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Numerische Simulation auf massiv parallelen Rechnern

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Eigenvibrations of a plate with elastically attached load

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Abstract This paper presents the investigation of the nonlinear eigenvalue problem describing natural oscillations of a plate with elastically attached load. We study properties of eigenvalues and eigenfunctions and prove the existence theorem for this eigenvalue problem. Theoretical results are illustrated by numerical experiments.

Key Words nonlinear eigenvalue problem, eigenvibrations of a plate, natural oscillations, eigenvalue, eigenfunction

AMS(MOS) subject classification 74H20, 74H45, 49R50, 65N25, 47J10, 47A75, 35P05, 35P30

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1 Introduction

Problems on eigenvibrations of mechanical structures with elastically attached loads have important applications. A survey of results in this direction is presented in [1]. An analytical method for solving some problems of this class is also described and analyzed in [1]. This method can be applied only in particular cases when we are known analytical formulae for eigenvalues and eigenfunctions of mechanical structures without loads.

In the present paper, to treat the general case we propose a new approach for describing and finding eigensolutions of problems on eigenvibrations of mechanical structures with elastically attached loads. To introduce our approach we shall study the problem on eigenvibrations of a plate with elastically attached load.

First let us describe eigenvibrations of the plate-spring-load system. We shall investigate the flexural vibrations of an isotropic elastic thin clamped plate with middle surface occupying the plane domain Ω with the boundary Γ [4]. Assume that $\rho = \rho(x)$ is the volume mass density, $D = D(x) = Ed^3/12(1-\nu^2)$ is the flexural rigidity of the plate, $E = E(x)$, $\nu = \nu(x)$, $d = d(x)$, are Young modulus, Poisson ratio, and the thickness of the plate at the point $x \in \Omega$, respectively, $0 < \nu < 1/2$. Assume that a load of mass M is joined by an elastic spring with the stiffness coefficient K at the point $x_0 \in \Omega$, i.e., a harmonic oscillator with the vibration frequency $\omega = \sqrt{\sigma}$, $\sigma = K/M$, is attached at the point $x_0 \in \Omega$ of the plate.

Denote by $w(x,t)$ the vertical deflection of the plate at the point $x \in \Omega$ at time t, by $\xi(t)$ the vertical displacement of the load of mass M at time t. These functions satisfy the following equations (see, for example, [1]):

$$\begin{aligned} &Lw(x,t) + \rho(x)d(x)w_{tt}(x,t) + M\xi_{tt}(t)\delta(x-x_0) = 0, \quad x \in \Omega, t > 0, \\ &w(x,t) = \partial_n w(x,t) = 0, \quad x \in \Gamma, t > 0, \\ &M\xi_{tt}(t) + K(\xi(t) + w(x_0,t)) = 0, \quad t > 0, \end{aligned} \tag{1}$$

where $\delta(x)$ is the delta function of Dirac, ∂_n is the outward normal derivative on Γ, L is the differential operator defined by the relation:

$$Lw = \partial_{11}D(\partial_{11}w + \nu\partial_{22}w) + \partial_{22}D(\partial_{22}w + \nu\partial_{11}w) + 2\partial_{12}D(1-\nu)\partial_{12}w,$$

where $\partial_{ij} = \partial_i\partial_j$, $\partial_i = \partial/\partial x_i$, $i,j = 1,2$.

The eigenvibrations of the plate-spring-load system are characterized by the functions $w(x,t)$ and $\xi(t)$ of the form:

$$w(x,t) = u(x)v(t), \quad x \in \Omega, \quad \xi(t) = c_0 u(x_0)v(t), \quad t > 0,$$

where $v(t) = a_0 \cos\sqrt{\lambda}t + b_0 \sin\sqrt{\lambda}t$, $t > 0$; a_0, b_0, c_0, and λ are constants.

From the third equation of (1), we conclude that $c_0 = \sigma/(\lambda - \sigma)$, $\sigma = K/M$. The first two equations of (1) lead to the following nonlinear eigenvalue problem: find values λ and nontrivial functions $u(x)$, $x \in \Omega$, such that

$$
\begin{aligned}
&Lu + \frac{\lambda\sigma}{\lambda-\sigma} M\delta(x-x_0)u = \lambda\,\rho d\,u, \quad x\in\Omega,\\
&u = \partial_n u = 0, \quad x\in\Gamma.
\end{aligned} \tag{2}
$$

The present paper is devoted to the investigation of nonlinear eigenvalue problem (2). In Section 2 we state the variational formulation for differential eigenvalue problem (2). In Section 3 we introduce parameter linear eigenvalue problems and study their properties. These parameter eigenvalue problems are used for proving the existence theorem in Section 4. In Section 5 we consider the nonlinear biharmonic eigenvalue problem and demonstrate numerical experiments. Similar results have been established in [24] for a beam with elastically attached load and in [23], [12], for a cylindrical shell with elastically attached load. In this paper we use an analysis analogous to [16].

2 Variational statement of the problem

By $\mathbb{R}$ denote the real axis. Let Ω be a plane domain with a Lipschitz-continuous boundary Γ. As usual, let $L_2(\Omega)$ and $W_2^2(\Omega)$ denote the real Lebesgue and Sobolev spaces, equipped with the norms $|.|_0$ and $\|.\|_2$:

$$
|u|_0 = \left(\int_\Omega u^2\,dx\right)^{1/2}, \quad \|u\|_2 = \left(\sum_{i=0}^{2} |u|_i^2\right)^{1/2},
$$

where

$$
|u|_1 = \left(\sum_{i=1}^{2} |\partial_i u|_0^2\right)^{1/2}, \quad |u|_2 = \left(\sum_{ij=1}^{2} |\partial_{ij} u|_0^2\right)^{1/2},
$$

$\partial_i = \partial/\partial x_i$, $\partial_{ij} = \partial_i\partial_j$, $i,j=1,2$. Denote by $\overset{\circ}{W}{}_2^2(\Omega)$ the space of functions u from $W_2^2(\Omega)$ such that $u = \partial_n u = 0$ on Γ, $\partial_n u$ is the outer normal derivative of u along the boundary Γ.

Put $\Lambda = (0,\infty)$, $H = L_2(\Omega)$, $V = \overset{\circ}{W}{}_2^2(\Omega)$. Note that the space V is compactly embedded into the space H, any function from V is continuous on $\overline{\Omega}$. The semi-norm $|.|_2$ is a norm over the space V, which is equivalent to the norm $\|.\|_2$.

