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A Dirichlet-Dirichlet DD-pre-conditioner for p-FEM

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Abstract

In this paper, a uniformly elliptic second order boundary value problem in 2D is discretized by the p-version of the finite element method. An inexact Dirichlet-Dirichlet domain decomposition pre-conditioner for the system of linear algebraic equations is investigated. The solver for the problem in the sub-domains and a pre-conditioner for the Schur-complement are proposed as ingredients for the inexact DD-pre-conditioner. Finally, several numerical experiments are given.

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Contents

1 Introduction **1**

2 Pre-conditioner for A_i **4**
2.1 Properties of $A_{\mathcal{R}_2}$ 4
2.2 A first pre-conditioner for the element stiffness matrix 5
2.2.1 Pre-conditioner of Jensen and Korneev 5
2.2.2 Modification of the pre-conditioner in 1D 6
2.2.3 Modification of the pre-conditioner in 2D and 3D 6
2.3 Similar systems of linear equations for other methods of discretization 6
2.3.1 Finite differences 7
2.3.2 h-version of the FEM, linear elements 8
2.4 Multi-level methods for the p-version block matrix A_i 8
2.5 Numerical results 11

3 Pre-conditioner for the Schur-complement **13**

4 Numerical results for the DD pre-conditioner **14**
4.1 Results for several problems with the same pre-conditioner 17
4.2 Influence of the parameter δ 19
4.3 Comparison of several pre-conditioners for one example. 19

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1 Introduction

We consider the following boundary value problem. Let $\Omega \subset \mathbb{R}^2$ be a domain which can be decomposed into quadrilaterals R_s. Find $u \in \hat{H}_0^1(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\}$, $\Gamma_1 \cap \Gamma_3 = \emptyset$, $\Gamma_1 \cup \Gamma_3 \subset \partial\Omega$ such that

$$a_\triangle(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v + \int_{\Gamma_3} f_1 v := \langle f, v\rangle + \langle f_1, v\rangle_{\Gamma_3} \tag{1}$$

for all $v \in \hat{H}_0^1(\Omega)$ holds. Problem (1) will be discretized by means of the p-version of the finite element method. Let $\mathcal{R}_2 = (-1, 1)^2$ be the reference element and $\Phi_s : \mathcal{R}_2 \to R_s$ be the bilinear mapping to the element R_s. We define the finite element space

$$\mathbb{M} := \{u \in \hat{H}_0^1(\Omega), u\mid_{R_s} = \tilde{u}(\Phi_s^{-1}(x, y)), \tilde{u} \in \mathbb{Q}_p\},$$

where $\mathbb{Q}_p$ is the space of all polynomials $p(x, y) = p_1(x)p_2(y)$ of maximal degree p in each variable. On the reference element $\mathcal{R}_2$, we choose the integrated Legendre polynomials $\hat{L}_{i,j}(x, y) = \hat{L}_i(x)\hat{L}_j(y)$ as basis, where

$$\hat{L}_i(x) = \frac{1}{2}\sqrt{(2i-3)(2i-1)(2i+1)} \int_{-1}^{x} L_{i-1}(s)\, \mathrm{d}s \quad \text{for} \quad i \geq 2, \quad \hat{L}_{0/1}(x) = \frac{1 \pm x}{2} \tag{2}$$

with the i-th Legendre polynomial $L_i(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i (x^2-1)^i$. Note that $\hat{L}_i(\pm 1) = 0$ for $i \geq 2$. Thus, the ansatz functions on the reference element can be split into the vertex-functions with $i, j = 0, 1$, the interior bubbles with $2 \leq i, j \leq p$ and the edge bubble functions.

In order to define a basis $(\psi_1, \ldots, \psi_{n_p})$ for $\mathbb{M}$, we can proceed as follows:

Let n_v, n_e and n_i be the number of vertices not having a Dirichlet boundary condition, number of edges not having a Dirichlet boundary condition, and number of elements. To each vertex corresponds 1, to each edge correspond $p-1$ and to each element correspond $(p-1)^2$ basis functions. Thus, the dimension of the ansatz space is $n_p = n_v + (p-1)n_e + (p-1)^2 n_i$. So, we define the functions $\psi_1, \ldots, \psi_{n_v}$ as the usual piecewise bilinear hat functions. The functions $\psi_{n_v+(j-1)(p-1)+1}, \ldots, \psi_{n_v+j(p-1)}$ correspond to the edge e_j of the mesh, and vanish on all other edges, i.e. satisfy the condition $\psi_{n_v+(j-1)(p-1)+k-1} \mid_{e_l} = \delta_{j,l}\hat{L}_k$, where $k = 2, \ldots, p$ and $j, l = 1, \ldots, n_e$. The support of an edge function is formed by those two elements, which have this edge e_j in common. The remaining basis functions $\psi_{n_v+(p-1)n_e+1}, \ldots, \psi_{n_p}$ are interior bubble functions consisting of a support containing one element only.

The Galerkin projection of (1) onto $\mathbb{M}$ leads to the linear system of algebraic finite element equations

$$A\underline{u}_p = \underline{f}_p, \quad \text{where} \quad A = [a_\Delta(\psi_j, \psi_i)]_{i,j=1}^{n_p}, \quad \text{and} \quad \underline{f}_p = [\langle f, \psi_i\rangle + \langle f_1, \psi_i\rangle_{\Gamma_3}]_{i=1}^{n_p}. \tag{3}$$

Efficient solvers for (3) can be built by domain decomposition techniques, [14], [5], [1]. We consider there an approach of Jensen/Korneev [18] and Ivanov/Korneev [16], [17]. For this purpose, the basis functions ψ_i are divided into three groups,

- the vertex functions, i.e. $\psi_1, \ldots, \psi_{n_v}$,
- the edge bubble functions $\psi_{n_v+1}, \ldots, \psi_{n_v+(p-1)n_e}$,
- the interior bubbles $\psi_{n_v+(p-1)n_e+1}, \ldots, \psi_{n_p}$.

Corresponding to the division of the shape functions, the matrix A is splitted into three blocks

$$A = \begin{bmatrix} A_v & A_{v,e} & A_{v,i} \\ A_{e,v} & A_e & A_{e,i} \\ A_{i,v} & A_{i,e} & A_i \end{bmatrix}, \tag{4}$$

where the indices v, e and i denote the blocks corresponding to the vertex, edge bubble and interior bubble functions, respectively. Considering the simpler matrix

$$C = \begin{bmatrix} A_v & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_e & A_{e,i} \\ \mathbf{0} & A_{i,e} & A_i \end{bmatrix}, \tag{5}$$

it has been proved in [5] that the condition number $\kappa(C^{-1}A)$ grows as $1 + \log p$. Therefore, the vertex unknowns can be determined separately. Computing the other unknowns, we factorize the remaining 2 by 2 block

$$A_{II} = \begin{bmatrix} A_e & A_{e,i} \\ A_{i,e} & A_i \end{bmatrix} = \begin{bmatrix} I & A_{e,i}A_i^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} S & \mathbf{0} \\ \mathbf{0} & A_i \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ A_i^{-1}A_{i,e} & I \end{bmatrix} \tag{6}$$

with the Schur-complement $S = A_e - A_{e,i}A_i^{-1}A_{i,e}$. The matrix A_i is a block diagonal matrix, each block A_{R_s} corresponds to one element R_s, i.e.

$$A_i = \mathrm{blockdiag}[A_{R_s}]_{s=1}^{n_i}. \tag{7}$$

Therefore, in order to compute the interior unknowns, we have to solve a Dirichlet problem on each quadrilateral. The edge unknowns are computed via the Schur-complement S.
An inexact DD-pre-conditioner for (4) includes a pre-conditioner for A_i, a pre-conditioner for the Schur-complement S and an extension operator from the edges of the quadrilateral into its interior.
In [18], Jensen/Korneev have proved the following result.

LEMMA 1.1. *Let $\partial R_s \in C^{(t)}$, $t \geq 2$, $s = 1, \ldots, n_i$, where $C^{(t)}$ denotes the class of all boundaries which consist of a finite number of t times continuously differentiable curves and the angles of these curves at their intersection points on ∂R_s are distinct from 0 and 2π. Let $\tilde{A}$ be the result of assembling the element stiffness matrices on the reference element $\mathcal{R}_2 = [-1,1]^2$ instead of the element stiffness matrix corresponding to the element R_s. Then, $\kappa\left(\tilde{A}^{-1}A\right) = \mathcal{O}(1)$.*

So, it is possible to restrict ourselves to the case of the reference element in order to derive a pre-conditioner for A_i and S.

