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Numerische Simulation auf massiv parallelen Rechnern

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Optimal preconditioners for the p-Version of the fem

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Abstract

In this paper, we consider domain decomposition preconditioners for a system of linear algebraic equations arising from the p-version of the fem. We propose several multi-level preconditioners for the Dirichlet problems in the sub-domains in two and three dimensions. It is proved that the condition number of the preconditioned system is bounded by a constant independent of the polynomial degree. The proof uses interpretations of the p-version element stiffness matrix and mass matrix on $[-1, 1]$ as h-version stiffness matrix and weighted mass matrix. The analysis requires wavelet methods.

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1 Introduction

1.1 Formulation of the problem

We consider the boundary value problem

$$
\begin{aligned}
-\triangle u &= f \quad \text{in} \quad \Omega_1, \\
u &= 0 \quad \text{on} \quad \Gamma_1, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \Gamma_2,
\end{aligned}
\tag{1}
$$

where $\Omega_1 \subset \mathbb{R}^2$ is a domain which can be decomposed into (straight-line) quadrilaterals and $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega_1$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. The weak formulation of this problem is:
Find $u \in H_0(\Omega_1) := \{u \in H^1(\Omega_1), u \mid_{\Gamma_1} = 0\}$ such that

$$
a_\triangle(u, v) := \int_{\Omega_1} u_x v_x + u_y v_y = \int_{\Omega_1} f v \quad \forall v \in H_0(\Omega_1) \tag{2}
$$

holds. Problem (1) will be discretized by means of the p-version of the finite element method using quadrilaterals R_s. Let $\mathcal{R}_2 = (-1, 1)^2$ be the reference element and $\Phi_s : \mathcal{R}_2 \to R_s$ be the bilinear mapping to the element R_s. We define the finite element space

$$
\mathbb{M} := \{u \in H_0(\Omega_1), u \mid_{R_s} = u(\Phi_s(\xi, \eta)) = \tilde{u}(\xi, \eta), \tilde{u} \in \mathbb{Q}_p\},
$$

where $\mathbb{Q}_p$ is the space of all polynomials $p(\xi, \eta) = p_1(\xi)p_2(\eta)$ of maximal degree p in each variable. Now, the discretized problem can be formulated: Find $u^p \in \mathbb{M}$ such that

$$
a_\triangle(u^p, v^p) = \int_{\Omega_1} f v^p \quad \forall v^p \in \mathbb{M} \tag{3}
$$

holds. Let $(\psi_1, \ldots, \psi_{n_p})$ be a basis of $\mathbb{M}$. Then, problem (3) is equivalent to solving the system of algebraic finite element equations

$$
A_p \underline{u}_p = \underline{f}_p, \tag{4}
$$

where

$$
\begin{aligned}
A_p &= [a_\triangle(\psi_j, \psi_i)]_{i,j=1}^{n_p}, \\
\underline{u}_p &= [u_i]_{i=1}^{n_p}, \\
\underline{f}_p &= \left[\int_{\Omega_1} f \psi_i\right]_{i=1}^{n_p}.
\end{aligned}
$$

Then, $u^p = \sum_i u_i \psi_i$ is the solution of (3). We are interested in finding an efficient solver for the system of linear algebraic equations (4).

1.2 Domain decomposition

Domain decomposition techniques, [19], [17], [18], [16], [15], are efficient iterative methods in order to solve linear systems of algebraic equations of the type (4). The approximation space $\mathbb{M}$ will be split into a direct sum $\mathbb{M} = \mathbb{M}_1 \oplus \ldots \oplus \mathbb{M}_k$. The efficient preconditioner

$$\mathcal{C}^{-1} = \sum_{i=1}^{k} V_i (V_i^T A_p V_i)^{-1} V_i^T$$

can be built, where V_i is the matrix representation of the orthogonal projection $\mathbb{M} \mapsto \mathbb{M}_i$ with respect to the energetic scalar product $a_\triangle(\cdot,\cdot)$.
For our purpose, we have to choose $k = 3$. The corresponding spaces are defined as follows:

- $\mathbb{M}_1 = \mathbb{M}_{vert}$ is the space of the vertex functions which are the usual piecewise bilinear functions of the h-version of the finite element method,
- $\mathbb{M}_2 = \mathbb{M}_{edg}$ is the space of the edge bubble functions,
- $\mathbb{M}_3 = \mathbb{M}_{int}$ is the space of the interior bubbles which are nonzero on one element only.

An edge bubble function corresponds to an edge e of the mesh. Its support is formed by those two elements which have this edge e in common. Corresponding to this splitting of the shape functions, the matrix A_p is split analogously into sub-blocks,

$$A_p = \begin{bmatrix} A_{vert} & A_{vert,edg} & A_{vert,int} \\ A_{edg,vert} & A_{edg} & A_{edg,int} \\ A_{int,vert} & A_{int,edg} & A_{int} \end{bmatrix}. \tag{5}$$

The indices $vert$, edg and int denote the blocks corresponding to the vertex, edge bubble and interior bubble functions, respectively. Jensen and Korneev, [13], and Ivanov and Korneev, [11], [12], developed preconditioners for the p-version of the finite element method in a two-dimensional domain using domain decomposition techniques, [1]. They proposed the preconditioning matrix

$$C_p = \begin{bmatrix} A_{vert} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{edg} & A_{edg,int} \\ \mathbf{0} & A_{int,edg} & A_{int} \end{bmatrix} \tag{6}$$

corresponding to the splitting $\mathbb{M}_{vert} \oplus (\mathbb{M}_{edg} \oplus \mathbb{M}_{int})$ which is considered in a first step. This splitting is nearly stable as the following lemma confirms.

LEMMA 1.1. *The condition number* $\kappa\left(C_p^{-1} A_p\right)$ *grows as* $(1 + \log p)$.

Proof: The proof can be found in [11], Lemma 2.3. □
Therefore, the vertex unknowns can be determined separately. Efficient solution methods are direct solvers in the case of the p-version of the fem, if the matrix A_{vert} is small, or multi-grid

methods, [10], in the hp-version. However, the splitting $\mathbb{M}_{edg} \oplus \mathbb{M}_{int}$ is not stable. Therefore, we can proceed as follows. The sub-block corresponding to $\mathbb{M}_{edg}$ and $\mathbb{M}_{int}$ is factorized as

$$\begin{bmatrix} A_{edg} & A_{edg,int} \\ A_{int,edg} & A_{int} \end{bmatrix} = \begin{bmatrix} I & A_{edg,int}A_{int}^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \hat{S} & \mathbf{0} \\ \mathbf{0} & A_{int} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ A_{int}^{-1}A_{int,edg} & I \end{bmatrix}$$

with the Schur complement

$$\hat{S} := A_{edg} - A_{edg,int}A_{int}^{-1}A_{int,edg}.$$

Thus for $\mathbb{M}_{int}$, the subproblem restricted to this space has to be solved, whereas for $\mathbb{M}_{edg}$ a modified problem is considered. The matrix A_{int} corresponds to the interior bubbles having a support containing one element only. Therefore, the matrix A_{int} is a block diagonal matrix, where each block corresponds to one element. Hence, in order to compute the interior unknowns, we have to solve a Dirichlet problem on each quadrilateral. The edge unknowns are computed via the Schur complement $\hat{S}$ and multiplications with the matrix $\begin{bmatrix} I \\ -A_{int}^{-1}A_{int,edg} \end{bmatrix}$ and its transpose. So, in addition to a solver for A_{vert}, three tools are required to define a preconditioner for the matrix of (6), namely

- a preconditioner for the interior problem,
- a preconditioner for the Schur complement $\hat{S}$ and
- an extension operator from the edges of a quadrilateral into its interior in order to replace the matrix $A_{int}^{-1}A_{int,edg}$.

