometry, which our physicysts call Pseudo-Euclidean geometry. The principle tells us: If we consider the world from a rocket, even if it is moving very fast, so we get new space coordinates and new time coordinates, that means a new relation system. We get coordinate transformations from the new coordinate system to the old one. By these space and time quantities are mixed. That must mean: Time and space are of the same kind, of the same nature. In visualization this is incomprehensible! And in relativity we obtain horrible results as the consequences of time dilatation.

However, the visualization possibilities are very much limited in oder to see deper mathematical relations. This is true already in 3–dimensional geometry. For instance look at the following drawing:



What figure does the plane in M, which is $\perp$ D, cut out of the cube?

The answer is: a regular hexagon. But this is not visible. We have to prove it.

There is the question: What is the method by which our neuro-physiological system produces the geometric visualization? The question is very difficult and unsolved up today. Certainly, no direct geometric picture is generated in the brain. Otherwise, some parts of the picture would disappear after a brain operation. Nobody realized a thing like that. But in our nervous system there are many waves, unfortunately of low frequency. So first the picture has to be analysed by a small set of characteristic features. Then these waves, may be, can serve for a transformation of the features of the geometric objects. Experiments could be done with the transformed objects. It may be that the transformations are distorted Fourier transformations like the holographic transforms.

As said already: the opinion of our great German philosopher Kant was that the human reason requires the Euclidean geometry. But it certainly was that geometry given by the geometric vision. Later on, mathematicians saw, that also other geometries can logically be constructed to be as consistent as the Euclidean is. For instance, the hyperbolic geometry is possible in this way. Gauss, Lobatchevsky and Bolyai pointed to this at the beginning of the 19^{th} century. In the hyperbolic geometry all axioms are valid, exept the axiom of the parallels: If you have a line g (in the plane) and a point P outside the line then there is not just one line but there are very many lines through P, which do not intersect g.

Gauss saw (in error) a contradiction to Kant, but he was in great fear of the big fuss, which the "Bœetians" (peasants from the people around Theben in ancient Greece) would have been made about his ideas. Therefore, he did not publish them.

6. Here, we shall develop the Euclidean geometry only. We restrict to the Euclidean geometry not, because it is more true, but because it leads to the notion of vectorspace and this is fundamental for mathematics.

We shall treat the geometry of visualization by logical methods. For doing this we take a system of axioms, which are visually evident. Then, we shall derive all other theorems from these axioms by logical deduction, only. We shall need 18 geometric axioms. We shall not work with lines, angles (like Euclides and Hilbert did) but with arrows. Once, J. Dieudonné called the use of the vector space the royal way.

I have to thank my collegue S.J. Patterson for correcting the manuscript.

1 Addition of Vectors in the Plane of Visualization

1. The Visual Plane. We always assume E to be the visual plane. We consider it to be a set of points.

Axiom I. *E is a non empty set. The elements of $P \in E$ are called* points.

Definition 1.1. *An* arrow *$x = (P, Q)$ is an ordered pair of points $P, Q \in E$.*

We draw x by an arrow, which reaches from P to Q. If $y = (Q, P)$ is the arrow with opposite direction and if $P \neq Q$, we have $x \neq y$. We shall write X for the set $\{x\}$.

The notion parallel is visually evident in small reagons, at least and then also in large ones, since these are isomorphic to small reagons. Its meanig for arrows is: same length and same direction. The evidence of the following axioms comes also from the evidence

in local domains.

Axiom II. *In X a binary relation $\parallel$ (in words:* parallel*) is given.*

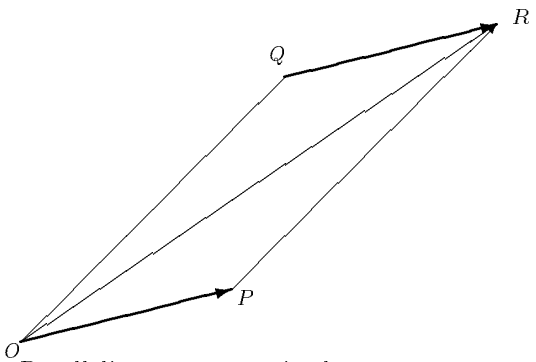
Axiom III. *Assume that $x = (P, Q) \in X$ and that $P_0 \in E$. Then, there is exactly one point $Q_0 \in E$ such that $y = (P_0, Q_0) \parallel x$.*

The mathematician is not concerned with the question how the visualization is created. He only needs to know that the axioms are valid. So we may go on.

Axiom IV. *The binary relation $\parallel$ is an equivalence relation (in the general sense of algebra).*

So the following laws are valid: We always have $x \parallel x$; if $x \parallel y$, then $y \parallel x$ also; if $x \parallel y$ and $y \parallel z$, then $x \parallel z$ (transitive law).

The next axiom is known to all physicists and engineers very well. They speak of the "paralellogram of forces".



Parallelism, commutative law

Axiom V. *(Parallelogram axiom) Assume that for $O, P, Q, R \in E$ the arrows $(O, P) \parallel (Q, R)$. Then it follows $(O, Q) \parallel (P, R)$.*

This axiom is evident for the following reason:

Assume that P_0, P_1 are two points. Then there is always a translation in E which

maps P_0 onto P_1. By this every vector is mapped onto a parallel vector. Assume now that F is a translation with with $F(O) = Q$. Then because of $(O,P) \parallel (Q,R)$ we have $F(P) = R$. But in the case of a translation also the connecting vectors of the points and the image points are parallel to each other. So we get that (O,Q) and (P,R) are parallel.

2. The Vector Axioms. Now for the vector addition the system of axioms is complete. From now on, we call our arrows also vectors. There are some consequences. These are mathematical propositions. To them belong also the following vector axioms.

We choose a fixed point $O \in E$, which we call "base point" and put $V = \{(O,P) : P \in E\} \subset X$. We call V the set of vectors in O.

We introduce the vector addition. Assume that $x = (O,P)$, $y = (O,Q)$ are in V. Then there is a well determined point $R \in E$ with $(O,Q) \parallel (P,R)$. We write $x + y = (O,P) + (O,Q) = (O,P) + (P,R) = (O,R) \in V$ and by this the addition is defined. Now, we get the commutative law:

Vector Axiom A. *We always have* $x + y = y + x$.

Proof. We use $x = (O,P)$, $y = (O,Q)$. We assume $(O,Q) \parallel (P,R)$. Then we get $(O,P) \parallel (Q,R)$ by axiom V. Therefore, it follows $y + x = (O,Q) + (O,P) = (O,Q) + (Q,R) = (O,R) = x + y$.

Definition 1.2. *We call the vector* (P,P) *the* null vector in P.

Theorem 1.3. *Always* $(P,P) \parallel (Q,Q)$ *is true.*

Proof. We have $(P,Q) \parallel (P,Q)$. So we get from the parallelogram axiom $(P,P) \parallel (Q,Q)$.

Definition 1.4. *$O = (O, O)$ is called the* null vector *in V.*

Vector Axiom *B. In V always $x + O = x$ is valid.*

Proof. We know $O \parallel (P, P)$. So we get $(O, P) + O = (O, P) + (P, P) = (O, P)$.

For every (P, O) there is exacly one point $Q \in E$ with $(O, Q) \parallel (P, O)$.

Definition 1.5. *The vector $-(O, P) = (O, Q) \in V$ is the negative of (O, P).*

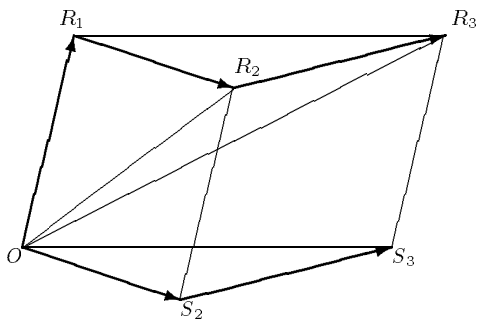
Vector Axiom *C. We always have in V the equation $x + (-x) = O$.*

Proof. It follows $(O, P) + (-(O, P)) = (O, P) + (O, Q) = (O, P) + (P, O) = (O, O) = O$.

Now we shall prove the associative law for the addition.

Vector Axiom *D. In V always the equation $(x + y) + z = x + (y + z)$ is true.*

Proof.



The associative law

Asume $x = (O, R_1), y = (O, S_2), z = (O, Z)$. Then there is a R_2 with $(O, S_2) \parallel (R_1, R_2)$ and a R_3 with $(O, Z) \parallel (R_2, R_3)$, moreover a S_3 with $(O, Z) \parallel (S_2, S_3)$.

By the parallelogram axiom we get $(O,R_1) \parallel (S_2,R_2)$ and then $(S_2,R_2) \parallel (S_3,R_3)$ and by this $(O,R_1) \parallel (S_3,R_3)$ and then $(O,S_3) \parallel (R_1,R_3)$. So we have $(x+y)+z = \Big((O,R_1)+(O,S_2)\Big)+(O,Z) = \Big((O,R_1)+(R_1,R_2)\Big)+(R_2,R_3) = (O,R_3)$ and $x+(y+z) = (O,R_1)+\Big((O,S_2)+(O,Z)\Big) = (O,R_1)+\Big((O,S_2)+(S_2,S_3)\Big) = (O,R_1)+(O,S_3) = (O,R_1)+(R_1,R_3) = (O,R_3)$ again.

By this is proved:

Theorem 1.6. *$(V,+)$ is an abelian group.*

Our original axioms were evident from vision. This is quite similar to the situation in set theory. But quite in the reverse the vector axioms, which appear here, are derived. They belong to the definition of the notion of vectorspace. So, this is completely different.

2 Multiplication by Scalars

1. Axioms. Naturally, we take for scalars the real numbers. We denote by $R_0^+ = \{a \in \mathbb{R} : a \geq 0\}$ the set of non-negative real numbers. In the spatial visualization is evident: If $x = (0,P) \in V$ is a vector, then it has a length $l \in \mathbb{R}_0^+$ and a direction. For $a \in \mathbb{R}_0^+$ we denote by $a \cdot x$ the vector of same direction but of length $a \cdot l$.

Axiom VI. *We have a binary operation $\mathbb{R}_0^+ \times V \to V : (a,x) \to a \cdot x \in V$.*

If $a,b \in \mathbb{R}_0^+$, then $(a+b) \cdot x$ and $ax+bx$ have the same direction as x. But the length' are $(a+b) \cdot l$ and $al+bl$. That is equal.

Axiom VII. *We always have $(a+b) \cdot x = ax+bx$.*

Also the associative law is valid:

Axiom VIII. *We always have $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.*

Axiom IX. *We have the unitary law* $1 \cdot x = x$.

In order to obtain the other laws of scalar multiplication we need the continuity. Also this is evident from visualization.

Assume that $x = ax_1 - ax_2 - ax_3$ with $x_1, x_2, x_3 \in V, a \in \mathbb{R}_0^+$. Then, if $b \sim a,\ b \in \mathbb{R}_0^+$, also $y = b \cdot x_1 - b \cdot x_2 - b \cdot x_3$ has to be very near to x.

Axiom X. *Assume that* $x = ax_1 - ax_2 - ax_3$ *and that in arbitrary neighborhood of* a *there are numbers* $r \in \mathbb{R}_0^+$ *with* $rx_1 - rx_2 - rx_3 = 0$. *Then also* x *is* 0.

2. Consequences.

We shall prove some consequences.

Proposition 2.1. *We have* $0 \cdot x = O$.

Proof. We have $(a + 0) \cdot x = ax + 0 \cdot x = ax$. By this we get $0 \cdot x = O$, like in every group.

Definition 2.2. *If* $a \leq 0$, *we put* $a \cdot x = -\Big((-a) \cdot x\Big)$.