In [16], [17] and [18], two pre-conditioners for the Schur-complement are proposed. The pre-conditioner uses basis transformations from the integrated Legendre polynomials to the Chebyshev polynomials or a Lagrange basis. The extension operator has been considered in [5].

Moreover, Jensen/Korneev, [18], derived a spectrally equivalent pre-conditioner $\tilde{C}$ for the matrix $A_{\mathcal{R}_2}$ resulting from the discretization of the Dirichlet problem in the element $\mathcal{R}_2 = [-1,1]^2$. This pre-conditioner has $\mathcal{O}(p^2)$ non-zero entries. In the case of parallelogram elements, the element stiffness matrix has $\mathcal{O}(p^2)$ non-zero entries, too. Otherwise, it is a dense matrix. However the algorithms presented in [18] in order to solve systems of algebraic equations of the type $\tilde{C}\underline{w} = \underline{r}$ require $\mathcal{O}(p^3)$ arithmetical operations. Therefore, it is important to find fast solvers, i.e. solvers which produce the solution by means of $\mathcal{O}(p^2)$ arithmetical operations.

In [7], a new pre-conditioner for $A_{\mathcal{R}_2}$ is defined which can be interpreted as a system matrix of a degenerated elliptic boundary value problem discretized by the h-version of the FEM or the method of finite differences. Linear elements on isosceles, right-angled triangles or finite differences on a regular grid are used. Another possibility are bilinear elements on squares [6]. This approach can be extended to the case of a pre-conditioner for the element stiffness matrix on the reference element $\mathcal{R}_3 = [-1,1]^3$ in 3D, see [10], [6]. The problem on the element $\mathcal{R}_2$ can be treated using multi-grid algorithms, see e.g. [21], [22], [23], [12], [11], [15], with special smoothers. A multi-grid convergence proof for linear elements in $2D$ is given in [7]. An alternative for the case of the reference elements $\mathcal{R}_2$ and $\mathcal{R}_3$ are wavelet methods, see [10].

Korneev, [20], found an optimal pre-conditioner for this degenerated problem by using Domain Decomposition techniques. For the subproblems, Fast Fourier Transforms, [13], are used and tridiagonal systems are solved.

In this paper, we define several pre-conditioners for S and A_i and give the main condition number estimates. In the main part of this paper, the performance of the proposed pre-conditioners is investigated in some numerical experiments.

The paper is organized as follows. In section 2, we consider the pre-conditioner for the matrix A_i. We introduce and modify the pre-conditioner of Jensen/Korneev, [18]. Moreover, it is shown that the modified pre-conditioner can be obtained by discretizing elliptic problems with variable coefficients using finite differences or the h-version of the finite element method. In section 3, the pre-conditioner of the Schur-complement is defined. In section 4, the performance of the DD pre-conditioner is shown in several numerical examples.

Throughout this paper, $\mathcal{R}_2$ will denote the unit square $(-1,1)^2$, Ω_1 the square $(0,1)^2$. The integer p is the polynomial degree, $\hat{L}_i$ the $i-$th integrated Legendre polynomial. The real number $\lambda_{max}(A)$ will denote the largest eigenvalue of a matrix A and $\lambda_{min}(A)$ the smallest eigenvalue of A. For a sequence of symmetric and positive definite matrices $A, B \in \mathbb{R}^{n\times n}$, the relation $A \preceq B$ means that $A - cB$ is positive definite, where c is a constant independent of n. The relation $A \sim B$ means $A \preceq B$ and $B \preceq A$, i.e. the matrices A and B are spectrally equivalent.

2 Pre-conditioner for A_i

Due to (7), we have $A_i = \text{blockdiag}\,[A_{R_s}]_{s=1}^{n_i}$. By Lemma 1.1 applied to $\tilde{A} = A_{R_s}$, $s = 1, \ldots, n_i$, one has $A_{R_s} \sim A_{\mathcal{R}_2}$. Thus, $A_i \sim \text{blockdiag}\,[A_{\mathcal{R}_2}]_{s=1}^{n_i}$. Therefore, in order to derive a pre-conditioner for A_i, it suffices to develop a pre-conditioner for $A_{\mathcal{R}_2}$. In subsection 2.1, the most important properties of $A_{\mathcal{R}_2}$ are proved. In subsection 2.2, a first pre-conditioner for $A_{\mathcal{R}_2}$ is proposed. This pre-conditioner can be interpreted as stiffness matrix of an h-version finite element discretization of a degenerated elliptic problem, cf. subsection 2.3. An optimal pre-conditioner for such a problem is a multi-grid pre-conditioner with line-smoother, which is proposed in subsection 2.4. In subsection 2.5, some numerical examples show the efficiency of the pre-conditioner for several elements R_s.

2.1 Properties of $A_{\mathcal{R}_2}$

We consider the model problem

$$-\triangle u = f \quad \text{in} \quad \mathcal{R}_2 = (-1,1)^2, \qquad u = 0 \quad \text{on} \quad \partial\mathcal{R}_2. \tag{8}$$

Problem (8) is solved by using the $p-$version of the finite element method with only one element $\mathcal{R}_2$. Problem (8) is the model problem for solving a linear system with the matrix $A_{\mathcal{R}_2}$. As finite element space, we choose $\mathbb{M}_0 = H_0^1 \cap \text{span}\{x^i y^j\}_{i,j=0}^p$. The discrete problem is: find $u_p \in \mathbb{M}_0$ such that

$$\int_{\mathcal{R}_2} \nabla u_p \cdot \nabla v_p = \int_{\mathcal{R}_2} f v_p \quad \forall v_p \in \mathbb{M}_0.$$

With the basis of the integrated Legendre polynomials (2), the stiffness matrix $A_{\mathcal{R}_2}$ for (8) is determined by

$$A_{\mathcal{R}_2} = [a_{ij,kl}]_{i,j=2;k,l=2}^p, \quad \text{where} \quad a_{ij,kl} = \int_{\mathcal{R}_2} \nabla \hat{L}_{ij}(x,y) \cdot \nabla \hat{L}_{kl}(x,y). \tag{9}$$

The matrix $A_{\mathcal{R}_2}$ can be written explicitely as

$$A_{\mathcal{R}_2} = F \otimes D + D \otimes F, \tag{10}$$

where

$$F = \begin{bmatrix} 1 & 0 & \mathfrak{c}_2 & 0 & \cdots \\ 0 & 1 & 0 & \mathfrak{c}_3 & \ddots \\ \mathfrak{c}_2 & 0 & 1 & 0 & \ddots \\ & \ddots & & \ddots & \ddots \\ \cdots & 0 & \mathfrak{c}_{p-2} & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \mathfrak{d}_2 & 0 & & \cdots & 0 \\ 0 & \mathfrak{d}_3 & 0 & \ddots & 0 \\ 0 & 0 & \mathfrak{d}_4 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \mathfrak{d}_p \end{bmatrix} \tag{11}$$

with the coefficients

$$\mathfrak{c}_i = -\frac{1}{2}\sqrt{\frac{(2i-3)(2i+5)}{(2i-1)(2i+3)}} \qquad \text{and} \qquad \mathfrak{d}_i = \frac{(2i-3)(2i+1)}{2},$$

cf. [18]. Applying a permutation P of rows and columns, we get

$$PA_{\mathcal{R}_2}P^T = \begin{bmatrix} \mathcal{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{A}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_4 \end{bmatrix}. \tag{12}$$

The four blocks correspond to the polynomials $\{\hat{L}_{2i,2j}\}$, $\{\hat{L}_{2i+1,2j}\}$, $\{\hat{L}_{2i,2j+1}\}$, and $\{\hat{L}_{2i+1,2j+1}\}$. If p is odd, all four blocks have the same size. We have to find a fast solver for a system of linear equations with the matrix $A_{\mathcal{R}_2}$, or equivalently, $\mathcal{A}_i$, $i = 1, 2, 3, 4$. The main properties of the blocks $\mathcal{A}_i$ are summarized in the following lemma.