Ivanov and Korneev, [11], [12], derived some preconditioners $C_{\hat{S}}$ for the Schur complement. The condition number of $C_{\hat{S}}^{-1}\hat{S}$ is $\mathcal{O}(1+\log^2 p)$ in the worst case, where p is the polynomial degree. The solution of $C_{\hat{S}}\underline{x} = \underline{y}$ can be done by solving triangular systems and fast Fourier transform, [8]. The problem of the extension operator was investigated by Babuška et. al, [1].
We focus now on a fast solver for $A_{int} = \mathrm{blockdiag}\,[A_{R_s}]_s$, where A_{R_s} is that block of the stiffness matrix A_{int} which corresponds to the element R_s. The following lemma is valid.

LEMMA 1.2. *Let $\partial R_s \in C^{(t)}$, $t \geq 2$, where $C^{(t)}$ denotes the class of all boundaries which consist of a finite number of t times continuously differentiable curves and the angles of these curves at their intersection points on ∂R_s are distinct from 0 and 2π. Then, $\kappa\left(A_{R_s}{}^{-1}A_{\mathcal{R}_2}\right) = \mathcal{O}(1)$, where $A_{\mathcal{R}_2} = (-1,1)^2$.*

Proof: The proof can be found in [13], Lemma 4.2. □
Hence, it is sufficient to investigate the matrix $A_{\mathcal{R}_2}$ in order to find a good preconditioner for A_{int}. This will be done in the next sections.

1.3 Properties of the element stiffness matrix

Let $d = 2$ be the dimension of the domain. By Lemma 1.2,

$$\begin{array}{rcll} -\triangle u & = & f & \text{in} \quad \mathcal{R}_d = (-1,1)^d, \\ u & = & 0 & \text{on} \quad \partial\mathcal{R}_d \end{array} \tag{7}$$

is the typical model problem in order to solve the system

$$A_{int}\underline{x} = \underline{y}$$

of linear algebraic finite element equations. Problem (7) will be investigated in the case $d = 3$ as well. Problem (7) is solved by the $p-$version of the finite element method with one element $\mathcal{R}_d$ only. As finite element space,

$$\mathbb{M}_p = \left\{ \begin{array}{ll} H_0^1(\mathcal{R}_2) \cap \operatorname{span}\{\phi_{ij}(x,y)\}_{i,j=0}^p & \text{for} \quad d = 2, \\ H_0^1(\mathcal{R}_3) \cap \operatorname{span}\{\phi_{ijk}(x,y,z)\}_{i,j,k=0}^p & \text{for} \quad d = 3 \end{array} \right.$$

is chosen, where $\phi_{ij}(x,y) = x^i y^j$ and $\phi_{ijk}(x,y,z) = x^i y^j z^k$, respectively. The discrete problem is: Find $u^p \in \mathbb{M}_p$ such that

$$\int_{\mathcal{R}_d} \nabla u^p \cdot \nabla v^p = \int_{\mathcal{R}_d} f v^p \quad \forall v^p \in \mathbb{M}_p. \tag{8}$$

In order to define a basis in $\mathbb{M}_p$, we choose tensor products of the integrated Legendre polynomials $\hat{L}_i$ [5]. More precisely, let

$$\begin{array}{rcll} \hat{L}_{ij}(x,y) & = & \hat{L}_i(x)\hat{L}_j(y) & 0 \le i,j \le p, \\ \hat{L}_{ijk}(x,y,z) & = & \hat{L}_i(x)\hat{L}_j(y)\hat{L}_k(z) & 0 \le i,j,k \le p. \end{array}$$

Since $\hat{L}_i(\pm 1) = 0$ for $i \ge 2$,

$$\mathbb{M}_p = \operatorname{span}\{\hat{L}_{ij}(x,y)\}_{i,j=2}^p$$

for $d = 2$ and

$$\mathbb{M}_p = \operatorname{span}\{\hat{L}_{ijk}(x,y,z)\}_{i,j,k=2}^p$$

for $d = 3$. The stiffness matrix $A_{\mathcal{R}_2}$ for (8) (with $d = 2$) is given by $A_{\mathcal{R}_2} = [a_{ij,kl}]_{i,j=2;k,l=2}^p$, where

$$a_{ij,kl} = \int_{\mathcal{R}_2} \nabla \hat{L}_{ij}(x,y) \cdot \nabla \hat{L}_{kl}(x,y) \, \mathrm{d}(x,y). \tag{9}$$

Analogously, the matrix $A_{\mathcal{R}_3}$ is defined. The matrices $A_{\mathcal{R}_d}$ can be written explicitely as

$$\begin{array}{rcl} A_{\mathcal{R}_2} & = & F \otimes D + D \otimes F \quad \text{and} \\ A_{\mathcal{R}_3} & = & F \otimes F \otimes D + F \otimes D \otimes F + D \otimes F \otimes F, \end{array} \tag{10}$$

where the matrices F and D are the one-dimensional mass matrix and stiffness matrix in the basis of the integrated Legendre polynomials $\{\hat{L}_i(x)\}_{i=2}^p$, i.e.

$$\begin{aligned} F &= \left[\int_{-1}^{1} \hat{L}_i(x)\hat{L}_k(x)\,\mathrm{d}x\right]_{i,k=2}^p, \\ D &= \left[\int_{-1}^{1} \hat{L}_i'(x)\hat{L}_k'(x)\,\mathrm{d}x\right]_{i,k=2}^p. \end{aligned}$$

Then, a simple calculation shows that the matrix F is a penta-diagonal matrix with the main diagonal $\mathfrak{e} = [1, 1, \ldots, 1]^T$, the first sub-diagonal $[0, \ldots, 0]^T$ and the second sub-diagonal $\mathfrak{p} = \left[-\frac{1}{2}\sqrt{\frac{(2i-3)(2i+5)}{(2i-1)(2i+3)}}\right]_{i=2}^{p-2}$ and that D is a diagonal matrix with the main diagonal $\mathfrak{d} = \left[\frac{(2i-3)(2i+1)}{2}\right]_{i=2}^p$. More precisely, one obtains

$$\begin{aligned} F &= \mathrm{pentdiag}[\mathfrak{e}, \mathbf{0}, \mathfrak{p}], \\ D &= \mathrm{diag}[\mathfrak{d}], \end{aligned} \tag{11}$$

cf. [13]. A reordering $\tilde{P}$ of the rows and columns of the matrices F and D gives

$$\tilde{P}F\tilde{P}^T = \begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix}, \tag{12}$$

where $F_1 = \mathrm{tridiag}[\mathfrak{e}, \mathfrak{p}_o]$ and $F_2 = \mathrm{tridiag}[\mathfrak{e}, \mathfrak{p}_e]$. Analogously, with the same permutation $\tilde{P}$, one easily derives

$$\tilde{P}D\tilde{P}^T = \begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix}, \tag{13}$$

where $D_1 = \mathrm{diag}[\mathfrak{d}_o]$ and $D_2 = \mathrm{diag}[\mathfrak{d}_e]$. The indices o and e denote the odd and even components of the vectors $\mathfrak{p}$ and $\mathfrak{d}$. The matrix $A_{\mathcal{R}_2}$ has some important properties which we summarize in a proposition.

PROPOSITION 1.3. *Let*

$$G_{2i+j-2} = F_j \otimes D_i + D_j \otimes F_i, \quad i, j = 1, 2. \tag{14}$$

Then, the following assertions are valid.

1. *There exists a permutation P_2 of rows and columns such that*

$$P_2 A_{\mathcal{R}_2} P_2^T = \mathrm{blockdiag}\,[G_i]_{i=1}^4 \tag{15}$$

holds.

2. *The matrices G_i, $i = 1, 2, 3, 4$ are sparse.*

3. *Moreover, each block G_i has a 5-point stencil structure.*

4. *The condition number of G_i is of order p^2.*

5. *The blocks G_i are spectrally equivalent to each other, i.e. $\kappa\left({G_i}^{-1}G_j\right) = \mathcal{O}(1)$ for $i,j = 1,\ldots,4$.*

Proof: This proposition is proved in [13]. □
Similar results are valid for $A_{\mathcal{R}_3}$. We introduce the matrices

$$H_{4i+2j+k-6} = F_i \otimes F_j \otimes D_k + F_i \otimes D_j \otimes F_k + D_i \otimes F_j \otimes F_k \tag{16}$$

for $i,j,k = 1,2$. Using similar arguments as in Proposition 1.3, the next proposition follows.