By this the product $a \cdot x$ is well defined for all $a \in \mathbb{R}$. Namely, if $a = 0$, then by the axiom and by the definition $a \cdot x = O$ follows.

Proposition 2.3. *We have for all* $a \in \mathbb{R}$ *the equation* $(-a) \cdot x = -ax$.

Proof. If $a \geq 0$ then $-a \leq 0$. By our definition follows $(-a) \cdot x = -\Big((--a) \cdot x\Big) = -a \cdot x$. If $a \leq 0$, we get $-\Big((-a) \cdot x\Big) = a \cdot x$ and like in every group $-a \cdot x = (-a) \cdot x$.

Proposition 2.4. *If* $n \in \mathbb{N}$, *then* $n \cdot x = x + \ldots + x$ n-times *follows.*

Proof. We perform a complete induction on n. The case $n = 1$ is trivial by axiom IX. Now, assume $n > 1$ and that the case of $n - 1$ is proved, already. The we obtain $n \cdot x = (n-1) \cdot x + x = \underbrace{x + \ldots + x}_{(n-1)\text{-times.}} + x = x + \ldots + x$ n-times.

Proposition 2.5. *We always have* $n \cdot (x+y) = nx + n \cdot y$.

Proposition 2.6. *For all* $n, m \in \mathbb{N}$ *the equation:*

$$\frac{n}{m} \cdot (x+y) = \frac{n}{m} \cdot x + \frac{n}{m} \cdot y$$

is valid.

Proof. We get $m \cdot (\frac{n}{m} \cdot x + \frac{n}{m} \cdot y) = m\frac{n}{m}x + m \cdot \frac{n}{m}y = nx + ny = n \cdot (x+y)$. By multiplication with $\frac{1}{m}$ we obtain: $\frac{n}{m} \cdot x + \frac{n}{m} \cdot y = \frac{n}{m} \cdot (x+y)$.

Using the axiom of continuity we get:

Theorem 2.7. *For all* $a \in \mathbb{R}_0^+$ *is* $a \cdot (x+y) = a \cdot x + a \cdot y$.

Proof. The number a is an infinite decimal fraction. There are arbitrarily near to a finite decimal fractions and so rational numbers $r = \frac{n}{m}$ with $n, m \in \mathbb{N}$. Assume that $z = a \cdot (x+y) - ax - ay$. But we always have $r \cdot (x+y) - rx - ry = 0$. Because of the continuity we get $z = 0$.

3. Vector Axioms.

Now we shall derive further axioms of the vector space.

Vector Axiom *E*. *The equation:* $a \cdot (x+y) = ax + ay$ *is valid for all real numbers* a *(distributive law).*

Proof. We just have to consider the case $a < 0$. We have $a \cdot (x+y) = -\Big((-a) \cdot (x+y)\Big) = -\Big((-a) \cdot x + (-a) \cdot y\Big) = -\Big(-a \cdot x + (-a \cdot y)\Big) = -\Big(-(ax + ay)\Big)$ like in every group $= ax + ay$.

Vector Axiom *F*. *We have the distributive law* $(a+b) \cdot x = ax + bx$ *for all real numbers* a, b.

Proof. 1) This is true by axiom VII in the case $a \geq 0, b \geq 0$. Now, we assume 2) $a < 0, b \geq 0, a + b \geq 0$. Then, $(a+b) \cdot x + (-a) \cdot x = (a + b - a)x = bx$ follows. That

means: we have $(a+b)\cdot x - ax = bx$. So we get $(a+b)x = ax + bx$. All other cases can be treated in an analogous way.

In order to obtain the associative law for the scalar multiplication we prove:

Proposition 2.8. *We have for all $a \in \mathbb{R}$ the equation $a \cdot O = O$.*

Proof. We get $a \cdot (x + O) = ax = ax + a \cdot O$ and by this $a \cdot O = O$, like in every group.

Proposition 2.9. *$a \cdot (-x) = -a \cdot x$ for all $a \in \mathbb{R}$.*

Proof. We have $a \cdot \Big(x + (-x)\Big) = a \cdot O = O = ax + a(-x)$. Hence, like in every group follows $a \cdot (-x) = -ax$.

Finally, we obtain:

Vector Axiom G. *We have $a \cdot (b \cdot x) = (ab) \cdot x$ for all real numbers a, b.*

Proof. If $a \geq 0, b \geq 0$, then this follows by axiom VIII. Now, we assume $a < 0$ and $b \geq 0$. Then, we get $a \cdot (b \cdot x) = -\Big((-a) \cdot (bx)\Big) = -\Big((-ab) \cdot x\Big) = --abx = abx$. The case $a < 0, b < 0$ can be treated in a similar way.

Now, we have proved that V satisfies the axioms of a real vector space.

Theorem 2.10. *V is a real vector space.*

4. Dimension We wish to see that V is 2-dimensional. For this we need the notion of line, which we have to define using the axioms. Asume that $x_0, x_1 \in V$ with $x_0 \neq 0$.

Definition 2.11. *The set $\mathcal{G} = \{x = x_1 + t \cdot x_0 : t \in \mathbb{R}\}$ is called the line through x_1 with direction vector x_0.*

Axiom XI. *There is a line $\mathcal{G}$ through 0 such that the set $V - \mathcal{G} \neq \emptyset$.*

Axiom XII. *Assume that $\mathcal{G}_1, \mathcal{G}_2$ are 2 different lines through $O \in V$. Then if $x \in V$, there are 2 vectors $x_1 \in \mathcal{G}_1, x_2 \in \mathcal{G}_2$ with $x = x_1 + x_2$.*

Theorem 2.12. *We have* $\dim V = 2$.

Proof. By axiom XI there is a line $\mathcal{G}_1$ through 0 such that $V - \mathcal{G}_1 \neq \emptyset$. We denote the direction vector by y_1. Then we choose a vector $y_2 \in V - \mathcal{G}_1$. We denote by $\mathcal{G}_2$ the line $\{x = t \cdot y_2 : t \in \mathbb{R}\}$. We get $\mathcal{G}_1 \neq \mathcal{G}_2$. The vectors y_1, y_2 are linearly independent. Namely, if $t_1 \cdot y_1 + t_2 \cdot y_2 = O$ and is not trivial, first we get $t_1 \neq 0, t_2 \neq 0$ and then that y_2 is a multitude of y_1 and vice versa. That is a contradiction, since $\mathcal{G}_1$ and $\mathcal{G}_2$ are different. On the other hand y_1, y_2 generate the whole vectorspace V. So they are a basis. We have $\dim V = 2$.

3 The Euclidean Plane

We shall not equip V with a scalar product but with a notion of angle. Then, there also is a scalar product. It is defined up to a gauge.

1. The Rotations. It is evident in geometric visualization that there are the rotations around $O \in E$. We shall put up further 6 axioms for these.

Axiom XIII. *To every $\Theta \in \mathbb{R}$ a map $D_\Theta : E \to E$ is given with $O \to O$.*

In *geometric vision* the map D_Θ means the rotation of E around O with the angle Θ. We shall measure the angle in degrees, where 360 is the full rotation, which is the identity again. The rotation always is of positive kind in the mathematical sense, if $\Theta > 0$. So, then it is negative to the rotation of the hand of a clock. If $\Theta < 0$, then we will rotate like the clock does.

The rotation D_Θ always defines a map of the vectors. We put: $\hat{D}_\Theta(P,Q) = \Big(D(P), D(Q)\Big)$ for $P, Q \in E$. We obtain $\hat{D}_\Theta(O,P) = \Big(O, D_\Theta(P)\Big) \in V$. So the map $\hat{D}_\Theta$ maps the vectorspace V onto itself.

It is evident that $\hat{D}_\Theta$ leaves "parallel" invariant:

Axiom XIV. *Assume that* $(P,Q) \parallel (P_0,Q_0)$. *Then, we also have* $\hat{D}_\Theta(P,Q) \parallel \hat{D}_\Theta(P_0,Q_0)$.

The lengthening by a factor a before the rotation leads to a lengthening by the factor a after the rotation.

Axiom XV. *If* $a \in \mathbb{R}_0^+$, *we get* $\hat{D}_\Theta\Big(a \cdot (O,P)\Big) = a \cdot \hat{D}_\Theta\Big((O,P)\Big)$.

We can prove that all rotations are linear maps $V \to V$.

Theorem 3.1. *Every rotation* $\hat{D}_\Theta : V \to V$ *is a linear map.*

Proof. 1) We have to prove that $\hat{D}_\Theta\Big((O,P)+(O,Q)\Big) = \hat{D}_\Theta\Big((O,P)\Big) + \hat{D}_\Theta\Big((O,Q)\Big)$. There is precisely a point $R \in E$ with $(O,Q) \parallel (P,R)$ as follows by axion III. We get $(O,P)+(O,Q) = (O,P)+(P,R) = (O,R)$. Moreover, we have $\Big(O, D(Q)\Big) \parallel \Big(D(P), D(R)\Big) = \hat{D}\Big((P,R)\Big)$. That means $\hat{D}\Big((O,P)\Big) + \hat{D}\Big((O,Q)\Big) = \Big(O, D(P)\Big) + \Big(D(P), D(R)\Big) = \Big(O, D(R)\Big) = \hat{D}\Big((O,R)\Big)$. So $\hat{D} : (V,+) \to (V,+)$ is a group homomorphism.

2) We still must prove: If $a \in \mathbb{R}$ we get $a \cdot \hat{D}\Big((O,P)\Big) = \hat{D}\Big(a \cdot (O,P)\Big)$. If $a \geq 0$ this is told to us by axiom XV. So we assume $a < 0$. Then, we have: $\hat{D}\Big(a \cdot (O,P)\Big) = \hat{D}\Big(-(-a) \cdot (O,P)\Big) = -\hat{D}\Big((-a) \cdot (O,P)\Big)$, since $\hat{D}$ is a group homomorphism. This is $-(-a) \cdot \hat{D}\Big((O,P)\Big) = a \cdot \hat{D}\Big((O,P)\Big)$.

In visualization also evident is:

a) If we do a rotation with the angle Θ_2 and then a rotation with the angle Θ_1, so we get a rotation with the angle $\Theta_1 + \Theta_2$.

b) Any two rotations D_{Θ_1} and D_{Θ_2} are equal if and only if $\Theta_2 - \Theta_1 = n \cdot 360$ with $n \in \mathbb{Z}$.

We obtain:

Axiom XVI. $D_{\Theta_1} \circ D_{\Theta_2} = D_{\Theta_1+\Theta_2}$.

Axiom XVII. *The rotation D_0 is the identity. We have $D_{\Theta_1} = D_{\Theta_2}$ if and only if $\Theta_2 = \Theta_1 + n \cdot 360$.*

We get the same propositions for the maps $\hat{D}_\Theta$. We obtain: $\hat{D}_{\Theta_1} \circ \hat{D}_{\Theta_2} = \hat{D}_{\Theta_1+\Theta_2}, \hat{D}_0 = id : V \to V$, $\hat{D}_{\Theta_1} = \hat{D}_{\Theta_2}$ if and only if $\Theta_2 = \Theta_1 + n \cdot 360$.

V is a 2-dimensional vectorspace. Therefore, there are isomorphisms $\alpha : \mathbb{R}^2 \xrightarrow{\sim} V$. If such an α is given, we call it a chart of V and the points of $\mathbb{R}^2$ coordinates in V. Every rotation $\hat{D} : V \to V$ transfers to a linear map $\varphi = \alpha^{-1} \circ D \circ \alpha : \mathbb{R}^2 \to \mathbb{R}^2$. We call it the rotation D in the coordinates of V. This map φ is given by a matrix:

$$A_\alpha(\Theta) = \begin{pmatrix} a_\alpha(\Theta), & b_\alpha(\Theta) \\ c_\alpha(\Theta), & d_\alpha(\Theta) \end{pmatrix}.$$

Axiom XVIII. *There is a chart $\alpha : \mathbb{R}^2 \xrightarrow{\sim} V$ such that $a_\alpha(\Theta), \ldots, d_\alpha(\Theta)$ are continuous in Θ.*

Actually, the continuity is independent of the chart.