LEMMA 2.1. *The condition number of $\mathcal{A}_i$ is of order p^2. The blocks $\mathcal{A}_i$ are spectrally equivalent to each other, i.e. $\mathcal{A}_i \sim \mathcal{A}_j$ for $i, j = 1, \ldots, 4$.*

Proof. See [18]. □

In the following, we will focus on finding an efficient pre-conditioner for $\mathcal{A}_1$. Via Lemma 2.3, the pre-conditioner for $A_{\mathcal{R}_2}$, and via relation (7) and Lemma 1.1, the pre-conditioner for A_i follows. Let $D_1 = \text{diag}\,[\mathfrak{d}_2, \mathfrak{d}_4, \mathfrak{d}_6, \ldots]$, $F_1 = \text{tridiag}[-c_e, 1, -c_e]$ with $c_e = [\mathfrak{c}_2, \mathfrak{c}_4, \mathfrak{c}_6, \ldots]$. Then, cf. (10), (11), we have

$$\mathcal{A}_1 = D_1 \otimes F_1 + F_1 \otimes D_1.$$

In the following, let us assume that p is odd. Moreover, let $n - 1 = \frac{p-1}{2}$ be the dimension of F_1 and D_1.

2.2 A first pre-conditioner for the element stiffness matrix

2.2.1 Pre-conditioner of Jensen and Korneev

In [18], Jensen and Korneev have derived a pre-conditioner C_1 for the matrix $\mathcal{A}_1$, or equivalently, for $A_{\mathcal{R}_2}$. The matrices

$$D_3 = 4\;\text{diag}\left[i^2\right]_{i=1}^{n-1}, \qquad T_1 = D_3^{-1} + \frac{1}{2}\text{tridiag}\left[-1, 2, -1\right],$$
$$\text{and} \quad C_1 \;=\; D_3 \otimes T_1 + T_1 \otimes D_3 \tag{13}$$

are introduced. Then, the following lemma holds.

LEMMA 2.2. *The spectral equivalence relations $D_1 \sim D_3$, $F_1 \sim T_1$ and $C_1 \sim \mathcal{A}_1$ are valid.*

Proof. The estimate $D_1 \sim D_3$ is trivial, $F_1 \sim T_1$ has been proved in [18], and the assertion $C_1 \sim \mathcal{A}_1$ follows by the properties of the Kronecker product from $D_1 \sim D_3$ and $F_1 \sim T_1$. □

In the matrix C_1, the same matrix entries are nonzero as in $\mathcal{A}_1$, but the structure of the nonzero elements is simpler. However, a fast solver for C_1 is required as well as for $\mathcal{A}_1$.

2.2.2 Modification of the pre-conditioner in 1D

Now, the pre-conditioners (13) are modified. The resulting matrices can be interpreted as stiffness matrices of discretizations of degenerated elliptic problems which will be shown in subsection 2.3. In a first step, the matrix T_1 is simplified. Let

$$T_2 = \frac{1}{2}\mathrm{tridiag}\left[-1, 2, -1\right]. \tag{14}$$

In [9], we have shown the following result.

LEMMA 2.3. *The eigenvalues of the matrix $T_2^{-1}T_1$ can be estimated by $\lambda_{min}\left({T_2}^{-1}T_1\right) \geq 1$ and $\lambda_{max}\left({T_2}^{-1}T_1\right) \leq c$, where the constant c is independent of the dimension of the matrices.*

Proof. The lower eigenvalue estimate is trivial. In [6], [10], we have proved $\lambda_{max}\left({T_2}^{-1}T_1\right) \preceq (1 + \log n)$, where $T_1 \in \mathbb{R}^{n-1 \times n-1}$. With methods of the multi-resolution analysis derived in [10], this result can be strengthened by the estimate $\lambda_{max}\left({T_2}^{-1}T_1\right) \preceq 1$. For more details, we refer to [9]. $\square$

In a second step, the diagonal matrix D_3 is modified. The matrix $D_4 = D_3 + \frac{2}{3}I$ is introduced, where I denotes the identity matrix. The proof of next lemma is trivial.

LEMMA 2.4. *The eigenvalue estimates $\lambda_{min}\left({D_4}^{-1}D_3\right) = \frac{6}{7}$ and $\lambda_{max}\left({D_4}^{-1}D_3\right) < 1$ are valid.*

2.2.3 Modification of the pre-conditioner in 2D and 3D

Via tensor product and by the relations (13) for D_3, and (14) for T_2, the matrices

$$C_3 = D_3 \otimes T_2 + T_2 \otimes D_3 \qquad \text{and} \qquad C_4 = D_4 \otimes T_2 + T_2 \otimes D_4 \tag{15}$$

are introduced. Then, the following theorem holds.

THEOREM 2.5. *For $i = 3, 4$, the spectral equivalence relations $C_i \sim \mathcal{A}_1$ are valid.*

Proof. Note that D_3, D_4, and T_2 are symmetric and positive definite. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, the assertions follow. $\square$

2.3 Similar systems of linear equations for other methods of discretization

In this subsection, we show interpretations of the matrices C_3 and C_4 as discretizations of a degenerated elliptic boundary value problem in the unit square by finite differences and finite elements.

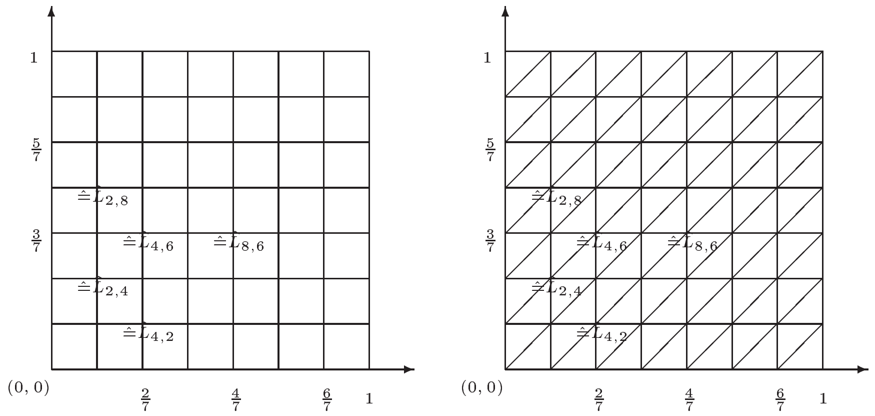

Figure 1: Mesh for h-version (right), grid (left) for $n = 7$.

2.3.1 Finite differences

We consider the problem

$$-2\left(y^2 u_{xx} + x^2 u_{yy}\right) = g \quad \text{in} \quad \Omega_1 = (0,1)^2 \qquad \text{and} \qquad u\mid_{\partial\Omega_1} = 0. \tag{16}$$

This problem is discretized by finite differences in the grid of Figure 1. Let $u^{i,j}$ be the approximation of u in $x_{ij} = \frac{1}{n}(i,j)$. The second derivatives in (16) are approximated by the usual second order central difference quotient. More precisely,

$$\begin{aligned} y^2 u_{xx}(x_{ij}) &\approx j^2(u^{i+1,j} - 2u^{i,j} + u^{i-1,j}), \\ x^2 u_{yy}(x_{ij}) &\approx i^2(u^{i,j+1} - 2u^{i,j} + u^{i,j-1}). \end{aligned} \tag{17}$$

Then, the following lemma holds.

LEMMA 2.6. *Let $u^{i,j}$ be the approximation of u in x_{ij} of problem (16). Then, the linear system $C_3\underline{u} = \underline{g}$ has to be solved, where C_3 is defined in (15), $\underline{u} = (u^{i,j})_{i,j=1}^{n-1}$ and $\underline{g} = (g(x_{ij}))_{i,j=1}^{n-1}$.*

Proof. From (17) follows that the linear equations

$$\begin{aligned} 4(i^2 + j^2)u^{i,j} - 2j^2(u^{i+1,j} + u^{i-1,j}) - 2i^2(u^{i,j+1} + u^{i,j-1}) &= g(x_{ij}), \\ u^{0,j} = 0, \quad u^{n,j} = 0, \quad u^{i,0} = 0, \quad u^{i,n} = 0 & \end{aligned}$$

have to be solved for $i, j = 1, \ldots, n-1$. These linear algebraic equations are equivalent to $C_3\underline{u} = \underline{g}$. $\square$

We note that this approach can be extended to the case of the element stiffness matrix on the reference element $\mathcal{R}_3 = [-1,1]^3$. Then, the 4th order differential equation

$$z^2 u_{xxyy} + y^2 u_{xxzz} + x^2 u_{yyzz} = g$$

has to be discretized.