Assume that $L_\infty(\Omega)$ is the space of measurable real functions u bounded almost everywhere on Ω with the norm

$$
|u|_{0,\infty} = \operatorname*{ess.sup}_{x\in\Omega} |u(x)|.
$$

Note that there exists c_0 such that

$$
|v|_{0,\infty} \le c_0 |v|_2 \quad \forall v \in W_2^2(\Omega).
$$

Introduce the numbers $K>0$, $M>0$, $\sigma = K/M$. Define functions E, ν, ρ, and d from $L_\infty(\Omega)$, for which there exist positive numbers E_1, E_2, ρ_1, ρ_2, d_1, d_2, such that

$$
\begin{aligned}
&E_1 \le E(x) \le E_2, && 0 < \nu(x) < 1/2,\\
&\rho_1 \le \rho(x) \le \rho_2, && d_1 \le d(x) \le d_2,
\end{aligned}
$$

for almost all $x \in \Omega$.

Set

$$D = \frac{Ed^3}{12(1-\nu^2)}.$$

Define the bilinear forms $a : V \times V \to \mathbb{R}$, $b : H \times H \to \mathbb{R}$, $c : V \times V \to \mathbb{R}$, and the functions $\xi(\mu)$, $\mu \in \Lambda$, $\zeta(\mu)$, $\mu \in \Lambda$, by the formulae:

$$\begin{aligned} a(u,v) &= \int\limits_{\Omega} D[(\partial_{11}u + \partial_{22}u)(\partial_{11}v + \partial_{22}v) + \\ &\quad +(1-\nu)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)]\,dx, \\ b(u,v) &= \int\limits_{\Omega} \rho d\, uv\, dx, \\ c(u,v) &= Mu(x_0)v(x_0), \quad u,v \in V, \\ \zeta(\mu) &= \frac{\sigma}{\sigma-\mu}, \quad \mu \in \Lambda, \\ \xi(\mu) &= \frac{\mu\sigma}{\mu-\sigma}, \quad \mu \in \Lambda, \end{aligned}$$

where x_0 is the fixed point on Ω.

Consider the following differential eigenvalue problem: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, $Lu \in H$, such that

$$Lu + \xi(\lambda) M\delta(x - x_0)u = \lambda\, \rho d\, u.$$

This differential problem is equivalent to the following variational eigenvalue problem: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, such that

$$a(u,v) + \xi(\lambda)\, c(u,v) = \lambda\, b(u,v) \quad \forall v \in V. \tag{3}$$

The number λ that satisfies (3) is called an eigenvalue, and the element u is called an eigenelement of problem (3) corresponding to λ. The set $U(\lambda)$ that consists of the eigenelements corresponding to the eigenvalue λ and the zero element is a closed subspace in V, which is called the eigensubspace corresponding to the eigenvalue λ. The dimension of this subspace is called a multiplicity of the eigenvalue λ. A pair λ and u is called the eigensolution or the eigenpair of problem (3).

Remark 1 Equation (3) can be written in the following equivalent form:

$$a(u,v) = \lambda(b(u,v) + \zeta(\lambda)\, c(u,v)) \quad \forall v \in V.$$

3 Parameter eigenvalue problems

Put $\Lambda_1 = (0,\sigma)$, $\Lambda_2 = (\sigma,\infty)$.

Let us write problem (3) for $\lambda \neq \sigma$ as two following problems on the intervals Λ_1 and Λ_2.

Find $\lambda \in \Lambda_1$, $u \in V \setminus \{0\}$, such that

$$a(u,v) = \lambda(b(u,v) + \zeta(\lambda)\, c(u,v)) \quad \forall v \in V. \tag{4}$$

Find $\lambda \in \Lambda_2$, $u \in V \setminus \{0\}$, such that

$$a(u,v) + \xi(\lambda)\, c(u,v) = \lambda\, b(u,v) \quad \forall v \in V. \tag{5}$$

For problems (4) and (5) we introduce parameter linear eigenvalue problems for fixed parameter $\mu \in \Lambda$.

Find $\varphi(\mu) \in \mathbb{R}$, $u \in V \setminus \{0\}$, such that

$$a(u,v) = \varphi(\mu)(b(u,v) + \zeta(\mu)c(u,v)) \quad \forall v \in V. \tag{6}$$

Find $\psi(\mu) \in \mathbb{R}$, $u \in V \setminus \{0\}$, such that

$$a(u,v) + \xi(\mu)c(u,v) = \psi(\mu)b(u,v) \quad \forall v \in V. \tag{7}$$

Define the subspace $V_0 = \{v : v \in V, v(x_0) = 0\}$ of the space V. Let us consider the following linear eigenvalue problems in the spaces V and V_0 for bilinear forms $a : V \times V \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$.

Find $\lambda^0 \in \mathbb{R}$, $u \in V \setminus \{0\}$, such that

$$a(u,v) = \lambda^0 b(u,v) \quad \forall v \in V. \tag{8}$$

Find $\lambda_0 \in \mathbb{R}$, $u \in V_0 \setminus \{0\}$, such that

$$a(u,v) = \lambda_0 b(u,v) \quad \forall v \in V_0. \tag{9}$$

The results on the existence and properties of eigensolutions for linear eigenvalue problems (6)–(9) will be obtained by using the following lemma.

Lemma 2 *Let W be an infinite dimensional real Hilbert space with the norm $\|.\|$. Assume that bilinear forms $a : W \times W \to \mathbb{R}$ and $b : W \times W \to \mathbb{R}$ are symmetric and satisfy the following properties:*

$$\begin{aligned} \alpha_1\|v\|^2 &\le a(v,v) \le \alpha_2\|v\|^2 && \forall v \in W, \\ 0 &< b(v,v) \le \beta_2\|v\|^2 && \forall v \in W \setminus \{0\}. \end{aligned}$$

Suppose that the following compactness property is valid: for any bounded sequence $v_i \in W$, $\|v_i\| \le C$, $i = 1,2,\ldots$, C is a constant, there exists a subsequence v_{i_j}, $j = 1,2,\ldots$, and an element $w \in W$ such that $\|v_{i_j} - w\|_b \to 0$ as $j \to \infty$, where $\|v\|_b = \sqrt{b(v,v)}$, $v \in W$.

Then the eigenvalue problem: find $\lambda \in \mathbb{R}$, $u \in W \setminus \{0\}$, *such that*

$$a(u,v) = \lambda b(u,v) \quad \forall v \in W$$

has a denumerable set of real eigenvalues of finite multiplicity λ_k, $k = 1, 2, \ldots$, *which are repeated according to their multiplicity:*

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le \ldots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

The corresponding eigenelements u_k, $k = 1, 2, \ldots$, *form a complete system in* W *such that*

$$b(u_i, u_j) = \delta_{ij}, \quad a(u_i, u_j) = \lambda_i \delta_{ij}, \quad i, j = 1, 2, \ldots$$

The following relations hold:

$$\lambda_k = R(u_k) = \min_{v \in S_k \setminus \{0\}} R(v) = \max_{v \in E_k \setminus \{0\}} R(v),$$

$$\lambda_k = \min_{W_k \subset W} \max_{v \in W_k \setminus \{0\}} R(v), \quad k = 1, 2, \ldots,$$

where $R(v) = a(v,v)/b(v,v)$, $v \in W \setminus \{0\}$, $S_1 = W$, $S_k = \{v : v \in W, b(v, u_i) = 0, i = 1, 2, \ldots, k-1\}$, $k = 2, 3, \ldots$, E_k *is the subspace spanned on the eigenelements* u_i, $i = 1, 2, \ldots, k$, W_k *is* k*-dimensional subspace of the space* W, $k = 1, 2, \ldots$

Proof Define the operator $A : W \to W$ by the following equality:

$$a(Au, v) = b(u, v) \quad \forall u, v \in W.$$

Then the variational eigenvalue problem can be written as the operator eigenvalue problem: find $\lambda \in \mathbb{R}$, $u \in W \setminus \{0\}$, such that

$$u = \lambda A u.$$

It is not difficult to verify that A is a self-adjoint compact operator. Using spectral theory of self-adjoint compact operators in the Hilbert space [2] we obtain all assertions of this lemma. □

The following lemma formulates the properties for bilinear forms of eigenvalue problems (3)–(9).