Theorem 3.2. *Assume that $\beta : \mathbb{R}^2 \xrightarrow{\sim} V$ is another chart of V. Then, also the $a_\beta(\Theta), \ldots, d_\beta(\Theta)$ are continuous in Θ.*

Proof. We have $\varphi_\beta = \beta^{-1} \circ \hat{D} \circ \beta = \beta^{-1} \circ \alpha \circ \alpha^{-1} \circ \hat{D} \circ \alpha \circ \alpha^{-1} \circ \beta = \gamma^{-1} \circ \varphi_\alpha \circ \gamma$ with $\gamma = \alpha^{-1} \circ \beta : \mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}^2$. The map γ is the coordinate transformation. It transfers the new coordinates (i.e. those given by β) into the old ones. It is written by a matrix C such that the following equation is valid:

$$A_\beta = C^{-1} \circ A_\alpha \circ C.$$

If A_α is continuous in Θ, then also A_β has to be continuous in Θ.

2. Suitable Coordinates. Assume that T is a symmetric matrix. We call such a matrix T positive definite, if the matrix product $x^t \circ T \circ x > 0$ for every vector $x \in \mathbb{R}^n - \{O\}$. If A is a non singular matrix, then $A^t \circ A = T$ is symmetric and positive definite. Namely, the product $x^t \circ T \circ x = (A \circ x)^t \circ (A \circ x)$ is the scalar product of $A \circ x$ with itself and we have $A \circ x \neq O$, if $x \neq O$.

We have $A_\alpha(\Theta) \circ A_\alpha(-\Theta) = A_\alpha(0) = E$, the unit matrix. So $A_\alpha(\Theta)$ never is singular. We put $T = \int_0^{360} A_\alpha^t(\Theta) \circ A_\alpha(\Theta) d\Theta$ and get $T^t = \int_0^{360} A_\alpha^t \circ (A_\alpha^t)^t \cdot d\Theta = T$. So T is symmetric. We have $x^t \circ T \circ x = \int_0^{360} x^t \circ A_\alpha^t \circ A_\alpha \circ x d\Theta > 0$, if $x \neq O$. So T is positive definite. Moreover we get: $A_\alpha^t(\Theta_0) \circ T \circ A_\alpha(\Theta_0) = \int_0^{360} \Big(A_\alpha(\Theta) \circ A_\alpha(\Theta_0)\Big)^t \circ \Big(A_\alpha(\Theta) \circ A_\alpha(\Theta_0)\Big) d\Theta = \int_0^{360} A_\alpha^t(\Theta + \Theta_0) \circ A_\alpha(\Theta + \Theta_0) d\Theta = \int_{\Theta_0}^{360+\Theta_0} A_\alpha^t(\Theta) \circ A_\alpha(\Theta) d\Theta = T$, because $A_\alpha(\Theta)$ is periodic. Therefore, the transformation $A(\Theta_0)$ leaves the quadratic form T invariant.

By a well known transformation $U^t \circ T \circ U$ with a positive orthogonal U we can bring the matrix T to a diagonal matrix. Since the matrix is positive definite all components in the diagonal are positive. Next we do a gauge transformation to pass the matrix to a unit matrix. We have got:

Proposition 3.3. *There is a non singular matrix U such that $\tilde{T} = U^t \circ T \circ U$ is the unit matrix.*

We do a simplification of our chart by introducing new coordinates using the transformation $x = U \circ \tilde{x}$. We consider the geometric rotations with respect to the new coordinates. The are given by a matrix $U^{-1} \circ A_\alpha(\Theta) \circ U = \tilde{A}(\Theta)$. We have $E = U^t \circ T \circ U = (U^{-1} \circ A_\alpha \circ U)^t \circ (U^t \circ T \circ U) \circ (U^{-1} \circ A_\alpha \circ U) = \tilde{A}^t \circ E \circ \tilde{A}$. Therefore, we get $\tilde{A}^t \circ \tilde{A} = E$ That means that $\tilde{A}(\Theta)$ always is an orthogonal transformation. It is continuous in Θ, we have $\tilde{A}(0) = E$. So $\det \tilde{A}(\Theta) = 1$ follows and $\tilde{A}(\Theta)$ is positive

orthogonal map i.e. an Euclidean rotation. We have $\tilde{A}(\Theta) \in SO(2)$.

Proposition 3.4. *Any matrix A is in $SO(2)$ if and only if $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a^2 + b^2 = 1$.*

Proof. 1) Assume that the condition is satisfied. Then $\det A = a^2 + b^2 = 1$ and by this

$$A^t \circ A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \circ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = E$$

follows.
2) If we have $A \in SO(2)$, then

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = E$$

is valid. This implies $ab + cd = 0$, and so $\det \begin{pmatrix} a & d \\ -c & b \end{pmatrix} = 0$. We do not have $(d, b) = O$, since A is non singular. Therefore there is a $\tau \in \mathbb{R}$ with $a = \tau d,\ c = -\tau b$. We further have $1 = ad - cb = \tau(d^2 + b^2)$. Then τ is equal 1, since $d^2 + b^2 = 1$ and we have $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$. So there is the desired form.

Proposition 3.5. *The group $SO(2)$ is commutative.*

Proof. We have

$$\begin{pmatrix} a_1, & -b_1 \\ b_1, & a_1 \end{pmatrix} \circ \begin{pmatrix} a_2, & -b_2 \\ b_2, & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2, & -a_1b_2 - b_1a_2 \\ b_1a_2 + a_1b_2, & -b_1b_2 + a_1a_2 \end{pmatrix},$$

which is independent of the oder.

Corollary 3.6. *We have the following theorems for addition:*

$$\begin{aligned} a(\Theta_1+\Theta_2) &= a(\Theta_1)\cdot a(\Theta_2) - b(\Theta_1)\cdot b(\Theta_2) \\ b(\Theta_1+\Theta_2) &= b(\Theta_1)\cdot a(\Theta_2) + a(\Theta_1)\cdot b(\Theta_2). \end{aligned}$$

Moreover we have $a(0)=1, b(0)=0$, *the periods are 360.*

Proof. The theorems for addition are equivalent with the theorem for addition for $D(\Theta)$ and this is equivalent with the theorem of addition for $A_\alpha(\Theta)$.

Finally, we also can apply a coordinate transformation $\tilde{x} = U \circ \tilde{\tilde{x}}$, where U is the matrix $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, we have $\det(U) = -1$. This transformation interchanges both coordinates. We get for the rotations in the new coordinates $\tilde{\tilde{A}}(\Theta) = U^{-1} \circ \tilde{A}(\Theta) \circ U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. So only an interchange in the antidigonal happens. The function $b(\Theta)$ can be replaced by its negative.

3. The Model. The question is: Is there a mathematical model to the axioms I - XVIII? We take $V = \mathbb{R}^2$ and put $a(\Theta) = \cos\left(\frac{2\pi}{360}\Theta\right), b(\Theta) = \sin\left(\frac{2\pi}{360}\Theta\right)$. Then the theorem of addition for the rotations will be valid, since we have the theorems of addition for the functions $\cos, \sin$. We get very easily that the axioms are satisfied. We obtain the map $\tilde{A}(\Theta)$. So we have a correct model. Our geometry is consistent! – In this case we shall write $A_0(\Theta)$ instead of $\tilde{A}(\Theta)$.

We shall prove that $\tilde{A}(\Theta)$ is uniquely determined. We can define $F(\Theta) = \frac{360}{2\pi}$ $\arccos(a(\Theta)) = \frac{360}{2\pi}\arcsin(b(\Theta))$ as a continuous function on $\mathbb{R}$ with $F(0) = 0$. It is uniquely determined. We use the matrix terminology. The map $\Theta \to A_0(\Theta)$ is locally invertible. There is an independent magnitude of the neighborhoods, where the inverse map exists. We get also $F(\Theta_1) + F(\Theta_2) = F(\Theta_1 + \Theta_2)$, because in the matrix terminology we have F as a product of a inverse map with a map, where both maps satisfy the same theorem of addition. We have to prove this statement only for small Θ_2. Using continuity of F we get $F(\Theta) = c \cdot \Theta$. The function F mod(360) has the period 360. This implies $F(\Theta) = \pm\,\Theta$. By employing an interchange of coordinates

we can obtain $F(\Theta) = \Theta$. Hence we get $a(\Theta) = cos(\frac{2\pi}{360}\Theta),\ b(\Theta) = sin(\frac{2\pi}{360}\Theta)$. So our rotations are those of our model. Our geometry is uniquely determined.

Corollary 3.7. *The Euclidean geometry is uniquely determined by the axioms.*

4. The Notion of Angle. Assume that $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ with $x_1^2 + x_2^2 = 1,\ y_1^2 + y_2^2 = 1$. The rotations are

$$A(\Theta) = \begin{pmatrix} \cos\left(\frac{2\pi}{360} \cdot \Theta\right), & -\sin\left(\frac{2\pi}{360} \cdot \Theta\right) \\ \\ \sin\left(\frac{2\pi}{360} \cdot \Theta\right), & \cos\left(\frac{2\pi}{360} \cdot \Theta\right) \end{pmatrix}.$$

We can determine Θ such that $y = A(\Theta) \circ x$. To do this, we first choose Θ_1 such that $x = A(x_1) \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This is true if and only if $x_1 = \cos\left(\frac{2\pi}{360} \cdot \Theta_1\right),\ x_2 = \sin\left(\frac{2\pi}{360} \cdot \Theta_2\right)$. As well known this equation is solvable. In the same way we put $y = A(\Theta_2) \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and obtain for $\Theta = \Theta_2 - \Theta_1$ the desired identity $y = A(\Theta) \circ x$.

If $y = A(\Theta) \circ x = A(\tilde{\Theta}) \circ x$, then $A^{-1}(\Theta_2) \circ A(\Theta) \circ A(\Theta_1), A^{-1}(\Theta_2) \circ A(\tilde{\Theta}) \circ A(\Theta_1)$ map the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto itself. Then we have

$$\cos\left(\tfrac{2\pi}{360}(\Theta + \Theta_1 - \Theta_2)\right) = \cos\left(\tfrac{2\pi}{360}(\tilde{\Theta} + \Theta_1 - \Theta_2)\right)$$

$$\sin\left(\tfrac{2\pi}{360}(\Theta + \Theta_1 - \Theta_2)\right) = \sin\left(\tfrac{2\pi}{360}(\tilde{\Theta} + \Theta_1 - \Theta_2)\right).$$

That are the same values of $\cos, \sin$ as for the argument 0. We get that the arguments differ only by $n \cdot 360$ mit $n \in \mathbb{Z}$. So we have $\tilde{\Theta} = \Theta + n \cdot 360$. That means that the angle $\sphericalangle(x, y)$ is determined.

The length $\| x \| = |x_1^2 + x_2^2|^{\frac{1}{2}}$ of any vector stays unchanged under any rotation. So every 2 of our suitable coordinate systems can be different just by a homothety $\tilde{x} = c \cdot x$ with $c \in \mathbb{R}^+$ or by a rotation. That means that the norm $\| x \|$ is determined up to a

factor c. It can be altered by a gauge only. Also the scalar product $x \cdot y$ is determined up to the factor c^2.

5. Classical Theorems. We can easily see that two vectors are rectangular to each other if and only if their scalar product vanishes. The ancient Greeks knew the theorem of Thales, that of Pythagoras and several theorems on angles coming from parallel lines. All these are trivial in Euclidean vector spaces. Since the plane given by our mental picture is isomorphic to an Euclidean vectorpace they are true for this visual plane. So all these theorems known to the Greeks are proved. It should be clear that only a proof coming from a system of axioms for the visual plan is really a proof.