2.3.2 h-version of the FEM, linear elements

We consider now problem (16) in the weak formulation:
Find $u \in \tilde{H}_0^1(\Omega_1) = \{u \in L^2(\Omega_1):\ xu_y, yu_x \in L^2(\Omega_1),\ u = 0 \text{ on } \partial\Omega_1\}$ such that

$$a(u, v) := \int_{\Omega_1} y^2 u_x v_x + x^2 u_y v_y = \int_{\Omega_1} gv =: \langle g, v\rangle \quad \forall v \in \tilde{H}_0^1(\Omega_1) \tag{18}$$

holds. The domain Ω_1 is the unit square $(0, 1)^2$.
Problem (18) is discretized by linear ansatz functions on the mesh τ_k of Figure 1, consisting of congruent, isosceles, right-angled triangles. Let k be the level of approximation, $n = 2^k$, $I = (n-1)(i-1) + j$ and $N = (n-1)^2$. The shape functions ϕ_I^k are the usual piecewise linear hat-functions associated to the nodes $x_{ij} = \frac{1}{n}(i, j)$. Let ω_k and ω_{k-1} be the index sets of the numbers of nodes in the meshes τ_k and τ_{k-1}, respectively. We define the approximation space

$$\mathbb{V}_k = \text{span}\{\phi_I^k, I \in \omega_k\}$$

and we introduce the subspace of nodal basis functions associated to the new nodes

$$\mathbb{W}_k = \text{span}\{\phi_I^k, I \in \omega_k \backslash \omega_{k-1}\}.$$

Then, the Galerkin projection of (18) onto $\mathbb{V}_k$ is: Find $u^k \in \mathbb{V}_k$ such that

$$a(u^k, v^k) = \langle g, v^k\rangle \quad \forall v^k \in \mathbb{V}_k \tag{19}$$

holds. Problem (19) is equivalent to solving the system of algebraic finite element equations $K_{h,k}\underline{u}_h = \underline{g}_h$ with

$$K_{h,k} = \left[a(\phi_J^k, \phi_I^k)\right]_{I,J=1}^N, \qquad \underline{g}_h = \langle g, \phi_J^k\rangle_{J=1}^N, \qquad \underline{u}_h = (u_I)_{I=1}^N. \tag{20}$$

Then, $u_h = \sum_{I=1}^N u_I \phi_I^k$ is the solution of (19).

LEMMA 2.7. *Let C_4 be the matrix defined in (15). Then, we have $K_{h,k} = \frac{1}{2n^2} C_4$.*

Proof. The proof is a simple integration, [7]. ☐

2.4 Multi-level methods for the p-version block matrix A_i

We are interested in an optimal pre-conditioner for the matrix A_i (7), or equivalently for $\mathcal{A}_1$ (9), the first block of the element stiffness matrix $A_{\mathcal{R}_2}$ for the interior unknowns on $(-1, 1)^2$ with respect to the basis of the integrated Legendre polynomials $\hat{L}_{ij}$, $2 \le i, j \le p$, cf. Lemma 1.1 and 2.1. By Theorem 2.5, the matrix C_4 (15) is a good pre-conditioner for each block $\mathcal{A}_j$, $j = 1, \ldots, 4$, of the matrix $PA_{\mathcal{R}_2}P^T$. By Lemma 2.7 we can conclude that the matrix C_4 can be interpreted as the stiffness matrix for $-x^2 u_{yy} - y^2 u_{xx}$ using piecewise linear shape functions on isosceles, right-angled, congruent triangles on the domain $\Omega_1 = (0,1)^2$ with Dirichlet boundary conditions, i.e.

$$K_{h,k} = \frac{1}{2n^2} C_4.$$

Now, we present a multi-grid algorithm in order to solve a system of algebraic finite element equations $K_{h,k}\underline{u}_h = \underline{f}_h$. We represent the space $\mathbb{V}_k$ as the direct sum

$$\mathbb{V}_k = \mathbb{V}_{k-1} \oplus \mathbb{W}_k.$$

Let u_0 be the initial guess. The new iterate u_1 is computed by the following recursive procedure $u_1 = MULT(k, u_0, g)$.

- Set $l = k$. If $l > 1$ then do

  1. Pre-smoothing on $\mathbb{W}_l$: Solve

     $$a(w,v) = \langle g, v\rangle - a(u_0, v) \quad \forall v \in \mathbb{W}_l$$

     approximately by using ν steps of a simple iterative method $\mathcal{S}$, the approximate solution is $\tilde{w}$. Set $u_0^1 = u_0 + \tilde{w}$.
  2. Coarse grid correction on $\mathbb{V}_{l-1}$: Find $w \in \mathbb{V}_{l-1}$ such that

     $$a(w,v) = \langle g, v\rangle - a(u_0^1, v) = \langle r, v\rangle \quad \forall v \in \mathbb{V}_{l-1}$$

     holds. Compute an approximate solution $\tilde{w}$ by using μ_{l-1} steps of the algorithm $MULT(l-1, 0, r)$. Set $u_0^2 = u_0^1 + \tilde{w}$.
  3. Post-smoothing on $\mathbb{W}_l$: Solve

     $$a(w,v) = \langle g, v\rangle - a(u_0^2, v) \quad \forall v \in \mathbb{W}_l$$

     approximately by using ν steps of a simple iterative method $\mathcal{S}$, the approximate solution is $\tilde{w}$. Set $u_1 = u_0^2 + \tilde{w}$.

- else

  - Solve $a(w,v) = \langle g, v\rangle - a(u_0, v)$ for all $v \in \mathbb{V}_1$ exactly.

- endif.

REMARK 2.8. *In a standard multi-grid algorithm the space $\mathbb{W}_l$ in 1. and 3. is replaced by $\mathbb{V}_l$, e.g. the smoother operates on the complete approximation space.*

In order to define a smoother $\mathcal{S}$, we consider the auxiliary bilinear form $\tilde{a} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$,

$$\tilde{a}(u,v) = \int_{\Omega_1} g_1(x,y) u_x v_x + g_2(x,y) u_y v_y, \tag{21}$$

where

$$g_1(x,y)=\begin{cases} y^2 & \text{if} \quad x\le y\\ 0 & \text{if} \quad x>y\end{cases} \quad \text{and} \quad g_2(x,y)=\begin{cases} x^2 & \text{if} \quad y\le x\\ 0 & \text{if} \quad y>x\end{cases}.$$

With the choice of $\mathbb{X}=\mathbb{W}_k$ and $\mathbb{X}=\mathbb{V}_k$ in the bilinear form $\tilde{a}(\cdot,\cdot)$ (21), efficient smoothers for the Algorithm $MULT$ and the multi-grid algorithm of Remark 2.8 can be defined. Let $K_{\mathbb{W}_k}=a(\phi_J^k,\phi_I^K)_{I,J\in(\omega_k\setminus\omega_{k-1})}$ be the stiffness matrix of the Galerkin projection of (18) onto $\mathbb{W}_k$. Furthermore let

$$T_1=[t_{IJ}^{(1)}]_{I,J=1}^N, \quad \text{where} \quad t_{IJ}^{(1)}=\begin{cases} a(\phi_J^k,\phi_I^k) & \text{if} \quad I=J\\ \tilde{a}(\phi_J^k,\phi_I^k) & \text{if} \quad I\ne J\end{cases} \quad \text{and}$$

$$T_0=[t_{IJ}^{(0)}]_{I,J\in(\omega_k\setminus\omega_{k-1})}, \quad \text{where} \quad t_{IJ}^{(0)}=\begin{cases} a(\phi_J^k,\phi_I^k) & \text{if} \quad I=J\\ \tilde{a}(\phi_J^k,\phi_I^k) & \text{if} \quad I\ne J\end{cases}.$$

We define now the preconditioned Richardson iterations

$$\mathcal{S}_1=I-\omega T_1^{-1}K_{h,k} \quad \text{on} \quad \mathbb{V}_k \quad \text{and} \quad \mathcal{S}_0=I-T_0^{-1}K_{\mathbb{W}_k} \quad \text{on} \quad \mathbb{W}_k, \tag{22}$$

with the parameter $\omega=0.8$ by their error transion operators. Moreover let $\mathcal{S}_2$ be the usual Gauss-Seidel smoother acting on the space $\mathbb{V}_k$. Because of the properties of the bilinear form (21), the matrices T_0 and T_1 are tridiagonal after a proper permutation of the unknowns. Thus, $\mathcal{S}_0$ and $\mathcal{S}_1$ are line smoothers acting on lines L_{2m-1}, $m=1,\dots,\frac{n}{2}-1$ and L_m, $m=1,\dots,n-1$, where

$$L_m=\{\phi_I^k, I=(n-1)(i-1)+j, \max\{i,j\}=m\}.$$

Hence, the solution operations $T_0\underline{x}=\underline{y}$ or $T_1\underline{x}=\underline{y}$ can be done using Cholesky-decomposition in $\mathcal{O}(N)$ operations. The lines to the corresponding tridiagonal systems are marked by a bold line in Figure 2. This construction follows an idea of Axelsson and Padiy, [2]. For more details,

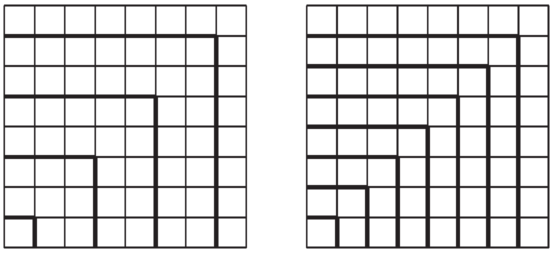

Figure 2: Lines for smoother $\mathcal{S}_0$ (left) and $\mathcal{S}_1$ (right).

see [7].