Lemma 3 *The following inequalities hold:*

$$\begin{aligned} \alpha_1 |v|_2^2 &\le a(v,v) \le \alpha_2 |v|_2^2 \qquad \forall v \in V, \\ \beta_1 |v|_0^2 &\le b(v,v) \le \beta_2 |v|_0^2 \qquad \forall v \in H, \\ 0 &\le c(v,v) \le \gamma_2 |v|_2^2 \qquad \forall v \in V, \end{aligned}$$

where $\alpha_i = D_i$, $\beta_i = \rho_i d_i$, $i = 1, 2$, $D_1 = E_1 d_1^3/12$, $D_2 = E_2 d_2^3/9$, $\gamma_2 = c_0^2 M$.

Proof The inequalities follow from the definitions of the bilinear forms and the assumptions on the coefficients. □

The Rayleigh quotients associated with eigenvalue problems (6)–(9) are defined by the formulae:

$$\begin{aligned} R_1(\mu, v) &= \frac{a(v,v)}{b(v,v)+\zeta(\mu)c(v,v)}, \quad v \in V\setminus\{0\}, \mu \in \Lambda_1, \\ R_2(\mu, v) &= \frac{a(v,v)+\xi(\mu)c(v,v)}{b(v,v)}, \quad v \in V\setminus\{0\}, \mu \in \Lambda_2, \\ R(v) &= \frac{a(v,v)}{b(v,v)}, \quad v \in V\setminus\{0\}. \end{aligned}$$

Problems (8) and (9) have a denumerable sets of real eigenvalues of finite multiplicity λ_k^0, $k=1,2,\ldots$, and λ_{0k}, $k=1,2,\ldots$, respectively, which are repeated according to their multiplicity:

$$\begin{aligned} &0<\lambda_1^0 \le \lambda_2^0 \le \ldots \le \lambda_k^0 \le \ldots, \quad \lim_{k\to\infty}\lambda_k^0 = \infty, \\ &0<\lambda_{01} \le \lambda_{02} \le \ldots \le \lambda_{0k} \le \ldots, \quad \lim_{k\to\infty}\lambda_{0k} = \infty. \end{aligned}$$

The corresponding eigenelements u_k^0, $k=1,2,\ldots$, and u_{0k}, $k=1,2,\ldots$, form a complete systems in V and V_0, respectively.

Let us introduce the following eigenvalue problem.

Find $\lambda^* \in \mathbb{R}$, $u \in V\setminus\{0\}$, such that

$$a(u,v) = \lambda^* c(u,v) \quad \forall v \in V. \tag{10}$$

It can be easily prove that eigenvalue problem (10) has only one eigenvalue λ^*, which is positive and simple. There exists a unique normalized eigenfunction u^*, $a(u^*,u^*)=1$, $u^*(x_0)>0$, corresponding to the eigenvalue λ^*.

Assume that there exists an eigenfunction u of eigenvalue problem (8) such that $u(x_0)\neq 0$. In the converse case, the existence of eigenvalues of problem (3) is stated by Remark 13.

Lemma 4 *For fixed $\mu\in\Lambda_1$ problem (6) has a denumerable set of real eigenvalues of finite multiplicity $\varphi_k(\mu)$, $k=0,1,\ldots$, which are repeated according to their multiplicity:*

$$0<\varphi_0(\mu)\le\varphi_1(\mu)\le\ldots\le\varphi_k(\mu)\le\ldots, \quad \lim_{k\to\infty}\varphi_k(\mu)=\infty.$$

The corresponding eigenelements v_k^μ, $k=0,1,\ldots$, form a complete system in V such that

$$b(v_i^\mu, v_j^\mu)+\zeta(\mu)c(v_i^\mu,v_j^\mu)=\varphi_i(\mu)^{-1}\delta_{ij}, \quad a(v_i^\mu,v_j^\mu)=\delta_{ij}, \quad i,j=0,1,\ldots$$

The following relations hold:

$$\begin{aligned} &\varphi_{k-1}(\mu) = R_1(\mu, v_{k-1}^\mu) = \min_{v\in S_k^1(\mu)\setminus\{0\}} R_1(\mu,v) = \max_{v\in E_k^1(\mu)\setminus\{0\}} R_1(\mu,v), \\ &\varphi_{k-1}(\mu) = \min_{W_k\subset V}\max_{v\in W_k\setminus\{0\}} R_1(\mu,v), \quad k=1,2,\ldots, \quad \mu\in\Lambda_1, \end{aligned}$$

where $S_1^1(\mu) = V$, $S_k^1(\mu) = \{v : v \in V, a(v, v_{i-1}^{\mu}) = 0, i = 1, 2, \ldots, k-1\}$, $k = 2, 3, \ldots$, $E_k^1(\mu)$ *is the subspace spanned on the eigenelements* v_{i-1}^{μ}, $i = 1, 2, \ldots, k$, W_k *is k-dimensional subspace of the space* V, $k = 1, 2, \ldots$

The following limit properties hold: $\varphi_0(\mu) \to 0$ *as* $\mu \to \sigma$, $\mu \in \Lambda_1$, $\varphi_k(\mu) \to \delta_k$ *as* $\mu \to \sigma$, $\mu \in \Lambda_1$, $\delta_k > 0$, $k = 1, 2, \ldots$ *Moreover, the following representation is valid:* $\varphi_0(\mu) = c_*(\sigma - \mu) + o(\sigma - \mu)$ *as* $\mu \to \sigma$, $\mu \in \Lambda_1$, *where* c_* *is a positive constant.*

Proof Let us first prove that all conditions of Lemma 2 are fulfilled for problem (6). By Lemma 3 we have the ellipticity and boundedness of the bilinear form $a : V \times V \to \mathbb{R}$ and we obtain the following property:

$$0 < b(v, v) + \zeta(\mu)c(v, v) \leq \tilde{\beta}_2 |v|_2^2 \quad \forall v \in V \setminus \{0\},$$

where $\tilde{\beta}_2 = \beta_2 + \zeta(\mu)\gamma_2$, β_2 and γ_2 are defined in Lemma 3.