We know that mathematicians doubted in the axiom of parallels. The reason that it is not completely evident by vision is that the space near to infinity is not visuable. In its origin the geometric vision gives only a mental picture of a very bounded space. The full space has to be defined by an infinite construction. We can do the construction so that it is no longer Euclidean. But then the visualibility is lost, completely.

II. Book

Discrete Geometry

Introduction

1. We shall consider a mathematical model, which simultaneously describes the wave nature and the particle nature of matter. An important new idea is the lattice structure of matter. By this those structures in quantum physics will be removed, which were called to be *crazy* by Einstein.

In order to say it very precisely already in the beginnig: all that which follows is mathematics and not (new) physics. Since Feynman we know that the physical vacuum has a chaotic behaviour. But the mathematics, which pysicists used, was always smooth. So the mathematical theory could only bring a description. This theory was far away from the reality. I feel that the kind of geometry, which we use, should already contain the chaos. The purpose of this article is to obtain such a geometry.

In every natural inertial coordinate system the points of space–time consist of spatial coordinate points $\vec{x}' = (x_1, x_2, x_3)$ and a time value $t = c^{-1} \cdot x_0$. Here c represents the velocity of light. Together they produce the 4-dimensional space $\mathbb{R}^4$ of vectors $\vec{x} = (x_0, \ldots, x_3)$. In the Michelson experiment natural changes from inertial coordinate system to inertial coordinate system occur. By such a transformation light rays $\{\vec{x} = \vec{x}_0 + \vec{e} \cdot ct \,:\, t \in \mathbb{R}\}$ are mapped into light rays. Here the spatial components of $\vec{e}$ form a Euclidean unit vector and e_0 is 1. For physical reasons we have to require that the transformations are continuous. So we get rather easily that the transformation is a composition of a Poincaré transformation and a homothety where a Poincaré transformation is a composition of a Lorentz transformation and a translation. Lorentz transformations are those linear transformations $\mathbb{R}^4 \to \mathbb{R}^4$, which leave the Minkowski metric unchanged. The Einstein principle of relativity tells us that all laws in physics are invariant precisely with respect to these transformations. Clearly, such a metric

has to be indefinite. It has to have 1 positive and 3 negative eigenvalues. It is certainly different from the Euclidean metric. We shall call the geometry which is defined by this metric the Pseudo-Euclidean geometry. In German it is known as Minkowski geometry. The Goettingen mathematician Minkowski did a lot of work on it.

Relativistic quantum physics uses fields of spinors. In every point of space–time the vector space of spinors is a representation of the simply connected group $SU(2)$. It is a 2–sheeted covering of the group of rotations $SO(3)$, but the difference is not too great. We can look at $SU(2)$ as to be a certain square root of $SO(3)$: if we do a rotation by 360 degrees the result in $SU(2)$ is not yet the identity but only its half.

2. Following E. Kaehler (1962) we can replace the spinors at a point by the objects of the so called inner calculus Φ of exterior forms. It is a Clifford algebra. The objects of Φ have 16 components. To begin we use them as complex vectors. But we require that they are invariant against the $*-operator$. This operator is purely imaginary. That means that we need the real part only: The imaginary part is the result of the $*-operator$. The objects of Φ are even $SO(3)$–invariant. Therefore, differently from the spinors they can be interpreted geometrically. However, the $SO(3)$–representation Φ is not irreducible. It decomposes into 2 direct summands. We shall interpret one for electric matter and the other one for completely unelectric matter (for instance neutrinos). There is a geometric reason for this interpretation. We just have to use a stochastical dynamics.

The following ideas are essential. The Pseudo-Euclidean metric is defined by a Riemannian metric. We do not use this metric but a system $\mathcal{V}$ of light vectors instead. There are minimal volume elements. The set $\mathcal{V}$ of light vectors is discrete. Hence the number of our light vectors which go from one minimal volume element to another is finite. The number is (nearly) independent of the direction but opposite proportional to the (Euclidean) length. We have that (up to the possibility of measurement) the system $\mathcal{V}$ is precisely invariant against Poincaré transformations. So the Pseudo-Euclidean geometry is also defined by it as well. We can form by our light vectors vortices of dimension 0 to 4. These are the components of the objects of Φ in complementary

dimension.

The dynamics is given in the following way. The distribution of our light vectors is connected with a stochastical process. In this light vectors of same direction are joined and light vectors of opposite direction cause a division of the larger one into two. The number of conversions per unit time is proportional to the density of the kind of light vectors used. These densities can really not be obtained as a Lebesgue measure, since such a set will not be measurable and this is the first time that we have to restrict ourselves to discreteness: the system of light vectors has to be discrete and the measure actually is given as an integral number. The distribution of $\mathcal{V}$, which gives the Pseudo-Euclidean geometry, is obtained by this process. This process will be extended to fields of Φ, then. We shall get equations of equilibrium. So the well known Dirac equation can be derived geometrically. – The density of light vectors should be very large. We think of about 10^{200} light vectors per (7-dimensional) volume unit. This means that all fields will be smooth within the range of posible measurement.

There is an integral equation which means precisely that the distribution of $\mathcal{V}$ is preserved by the stochastic process. Unfortunately, this integral equation has other solutions, completely different from the distribution of $\mathcal{V}$. Finally, another discretization lead to uniqueness: We have to take a fixed inertial coordinate system and to assume that every arrow in $\mathcal{V}$ is the union of finitely many *elementary* light arrows of same direction. All elementary light arrows have the same length $\epsilon > 0$. It is a universal constant and should have a value between $10^{-25}m$ and the Planck length of $10^{-35}m$. The stochastic process is not essentially changed. It will be practically so as described. The result of the process only has to be moved in a positive time direction by a fixed value. This means that the stochastic time is identical with our common time. By this it also becomes clear that time has a direction.

Since we get the same equilibrium, when we require restrictively that at most n elementary light vectors can joint (with $n \in \mathbb{N}$ = positive integer) the game can be played on a very simple computer.

3. Our fields have a lattice structure. When we have several lattices, they may generate a joint lattice, which may be interpreted as a real tensor product. But in physics a complex tensor product is required. We can obtain it by assuming that space–time is oriented: that there is a maximal field of 4–dimensional vortices. It follows that the parity has to be given up. – By this tensor product a huge multitude of fields can be constructed.

By a process of damping and generation (this only in a point where the real part of the field and its imaginary part coincide exactly) the dualism of wave and particle can be explained. The Copenhagen model was still a child of German idealism (in epistemology). Since nowadays we can guess how the instrument of cognition, the Aristotelean mind, functions, we see that this standpoint is out–dated. At least in the case of data preparation, which is visible from outside, scientific possibilities are used only. Therefore, also the dualism has to be expained in the way of physics.

Basic in our model is space–time but not matter. Matter is just a structure in space which stays preserved because of the equilibrium which is described by the Dirac equation. It is a structure of relative order in the totally chaotic space. It has a lattice structure. Some of the light vectors of space–time are bounded to it and so taken away from space–time. This means an enlargement of other particles and so a read shift. We get a curvature of space–time and by this the gravity.

Certainly, our model is terribly complex, but a monstrous land scape of possibilities has opened. We are very far from having a view of it or being able to master it properly. New experiments show that the complexity is there. We have to part from the scientific method that everything must be given simple explanations.

Our model contains a discrete structure. This structure is so fine that everything is practically smooth. In order to obtain measurable quantities we have to approximate by smooth mathematics: We need a mathematical model of the discrete model! None of the old methods becomes superfluous.

1 Sets

1. Integrability. We assume that X is a complete topological vectorspace, that $Q \subset \mathbb{R}^n$ is a compact cuboid, that t is a stepfunction on Q with values in X and that $\sum(t) \in X$ is the Riemann sum:

$$\sum_{j_1,\ldots,j_o} t(Q_{j_1\ldots j_o}) \cdot I(Q_{j_1\ldots j_o}).$$

We consider open tubes $\mathcal{F} \subset Q \times X$ over Q. They are mapped by the projection onto Q. We say that a stepfunction $t \in\in \mathcal{F}$, if the closure of its graph is in $\mathcal{F}$.

The set $U(O) \subset X$ denotes a neighborhood. We define:

Definition 1.1. *$\mathcal{F}$ has its* width *in U, if the following property is true:*
If $t_1, t_2 \in\in \mathcal{F}$, then always $\sum(t_1) - \sum(t_2) = \sum(t_1 - t_2) \in U$.

Now, a function $f : Q \to X$ is called integrable, if for every neighborhood $U(O) \subset X$ there is a tube $\mathcal{F}$ around (the graph of) f with width in U, i.e. if there are arbitrary narrow tubes around f. This will be only possible, if the 0-function is integrable, which is equivalent to the property that X is locally convex (see: H. Grauert, Analysis, Funktionale und Geometrie. Mathematisches Institut Gttingen 1995, an exposé which can be bought there).

In the case of integrability the integral is defined. We take a countable neighborhood basis $\{U_i : i = 1, 2, 3, \ldots\}$ of the point $O \in X$ with $U_{i+1} \subset U_i$ and then tubes $\mathcal{F}_i$ around f with width in U_i and $\mathcal{F}_{i+1} \subset \mathcal{F}_i$. If the $t_i \in\in \mathcal{F}_i$ are step functions, we always have for $j \geq i$ the relation $t_i, t_j \in\in \mathcal{F}_i$ and therefore $\sum(t_i) - \sum(t_j) \in U_i$. So $\sum(t_i)$ is a Cauchy sequence. We get $\sum(t_i) \to a \in Y$. Furthermore we have:

1. The vector a is defined independently of t_i: If we take another sequence of step functions $t_i^* \in\in \mathcal{F}_i$, then we have $\sum(t_i) - \sum(t_i^*) \in\in \mathcal{F}_i$. Therefore, $\sum(t_i), \sum(t_i^*)$ converge simultaneously towards a.

2. The vector a is defined independently of the sequence $\mathcal{F}_i$: Namely, if $\mathcal{F}_i^*$ is another sequence, we may pass to the intersections $\mathcal{F}_i \cap \mathcal{F}_i^*$.

Definition 1.2. *Denote by $d\mu$ the measure $t \to \sum(t)$. Then a is the* integral $\int_Q f(x)d\mu$.

The integral is linear, of course. It is nothing else but the Lebesgue integral (see: exposè Gttingen).

We get the volume of a set M by integration on the characteristic function

$$\chi_M(x) = \begin{cases} 1 & : x \in M \\ 0 & : \text{ otherwise.} \end{cases}$$

From now on always we denote by $d\mu = dx$ this euclidian measure. We shall define:

Definition 1.3. *M is called* measurable, *if χ_M is integrable. Then we have the volume $I(M) = \int\limits_Q \chi_M dx$.*

2. Non-Measurable Sets.

We put $\mathcal{I} = [0,2] \subset \mathbb{Q}$ and shall decompose $\mathcal{I}$ into 2 congruent, homogeneously lying subsets M_+, M_-. Both will not be measurable. We denote $\mathbb{Q}_0 = \{2^\nu \cdot \frac{p}{q} : q \in \mathbb{N} \text{ odd}, p \in \mathbb{Z} \text{ odd or } 0, \nu \geq 0\}$. Then $\mathbb{Q}_0$ is an abelian group with resp. to the addition. The quotient projection $\mathbb{R} \to \mathbb{R}/\mathbb{Q}_0$ gives a fibration over $\mathbb{R}/\mathbb{Q}_0$. By the axiom of choice there is a fibre preserving map $f : \mathbb{R}/\mathbb{Q}_0 \to \mathbb{R}$. We put $M = f(\mathbb{R}/\mathbb{Q}_0) \subset \mathbb{R}$. If we have $c \in \mathbb{Q}_0$, we get by translation $M + c \subset \mathbb{R}$. Then, the system $\{M + c : c \in \mathbb{Q}_0\}$ is a decomposition of $\mathbb{R}$ into countably many subsets.

