Algebraic K-Theory of Non-Linear Projective Spaces

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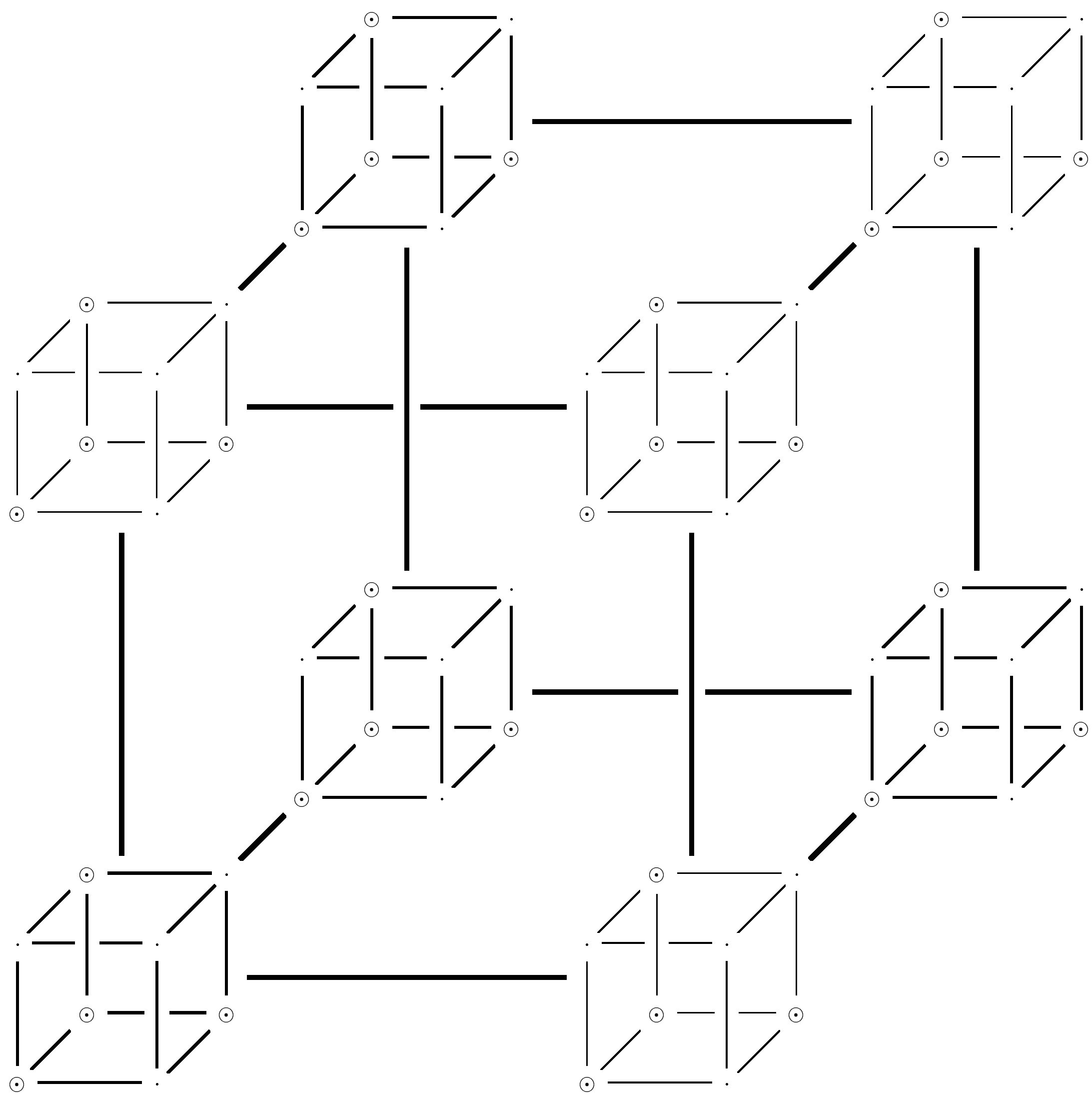
We extend the projective line category of [KVWW] to higher dimensions and present a calculation of its K-theory, a "non-linear" analogue of QUILLEN's isomorphism $K_k(\mathbf{P}^n_R) \cong \bigoplus_0^n K_k(R)$.



Contents

1. Preliminaries

1.1. Introduction

Let R denote a commutative ring. QUILLEN has proved ([Q2], § 8, theorem 2.1) that there is an isomorphism of K-groups $K_i\mathbf{P}^n_R \cong \bigoplus_0^n K_i(R)$ for all $i \geq 0$ where $\mathbf{P}^n_R = \mathrm{Proj}\ R[X_0,\ X_1,\ \ldots,\ X_n]$ is the n-dimensional projective space over R.

This paper is concerned with an analogous result for the algebraic K-theory of spaces in the sense of [W]. We will define a category $\mathbf{P}^n$ of "quasi-coherent sheaves" on projective n-space and a notion of twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n}(j)$. If Y is a pointed space, we can form the "tensor product" $Y \wedge \mathcal{O}_{\mathbf{P}^n}(j)$. Thus there is a map $\prod_0^n A^{sfd}(*) \longrightarrow \Omega|h\mathcal{S}_\bullet\mathbf{P}^n|$ (where $A^{sfd}(*)$ is the version of WALDHAUSEN's algebraic K-theory of spaces functor using stably finitely dominated spaces, and the target is the algebraic K-theory of non-linear projective n-space) induced from

$$(Y_0,\ Y_1,\ \ldots, Y_n) \mapsto \bigvee_{j=0}^{n} Y_j \wedge \mathcal{O}_{\mathbf{P}^n}(-j)$$

and it turns out that this map is a weak homotopy equivalence.

This program has been carried out in [KVWW] for $n = 1$. We will use this reference as a guideline. However, the present account takes full advantage of categorical techniques and the existence of model structures in the sense of [Q1].

The first chapter will contain definitions and results collected from various sources. We assume that the reader is familiar with the basic notions of model categories ([Q1], [DS]).

In § 2 we define the category $\mathbf{P}^n$ and develop some machinery. Terminology is taken from algebraic geometry, and most constructions have a "linear" analogue in the world of rings and modules.

Chapter 3 is concerned with the K-theory of $\mathbf{P}^n$. As usual one has to impose some finiteness condition on the objects under consideration. Our choice are "stably finitely dominated objects". We indicate briefly how they compare to objects with a more restrictive finiteness condition.

Appendix A contains a description of model structures for generalized diagram categories. The framework presented there is general enough to include the cases of cubical diagrams of topological spaces and (a variant of) the category $\mathbf{P}^n$. However, the reader should be aware that in both cases one can check the existence of model structures without the abstract setup of appendix A. The amount of work to be done is about the same for the examples as for the general case.

Finally, appendix B contains a list of some of the used (non-standard) symbols and words. It is not intended as a complete and comprehensive index, but provides references for the most important topics.

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All diagrams are made with Paul Taylor's diagrams macro package for TEX ([T]).

1.2. Equivariant Spaces

Let $k\,Top_*$ denote the category of pointed (or based) Kelley spaces (or k-spaces) in the sense of [Ho], definition 2.4.21 (3): a space Y is a Kelley space if every compactly open subset $U \subseteq Y$ is open (here U is compactly open if for all compact Hausdorff spaces K and all continuous maps $f : K \longrightarrow Y$, the set $f^{-1}(U)$ is open in K). According to [Ho, 2.4.24] this category has a model structure where a map f is a weak equivalence if and only if it is a weak homotopy equivalence, and f is a fibration if and only if it is a Serre fibration. All k-spaces are fibrant. Cofibrations are retracts of generalized CW-inclusions ([DS], 8.8 and 8.9) where we have to use the pointed cells Δ^n_+. Colimits agree in $k\,Top_*$ and the category of pointed topological spaces. Limits can be computed by first calculating the limit in the category of topological spaces, then applying the "Kelleyfication" functor k. In particular, all smash products occuring in this paper will bear this modified product topology.

Since the forgetful functor from k-spaces to topological spaces preserves fibrations and cofibrations, since weak equivalences are detected on underlying topological spaces, and since the map $k(Y) \longrightarrow Y$ is a weak equivalence for all Kelley spaces (where k denotes the Kelleyfication functor), $k\,Top_*$ is a proper model category. In particular, the gluing lemma holds.

A monoid is a (multiplicative) semigroup with identity element 1. A *monoid with zero* is a monoid M with a distinguished element $0 \in M$ such that $m \cdot 0 = 0 \cdot m = 0$ for all $m \in M$. A map of monoids with zero is a monoid homomorphism preserving the zero element. A *topological monoid* with zero is a monoid with zero which is also a pointed Kelley space with 0 as basepoint such that the multiplication is continuous.

Note that the zero element of a monoid with zero is unique. Furthermore, the equality $0 = 1$ holds if and only if the monoid in question is the trivial monoid with zero consisting of one element, a final object in the category of monoids with zero. We can consider S^0 as a monoid with zero, it is inital in the category of monoids with zero. If M is a topological monoid, one can add a disjoint zero element. This gives a functor $M \mapsto M_+$ left adjoint to the forgetful functor (forgetting the zero element).

Suppose M is a topological monoid with zero. A *right action* of M on a space $Y \in k\,Top_*$ is a continuous map $Y \wedge M \longrightarrow Y$ satisfying the usual associativity and unitality condition. Similarly we can define left actions. If M happens to be commutative every right action determines a left action and *vice versa*. Let M-$k\,Top_*$ denote the category of pointed topological spaces with a right action of M; morphisms are M-equivariant pointed continuous maps.

Proposition 1.2.1.

(1) The category M-$kTop_*$ has the structure of a topological model category where a map is a weak equivalence (resp. fibration) if and only if it is a weak homotopy equivalence (resp. fibration) of underlying k-spaces, and a cofibration if and only if it is a retract of a generalized CW-inclusion in the sense of [DS, 8.8] (cells are of the form $\Delta^n_+ \wedge M$). Furthermore, all objects are fibrant.

(2) If $M = G \wedge \bar{M}$ is the smash product of a topological monoid with zero G and a discrete monoid with zero $\bar{M}$, the forgetful functor $(G\wedge\bar{M})$-$kTop_* \longrightarrow G$-$kTop_*$ preserves cofibrations.

(3) If M is cofibrant as an object of $kTop_*$, the forgetful functor M-$kTop_* \longrightarrow kTop_*$ preserves cofibrations.

(4) If M is cofibrant as an object of $kTop_*$, the category M-$kTop_*$ is a proper model category.

Proof.

Existence of the model structure in (1) follows from theorem B of [SV] for an appropriate choice of a (pointed, topological) theory Θ, the theory "associated to M". We omit the details. Proposition (5), *loc. cit.*, provides the characterization of cofibrations (in [SV], generalized CW-inclusions are called "special cofibrations"). Finally, proposition (6), *loc. cit.*, asserts that all objects are fibrant.

For (2), assume f is a cofibration in $(G\wedge\bar{M})$-$kTop_*$. By (1) we know that f is a retract of a generalized CW-inclusion g, *i.e.*, g is a colimit of a (possibly transfinite) sequence of maps $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots$ such that each X_{i+1} is obtained from X_i by attaching a one-point union of free $(G\wedge\bar{M})$-equivariant cells. But since $\bar{M}$ is discrete, $G \wedge \bar{M} \cong \bigvee_{\bar{M}\setminus\{0\}} G$ as a G-space. Thus X_{i+1} is obtained from X_i by attaching a (bigger) one-point union of free G-equivariant cells. Hence f is a cofibration in G-$kTop_*$.

To prove (3), suppose f is a cofibration in G-$kTop_*$. Then f is a retract of a generalized CW-inclusion g, *i.e.*, g is a colimit of a (possibly transfinite) sequence of maps $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots$ such that each X_{i+1} is obtained from X_i by attaching a one-point union of free M-equivariant cells. Without loss, we may assume that X_{i+1} is obtained from X_i by attaching one single cell (make an even longer sequence). Thus it suffices to show that each map $X_i \longrightarrow X_i \cup_{M\wedge\partial\Delta^k_+} M \wedge \Delta^k_+ = X_{i+1}$ is a cofibration in $kTop_*$. But this map is a cobase change of the inclusion $M \wedge \partial\Delta^k_+ \longrightarrow M \wedge \Delta^k_+$, hence it is enough to show that this inclusion is a cofibration in $kTop_*$.

Now $M \wedge \cdot : kTop_* \longrightarrow kTop_*$ has a right adjoint $\hom_{kTop_*}(M, \cdot)$, and $kTop_*$ is a topological model category. Hence we know that the latter functor preserves acyclic fibrations since M is cofibrant, *i.e.*, the map $* \longrightarrow M$ is a cofibration (this is a special case of axiom SM7 of [Q1]). Hence the former preserves cofibrations by adjointness ([DS], 9.8). Since $\partial\Delta^k_+ \longrightarrow \Delta^k_+$ is a cofibration in $kTop_*$, this proves the assertion.

Finally, statement (4) means the following two things: a base change of a weak equivalence along a fibration is weak equivalence, and the cobase change of a weak equivalence along a cofibration is a weak equivalence. The first is true since weak equivalences are defined on underlying objects of $k\,Top_*$, and the forgetful functor preserves fibrations. The second follows from a similar argument, using the fact that the forgetful functor preserves cofibrations by (3) under the given assumptions. $\square$

From now on, "topological space", "topological monoid" *etc.* **will always refer to k-spaces.**

The category M-$k\,Top_*$ has a suspension functor $\Sigma Y := S^1 \wedge Y$. Suspension preserves cofibrations, acyclic cofibrations and hence all weak equivalences between cofibrant objects. (This follows from [DS], 9.8 and 9.9 since Σ has a right adjoint Ω, the loop space functor, which preserves fibrations and acyclic fibrations since these notions are defined on underlying topological spaces.) Note that this is still true if "cofibration" (and "cofibrant") refers to the underlying objects in $k\,Top_*$.

Proposition 1.2.1 implies that an object of M-$k\,Top_*$ is cofibrant if and only if it is a retract of a generalized M-free pointed CW-complex. Let $\mathbf{C}(M)$ denote the full subcategory of cofibrant objects in M-$k\,Top_*$.

An object $Y \in M$-$k\,Top_*$ is called *finite* if it is obtained from a point by attaching finitely many free M-cells (in particular, Y is cofibrant). It is called *homotopy finite* if it is connected by a chain (or zigzag) of weak equivalences in M-$k\,Top_*$ to a finite object. By [DS] 5.8 and 5.11, this is equivalent to the existence of a finite object Z and a weak equivalence $Z \longrightarrow Y$. If in addition Y is cofibrant, the WHITEHEAD theorem ([DS], lemma 4.24) applies: Y is homotopy finite if and only if Y is homotopy equivalent, in the strong sense, to a finite object. A space Y is called *finitely dominated* if it is a retract of a homotopy finite object. Finally Y is said to be *stably finitely dominated* if some suspension of Y is finitely dominated. The full subcategories of $\mathbf{C}(M)$ consisting of the finite, homotopy finite, finitely dominated and stably finitely dominated objects will be denoted by $\mathbf{C}_f(M)$, $\mathbf{C}_{hf}(M)$, $\mathbf{C}_{fd}(M)$ and $\mathbf{C}_{sfd}(M)$, respectively.

Suppose Y and Z are objects of $k\,Top_*$ with an action of M from the right *resp.* left, we can form their *tensor product* $Y \wedge_M Z \in k\,Top_*$ defined as the coequalizer (in $k\,Top_*$) of the two maps $Y \wedge M \wedge Z \longrightarrow Y \wedge Z$ given by the action of M on Y and Z. If Z has an additional right $\bar{M}$-action for some topological monoid with zero $\bar{M}$ (compatible with the left M-action), the tensor product $Y \wedge_M Z$ is an $\bar{M}$-equivariant space.

Let $I := [0,\ 1]_+$ denote the unit interval with a disjoint basepoint. If $Y \in M$-$k\,Top_*$ is cofibrant, $Y \wedge I$ is a good cylinder object for Y ([DS], definition 4.2), *i.e.*, the map $Y \vee Y \longrightarrow Y \wedge I$ (inclusion of top and bottom into the cylinder) is a cofibration of M-spaces. (Sketch of proof: If Y is a generalized free M-equivariant CW-complex, one can explicitly construct $Y \wedge I$ from $Y \vee Y$ by attaching free M-cells, hence the map is a cofibration. If Y is a retract of a generalized CW-complex Z, then $Y \vee Y \longrightarrow Y \wedge I$ is a retract of the cofibration $Z \vee Z \longrightarrow Z \wedge I$.)

Lemma 1.2.2. *Suppose $f\colon M \longrightarrow \bar{M}$ is a morphism of topological monoids with zero. We consider $\bar{M}$ as an object of $k\,Top_*$ with M acting via f from the left.*

(1) The functor $\cdot \wedge_M \bar{M} : M\text{-}k\,Top_* \longrightarrow \bar{M}\text{-}k\,Top_*$ has a right adjoint $\bar{Y} \mapsto Y$ where Y is $\bar{Y}$ as a topological space, but with M acting via f.

(2) The functor $\cdot \wedge_M \bar{M}$ preserves cofibrations and acyclic cofibrations. It maps weak equivalences between cofibrant objects to weak equivalences.

(3) The functor $\cdot \wedge_M \bar{M}$ maps cells to cells. It restricts to a functor $\mathbf{C}_?(M) \longrightarrow \mathbf{C}_?(\bar{M})$ where ? may denote any of the decorations f, hf, fd or sfd.

Proof. Assertion (1) is obvious. The right adjoint preserves fibrations and acyclic fibrations (since both notions are defined on underlying topological spaces, cf. 1.2.1 (1)). Hence the left adjoint functor preserves cofibrations and acyclic cofibrations by [DS], 9.8. In particular, it maps acyclic cofibrations between cofibrant objects to weak equivalences, and Brown's lemma ([DS], 9.9) shows that $\cdot \wedge_M \bar{M}$ preserves all weak equivalences between cofibrant objects. This proves (2).