PROPOSITION 1.4. *There exists a permutation P_3 of rows and columns such that*

$$P_3 A_{\mathcal{R}_3} P_3^T = \text{blockdiag}\left[H_i\right]_{i=1}^{8}$$

holds. The blocks H_i are spectrally equivalent to each other, i.e. $\kappa\left({H_i}^{-1}H_j\right) = \mathcal{O}(1)$ for all $i,j = 1,\ldots,8$.

In the following, we will focus on finding efficient preconditioners for G_1, and H_1. Via Propositions 1.3 and 1.4, the preconditioners for $A_{\mathcal{R}_d}$, $d = 2,3$ can be constructed. For reasons of simplicity, we assume that p is odd. Furthermore, let $n-1 = \frac{p-1}{2}$ be the dimension of F_1, and D_1.
In order to define a preconditioner for G_1, we introduce the matrices

$$\begin{aligned} T_2 &= \frac{1}{2}\text{tridiag}\left[2\mathfrak{e}, -\mathfrak{e}\right], \\ D_4 &= 4\,\text{diag}\left[\mathfrak{q}\right] \end{aligned} \tag{17}$$

with $\mathfrak{q} = \left[i^2 + \frac{1}{6}\right]_{i=1}^{n-1}$ and

$$K_k = T_2 \otimes D_4 + D_4 \otimes T_2. \tag{18}$$

LEMMA 1.5. *Let $n-1$ be the dimension of the matrices. The eigenvalue estimates*

$$\begin{aligned} \lambda_{min}\left({D_1}^{-1}D_4\right) &\geq c, \qquad \lambda_{max}\left({D_1}^{-1}D_4\right) \leq c, \\ \lambda_{min}\left({T_2}^{-1}F_1\right) &\geq c, \qquad \lambda_{max}\left({T_2}^{-1}F_1\right) \leq c(1+\log n) \end{aligned}$$

are valid.

Proof: The assertion has been proved in [2]. □
By tensor product arguments, the estimate $\kappa\left({K_k}^{-1}G_1\right) \preceq (1+\log p)$ follows. Analogously, we introduce the matrix $K_{k,3} = D_4 \otimes T_2 \otimes T_2 + T_2 \otimes D_4 \otimes T_2 + D_4 \otimes T_2 \otimes T_2$. This matrix is a preconditioner for H_1 with $\kappa\left({K_{k,3}}^{-1}H_1\right) \preceq (1+\log p)^2$. The matrices K_k and $K_{k,3}$ can be interpreted as piecewise linear, bilinear, or trilinear finite element discretization matrices of degenerated problems on regular tensor product like meshes, [3], [14].
The corresponding system $K_k \underline{u} = \underline{f}$ can be solved by a preconditioned conjugate gradient method with

- an Algebraic Multi-Level Iteration (AMLI) preconditioner $\hat{K}_{k,2} = L_k$, [6],
- a multi-grid preconditioner $\hat{K}_{k,2} = M_k$, [5],
- a wavelet preconditioner $\hat{K}_{k,2} = W_{k,2}$, [7].

In the corresponding papers, [6], [5], [7], the condition number estimates $\kappa\left(\hat{K}_{k,2}^{-1}K_k\right) \leq c$ has been shown for $\hat{K}_{k,2} = L_k$, $\hat{K}_{k,2} = M_k$, and $\hat{K}_{k,2} = W_{k,2}$.
In order to solve $K_{k,3}\underline{w} = \underline{r}$, efficient preconditioners are derived by wavelets bases, [7].
Using Propositions 1.3 and 1.4, preconditioners $\hat{\mathcal{K}}_{k,d} = P_d \text{blockdiag}\left[\hat{K}_{k,d}\right]_{i=1}^{2^d} P_d^T$, for $A_{\mathcal{R}_d}$, $d = 2, 3$, follow. Since $\kappa\left({K_k}^{-1}G_1\right) \preceq (1 + \log p)$ and $\kappa\left({K_{k,3}}^{-1}H_1\right) \preceq (1 + \log p)^2$, we can prove the result $\kappa\left(\hat{\mathcal{K}}_{k,d}^{-1}A_{\mathcal{R}_d}\right) \preceq (1 + \log p)^{d-1}$ only, [6], [7]. The solution of $\hat{\tilde{K}}_{k,d}\underline{w} = \underline{r}$ is arithmetically optimal. In [14], a preconditioner $\tilde{C}$ for $A_{\mathcal{R}_2}$ is derived with $\kappa\left(\tilde{C}^{-1}A_{\mathcal{R}_2}\right) \leq c$. However, the solution of $\tilde{C}\underline{w} = \underline{r}$ requires $\mathcal{O}(p^2(1 + \log p))$ floating point operations.
The aim of this paper is to prove the stronger estimate $\kappa\left(\hat{\mathcal{K}}_{k,d}^{-1}A_{\mathcal{R}_d}\right) \leq c$ for the mentioned preconditioners with $\hat{K}_{k,2} = L_k$, $\hat{K}_{k,2} = M_k$, and $\hat{K}_{k,2} = W_{k,2}$, where the constant c is independent of p for $d = 2, 3$. The key in order to obtain this result is the proof of the stronger estimate $\lambda_{max}\left({T_2}^{-1}F_1\right) \leq c$. We will conclude this result as a corollary of the multi-resolution weighted norm equivalences given in [7]. Moreover, the preconditioning operation $\hat{\mathcal{K}}_{k,d}^{-1}\underline{w} = \underline{r}$ can be applied in $\mathcal{O}(p^d)$ operations.
This paper is organized as follows. In section 2, we define preconditioners D_5 for D_1 and T_3 for F_1. In section 3, we show that D_5 and T_3 can be interpreted as finite element discretization matrices of several auxiliary problems. In section 4, we start with a short motivation of the purpose of wavelet preconditioners. Furthermore, we define wavelet preconditioners for D_5 and T_3. The main condition number estimates are proved. Finally, we show $\lambda_{max}\left({T_2}^{-1}F_1\right) \leq c$ and strengthen the condition number estimates for the matrix $(\hat{\mathcal{K}}_{k,d})^{-1}A_{\mathcal{R}_d}$ given in [6], [5], [7]. In section 5, a numerical example is given.

2 Preconditioners for the one-dimensional mass and stiffness matrix

The aim of this section is to derive preconditioners T_3 for F_1 (12) and D_5 for D_1 (13) such that the condition numbers of $T_3^{-1}F_1$ and $D_5^{-1}D_1$ are bounded by a constant c independent of the dimension of the matrix. The resulting matrices T_3 and D_5 will be interpreted as piecewise linear finite element discretizations of auxiliary bilinear forms, see section 3. For this purpose, the matrices

$$D_3 = 4\,\text{diag}\left[\mathfrak{p}\right], \tag{19}$$

$$D_5 = \text{tridiag}[\mathfrak{b}, \mathfrak{a}] \tag{20}$$

with $\mathfrak{p} = [i^2]_{i=1}^{n-1}$, $\mathfrak{a} = \left[i^2 + i + \frac{3}{10}\right]_{i=1}^{n-2}$, and $\mathfrak{b} = \left[4i^2 + \frac{2}{5}\right]_{i=1}^{n-1}$ are introduced.

LEMMA 2.1. *Let $n-1$ be the dimension of the matrices. The eigenvalue estimates*

$$\lambda_{min}\left({D_1}^{-1}D_i\right) \geq c, \qquad \lambda_{max}\left({D_1}^{-1}D_i\right) \leq c, \quad i = 3, 5, \tag{21}$$

are valid. Moreover, let

$$T_1 = D_3^{-1} + T_2. \tag{22}$$

Then, $\lambda_{min}\left({T_1}^{-1}F_1\right) \geq c$ and $\lambda_{max}\left({T_1}^{-1}F_1\right) \leq c$ hold, where c is independent of n.