Using the multi-grid method with μ cycles and the smoother $\mathcal{S}$, we can derive a pre-conditioner $M_\mu^{\mathcal{S}}$ in order to solve a linear system with the matrix $K_{h,k}$ using the pcg-method, [19]. Furthermore, let

$$\tilde{M}_\mu^{\mathcal{S}}=P^T\text{blockdiag}\left[M_\mu^{\mathcal{S}}\right]_{i=1}^4 P \tag{23}$$

with the permutation P of (12).

THEOREM 2.9. *Let $A_{\mathcal{R}_2}$ be the matrix (10), let $\mathcal{S}_0$ be defined in (22) and let $\mu = 3$ and $\nu \geq 3$. The statement $\tilde{M}_\mu^{\mathcal{S}_0} \sim \frac{1}{p^2} A_{\mathcal{R}_2}$ is valid.*

Proof. In [7], the result $M_\mu^{\mathcal{S}} \sim K_{h,k}$ has been proved for $\mu \geq 3$. By (23), (12), Theorem 2.5 and Lemma 2.7, the assertion follows. □

Hence, we have found an asymptotically optimal method. The polynomial degree μ can be reduced from 3 to 2 by using AMLI-pre-conditioners $C_{h,k}$, [3], [4], [2] as pre-conditioner for the matrix $K_{h,k}$ (20). We refer to [8] for more details.
Due to Lemma 1.1 and relation (7), we propose the pre-conditioner

$$\hat{M}_\mu^{\mathcal{S}} = 2p^2 \operatorname{blockdiag}\left[\tilde{M}_\mu^{\mathcal{S}}\right]_{s=1}^{n_i} \tag{24}$$

for A_i.

THEOREM 2.10. *The spectral equivalence relation $\hat{M}_\mu^{\mathcal{S}_0} \sim A_i$ is valid for $\mu \geq 3$.*

Proof. Apply Lemma 1.1. By relations (7) and Theorem 2.9 the assertions follow. □

2.5 Numerical results

In this subsection, some numerical results show the performance of the pre-conditioner $\tilde{M}_\mu^{\mathcal{S}}$ for the element stiffness matrix A_{R_s}. The following examples of elements R are considered, cf. Figure 3,

(a) R is the unit square $[0,1]^2$, similar to the reference element $\mathcal{R}_2$,

(b) R is the trapezoid with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,0.4)$,

(c) R is the trapezoid with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,0.02)$,

(d) R is the quadrilateral with the vertices $(0,0)$, $(1,0)$, $(2,2)$ and $(0,1)$,

(e) R is a rectangle with lengths $a = 0.5$ and $b = 5$.

The linear system $A_R \underline{u} = \underline{f}$ is solved by a preconditioned conjugate gradient method with the pre-conditioners $\tilde{M}_\mu^{\mathcal{S}}$, namely

- $\tilde{M}_1^{\mathcal{S}_1}$, $\tilde{M}_1^{\mathcal{S}_0}$, $\tilde{M}_3^{\mathcal{S}_0}$ and $\tilde{M}_1^{\mathcal{S}_2}$ for example (a),
- $\tilde{M}_1^{\mathcal{S}_1}$ for the examples (b), (c), (d), and (e).

In all examples, the algorithm is stopped if the error in the preconditioned energy norm is reduced by a factor 10^{-9}. Table 1 displays the numbers of iterations for example (a) with $\underline{f} = [1,\ldots,1]^T$. In the case of the multi-grid pre-conditioners $\tilde{M}_1^{\mathcal{S}_0}$ and the multi-grid pre-conditioner with Gauss-Seidel smoother $\tilde{M}_1^{\mathcal{S}_2}$, the numbers of iterations grow. For $\tilde{M}_1^{\mathcal{S}_1}$ and $\tilde{M}_3^{\mathcal{S}_0}$, the numbers of iterations increase very slowly. Note that by Theorem 2.9, the spectral equivalence $A_R \sim p^2 \tilde{M}_3^{\mathcal{S}_0}$

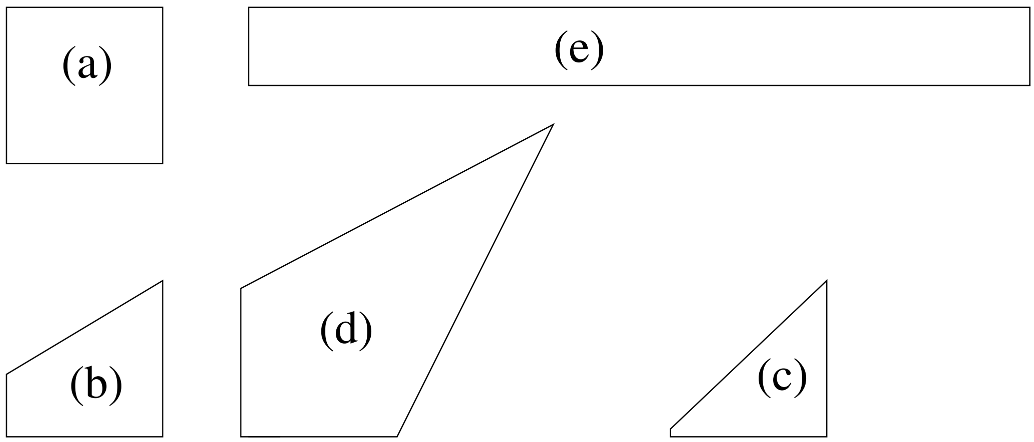

Figure 3: Plots of the elements R for examples (a), (b), (c), (d), (e).

p	$\tilde{M}_1^{S_1}$	$\tilde{M}_1^{S_0}$	$\tilde{M}_3^{S_0}$	$\tilde{M}_1^{S_2}$
3	2	2	2	2
7	15	15	16	14
15	17	20	20	16
31	20	26	23	19
63	21	31	24	24
127	22	36	25	31

Table 1: Numbers of iterations of the pcg-method for example (a) using several multi-grid pre-conditioners.

p	3	5	7	11	15	21	27	33	45	63
(b)	5	16	22	27	30	33	35	36	39	42
(c)	5	18	31	58	91	145	208	266	377	463
(d)	4	11	21	25	27	28	29	30	32	34
(e)	2	5	10	27	53	92	108	112	124	132

Table 2: Numbers of iterations of the pcg-method for examples (b), (c) and (d) using the multi-grid pre-conditioner $\tilde{M}_1^{S_1}$.

is valid. However, for polynomial degrees $p < 50$, the multi-grid pre-conditioner with Gauss-Seidel smoother has about as many iterations as the pre-conditioners $\tilde{M}_1^{S_1}$ and $\tilde{M}_3^{S_0}$. Table 2 displays the numbers of iterations for examples (b), (c), (d), and (e) with the multi-grid pre-conditioner $\tilde{M}_1^{S_1}$. Here, the right hand side $f(x,y) \equiv 1$ in (8) is chosen. For the examples (b) and (d), the numbers of iterations grow moderately. In comparison to example (a), the absolute values of the numbers of iterations are larger. For example (e), the numbers of iterations are very large. The reason is the geometry of the element R with two edges of length 0.5 and two edges of length 5. For elements which are similar to a rectangle with lengths a and b, where $a >> b$, the proposed multi-grid pre-conditioners can be modified such that the condition number estimate is independent of the parameters a and b and the polynomial degree p, see e.g. [10].
The trapezoid with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,0.02)$ is very close to the reference triangle with the vertices $(0,0)$, $(0,1)$ and $(1,1)$. This explains the fast increasing numbers of iterations in example (c).