If $v_i \in V$, $|v_i|_2 \leq C$, $i = 1, 2, \ldots$, C is a constant, then there exists a subsequence v_{i_j}, $j = 1, 2, \ldots$, and an element $w \in V$ such that $\|v_{i_j} - w\|_{\tilde{b}} \to 0$ as $j \to \infty$, where $\|v\|_{\tilde{b}} = \sqrt{b(v, v) + \zeta(\mu)c(v, v)}$, $v \in V$. Here we take into account that the space V is compactly embedded into the space H and that any closed interval is a compact set on $\mathbb{R}$.

Thus, all conditions of Lemma 3 are fulfilled and we obtain the result on the existence of eigensolutions for problem (6).

Let us now prove limit properties for eigenvalues of problem (6). We start by writing the eigenelement u, $u(x_0) \neq 0$, of problem (6) in form of the expansion:

$$u(x) = \sum_{i=1}^{\infty} c_i u_i^0(x) = \sum_{i=1}^{\infty} \frac{\varphi(\mu)\zeta(\mu)}{\lambda_i^0 - \varphi(\mu)} c(u, u_i^0) u_i^0(x),$$

where $\varphi(\mu)$ is the eigenvalue corresponding to u, λ_i^0 and u_i^0, $i = 1, 2, \ldots$, are eigensolutions of problem (8). Therefore, for $x = x_0$ we get

$$u(x_0) = \sum_{i=1}^{\infty} \frac{\varphi(\mu)\zeta(\mu)}{\lambda_i^0 - \varphi(\mu)} M (u_i^0(x_0))^2 u(x_0).$$

Consequently, the eigenvalue $\varphi(\mu)$ is the root of the equation

$$\frac{\sigma - \mu}{\sigma M} \frac{1}{\varphi(\mu)} = G(\varphi(\mu)),$$

where

$$G(\varphi(\mu)) = \sum_{i=1}^{\infty} \frac{(u_i^0(x_0))^2}{\lambda_i^0 - \varphi(\mu)}.$$

This equation implies the limit properties for eigenvalues. □

Lemma 5 *For fixed $\mu \in \Lambda_2$ problem (7) has a denumerable set of real eigenvalues of finite multiplicity $\psi_k(\mu)$, $k = 1, 2, \ldots$, which are repeated according to their multiplicity:*

$$0 < \psi_1(\mu) \le \psi_2(\mu) \le \ldots \le \psi_k(\mu) \le \ldots, \quad \lim_{k\to\infty} \psi_k(\mu) = \infty.$$

The corresponding eigenelements w_k^μ, $k = 1, 2, \ldots$, form a complete system in V such that

$$b(w_i^\mu, w_j^\mu) = \delta_{ij}, \quad a(w_i^\mu, w_j^\mu) + \xi(\mu)c(w_i^\mu, w_j^\mu) = \psi_i(\mu)\delta_{ij}, \quad i, j = 1, 2, \ldots$$

The following relations hold:

$$\psi_k(\mu) = R_2(\mu, w_k^\mu) = \min_{v\in S_k^2(\mu)\setminus\{0\}} R_2(\mu, v) = \max_{v\in E_k^2(\mu)\setminus\{0\}} R_2(\mu, v),$$

$$\psi_k(\mu) = \min_{W_k\subset V} \max_{v\in W_k\setminus\{0\}} R_2(\mu, v), \quad k = 1, 2, \ldots, \quad \mu \in \Lambda_2,$$

where $S_1^2(\mu) = V$, $S_k^2(\mu) = \{v : v \in V, b(v, w_i^\mu) = 0, i = 1, 2, \ldots, k-1\}$, $k = 2, 3, \ldots$, $E_k^2(\mu)$ is the subspace spanned on the eigenelements w_i^μ, $i = 1, 2, \ldots, k$, W_k is k-dimensional subspace of the space V, $k = 1, 2, \ldots$

Proof The assertions of this lemma follow from Lemmata 2 and 3. □

As usual, let $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively. For brevity, we shall use the same notation for a family and for its subsequence.

Lemma 6 *The functions $\varphi_k(\mu)$, $\mu \in \Lambda_1$, $k = 0, 1, \ldots$, are continuous nonincreasing functions. The following relations hold: $\varphi_0(\mu) \to 0$ as $\mu \to \sigma$, $\mu \in \Lambda_1$, $\varphi_k(\mu) \to \lambda_{0k}$ as $\mu \to \sigma$, $\mu \in \Lambda_1$, $k = 1, 2, \ldots$ Moreover, the following representation is valid: $\varphi_0(\mu) = \lambda^*(\sigma - \mu)/\sigma + o(\sigma - \mu)$ as $\mu \to \sigma$, $\mu \in \Lambda_1$. From the family v_k^μ, $a(v_k^\mu, v_k^\mu) = 1$, $\mu \in \Lambda_1$, one can extract a subsequence such that $v_k^\mu \to u_{0k}$ in V as $\mu \to \sigma$, $\mu \in \Lambda_1$, $a(u_{0k}, u_{0k}) = 1$, $k = 1, 2, \ldots$ If λ_{0k} is a simple eigenvalue, then the convergence can be established for the whole family v_k^μ with $a(v_k^\mu, u_{0k}) > 0$, $\mu \in \Lambda_1$, $k = 1, 2, \ldots$ For the whole family v_0^μ, $a(v_0^\mu, v_0^\mu) = 1$, $v_0^\mu(x_0) > 0$, $\mu \in \Lambda_1$, we have the convergence $v_0^\mu \to u^*$ in V as $\mu \to \sigma$, $\mu \in \Lambda_1$, where $a(u^*, u^*) = 1$, $u^*(x_0) > 0$.*

Proof (1) By Lemma 4 and the definition of the function $\zeta(\mu)$, $\mu \in \Lambda_1$, we obtain that the functions $\varphi_k(\mu)$, $\mu \in \Lambda_1$, $k = 0, 1, \ldots$, are continuous nonincreasing functions. Moreover, the following limit properties are valid: $\varphi_0(\mu) \to 0$ as $\mu \to \sigma$, $\mu \in \Lambda_1$, $\varphi_0(\mu) = c_*(\sigma - \mu) + o(\sigma - \mu)$ as $\mu \to \sigma$. From the family v_0^μ, $a(v_0^\mu, v_0^\mu) = 1$, $\mu \in \Lambda_1$, one can extract a subsequence such that $v_0^\mu \rightharpoonup w$ in V, $v_0^\mu \to w$ in H, $v_0^\mu(x_0) \to w(x_0)$, as $\mu \to \sigma$, $\mu \in \Lambda_1$, where $w \in V$.