We have $(\Delta^k_+ \wedge M) \wedge_M \bar{M} \cong \Delta^k_+ \wedge \bar{M}$, *i.e.*, the image of a cell is a cell. Since $\cdot \wedge_M \bar{M}$ has a right adjoint by (1), it commutes with colimits and in particular with cell attachment. Thus we get a functor $\mathbf{C}_f(M) \longrightarrow \mathbf{C}_f(\bar{M})$.

If $Y \in \mathbf{C}_{hf}(M)$ is homotopy finite, we know that there is a space $Z \in \mathbf{C}_f(M)$ and a weak equivalence $f\colon Z \longrightarrow Y$. By (2), application of $\cdot \wedge_M \bar{M}$ yields a weak equivalence $Z \wedge_M \bar{M} \longrightarrow Y \wedge_M \bar{M}$, and by what we have just shown, the source is finite. Hence the target is homotopy finite by definition.

Now assume $Y \in \mathbf{C}(M)$ is a retract of a homotopy finite space Z. Call section and retraction s and r, respectively. Factor the map $* \longrightarrow Z$ as a cofibration $* \longrightarrow Q$ followed by an acyclic fibration $\pi\colon Q \longrightarrow Z$. Then Q is homotopy finite since Z is. Consider the following commutative diagram:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Q \arrow[d, "\pi"] \arrow[ddr, "r\circ\pi"] & \\
Y \arrow[r, "s"] \arrow[ur, dotted, "\ell"] \arrow[drr, "id_Y"'] & Z \arrow[dr, "r"] & \\
& & Y
\end{tikzcd}$$

Since the left vertical map in the square is a cofibration, and the right vertical map is an acyclic fibration, the dotted lift ℓ exists. Hence Y is a retract of the homotopy finite cofibrant space Q (with section ℓ and retraction $r \circ \pi$). Thus $Y \wedge_M \bar{M}$ is a retract of $Q \wedge_M \bar{M}$. The latter space is known to be homotopy finite by the previous case, hence the former is finitely dominated.

Finally, suppose $Y \in \mathbf{C}_{sfd}(M)$. Then there exists $k \geq 0$ with $\Sigma^k Y$ finitely dominated. But this implies that $\Sigma^k(Y \wedge_M \bar{M}) \cong (\Sigma^k Y) \wedge_M \bar{M}$ is finitely dominated by the previous case, *i.e.*, $Y \wedge_M \bar{M}$ is stably finitely dominated. □

Using the cylinder objects $Y \wedge I$ introduced above, we obtain a functorial mapping cylinder construction for maps in M-$k\,Top_*$ with the following well-known properties:

Lemma 1.2.3. *Suppose $g\colon X \longrightarrow Y$ is a map in M-$k\,Top_*$, and let Z_g denote its mapping cylinder.*

(1) The mapping cylinder construction is a cylinder functor in the sense of [W], 1.6 for maps between cofibrant objects. In particular, if X and Y are cofibrant, so is Z_g. The cylinder axiom is satisfied, i.e., the canonical projection $Z_g \longrightarrow Y$ is a weak equivalence.

(2) Let ? denote any of the subscripts f, hf, fd or sfd, and suppose X and Y are objects of $\mathbf{C}_?(M)$. Then Z_g is an object of $\mathbf{C}_?(M)$, i.e., the mapping cylinder respects finiteness.

(3) The functor $Ar(M\text{-}k\,Top_*) \longrightarrow M\text{-}k\,Top_*$, $g \mapsto Z_g$ commutes with colimits (since it is defined via pushouts and the smash product functor $\cdot \wedge I$).

□

Being a cylinder functor in the sense of [W] means the following:

(1) If $f : X \longrightarrow Y$ is a map of cofibrant objects, the mapping cylinder Z_g is cofibrant, and the canonical map $X \vee Y \longrightarrow Z_g$ is a cofibration.

(2) Given a commutative square

$$\begin{array}{ccc} P & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Q & \longrightarrow & Y \end{array}$$

of cofibrant objects with both horizontal map cofibrations, the induced diagram

$$\begin{array}{ccc} P \vee Q & \longrightarrow & X \vee Y \\ \downarrow & & \downarrow \\ Z_f & \longrightarrow & Z_g \end{array}$$

has the property that the upper horizontal map is a cofibration, and the induced map $Z_f \cup_{P\vee Q} (X \vee Y) \longrightarrow Z_g$ from the pushout into the terminal vertex is a cofibration. (We say that the latter square is obtained from the former by "applying the mapping cylinder to the vertical maps".)

We will also have occasion to apply the following technical result:

Lemma 1.2.4. *Let G denote a topological monoid with zero. Suppose M is a discrete monoid with zero, and $t \in M$ is an element such that right translation by t (i.e., the map $\rho_t : M \longrightarrow M$, $m \mapsto mt$) is injective. Then for each $Y \in \mathbf{C}(G \wedge M)$, the selfmap $Y \xrightarrow{t} Y$, $y \mapsto yt$ is a cofibration in G-$k\,Top_*$ (though not necessarily in $(G \wedge M)$-$k\,Top_*$).*

Proof. Let $Q := M \setminus \rho_t(M)$ denote the subset of those elements which are not a translate of t. Assume first that Y is a finite generalized free $(G \wedge M)$-equivariant CW-complex, *i.e.*, Y can be obtained from a point by attaching finitely many free cells. Write $Y = \bar{Y} \cup_{\partial C} C$ where $C = \Delta^k_+ \wedge G \wedge M$ is a free cell with boundary ∂C, and consider the following diagram:

$$\begin{tikzcd}
\Delta^k_+ \wedge G \wedge M \arrow[d, "t"'] & \partial\Delta^k_+ \wedge G \wedge M \arrow[l] \arrow[r] \arrow[d, "t"] & \bar{Y} \arrow[d, "t"] \\
\Delta^k_+ \wedge G \wedge M & \partial\Delta^k_+ \wedge G \wedge M \arrow[l] \arrow[r] & \bar{Y}
\end{tikzcd}$$

We may assume by induction that the right vertical map is a cofibration. We claim that the map from the pushout of the left square into its terminal vertex is a cofibration. Indeed, the pushout and the terminal vertex differ by a one-point union of $\Delta^k_+ \wedge G$, indexed over Q, with attachment done over a one-point union of $\partial\Delta^k_+ \wedge G$, indexed over Q. (This is true since right translation by t is assumed to be injective.)

In this situation, Reedy's patching lemma ([BF], lemma 3.8) asserts that the induced map from the pushout of the top row into the pushout of the bottom row is a cofibration. Hence the lemma is true for finite Y.

By transfinite induction, this proves the lemma for (not necessarily finite) generalized free CW-complexes (note that a transfinite composition of cofibrations is a cofibration).

Finally, if Y is a retract of a generalized free CW-complex Z, the map $Y \xrightarrow{t} Y$ is a retract of the map $Z \xrightarrow{t} Z$. But the latter was shown to be a cofibration, hence so is the former. □

1.3. Iterated Homotopy Cofibres

Definition 1.3.1. Assume $\mathcal{C}$ is a category. For any object $c \in \mathcal{C}$ let $\mathcal{C} \downarrow c$ denote the category of objects over c, and define $\mathcal{C} \downarrow \hat{c}$ as the full subcategory of objects over c without the identity of c. There is a functor $j : \mathcal{C} \downarrow \hat{c} \longrightarrow \mathcal{C}$ defined by $(a \longrightarrow c) \mapsto a$. If Y is a functor $\mathcal{C} \longrightarrow G\text{-}k\mathit{Top}_*$ we define the *latching space of* Y *at* c as

$$L_c Y := \operatorname*{colim}_{\mathcal{C}\downarrow\hat{c}} Y \circ j$$

(cf. [DHK], 11.3). Note that $L_c Y$ comes equipped with a canonical map to $Y(c)$ induced by the structure maps $Y(a) \longrightarrow Y(c)$. Given an object of $\mathcal{C} \downarrow \hat{c}$, *i.e.*, a morphism $b \longrightarrow c$ in $\mathcal{C}$ different from id_c, we obtain a map $Y(b) \longrightarrow L_c Y$ since $Y(b)$ appears in the diagram defining the latching space.

For a set N let $\langle N \rangle$ denote its power set regarded as a category with inclusions as morphisms. For all $A \subseteq N$ we identify $\langle N \rangle \downarrow \hat{A}$ with a full subcategory of $\langle N \rangle$.

Let $\mathcal{C}^{\langle N \rangle} = \mathit{Func}(\langle N \rangle, \mathcal{C})$ denote the category of functors $\langle N \rangle \longrightarrow \mathcal{C}$. If G denotes a topological monoid with zero, we have defined $G\text{-}k\mathit{Top}_*^{\langle N \rangle} := \mathit{Func}(\langle N \rangle, G\text{-}k\mathit{Top}_*)$, the category of N-cubical diagrams in $G\text{-}k\,Top_*$. If Y is an object of $G\text{-}k\,Top_*^{\langle N\rangle}$ we write Y^A for $Y(A)$. The latching space L_AY has a canonical map to Y^A. We say that Y *satisfies the latching space condition at* A if this map is a cofibration in $G\text{-}k\,Top_*$. More generally if $f : Y \longrightarrow Z$ is a map of cubes, we say that f *satisfies the latching space condition at* A if the induced map $L_AZ \cup_{L_AY} Y^A \longrightarrow Z^A$ is a cofibration.

For $0 \leq k \leq \#N$ (this k is different from the k appearing in $k\,Top_*$) define a functor

$$e_k : G\text{-}k\,Top_*^{\langle N\rangle} \longrightarrow G\text{-}k\,Top_*^{\langle N\rangle}, \quad e_k(Y)^A := \begin{cases} Cyl(L_AY \longrightarrow Y^A), & \text{if } k = \#A \\ Y^A, & \text{if } k \neq \#A \end{cases}$$

where Cyl denotes the mapping cylinder construction, and $e_k(Y)$ has the obvious structure maps: for $B = A \amalg \{i\} \subseteq N$, the map $e_k(Y)^A \longrightarrow e_k(Y)^B$ is given by the structure map $Y^A \longrightarrow Y^B$ of Y if neither A nor B has k elements; by the composite $Y^A \longrightarrow L_BY \longrightarrow Cyl(L_BY \longrightarrow Y^B)$ if $\#B = k$; and by the composite $Cyl(L_AY \longrightarrow Y^A) \longrightarrow Y^A \longrightarrow Y^B$ if $\#A = k$.

Let $\gamma : G\text{-}k\,Top_*^{\langle N\rangle} \longrightarrow G\text{-}k\,Top_*$ denote the functor taking Y to the strict cofibre of the map $L_NY \longrightarrow Y^N$. We define the *iterated homotopy cofibre of* Y as

$$\Gamma(Y) := \gamma \circ e_{\#N} \circ \ldots \circ e_1 \circ e_0(Y) \; . \qquad (*)$$

Finally we define the KRONECKER *delta cube* δ_C (for a subset $C \subseteq N$) as the functor $\delta_C : G\text{-}k\,Top_* \longrightarrow G\text{-}k\,Top_*^{\langle N\rangle}$ with $\delta_C(K)^A = *$ if $A \neq C$ and $\delta_C(K)^C = K$.

Remark 1.3.2.

(1) We will only be interested in $\Gamma(Y)$ if all the spaces Y^A are cofibrant in $G\text{-}k\,Top_*$. In this case, the following description holds: calculate the homotopy colimit C of the diagram obtained from Y by deleting the terminal vertex Y^N; there is a canonical map $C \longrightarrow Y^N$, its homotopy cofibre is $\Gamma(Y)$. The given definition of Γ is a factorization of this procedure into simple steps; philosophically speaking: the functors e_j "make Y cofibrant as a cube" which guarantees that application of γ "gives the correct homotopy type". Technically speaking, Γ is a model for the total left derived of γ in the sense of [Q1] on the subcategory of objects with cofibrant components.

(2) Since both the mapping cylinder construction and formation of latching spaces are compatible with smash products, it is true that Γ commutes with smash products with spaces. Explicitly, if Y is an object of $k\,Top_*^{\langle N\rangle}$ and $K \in G\text{-}k\,Top_*$, there is a canonical isomorphism of G-spaces $\Gamma(K \wedge Y) \cong K \wedge \Gamma(Y)$ where $K \wedge Y$ denotes the cubical diagram $A \mapsto K \wedge Y^A$ in $G\text{-}k\,Top_*$.

(3) There are natural transformations $e_k \longrightarrow \mathrm{id}$ which are weak equivalences on each component. Explicitly, the A-component is given by id_{Y^A} if $\#A \neq k$, and is the projection from the mapping cylinder of $L_AY \longrightarrow Y^A$ to Y^A if A has k elements. Moreover $e_0 \cong \mathrm{id}$.

(4) The functor Γ admits a recursive definition. If we write $N = M \amalg \{j\}$, we can regard an N-cube as a map "in j-direction" of two M-cubes and compute the pointwise homotopy cofibre. The iterated homotopy cofibre of the resulting M-cube is isomorphic to the iterated homotopy cofibre of the original N-cube. (Here "homotopy cofibre of the map f" always means the (functorial) mapping cone of f.) This means that $\Gamma(Y)$ can be computed by iteratively calculating pointwise homotopy cofibres of maps between cubes. The proof of this is only a matter of explicitly spelling out the "symmetric" definition of Γ as given in $(*)$ above, and comparing this to the recursive version. We omit the details.

Example 1.3.3. For $N = \{0,\ 1\}$, consider the following N-cube Y:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\}} \arrow[d] \\ Y^{\{1\}} \arrow[r] & Y^{\{0,1\}} \end{tikzcd}$$

Since $e_0 \cong \mathrm{id}$, application of this functor has no visible effect. Applying e_1 means replacing $Y^{\{0\}}$ with the (reduced) mapping cylinder of the map $* \longrightarrow Y^{\{0\}}$, and similarly for $Y^{\{1\}}$, thus again nothing happens (up to canonical isomorphism) because the initial vertex is a point. Now things get a bit more interesting: the latching space $L_{\{0,1\}}(e_1 \circ e_0(Y))$ is given by the colimit of $Y^{\{1\}} \longleftarrow * \longrightarrow Y^{\{0\}}$, *i.e.*, the space $Y^{\{0\}} \vee Y^{\{1\}}$. Let f denote the map $Y^{\{0\}} \vee Y^{\{1\}} \longrightarrow Y^{\{0,1\}}$ with mapping cylinder Z_f. Then $e_2 \circ e_1 \circ e_0(Y)$ is given by the following diagram:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\}} \arrow[d] \\ Y^{\{1\}} \arrow[r] & Z_f \end{tikzcd}$$

The right vertical map is given by the composite $Y^{\{0\}} \longrightarrow Y^{\{0\}} \vee Y^{\{1\}} \longrightarrow Z_f$, and similarly for the lower horizontal map. Application of γ yields the (strict) cofibre of the map $Y^{\{0\}} \vee Y^{\{1\}} \longrightarrow Z_f$, *i.e.*, the mapping cone of f; this space is $\Gamma(Y)$.

The category $G\text{-}k\mathit{Top}_*^{\langle N\rangle}$ admits two model structures with pointwise weak equivalences. Using the terminology of appendix A, define a (trivial) adjunction bundle indexed over $\mathcal{I} = \langle N\rangle$ with $\mathcal{C}_A = G\text{-}k\mathit{Top}_*$ for all $A \subseteq N$, and all functors F_σ and U_σ are the identity of $G\text{-}k\mathit{Top}_*$. The *f-structure* (called *inverse structure* in [DHK]) is obtained from theorem A.3.13 if $\mathcal{I}$ is equipped with degree function $d(A) := -\#A$. A map of cubes is an *f-cofibration* if it is a pointwise cofibration. The *c-structure* (or *direct structure*) has pointwise fibrations. The corresponding cofibrations will be denoted by *c-cofibrations*; explicitly a map f is a c-cofibration if and only if it satisfies the latching space condition at A for all $A \subseteq N$. This follows from A.3.8 using the degree function $d(A) := \#A$.

Lemma 1.3.4.

(1) A map $f\colon Y \longrightarrow Z$ in G-$k\,Top_*^{\langle N\rangle}$ is a weak equivalence if and only if $e_k(f)$ is a weak equivalence. If f is an f-cofibration and Y is f-cofibrant, $e_k \circ e_{k-1} \circ \ldots \circ e_0(f)$ satisfies the latching space condition at A if $A \subseteq N$ with $\#A \le k$; in particular, $e_{\#N} \circ \ldots \circ e_1 \circ e_0(f)$ is a c-cofibration.

(2) The functor e_k commutes with colimits.

(3) The functor γ maps (acyclic) c-cofibrations to (acyclic) cofibrations in G-$k\,Top_*$.

(4) The functor γ commutes with colimits.

Proof. *Ad* (1). By 1.3.2, there is a commutative square

$$\begin{tikzcd} e_k(Y) \arrow[r, "e_k(f)"] \arrow[d] & e_k(Z) \arrow[d] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

where the vertical maps are weak equivalences. Hence f is a weak equivalence if and only if $e_k(f)$ is.

Now suppose f is an f-cofibration. We prove the second claim of (1) by induction on k. For $k = 0$ the assertion is true since $e_0 \cong \mathrm{id}$.

Let $A \subseteq N$ with $\#A \le k$ be given. Define $\bar{e} := e_{k-1} \circ \ldots \circ e_0$. We have to check that the induced map

$$L_A\big(e_k \circ \bar{e}(Z)\big) \cup_{L_A(e_k \circ \bar{e}(Y))} \big(e_k \circ \bar{e}(Y)\big)^A \longrightarrow \big(e_k \circ \bar{e}(Z)\big)^A \qquad (*)$$

is a cofibration.