3 Pre-conditioner for the Schur-complement

For the Schur-complement S in (6), Korneev and coauthors, [18], [17], have derived several pre-conditioners using basis transformations from the integrated Legendre polynomials $\left\{\hat{L}_i\right\}_{i=2}^p$ (2) to the Chebyshev polynomials or a Lagrange basis. The Schur-complement S corresponds to the edges of the mesh not having a Dirichlet boundary condition. On each edge e_j, $j = 1, \ldots, n_e$, the number of degrees of freedom is $p-1$.
Let for $j = 0, \ldots, p$,

$$T_j(x) = \cos(j \arccos x) \tag{25}$$

be the j-th Chebyshev polynomial first kind. Moreover, let $\xi_i^p = \cos\left(\frac{i\pi}{p}\right)$, $i = 0, \ldots, p$, be a set of grid-points and let

$$\ell_i^p(x) = \prod_{\substack{j=0 \\ j \neq i}}^{p} \frac{x - \xi_j^p}{\xi_i^p - \xi_j^p}, \quad i = 1, \ldots, p-1, \tag{26}$$

the Lagrangian interpolation polynomials according to the grid-points $\{\xi_i^p\}_{i=0}^p$. Moreover, let $W \in \mathbb{R}^{p \times p-2}$ and $V \in \mathbb{R}^{p-2 \times p-2}$ be the basis transformation matrices between $\{\hat{L}_j(x)\}_{j=2}^p$ and $\{T_j(x)\}_{j=0}^p$, and between $\{\hat{L}_j(x)\}_{j=2}^p$ and $\{\ell_j^p(x)\}_{j=1}^{p-1}$, i.e. for $\underline{a} = [a_j]_{j=2}^p$, $\underline{b} = [b_j]_{j=0}^p$, $\underline{c} = [c_j]_{j=1}^{p-1}$ and

$$p(x) = \sum_{j=2}^p a_j \hat{L}_j(x) = \sum_{j=0}^p b_j T_j(x) = \sum_{j=1}^{p-1} c_j \ell_j^p(x),$$

we have $\underline{b} = W\underline{a}$ and $\underline{c} = V\underline{a}$. With the diagonal matrix $\hat{D} = \mathrm{diag}[1, 2, \ldots, p+1] \in \mathbb{R}^{p+1 \times p+1}$

and T_2 defined in (14), we introduce the pre-conditioners

$$\hat{S}_j = \text{blockdiag}\left[\tilde{S}_j\right]_{i=1}^{n_v}, \quad j = 1, 2, \tag{27}$$

where $\tilde{S}_1 = \frac{2}{1+\log p} W^T \hat{D} W$ and $\tilde{S}_2 = V^T (2T_2)^{\frac{1}{2}} V$.

LEMMA 3.1. *The condition number estimates* $\kappa\left(\hat{S}_1^{-1} S\right) \preceq (1+\log p)^3$ *and* $\kappa\left(\hat{S}_2^{-1} S\right) \preceq (1+\log p)$ *are valid.*

Proof. The proof is given in [18]. □

The matrix W is a matrix of the form $W^T = \left[\begin{array}{ccc} v_1 & v_2 & \tilde{W}^T \end{array}\right]$, where $\tilde{W} \in \mathbb{R}^{p-2 \times p-2}$ is a lower triangular matrix, see [24]. Thus, the linear system $\hat{S}_1 \underline{w} = \underline{r}$ can be solved by forwards and backwards elimination. The matrix V can be factorized as $V = UW$, where $U \in \mathbb{R}^{p-2\times p}$ is the matrix of the discrete cosine transform, i.e. $U = \left(\cos \frac{jk\pi}{p}\right)_{j=1,k=0}^{p-1,p}$. Since $T_2 = \tilde{U}\Lambda\tilde{U}$, where $\tilde{U}$ is the matrix of the discrete sinus transform, i.e. $\tilde{U} = \left(\sin \frac{jk\pi}{p}\right)_{j,k=1}^{p-1}$ and $\Lambda = \frac{4}{p}\text{diag}\left(\sin^2 \frac{j\pi}{2p}\right)_{j=1}^{p-1}$, the system $\hat{S}_2 \underline{w} = \underline{r}$ can be solved by forwards and backwards elimination and Fast Fourier Transforms (FFT), [13].

4 Numerical results for the DD pre-conditioner

In this section, several numerical test examples are given. The linear system (3) is the result of the hp-version fem discretization of (1) for the following examples:

- `hufen`,
- `magnet1`,
- `sechseck`,
- `schlitz4`,
- `qua1`,
- `swing1`.

In each example, the right hand side $f(x, y) \equiv 1$ is chosen. Figure 4 shows the domains, the coarse finite element meshes on level 0 and the boundary conditions of the considered examples. A black line corresponds to an interior edge of the coarse mesh, a red line to homogenous Neumann boundary conditions, i.e. to $\partial\Omega \backslash (\Gamma_1 \cup \Gamma_3)$, a blue line to inhomogeneous Neumann boundary conditions, i.e. to Γ_3 and a green line to (possibly inhomogeneous) Dirichlet boundary conditions, i.e. to Γ_1. Figure 5 displays the behaviour of the approximate solution of (1)

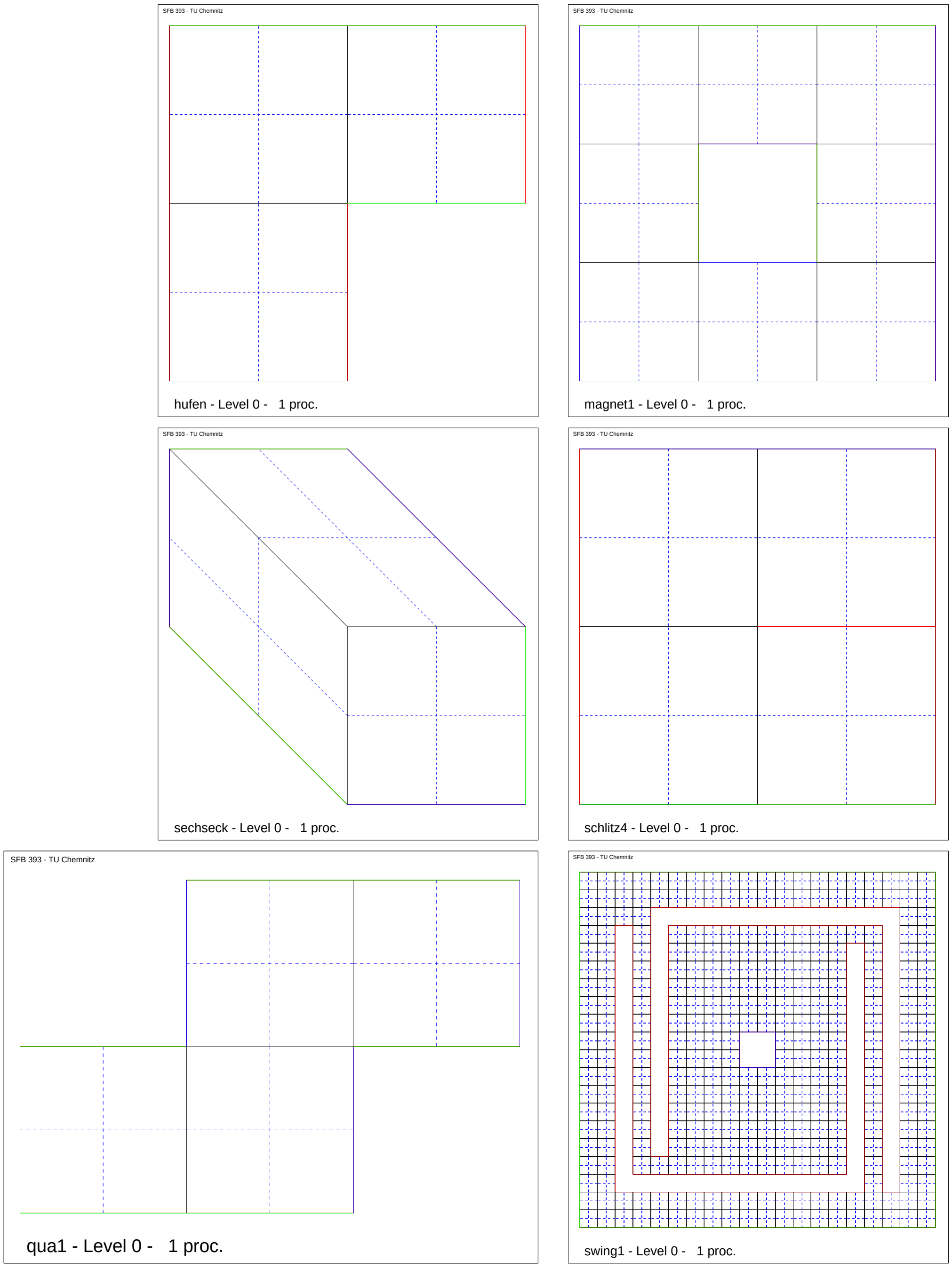

Figure 4: Finite element coarse meshes for the examples.