Proof: The proof of (21) for $i = 3$ has been given in [2]. In order to prove (21) in the case $i = 5$, we refer to [4]. The assertion $\kappa\left({T_1}^{-1}F_1\right) \leq c$ has been proved in [13]. □
In previous papers, [5], [6], [7], the matrix T_2 (17) is used as preconditioner for F_1 (12). This matrix has a simple interpretation as the discretization of the one-dimensional Laplacian, see section 3. For T_1 (22), or D_3^{-1}, such an interpretation is not obvious in the case of finite element discretizations. However, we are able to prove $\kappa\left({T_1}^{-1}F_1\right) = \mathcal{O}(1)$ in comparison to the old result $\kappa\left({T_2}^{-1}F_1\right) \preceq \mathcal{O}(1 + \log n)$ given in [2]. Now, we introduce a tridiagonal matrix D_6 from which we will show that $\kappa\left(D_3D_6\right) \leq c$. Let

$$D_6 = \mathrm{tridiag}[\mathfrak{h}, \mathfrak{r}], \tag{23}$$

where

$$\begin{aligned} \mathfrak{h} &= \frac{1}{2}\left[(j-1)\ln(j-1) - (j+1)\ln(j+1) + 2\ln(j) + 2\right]_{j=1}^{n-1}, \\ \mathfrak{r} &= \frac{1}{4}\left[-2 + (2j+1)\ln(j+1) - (2j+1)\ln(j)\right]_{j=1}^{n-2}. \end{aligned}$$

For reasons of simplicity, the undefined value "$0 \ln 0$" is 0 by definition. It will be shown in the next section that the matrix D_6 can be interpreted as a weighted mass-matrix.
The following result is valid.

LEMMA 2.2. *The condition number of D_3D_6 is bounded by a constant independent of n, i.e. $\kappa\left(D_3D_6\right) \leq c$.*

Proof: The proof is similar to the proof of Lemma 2.1 in [4]. More precisely, we determine the entries of the symmetric tridiagonal matrix

$$H_2 = [h_{ij}^{(2)}]_{i,j=1}^{n-1} = D_3^{\frac{1}{2}} D_6 D_3^{\frac{1}{2}}$$

and take Gerschgorin disks. Then, one easily checks

$$\begin{aligned} h_{jj}^{(2)} &= 4j^2 + 4j^2\ln j + 2j^2(j-1)\ln(j-1) - 2j^2(j+1)\ln(j+1), \\ h_{j+1,j}^{(2)} = h_{j,j+1}^{(2)} &= (2j+1)j(j+1)\ln(j+1) - (2j+1)j(j+1)\ln j - 2j(j+1). \end{aligned}$$

One easily verifies that $h_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. Moreover, we obtain

$$\begin{aligned} h^{(2)}_{j,j-1} + h^{(2)}_{jj} + h^{(2)}_{j,j+1} &= 2\left(j^2 \ln \frac{j^2-1}{j^2} + j \ln \frac{j+1}{j-1}\right), \\ -h^{(2)}_{j,j-1} + h^{(2)}_{jj} - h^{(2)}_{j,j+1} &= 2\left(5j^2 \ln \frac{j^2}{j^2-1} + 8j^2 + j \ln \frac{j-1}{j+1} + 4j^2 \ln \frac{j-1}{j+1}\right) \end{aligned}$$

for $j \geq 2$. The function $f : (1, \infty) \mapsto \mathbb{R}$,

$$f(x) = 2\left(x^2 \ln \frac{x^2-1}{x^2} + x \ln \frac{x+1}{x-1}\right)$$

is monotonic decreasing for $x \geq 2$. It attains its maximum on $[2, \infty)$ at $x = 2$, where

$$\max_{x \in [2,\infty)} f(x) = f(2) = 12 \ln 3 - 16 \ln 2. \tag{24}$$

The function $g : (1, \infty) \mapsto \mathbb{R}$,

$$g(x) = 2\left(5x^2 \ln \frac{x^2}{x^2-1} + 8x^2 + x \ln \frac{x-1}{x+1} + 4x^2 \ln \frac{x-1}{x+1}\right)$$

is monotonic decreasing for $x \geq 2$ and satisfies

$$\inf_{x \in [2,\infty)} g(x) = \lim_{x \to \infty} g(x) = \frac{2}{3}, \tag{25}$$

which is its infimum on the interval $[2, \infty)$. Moreover, by a direct calculation, the relations $h^{(2)}_{11} = 4 - 4 \ln 2$ and $h^{(2)}_{12} = 6 \ln 2 - 4$ are valid. Thus,

$$h^{(2)}_{11} + h^{(2)}_{12} = 2 \ln 2, \tag{26}$$

$$h^{(2)}_{11} - h^{(2)}_{12} = 8 - 10 \ln 2. \tag{27}$$

By (24) and (26), the lower eigenvalue estimate

$$\lambda_{min}\left(D_3 D_6\right) \leq 12 \ln 3 - 16 \ln 2$$

follows. By (25) and (27), one obtains the upper eigenvalue estimate

$$\lambda_{max}\left(D_3 D_6\right) \leq \frac{2}{3}$$

which proves the lemma. □

Now, we introduce the matrix

$$T_3 = D_6 + T_2. \tag{28}$$

Then by Lemma 2.2, the following conclusion can be drawn.

COROLLARY 2.3. *The matrix* $T_1 = D_3^{-1} + T_2$ *is spectrally equivalent to the matrix* T_3*, i.e.* $\kappa\left({T_1}^{-1} T_3\right) \leq c$.

Proof: Use Lemma 2.2 and the fact that D_3, D_6 and T_2 are symmetric and positive definite matrices. □

3 Interpretation of the preconditioners

In the previous section, several preconditioners for the matrices F_1 and D_1, cf. (12) and (13) are derived. In this chapter, we show that these preconditioners can be interpreted as matrices resulting from the discretization of several one dimensional auxiliary problems. Consider the following problem:
Find $u \in H_0^1((0,1)) \cap L^2_\omega((0,1)) \cap L^2_{\omega^{-1}}((0,1))$ such that

$$a_1(u,v) = \langle u', v' \rangle + \langle u, v\rangle_\omega + \langle u, v\rangle_{\omega^{-1}} = \langle g, v\rangle \tag{29}$$

holds for all $v \in H_0^1((0,1)) \cap L^2_\omega((0,1)) \cap L^2_{\omega^{-1}}((0,1))$, where

$$\langle u, v\rangle_\omega = \int_0^1 \omega^2(x)u(x)v(x)\,\mathrm{d}x.$$

The weight function is specified later. This one-dimensional problem (29) is discretized by linear finite elements on the equidistant mesh

$$T_k = \bigcup_{i=0}^{n-1} \tau_i^k,$$

where

$$\tau_i^k = \left(\frac{i}{n}, \frac{i+1}{n}\right).$$

The parameter k denotes the level number. On this mesh, we introduce the one-dimensional hat-functions

$$\phi_i^{(1,k)}(x) = \begin{cases} nx-(i-1) & \text{on} \quad \tau_{i-1}^k \\ (i+1)-nx & \text{on} \quad \tau_i^k \\ 0 & \text{else} \end{cases}, \quad i = 1,\dots,n-1, \tag{30}$$

where $n = 2^k$. Let $\mathbb{V}_k^{(1)} = \mathrm{span}\{\phi_i^{(1,k)}\}_{i=1}^{n-1}$ be the corresponding finite element space. Then, the Galerkin projection of (29) onto $\mathbb{V}_k^{(1)}$ is:
Find $u^k \in \mathbb{V}_k^{(1)}$ such that

$$a_1(u^k, v^k) = \langle g, v^k\rangle \quad \forall v^k \in \mathbb{V}_k^{(1)}. \tag{31}$$

Then using (17), we obtain

$$T^\phi_{\omega=1} := \left[\langle (\phi_j^{(1,k)})', (\phi_i^{(1,k)})'\rangle\right]_{i,j=1}^{n-1} = 2n\, T_2 = n\,\mathrm{tridiag}[2\mathfrak{e}, -\mathfrak{e}]. \tag{32}$$