If A has strictly less than k elements, the induction hypothesis immediately implies the assertion: we have $L_A\big(e_k \circ \bar{e}(Y)\big) = L_A\big(\bar{e}(Y)\big)$, and similarly $L_A\big(e_k \circ \bar{e}(Z)\big) = L_A\big(\bar{e}(Z)\big)$. Moreover, $\big(e_k \circ \bar{e}(Y)\big)^A = \bar{e}(Y)^A$, and similarly for Z. By induction hypothesis, the map

$$L_A\big(\bar{e}(Z)\big) \cup_{L_A(\bar{e}(Y))} \bar{e}(Y)^A \longrightarrow \bar{e}(Z)^A$$

is a cofibration. But this map is the same as the map $(*)$.

So assume $\#A = k$. By construction, we know that the object $\bar{e}(Y)$ satisfies the latching space condition at B for all proper subsets $B \subset A$. The latching space functor L_A, considered as a functor G-$k\,Top_*^{\langle A\rangle} \longrightarrow G$-$k\,Top_*$, has a right adjoint (the “diagonal” or “constant diagram” functor) which maps acyclic fibrations to acyclic c-fibrations (cf. A.3.11). Hence L_A preserves c-cofibrations, hence maps cofibrant objects to cofibrant objects. This shows that $L_A\big(\bar{e}(Z)\big)$ is cofibrant, and that the map $L_A\big(\bar{e}(f)\big)$ is a cofibration.

By definition, the space $e_k \circ \bar{e}(Y)^A$ is given by the mapping cylinder of the map

$L_A(\bar{e}(Y)) \longrightarrow Y^A$, and similarly for Z. Consider the following commutative square:

$$\begin{tikzcd} L_A(\bar{e}(Y)) \arrow[r, "L_A(\bar{e}(f))"] \arrow[d] & L_A(\bar{e}(Z)) \arrow[d] \\ Y^A \arrow[r, "f^A"'] & Z^A \end{tikzcd}$$

The lower horizontal map is a cofibration since f is an f-cofibration, the upper horizontal map is a cofibration by the above. Application of the mapping cylinder functor (to the vertical arrows) yields

$$\begin{tikzcd} L_A(\bar{e}(Y)) \arrow[r, "L_A(\bar{e}(f))"] \arrow[d] & L_A(\bar{e}(Z)) \arrow[d] \\ (e_k \circ \bar{e}(Y))^A \arrow[r] & (e_k \circ \bar{e}(Z))^A \end{tikzcd}$$

and the induced map from the pushout into the terminal vertex is the map $(*)$ (note that $L_A(e_k \circ \bar{e}(Y)) = L_A(\bar{e}(Y))$ and similarly for Z since the A-components do not appear in the colimit defining the latching space). Since the mapping cylinder construction is a cylinder functor in the sense of [W] by 1.2.3 (1), this is a cofibration.

Ad (2). The functor e_k is defined using mapping cylinders. But this construction is compatible with colimits by 1.2.3 (3).

Ad (3) and (4). Observe that γ is a left adjoint of the KRONECKER functor δ_N which proves (4). The latter functor maps (acyclic) fibrations to (acyclic) c-fibrations since they are defined pointwise, hence (3) holds (cf. [DS], remark 9.8). □

Corollary 1.3.5.

(1) The functor Γ maps an (acyclic) f-cofibration between f-cofibrant objects to an (acyclic) cofibration in G-$k\,Top_*$; in particular $\Gamma(Y)$ is cofibrant if Y is f-cofibrant.

(2) The functor Γ preserves weak equivalences between f-cofibrant objects.

(3) The functor Γ commutes with colimits.

Proof. If f is an f-cofibration, the map $e_{\#N} \circ \ldots \circ e_0(f)$ is a c-cofibration by part (1) of 1.3.4. By 1.3.4 (3) we infer that $\Gamma(f)$ is a cofibration. This proves (1). Application of BROWN's lemma ([DS], 9.9) shows that Γ preserves all weak equivalences between f-cofibrant cubes, hence (2) holds. The third assertion is a summary of 1.3.4 (2) and (4). □

In what follows we will be interested in cubes with trivial initial vertex. We call such a cube *special* if it has the additional property that all structure maps away from the inital vertex are weak homotopy equivalences.

Lemma 1.3.6. *Suppose N is a finite non-empty set and Y is an f-cofibrant special N-cube.*

(1) For all $k \in N$, the iterated homotopy cofibre of Y is weakly equivalent to $\Sigma^n Y^{\{k\}}$ (where $n = \#N - 1$).

(2) If Y has contractible iterated homotopy cofibre and the spaces $Y^{\{k\}}$ are simply connected for all $k \in N$, then Y has weakly contractible components.

Proof.

(1) Compute homotopy cofibres in k-direction as explained in remark 1.3.2 (4). The resulting $\big(N \setminus \{k\}\big)$-cube Z is weakly equivalent to $\delta_\emptyset Y^{\{k\}}$. This is true since all structure maps starting at a non-initial vertex are weak equivalences and consequently their homotopy cofibres are weakly contractible. Next, computing in ℓ-direction (where $\ell \neq k$), we see that $\Gamma(Z)$ is weakly equivalent to the iterated homotopy cofibre of the $\big(N \setminus \{k, \ell\}\big)$-cube $\delta_\emptyset(\Sigma Y^{\{k\}})$ since the homotopy cofibre of a map $K \longrightarrow *$ is one possible model for ΣK. Continuing in this manner, we obtain a chain of isomorphisms and weak equivalences

$$\Gamma(Y) \cong \Gamma(Z) \simeq \Gamma(\delta_\emptyset Y^{\{k\}}) \simeq \Sigma^n Y^{\{k\}} \ .$$

(2) By (1), all the spaces $\Sigma^n Y^{\{k\}}$ for $k \in N$ are contractible. This implies that $Y^{\{k\}}$ is contractible for all k since $Y^{\{k\}}$ is simply connected. Hence all components of Y are contractible because Y is special. □

2. Non-Linear Projective Space

2.1. A short Review of Projective Spaces

This section is intended to serve as an informal motivation for the setup presented in the rest of these notes.

Let R be a commutative ring. Algebraic geometers define the n-dimensional projective space over R as the (projective) scheme $P^n_R := \text{Proj } S$ where S is the polynomial ring $S := R[X_0,\ X_1,\ \ldots,\ X_n]$. Note that S is a graded ring with the usual total degree of polynomials; indeterminates have degree 1.

There is another description of the scheme P^n_R: It can be obtained by gluing certain affine schemes. Let $S^i := R[X_0,\ X_1,\ \ldots,\ X_n,\ X_i^{-1}]$ denote the ring S with X_i inverted, and let similarly S^{ij} be S with both X_i and X_j inverted. The rings S^i and S^{ij} are graded rings again (with X_i^{-1} having degree -1) and we can define R^i and R^{ij} to be their degree 0 subrings. There exist inclusion maps $R^i \longrightarrow R^{ij} \longleftarrow R^j$. Passage to spectra (in the sense of algebraic geometry) yields

$$\text{Spec } R^i \longleftarrow \text{Spec } R^{ij} \longrightarrow \text{Spec } R^j\ ,$$

and it can be shown that the scheme obtained by gluing all the affine schemes Spec R^i for $i = 0,\ 1,\ \ldots,\ n$ along the schemes Spec R^{ij} is isomorphic to Proj S (cf. [I], §§1.18 and 3.4). Thus Spec R^{ij} is the intersection of Spec R^i and Spec R^j inside P^n_R. We call Spec R^i the i-th canonical open set.

We can characterize the intersections of more than two of the canonical sets. Let $\langle n\rangle$ denote the set $\{0,\ 1,\ \ldots,\ n\}$. For $A \subseteq \langle n\rangle$ we define S^A as S with $\prod_{i\in A} X_i$ inverted (*i.e.*, with all X_i, $i \in A$, inverted), and let R^A denote the degree 0 subring of S^A. Then we have inclusion maps $R^A \longrightarrow R^B$ whenever $A \subseteq B \subseteq \langle n\rangle$. By applying the functor Spec, we obtain a collection of subschemes of P^n_R, and for non-empty $A \subseteq \langle n\rangle$ we see that Spec R^A is the intersection $\bigcap_{i\in A}$ Spec R^i inside $\mathbf{P}^n_R$.

The reader should note at this point that all the rings constructed from R are monoid rings of a very special kind. Let $\mathbf{N}$ denote the natural numbers including 0 considered as a monoid with respect to the sum. Then the polynomial ring $R[X]$ is, by definition, the monoid ring $R[\mathbf{N}]$ with X corresponding to $1 \in \mathbf{N}$ (the generator of $\mathbf{N}$). More generally, we have

$$R[X_0,\ X_1,\ \ldots,\ X_n] = R[\underbrace{\mathbf{N} \oplus \cdots \oplus \mathbf{N}}_{n+1 \text{ summands}}]$$

where $\oplus$ denotes the sum of abelian monoids (finite sums and products agree and are given by cartesian product of underlying sets). Inverting X_i amounts then to changing the corresponding factor $\mathbf{N}$ to $\mathbf{Z}$; thus $R[X_0,\ X_1,\ X_0^{-1}] = R[\mathbf{Z} \oplus \mathbf{N}]$ and $R[X_0,\ X_1,\ X_1^{-1}] = R[\mathbf{N} \oplus \mathbf{Z}]$.

To introduce some notation, we denote by $\widetilde{M}_n^A$ the monoid described above occuring in the definition of S^A, *i.e.*, a sum of $n+1$ copies of $\mathbf{N}$ or $\mathbf{Z}$ (a more formal definition will be given later). Then we have the short formula $S^A = R[\widetilde{M}_n^A]$.

There is a similar description of the rings R^A. Let M_n^A denote the set of those elements of $\widetilde{M}_n^A$ having sum zero. Then M_n^A is a submonoid of $\widetilde{M}_n^A$ and $R^A = R[M_n^A]$. This is a special case of the following lemma.

A *graded monoid* is a (multiplicative) monoid M together with a $\mathbf{Z}$-indexed collection of subsets $M_i \subseteq M$ such that as a set M is the disjoint union of the M_i, and such that $M_i \cdot M_j \subseteq M_{i+j}$.

Lemma 2.1.1. *Suppose M is a graded monoid and R is a ring with 1. Then the monoid ring $R[M]$ is a graded ring, its degree-i-part is given by $R[M]_i = R[M_i]$, the free R-module on M_i.* □

This lemma is a first hint that certain constructions arising in connection with the study of projective spaces can be performed in a "non-linear" context, using only the multiplicative monoid structure.

Back to the situation at hand, we can view $\widetilde{M}_n^A$ as a graded monoid; the grading is given by the sum of elements. Then M_n^A is indeed the submonoid of degree 0 elements, and the lemma applies.

We are interested in quasi-coherent sheaves on $\mathbf{P}_R^n$. Such a sheaf is determined by its sections over the canonical open sets Spec (R^i) and its behaviour on intersections of these. In more detail, suppose we have, for all non-empty $A \subseteq \langle n \rangle$, an R^A-module M^A, and for each inclusion $A \subseteq B$ an R^A-equivariant additive map $M^A \longrightarrow M^B$ which becomes an isomorphism after "inverting the action of X_i for $i \in B \setminus A$ ". Then there is a unique quasi-coherent $\mathcal{O}_{\mathbf{P}_R^n}$-module $\mathcal{F}$ with $\Gamma(\mathcal{F}, \text{Spec } (R^A)) = M^A$. The category of "diagrams" of this kind (they are examples of "twisted diagrams" in the sense of appendix A) is equivalent to the category of quasi-coherent sheaves on $\mathbf{P}_R^n$.

2.2. Non-Linear Polynomial Rings

By "forgetting the linear structure", *i.e.*, forgetting the ring R, we pass from monoid rings to monoids (with zero) which can be thought of as "non-linear rings".

Definition 2.2.1. Suppose N is a non-empty finite set. For $A \subseteq N$, define the monoid with zero

$$\widetilde{M}^A = \widetilde{M}_N^A := \left\{ (m_i)_{i \in N} \in \mathbf{Z}^N \;\middle|\; \forall i \in N \setminus A \colon m_i \geq 0 \right\}_+ .$$

The homomorphism $\deg \colon \widetilde{M}^A \longrightarrow \mathbf{Z}_+$, $(a_i)_{i \in N} \mapsto \sum_{i \in N} a_i$ is called "degree map" (here $\mathbf{Z}_+$ is the set of all integers with a disjoint basepoint considered as a monoid with zero; "multiplication" is given by the usual sum of integers).

Define subsets $\widetilde{M}^A(j) = \widetilde{M}^A_N(j) := \deg^{-1}(j)$ for all $j \in \mathbf{Z}$, and note that $\widetilde{M}^A(0)$ is really a monoid with zero which acts on the pointed sets $\widetilde{M}^A(j)$. It is convenient to introduce the notation $M^A = M^A_N := \widetilde{M}^A(0)$.

There is a convenient class of finite sets, the standard sets $\langle n\rangle := \{0,\ 1,\ \ldots,\ n\}$ for $n \in \mathbf{Z}$, and the even-more-standard sets $[n]$ which are $\langle n\rangle$ as sets again, but equipped with the natural order. When using the standard sets, we write M_n for $M_{\langle n\rangle}$ and $M_{[n]}$.

We think of the above monoids as multiplicative monoids: write t_i for the element which contains a $1 \in \mathbf{Z}$ in the i-th place and $0 \in \mathbf{Z}$ everywhere else, then the collection of the t_i and t_i^{-1} generates $\widetilde{M}^N_N = (\mathbf{Z}^{\# N})_+$ as an abelian monoid with zero. The monoids M^N_N are generated by the compound symbols $t_i t_j^{-1}$ for $i \neq j$.—The symbol t_i should be thought of as the indeterminate X_i, and M^A (*resp.* $\widetilde{M}^A$) corresponds to the ring R^A (*resp.* S^A) of the previous subsection.

Example 2.2.2. The monoid $M_1^{\{0\}}$ is isomorphic to $\mathbf{N}_+$ (natural numbers with disjoint basepoint):

$$\mathbf{N}_+ \xrightarrow{\cong} M_1^{\{0\}} = \{(a,\ b) \in \mathbf{Z}\times\mathbf{N} \mid a+b=0\}_+\,, \quad t \mapsto (-t,\ t)$$

This corresponds, in the linear setting, to the isomorphism of rings

$$R[T_0,\ T_0^{-1},\ T_1]_0 \cong R[X], \quad T_0^{-1}T_1 \mapsto X\ .$$

The map $t \mapsto (t,\ -t)$ is an isomorphism $\mathbf{N}_+ \longrightarrow M_1^{\{1\}}$. Finally, the monoid with zero $M_1^{\{0,1\}} = M_1^{\langle 1\rangle}$ is isomorphic to $\mathbf{Z}_+$ via $t \mapsto (t,\ -t)$.

For $n = 2$, we find the following isomorphisms:

$$\begin{array}{lll}
\mathbf{N}\times\mathbf{N} \longrightarrow M_2^{\{0\}}, & & (a,\ b) \mapsto (-a-b,\ a,\ b)\\
\mathbf{Z}\times\mathbf{N} \longrightarrow M_2^{\{0,1\}}, & & (a,\ b) \mapsto (-a-b,\ a,\ b)\\
\mathbf{Z}\times\mathbf{Z} \longrightarrow M_2^{\{0,1,2\}}, & & (a,\ b) \mapsto (-a-b,\ a,\ b)
\end{array}$$

Definition 2.2.3. Suppose M is a monoid with zero. A subset $I \subseteq M$ is called a *(twosided) ideal* of M, denoted $I \triangleleft M$, if $0 \in I$, and for all $(m, a) \in M \times I$ the elements $a \cdot m$ and $m \cdot a$ are contained in I.

If $I \triangleleft M$ we can define a new monoid with zero M/I. As a set, it is given by $(M \setminus I)_+$ (this is the quotient $M/I = M \cup_I *$ in the category of pointed sets). The monoid structure is induced by that of M. To be precise, let $a \odot b$ denote the product of $a,\ b \in M/I$. Then we have

$$a \odot b = \begin{cases} a\cdot b, & \text{if } a\cdot b \notin I\\ *, & \text{otherwise}\end{cases}$$

where $*$ denotes the (added) zero element of M/I. The monoid M/I is abelian if M is. We will drop the notation $a \odot b$ for the product in M/I from now on.

There is an obvious map $M \longrightarrow M/I$, sending $m \in M \setminus I$ to $m \in M/I$ and mapping all of $I \subseteq M$ to $* \in M/I$. It is readily verified that this map is a map of monoids with zero. Its kernel, *i.e.*, the preimage of the zero element (not of the identity) is I. It is left to the reader to verify the following statement which links the notions of ideals in monoids and rings.

Remark 2.2.4. If M is a monoid with zero, $I \lhd M$ is an ideal, and R is a ring, the rings $\widetilde{R}[M/I]$ and $\widetilde{R}[M]/\widetilde{R}[I]$ are naturally isomorphic. (Here the functor $\widetilde{R}$ is the reduced free R-module functor.) □

Example 2.2.5. Suppose N is a finite set, $A \subset N$ a subset, and $j \in N \setminus A$. We define the ideal "generated by t_j"

$$I^A_{j,N} := \left\{(a_i)_{i\in N} \in M^A_N \;\middle|\; a_j \neq 0\right\}_+$$

of M^A_N. Then there is an isomorphism $M^A_N/I^A_{j,N} \cong M^A_{N\setminus\{j\}}$, mapping $(a_i)_{i\in N}$ to $(a_i)_{i\in N\setminus\{j\}}$. In particular, M^A_N acts on $M^A_{N\setminus\{i\}}$ via the projection $M^A_N \longrightarrow M^A_N/I^A_{j,N}$. We will repeatedly make use of this fact.