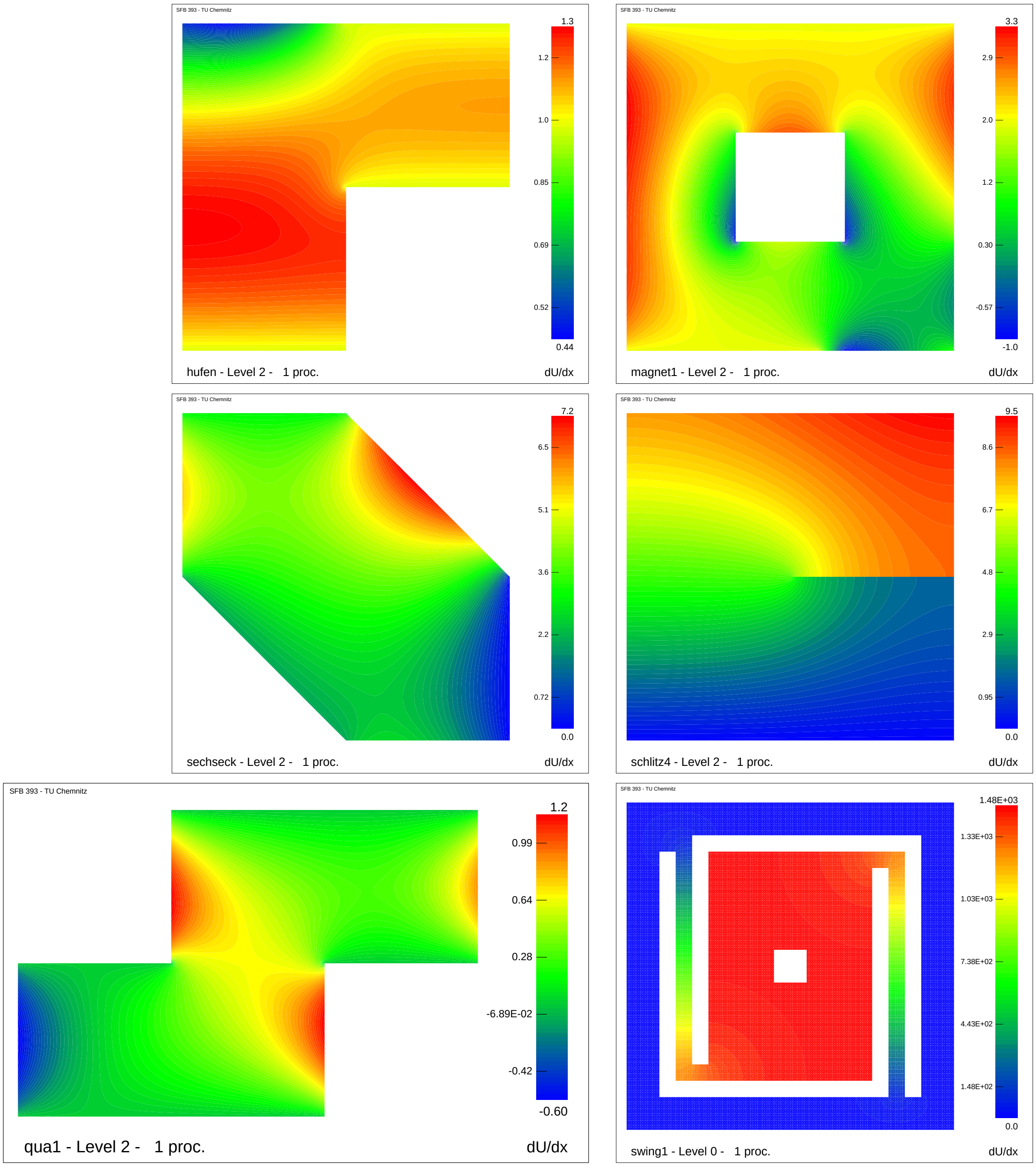

Figure 5: Approximate solution for the examples.

discretized by the hp-version of the finite element method with $p = 7$ and a two-times uniform refinement of the coarse mesh.
The system (3) is solved by a precondioned conjugate gradient method with the domain decomposition pre-conditioner

$$\hat{C}_{k,j,\delta} = \begin{bmatrix} \delta C_v & \mathbf{0} \\ \mathbf{0} & C_{II}^{(k,j)} \end{bmatrix}, \quad \text{where} \quad C_{II}^{(k,j)} = \begin{bmatrix} I & A_{e,i}A_i^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \hat{S}_j & \mathbf{0} \\ \mathbf{0} & E_k \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ A_i^{-1}A_{i,e} & I \end{bmatrix} \tag{28}$$

is the pre-conditioner for A_{II} (6). The matrix $\hat{S}_j$, $j = 1, 2$ is the Schur-complement pre-conditioner (27), the matrix C_v is the Yserentant pre-conditioner, [25], with coarse grid solver. The matrix E_k, $k = 1, 2$ denotes the multi-grid pre-conditioners $E_1 = \hat{M}_1^{\mathcal{S}_1}$, and $E_2 = \hat{M}_1^{\mathcal{S}_2}$ for A_i, cf. (24), whereas $E_3 = A_i$. Thus, in the case E_3, the system $A_i \underline{w}_i = \underline{r}_i$ is solved exactly. For the multiplications $\underline{w}_e = A_{e,i}A_i^{-1}\underline{r}_i$ and $\underline{w}_i = A_i^{-1}A_{i,e}\underline{r}_e$, a system solve with the matrix A_i has to be done. This is performed by a pcg-method with a relative accuracy of 10^{-9}, i.e. the system is solved nearly exactly.
For the inexact DD-pre-conditioner (28), the condition number of the preconditioned system matrix depends on the lower and upper eigenvalue bounds of $(\delta C_v)^{-1}A_v$, and $(C_{II}^{k,j})^{-1}A_{II}$ and of $C^{-1}A$ cf. (5), i.e.

$$\begin{aligned} \lambda_{min}\left(\hat{C}_{k,j,\delta}^{-1}A\right) &= \lambda_{min}\left(C^{-1}A\right)\min\{\lambda_{min}\left((\delta C_v)^{-1}A_v\right), \lambda_{min}\left(\left(C_{II}^{(k,j)}\right)^{-1}A_{II}\right)\} \quad \text{and} \\ \lambda_{max}\left(\hat{C}_{k,j,\delta}^{-1}A\right) &= \lambda_{max}\left(C^{-1}A\right)\max\{\lambda_{max}\left((\delta C_v)^{-1}A_v\right), \lambda_{max}\left(\left(C_{II}^{(k,j)}\right)^{-1}A_{II}\right)\}. \end{aligned}$$

So, it is important to choose the parameter δ in such a way that either $\lambda_{min}\left((\delta C_v)^{-1}A_v\right) \approx \lambda_{min}\left({C_{II}^{(k,j)}}^{-1}A_{II}\right)$, or $\lambda_{max}\left((\delta C_v)^{-1}A_v\right) \approx \lambda_{max}\left({C_{II}^{(k,j)}}^{-1}A_{II}\right)$. Then, we obtain the lowest condition numbers for $\hat{C}_{k,j,\delta}^{-1}A$. In most experiments, we have checked that $\delta = 4$ is a good choice if the pre-conditioner for A_v is the Yserentant-pre-conditioner, and the pre-conditioner $C_{II}^{(1,2)}$ is chosen for A_{II}, i.e. $\hat{S}_2$ and $M_1^{\mathcal{S}_1}$ are chosen as pre-conditioners for S and A_i.
All calculations are done on a LINUX cluster, where each machine is a Pentium III, 800 MHz. The pcg-algorithm is stopped if the relative accuracy in the preconditioned energy norm is reduced up to a factor of $\varepsilon = 10^{-5}$.