Moreover, an easy calculation shows

$$M^\phi_{\omega=x} := \left[\langle \phi_j^{(1,k)}, \phi_i^{(1,k)}\rangle_{\omega=x}\right]_{i,j=1}^{n-1} = \frac{1}{6n^3} D_5 \tag{33}$$

and

$$M^{\phi}_{\omega=x^{-1}} = \left[\langle \phi_j^{(1,k)}, \phi_i^{(1,k)} \rangle_{\omega=x^{-1}}\right]_{i,j=1}^{n-1} = 4nD_6. \tag{34}$$

By (32), (34), and (28), one checks

$$\left[2\langle (\phi_j^{(1,k)})', (\phi_i^{(1,k)})' \rangle + \langle \phi_j^{(1,k)}, \phi_i^{(1,k)} \rangle_{\omega=x^{-1}}\right]_{i,j=1}^{n-1} = 4nT_3. \tag{35}$$

Hence, interpretations of the matrices $T_2 \in \mathbb{R}^{n-1\times n-1}$ (17), $T_3 \in \mathbb{R}^{n-1\times n-1}$ (28), $D_5 \in \mathbb{R}^{n-1\times n-1}$ (19) and $D_6 \in \mathbb{R}^{n-1\times n-1}$ (23) have been given.

4 Wavelet preconditioners

In section 3, we have considered a finite element discretization of an auxiliary problem in one space dimension. The discretization matrices $T_3 \in \mathbb{R}^{n-1\times n-1}$ (28), and $D_5 \in \mathbb{R}^{n-1\times n-1}$ (20) are preconditioners for the matrices F_1 (12) and D_1 (13).
Due to (10), we primarily are interested in finding fast solvers for tensor products of the matrices T_3 and D_5, or F_1 and D_1. Moreover, we propose preconditioners for the element stiffness matrices of the p-version of the fem, $A_{\mathcal{R}_2}$ and $A_{\mathcal{R}_3}$ (10) such that the condition numbers of the preconditioned systems are bounded by a constant independent of the polynomial degree p.

4.1 1D case, motivation

We consider problem (29) with the weight function $\omega(\xi)$. The matrices D_5, and D_6 corresponding to the mass parts $\langle\cdot,\cdot\rangle_\omega$, and $\langle\cdot,\cdot\rangle_{\omega^{-1}}$ of the bilinear form $a_1(\cdot,\cdot)$ (29) are spectrally equivalent to the diagonal matrix D_3 (19), and its inverse D_3^{-1}, cf. Lemma 2.1, and Lemma 2.2. However, for the matrix $T_2 \in \mathbb{R}^{n-1\times n-1}$ corresponding to the stiffness part in the bilinear form $a_1(\cdot,\cdot)$, it does not exist a diagonal matrix $D \in \mathbb{R}^{n-1\times n-1}$ such that the condition number of $D^{-1}T_2$ is bounded by a constant independent of the dimension $n-1$. Let $\{\phi_i^{(1,l)}\}_{(i,l)\in\hat{I}_k}$ be the hierarchical basis, see [21], on level k. The index set $\hat{I}_k$ is given by

$$\hat{I}_k = \left\{(i,l) \in \mathbb{N}^2, 1 \le l \le k, i = 2m-1, 1 \le m \le 2^{l-1}, m \in \mathbb{N}\right\}.$$

Let

$$T^{\phi,h}_{\omega=1} = \left[\langle (\phi_i^{(1,l)})', (\phi_j^{(1,l')})' \rangle_{\omega=1}\right]_{(j,l'),(i,l)\in\hat{I}_k}$$

be the matrix corresponding to the stiffness part of the bilinear form (29) with respect to the hierarchical basis $\{\phi_i^{(1,l)}\}_{(i,l)\in\hat{I}_k}$. Then, by a simple calculation, the matrix $T^{\phi,h}_{\omega=1}$ is a diagonal matrix. More precisely, one obtains

$$\langle (\phi_i^{(1,l)})', (\phi_j^{(1,l')})' \rangle_{\omega=1} = 2^l \delta_{ll'}\delta_{ij}.$$

Thus, we have found a basis in which the stiffness part of the bilinear form $a_1(\cdot,\cdot)$ is spectrally equivalent to a diagonal matrix. However, a diagonal matrix D is not known such that the mass matrix

$$M_\omega^{\phi,h} = \left[\langle \phi_i^{(1,l)}, \phi_j^{(1,l')}\rangle_\omega\right]_{(j,l'),(i,l)\in \hat{I}_k}$$

with respect to the hierarchical basis satisfies the condition number estimate $\kappa\left(D^{-1}M_\omega^{\phi,h}\right) < c$ independently of the dimension of the matrices.

Consider (29) with the weight function $\omega(x) = 1$. In the wavelet theory, see e.g. [9], [20], it is known that it can be constructed a basis $\{\psi_j^l\}_{l\le k}$ with $\operatorname{span}\{\psi_j^l\}_{l\le k} = \operatorname{span}\{\phi_i^{(1,k)}\}_{i=1}^{n-1}$ such that the matrices

$$\begin{aligned} M_{\omega=1}^{\psi} &= \left[\langle \psi_{j'}^{l'}, \psi_j^l\rangle_{\omega=1}\right]_{(j,l),(j',l')} \quad \text{and} \\ T_{\omega=1}^{\psi} &= \left[\langle (\psi_{j'}^{l'})', (\psi_j^l)'\rangle_{\omega=1}\right]_{(j,l),(j',l')} \end{aligned}$$

are spectrally equivalent to diagonal matrices. More precisely, let $D_{M_{\omega=1}^\psi}$ be the identity matrix I and $D_{T_{\omega=1}^\psi} = \operatorname{diag}[\mathfrak{u}]$, where $\mathfrak{u} = \left[2^{2l}\right]_{(j,l)}^T$. Then, see [9], [20], there

$$\kappa\left((D_{M_{\omega=1}^\psi})^{-1}M_{\omega=1}^\psi\right) = \mathcal{O}(1), \tag{36}$$

$$\kappa\left((D_{T_{\omega=1}^\psi})^{-1}T_{\omega=1}^\psi\right) = \mathcal{O}(1) \tag{37}$$

holds. These facts can be used to derive a preconditioner for $T_{\omega=1}^\phi$ and $M_{\omega=1}^\phi$. Let Q be the basis transformation from the nodal basis $\{\phi_i^{(1,k)}\}_{i=1}^{2^k-1}$ to the wavelet basis $\{\psi_j^l\}_{l\le k}$. Then,

$$T_{\omega=1}^\psi = Q^T T_{\omega=1}^\phi Q.$$

By $\kappa\left((D_{T_{\omega=1}^\psi})^{-1}T_{\omega=1}^\psi\right) = \mathcal{O}(1)$, the condition number estimates

$$\kappa\left((D_{T_{\omega=1}^\psi})^{-1}Q^T T_{\omega=1}^\phi Q\right) = \mathcal{O}(1) \quad \Longleftrightarrow \quad \kappa\left(Q(D_{T_{\omega=1}^\psi})^{-1}Q^T T_{\omega=1}^\phi\right) = \mathcal{O}(1)$$

are valid. Similarly, $\kappa\left(Q(D_{M_{\omega=1}^\psi})^{-1}Q^T M_{\omega=1}^\phi\right) = \mathcal{O}(1)$ is valid. Thus, we have found preconditioners for $T_{\omega=1}^\phi$, and $M_{\omega=1}^\phi$.

In (31), we consider the case of the singular weight functions $\omega(x) = x$ and $\omega(x) = \frac{1}{x}$. In [7], the result $\kappa\left(Q(D_{M_\omega^\psi})^{-1}Q^T M_\omega^\phi\right) = \mathcal{O}(1)$ is shown under certain assumptions concerning the weight function ω and the wavelets ψ_j^l which we formulate now.