For the rest of this paper we will consider M^A as a topological monoid with zero having the discrete topology. Similarly we regard $\widetilde{M}^A(j)$ as an object of M^A-$k\,Top_*$. **From now on, let G denote a topological monoid with zero, and suppose N is a non-empty finite set with $n+1$ elements.**

Definition 2.2.6. Suppose we are given a (possibly empty) subset $A \subseteq N$, an element $i \in N$, and a space $Y \in (G\wedge M^A)$-$k\,Top_*$. Then we can form the space

$$Y[t_i^{-1}] := Y \underset{M^A}{\wedge} M^{A\cup\{i\}} \in (G\wedge M^{A\cup\{i\}})\text{-}k\,Top_*$$

and call $Y[t_i^{-1}]$ obtained by inverting the action of t_i on Y. This construction is functorial in Y. More generally, given subsets A and B of N, there is a functor

$$\cdot[t_B^{-1}] = \cdot \underset{M^A}{\wedge} M^{A\cup B} \colon (G\wedge M^A)\text{-}k\,Top_* \longrightarrow (G\wedge M^B)\text{-}k\,Top_*, \quad Y \mapsto Y[t_B^{-1}]\ .$$

The notation is intuitive, but also slightly misleading. Recall that $t_i \in \widetilde{M}^A$ has degree 1, hence is not an element of M^A. Instead of inverting t_i, we have to invert all those generators of M^A which contain t_i, *i.e.*, all the symbols $t_it_j^{-1}$ for $j \in A$; cf. the telescope construction below.

The construction of 2.2.6 is the non-linear analogue of inverting indeterminates in a Laurent ring. As in the linear case, there are alternative descriptions using mapping telescopes. One can check that all three constructions have the same universal property and hence are canonically isomorphic; we omit the details.

Lemma 2.2.7. *(Telescope construction)*

Suppose $A \subseteq N$ is not empty; write $a = \#A$ and $m = \max(A)$.

(1) The space $Y[t_i^{-1}] = Y \wedge_{M^A} M^{A\cup\{i\}}$ is isomorphic to the colimit (in the category G-$k\,Top_*$) of the sequence

$$Y \xrightarrow{t_i^a \cdot t_A^{-1}} Y \xrightarrow{t_i^a \cdot t_A^{-1}} Y \xrightarrow{t_i^a \cdot t_A^{-1}} \cdots$$

where we have used the multi-index notation $t_A^{-1} := \prod_{j\in A} t_j^{-1}$. In particular, the colimit admits a canonical action of $M^{A\cup\{i\}}$.

(2) The space $Y[t_i^{-1}] = Y \wedge_{M^A} M^{A\cup\{i\}}$ is isomorphic to the colimit (in the category G-$k\,Top_*$) of the following sequence:

$$Y \xrightarrow{t_i \cdot t_m^{-1}} Y \xrightarrow{t_i \cdot t_m^{-1}} Y \xrightarrow{t_i \cdot t_m^{-1}} \cdots$$

In particular, the colimit admits a canonical action of $M^{A\cup\{i\}}$.

□

By comparing universal properties again, we see that the result of consecutively inverting several "indeterminates" is independent, up to canonical isomorphism, of the order in which they are inverted:

$$Y[t_i^{-1}][t_k^{-1}] \cong Y[t_{\{i,k\}}^{-1}] \cong Y[t_k^{-1}][t_i^{-1}]$$

For easier reference, we restate lemma 1.2.2, applied to the inclusion $M^A \longrightarrow M^B$ of monoids with zero:

Lemma 2.2.8. *Suppose $A \subseteq B$ are non-empty subsets of N.*

(1) The functor $\cdot[t_B^{-1}] : (G\wedge M^A)$-$k\,Top_* \longrightarrow (G\wedge M^B)$-$k\,Top_*$ has a right adjoint functor $\bar{Y} \mapsto Y$ where Y is $\bar{Y}$ as a topological space with action restricted to $G \wedge M^A$.

(2) The functor $\cdot[t_B^{-1}]$ preserves cofibrations and acyclic cofibrations. It maps weak equivalences between cofibrant objects to weak equivalences.

(3) The functor $\cdot[t_B^{-1}]$ maps cells to cells (where cells are of the form $G\wedge M^A \wedge \Delta_+^k$ resp. $G \wedge M^B \wedge \Delta_+^k$). It restricts to a functor $\mathbf{C}_?(G\wedge M^A) \longrightarrow \mathbf{C}_?(G\wedge M^B)$ where $?$ may denote any of the decorations f, hf, fd or sfd.

In the rest of this subsection we present two technical results about equivariant spaces which will be used when dealing with finiteness conditions.

Lemma 2.2.9. (*Smallness of finite equivariant spaces.)*

Suppose G is cofibrant as an object of $k\,Top_*$. Let $A \subseteq N$, $j \in N \setminus A$ and spaces $Z \in \mathbf{C}(G\wedge M^A)$ and $Y \in \mathbf{C}_f(G\wedge M^A)$ be given; define $a := \#A$. Let s_0 denote the canonical $(G\wedge M^A)$-equivariant inclusion $Z \longrightarrow Z[t_j^{-1}] = Z \wedge_{M^A} M^{A\amalg\{j\}}$ (the unit of the adjunction described in 2.2.8 (1)). For any map $f : Y \longrightarrow Z[t_j^{-1}]$ in $(G\wedge M^A)$-$k\,Top_*$ there is a $(G\wedge M^A)$-equivariant map $g : Y \longrightarrow Z$ and an integer $k \geq 0$ such that $f = t_j^{-ak} t_A^k \circ s_0 \circ g$. Moreover, k can be enlarged arbitrarily.

Proof. We proceed by induction on the number of (equivariant) cells in Y. For $Y = *$ we can choose $k = 0$ and $g = f$.

Identify $Z[t_j^{-1}]$ with the colimit of the telescope construction (2.2.7 (1)). The top

$$\begin{tikzcd}
Z \arrow[r, "t_j^a \cdot t_A^{-1}"] \arrow[d, "s_0"'] & Z \arrow[r, "t_j^a \cdot t_A^{-1}"] \arrow[d, "s_1"'] & \dots \arrow[r, "t_j^a \cdot t_A^{-1}"] & Z \arrow[d, "s_k"] & \dots \\
Z[t_j^{-1}] \arrow[r, equal] & Z[t_j^{-1}] \arrow[r, equal] & \dots \arrow[r, equal] & Z[t_j^{-1}] \dots &
\end{tikzcd}$$

row of the diagram is the telescope. The vertical arrows are the canonical maps into the colimit, where s_0 is as above. The maps in the telescope are cofibrations of G-spaces (this follows from lemma 1.2.4 since M^A is a discrete monoid, and right translation by $t_j^a \cdot t_A^{-1}$ is injective), and since G is cofibrant, the telescope consists of cofibrations in $k\mathit{Top}_*$ (1.2.1 (3)).

Let $C := \Delta^\ell_+ \wedge G \wedge M^A$ denote a cell with boundary ∂C, and suppose $Y = \bar{Y} \cup_{\partial C} C$. By induction, we find $\bar{k}$ and a map $\bar{g}\colon \bar{Y} \longrightarrow Z$ with $f|_{\bar{Y}} = t_j^{-a\bar{k}} t_A^{\bar{k}} \circ s_0 \circ \bar{g} = s_{\bar{k}} \circ \bar{g}$. Let $\alpha\colon \Delta^\ell \longrightarrow Z[t_j^{-1}]$ denote the restriction of f to the "generating" non-equivariant cell of C. Since Δ^ℓ is compact, this map factors through some finite stage ℓ of the telescope construction. By forcing $(G \wedge M^A)$-equivariance we obtain a map $\beta\colon C \longrightarrow Z$ such that $f|_C = t_j^{-a\ell} t_A^\ell \circ s_0 \circ \beta = s_\ell \circ \beta$. Since $s_{\bar{k}+\ell}$ is injective (this is similar to the proof of 3.2.2 (2)—we omit the details) the following diagram commutes:

$$\begin{tikzcd}
 & \bar{Y} \arrow[r, "\bar{g}"] \arrow[dr, "f|_{\bar{Y}}"] & Z \arrow[r, "t_j^{a\ell} t_A^{-\ell}"] \arrow[d, "s_{\bar{k}}"] & Z \arrow[d, equal, "\mathrm{id}"] \\
\partial C \arrow[ur] \arrow[dr] & & Z[t_j^{-1}] & Z \arrow[l, "s_{\bar{k}+\ell}"'] \arrow[d, equal, "\mathrm{id}"] \\
 & C \arrow[r, "\beta"'] \arrow[ur, "f|_C"] & Z \arrow[r, "t_j^{a\bar{k}} t_A^{-\bar{k}}"] \arrow[u, "s_\ell"] & Z
\end{tikzcd}$$

Let $k := \bar{k} + \ell$, and define g as the induced map from $Y = \bar{Y} \cup_{\partial C} C$ to the rightmost Z in the diagram. Then by construction $f = s_k \circ g = t_j^{-ak} t_A^k \circ s_0 \circ g$.

To enlarge k by $m \geq 0$, note that s_0 is M^A-equivariant, hence:

$$\begin{aligned}
f = t_j^{-ak} t_A^k \circ s_0 \circ g &= t_j^{-a(k+m)} t_A^{(k+m)} \circ t_j^{am} t_A^{-m} \circ s_0 \circ g \\
&= t_j^{-a(k+m)} t_A^{(k+m)} \circ s_o \circ (t_j^{am} t_A^{-m} \circ g)
\end{aligned}$$

□

Lemma 2.2.10. (*Smallness of finite equivariant spaces—alternative version.*)

Suppose G is cofibrant as an object of $k\mathit{Top}_*$. Let $A \subseteq N$, $j \in N \setminus A$ and spaces $Z \in \mathbf{C}(G \wedge M^A)$ and $Y \in \mathbf{C}_f(G \wedge M^A)$ be given; define $m := \max(A)$. Let s_0 denote the canonical $(G \wedge M^A)$-equivariant inclusion $Z \longrightarrow Z[t_j^{-1}] = Z \wedge_{M^A} M^{A \amalg \{j\}}$ (the unit of the adjunction described in 2.2.8 (1)). For any map $f : Y \longrightarrow Z[t_j^{-1}]$ in $(G \wedge M^A)$-$k\mathit{Top}_*$ there is a $(G \wedge M^A)$-equivariant map $g : Y \longrightarrow Z$ and an integer $k \geq 0$ such that $f = t_j^{-k} t_m^k \circ s_0 \circ g$. Moreover, k can be enlarged arbitrarily.

Proof. This is similar to the previous lemma, except that one uses the second telescope construction (2.2.7 (2)) instead of the first. □

Corollary 2.2.11. *Suppose G is cofibrant as an object of $k\mathit{Top}_*$. Let $A \subseteq N$, $j \in N \setminus A$ and $Y \in \mathbf{C}_f(G \wedge M_N^A)$ be given. If $Y[t_j^{-1}] \simeq *$, the space $\Sigma Y \vee Y$ is homotopy finite as a $(G \wedge M_{N\setminus\{j\}}^A)$-space (with the restricted action); hence Y is finitely dominated as a $(G \wedge M_{N\setminus\{j\}}^A)$-space.*

Proof. Since $Y[t_j^{-1}] \simeq *$, the inclusion $s_0 : Y \longrightarrow Y[t_j^{-1}]$ is null homotopic (where s_0 is as in 2.2.9). Choose such a homotopy $H : Y \wedge I \longrightarrow Y[t_j^{-1}]$ from s_0 to the trivial map.

The space $Y \wedge I$ is finite as a $(G \wedge M_N^A)$-space, hence we know by 2.2.9 that H factors through some finite stage of the telescope: there is a map $F : Y \wedge I \longrightarrow Y$ and an integer $m \geq 0$ with $H = t_j^{-am} t_A^m \circ s_0 \circ F$ where $a := \#A$, and consequently $t_j^{am} t_A^{-m} \circ H = s_0 \circ F$. By choice of H, the map $s_0 \circ F$ is a homotopy from $t_j^{am} t_A^{-m} \circ s_0 = s_0 \circ t_j^{am} t_A^{-m}$ to the trivial map. But s_0 is an injective map (this is similar to the proof of 3.2.2 (2)—we omit the details), so F is a null homotopy of $Y \xrightarrow{t_j^{am} t_A^{-m}} Y$. So we have a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "i_0"] \arrow[d, "t_j^{am} t_A^{-m}"'] & Y \wedge I \arrow[d, "F"'] & Y \arrow[l, "i_1"'] \arrow[d, "*"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd} \qquad (*)$$

where i_0 and i_1 denote the inclusion of Y as top and bottom into the cylinder. Application of the homotopy cofibre (mapping cone) functor to the vertical maps yields a sequence of weak equivalences

$$\text{hocofibre}\,(Y \xrightarrow{t_j^{am} t_A^{-m}} Y) \xrightarrow{\sim} \text{hocofibre}\,(F) \xleftarrow{\sim} \text{hocofibre}\,(Y \xrightarrow{*} Y)\ .$$

The last space is (one possible model for) $\Sigma Y \vee Y$. Note that these weak equivalences are $(G \wedge M_{N\setminus\{j\}}^A)$-equivariant since they are induced by the $(G \wedge M_N^A)$-equivariant horizontal maps in diagram $(*)$.

On the other hand, the left vertical map in $(*)$ is a cofibration in $G\text{-}k\mathit{Top}_*$ by lemma 1.2.4. Hence the canonical map from the mapping cone into the strict cofibre $Y/t_j^{am}t_A^{-m}(Y)$ is a weak homotopy equivalence. It is a $(G\wedge M^A_{N\setminus\{j\}})$-equivariant map since it is induced by $(G\wedge M^A_N)$-equivariant maps.

It remains to note that the space $Y/t_j^{am}t_A^{-m}(Y)$ is finite as a $(G\wedge M^A_{N\setminus\{j\}})$-space. This is shown by induction on the number of cells in Y. Since formation of the quotients commutes with cell attachment, it suffices to show that the cofibre of

$$M^A_N \xrightarrow{t_j^{am}t_A^{-m}} M^A_N \qquad (**)$$

is isomorphic, as an $M^A_{N\setminus\{j\}}$-space, to a finite one-point union of copies of $M^A_{N\setminus\{j\}}$; then a free $(G\wedge M^A_N)$-equivariant cell of Y gives rise to a finite one-point union of free $(G\wedge M^A_{N\setminus\{j\}})$-equivariant cells of the quotient space. But we can partition the set M^A_N into subsets according to the value of the j-th component; explicitly, we have an isomorphism

$$M^A_N \cong \bigvee_{i\geq 0} \widetilde{M}^A_{N\setminus\{j\}}(-i)\wedge\{i\}_+$$

of $M^A_{N\setminus\{j\}}$-spaces. Each of the subsets $\widetilde{M}^A_{N\setminus\{j\}}(-i)$ is (non-canonically) equivariantly isomorphic to $M^A_{N\setminus\{j\}}$, an isomorphism is given by t_b^i for any $b\in A$. Now the cofibre of the map $(**)$ is seen to be $\bigvee_{i=0}^{am-1}\widetilde{M}^A_{N\setminus\{j\}}(-i)$ which is isomorphic to $\bigvee_{i=0}^{am-1}M^A_{N\setminus\{j\}}$.

Since the chain of weak equivalences connecting the finite space $Y/t_j^{am}t_A^{-m}(Y)$ with $\Sigma Y\vee Y$ consists of $(G\wedge M^A_{N\setminus\{j\}})$-equivariant maps, $\Sigma Y\vee Y$ is homotopy finite as claimed.

Finally, Y is a retract of $\Sigma Y\vee Y$, and it follows that Y is finitely dominated as a $(G\wedge M^A_{N\setminus\{j\}})$-space. □

2.3. Non-Linear Sheaves on Projective Space

In section 2.1 we indicated how to describe quasi-coherent sheaves by certain diagrams of modules. We want to "forget the linear structure", *i.e.*, replace rings by monoids and modules by equivariant spaces. Thus we obtain a non-linear version of sheaves on projective space. As before, we assume that G is a topological monoid with zero, and N is a non-empty finite set with $n+1$ elements.

Definition 2.3.1. *(Presheaves on projective space.)*

We define the category $\mathbf{pP}^N(G)$ of *(G-equivariant quasi-coherent) presheaves* on projective N-space to be the following subcategory of $k\mathit{Top}_*^{\langle N\rangle}$: objects are the functors $Y\colon\langle N\rangle\longrightarrow k\mathit{Top}_*$ with $Y(\emptyset)=*$ such that for each $A\subseteq N$, the space $Y^A:=Y(A)$ has a (right) $(G\wedge M^A_N)$-action, and for each morphism $\sigma\colon A\longrightarrow B$ in $\langle N\rangle$, the associated map $Y^\flat_\sigma\colon Y^A\longrightarrow Y^B$ is $(G\wedge M^A)$-equivariant. We will sometimes refer to the map $Y^\flat_\sigma$ as a "$\flat$-type structure map" of Y. The space Y^A is called the *A-component* of Y. A morphism $f: Y \longrightarrow Z$ is a natural transformation of diagrams, consisting of $(G \wedge M^A)$-equivariant maps $f^A: Y^A \longrightarrow Z^A$ called *components* of f.—If $N = [n]$ or $N = \langle n \rangle$, we write $\mathbf{pP}^n(G)$ instead of $\mathbf{pP}^N(G)$.—Note that $\mathbf{pP}^0(G) = G\text{-}k\,Top_*$, and every choice of a bijection $N \cong \langle n \rangle$ defines an isomorphism of categories $\mathbf{pP}^N(G) \cong \mathbf{pP}^n(G)$.