We denote by $\mathbb{Q}_+ = \{2^\nu \frac{p}{q} : \nu \geq 1\}$ the set of even numbers in $\mathbb{Q}_0$ and by $\mathbb{Q}_- = \{2^\nu \frac{p}{q} : \nu = 0, p \neq 0\}$ the set of odd. We put $M_+ = \bigcup\limits_{c \in \mathbb{Q}_+} (M + c) \supset M$ and $M_- = \bigcup\limits_{c \in \mathbb{Q}_-} (M + c)$ and obtain a (disjoint) decomposition of $\mathbb{R}$. It has the period 2, M_+ and M_- are lying in $\mathbb{R}$ everywhere dense. The addition of $c \in \mathbb{Q}_+$ gives translations $M_+ \xrightarrow{\sim} M_+, M_- \xrightarrow{\sim} M_-$. That means that M_+, M_- everywhere are of the same distribution, in other words

it means the homogeneity. We obtain a congruence $M_+ \stackrel{\sim}{\to} M_-$ by arbitrary small translations with $c \in Q_-$. Finally, we replace M_+ and M_- by their intersections with $\mathcal{I}$.

If M_+ or M_- were measurable, we would have $I(M_+) = I(M_-) = 1$. From this we will get a contradiction, however.

Now, assume that M_+ is measurable. Then, there is a tube $\mathcal{F}[h, g]$ around χ_{M_+} with arbitrary small width. If $x \in M_+$, there would be an interval $U \subset \mathcal{I}$ around x, which is open with respect to the relative topology such that the step function $\chi_U \in\in \mathcal{F}$. We always can take U out of a fixed countable basis of the topology of $\mathcal{I}$. The (countable) union of these $U(x)$ then is an open set $U_+ \subset \mathcal{I}$ around M_+. There are step functions $t \in\in \mathcal{F}$ with values 0 or 1, whose support is in U_+ such that $\sum(t)$ is arbitrary near to $I(U_+)$. We take an $\epsilon > 0$ arbitrary small, we can get

$$I(M_+) \leq I(U_+) \leq I(M_+) + \epsilon.$$

By an arbitrary small translation we get from U_+ a neighborhood U_- of M_-. There is the same inequality. The sets U_+, U_- can be as equal as we like it. So we obtain $2 = I(U_+ \cup U_-) \sim I(U_+) \leq I(M_+) + \epsilon = 1 + \epsilon$. That is a contradiction.

It is easily to see, more generally: Assume that the distribution of M_+ in $\mathcal{I}$ is homogeneous and that this set is measurable, then we never have $I(M_+) > 0$ and simultaneously $I(M_+) < 2$. That means that all the good "smooth" subsets of I are not understood by the integration theory of Lebesgue.

2 The Pseudo-Euclidean Geometry

The old Greeks did not yet know the principle of inertia. They believed, if a body is in motion, then it is always pulled or pushed. Around 390 BC Greek mathematicians (so called Pythagoreans) proved that the earth is a ball. (Parmenides did speak of that, already before). Very soon these people knew that the air is in a very narrow

shell around the earth, only. The planets must move outside this shell! Therefore, they could not be pushed or pulled. They were moved by a force coming from inside. This being "moved from the inside" the Greeks called "life". Aristotle thought it to be an entelechy, i.e. it is given by a form, which is embossed on the bodies (on the matter). This form must also perish with the bodies. He spoke of "soul". But this Aristotelian word had nothing in common with the later christian (and also the early Greek) concept. – Similarly the biological life, of which we speak today, consists in a form, which is embossed on the (chemical) matter. There is the DNA, there are the enzymes, the organelles, the cells.

Not before Galilei the principle of inertia was accepted. However, Newton was the first, who worked it into a true theory. Then it became possible, to establish the principle of relativity in mechanics: If you look at the bodies from a point, which is moved uniformly, than you have the same laws as if you look at it from a resting point. The laws can be described in these 2 situations using 2 coordinate systems. There is a coordinate transformation from second to the first. When the point moves with the velocity v, then these are:

$$t = \tilde{t},\ x = \tilde{x} + v \cdot t.$$

Here, t is the time and $x = (x_1, x_2, x_3)$ are the space coordinates.

In the year 1881 the famous Michelson experiment on the spreading of light was done in the USA. Michelson was born in Strelno near Poznam (Poland). He showed that the spreading is the same in all the coordinate systems put from any uniformly moved point into time and space in the usual way. That meant, that also optics had to be included into the principle of relativity. This got possible by altering the Galilei coordinate transformations to the Lorentz transformations. These are just those linear transformations, which leave invariant the Minkowski metric. Then, there were new consequences for mechanics with high velocity. All these laws were invented by A. Einstein. Therefore, we speak of the Einstein principle of relativity.

1. The Minkowski geometry. In the following text we shall call it the Pseudo-euclidean geometry, as usual. We shall permit at the end of this article that

there is a curvature in that geometry.

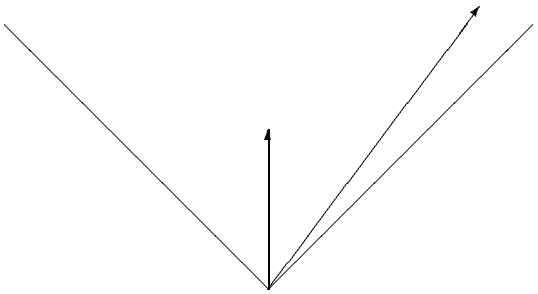
We shall consider the space $\mathbb{R}^4 = \{(x_0, x_1, \ldots, x_3) : x_i \in \mathbb{R}\}$ with $x_0 = c \cdot t$, where t is the time and c is the velocity of light. We define $x' = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\parallel x \parallel' = (x_1^2 + \ldots + x_3^2)^{\frac{1}{2}} = r \geq 0$. The Minkowski norm is $\parallel x \parallel = (x_0^2 - r^2)^{\frac{1}{2}}$. It is real only if $\mid x_0 \mid \geq r$. We are only interested in this case. We have $\parallel x \parallel = 0$ if and only if $r = ct$ that means if $x' = c \cdot t \cdot e$, where $e = (e_1, e_2, e_3)$ denotes a spatial vector with $\parallel e \parallel' = 1$. Therefore the Minkowsky norm $\parallel x \parallel$ is 0, only if x is on a light ray. We denote by $S_O = \{x : \parallel x \parallel = 0\}$ the union set of all light rays through $O \in \mathbb{R}^4$ and call S_O the light cone in O. There is a light cone S_x in every point $x \in \mathbb{R}^4$, correspondingly. We call the set $\hat{S}_O = \{x : \mid x_0 \mid \geq r\}$ the full light cone in O. It consists of a positive cone $\hat{S}_O^+ = \{x_0 \geq r\}$ and a negative cone $\hat{S}_O^- = \{x_0 \leq -r\}$. We have correspondingly $\hat{S}_x, \hat{S}_x^+, \hat{S}_x^-$.

The 2-dimensional case is more simple: $\mathbb{R}^2 = \{x = (x_0, x_1)\} = \mathbb{R}^4 \cap \{x_2 = x_3 = 0\}$. The the Minkowski norm is $\parallel x \parallel = (x_0^2 - x_1^2)^{\frac{1}{2}}$. We have 2 light rays, one left the other right from the positive x_0–axis. We project vertically onto these light rays and obtain (after an application of an gauge factor) new coordinates: $y_l = \frac{1}{2}(x_0 - x_1)$, $y_r = \frac{1}{2}(x_0 + x_1)$. The inversion is: $x_0 = y_r + y_l$, $x_1 = y_r - y_l$. We get $\parallel x \parallel = \frac{1}{4} y_l \cdot y_r$. We define $y_l = a\tilde{y}_l, y_r = \frac{1}{a}\tilde{y}_r$ for $a > 0$. Then, we get $\parallel x \parallel = \parallel \tilde{x} \parallel$. So we have a Lorentz transformation.

We get for the resting point the equation $\{x_1 = 0\}$ after the transformation

$$\begin{aligned}
\tilde{x}_1 &= \tilde{y}_r - \tilde{y}_l = a y_r - \tfrac{1}{a} y_l = \tfrac{1}{2}(a - \tfrac{1}{a})x_0 + \tfrac{1}{2}(a + \tfrac{1}{a})x_1 \\
&= \tfrac{1}{2}(a - \tfrac{1}{a})x_0 \quad \text{and} \\
\tilde{x}_0 &= \tilde{y}_r + \tilde{y}_l = a y_r + \tfrac{1}{a} y_l = \tfrac{1}{2}(a + \tfrac{1}{a})x_0, \quad \text{and by this} \\
\tilde{x}_1 &= \Big((a - \tfrac{1}{a})/(a + \tfrac{1}{a})\Big)\tilde{x}_0.
\end{aligned}$$

That means that the velocity of the transformed point goes near to 1, as $a > 1$ gets very large. The velocity of the point is nearly the velocity of light, then.



Motion in x_1–direction

The transformations belonging to a just are the proper Lorentz transformations. They are marked by this property: They map the positive and the negative full light cone onto itself without changing the orientation of time–space. In physics we have for every matter point the energy–momentum form:

$$\mathcal{P} = -Edt + p_1 dx_1 + p_2 dx_2 + p_3 dx_3.$$

Here, E is the energy of the matter point and (p_1, p_2, p_3) is its momentum. The form $\mathcal{P}$ transforms like an exterior form (of differentials) with all proper Lorentz transformations. If the point has mass m (with resp. to its motion), we can apply Einstein's equation $E = mc^2$. Therefore, we also have the equation

$$\mathcal{P} = -mcdx_0 + p_1 dx_1 + \ldots + p_3 dx_3.$$

If we have to consider several matter points, then the sum of the energy–momentum forms stays unaltered, when local procedures occur (preservation of the energy and the momentum).

If the matter point is resting, we have $\mathcal{P} = -mcdx_0$. Then, if it gets in motion, we obtain $\mathcal{P} = -\frac{mc}{2}(a + \frac{1}{a})d\tilde{x}_0 + \frac{mc}{2}(a - \frac{1}{a})d\tilde{x}_1$. We see that energy and momentum become arbitrary large as $a \to \infty$. It is impossible to exceed the velocity of light.

Now, we shall consider the Lorentz group in $\mathbb{R}^4$, again. It consists of four connected components. These can be obtained in each case by the composition of one basic transformation and all proper Lorentz transformations.

1) In the case of the P-transformations (the change of the parity) we take for a basic transformation

$$x_0 \;=\; \tilde{x}_0, \quad x_i \;=\; -\tilde{x}_i \text{ for } i = 1, 2, 3.$$

So the P–transformations change the orientation of $\mathbb{R}^4$, since the Jacobian is $= -1$.

2) In the case of the T-transformations (the time inverse) we take for a basic transformation

$$x_0 \;=\; -\tilde{x}_0, \quad x_i \;=\; \tilde{x}_i \text{ for } i = 1, \ldots, 3.$$

Also here the orientation of $\mathbb{R}^4$ will be changed.

3) The compositions PT of P-transformations and T-transformations does not change the orientation. A basic transformation is

$$x_0 = -\tilde{x}_0, \ldots, x_3 = -\tilde{x}_3.$$

Now, there is the famous TCP-theorem of G. Lüders. It tells us that a PT-transformation does not change physics, if we moreover replace particles by antiparticles. This causes us *to interpret the PT- transform of the wave of the particle as the wave of the antiparticle.*

In the case of a PT-transformation F we transform $\mathcal{P}$ into $-\mathcal{P} \circ F$, where $\mathcal{P} \circ F$ means the transform of $\mathcal{P}$, when we consider it as an exterior form. That means that $\mathcal{P}$ is not a true 1–form.