ASSUMPTION 4.1. *The nonnegative weight function $\omega(x)$ is assumed to belong to the space $W^{1,\infty}((\delta,1))$ for every $\delta > 0$ and to satisfy*

$$C_\omega^{-1} \le \frac{\omega(x)}{x^\alpha} \le C_\omega, \quad C_\omega^{-1} \le \frac{\omega'(x)}{x^{\alpha-1}} \le C_\omega,$$

for some $C_\omega > 0$ and some $\alpha \in \mathbb{R}$.

Here and in the following, C_ω denotes a generic positive constant depending only on the weight function $\omega(x)$ which can take different values in different places. The parameter α will be specified in the next assumption. At the boundary $x = 0$, we consider the following kind of multiresolution spaces.

ASSUMPTION 4.2. $\psi_k^l \in \mathbb{W}^0 \subset W^{1,\infty}((0,1))$ *with* $0 \in \operatorname{supp} \psi_k^l$ *satisfies*

$$|\psi_k^l(x)| \leq C_\psi 2^{l/2}(2^l x)^\beta, \quad |(\psi_k^l)'(x)| \leq C_\psi 2^{3l/2}(2^l x)^{\beta-1}, \tag{38}$$

for $x \in [0, 2^{-l}]$, $\beta \in \mathbb{N} \cup \{0\}$. *We assume that* $\alpha + \beta > -\frac{1}{2}$, *or, equivalently,* $2\alpha + 2\beta + 1 > 0$.

ASSUMPTION 4.3. *We suppose that there exists also a biorthogonal, or dual, Riesz basis*

$$\tilde{\Psi} = \operatorname{span}\left\{\tilde{\psi}_j^l\right\}$$

such that $\langle \tilde{\psi}_j^l, \psi_{j'}^{l'} \rangle = \delta_{j,j'}\delta_{l,l'}$ *and every* $v \in L^2((0,1))$ *has a representation*

$$v = \sum_{l=1}^{\infty} \sum_j \langle v, \psi_j^l \rangle \tilde{\psi}_j^l = \sum_{l=1}^{\infty} \sum_j \langle v, \tilde{\psi}_j^l \rangle \psi_j^l$$

and that the norm equivalences

$$\|v\|_0^2 \sim \sum_{l=1}^{\infty} \sum_j |\langle v, \psi_j^l \rangle|^2 \sim \sum_{l=1}^{\infty} \sum_j |\langle v, \tilde{\psi}_j^l \rangle|^2$$

$$\|v\|_1^2 \sim \sum_{l=1}^{\infty} 2^{2l} \sum_j |\langle v, \tilde{\psi}_j^l \rangle|^2$$

hold.

In [7], Theorem 3.3, it has been proved the equivalence of the L^2_ω norm of a function

$$u = \sum_{l=1}^{\infty} \sum_j u_j^l \psi_j^l \in L^2_\omega\left((0,1)\right)$$

with its discrete l^2_ω norm of the coefficients $u_j^l \in \mathbb{R}$, i.e.

$$|||u_j^l|||_w^2 := \sum_{l=1}^{\infty} \sum_j \omega^2(2^{-l}j)|u_j^l|^2.$$

THEOREM 4.4. *Let us assume that the Assumptions 4.1, 4.2, and 4.3 are satisfied. Let* $\| \, u \, \|_\omega^2 = \langle u, u \rangle_\omega$. *For any* $u = \sum_{l=1}^{\infty} \sum_j u_j^l \psi_j^l \in L^2_\omega\left((0,1)\right)$ *holds*

$$\| \, u \, \|_\omega^2 \approx |||u_j^l \, |||_w^2 \, .$$

Let us introduce the diagonal matrices

$$D_{T^{\psi}_{\omega=1}} = \mathrm{diag}[\mathfrak{v}] \qquad \text{with} \qquad \mathfrak{v} = \left[2^{2l}\right]^{T}_{(j,l)} \quad \text{and}$$
$$D_{M^{\psi}_{\omega}} = \mathrm{diag}[\mathfrak{t}] \qquad \text{with} \qquad \mathfrak{t} = \left[\omega^2(2^{-l}j)\right]^{T}_{(j,l)}.$$

COROLLARY 4.5. *Let* $\omega(\xi) = \xi^{\alpha}$, $\alpha = \pm 1$. *Let* $\{\psi^l_j\}_{l\le k} \subset \mathbb{V}_k$ *be a wavelet basis with* $\{\psi^l_j\}_{l\le k} \in W^{1,\infty}((0,1))$ *and* $\{\tilde{\psi}^l_j\}_{l\le k} \in W^{1,\infty}((0,1))$. *The basis transformation from the nodal basis* $\{\phi^{(1,k)}_i\}^{2^k-1}_{i=1}$ *to the wavelet basis* $\{\psi^l_j\}_{l\le k}$ *is denoted by* Q. *Then, the following eigenvalue estimates are valid:*

$$\lambda_{min}\left(Q\left(D_{T^{\psi}_{\omega=1}}\right)^{-1} Q^T T^{\phi}_{\omega=1}\right) \ge c, \qquad \lambda_{max}\left(Q\left(D_{T^{\psi}_{\omega=1}}\right)^{-1} Q^T T^{\phi}_{\omega=1}\right) \le c,$$
$$\lambda_{min}\left(Q\left(D_{M^{\psi}_{\omega}}\right)^{-1} Q^T M^{\phi}_{\omega}\right) \ge c, \qquad \lambda_{max}\left(Q\left(D_{M^{\psi}_{\omega}}\right)^{-1} Q^T M^{\phi}_{\omega}\right) \le c,$$

where c *is a constant independent of* n.

Proof: Use Theorem 4.4. Since the functions $\phi^{1,k}_i$ are piecewise linear and $u(0) = 0$ for all $u \in \mathbb{V}_k$, the relation $\beta = 1$ holds at the boundary $x = 0$, cf. Assumption 4.2. Thus, the assumptions of Theorem 4.4 are satisfied for $\omega(x) = x^{\alpha}$ with $\alpha = \pm 1$. Hence, we have

$$\lambda_{min}\left(\left(D_{T^{\psi}_{\omega=1}}\right)^{-1} T^{\psi}_{\omega=1}\right) = \mathcal{O}(1), \qquad \lambda_{max}\left(\left(D_{T^{\psi}_{\omega=1}}\right)^{-1} T^{\psi}_{\omega=1}\right) = \mathcal{O}(1),$$
$$\lambda_{min}\left(\left(D_{M^{\psi}_{\omega}}\right)^{-1} M^{\psi}_{\omega}\right) = \mathcal{O}(1), \qquad \lambda_{max}\left(\left(D_{M^{\psi}_{\omega}}\right)^{-1} M^{\psi}_{\omega}\right) = \mathcal{O}(1).$$

Since $T^{\psi}_{\omega} = Q^T T^{\phi}_{\omega} Q$ and $M^{\psi}_{\omega} = Q^T M^{\phi}_{\omega} Q$, the assertion has been proved. □

4.2 Applications for the p-version of the fem in two and three dimensions

In this subsection, we define wavelet preconditioners $\mathcal{W}_{k,d}$ for the matrices $A_{\mathcal{R}_d}$, $d = 2, 3$, with $\kappa\left(\mathcal{W}_{k,d}{}^{-1} A_{\mathcal{R}_d}\right) \le c$. Note that $A_{\mathcal{R}_d}$, or equivalently, G_1, and H_1, are tensor products of the matrices F_1 (13) and D_1 (12), cf. (14) and (16). Thus, we propose wavelet preconditioners for F_1 and D_1 in a first step.
Due to Corollary 4.5, Lemma 2.1, Lemma 2.2 and (35), we propose the preconditioner

$$F_3^{-1} = Q\left(2D_{T^{\psi}_{\omega=1}} + D_{M^{\psi}_{\omega=x^{-1}}}\right)^{-1} Q^T$$

for F_1. The estimate $\kappa\left(F_3{}^{-1} F_1\right) = \mathcal{O}(1)$ is valid. The preconditioner

$$F_4^{-1} = \frac{1}{2} Q\left(D_{T^{\psi}_{\omega=1}}\right)^{-1} Q^T \tag{39}$$

can be chosen as well as F_3 and is simpler than F_3. Here, the matrix $2D_{T^{\psi}_{\omega=1}} + D_{M^{\psi}_{\omega=x^{-1}}}$ is replaced by the matrix $2D_{T^{\psi}_{\omega=1}}$.