One can think of non-linear projective space as "relative projective space over BG" in the sense of [Ha, II.7, p. 162] (cf. the remark at the end of § 3).

The reader should be warned that the linear analogue of such a diagram does not determine a presheaf in the sense of algebraic geometry. We ask the reader to apologize this abuse of terminology.

Definition 2.3.1 says that a presheaf is a twisted diagram in the sense of appendix A over an appropriate adjunction bundle. We will exploit this fact in the next subsection to construct "cofibrant" sheaves. We will sometimes use the category $\langle N \rangle_0$ of non-empty subsets of N as indexing category (omitting the redundant one-point space corresponding to $\emptyset \subseteq N$). If $G = S^0$ (the "trivial" monoid with $0 \neq 1$), we write $\mathbf{pP}^N$ omitting G from the notation.

The category $\mathbf{pP}^N(G)$ has a zero object given by $A \mapsto *$, the constant functor with the one point space as value. We call this the zero presheaf, sometimes denoted $*$.

For any object $Y \in \mathbf{pP}^N(G)$ we define the suspension of Y, denoted $\Sigma(Y)$, as the functor $A \mapsto \Sigma(Y^A)$ with M^A acting trivially on the suspension coordinate (componentwise suspension). Similarly, we can define a mapping cylinder by applying the mapping cylinder construction componentwise.

Each of the structure maps $Y^\flat_\sigma$ has a corresponding adjoint map

$$Y^\sharp_\sigma : Y^A[t_B^{-1}] = Y^A \underset{M^A}{\wedge} M^B \longrightarrow Y^B$$

since the functor $\cdot \wedge_{M^A} M^B : (G \wedge M^A)\text{-}k\,Top_* \longrightarrow (G \wedge M^B)\text{-}k\,Top_*$ is left adjoint to the functor restricting the $(G \wedge M^B)$-action to $G \wedge M^A$ along the inclusion $M^A \longrightarrow M^B$. Sometimes we will call $Y^\sharp_\sigma$ a "$\sharp$-type structure map" of Y. These structure maps will be used to formulate a "sheaf condition" (see below).

An object of $\mathbf{pP}^N(G)$ can be visualized as an n-simplex with a space attached to each of its faces, and maps corresponding to inclusion of faces (suppressing the redundant one point space corresponding to $\emptyset \subseteq N$). In the case $N = \langle 1 \rangle$, a typical object Y is depicted

$$Y^{\{0\}} \longrightarrow Y^{\{0,1\}} \longleftarrow Y^{\{1\}}$$

(the arrows indicate $\flat$-type structure maps). For $N = \langle 2 \rangle$, we have the following picture:

$$\begin{tikzcd}
 & & Y^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & Y^{\{0,2\}} \arrow[dr] & & Y^{\{1,2\}} \arrow[dl] & \\
 & & Y^{\{0,1,2\}} & & \\
Y^{\{0\}} \arrow[uur] \arrow[rr] & & Y^{\{0,1\}} \arrow[u] & & Y^{\{1\}} \arrow[ll] \arrow[uul]
\end{tikzcd}$$

Equivalently, by subdivision, we can regard an object of $\mathbf{pP}^N(G)$ as an $(n{+}1)$-cubical diagram with a point as initial vertex. For $N = \langle 2 \rangle$ this yields the following picture:

$$\begin{tikzcd}
 & Y^{\{2\}} \arrow[rr] \arrow[dd] & & Y^{\{0,2\}} \arrow[dd] \\
* \arrow[ur] \arrow[rr] \arrow[dd] & & Y^{\{0\}} \arrow[ur] \arrow[dd] & \\
 & Y^{\{1,2\}} \arrow[rr] & & Y^{\{0,1,2\}} \\
Y^{\{1\}} \arrow[ur] \arrow[rr] & & Y^{\{0,1\}} \arrow[ur] &
\end{tikzcd}$$

Definition 2.3.2. *(Structure presheaves of projective spaces.)*

We define the *structure presheaf of projective N-space* $\mathcal{O} = \mathcal{O}_{\mathbf{P}^N}$ to be the functor $A \mapsto M_N^A$ (for $A \neq \emptyset$); structure maps are given by inclusions, and M_N^A acts on itself by right translation.

If $N = \langle 2 \rangle$, we have the following picture for $\mathcal{O}_{\mathbf{P}^2}$:

$$\begin{tikzcd}
 & (\mathbf{N} \times \mathbf{N} \times \mathbf{Z})^0_+ \arrow[rr] \arrow[dd] & & (\mathbf{Z} \times \mathbf{N} \times \mathbf{Z})^0_+ \arrow[dd] \\
* \arrow[ur] \arrow[rr] \arrow[dd] & & (\mathbf{Z} \times \mathbf{N} \times \mathbf{N})^0_+ \arrow[ur] \arrow[dd] & \\
 & (\mathbf{N} \times \mathbf{Z} \times \mathbf{Z})^0_+ \arrow[rr] & & (\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z})^0_+ \\
(\mathbf{N} \times \mathbf{Z} \times \mathbf{N})^0_+ \arrow[ur] \arrow[rr] & & (\mathbf{Z} \times \mathbf{Z} \times \mathbf{N})^0_+ \arrow[ur] &
\end{tikzcd}$$

(The lower index "+" means adding a disjoint basepoint, and the upper index 0 denotes the subset of tuples with sum 0. That is, $(\mathbf{Z}\times\mathbf{Z}\times\mathbf{N})_+^0 = M_2^{\{0,1\}}$, and similarly for the other spaces in the diagram.)

For $N = \langle 1\rangle$, the structure presheaf looks like this (with the same conventions for notation as before):

$$(\mathbf{Z}\times\mathbf{N})_+^0 \longrightarrow (\mathbf{Z}\times\mathbf{Z})_+^0 \longleftarrow (\mathbf{N}\times\mathbf{Z})_+^0$$

Our next task is to define a notion of sheaves. This really is the non-linear analogue of the algebraic geometers' notion of quasi-coherent sheaves on projective n-space as indicated in section 2.1.

Definition 2.3.3. *(Sheaves on projective space.)*

We define the category $\mathbf{P}^N(G)$ of *(G-equivariant quasi-coherent) sheaves* on projective N-space to be the full subcategory of $\mathbf{pP}^N(G)$ consisting of those objects Y which satisfy the following *sheaf condition*: for every inclusion $\sigma\colon A \longrightarrow B$ of non-empty subsets of N, there is an object $\bar{Y}^A \in \mathbf{C}(G\wedge M^A)$ and a weak equivalence $\bar{r}_\sigma\colon \bar{Y}^A \longrightarrow Y^A$ such that the map

$$\bar{Y}^A[t_B^{-1}] \longrightarrow Y^B$$

adjoint to the composite map $\bar{Y}^A \xrightarrow{\bar{r}_\sigma} Y^A \xrightarrow{Y_\sigma^\flat} Y^B$ is a weak equivalence.

The sheaf condition is homotopy invariant in a sense described below.—The choice of the "cofibrant replacement" $\bar{Y}^A$ of Y^A is not important. For suppose $r\colon P \longrightarrow Y^A$ is another such choice. Then the map $\bar{r}_\sigma^{-1}\circ r\colon P \longrightarrow \bar{Y}^A$ exists in *Ho* $(G\wedge M^A)$-$k\,Top_*$, and since P is cofibrant by hypothesis, there is a map $f \in (G\wedge M^A)$-$k\,Top_*$ representing it. This map is a weak equivalence because its image in the homotopy category is an isomorphism. Thus $f[t_B^{-1}]$ is a weak equivalence since the functor $\cdot[t_B^{-1}]$ preserves weak equivalences between cofibrant objects by 2.2.8 (2). On the other hand, the diagram

$$\begin{tikzcd} P[t_B^{-1}] \arrow[r, "{f[t_B^{-1}]}"] \arrow[dr] & \bar{Y}^A[t_B^{-1}] \arrow[d] \\ & Y^B \end{tikzcd}$$

commutes up to homotopy since both composites from top left to bottom right have the same image in *Ho* $(G\wedge M^B)$-$k\,Top_*$ by choice of f. Hence the diagonal map is a weak equivalence if and only if the vertical map is. This shows that in the definition of the sheaf condition above, we could have worked with some *fixed* cofibrant replacement, or equivalently, we could have asked for the condition to be satisfied for *all* cofibrant replacements instead of just one. In particular, we can choose $\bar{Y}^A = Y^A$ if Y^A is cofibrant. We have shown:

Corollary 2.3.4. *Suppose $Y \in \mathbf{pP}^N(G)$ is locally cofibrant in the sense that the space Y^A is cofibrant in $(G \wedge M^A)$-$k\,Top_*$ for all non-empty $A \subseteq N$. Then Y is a sheaf if and only if for all inclusions $\sigma\colon A \longrightarrow B$ of non-empty subsets of N, the map $Y^\sharp_\sigma\colon Y^A[t_B^{-1}] \longrightarrow Y^B$ (adjoint to the structure map $Y^\flat_\sigma\colon Y^A \longrightarrow Y^B$) is a weak equivalence.* □

Corollary 2.3.5. *(Homotopy invariance of the sheaf condition.)*

Suppose Y and Z are objects of $\mathbf{pP}^N(G)$. Assume that there is a weak equivalence $f\colon Y \longrightarrow Z$ (i.e., all components of f are weak equivalences). Then Y is a sheaf if and only if Z is a sheaf.

Proof. Let $\sigma\colon A \longrightarrow B$ denote an inclusion of subsets of N. Since $(G \wedge M^A)$-$k\,Top_*$ is a model category, there is a cofibrant space $Q \in \mathbf{C}(G \wedge M^A)$ and a weak equivalence $\alpha\colon Q \longrightarrow Y^A$. Define $\beta := f^A \circ \alpha\colon Q \longrightarrow Z^A$. Since f^A is a weak equivalence, so is β. By composing with structure maps of Y and Z, we obtain a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{Y^\flat_\sigma \circ \alpha} & Y^B \\ {\scriptstyle \mathrm{id}}\Big\| & & \Big\downarrow {\scriptstyle f^B} \\ Q & \xrightarrow[Z^\flat_\sigma \circ \beta]{} & Z^B \end{array}$$

and, by passing to adjoint maps, we see that the following square commutes:

$$\begin{array}{ccc} Q[t_B^{-1}] & \longrightarrow & Y^B \\ {\scriptstyle \mathrm{id}}\Big\| & & \Big\downarrow {\scriptstyle f^B} \\ Q[t_B^{-1}] & \longrightarrow & Z^B \end{array}$$

The right vertical map is a weak equivalence by hypothesis on f. Hence the upper map is a weak equivalence if and only if the lower map is. □

We will use this corollary without further comment.

Remark 2.3.6. The structure presheaf $\mathcal{O}_{\mathbf{P}^N}$ defined above is in fact a sheaf, called the *structure sheaf of projective N-space*. This follows from the canonical isomorphism $M^A[t_B^{-1}] = M^A \wedge_{M^A} M^B \cong M^B$ and 2.3.4.

2.4. Model Structures

Our next goal is to establish two model structures on $\mathbf{pP}^N(G)$ sharing the same weak equivalences, but having a different class of cofibrations. (Both kinds of cofibrations are possible choices for defining the K-theory of projective space. We will discuss later the expected result that they lead, up to weak equivalence, to the same K-theory spaces.) One of the model structures has the advantage that all objects are fibrant.

The interplay of the different notions of cofibrations and fibrations will be an important feature for handling finiteness conditions in $\mathbf{P}^N(G)$.

As before, assume that G is a topological monoid with zero, and let N denote a non-empty finite set with power set $\langle N\rangle$. The symbol $\langle N\rangle_0$ means the set of non-empty subsets of N.

We want to identify $\mathbf{pP}^N(G)$ as a category of twisted diagrams in the sense of appendix A and apply theorems A.3.8 and A.3.13 to obtain two different model structures on $\mathbf{pP}^N(G)$. However, we will give an explicit description of the model structures, so the reader may safely ignore the abstract setup of appendix A.

Define an adjunction bundle $\mathfrak{P}^N(G) = (\mathcal{C},\ F,\ U)$ over $\langle N\rangle_0$, the "G-equivariant projective N-space adjunction bundle", in the following way: for a non-empty subset $A \subseteq N$ let $\mathcal{C}_A := (G\wedge M^A)\text{-}k\,Top_*$, for an inclusion $\iota\colon A \longrightarrow B$ let

$$\mathcal{C}_B = (G\wedge M^B)\text{-}k\,Top_* \xrightarrow{U_\iota} (G\wedge M^A)\text{-}k\,Top_* = \mathcal{C}_A$$

denote the restriction functor (restrict action to smaller monoid), and as left adjoints $F_\iota\colon (G\wedge M^A)\text{-}k\,Top_* \longrightarrow (G\wedge M^B)\text{-}k\,Top_*$ we choose

$$F_\iota = \cdot[t_B^{-1}] = \cdot \underset{M^A}{\wedge} M^B$$

if $A \neq B$, and define $F_{\mathrm{id}_A} = \mathrm{id}_{\mathcal{C}_A}$ for all non-empty $A \subseteq N$. Then we recognize $\mathbf{pP}^N(G)$ as the category of twisted diagrams with coefficients in $\mathfrak{P}^N(G)$.

Suppose Y is an object of $\mathbf{pP}^N(G)$ and $A \subseteq N$ is not empty. The latching space L_AY (definition A.3.4) is given by

$$L_AY := \operatorname*{colim}_{\langle A\rangle_0^1} R(Y)$$

(colimit in $(G\wedge M^A)\text{-}k\,Top_*$) where $\langle A\rangle_0^1$ is the subcategory of non-empty proper subsets of A, and $R(Y)$ is the diagram $B \mapsto Y^B[t_A^{-1}]$ (this is a diagram in $(G\wedge M^A)\text{-}k\,Top_*$). If $\sigma\colon B \longrightarrow C$ is an inclusion of proper subsets of A, *i.e.*, a morphism in $\langle A\rangle_0^1$, the structure map $R(Y)(\sigma)\colon R(Y)^B \longrightarrow R(Y)^C$ is given by the composite

$$\begin{aligned} R(Y)^B = Y^B[t_A^{-1}] = Y^B \underset{M^B}{\wedge} M^A \cong (Y^B \underset{M^B}{\wedge} M^C) \underset{M^C}{\wedge} M^A \\ \xrightarrow{Y_\sigma^\sharp\wedge\mathrm{id}} Y^C \underset{M^C}{\wedge} M^A = Y^C[t_A^{-1}] = R(Y)^C \end{aligned}$$

where $Y_\sigma^\sharp\colon Y^B[t_C^{-1}] \longrightarrow Y^C$ is adjoint to the structure map $Y_\sigma^\flat\colon Y^B \longrightarrow Y^C$. It is shown in appendix A (lemma A.3.5) that this construction yields a commutative diagram in $(G\wedge M^A)\text{-}k\,Top_*$.

The latching space L_AY comes equipped with a canonical map in $(G\wedge M^A)\text{-}k\,Top_*$ to Y^A, induced by the structure maps $Y_\tau^\sharp\colon Y^B[t_A^{-1}] \longrightarrow Y^A$. If $\tau\colon B \longrightarrow A$ is an inclusion of a proper subset, there is a $(G\wedge M^B)$-equivariant map $Y^B \longrightarrow U_\tau(L_AY)$ given by the composite $Y^B \longrightarrow U_\tau \circ F_\tau(Y^B) \longrightarrow L_AY$ (the first map is the unit of the adjunction of F_τ and U_τ, the second map exists since $F_\tau(Y^B) = Y^B[t_A^{-1}]$ appears in the diagram defining the latching space, hence maps to L_AY).

Proposition 2.4.1. *(The c-structure of* $\mathbf{pP}^N(G)$*)*

The category $\mathbf{pP}^N(G)$ *has the structure of a model category where a map is a weak equivalence (resp. fibration) if each of its components is a weak equivalence (resp. fibration) in its respective category. Furthermore the map* $Y \longrightarrow Z$ *is a cofibration if and only if the induced maps* $L_A Z \cup_{L_A Y} Y^A \longrightarrow Z^A$ *are cofibrations in* $G \wedge M^A$*-*$k\,Top_*$ *for all non-empty* $A \subseteq N$*. If* G *is cofibrant as an object of* $k\,Top_*$*, the category* $\mathbf{pP}^N(G)$ *is a proper model category.*

Proof. Consider $\langle N \rangle_0$ as a direct poset with degree function $d(A) := \# A$ and apply theorem A.3.8 (1, 4). (For properness, cf. 1.2.1 (4).) $\square$

Corollary 2.4.2. *All objects of* $\mathbf{pP}^N(G)$ *are fibrant with respect to the c-structure.*

Proof. The object Y is fibrant if the map $Y \longrightarrow *$ into the terminal object is a c-fibration. But by definition, this is the case if and only if for all non-empty $A \subseteq N$, the map $Y^A \longrightarrow *$ is a fibration in $(G \wedge M^A)$-$k\,Top_*$. But all equivariant spaces are fibrant by 1.2.1 (1). $\square$

Example 2.4.3. *(The c-structure of* $\mathbf{pP}^1(G)$*)*

Let $f : Y \longrightarrow Z$ denote a map in $\mathbf{pP}^1(G)$, *i.e.*, we have a commutative diagram of the following kind:

$$\begin{tikzcd}
Y^{\{0\}} \arrow[r] \arrow[d, "f^{\{0\}}"'] & Y^{\{0,1\}} \arrow[d, "f^{\{0,1\}}"'] & Y^{\{1\}} \arrow[l] \arrow[d, "f^{\{1\}}"] \\
Z^{\{0\}} \arrow[r] & Z^{\{0,1\}} & Z^{\{1\}} \arrow[l]
\end{tikzcd}$$

Then f is a weak equivalence if and only if its components f^A are weak equivalences in $(G \wedge M^A)$-$k\,Top_*$ for all non-empty $A \subseteq \langle 1 \rangle$. The map f is a fibration if and only if f^A is a fibration in $(G \wedge M^A)$-$k\,Top_*$ for all non-empty $A \subseteq \langle 1 \rangle$. Finally, f is a cofibration if and only if $f^{\{0\}}$ is a cofibration in $(G \wedge M^{\{0\}})$-$k\,Top_*$, $f^{\{1\}}$ is a cofibration in $(G \wedge M^{\{1\}})$-$k\,Top_*$, and the induced map

$$Y^{\{0,1\}} \cup_{L_{\{0,1\}} Y} L_{\{0,1\}} Z = Y^{\{0,1\}} \cup_{Y^{\{0\}}[t_1^{-1}] \vee Y^{\{1\}}[t_0^{-1}]} (Z^{\{0\}}[t_1^{-1}] \vee Z^{\{1\}}[t_0^{-1}]) \longrightarrow Z^{\{0,1\}}$$

is a cofibration in $(G \wedge M^{\{0,1\}})$-$k\,Top_*$. In particular, a sheaf Y is cofibrant if and only if $Y^{\{0\}} \in \mathbf{C}(G \wedge M^{\{0\}})$, $Y^{\{1\}} \in \mathbf{C}(G \wedge M^{\{1\}})$, and the map

$$Y^{\{0\}}[t_1^{-1}] \vee Y^{\{1\}}[t_0^{-1}] \longrightarrow Y^{\{0,1\}}$$

is a cofibration in $(G \wedge M^{\{0,1\}})$-$k\,Top_*$.—This model structure is used implicitly for the category $\mathbb{P}_f(G)'$ in [KVWW, proof of 3.3 (1)].