Using the equation

$$a(v_0^\mu, v) = \varphi_0(\mu)(b(v_0^\mu, v) + \zeta(\mu)c(v_0^\mu, v)) \quad \forall v \in V$$

as $\mu \to \sigma$, $\mu \in \Lambda_1$, we get the equality

$$a(w, v) = \alpha c(w, v) \quad \forall v \in V,$$

where $w \in V$, $a(w,w) = 1$, $\alpha = \lambda^* = c_*\sigma$, $w = u^*$.

The strong convergence for the subsequence $v_0^\mu \to w$ in V as $\mu \to \sigma$, $\mu \in \Lambda_1$, can be derived from the relations:

$$\begin{aligned} \alpha_1 |v_0^\mu - w|_2^2 &\leq a(v_0^\mu - w, v_0^\mu - w) = \\ &= a(v_0^\mu, v_0^\mu) - 2a(v_0^\mu, w) + a(w,w) = \\ &= 2 - 2a(v_0^\mu, w) \to \\ &\to 2 - 2a(w,w) = 0 \end{aligned}$$

as $\mu \to \sigma$, $\mu \in \Lambda_1$.

Since λ^* is a simple eigenvalue, we obtain the strong convergence $v_0^\mu \to u^*$ in V as $\mu \to \sigma$, $\mu \in \Lambda_1$, for the whole family v_0^μ, $a(v_k^\mu, v_k^\mu) = 1$, $v_0^\mu(x_0) > 0$, $\mu \in \Lambda_1$.

(2) Let us prove that $\varphi_n(\mu) \to \lambda_{0n}$, $v_n^\mu \to u_{0n}$ in V, as $\mu \to \sigma$, $\mu \in \Lambda_1$, $n = 1$. Set $\lambda_{00} = 0$.

By Lemma 4 we have

$$0 < \varphi_n(\mu) = R_1(\mu, v_n^\mu) = \min_{v \in S_{n+1}^1(\mu) \setminus \{0\}} R_1(\mu, v) \leq R_1(\mu, w^\mu) = \eta_n^\mu,$$

where $S_{n+1}^1(\mu) = \{v : v \in V, a(v, v_i^\mu) = 0, i = 0, 1, \ldots, n-1\}$,

$$w^\mu = u_{0n} - \sum_{i=0}^{n-1} a(u_{0n}, v_i^\mu) v_i^\mu.$$

Note that $w^\mu \in S_{n+1}^1(\mu)$ and $w^\mu \not\equiv 0$ as $\mu \to \sigma$, $\mu \in \Lambda_1$, $w^\mu \to u_{0n}$ in V, $\eta_n^\mu = R_1(\mu, w^\mu) \to \lambda_{0n}$ as $\mu \to \sigma$, $\mu \in \Lambda_1$.

Now we derive

$$0 < \varphi_n(\mu) \leq \eta_n^\mu \to \lambda_{0n}$$

as $\mu \to \sigma$, $\mu \in \Lambda_1$. Hence $\varphi_n(\mu) \to \alpha$ as $\mu \to \sigma$, $\mu \in \Lambda_1$, $\lambda_{0,n-1} \leq \alpha \leq \lambda_{0n}$.

From the family v_n^μ, $a(v_n^\mu, v_n^\mu) = 1$, $\mu \in \Lambda_1$, $\mu \to \sigma$, one can extract a subsequence such that $v_n^\mu \rightharpoonup w$ in V, $v_n^\mu \to w$ in H, $v_n^\mu(x_0) \to w(x_0)$, as $\mu \to \sigma$, $\mu \in \Lambda_1$, where $w \in V$.

Since $a(v_n^\mu, v_n^\mu) = 1$, we get

$$\varphi_n(\mu)(b(v_n^\mu, v_n^\mu) + \zeta(\mu)\, c(v_n^\mu, v_n^\mu)) = 1$$

for $\mu \in \Lambda_1$. Hence

$$(v_n^\mu(x_0))^2 \leq \frac{1}{\varphi_n(\mu)} \frac{1}{M} \frac{\sigma - \mu}{\sigma} \to 0$$

as $\mu \to \sigma$, $\mu \in \Lambda_1$. Therefore, $w(x_0) = 0$ and $w \in V_0$.

Thus, there exists $w \in V_0$ such that $v_n^\mu \rightharpoonup w$ in V, $v_n^\mu \to w$ in H, $v_n^\mu(x_0) \to 0$, as $\mu \to \sigma$, $\mu \in \Lambda_1$.

Using the equation

$$a(v_n^\mu, v) = \varphi_n(\mu) b(v_n^\mu, v) \quad \forall v \in V_0$$

as $\mu \to \sigma$, $\mu \in \Lambda_1$, we get the equality

$$a(w,v) = \alpha b(w,v) \quad \forall v \in V_0,$$

where $w \in V_0$, $\lambda_{0,n-1} \le \alpha \le \lambda_{0n}$, $a(w,w) = 1$. Hence $\alpha = \lambda_{0n}$ and $w = u_{0n}$, $a(u_{0n}, u_{0n}) = 1$, $n = 1$.

The strong convergence for the subsequence $v_n^\mu \to w$ in V as $\mu \to \sigma$, $\mu \in \Lambda_1$, can be derived from the relations:

$$\begin{aligned} \alpha_1 |v_n^\mu - w|_2^2 &\le a(v_n^\mu - w, v_n^\mu - w) = \\ &= a(v_n^\mu, v_n^\mu) - 2a(v_n^\mu, w) + a(w,w) = \\ &= 2 - 2a(v_n^\mu, w) \to \\ &\to 2 - 2a(w,w) = 0 \end{aligned}$$

as $\mu \to \sigma$, $\mu \in \Lambda_1$.

Repeating the above proof for $n \ge 2$ we obtain the assertions of this lemma. □

Lemma 7 *The functions $\psi_k(\mu)$, $\mu \in \Lambda_2$, $k = 1, 2, \ldots$, are continuous nonincreasing functions. The following relations hold: $\psi_k(\mu) \to \lambda_{0k}$ as $\mu \to \sigma$, $\mu \in \Lambda_2$, $k = 1, 2, \ldots$ From the family w_k^μ, $b(w_k^\mu, w_k^\mu) = 1$, $\mu \in \Lambda_2$, one can extract a subsequence such that $w_k^\mu \to u_{0k}$ in V as $\mu \to \sigma$, $\mu \in \Lambda_2$, $b(u_{0k}, u_{0k}) = 1$, $k = 1, 2, \ldots$ If λ_{0k} is a simple eigenvalue, then the convergence can be established for the whole family w_k^μ with $b(w_k^\mu, u_{0k}) > 0$, $\mu \in \Lambda_2$.*

Proof (1) Using the minimax principle of Lemma 5 and the definition of the function $\xi(\mu)$, $\mu \in \Lambda_2$, we obtain that the functions $\psi_k(\mu)$, $\mu \in \Lambda_2$, $k = 1, 2, \ldots$, are continuous nonincreasing functions.

(2) Let us prove that $\psi_k(\mu) \to \lambda_{0k}$, $w_k^\mu \to u_{0k}$ in V, as $\mu \to \sigma$, $\mu \in \Lambda_2$, $k = 1$.