On the following pages the $*$-operator for exterior forms will play an essential role. It is precisely invariant with resp. to the proper Lorentz transformations and the PT-transformations.

2. The Invariant Measure. We shall define a Radon measure in $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4 = \{x, y\}$ that stays invariant under the following transformations: $(x, y) = \hat{F}(\tilde{x}, \tilde{y}) = \Big(F(\tilde{x}), F(\tilde{y})\Big)$, where $F : \mathbb{R}^4 \to \mathbb{R}^4$ is a Poincaré transformation. That is a composition of a Lorentz transformation and a translation.

Clearly, the total derivative $-\frac{1}{2}d(x_0^2 - x_1^2 - \ldots - x_3^2)$ is a Lorentz invariant 1-form in $\mathbb{R}^4$. Also the volume element $dx_0 \wedge \ldots \wedge dx_3$ is invariant. Assume that
$\Phi^* = dx_0 \wedge \ldots \wedge dx_3 \ / \ -\frac{1}{2}d(x_0^2 - x_1^2 - \ldots - x_3^2) \mid S_O - \{O\}$.
Here the quotient first is a 3-form φ with $\varphi \wedge (-\frac{1}{2}d(x_0^2 - \ldots - x_3^2)) = dx_0 \wedge \ldots \wedge dx_3$. We can define this form φ locally in the points of $S_O - \{O\}$ in the $\mathbb{R}^4$. It is determined up to a multiple of $-\frac{1}{2}d(x_0^2 - \ldots - x_3^2)$. Then we take the inverse image by the embedding map on the 3–fold $S_O - \{O\}$. Doing this every multiple of $-\frac{1}{2}d(x_0^2 - \ldots - x_3^2)$ goes to 0. So the inverse image is well defined.

We shall prove:

Theorem 2.1. *We have*

$$\Phi^*(x_0, \ldots, x_3) = \frac{dx_1 \wedge \ldots \wedge dx_3}{r}$$

with $r = \| \ x \ \|'$.

Proof. Because everything is invariant under spatial rotations and PT- tranformations, we may assume $x_2 = x_3 = 0, \ x_1 > 0, \ x_0 > 0$. So we are in the case of $\mathbb{R}^2$. Then, we have $-\frac{1}{2}d(x_0^2 - x_1^2) = x_1 dx_1 - x_0 dx_0$ and $dx_0 \wedge dx_1$ is $\frac{dx_1}{x_0} \wedge (x_1 dx_1 - x_0 dx_0) = \frac{dx_1}{r} \wedge \left(-\frac{1}{2}d(x_0^2 - x_1^2)\right)$. That means $\Phi^* = \frac{dx_1 \wedge \ldots \wedge dx_3}{r}$.

Now, we define $S = \bigcup\limits_{y \in \mathbb{R}^4} S_y \times y \subset \mathbb{R}^8$. It is the union of all light cones. Then, S almost everywhere is a submanifold of codimension 1. We define Φ^* for all points $y \in \mathbb{R}^4$ instead of for O, only. Then, we obtain a positive volume form

$$\Phi = \frac{dx_1 \wedge \ldots \wedge dx_3}{r} dy_0 \wedge \ldots \wedge dy_3, \ r = \| \ x' - y' \ \|'$$

on the smooth submanifold S — the diagonal. It is a Radon measure in $\mathbb{R}^8$: If τ is a step function on a cuboid $Q \subset \mathbb{R}^8$, we put $\sum(\tau) = \int\limits_S \tau(x,y) \cdot \Phi(x,y)$. Of course, the support of this Radon measure is S.

Assume that $F : \mathbb{R}^4 \xrightarrow{\sim} \mathbb{R}^4$ is an invertible continuously differentiable map. Then, we put $\hat{F}(\tilde{x}, \tilde{y}) = \big(F(\tilde{x}), F(\tilde{y})\big) : \mathbb{R}^8 \xrightarrow{\sim} \mathbb{R}^8$. Also this map is invertible continuously differentiable. Then $\Phi \circ \hat{F}$ again is a Radon measure in $\mathbb{R}^8$.

Theorem 2.2. *The Radon measure Φ is Lorentz invariant. That means, we always have $\Phi \circ \hat{F} = \Phi$ for any Poincaré transformation F.*

Proof. We have $x = F(\tilde{x}) = L(\tilde{x}) + x_0$. In this L is a Lorentz transformation and x_0 is a constant vector. The translations do not change anything. Therefore, we may assume $x_0 = O$ and $y = \tilde{y} = O$ The jacobian of L is 1. So we get in $y = O$:

$$\begin{aligned} \Phi \circ \hat{F} &= \left(\frac{dx_1 \wedge \ldots \wedge dx_3}{r}\right) \circ L \wedge (dy_0 \wedge \ldots \wedge dy_3) \circ L \\ &= \frac{d\tilde{x}_1 \wedge \ldots \wedge d\tilde{x}_3}{r} \wedge d\tilde{y}_0 \wedge \ldots \wedge d\tilde{y}_3 = \Phi. \end{aligned}$$

Proposition 2.3. *The points x, y are on the same light ray if and only if $(x, y) \in S$ and that if and only if $\Phi(x, y) \neq 0$.*

Proof. Since the Radon measure is defined by integration on S and we have a positive volume element on S we get $\Phi(x, y) \neq 0$ if and only if $(x, y) \in S$.

Theorem 2.4. *Assume that $F : \mathbb{R}^4 \xrightarrow{\sim} \mathbb{R}^4$ is invertibly differentiable and that the map $\hat{F}$ leaves Φ invariant. Then F is a Poincaré transformation.*

Proof. We may again assume that $F(O) = O$. Now, $\tilde{x}, O$ are on the same light ray if and only if $\Phi(\tilde{x}, O) \neq 0$. Then the same is true for $x = F(\tilde{x}), O$. So, every light cone is mapped into a light cone. Assume that $\tilde{x}_1$ is on the line segment connecting O and $\tilde{x}$ and that $x_1 = F(\tilde{x}_1)$. Then x is in the intersection $F(S_O) \cap F(S_{\tilde{x}_1}) = S_O \cap S_{x_1}$. That means that also x_1 is in the line segment from O to x. Every light ray through O is mapped into a light ray through O. Then, for F this is true for any light ray. So

by a theorem of Zeeman (see Borchers and Hegerfeld: Nachr. Akad. Wissenschaften Göttingen 10) the map F is the composition of a Poincaré transformation with a homothety. But the homotheties change the measure Φ, if they are different from the identity. So F is a Poincaré transformation.

Corollary 2.5. *The Pseudo-Euclidean geometry is given by the Radon measure* Φ *as well as by the Minkowski metric.*

3 Geometric Interpretation: Fight the Continuum!

1. Light Vectors. Light vectors are those vectors of $\mathbb{R}^4$, which are in the set S. We assume that space-time is equipped with a set $\mathcal{L}$ of light vectors.

Exactly, if $(x, y) \in S$, we have for the measure $\Phi(x, y)) \neq 0$. So, we should see $\Phi(x, y)$ as a measure for the set of light vectors $\in \mathcal{L}$ running from y to x. More precisely, we have to take infinitesimally small neighborhoods: $dy_0 \wedge \ldots \wedge dy_3$ and $dx_1 \wedge \ldots \wedge dx_3$ and Φ should be the measure for the light vectors running from one neighborhood to the other. The measure must be constant for constant r. In this sense the distribution of the light vectors has to be homogeneous. But this will imply that our set $\mathcal{L}$ certainly is not measurable (see 2.4.6).

How can we solve this problem? The answer is: We have to make everything discrete. We have to assume that set of light vectors in $\mathcal{L}$ running form the first neighborhood to the second has a finite number dN (Here, at the moment, we should consider the 2 volumes as to be unit volumes, so being not infinitesimal). Then we get

$$dN = \gamma \cdot \frac{dx_1 \wedge \ldots \wedge dx_3}{r} \wedge dy_0 \wedge \ldots \wedge dy_3,$$

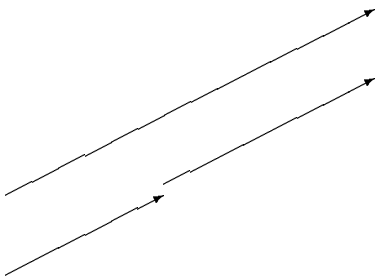
where γ is a very big measure factor. We could think about $\gamma = 10^{200}$. The measure

then is the number dN. It is also true in the infinitesimal case. It means something, which is very exact, certainly far beyond every possibility of measurement.

In the last part of the book we may assume that even the form for dN depends a little bit on the points x, y. It might be impossible to find a coordinate system such that for constant x_0 the form belonging to y and $x = y + x_0$ is independent of y. Then space–time $\mathbb{R}^4$ has a curvature. We are in the case of general relativity.

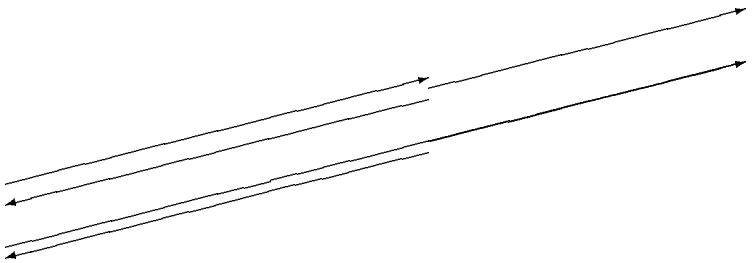
The number of vectors of our set going from one infinitesimal volume to another is proportional to $\frac{1}{r}$. So we have a well defined distribution in space -time. The question is: Is there a method which does produce it? We shall see that the method is a *stochastic process.* We shall describe it now:

1) Assume that v_1, v_2 are two lightvectors of same direction and that the second vector starts nearly where the first one ends. Then both vectors can be jont to a light vector $v_1 + v_2$. This has the same direction. Its length is the sum of the length'.



The stochastic process: junction

2) Assume that the vectors v_1, v_2 are of opposite direction and that the second vector v_2 ends nearly at the beginning of the first v_1 and is shorter than v_1. Then a division of v_1 into vectors v_1' and v_1'' may happen. The vector v_2 is still there. Also end and beginning can be interchanged.



The stochastic process: disjunction

The relative frequency of the stochastic processes will be given by the product of the densities of the set of those lightvectors which enter in the process. There is an equilibrium in this stochastic process, which consists of junction and disjunction. It is given by an integral equality. Unfortunately, this has more solutions, than that which we need, here. So the question is: What can be done? The answer again is in a discretization. We have to use the notion of elementary vector and that of elementary length $\epsilon > 0$. Moreover, we have to assume that locally there always is given a well determined reference (coordinate) system for this purpose. In this reference system every light vector v is the unique junction of $n, n \in \mathbb{N}$ elementary vectors of same direction (we have to assume that the "same direction" is defined!) All elementary vectors have the same Euclidean length in that reference system, always $\epsilon > 0$. This ϵ should be much larger than the Planck length of 10^{-35}m. However, it has to be much smaller than the length af any wave, which might appear by an elementary particle. Everything has to be smooth within the limits of measure possibilities. This follows from experience! We call ϵ the elementary length. It is an universal constant.

We assume that the junction and the disjunction respect the subdivision in elementary vectors and moreover that a translation of the result by a quantity into a positive time direction happens. This quantity is given by the elementary length. In the case of the junction 1) the vector v_1 (or the vector v_2) is in a translation in negative time direction against the other. The result $v_1 + v_2$ then is chained dicectly up to v_2, which will not be translated. In the case of the disjunction 2) we have to follow the drawing. Only

the result v_1' will be translated into a positive time direction.

The shift into positive time direction means that the stochastical time equals our usual time. Our usual time has a direction, therefore. There also is a conservation law: The number of the elementary light vectors in the whole space is constant.