Proposition 2.4.4. *(The f-structure of $\mathbf{pP}^N(G)$)*

The category $\mathbf{pP}^N(G)$ has the structure of a model category where a map is a weak equivalence (resp. cofibration) if each of its components is a weak equivalence (resp. cofibration) in its respective category. If G is cofibrant as an object of $k\,Top_*$, the category $\mathbf{pP}^N(G)$ is a proper model category.

Proof. Consider $\langle N\rangle_0$ as an inverse poset with degree function $d(A) := -\#A$ and apply theorem A.3.13 (1, 4). (For properness, cf. 1.2.1 (4).) $\square$

Example 2.4.5. *(The f-structure of $\mathbf{pP}^1(G)$)*

Let $f : Y \longrightarrow Z$ denote a map in $\mathbf{pP}^1(G)$. Then f is a weak equivalence if and only if its components f^A are weak equivalences in $(G\wedge M^A)$-$k\,Top_*$ for all non-empty $A \subseteq \langle 1\rangle$. The map f is a cofibration if and only if f^A is a cofibration in $(G\wedge M^A)$-$k\,Top_*$ for all non-empty $A \subseteq \langle 1\rangle$.

Finally, f is a fibration if and only if $f^{\{0,1\}}$ is a fibration in $(G\wedge M^{\{0,1\}})$-$k\,Top_*$, the induced map

$$Y^{\{0\}} \longrightarrow Z^{\{0\}} \times_{Z^{\{0,1\}}} Y^{\{0,1\}}$$

is a fibration in $(G\wedge M^{\{0\}})$-$k\,Top_*$ (where $Y^{\{0,1\}}$ and $Z^{\{0,1\}}$ are considered as $(G\wedge M^{\{0\}})$-equivariant spaces with the restricted action), and if the map

$$Y^{\{1\}} \longrightarrow Z^{\{1\}} \times_{Z^{\{0,1\}}} Y^{\{0,1\}}$$

is a fibration in $(G\wedge M^{\{1\}})$-$k\,Top_*$. We will not need this explicit description of fibrations.

Both model structures share the same weak equivalences, called *h-equivalences*, and hence have the same homotopy category $Ho\,\mathbf{pP}^N(G)$ ([DS], theorem 6.2).

We will need yet another notion of cofibrations which belongs to one of the model structures of the category G-$k\,Top_*^{\langle N\rangle}$. (Again, these cofibrations will give the same K-theory as the other two notions.)

Definition 2.4.6. A map $f : Y \longrightarrow Z$ in $\mathbf{pP}^N(G)$ is a *weak cofibration* if it is an f-cofibration when considered as a map in G-$k\,Top_*^{\langle N\rangle}$, *i.e.*, if all its components are cofibrations in G-$k\,Top_*$.

By theorem A.3.8 (2), any c-cofibration is an f-cofibration, and any f-cofibration is a weak cofibration since forgetting the M^A-actions preserves cofibrations by 1.2.1 (2).

Definition 2.4.7. The presheaf Y is called *strongly cofibrant* if it is cofibrant with respect to the c-structure. Explicitly Y is strongly cofibrant if and only if the map $L_AY \longrightarrow Y^A$ is a cofibration in $G\wedge M^A$-$k\,Top_*$ for all non-empty $A \subseteq N$. An object $Y \in \mathbf{pP}^N(G)$ is said to be *locally cofibrant* if it is cofibrant with respect to the f-structure. Explicitly Y is locally cofibrant if and only if $Y^A \in \mathbf{C}(G\wedge M^A)$ for all $A \subseteq N$. Finally Y is called *weakly cofibrant* if it is f-cofibrant as on object of G-$k\,Top_*^{\langle N\rangle}$, *i.e.*, if all its components are cofibrant in G-$k\,Top_*$.

Any strongly cofibrant presheaf is locally cofibrant, and a locally cofibrant presheaf is weakly cofibrant.

Lemma 2.4.8.

(1) If Y is locally cofibrant, the object $Y \wedge I$ is a good cylinder object in the sense of [DS, 4.2] for the f-structure; i.e., the canonical map $Y \vee Y \longrightarrow Y \wedge I$ is an f-cofibration.

(2) The mapping cylinder construction is a cylinder functor in the sense of [W] on the subcategory of locally cofibrant objects. The cylinder axiom is satisfied.

(3) If Y is strongly cofibrant, the object $Y \wedge I$ is a good cylinder object in the sense of [DS, 4.2] for the c-structure; i.e., the canonical map $Y \vee Y \longrightarrow Y \wedge I$ is a c-cofibration.

(4) The mapping cylinder construction is a cylinder functor in the sense of [W] on the subcategory of strongly cofibrant objects. The cylinder axiom is satisfied.

Proof. *Ad 1.* The map $Y \vee Y \longrightarrow Y \wedge I$ is an f-cofibration if and only if for all non-empty $A \subseteq N$ its A-component $Y^A \vee Y^A \longrightarrow Y^A \wedge I$ is a cofibration in $(G \wedge M^A)$-$k\,Top_*$. But this is true since $Y^A \in \mathbf{C}(G \wedge M^A)$, and $Y^A \wedge I$ is a good cylinder object for Y^A.

Ad 2. We have to show the following: if

$$\begin{array}{ccc} P & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Q & \longrightarrow & Z \end{array}$$

is a diagram of locally cofibrant objects in $\mathbf{pP}^N(G)$, and if both horizontal maps are f-cofibrations, then in the induced diagram

$$\begin{array}{ccc} P \vee Q & \longrightarrow & Y \vee Z \\ \downarrow & & \downarrow \\ Z_f & \longrightarrow & Z_g \end{array}$$

both vertical maps are f-cofibrations, the upper horizontal map is an f-cofibration, and the map from the pushout into the terminal vertex is an f-cofibration. (Here $Z_f := (P \wedge I) \cup_{P \vee P} (P \vee Q)$ denotes the mapping cylinder of f, and similarly for Z_g.)

The left vertical map is a cobase change of the f-cofibration $P \vee P \longrightarrow P \wedge I$ (cf. (1)), hence is an f-cofibration. A similar argument holds for the right vertical map. The upper horizontal map can be factored as $P \vee Q \longrightarrow Y \vee Q \longrightarrow Y \vee Z$. Both maps are cobase changes of f-cofibrations, hence their composition is an f-cofibration.

Finally, the map $Z_f \cup_{P \vee Q} (Y \vee Z) \longrightarrow Z_g$ is an f-cofibration if and only if its A-components are cofibrations in $(G \wedge M^A)$-$k\,Top_*$ for all non-empty subsets $A \subseteq N$. But the mapping cylinder construction is applied pointwise, *i.e.*, $(M_f)^A = M_{(f^A)}$. Hence the A-component is given by the map $Z_{f^A} \cup_{P^A \vee Q^A} (Y^A \vee Z^A) \longrightarrow Z_{g^A}$ which is known to be a cofibration since the mapping cylinder construction is a cylinder functor for the category $\mathbf{C}(G \wedge M^A)$ (1.2.3 (1)).

Ad 3. We have to check that for all non-empty subsets $A \subseteq N$, the map from the pushout to the terminal vertex in the follwing diagram

$$\begin{tikzcd} L_A(Y \vee Y) \arrow[r] \arrow[d] & L_A(Y \wedge I) \arrow[d] \\ (Y \vee Y)^A \arrow[r] & (Y \wedge I)^A \end{tikzcd}$$

is a cofibration of $(G \wedge M^A)$-spaces. But the functor L_A commutes with colimits and with $\cdot \wedge I$ since it is defined using colimits and smash products. Hence the above diagram is isomorphic to the follwing square:

$$\begin{tikzcd} L_A(Y) \vee L_A(Y) \arrow[r] \arrow[d] & L_A(Y) \wedge I \arrow[d] \\ Y^A \vee Y^A \arrow[r] & Y^A \wedge I \end{tikzcd} \tag{$*$}$$

This square can be obtained by applying the mapping cylinder construction to the horizontal maps of the follwing diagram:

$$\begin{tikzcd} L_A Y \arrow[r, "\mathrm{id}"] \arrow[d] & L_A Y \arrow[d] \\ Y^A \arrow[r, "\mathrm{id}"'] & Y^A \end{tikzcd}$$

The vertical maps are cofibrations since Y is strongly cofibrant by hypothesis. Since the mapping cylinder construction is known to be a cylinder functor for $\mathbf{C}(G \wedge M^A)$, it follows that the induced map in $(*)$ from the pushout to the terminal vertex is a cofibration.

Ad 4. We have to show the following: if

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Q \arrow[r] & Z \end{tikzcd}$$

is a diagram of strongly cofibrant objects in $\mathbf{pP}^N(G)$, and if both horizontal maps are c-cofibrations, then in the induced diagram

$$\begin{tikzcd} P \vee Q \arrow[r] \arrow[d] & Y \vee Z \arrow[d] \\ Z_f \arrow[r] & Z_g \end{tikzcd}$$

both vertical maps are c-cofibrations, the upper horizontal map is a c-cofibration, and the map from the pushout into the terminal vertex is a c-cofibration. (Here $Z_f := (P \wedge I) \cup_{P \vee P} (P \vee Q)$ denotes the mapping cylinder of f, and similarly for Z_g.)

The left vertical map is a cobase change of the c-cofibration $P \vee P \longrightarrow P \wedge I$ (cf. (3)), hence is a c-cofibration. A similar argument holds for the right vertical map. The upper horizontal map can be factored as $P \vee Q \longrightarrow Y \vee Q \longrightarrow Y \vee Z$. Both maps are cobase changes of c-cofibrations, hence their composition is a c-cofibration.

Finally, the map $Z_f \cup_{P \vee Q} (Y \vee Z) \longrightarrow Z_g$ is a c-cofibration if and only if for all non-empty subsets $A \subseteq N$, the induced map from the pushout T into the terminal vertex of the diagram

$$\begin{tikzcd}
L_A\big(Z_f \cup_{P \vee Q} (Y \vee Z)\big) \arrow[r] \arrow[d] & L_A(Z_g) \arrow[d] \\
\big(Z_f \cup_{P \vee Q} (Y \vee Z)\big)^A \arrow[r] & (Z_g)^A
\end{tikzcd}$$

is a cofibration of $(G \wedge M^A)$-spaces. We want to write this map $T \longrightarrow (Z_g)^A$ in a different way. First note that L_A commutes with colimits since it is defined using colimits and left adjoints. Consider the following cubical diagram:

$$\begin{tikzcd}
 & P^A \vee Q^A \arrow[rr] \arrow[dd] & & Y^A \vee Z^A \arrow[dd] \\
L_A(P \vee Q) \arrow[ur] \arrow[rr] \arrow[dd] & & L_A(Y \vee Z) \arrow[ur] \arrow[dd] & \\
 & (Z_f)^A \arrow[rr] & & (Z_g)^A \\
L_A(Z_f) \arrow[ur] \arrow[rr] & & L_A(Z_g) \arrow[ur] &
\end{tikzcd}$$

The above square is formed by the pushouts of front and back face together with the terminal vertices of front and back face. But since colimits may be computed in any order, we obtain T by calculating the colimit of the above cube with terminal vertex deleted. Again since colimits may be computed in any order, we could just as well form pushouts on top and bottom face first. Thus the claim that $T \longrightarrow (Z_g)^A$ is a cofibration is equivalent to the claim that in the following square, the induced map from the pushout (which is T again) to the terminal vertex is a cofibration of $(G \wedge M^A)$-spaces:

$$\begin{tikzcd}
L_A(Y \vee Z) \cup_{L_A(P \vee Q)} (P^A \vee Q^A) \arrow[r] \arrow[d] & Y^A \vee Z^A \arrow[d] \\
L_A(Z_g) \cup_{L_A(Z_f)} (Z_f)^A \arrow[r] & (Z_g)^A
\end{tikzcd}$$

But L_A and the mapping cylinder construction commute (all are defined via colimits and smash products), and mapping cylinders commute with colimits. Hence this square is obtained from the following diagram by application of the mapping cylinder construction to the vertical maps:

$$\begin{tikzcd} L_A Y \cup_{L_A P} (P^A) \arrow[r] \arrow[d] & Y^A \arrow[d, "g^A"] \\ L_A Z \cup_{L_A Q} Q^A \arrow[r] & Z^A \end{tikzcd}$$

The horizontal maps are cofibrations since $P \longrightarrow Y$ and $Q \longrightarrow Z$ were assumed to be c-cofibrations. Since the mapping cylinder construction is a cylinder functor for $\mathbf{C}(G \wedge M^A)$, we have shown that $T \longrightarrow Z_{g^A}$ is a cofibration as required. $\square$

Let $Y \in \mathbf{pP}^N(G)$ be a presheaf, and fix $j \in N$. We can consider Y as an N-cubical diagram in G-$k\mathit{Top}_*$; then its j-th face is an $N \setminus \{j\}$-cubical diagram consisting of those components Y^B with $j \in B \subseteq N$. More generally, a subset $C \subseteq N$ determines an $N \setminus C$-cubical diagram formed by those components Y^B with $C \subseteq B$.

Definition 2.4.9. Let $Y \in \mathbf{pP}^N(G)$ and $C \subseteq A \subseteq N$ be given. The *restricted latching space* $L_A^{+C} Y$ is defined as

$$L_A^{+C} Y := \operatorname*{colim}_{\langle A \rangle^1_{0,C}} R(Y)$$

(colimit in $(G \wedge M^A)$-$k\mathit{Top}_*$) where $\langle A \rangle^1_{0,C}$ is the subcategory of non-empty proper subsets of A containing C, and $R(Y)$ is defined as in the case of (unrestricted) latching spaces L_A, *i.e.*, $R(Y)$ is the diagram $B \mapsto Y^B[t_A^{-1}]$ (this is a diagram in $(G \wedge M^A)$-$k\mathit{Top}_*$) with structure maps as defined earlier.

In effect, the space $L_A^{+C} Y$ is the latching space at $A \setminus C$ of the restricted $(N \setminus C)$-cubical (twisted) diagram determined by C (given by $B \mapsto Y^{B \cup C}$).

The following lemma is a "twisted" version of the familiar fact that if a cube is "cofibrant as a cube", the same is true for all its faces.

Lemma 2.4.10. *Let $Y \in \mathbf{pP}^N(G)$, $C \subseteq A \subseteq N$ and $j \in A \setminus C$ be given.*

(1) The restricted latching space comes equipped with a map $L_A^{+C} Y \longrightarrow Y^A$.

(2) For $D \subseteq C$, there is a canonical map $L_A^{+C} Y \longrightarrow L_A^{+D} Y$, and the composite $L_A^{+C} Y \longrightarrow L_A^{+D} Y \longrightarrow Y^A$ is the map of (1). In particular, there is a canonical map $L_A^{+C} Y \longrightarrow L_A Y$, and the composite $L_A^{+C} Y \longrightarrow L_A Y \longrightarrow Y^A$ is the map of (1).

(3) Let σ denote the inclusion $A \setminus \{j\} \longrightarrow A$. The following square is a pushout:

$$\begin{tikzcd} F_\sigma(L^{+C}_{A \setminus \{j\}} Y) \arrow[r] \arrow[d] & F_\sigma(Y^{A \setminus \{j\}}) \arrow[d] \\ L_A^{+C \amalg \{j\}} Y \arrow[r] & L_A^{+C} Y \end{tikzcd}$$

(4) If Y is strongly cofibrant, the natural map $L_A^{+\{j\}} Y \longrightarrow Y^A$ of (1) is a cofibration.

Proof. *Ad (1).* The objects occuring in the definition of the restricted latching space are of the form $Y^C[t_A^{-1}]$. The $\sharp$-type structure maps $Y^C[t_A^{-1}] \longrightarrow Y^A$ are compatible with the structure maps of $R(Y)$. Hence we have an induced map $L_A^{+j}Y \longrightarrow Y^A$.