LEMMA 4.6. *The estimate* $\lambda_{min}\left(\left(D_{M^{\psi}_{\omega=x^{-1}}}\right)^{-1} D_{T^{\psi}_{\omega=1}}\right) = 1$ *is satisfied.*

Proof: Note that

$$\begin{aligned} D_{M_{\omega=x^{-1}}} &= \operatorname{diag}[\mathfrak{t}] \quad \text{with} \quad \mathfrak{t} = [\omega^2(2^{-l}j)]_{(j,l)} = [2^{2l}j^{-2}]_{(j,l)} \quad \text{and} \\ D_{T_{\omega=1}} &= \operatorname{diag}[\mathfrak{v}] \quad \text{with} \quad \mathfrak{v} = [2^{2l}]_{(j,l)}. \end{aligned}$$

Thus, $(D_{M_{\omega=x^{-1}}})^{-1} D_{T_{\omega=1}} = \operatorname{diag}[\mathfrak{c}]$ with $\mathfrak{c} = [j^2]_{(j,l)}$. Since $j \geq 1$, each entry of the diagonal matrix $\operatorname{diag}[\mathfrak{c}]$ is bounded by 1 from below. Hence, the assertion follows. □
A direct consequence of this lemma is the following corollary.

COROLLARY 4.7. *Let Q denote the basis transformation from the nodal basis $\{\phi_i^{(1,k)}\}_{i=1}^{2^k-1}$ to the wavelet basis $\{\psi_j^l\}_{l\leq k}$. Then, the condition number estimates*

$$\begin{aligned} \kappa\left(Q(D_{M^{\psi}_{\omega=x}})^{-1}Q^T D_1\right) &< c \quad \textit{and} \\ \kappa\left({F_4}^{-1}F_1\right) &< c \end{aligned} \tag{40}$$

are valid, where c is independent of p.

Proof: The first assertion follows from Corollary 4.5, (33) and (21). In order to prove (40), we have

$$\begin{aligned} (F_1\underline{v},\underline{v}) \asymp \left((D_3^{-1}+T_2)\underline{v},\underline{v}\right) &\asymp \frac{1}{n}\left((M^{\phi}_{\omega=x^{-1}}+T^{\phi}_{\omega=1})\underline{v},\underline{v}\right) \\ &\asymp \frac{1}{n}\left(Q^{-T}(D_{M^{\psi}_{\omega=x^{-1}}}+D_{T^{\psi}_{\omega=1}})Q^{-1}\underline{v},\underline{v}\right) \\ &\asymp \frac{1}{n}\left(Q^{-T}D_{M^{\psi}_{\omega=x^{-1}}}Q^{-1}\underline{v},\underline{v}\right) \asymp \frac{1}{n}\left(F_4\underline{v},\underline{v}\right) \end{aligned}$$

by Lemma 2.1, (32), (34), Corollary 4.5, Lemma 4.6 and (39) for all $\underline{v}$. □
Now, preconditioners for G_1 and H_1 are proposed, i.e.

$$\begin{aligned} W_{2,k} &= 2\left(Q^{-T}\otimes Q^{-T}\right)\left(D_{T^{\psi}_{\omega=1}}\otimes D_{M^{\psi}_{\omega=x}} + D_{M^{\psi}_{\omega=x}}\otimes D_{T^{\psi}_{\omega=1}}\right)\left(Q^{-1}\otimes Q^{-1}\right), \\ W_{3,k} &= 4\left(Q^{-T}\otimes Q^{-T}\otimes Q^{-T}\right) \\ &\quad \left(D_{T^{\psi}_{\omega=1}}\otimes D_{T^{\psi}_{\omega=1}}\otimes D_{M^{\psi}_{\omega=x}} + D_{T^{\psi}_{\omega=1}}\otimes D_{M^{\psi}_{\omega=x}}\otimes D_{T^{\psi}_{\omega=1}}\right. \\ &\quad \left.+D_{M^{\psi}_{\omega=x}}\otimes D_{T^{\psi}_{\omega=1}}\otimes D_{T^{\psi}_{\omega=1}}\right) \\ &\quad \left(Q^{-1}\otimes Q^{-1}\otimes Q^{-1}\right). \end{aligned} \tag{41}$$

LEMMA 4.8. *The condition number estimates $\kappa\left({W_{k,2}}^{-1}G_1\right) \leq c$ and $\kappa\left({W_{k,3}}^{-1}H_1\right) \leq c$ are valid.*

Proof: Use Corollary 4.7 and the properties of the Kronecker product. □
In a last step, the preconditioners for $A_{\mathcal{R}_d}$ are defined. Let

$$\mathcal{W}_{k,d} = P_d^T \text{blockdiag}\,[W_{k,d}]_{j=1}^{2^d} P_d \quad \text{for} \quad d = 2, 3, \tag{42}$$

where P_d denotes the permutation of Propositions 1.3 and 1.4. Now, we can formulate the main theorem of this section.

THEOREM 4.9. *The condition number estimates $\kappa\left(\mathcal{W}_{k,d}{}^{-1} A_{\mathcal{R}_d}\right) \le c$ are satisfied. The parameter c denotes a constant which is independent of the polynomial degree p.*

Proof: The result follows from Propositions 1.3, 1.4, and Lemma 4.8.□
In [7], Theorem 4.2, the weaker result $\kappa\left(\mathcal{W}_{k,d}{}^{-1} A_{\mathcal{R}_d}\right) \le c(1 + \log p)^{d-1}$ has been shown. The reason for the logarithmic term is the stronger estimate in Lemma 4.6.
Moreover, the preconditioning operation $\underline{w} = \mathcal{W}_{k,d}^{-1} \underline{r}$ is arithmetically optimal, i.e. it requires $\mathcal{O}(p^d)$ floating point operations, [7]. Hence, a preconditioned conjugate gradient method with the preconditioner $\mathcal{W}_{k,d}$ is an arithmetically optimal solver for a system with the matrix (10).
For $A_{\mathcal{R}_2}$, we propose other methods in the following subsection.

4.3 Further conclusions

In Lemma 1.5, we have proved the result $\lambda_{max}\left(T_2{}^{-1} F_1\right) \le c(1 + \log n)$. Using the wavelet approach, a stronger result for the largest eigenvalue of the matrix $T_2^{-1} F_1$ is valid.

LEMMA 4.10. *The eigenvalue estimate $\lambda_{max}\left(T_2{}^{-1} F_1\right) \le c$ is satisfied.*

Proof: By Lemma 4.6, we have

$$\left(D_{M^{\psi}_{\omega=x^{-1}}} \underline{v}, \underline{v}\right) \le \left(D_{T^{\psi}_{\omega=1}} \underline{v}, \underline{v}\right) \quad \forall \underline{v} \in \mathbb{R}^{n-1}.$$

Using Corollary 4.5, one obtains

$$\left(Q M^{\phi}_{\omega=x^{-1}} Q^T \underline{v}, \underline{v}\right) \le c_1 \left(Q T^{\phi}_{\omega=1} Q^T \underline{v}, \underline{v}\right) \quad \forall \underline{v} \in \mathbb{R}^{n-1},$$

where Q denotes the matrix representation of the fast wavelet transformation. With the substitution $Q^T \underline{v} = \underline{w}$, one concludes

$$\left(M^{\phi}_{\omega=x^{-1}} \underline{w}, \underline{w}\right) \le c_1 \left(T^{\phi}_{\omega=1} \underline{w}, \underline{w}\right) \quad \forall \underline{w} \in \mathbb{R}^{n-1}.$$