By our necessary discretization the integral equation is transformed into a system of very very many quadratic equations. Nevertheless, it can be proved that it has the correct solution, only. See: H. Grauert, Selected Papers, Vol. II, Part X. Springer Heidelberg 1994 and S. Leykum, Ein Satz aus der statistischen Geometrie. Thesis 1980, University of Goettingen.

The stochastics are somewhat different from the usual. Here we have to apply the law of big numbers in writable (and not infinite) time intervals. If the equilibrium is disturbed, it always will be restored in such intervals. However, in infinite intervals there must be a degeneration. – We see that in our theory time is somewhat more than a coordinate. The Becoming has an essential role.

It is easily possible to play the stochastical process on a computer. This comes from the fact that we may assume that there is a maximal number n of elementary light vectors, which can be jont. The distribution of the light vectors with a number not bigger than n then will be the same as in the case $n = \infty$. If, e.g. we take $n = 5$ and a set of 32 elementary light vectors it will be very easy to acchieve degeneration. But already, if we have about 100 light vectors this will become difficult. The equilibrium always will be restored.

There is some literature on these things:

1. H. Grauert: Statistische Geometrie. Ein Versuch zur geometrischen Deutung physikalischer Felder. Nachr. Akad. Wiss. Gttingen Math.-Phys. Kl. **2** (1976).

2. H. Grauert: Statistical geometry and spacetime. Commun. Math. Phys. **49** (1976).

3. H. Grauert and S. Leykum: Die pseudoeuklidische Geometrie als statistisches Gleichgewicht . Math. Ann. **255** (1981).

4. H. Gauert: Selected Papers. Volume II, p. 907-913. Springer Heidelberg 1994.

2. The Clifford Algebra. From now on we shall idealize the geometry given by the set $\mathcal{L}$. We pass with the elementary length $\epsilon \to 0$ and with the constant $\gamma \to \infty$. Moreover, this means that we do so as if we had a continuum everywhere. But we keep the distribution of light vectors in $\mathcal{L}$ unchanged (up to the necessary gauge factor). So our L defines the Minkowski geometry. This is not logically possible, but nevertheless a physicist can do it.

We take 16 light vectors out of our system, which have equal length in a suitable reference system and some parallelity. Using such vectors we can form vortices of dimension $4, \ldots, 0$. We shall consider fields of very, very many of such vortices, such that the fields become practically smooth. Such fields are "currents" of complementary dimension, that means dual real exterior forms. If we consider only differentiable transformations, which do not change the orientation of space–time the currents are identical with the exterior forms. In the 1950^{th} the famous Cobalt-60-experiment was done. By this it was proved that space–time has an orientation. We may restrict ourselves to transformations without change of the orientation, therefore. So our currents are exterior forms.

We put $\gamma_\nu = dx_\nu$ for $\nu = 0, \ldots, 3$. Then, we define $\gamma_\nu \cdot \gamma_\mu = \gamma_\nu \wedge \gamma_\mu = -\gamma_\mu \cdot \gamma_\nu$ if $\nu \neq \mu$ and $\gamma_0^2 = -1$ and $\gamma_\nu^2 = 1$ for $\nu = 1, \ldots, 3$ and obtain the so called inner product. We consider the forms

$$\varphi = \sum_{n=0}^{4} \sum_{0 \leq \iota_1 < \ldots < \iota_n \leq 3} a_{\iota_1 \ldots \iota_n} \gamma_{\iota_1} \cdot \ldots \cdot \gamma_{\iota_n}, \; a_{\iota_1 \ldots \iota_n} \in \mathbb{C}$$

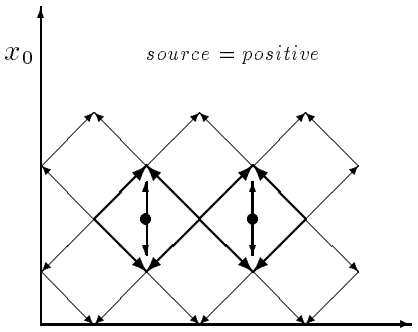
and obtain a Clifford algebra (See: B.L. van der Waerden: Algebra, Vol. II, §93.5. Heidelberger Taschenbuch 23. Springer Heidelberg 1967). The inner product comes from the quadratic form $x_0^2 - x_1^2 - \ldots - x_3^2$. So, if we transform φ like an exterior form,

we get that the Clifford algebra $\mathcal{C} = \{\varphi\}$ is Lorentz invariant. As said already we use complex coefficients.

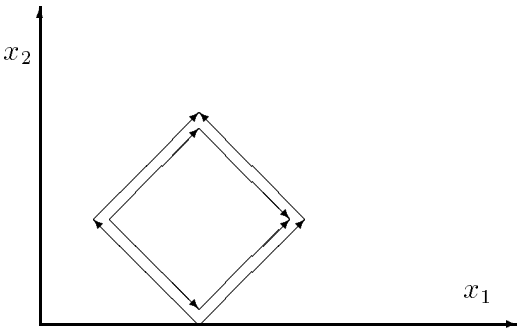
In relativistic quantum physics we shall replace the spinors by our φ. This goes back to E. Khler: Der innere Differentialkalkl (see: Abh. Math. Sem. Hamburg **25**, 192-205 (1962) and Rendiconti di Matematica **21**, 425-523 (1962)). However, a φ has 16 components and the spinors have just 4. We define $\gamma = \gamma_0 \cdot \ldots \gamma_3$. Then, we get $\gamma^2 = -1$ and for $\sigma = -i \cdot \gamma$ we obtain $\sigma^2 = 1$. The $*$–operator is well known for exterior forms. It is invariant under those Lorentz transformations, which preserve the orientation. We see immediately that $*\varphi = \varphi \cdot \sigma$. We shall call the forms φ self dual, if $*\varphi = \varphi$. We shall use self dual φ, only.

So, from now on, we always assume that φ is self dual. Then, for $a \in \mathbb{R}$ we get $*ia\gamma_{\iota_1} \cdot \ldots \cdot \gamma_{\iota_n} = ia \cdot (-i)\gamma_{\iota_1} \cdot \ldots \cdot \gamma_{\iota_n} \cdot \gamma = a\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_{4-n}}$. So, the imaginary part of a monomial $\gamma_{\iota_1} \cdot \ldots \cdot \gamma_{\iota_n}$ is the real part of the dual monomial. The imaginary parts are superfluous. Therefore in general, we shall write φ with real coefficients. We have 16 real = 8 complex components. This number still is twice as high as in the case of spinors. Later on, we shall prove using the stochastical equilibrium that the reason for this is that the world consists of electric matter and a completely unelectric kind.

3. The Dirac Equation. To define the vortices in $\mathbb{R}^4$ we first have to do it in $\mathbb{R}^2$. We take 4 light vectors in $\mathbb{R}^2$ and form a square, which stands on a vertex. If the orientation of the vectors is suitable, we get vortices $\pm 1, \pm\gamma_0, \pm\gamma_1 \pm \gamma_0 \cdot \gamma_1$. Then, a vortex in $\mathbb{R}^4$ is constructed by linking vortices in 2 dimensional planes. We shall consider lattices of vortices of the same kind. In the beginning we do it in $\mathbb{R}^2$.

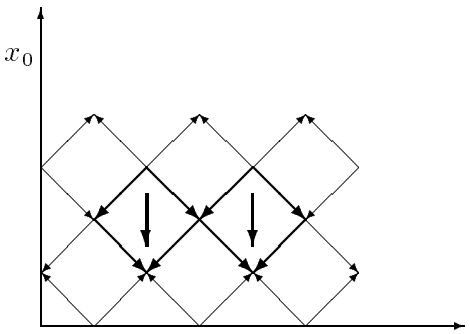


Dirac equation for γ_1



Disjunction of the vortices into its vectors by spin

First we have the γ_1-vortices in the upper and the lower line. The particle has to have a spin in x_3-direction. If this is $\frac{1}{2}$ and the particle is simple, we must have $i\gamma_1 \cdot \gamma_2 \cdot \varphi = \varphi$, if we have choosen the complex style (to write the wave function). We assume this. We can obtain the vortices in the upper and the lower line also by linking a x_2-stream ($= \gamma_1$-vortex) in the (x_1, x_2)-plane with a 2-dimensional vortex in the (x_0, x_3)- plane. Then, because of the spin there are linkings of x_1-streams with (x_0, x_3)-vortices in the middle. These cause a division of the x_2-streams into its vectors. And these can junct again in the middle. We get lattices of linkings of sources in the (x_0, x_1)-plane with 2-dimensional vortices in the (x_2, x_3)-plane, that means a lattice of $\gamma_0\gamma_1$-vortices.



Dirac equation for $\gamma_0\gamma_1$

Assume, conversely, that $\gamma_0\gamma_1$-vortices are given in the upper and lower line. Then, in the same way, a field of γ_1-vortices is formed in the middle.

We shall consider very dense fields φ, which are the superposition of many lattices sucht that exact sinus–vibrations are obtained. We denote the wave length by $l = \frac{1}{m}$ and by $D = \gamma_0 \frac{\partial}{\partial x_0} + \ldots + \gamma_3 \frac{\partial}{\partial x_3}$ the Dirac operator. In the drawn case, and as it can be seen rather easily, also in general the Dirac equation $D\varphi = m \cdot \varphi$ is valid. So we got a geometric deduction of this equation. It is important that for every lattice we have the spin lattice, which has to be determined very exactly. So 4 real components are fixed.

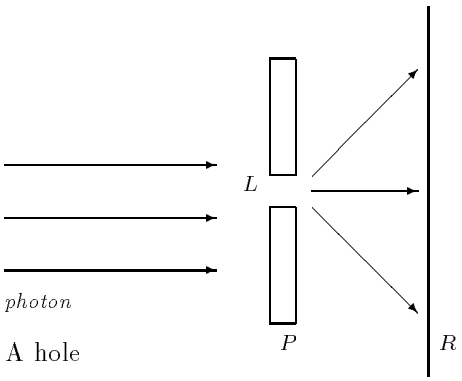
Also the imaginary part is essential for the particle. Its components are given by the complementary monomials. The imaginary part decides if we have a particle or an anti-particle. But, what is the proper sense of the imaginary part? The real part and the imaginary part consist of lattices with four components. It can happen that at a given time two such lattices go together at exactly one place, as is the case in a vernier scale. In time this place may move through the particle wave even with a speed exceeding the speed of light. There are reasons to presume that at such places the particle wave always is generated. So we shall speak of generating points. Outside of these generating points an attenuation will occur, since the recombination, which has to take place by the Dirac equation, not always will be successful. The wave itself leads the movement of the generating points. So we get propositions on the probability of the place of residence of the particle. This place always is a generating point, we can measure nothing else, since the action of two particles onto each other is a tensor product in generating points.

If there are two particles, it can happen that the lattices link in generating points, or better that they generate a united wave there. First, that means a real tensor product (in each point). If for instance $\varphi = \sin(\omega_1 \cdot t) \cdot \gamma_0$ and $\psi = \cos(\omega_2 t) \cdot \gamma_0$, so $\varphi \otimes \psi = \sin(\omega_1 t) \cdot \cos(\omega_2 \cdot t)\gamma_0 \otimes \gamma_0 = \frac{1}{2} \sin\Big((\omega_1 + \omega_2)t\Big)\gamma_0 \otimes \gamma_0$ will be generated. But, space-time is oriented. There is a maximal field of 4-dimensional vortices. Using this, we can give reasons for a complex product as it is necessary in physics. Only the components in even dimension are used. Also the time direction is essential. The opinion that an anti-particle moves into negative time direction is a total nonsense. It moves in ordinary direction. All this means that the tensor product is Hermitian for anti particles. We

get the conversation of energy–momentum.