By Lemma 5 we have

$$\begin{aligned} &0 < \psi_k(\mu) = R_2(\mu, w_k^\mu) = \min_{v \in V \setminus \{0\}} R_2(\mu, v) \le \\ &\le R_2(\mu, u_{0k}) = R(u_{0k}) = \lambda_{0k} \end{aligned}$$

for $\mu \in \Lambda_2$. Hence $\psi_k(\mu) \to \alpha$ as $\mu \to \sigma$, $\mu \in \Lambda_2$, where $0 < \alpha \le \lambda_{0k}$.

For eigenelements w_k^μ, $b(w_k^\mu, w_k^\mu) = 1$, we derive

$$a(w_k^\mu, w_k^\mu) + \xi(\mu)\, c(w_k^\mu, w_k^\mu) = \psi_k(\mu),$$

where $\mu \in \Lambda_2$. Consequently, $|w_k^\mu|_2^2 \le \lambda_{0k}/\alpha_1$ for $\mu \in \Lambda_2$. Therefore, from the family w_k^μ, $\mu \to \sigma$, one can extract a subsequence such that $w_k^\mu \rightharpoonup w$ in V, $w_k^\mu \to w$ in H, $w_k^\mu(x_0) \to w(x_0)$, as $\mu \to \sigma$, $\mu \in \Lambda_2$, where $w \in V$.

Since $1 = b(w_k^\mu, w_k^\mu) \to b(w,w)$ as $\mu \to \sigma$, $\mu \in \Lambda_2$, we obtain $b(w,w) = 1$.

Applying the relations

$$\begin{aligned} &0 < \psi_k(\mu) = R_2(\mu, w_k^\mu) = \\ &= a(w_k^\mu, w_k^\mu) + \xi(\mu)\, c(w_k^\mu, w_k^\mu) \le \lambda_{0k}, \end{aligned}$$

we get

$$\xi(\mu)\, c(w_k^\mu, w_k^\mu) = \xi(\mu)\, M\, (w_k^\mu(x_0))^2 \leq \lambda_{0k}$$

for $\mu \in \Lambda_2$. Hence

$$(w_k^\mu(x_0))^2 \leq \frac{\lambda_{0k}}{M} \frac{\mu - \sigma}{\mu\sigma} \to 0$$

as $\mu \to \sigma$, $\mu \in \Lambda_2$. Therefore, $w(x_0) = 0$ and $w \in V_0$.

Thus, there exists $w \in V_0$ such that $w_k^\mu \rightharpoonup w$ in V, $w_k^\mu \to w$ in H, $w_k^\mu(x_0) \to 0$, as $\mu \to \sigma$, $\mu \in \Lambda_2$.

Using the equation

$$a(w_k^\mu, v) = \psi_k(\mu) b(w_k^\mu, v) \quad \forall v \in V_0$$

as $\mu \to \sigma$, $\mu \in \Lambda_2$, we get the equality

$$a(w, v) = \alpha b(w, v) \quad \forall v \in V_0,$$

where $w \in V_0$, $0 < \alpha \leq \lambda_{0k}$, $b(w, w) = 1$. Hence $\alpha = \lambda_{0k}$ and $w = u_{0k}$, $b(u_{0k}, u_{0k}) = 1$, $k = 1$.

The strong convergence for the subsequence $w_k^\mu \to w$ in V as $\mu \to \sigma$, $\mu \in \Lambda_2$, can be derived from the relations:

$$\begin{aligned}
\alpha_1 |w_k^\mu - w|_2^2 &\leq a(w_k^\mu - w, w_k^\mu - w) + \xi(\mu)\, c(w_k^\mu, w_k^\mu) = \\
&= a(w_k^\mu, w_k^\mu) - 2a(w_k^\mu, w) + a(w, w) + \xi(\mu)\, c(w_k^\mu, w_k^\mu) = \\
&= \psi_k(\mu) - 2\psi_k(\mu) b(w_k^\mu, w) + \lambda_{0k} \to \\
&\to \lambda_{0k} - 2\lambda_{0k} + \lambda_{0k} = 0
\end{aligned}$$

as $\mu \to \sigma$, $\mu \in \Lambda_2$.

(3) Assume that $\psi_i(\mu) \to \lambda_{0i}$, $w_i^\mu \to u_{0i}$ in V, as $\mu \to \sigma$, $\mu \in \Lambda_2$, $i = 1, 2, \ldots, k-1$, $k \geq 2$. Let as prove that $\psi_k(\mu) \to \lambda_{0k}$, $w_k^\mu \to u_{0k}$ in V, as $\mu \to \sigma$, $\mu \in \Lambda_2$, $k \geq 2$.

By Lemma 5 we get

$$0 < \psi_k(\mu) = R_2(\mu, w_k^\mu) = \min_{v \in S_k^2(\mu) \setminus \{0\}} R_2(\mu, v) \leq R_2(\mu, w^\mu) = \eta_k^\mu,$$

where $S_k^2(\mu) = \{v : v \in V, b(v, w_i^\mu) = 0, i = 1, 2, \ldots, k-1\}$, $k = 2, 3, \ldots$, $\mu \in \Lambda_2$,

$$w^\mu = u_{0k} - \sum_{i=1}^{k-1} b(u_{0k}, w_i^\mu) w_i^\mu.$$

Note that $w^\mu \in S_k^2(\mu)$ and $w^\mu \not\equiv 0$ as $\mu \to \sigma$, $\mu \in \Lambda_2$, $w^\mu \to u_{0k}$ in V, $\eta_k^\mu = R_2(\mu, w^\mu) \to \lambda_{0k}$ as $\mu \to \sigma$, $\mu \in \Lambda_2$.

We have

$$0 < \psi_k(\mu) \leq \eta_k^\mu \to \lambda_{0k}$$

as $\mu \to \sigma$, $\mu \in \Lambda_2$. Hence $\psi_k(\mu) \to \alpha$ as $\mu \to \sigma$, $\mu \in \Lambda_2$, $\lambda_{0,k-1} \leq \alpha \leq \lambda_{0k}$.

By analogy with part 2) of this proof, we obtain $\psi_k(\mu) \to \lambda_{0k}$, $w_k^\mu \to u_{0k}$ in V, as $\mu \to \sigma$, $\mu \in \Lambda_2$, $k \geq 2$. □

Lemma 8 *A number $\lambda \in \Lambda_1$ is an eigenvalue of problem* (4) *if and only if the number $\lambda \in \Lambda_1$ is a solutions of an equation from the set:*

$$\mu - \varphi_k(\mu) = 0, \quad \mu \in \Lambda_1, \quad k = 0, 1, \ldots$$

A number $\lambda \in \Lambda_2$ is an eigenvalue of problem (5) *if and only if the number $\lambda \in \Lambda_2$ is a solutions of an equation from the set:*

$$\mu - \psi_k(\mu) = 0, \quad \mu \in \Lambda_2, \quad k = 1, 2, \ldots$$

Proof The assertions follow directly from variational statements (4)–(7). □

4 Existence of eigensolutions

Set $\psi_0(\mu) = 0$, $\mu \in (\sigma, \infty)$, and define the functions $\gamma_k(\mu)$, $\mu \in \Lambda$, $k = 0, 1, \ldots$, by the formula:

$$\gamma_k(\mu) = \begin{cases} \varphi_k(\mu), & \mu \in (0, \sigma), \\ \lambda_{0k}, & \mu = \sigma, \\ \psi_k(\mu), & \mu \in (\sigma, \infty), \end{cases}$$

$k = 0, 1, \ldots$, $\lambda_{00} = 0$.