Ad (2). The diagram used for defining $L_A^{+C}Y$ includes into the diagram used for defining $L_A^{+D}Y$, and the map from the former to Y^A is the restriction of the map from the latter to Y^A. Moreover $L_A^{+\emptyset}Y = L_A Y$. Hence (2) holds.

Ad (3). Since F_σ is a left adjoint, it commutes with colimits. By explicitly spelling out the definitions, one realizes that the pushout in the above square and $L_A^{+C}Y$ really are colimits of the same diagram, hence are isomorphic.

Ad (4). By (2), we have a factorization $L_A^{+j}Y \longrightarrow L_A Y \longrightarrow Y^A$. The second map is a cofibration since Y is strongly cofibrant. Part (3) asserts that the first map is a cobase change of the image of $L_{A\setminus\{j\}}Z \longrightarrow Z^{A\setminus\{j\}}$ under F_σ. But this is a cofibration since Y is strongly cofibrant, and F_σ preserves cofibrations by definition of adjunction bundles. □

2.5. Restriction and Extension by Zero

In algebraic geometry the scheme "projective n-space over the commutative ring R" can be defined by applying the functor "Proj" to the ring $R[X_0,\ X_1,\ \ldots,\ X_n]$. The $n+1$ projections

$$R[X_0,\ X_1,\ \ldots,\ X_n] \longrightarrow R[X_0,\ X_1,\ \ldots,\ X_n]/\langle X_j\rangle \cong R[Y_0,\ Y_1,\ \ldots,\ Y_{n-1}]$$

induce $n+1$ different closed immersions of projective $(n-1)$-space into projective n-space. Restriction along these immersions induces $n+1$ different functors mapping quasi-coherent sheaves on $\mathbf{P}_R^n$ to quasi-coherent sheaves on $\mathbf{P}_R^{n-1}$.

An analogous construction can be made in the non-linear context.

Definition 2.5.1. Given $j \in N$, we define the j-th restriction functor

$$\rho_j = \rho_j^N : \mathbf{pP}^N(G) \longrightarrow \mathbf{pP}^{N\setminus\{j\}}(G)$$

by the equation $\rho_j(Y)^A := Y^A \wedge_{M_N^A} M_{N\setminus\{j\}}^A$ using the canonical map $M_N^A \longrightarrow M_{N\setminus\{j\}}^A$ of example 2.2.5 (whose effect is to get rid of the generator t_j by dividing it out).

Example 2.5.2. There is a natural isomorphism $\rho_j^N(\mathcal{O}_{\mathbf{P}^N}) \cong \mathcal{O}_{\mathbf{P}^{N\setminus\{j\}}}$ since

$$\rho_j^N(\mathcal{O}_{\mathbf{P}^N})^A = M_N^A \underset{M_N^A}{\wedge} M_{N\setminus\{j\}}^A \cong M_{N\setminus\{j\}}^A$$

for all non-empty $A \subseteq N \setminus \{j\}$.

If we represent an object $Y \in \mathbf{pP}^n(G)$ by a diagram having the shape of an n-simplex, the restricted sheaf $\rho_j(Y)$ is given by the j-th $(n-1)$-dimensional face of the diagram after dividing out the action of t_j.

Lemma 2.5.3. *The functor ρ_j^N has a right adjoint $\zeta_j^N : \mathbf{pP}^{N\setminus\{j\}}(G) \longrightarrow \mathbf{pP}^N(G)$ called "extension by zero". It is given by*

$$\zeta(Y)^A = \begin{cases} \bar{Y}^A , & \text{if } j \notin A \\ * , & \text{otherwise} \end{cases}$$

where $\bar{Y}^A$ is Y^A as a space with M_N^A acting via the canonical map $M_N^A \longrightarrow M_{N\setminus\{j\}}^A$ (cf. 2.2.5), i.e., with t_j acting trivially.

Proof. Given $Y \in \mathbf{pP}^N(G)$ and $Z \in \mathbf{pP}^{N\setminus\{j\}}(G)$, we have to show that there is a natural bijection

$$\hom_{\mathbf{pP}^{N\setminus\{j\}}(G)}(\rho_j(Y), Z) \cong \hom_{\mathbf{pP}^N(G)}(Y, \zeta_j(Z)) \ .$$

Let $f : \rho_j(Y) \longrightarrow Z$ be an element in the set on the left. Fix $A \subseteq N$. We want to define a map $g^A : Y^A \longrightarrow \zeta_j(Z)^A$. If $j \in A$, we have $\zeta_j(Z)^A = *$; let g^A denote the unique map into the point in this case. If $j \notin A$, we have a map

$$f^A : \rho_j(Y)^A = Y^A \underset{M_N^A}{\wedge} M_{N\setminus\{j\}}^A \longrightarrow Z^A$$

which, by adjointness, corresponds to a map $g^A : Y^A \longrightarrow \zeta_j(Z)^A$ (apply 1.2.2 (1) to the monoid homomorphism $M_N^A \longrightarrow M_{N\setminus\{j\}}^A$).—By naturality, the g^A assemble to a morphism $g : Y \longrightarrow \zeta_j(Z)$.

Conversely, given $g : Y \longrightarrow \zeta_j(Y)$, we define a morphism $f : \rho_j(Y) \longrightarrow Z$ by the requirement that f^A is adjoint to the map g^A (for $A \subseteq N \setminus \{j\}$).

It is clear that these constructions are mutually inverse because on each component they are induced by the adjunction 1.2.2 (1). □

For an object $Y \in \mathbf{P}^1$, the sheaf $\zeta_2(Y)$ is described by the following diagram (together with the convention that the additional "indeterminate" t_2 acts trivially on all spaces):

$$\begin{tikzcd}
 & & * \arrow[dl] \arrow[dr] & & \\
 & * \arrow[dr] & & * \arrow[dl] & \\
 & & * & & \\
Y^{\{0\}} \arrow[uur] \arrow[rr] & & Y^{\{0,1\}} \arrow[u] & & Y^{\{1\}} \arrow[uul] \arrow[ll]
\end{tikzcd}$$

Lemma 2.5.4. *The restriction functors ρ_j^N preserve all (acyclic) f-cofibrations and in addition weak equivalences of locally cofibrant objects. Moreover, they map locally cofibrant objects of $\mathbf{P}^N(G)$ to objects of $\mathbf{P}^{N\setminus\{j\}}(G)$.*

Proof. Let $f: Y \longrightarrow Z$ denote an f-cofibration. The map $\rho_j(f)$ is an f-cofibration if and only if $\rho_j(f)^A$ is a cofibration in $(G\wedge M^A_{N\setminus\{j\}})$-$k\,Top_*$ for all non-empty $A \subseteq N\setminus\{j\}$. But $\rho_j(f)^A$ is the image of f^A under the functor $\cdot \wedge_{M^A_N} M^A_{N\setminus\{j\}}$ which is known to preserve cofibrations (lemma 1.2.2 (2)). By two more applications of 1.2.2 (2) to the components of f we see that ρ_j also preserves acyclic cofibrations and weak equivalences between locally cofibrant objects.

Now assume Y is a locally cofibrant sheaf. Let $\sigma: A \longrightarrow B$ denote an inclusion of non-empty subsets of $N \setminus \{j\}$. Since $\rho_j(Y)$ is locally cofibrant by the above, corollary 2.3.4 asserts that it suffices to show that the map

$$\rho_j^N(Y)^\sharp_\sigma : \rho_j^N(Y)^A[t_B^{-1}] = \rho_j^N(Y)^A \underset{M^A_{N\setminus\{j\}}}{\wedge} M^B_{N\setminus\{j\}} \longrightarrow \rho_j^N(Y)^B \qquad (*)$$

is a weak homotopy equivalence. By definition of ρ_j^N the source of this map is given by $Y^A \wedge_{M^A_N} M^A_{N\setminus\{j\}} \wedge_{M^A_{N\setminus\{j\}}} M^B_{N\setminus\{j\}}$. There is a chain of canonical isomorphisms

$$Y^A \underset{M^A_N}{\wedge} M^A_{N\setminus\{j\}} \underset{M^A_{N\setminus\{j\}}}{\wedge} M^B_{N\setminus\{j\}} \cong Y^A \underset{M^A_N}{\wedge} M^B_{N\setminus\{j\}} \cong Y^A \underset{M^A_N}{\wedge} M^B_N \underset{M^B_N}{\wedge} M^B_{N\setminus\{j\}}$$

where the last space is $Y^A[t_B^{-1}] \wedge_{M^B_N} M^B_{N\setminus\{j\}}$. By definition of ρ_j, the target of $(*)$ is the space $Y^B \wedge_{M^B_N} M^B_{N\setminus\{j\}}$, and the map $(*)$ is obtained from the structure map

$$Y^\sharp_\sigma : Y^A[t_B^{-1}] \longrightarrow Y^B$$

by applying $\cdot \wedge_{M^B_N} M^B_{N\setminus\{j\}}$. But this functor is known to preserve weak equivalences between cofibrant spaces by 1.2.2 (2), and $Y^\sharp_\sigma$ is a weak equivalence since Y is a sheaf. $\square$

2.6. Twists and Canonical Sheaves

The definition of the twist functor as given in [KVWW] is not symmetric. It turns out that for twists and related constructions, it is convenient to have both symmetric and asymmetric descriptions, the latter arising from a choice of a total order on the indexing set. (In this case, it is enough to restrict attention to the standard ordered sets $[n]$.)

Definition 2.6.1. *(Tensor product of sheaves.)*

Suppose Y is an object of $\mathbf{pP}^N(G)$ and Z is an object of $\mathbf{pP}^N$. The *(non-linear) tensor product* $Y \wedge_{\mathcal{O}} Z$ of Y and Z is the presheaf given by $A \mapsto Y^A \wedge_{M^A} Z^A$ with M^A acting from the right on Z^A. (This definition makes sense because M^A is abelian, hence acts from the left and from the right on Z^A.) The tensor product is an object of $\mathbf{pP}^N(G)$ (where G acts on the components of Y).

Definition 2.6.2. *(Symmetric description of twisting.)*

The *j-th* SERRE *twisting sheaf* $\mathcal{O}_{\mathbf{P}^N}(j) = \mathcal{O}(j) \in \mathbf{P}^N$ is given by

$$A \mapsto \widetilde{M}_N^A\big(j \cdot (1 - \#A)\big) \qquad (A \neq \emptyset)$$

and structure maps $\mathcal{O}(j)^A \xrightarrow{t_{B\setminus A}^{-j}} \mathcal{O}(j)^B$ for non-empty subsets $A \subseteq B \subseteq N$. For $Y \in \mathbf{pP}^N(G)$ and $j \in \mathbf{Z}$, the *j-th twist functor* $\theta_j = \theta_j^N : \mathbf{pP}^N(G) \longrightarrow \mathbf{pP}^N(G)$ is defined as $\theta_j(Y) := Y \wedge_{\mathcal{O}} \mathcal{O}(j)$.

Example 2.6.3. Here is a picture of $\mathcal{O}_{\mathbf{P}^1}(j)$:

$$(\mathbf{Z}\times\mathbf{N})_+^0 \xrightarrow{(0,-j)} (\mathbf{Z}\times\mathbf{Z})_+^{-j} \xleftarrow{(-j,0)} (\mathbf{N}\times\mathbf{Z})_+^0$$

The lower index "+" means adding a disjoint basepoint, and an upper index a denotes the subset of tuples with sum a. With the same convention, the following diagram represents $\mathcal{O}_{\mathbf{P}^2}(j)$:

$$\begin{tikzcd}
 & (\mathbf{N}\times\mathbf{N}\times\mathbf{Z})_+^0 \arrow[rr, "{(-j,0,0)}"] \arrow[dd, "{(0,-j,0)}"] & & (\mathbf{Z}\times\mathbf{N}\times\mathbf{Z})_+^{-j} \arrow[dd, "{(0,-j,0)}"] \\
* \arrow[ur] \arrow[rr] \arrow[dd] & & (\mathbf{Z}\times\mathbf{N}\times\mathbf{N})_+^0 \arrow[ur, "{(0,0,-j)}"] \arrow[dd, "{(0,-j,0)}"] & \\
 & (\mathbf{N}\times\mathbf{Z}\times\mathbf{Z})_+^{-j} \arrow[rr, "{(-j,0,0)}"] & & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z})_+^{-2j} \\
(\mathbf{N}\times\mathbf{Z}\times\mathbf{N})_+^0 \arrow[ur, "{(0,0,-j)}"] \arrow[rr, "{(-j,0,0)}"] & & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{N})_+^{-j} \arrow[ur, "{(0,0,-j)}"] &
\end{tikzcd}$$

(The 0-direction is left to right, 1-direction is top to bottom.)

Definition 2.6.4. *(Asymmetric description of twisting.)*

The *j-th twist functor* $\vartheta_j = \vartheta_j^n : \mathbf{pP}^n(G) \longrightarrow \mathbf{pP}^n(G)$ (for $j \in \mathbf{Z}$) assigns to an object $Y \in \mathbf{pP}^n(G)$ the object $Z := \vartheta_j(Y)$ in the following way: for all $A \subseteq [n]$, we define $Z^A := Y^A$, and for non-empty subsets $A \subseteq B$ of $[n]$ we define the structure map $Z^A \longrightarrow Z^B$ as the composite

$$Z^A = Y^A \longrightarrow Y^B \xrightarrow{t_m^j t_k^{-j}} Y^B = Z^B \qquad (*)$$

where $m := \max(A)$ and $k := \max(B)$. A morphism $f : Y \longrightarrow \bar{Y}$ in $\mathbf{pP}^n(G)$ (with components f^A) gives rise to a morphism of twisted objects with the unchanged components f^A.

In other words: a presheaf Y and its twisted version $\vartheta_j(Y)$ are *locally* the same, but the gluing is different. One may think of twisting as "tensoring with a line bundle".

Remark 2.6.5.

(1) We have $\vartheta_0^n = \mathrm{id}$.

(2) The twist functor ϑ_j restricts to an endofunctor of $\mathbf{P}^n(G)$ since $t_m^j t_k^{-j}$ is an invertible map and hence a weak homotopy equivalence.

To make the asymmetric definition of twisting less obscure, we include the diagrams for the projective line ($n = 1$) and the projective plane ($n = 2$). An object Y of $\mathbf{pP}^1(G)$ is a diagram $Y^{\{0\}} \longrightarrow Y^{\{01\}} \longleftarrow Y^{\{1\}}$, the twisted object $\vartheta_j^1(Y)$ can be pictured as $Y^{\{0\}} \xrightarrow{t_0^j t_1^{-j}} Y^{\{01\}} \longleftarrow Y^{\{1\}}$. (Here we abbreviated the above composition $(*)$ to a single map.) In the second case, $Y \in \mathbf{pP}^2(G)$, the twisted object is given by the following diagram:

$$\begin{tikzcd}
 & & Y^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & Y^{\{0,2\}} \arrow[dr] & & Y^{\{1,2\}} \arrow[dl] & \\
 & & Y^{\{0,1,2\}} & & \\
Y^{\{0\}} \arrow[uur, "t_0^j t_2^{-j}"] \arrow[rr, "t_0^j t_1^{-j}"'] & & Y^{\{0,1\}} \arrow[u, "t_1^j t_2^{-j}"] & & Y^{\{1\}} \arrow[ll] \arrow[uul, "t_1^j t_2^{-j}"']
\end{tikzcd}$$

There are other (isomorphic) descriptions of the twist maps, which can be obtained by using a different order on the indexing set. In the case of the projective line, both of the objects

$$Y^{\{0\}} \xrightarrow{t_0^j t_1^{-j}} Y^{\{01\}} \longleftarrow Y^{\{1\}} \quad \text{and} \quad Y^{\{0\}} \longrightarrow Y^{\{01\}} \xleftarrow{t_0^{-j} t_1^j} Y^{\{1\}}$$

describe the j-th twist of Y. The left diagram corresponds to the natural order $\{0 < 1\}$, the right one to the reversed order $\{1 < 0\}$. An isomorphism between these two objects is given by the following diagram:

$$\begin{tikzcd}
Y^{\{0\}} \arrow[r, "t_0^j t_1^{-j}"] \arrow[d, "\mathrm{id}"'] & Y^{\{01\}} \arrow[d, "t_0^{-j} t_1^j"'] & Y^{\{1\}} \arrow[l] \arrow[d, "\mathrm{id}"] \\
Y^{\{0\}} \arrow[r] & Y^{\{01\}} & Y^{\{1\}} \arrow[l, "t_0^{-j} t_1^j"']
\end{tikzcd}$$

Lemma 2.6.6. *(Properties of the twist functor.)*

(1) We have $\mathcal{O}(i) \wedge_{\mathcal{O}} \mathcal{O}(j) \cong \mathcal{O}(i+j)$ and $\mathcal{O} = \mathcal{O}(0)$.

(2) There are natural isomorphisms $\theta_{j+k} \cong \theta_j \circ \theta_k$ and $\theta_0 \cong \mathrm{id}$. In particular, θ_k is an equivalence of categories. Similarly, $\vartheta_{j+k} \cong \vartheta_j \circ \vartheta_k$ and $\vartheta_0 = \mathrm{id}$.

(3) For all $Y \in \mathbf{pP}^n(G)$, the presheaves $\theta^n_j(Y)$ and $\vartheta^n_j(Y)$ are naturally isomorphic.

(4) If $Y \in \mathbf{P}^N(G)$, then $\theta_j(Y) \in \mathbf{P}^N(G)$.

Proof. Claim (1): the equality $\mathcal{O} = \mathcal{O}(0)$ is a restatement of the definition of the structure sheaf. To prove $\mathcal{O}(i) \wedge_{\mathcal{O}} \mathcal{O}(j) \cong \mathcal{O}(i+j)$ we have to construct a natural bijection (natural in A)

$$\beta\colon \big(\mathcal{O}(i) \underset{\mathcal{O}}{\wedge} \mathcal{O}(j)\big)^A = \widetilde{M}^A(i) \underset{M^A}{\wedge} \widetilde{M}^A(j) \longrightarrow \widetilde{M}^A(i+j) = \big(\mathcal{O}(i+j)\big)^A$$

for all non-empty $A \subseteq N$.