By (32), (34), and Lemma 2.2, one easily checks the relation

$$\frac{1}{c_2} \left(D_3^{-1} \underline{w}, \underline{w}\right) \le (D_6 \underline{w}, \underline{w}) \le \frac{c_1}{2} (T_2 \underline{w}, \underline{w}) \quad \forall \underline{w} \in \mathbb{R}^{n-1}. \tag{43}$$

Moreover, we have

$$(F_1 \underline{w}, \underline{w}) \le c_3 \left((D_3^{-1} + T_2) \underline{w}, \underline{w}\right) \quad \forall \underline{w} \in \mathbb{R}^{n-1} \tag{44}$$

by Lemma 2.1 with $c = c_3(1 + \frac{c_1 c_2}{2})$. Combining (43), (44), and the positive definiteness of all involved matrices, the assertion follows. $\square$

In a last step, we strengthen the condition number estimates for the preconditioners of $A_{\mathcal{R}_2}$ given in [5], [6]. Let L_k be the AMLI preconditioner $C_{h,k}$ in [6], and let M_k be the multi-grid preconditioner for K_k proposed in [5]. Moreover, let

$$\begin{aligned} \mathcal{L}_k &= P_2^T \text{blockdiag}\,[L_k]_{i=1}^4\, P_2, \\ \mathcal{M}_k &= P_2^T \text{blockdiag}\,[M_k]_{i=1}^4\, P_2, \end{aligned} \tag{45}$$

where P_2 is the permutation matrix in Proposition 1.3.

In [5], we have proved the condition number estimate $\kappa\left(\mathcal{M}_k{}^{-1} A_{\mathcal{R}_2}\right) \leq c(1+\log p)$. By Theorem 5.1 in [6], the estimate $\kappa\left(\mathcal{L}_k{}^{-1} A_{\mathcal{R}_2}\right) \leq c(1 + \log p)$ holds. For both condition numbers, the following theorem gives a more rigorous estimate.

THEOREM 4.11. *The condition number estimates* $\kappa\left(\mathcal{L}_k{}^{-1} A_{\mathcal{R}_2}\right) \leq c$ *and* $\kappa\left(\mathcal{M}_k{}^{-1} A_{\mathcal{R}_2}\right) \leq c$ *are valid.*

Proof: We start with $\kappa\left(\mathcal{L}_k{}^{-1} A_{\mathcal{R}_2}\right) \leq c$. By Theorem 3.2 in [6], we have $\kappa\left(L_k{}^{-1} K_k\right) \leq c$. Moreover, by Lemma 4.10, Lemma 1.5, relations (14) and (18) and tensor product arguments one easily checks the relation $\kappa\left(K_k{}^{-1} G_1\right) \leq c$. Thus, we have $\kappa\left(L_k{}^{-1} G_1\right) \leq c$. Hence, the assertion follows by Proposition 1.3 and (45). The estimate $\kappa\left(\mathcal{M}_k{}^{-1} A_{\mathcal{R}_2}\right) \leq c$ can be proved with similar arguments, see e.g. the proof of Theorem 7.1 in [3]. $\square$

Thus, we have two preconditioners $\mathcal{M}_k$ and $\mathcal{L}_k$ for $A_{\mathcal{R}_2}$ such that the condition number of the preconditioned system matrix is bounded by a constant independent of the polynomial degree. The cost in order to apply the preconditioning operations $\underline{w} = \mathcal{M}_k^{-1} \underline{r}$ and $\underline{w} = \mathcal{L}_k^{-1} \underline{r}$ is $\mathcal{O}(p^2)$. For more details, we refer to [6], [5]. So, we have two another arithmetically optimal solvers for the system of linear algebraic equations $A_{\mathcal{R}_2} \underline{u} = \underline{f}$.

5 Numerical example

Now, we give a numerical example. The condition numbers of the weighted mass matrices M_ω^ψ with $\omega(\xi) = \xi^{\pm 1}$ are considered. The wavelets are generated by the piecewise linear functions ψ_{2l} with two vanishing moments on the primal and $d = 2, 4, 6$ vanishing moments on the dual side. Figure 1 shows one wavelet of the families ψ_{22} and ψ_{24}. Since $\tilde{\psi}_{22} \notin W^{1,\infty}((0,1))$, the wavelet ψ_{22} does not satisfy the assumptions of Theorem 4.4.

Figure 2 displays the condition numbers of the scaled weighted mass matrices $(D_{M_\omega^\psi})^{-1} M_\omega^\psi$, where

$$M_\omega^\psi = \left[\langle \psi_j^l, \psi_{j'}^{l'} \rangle_\omega\right]$$

for $\omega(\xi) = \xi^{\pm 1}$ and $D_{M_\omega^\psi} = \text{diag}[\mathfrak{d}]$ with $\mathfrak{d} = [\omega^2(2^{-l} j)]$. For $\omega(\xi) = \xi$, [7], the wavelet basis generated by the function ψ_{22} shows the lowest condition numbers. For the wavelet ψ_{26}, the condition numbers are very large.

Using the weight function $\omega(\xi) = \xi^{-1}$, this behaviour is different. Now, the results for the mass matrices using the wavelet bases for ψ_{24} and ψ_{26} beat the wavelet basis generated by the

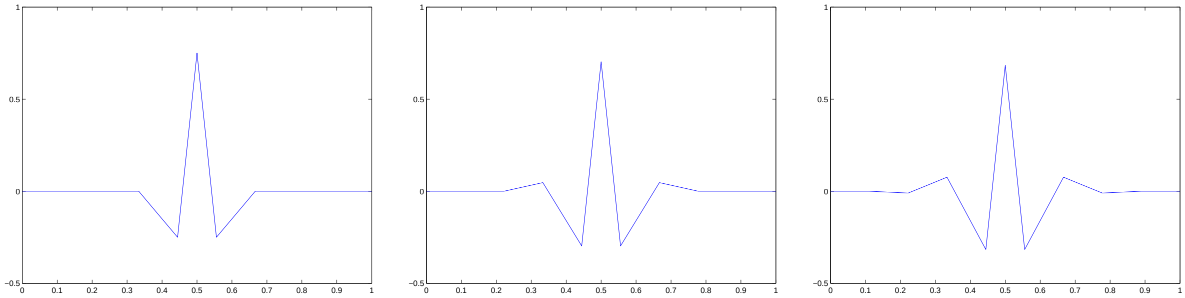

Figure 1: Wavelets ψ_{22} (left), ψ_{24} (middle) and ψ_{26} (right).

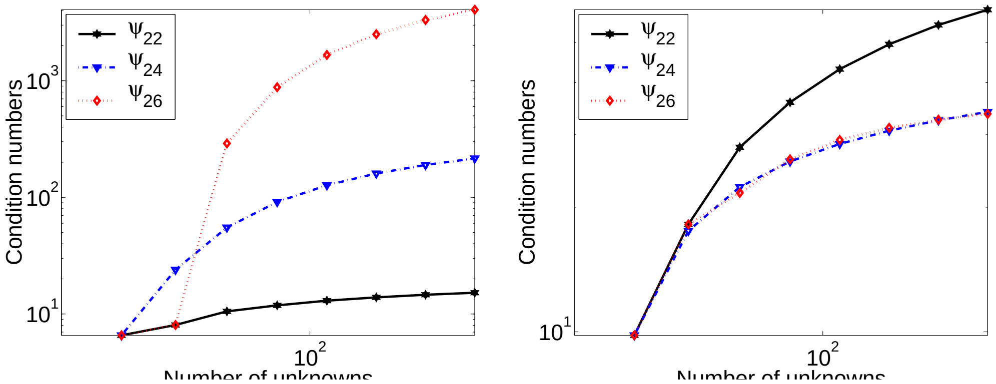

Figure 2: Condition numbers of the weighted mass matrix for $\omega(\xi) = \xi$ (left) and $\omega(\xi) = \xi^{-1}$ (right).

wavelet ψ_{22}. For ψ_{24} and ψ_{26}, the condition numbers are bounded by a constant independent of the level number. For ψ_{22}, the numerical experiments indicate a growth of the condition number proportionally to the level number.

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