Lemma 9 *The functions $\gamma_k(\mu)$, $\mu \in \Lambda$, $k = 0, 1, \ldots$, are continuous nonincreasing functions.*

Proof The assertions of this lemma follow from Lemmata 6 and 7 and the definitions of the functions $\gamma_k(\mu)$, $\mu \in \Lambda$, $k = 0, 1, \ldots$ □

Lemma 10 *A number $\lambda \in \Lambda$ is an eigenvalue of problem* (3) *if and only if the number $\lambda \in \Lambda$ is a solution of an equation from the set:*

$$\mu - \gamma_k(\mu) = 0, \quad \mu \in \Lambda, \quad k = 0, 1, \ldots$$

Proof The assertion of this lemma follows from Lemma 8. □

Theorem 11 *Let $\lambda_{0,i-1} < \sigma < \lambda_{0i}$, $i \geq 1$, where λ_{0k}, $k = 1, 2, \ldots$, are eigenvalues of eigenvalue problem* (9), *$\lambda_{00} = 0$. Then eigenvalue problem* (3) *has a denumerable set of real eigenvalues of finite multiplicity λ_k, $k = 0, 1, \ldots$, which are repeated according to their multiplicity:*

$$\begin{aligned} &0 < \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{i-1} < \sigma, \\ &\sigma < \lambda_i \leq \lambda_{i+1} \leq \ldots \leq \lambda_k \leq \ldots, \quad \lim_{k \to \infty} \lambda_k = \infty. \end{aligned}$$

Each eigenvalue λ_k*,* $k \geq 0$*, is a unique root of the equation:*

$$\mu - \gamma_k(\mu) = 0, \quad \mu \in \Lambda, \quad k \geq 0.$$

The eigensubspace $U(\lambda_k)$ *of nonlinear eigenvalue problem* (3) *is*

(a) *the eigensubspace corresponding to the eigenvalue* $\varphi_k(\mu)$ *of linear eigenvalue problem* (6) *for* $\mu = \lambda_k$*,* $\lambda_k \in \Lambda_1$*, or*

(b) *the eigensubspace corresponding to the eigenvalue* $\psi_k(\mu)$ *of linear eigenvalue problem* (7) *for* $\mu = \lambda_k$*,* $\lambda_k \in \Lambda_2$*.*

Proof By Lemma 9 each equation of the set

$$\mu - \gamma_k(\mu) = 0, \quad \mu \in \Lambda, \quad k \geq 0.$$

has a unique solution. Denote these solutions by λ_i, $i = 0, 1, \ldots$, i.e., $\lambda_i - \gamma_i(\lambda_i) = 0$, $i = 0, 1, \ldots$ To check that the numbers λ_i, $i = 0, 1, \ldots$, are put in a nondecreasing order, let us assume the opposite, i.e., $\lambda_i > \lambda_{i+1}$. Then, according to Lemma 9, we obtain a contradiction, namely

$$\lambda_i = \gamma_i(\lambda_i) \leq \gamma_i(\lambda_{i+1}) \leq \gamma_{i+1}(\lambda_{i+1}) = \lambda_{i+1}.$$

By Lemma 10, the numbers λ_i, $i = 0, 1, \ldots$, are eigenvalues of problem (3). Using Lemmata 6, 7, and inequalities $\lambda_{0,i-1} < \sigma < \lambda_{0i}$, we obtain the desired inequalities for eigenvalues. Thus, the theorem is proved. □

Remark 12 Let $\sigma = \lambda_{0i}$, $i \geq 2$, where λ_{0k}, $k = 1, 2, \ldots$, are eigenvalues of eigenvalue problem (9), $\lambda_{00} = 0$. Then eigenvalue problem (3) has a denumerable set of real eigenvalues of finite multiplicity λ_k, $k = 0, 1, \ldots$, which are repeated according to their multiplicity:

$$\begin{aligned} &0 < \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{i-1} < \sigma, \\ &\sigma = \lambda_i \leq \lambda_{i+1} \leq \ldots \leq \lambda_k \leq \ldots, \quad \lim_{k \to \infty} \lambda_k = \infty. \end{aligned}$$

Each eigenvalue λ_k, $k \geq 0$, is a unique root of the equation:

$$\mu - \gamma_k(\mu) = 0, \quad \mu \in \Lambda, \quad k \geq 0.$$

The eigensubspace $U(\lambda_k)$ of nonlinear eigenvalue problem (3) is

(a) the eigensubspace corresponding to the eigenvalue $\varphi_k(\mu)$ of linear eigenvalue problem (6) for $\mu = \lambda_k$, $\lambda_k \in \Lambda_1$, or

(b) the eigensubspace corresponding to the eigenvalue $\psi_k(\mu)$ of linear eigenvalue problem (7) for $\mu = \lambda_k$, $\lambda_k \in \Lambda_2$, or

(c) the eigensubspace corresponding to the eigenvalue $\lambda_{0i} = \lambda_i = \sigma$ of linear eigenvalue problem (9) for $k = i$.

Remark 13 Assume that any eigenfunction u of eigenvalue problem (8) satisfies the relation $u(x_0) = 0$. Then eigenvalue problems (6) and (7) have the eigenvalues $\varphi_{i-1}(\mu) = \psi_i(\mu) = \lambda_i^0$, $i = 1, 2, \ldots$ Hence eigenvalue problem (3) has eigenvalues $\lambda_i = \lambda_i^0$, $i = 1, 2, \ldots$ In this case, any eigenfunction u of eigenvalue problems (3), (6), (7), satisfies the equality $u(x_0) = 0$.