4.1 Results for several problems with the same pre-conditioner

In this subsection, cf. Tables 3 and 4, several examples with the pre-conditioner $\hat{C}_{1,2,4}$ are given. In all test examples considered, the number of iterations grow moderately if the polynomial degree is increased or the mesh is refined.
However, the absolute numbers of iterations are relatively different, i.e. for the case `qua1` in level 0 we have 15 iterations for $p = 15$, whereas the numbers of iterations is 69 in the case of `swing1` and $p = 15$. In general, the numbers of iterations depend on the geometry of the elements R_s. For meshes with elements having a regular geometry, the numbers of iterations are lower than for meshes with elements with an irregular geometry.
On the homepage

	Levels of refinement				
p	0	1	2	3	4
3	23	29	35	39	44
5	20	32	37	43	48
7	21	34	40	46	51
9	21	36	42	49	54
11	22	38	45	51	56
13	22	40	47	52	57
15	23	41	48	54	59

	Levels of refinement				
p	0	1	2	3	4
3	42	57	60	64	69
5	54	62	69	74	76
7	64	69	77	81	85
9	65	71	82	84	87
11	66	74	84	87	92
13	68	77	87	91	96
15	69	83	91	96	

Table 3: Numbers of iterations of the pcg-method for the problem `sechseck` (left) and `swing1` (right).

	Levels of refinement				
p	0	1	2	3	4
1	8	21	26	29	36
3	19	29	33	37	38
5	26	32	38	41	44
9	31	36	44	46	49
15	35				
25	41				
33	43				
45	47				
63	49				
125	56				
243	56				
513	61				

	Levels of refinement				
p	0	1	2	3	4
1	2	10	16	20	25
3	10	20	28	32	35
5	13	25	31	36	40
9	14	28	34	41	44
15	15				
25	17				
33	18				
45	19				
63	21				
125	23				
243	24				
513	26				

Table 4: Numbers of iterations of the pcg-method for the problem `schlitz4` (left) and `qua1` (right).

http://www.tu-chemnitz.de/sfb393/software/p-version-2D/,

more test examples are given.

4.2 Influence of the parameter δ

In this subsection, we give some numerical examples showing the influence of the parameter δ. As in the previous subsection, the pre-conditioners $\hat{S}_2$ for the Schur-complement and the pre-conditioner $E_2 = \hat{M}_1^{S_1}$ are chosen. The test example is `magnet1`. Three values for δ are considered, $\delta = 1$, $\delta = 4$ and $\delta = 10$.

	$\delta = 1$ Levels of refinement					$\delta = 4$ Levels of refinement					$\delta = 10$ Levels of refinement				
p	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	4	14	17	21	24	4	14	17	21	24	4	14	17	21	24
3	15	28	31	35	38	15	24	28	30	34	15	30	35	37	39
5	19	32	36	40	44	18	27	32	34	38	22	34	41	43	46
9	21	36	42	46	50	19	31	35	39	42	25	38	48	51	52
15	23	40	46	50	55	20	33	38	41	46	26	41	52	55	57
25	27	44	47			22	36	41			31	44	53		
33	29	48	51			24	37	42			34	46	55		
45	32	52	55			26	39	44			35	48	57		
63	35	52	60			27					37	49	57		

Table 5: Influence of the parameter δ.

Table 5 displays the numbers of iterations of the pcg-method for solving the system $A\underline{u} = \underline{f}$. From the results, one can see that the numbers of iterations of the pcg-method depend on the choice of the parameter δ. Moreover, one can conclude that the function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$: $\delta \mapsto \kappa\left(\hat{C}_{1,2,\delta}^{-1} A\right)$ has a local minimum in the interval $(1, 10)$.

4.3 Comparison of several pre-conditioners for one example.

In this subsection, the L-shaped test example `hufen` is considered. Several pre-conditioners are chosen, the pre-conditioners $\hat{C}_{1,2,4}$, $\hat{C}_{1,1,4}$, $\hat{C}_{2,2,4}$, $\hat{C}_{3,2,4}$, and $\hat{C}_{3,1,4}$. Note that in the cases $\hat{C}_{3,1,4}$ and $\hat{C}_{3,2,4}$, the pre-conditioner for the matrix A_i is the matrix A_i itself. In this case, the performance of the Schur-complement pre-conditioners $\hat{S}_1$ and $\hat{S}_2$ can be investigated.
Table 6 displays the numbers of iterations for solving $A\underline{u} = \underline{f}$ by the pcg-method with several pre-conditioners. For $p \leq 25$, the numbers of iterations of the pcg-method using the pre-conditioners $\hat{C}_{k,1,\delta}$ (i.e. the Schur-complement pre-conditioner $\hat{S}_1$) are about the same as for the pre-conditioners $\hat{C}_{k,2,\delta}$ with the Schur-complement pre-conditioner $\hat{S}_2$. For $p > 25$, the pre-conditioners $\hat{C}_{k,2,\delta}$ using the basis transformation to the Lagrangian polynomials (26) beat the

	$\hat{C}_{1,2,4}$ Levels of refinement					$\hat{C}_{1,1,4}$ Levels of refinement					$\hat{C}_{2,2,4}$ Levels of refinement				
p	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	2	10	16	20	24	2	10	16	20	24	2	10	16	20	24
3	11	21	27	31	34	11	20	27	31	33	11	21	27	31	34
5	15	25	32	35	38	13	25	32	36	38	15	25	32	35	38
9	16	29	36	40	43	14	30	37	42	45	16	28	36	40	43
15	18	31	37	44	46	17	34	40	47	49	18	31	37	44	46
25	20	34	39	47	50	25	38	44	52	55	19	32	39	47	50
33	22	35	40	49	52	28	41	46	55	58	21	35	40	49	52
45	24	36	42			30	43	48			23	35	42		
63	27	40	43			33	47	51			25	39	44		
125	30					39					29				
243	34					44					32				
513	36					51					35				

	$\hat{C}_{3,2,4}$ Levels of refinement					$\hat{C}_{3,1,4}$ Levels of refinement				
p	0	1	2	3	4	0	1	2	3	4
1	2	10	16	20	24	2	10	16	20	24
3	11	21	27	31	34	11	20	27	31	33
5	14	25	31	35	38	13	25	32	36	38
9	16	28	36	40	42	14	30	36	42	45
15	17	31	37	44	46	16	34	40	47	49
25	18	33	39	47	50	17	38	44	52	55
33	19	34	40	49	51	22	41	46	55	58
45	21	35	40			23	42	48		
63	22	37	43			25	45	50		
125	23					28				
243	25					32				
513	27					36				

Table 6: Numbers of iterations of the pcg-method for the test example `hufen`.

pre-conditioners $\hat{C}_{k,1,\delta}$ which use the basis transformation to the Chebyshev polynomials (25). The reason is the difference in the condition number estimates $\kappa\left(\hat{S}_1^{-1}S\right) \preceq (1+\log p)^3$ and $\kappa\left(\hat{S}_2^{-1}S\right) \preceq (1+\log p)$.
Moreover one can see that the pcg-method with the pre-conditioners $\hat{C}_{1,2,4}$ and $\hat{C}_{2,2,4}$, i.e. for A_i the multi-grid pre-conditioners $\hat{M}_1^{\mathcal{S}_1}$ and $\hat{M}_1^{\mathcal{S}_2}$ are chosen, have nearly the same numbers of iterations. Thus, the influence of the smoother $\mathcal{S}$ is not significantelly. Note that in numerical experiments the multi-grid convergence rate for solving the system with the matrix $K_{h,k}$ (20) using the V-cycle and the smoother $\mathcal{S}_1$ is bounded by a value of about 0.4, see [7], whereas the multi-grid convergence rate using the V-cycle and the Gauss-Seidel smoother $\mathcal{S}_2$ tends to 1.
The comparison of the pre-conditioners $\hat{C}_{3,k,4}$ and $\hat{C}_{1,k,4}$, $k=1,2$, i.e for A_i are used A_i itself and $\hat{M}_1^{\mathcal{S}_1}$ as pre-conditioner, shows that the replacement of $\underline{w}_i = A_i^{-1}\underline{r}_i$ in (28) by the preconditioning operation $\underline{w}_i = (\hat{M}_1^{\mathcal{S}_1})^{-1}\underline{r}_i$ increases the numbers of iterations on a factor of about 1.3 on level 0 of refinement of the mesh `hufen`. On levels $1,2,3,4$, the numbers of iterations are about the same.
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