4. Experiments for Quantum Philosophy. During the last 15 years in many countries experiments were carried out, which show that the curiosities of classical quantum physics really exist. In the early period after the invention of quantum physics these curiosities were explained using the epistemology, especially the Mach positivism. It should be clear that modern thinking requires that this should be done by the methods of science, only.

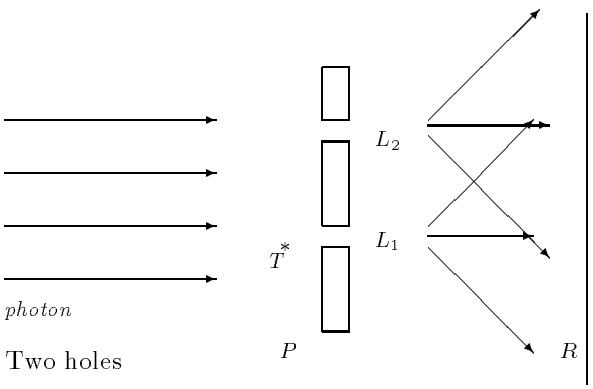
1. Arrangement of Experiments. Assume that P is a panel with a hole L and that R is a screen. When photons are sent to the panel from the left side, some of them will move through the channel L and will produce a pattern of interference on R. It is so as if the photons started from L.



Explanation. A photon will move through L only, if the generating center entered at L. The massive part of the panel then deadens the wave in P. In L a wave for guidance of the genrating center will be produced. This guidance wave prevents the generating centers to enter into P (We can say that P is a sleeping watchdog). So the particle moves through the channel and finally spreads from the end of L.

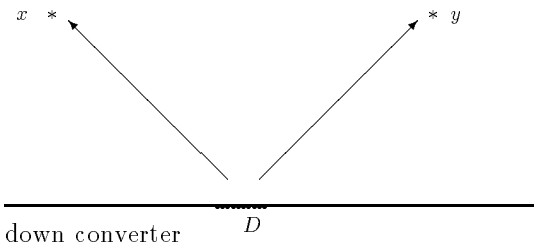
2. Arrangement of Experiments. Now the panel P has 2 holes L_1, L_2. The photons, which moved from the left side onto the panel P, then generate a pattern of interference on the screen R, as if every photon would spread from L_1 and L_2 simultaneously. Now,

assume that T is a particle very near to L_1 such that the photons can react very weakly with T. A photon will not be altered by this in a measurable sense. But we get knowledge of the place of its residence. If we restrict to photons, which did react with T, then the interference is so as if the wave had spread from L_1, alone.



Explanation. As weak the reaction with T is, we get a tensor product with T. This decays again and the wave spreads from T.

3. Arrangement of Experiments. Assume that D is a "down converter" and that here a linear polarized particle decays into two linear polarized photons whose energy is $\frac{1}{2}$. Then the direction of the polarization of the two photons will be indetermined. But by classical quantum physics their sum has to be 0. Experiments do confirm this: When you measure the direction of polarization of the first photon in a point x, then the second photon does know it immediately, even if it is in a far point y, and the sum of both directions of polarization is 0.



Explanation. Every particle has a lattice structure. There are the maximal lattices, whose wave length is that of the particle. That produces a certain rigidity in the whole system. The maximal lattices of each particle are connected with each other via D. They form a double lattice. So we get a tensor product T_{xy} of a lattice of the first particle in a generating point x with a latice of the second particle in a generating point y under the condition that both lattices belong to the same double lattice. The measurement uses lattice points G_1, G_2 of particles in x, y. A tensor product of T_{xy} with (G_1, G_2) will be formed. This is really a tensor product with a pair. It gives a generating point of a wave in x and a generating point of a wave in y and by this we obtain two values in the measurement of direction of polarization, one in x, the other in y. Because of the condition for the lattices we also get the relation for the directions of polarization.

In classical quantum physics the situation of the two particles is described by one wave function $\varphi(x, y, t)$. For a fixed t the function $\varphi(x, y, t)$ practically is the tensor product of a wave $\varphi_1(x)$ and a wave $\varphi_2(y)$. But in the classical description x, y are made to one point (x, y). In our theory, of this is given account by the fact that both lattices belong to the same double lattice.

Our discrete waves are very distinct from the Schroedinger waves. In the case of a free particle both waves will be totally different. But the Schroedinger wave will give an average. In the case of electrons around the kernel of a atom the difference will be less

strict. But the discrete wave will be very restless in a stochastic sense. So it might jump from one state into another. But we may consider our discrete waves as to be real. For the Schroedinger waves this would be idiotic. So Einstein called that theory (in German) "hirnrissig".

The world is a chaos. But the stochastical processes respect some laws. Therefore, M. Gellmann speaks of an *adaptive chaos.*– Of course, the smooth number space $\mathbb{R}^4$ is superfluous. It just serves for visuability. All the geometry is given by the system $\mathcal{L}$ of arrows. If we look at distances smaller than the elementary length, probably, we can describe the situation by a narrow tube around something which is 4–dimensional (within the possibilities of mesurement). This could lead to string theory.

4 Elementary Particles.

1. Basis Waves. Assume that φ is a wave, which satisfies the Dirac equation with a mass m. Then φ is a superposition of lattices. This superposition depends on the energy–momentum and the points of $\mathbb{R}^4$. The distribution of lattices is given by φ.

Definition 4.1. *1) In the case $m \neq 0$ the wave φ is called a basis wave, if there are $m = 1$, the energy $= 1$, the momentum $= 0$ and the spin can be given by a vector in positive x_3-direction.*

2) In the case $m = 0$ the wave φ is a basis wave, if energy–momentum equals $(1, 1, 0, 0)$ and the spin again is in positive x_3-direction.

Every wave is superposition of transforms of basis waves. These always are homogeneous waves. It is assumed that energy–momentum is sharp. For every elementary particle the basis wave is well defined.

First we shall discuss the basis wave of simple particles with $m \neq 0$ and spin $= \frac{1}{2}$. We choose the complex style:

I)

$$\varphi = e^{ix_0} \cdot [\gamma_1 - i\gamma_2 + i\gamma_0\gamma_1 + \gamma_0\gamma_2 + i\gamma_0\gamma_2\gamma_3 - \gamma_0\gamma_1\gamma_3 + \gamma_2\gamma_3 + i\gamma_1\gamma_3]$$

II)

$$\psi = e^{ix_0}[1 + i\gamma_1\gamma_2 + i\gamma_0 - \gamma_0\gamma_1\gamma_2 - i\gamma_0\gamma_1\gamma_2\gamma_3 - \gamma_0\gamma_3 - \gamma_1\gamma_2\gamma_3 + i\gamma_3].$$

If we consider φ there are always streams into direction of a vector v of the (x_1, x_2)-plane and orthogonal to it 2-dimensional vortices in that plane, which is spanned by the orthogonal to v in the (x_1, x_2)-plane and by x_3. These generate 3-dimensional vortices in the (x_1, x_2, x_3)-plane. That is a current $a \cdot \gamma_0$ and the potential of an electric field.

If we take the basis wave ψ, no such electric field is generated. – If we would use the terminology of spinors, we would have only one basis wave. Here we get two of them and they are very different.

2. Interpretation. In classical quantum theory in a point $x \in \mathbb{R}^4$ spinors have just 4 complex components. The set of spinors is an irreducible representation of the Lie group $SU(2)$, which is simply connected. This group is the 2-sheeted universal covering of the Lie group $SO(2)$. Instead of the spinors we use here the Clifford algebra. It is a representation of the $SO(2)$. But this representation is not irreducible. It is the direct sum of two representations of the complex dimension 4. That is related to the two basis waves φ und ψ. We can say that the representation cannot be irreducible since there is electric matter and total unelectric matter (like the neutrinos).

If a particle has a larger mass in rest and is sufficiently stable this should come from the electricity. The wave φ should be the basis wave of the electron. There will be an instable equlibrium between the wave and the electric field. But the particle will send energy into the vacuum. On the other hand there are virtual particles in the vacuum. These are particles, which have a generating center, but whose wave is not developed. They have an energy and might bring them to the particle. Probably, they are photons. So an equilibrium will be produced. It will bring the latitude of the electron. Of course this latitude comes from the elementary length and the density of $\mathcal{L}$.

A particle in rest with wave ψ should not exist.

3. Particles of Mass 0. We accelerate ψ in x_3-direction going towards the velocity of light. We obtain:

$$\psi_I = e^{i(x_0 - x_3)} \cdot [-i\gamma_0 + \gamma_0\gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_3 - i\gamma_3]$$

with spin $\frac{1}{2}$ and

$$\psi_{II} = e^{i(x_0 - x_3)} \cdot [1 - i\gamma_1\gamma_2 - i\gamma_0\gamma_1\gamma_2\gamma_3 + \gamma_0\gamma_3]$$

with spin $-\frac{1}{2}$. The anti-particles have the opposite spin. So the spin is given by direction of spreading. We have the so called helicity. It is possible to see that the waves ψ_I belong to the electron neutrinos and the ψ_{II} the μ-neutrinos. The neutrinos are not invariant under parity transfomations.

We can do the same thing in the case of the electron. We obtain:

$$\varphi_I = e^{i(x_0 - x_3)}[\gamma_1 - i\gamma_2 + i\gamma_0\gamma_2\gamma_3 - \gamma_0\gamma_1\gamma_3]$$

and

$$\varphi_{II} = e^{i(x_0 - x_3)}[i\gamma_0\gamma_1 - \gamma_0\gamma_2 + \gamma_2\gamma_3 - i\gamma_1\gamma_3].$$

But the case $m = 0$ can occur only, if the electric field vanishes. So we have to form the tensor product of φ_I and the anti-particle of φ_{II}. We obtain a neutral particle with spin ± 1. That is the photon. In this case we have particle = anti-particle. There is no helicity, any longer.

4. The μ-Particle. We shall define the basis wave of an electric particle with spin $\frac{1}{2}$. We define $\psi = \varphi_1 + *\varphi_1$ with $\varphi_1 = (x_1 - ix_2)\gamma_1 + (ix_1 + x_2)\gamma_2 + i(x_1 - ix_2)\gamma_0\gamma_1 + i(ix_1 + x_2)\gamma_0\gamma_2$. This (quasi-) particle is circular symmetric with respect to x_1, x_2. Its spin is $\frac{1}{2}$. But it satisfies the time part of the Dirac operator only. The rest is given by $D' = \frac{\partial}{\partial x_1}\gamma_1 + \ldots + \frac{\partial}{\partial x_3}\gamma_3$. So we have $D = \frac{\partial}{\partial x_0}\partial_0 + D'$. The wave $D'\varphi_1$ consists of $\varphi_2 = 2 + 2i\gamma_1\gamma_2$ and $\varphi_3 = -2i\gamma_0 + 2\gamma_0\gamma_1\gamma_2$. The spin is $\frac{1}{2}$, too.

The whole situation can be interpreted so: ψ is an electron with spin $-\frac{1}{2}$, which is moved with spin 1 around the origin of the $(x_1, x_2)-$plane. Then, there is an interaction of the wave in all points symmetric to O. This is a force, which produces the large mass of the particle in rest.

ψ should really give an electron and $D'\psi$ its motion in space. Unfortunately, the Dirac equation is not satisfied for $\psi + D'\psi$. But it is for $\psi + D'\psi - \frac{1}{2}D'\psi$. This means that we have to take the tensor product at φ_2 with a μ-neutrino (or anti neutrino) and at φ_3 with an e-neutrino (or an e-anti–neutrino). Both must have the spin $\frac{1}{2}$. One of the neutrinos has to be a particle, the other one an anti particle. We have to add these two particles. In the decay we will get back all three parts. The electron will move into one x_3-direction, the neutrinos both into the opposite. Then, it can be seen that the e-neutrino must be an anti-particle. So, the kind of decay will be determined in our theory.