Remark 14 To solve problems (4) and (5) one can use the finite element method or the finite difference method. The abstract convergence results for the finite element method for problems (4) and (5) are presented in the papers [13], [14], [15], [16]. The matrix nonlinear eigenvalue problem of the finite element method [16] has the form: find $\lambda \in \Lambda_0$, $y \in \mathbb{R}^N \setminus \{0\}$, such that

$$A(\lambda)y = \lambda B(\lambda)y \tag{11}$$

with large sparse symmetric positive definite matrices $A(\lambda)$ and $B(\lambda)$ of order N, $\Lambda_0 = \Lambda_1$ or $\Lambda_0 = \Lambda_2$. The matrix functions $A(\mu)$, $\mu \in \Lambda_0$, and $B(\mu)$, $\mu \in \Lambda_0$, has the monotonicity property $(A(\mu)y, y) \geq (A(\eta)y, y)$, $(B(\mu)y, y) \geq (B(\eta)y, y)$, for $\mu < \eta$, $\mu, \eta \in \Lambda_0$, $y \in \mathbb{R}^N$. Note that the matrix $A(\mu)$ is spectrally equivalent [5] to the matrix C of the grid approximation of the biharmonic operator. Therefore, the matrix C can be chosen as the preconditioner for $A(\mu)$. Efficient preconditioned iterative methods for solving large monotone nonlinear eigenvalue problems of the form (11) have been suggested in the papers [17], [18], [19], [20], [21]. A similar approach for solving the eigenvalue problem describing the guided modes of an optical fiber has been proposed in [22].

Remark 15 The discretization methods for linear eigenvalue problems (6), (7), (8), (9), lead to the matrix linear eigenvalue problem: find $\eta \in \mathbb{R}$, $y \in \mathbb{R}^N \setminus \{0\}$, such that

$$A_0 y = \eta B_0 y$$

with large sparse symmetric positive definite matrices A_0 and B_0 of order N. This problem can be solved by efficient preconditioned eigensolvers suggested and investigated in the recent papers [9], [6], [7], [8]. By analogy with Remark 14, the matrix C of the grid approximation of the biharmonic operator can be chosen as the preconditioner.

5 Nonlinear biharmonic eigenvalue problem

Let us consider problem (2) for the constant coefficients D, ν, ρ, and d. For this case problem (2) can be formulated in the following form: find values $\tilde{\lambda}$ and nontrivial functions $u(x)$, $x \in \Omega$, such that

$$\begin{aligned} &\Delta^2 u + \frac{\tilde{\lambda}\tilde{\sigma}}{\tilde{\lambda} - \tilde{\sigma}} \tilde{M} \delta(x - x_0) u = \tilde{\lambda}\, u, \quad x \in \Omega, \\ &u = \partial_n u = 0, \quad x \in \Gamma, \end{aligned} \tag{12}$$

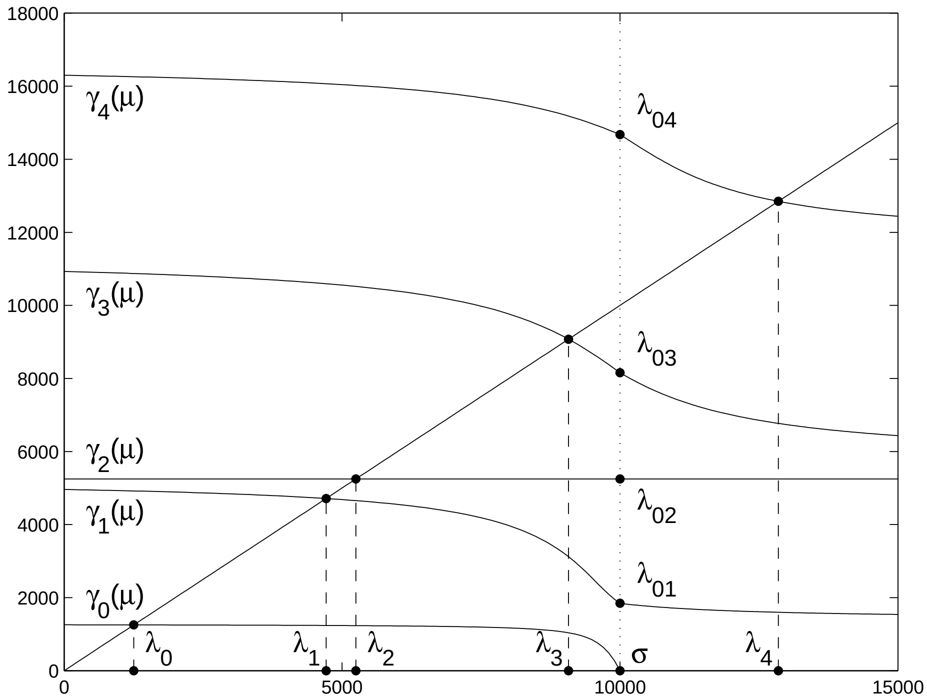

Figure 1: Five minimal eigenvalues of nonlinear biharmonic eigenvalue problem

where $\Delta^2 = \partial_1^4 + \partial_2^4 + 2\partial_1^2\partial_2^2$ denotes the biharmonic operator,

$$\tilde{\lambda} = \frac{\lambda \rho d}{D}, \quad \tilde{\sigma} = \frac{\sigma \rho d}{D}, \quad \tilde{M} = \frac{M}{\rho d}.$$

Note that the problem of form (12) can be derived from problem (2) when we put $D = 1$, $\rho d = 1$. Therefore, as a simple example of problem (2) we can take the following nonlinear biharmonic eigenvalue problem: find values λ and nontrivial functions $u(x)$, $x \in \Omega$, such that

$$\begin{aligned} &\Delta^2 u + \frac{\lambda\sigma}{\lambda - \sigma} M \delta(x - x_0) u = \lambda\, u, \quad x \in \Omega, \\ &u = \partial_n u = 0, \quad x \in \Gamma. \end{aligned} \tag{13}$$

Here we set $\Omega = (0,1)^2$, $M = 0.01$, $K = 100$, $\sigma = 10000$, $x_0 = (9/26, 19/26)^\top$. For problem (13) we can formulate problems (4)–(9), if we put $D = 1$, $\rho d = 1$. These problems have been solved numerically by applying the finite difference method. We use the standard thirteen-point finite difference approximation of the biharmonic operator on the uniform mesh [3], [25], [26], [10], [11]. Results of numerical experiments are demonstrated by Figure 1. We show the functions $\gamma_i(\mu)$, $i = 0, 1, 2, 3, 4$, eigenvalues λ_{0i}, $i = 1, 2, 3, 4$, of problem (9), and eigenvalues λ_i, $i = 0, 1, 2, 3, 4$, of nonlinear biharmonic eigenvalue problem (13). Thus, Figure 1 illustrates the existence result of Theorem 11 for nonlinear biharmonic eigenvalue problem (13).

6 Conclusion

This paper is devoted to the theoretical investigation of an eigenvalue problem describing the natural oscillations of a plate with elastically attached load. We consider the questions on the existence of eigenvalues and eigenfunctions and study their properties. We show that this problem belongs to the class of monotone positive definite nonlinear eigenvalue problems. This allows to apply the discretization methods for solving the problem and to use the theoretical results on the convergence and error estimates obtained by the author in the previous papers. The discretization methods lead to a monotone nonlinear matrix eigenvalue problem with large sparse matrices. We show that for solving this matrix eigenvalue problem one can apply efficient preconditioned iterative methods, which have been suggested in the previous papers of the author.

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