For this, we need an explicit description of $\widetilde{M}^A(i) \wedge_{M^A} \widetilde{M}^A(j)$. Elements are equivalence classes of pairs $(a,b) \in \widetilde{M}^A(i) \times \widetilde{M}^A(j)$ subject to the relations

$$(a,*) \sim * \sim (*,b) \qquad \text{and} \qquad (a+x,b) \sim (a,x+b)$$

for all $x \in M^A$.

Choose $k \in A$. Given $a = (a_\nu)_{\nu\in N} \in \widetilde{M}^A(i)$ and $b = (b_\nu)_{\nu\in N} \in \widetilde{M}^A(j)$, we claim that the element represented by $a \wedge b$ has a "special" representative $\bar{a} \wedge \bar{b}$ where $\bar{b}$ has j at the k-th place and zero everywhere else, *i.e.*, $\bar{b} = jt_k$. To see this, define $x := b - jt_k \in M^A$ (this element has degree 0 since b has degree j and t_k has degree 1); then by the description given before, $a \wedge b \sim (a+x) \wedge (b-x) = (a+x) \wedge (jt_k)$.

Now we define $\beta(a \wedge b) := a + b$ (componentwise sum of integers), and let

$$\alpha\colon \widetilde{M}^A(i+j) \longrightarrow \widetilde{M}^A(i) \underset{M^A}{\wedge} \widetilde{M}^A(j)$$

denote the map $y \mapsto (y - jt_k) \wedge (jt_k)$. Then α and β are mutually inverse on the level of "special" representatives in the sense above. This proves (1).

Statement (2) follows from (1): for $Y \in \mathbf{pP}^N(G)$, we have natural isomorphisms

$$\theta_j \circ \theta_k(Y) = \big(Y \underset{\mathcal{O}}{\wedge} \mathcal{O}(k)\big) \underset{\mathcal{O}}{\wedge} \mathcal{O}(j) \cong Y \underset{\mathcal{O}}{\wedge} \big(\mathcal{O}(k) \underset{\mathcal{O}}{\wedge} \mathcal{O}(j)\big) \overset{(1)}{\cong} Y \underset{\mathcal{O}}{\wedge} \mathcal{O}(j+k) = \theta_{j+k}(Y)$$

and $\theta_0(Y) = Y \wedge_{\mathcal{O}} \mathcal{O}(0) \cong Y$. The corresponding statements for ϑ_j are a direct consequence of its definition and the rule $(t^j_m t^{-j}_i)(t^k_m t^{-k}_i) = t^{j+k}_m t^{-j-k}_i$.

To prove (3) we show how to construct a natural isomorphism $\theta_1(Y) \longrightarrow \vartheta_1(Y)$. Then for $k > 0$ we infer from (2) that there is a natural chain of isomorphisms

$$\theta_k \cong \underbrace{\theta_1 \circ \ldots \circ \theta_1}_{k \text{ times}} \cong \underbrace{\vartheta_1 \circ \ldots \circ \vartheta_1}_{k \text{ times}} \cong \vartheta_k \ ,$$

and similarly for $k < 0$.

For $A \subseteq [n]$ with $\emptyset \neq A$, write $m := \max(A)$. Consider the map

$$\theta_1(Y)^A = Y^A \underset{M^A}{\wedge} \widetilde{M}^A(1-\#A) \xrightarrow{\mathrm{id}\wedge(t_A t_m^{-1})} Y^A \underset{M^A}{\wedge} M^A \cong Y^A = \vartheta_1(Y)^A \ .$$

Since t_i acts invertibly on Y^A for $i \in A$, this map is invertible. It remains to show that these maps assemble, for different A, to a natural transformation of diagrams $\theta_1(Y) \longrightarrow \vartheta_1(Y)$. Explicitly, we have to show that for all non-empty $A \subseteq B \subseteq [n]$ the following diagram commutes:

$$\begin{tikzcd}
Y^A \underset{M^A}{\wedge} \widetilde{M}^A(1-\#A) \arrow[r, "t_A t_m^{-1}"] \arrow[d, "t_{B\setminus A}^{-1}"'] & Y^A \arrow[d, "t_m t_b^{-1}"] \\
Y^B \underset{M^B}{\wedge} \widetilde{M}^B(1-\#B) \arrow[r, "t_B t_b^{-1}"'] & Y^B
\end{tikzcd}$$

(Here $m := \max(A)$, $b := \max(B)$, the left vertical map is the structure map in $\theta_1(Y)$, and the right vertical map is the corresponding structure map in $\vartheta_1(Y)$.) But this is obvious since $t_B t_{B\setminus A}^{-1} = t_A$.

In a similar way one shows that the inverse maps form a natural transformation. Thus (3) holds.

To prove (4), assume that Y is a sheaf. Then $\vartheta_k(Y)$ is a sheaf (2.6.5 (2)), and since $\theta_k(Y) \cong \vartheta_k(Y)$ by what we have just shown, we know that $\theta_k(Y)$ is a sheaf. $\square$

There is a functor from the category of **Z**-graded modules over $R[T_0, T_1, \ldots, T_n]$ to the category of quasi-coherent sheaves on projective n-space, mapping the module M to the sheaf $\widetilde{M}$ (cf. [Ha], II.5, p. 116). On the level of modules, twisting amounts to shifting the degree. Formally, for a graded module $M = (M_i)_{i\in\mathbf{Z}}$ we define the module $M(1)$ by $M(1)_i := M_{i+1}$, and it can be shown that $\widetilde{M(1)}$ is naturally isomorphic to the twisted sheaf $\widetilde{M}(1)$. In particular, the $n+1$ obvious maps $M \longrightarrow M(1)$ given by multiplication with T_i (for $i \in \langle n\rangle$) induce natural maps of sheaves $\widetilde{M} \longrightarrow \widetilde{M}(1)$. We have an analogous set of maps in the non-linear context:

Definition 2.6.7. Let $i \in N$ be given. The natural transformation

$$\kappa_i = \kappa_i^N : \theta_k^N \longrightarrow \theta_{k+1}^N$$

is given by morphisms $\theta_k(Y) \longrightarrow \theta_{k+1}(Y)$ for each object $Y \in \mathbf{pP}^N(G)$ described by

$$\theta_k(Y)^A \xrightarrow{t_A^{-1}t_i} \theta_{k+1}(Y)^A$$

for $\emptyset \neq A \subseteq N$.

Definition 2.6.8. The natural transformation

$$\bar{\kappa}_i = \bar{\kappa}_i^n : \vartheta_k^n \longrightarrow \vartheta_{k+1}^n$$

is given by morphisms $\vartheta_k(Y) \longrightarrow \vartheta_{k+1}(Y)$ for each object $Y \in \mathbf{pP}^n(G)$ described by

$$\vartheta_k(Y)^A = Y^A \xrightarrow{t_i t_m^{-1}} Y^A = \vartheta_{k+1}(Y)^A$$

with $m := \max(A)$ (where $\emptyset \neq A \subseteq [n]$).

The transformations κ_j and $\bar{\kappa}_j$ correspond under the isomorphism of θ_k and ϑ_k constructed above.

Definition 2.6.9. For $K \in G\text{-}k\mathit{Top}_*$ we define the *canonical sheaf associated to* K as the object $\psi_0(K) = \psi_0^N(K) \in \mathbf{P}^N(G)$ given by

$$K \wedge \mathcal{O}_{\mathbf{P}^N} : A \mapsto K \wedge M_N^A$$

($\emptyset \neq A \subseteq N$) with structure maps induced by inclusions of submonoids. The assignment $K \mapsto \psi_0(K)$ is functorial in K. For convenience, we introduce the *twisted canonical sheaf functor* $\psi_j = \psi_j^N := \theta_j^N \circ \psi_0^N$.

As a first example, we note that $\psi_0^N(S^0) \cong \mathcal{O}_{\mathbf{P}^N}$, the structure sheaf of projective N-space, and $\psi_j^N(S^0) \cong \mathcal{O}(j)$.

Recall that $\langle N \rangle_0$ is the set of non-empty subsets of N. Let $G\text{-}k\mathit{Top}_*^{\langle N \rangle_0}$ denote the functor category $\mathit{Func}(\langle N \rangle_0,\ G\text{-}k\mathit{Top}_*)$.

Lemma 2.6.10. *Let V denote the forgetful functor $\mathbf{pP}^N(G) \longrightarrow G\text{-}k\mathit{Top}_*^{\langle N \rangle_0}$. The functor $\Psi_{-k} : G\text{-}k\mathit{Top}_* \longrightarrow \mathbf{pP}^N(G)$ is left adjoint to the functor $\varprojlim \circ V \circ \theta_k$.*

Proof. For $k = 0$ this is a restatement of proposition A.3.11 where $P^N(G)$ has to be extended to an adjunction bundle over $\langle N \rangle$ in the obvious way: choose $\mathcal{C}_\emptyset = G\text{-}k\mathit{Top}_*$, the U's are given by forgetful functors $(G \wedge M^A)\text{-}k\mathit{Top}_* \longrightarrow G\text{-}k\mathit{Top}_*$. Then with the notation of A.3.10, $M^\emptyset = \varprojlim \circ V$ and $\beta_\emptyset = \psi_0$.

Since θ_k is an equivalence of categories with inverse given by θ_{-k} (2.6.6 (2)) the general case follows. □

The functor $V \circ \varprojlim$ (this is the case case $k = 0$ from the lemma) is the analogue of the algebraic geometers' global sections functor.

Lemma 2.6.11. *Suppose $Y \in \mathbf{pP}^N(G)$ is a locally cofibrant presheaf, and let $j \in N$ and $k \in \mathbf{Z}$ be given. Then $\kappa_j : \theta_k(Y) \longrightarrow \theta_{k+1}(Y)$ is a weak cofibration, its (strict) cofibre is isomorphic to $\zeta_j \circ \rho_j \circ \theta_{k+1}(Y)$, and the projection $\theta_{k+1}(Y) \longrightarrow \zeta_j \circ \rho_j \circ \theta_{k+1}(Y)$ is isomorphic to the $\theta_{k+1}(Y)$-component of the unit of the adjunction of ζ_j and ρ_j.*

Proof. We begin by showing that κ_j is a weak cofibration. By definition, we have to prove that for all non-empty $A \subseteq N$ (with a elements) its A-component

$$\kappa_j^A = \mathrm{id} \wedge (t_A^{-1} t_j) : Y^A \underset{M^A}{\wedge} \widetilde{M}^A\big(k(1-a)\big) \longrightarrow Y^A \underset{M^A}{\wedge} \widetilde{M}^A\big((k+1)(1-a)\big)$$

is a cofibration in $G\text{-}k\,Top_*$. Choose an element $e \in A$. We can factor the map

$$\widetilde{M}^A\big(k(1-a)\big) \xrightarrow{t_A^{-1} t_j} \widetilde{M}^A\big((k+1)(1-a)\big)$$

in the following way:

$$\widetilde{M}^A\big(k(1-a)\big) \xrightarrow{t_A^k t_e^{-k}} M^A \xrightarrow{t_A^{-1} t_j t_e^{a-1}} M^A \xrightarrow{t_A^{-k} t_e^{1+k-a}} \widetilde{M}^A\big((k+1)(1-a)\big)$$

The first and third map are isomorphisms of discrete M^A-spaces, the map in the middle is injective. Application of the functor $Y^A \wedge_{M^A} \cdot$ yields a factorization of κ_j^A as a composite of three maps, two of them being isomorphisms (the first and the third), the second one being a cofibration in $G\text{-}k\,Top_*$ by lemma 1.2.4. Hence κ_j is a weak cofibration.

Since θ_k is an equivalence of categories (2.6.6 (2)) it commutes with restriction, extension by zero and taking cofibres. Thus it suffices to prove the remaining claims for $k = -1$ only.

Fix a non-empty subset $A \subseteq N$ with a elements, and consider the A-component

$$\kappa_j^A : Y^A \underset{M_N^A}{\wedge} \widetilde{M}_N^A(a-1) \longrightarrow Y^A \underset{M_N^A}{\wedge} M_N^A \cong Y^A$$

of κ_j. We claim that its cofibre is isomorphic to $\zeta_j \circ \rho_j(Y)^A = Y^A \wedge_{M_N^A} M_{N\setminus\{j\}}^A$.

The functor $Y^A \wedge_{M_N^A} \cdot : M_N^A\text{-}k\,Top_* \longrightarrow M_N^A\text{-}k\,Top_*$ commutes with taking cofibres since colimits commute among themselves (taking cofibres means formation of pushouts of the form $B \cup_A *$, and the tensor product is a certain coequalizer, hence the colimit of a functor defined on the category with two objects and two parallel distinct non-identity morphisms). Thus it suffices to show that the cofibre of the map $\widetilde{M}_N^A(1-a) \xrightarrow{t_A^{-1} t_j} M_N^A$ is isomorphic to $M_{N\setminus\{j\}}^A$. But this is clear since its image is precisely the ideal $I_{j,N}^A$ and $M_N^A / I_{j,N}^A \cong M_{N\setminus\{j\}}^A$ by 2.2.5.

It is only a matter of unravelling the definitions to see that the natural isomorphisms given above assemble to a natural isomorphism of presheaves between the cofibre of κ_j and $\zeta_j \circ \rho_j \circ \theta_j(Y)$. □

2.7. Global Sections

We have seen above (2.6.10) that there is a non-linear analogue of global sections of sheaves. However, it turns out that this functor is not suitable for K-theory calculations. Hence we introduce a functor Γ which (philosophically speaking) captures global sections and higher sheaf cohomology at the same time. As explained in [KVWW], the idea behind the definition of Γ is similar to the idea of global sections.

Definition 2.7.1. *(Global sections of sheaves.)*
For any object $Y \in \mathbf{pP}^N(G)$ we define the global sections of Y, denoted

$$\Gamma(Y) = \Gamma^N(Y) \in G\text{-}k\,Top_* \ ,$$

as the iterated homotopy cofibre of Y in the sense of definition 1.3.1 where Y is considered as a functor $\langle N \rangle \longrightarrow G\text{-}k\,Top_*$ (with $\flat$-type structure maps).

For an object $Y \in \mathbf{pP}^1(G)$, the space $\Gamma(Y)$ is given by the mapping cone of the map $Y^{\{0\}} \vee Y^{\{1\}} \longrightarrow Y^{\{0,1\}}$ as explained in 1.3.3. An explicit model (as given in [KVWW]) for this is

$$\Gamma(Y) = CY^{\{0\}} \cup_{Y^{\{0\}}} Y^{\{0,1\}} \cup_{Y^{\{1\}}} CY^{\{1\}}$$

where $CK = K \wedge I/(K \times \{0\})$ denotes the (reduced) cone on the pointed space K.

Unfortunately, we do not have such a simple model for $\Gamma(Y)$ if $n > 1$. Still it is instructive to unravel the definition for the projective plane. For an object Y of $\mathbf{pP}^2(G)$, we proceed as follows: we have

$$\Gamma(Y) = \gamma \circ e_3 \circ e_2 \circ e_1 \circ e_0(Y)$$

with functors as defined in 1.3.1. Since $e_0 \cong \mathrm{id} \cong e_1$ (the second isomorphism comes from the fact that Y, considered as a cubical diagram, has a point as initial vertex, and the reduced mapping cylinder of a map $* \longrightarrow K$ is isomorphic to K), the objects Y, $e_0(Y)$ and $e_1(Y)$ are all represented by the following diagram:

$$\begin{tikzcd}
 & & Y^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & Y^{\{0,2\}} \arrow[dr] & & Y^{\{1,2\}} \arrow[dl] & \\
 & & Y^{\{0,1,2\}} & & \\
Y^{\{0\}} \arrow[uur] \arrow[rr] & & Y^{\{0,1\}} \arrow[u] & & Y^{\{1\}} \arrow[ll] \arrow[uul]
\end{tikzcd}$$

(This is just the picture of a "generic" object of $\mathbf{pP}^2(G)$.)

Application of e_2 replaces the space $Y^{\{0,1\}}$ by the mapping cylinder $Z^{\{0,1\}}$ of the map $Y^{\{0\}} \vee Y^{\{1\}} \longrightarrow Y^{\{0,1\}}$, and similarly for $Y^{\{0,2\}}$ and $Y^{\{1,2\}}$:

$$
\begin{tikzcd}
& & Y^{\{2\}} \arrow[dl] \arrow[dr] & & \\
& Cyl\left(\begin{smallmatrix} Y^{\{0\}}\vee Y^{\{2\}} \\ \to Y^{\{0,2\}} \end{smallmatrix}\right) \arrow[dr] & & Cyl\left(\begin{smallmatrix} Y^{\{1\}}\vee Y^{\{2\}} \\ \to Y^{\{1,2\}} \end{smallmatrix}\right) \arrow[dl] & \\
& & Y^{\{0,1,2\}} & & \\
Y^{\{0\}} \arrow[uur] \arrow[rr] & & Cyl\left(\begin{smallmatrix} Y^{\{0\}}\vee Y^{\{1\}} \\ \to Y^{\{0,1\}} \end{smallmatrix}\right) \arrow[u] & & Y^{\{1\}} \arrow[uul] \arrow[ll]
\end{tikzcd}
$$

If Y is weakly cofibrant, *i.e.*, all components of Y are cofibrant as G-spaces, we have achieved that the map $Y^{\{0\}} \vee Y^{\{1\}} \longrightarrow Z^{\{0,1\}}$ is a cofibration in G-$k\,Top_*$ (and similarly for the other two edges of the diagram). (The above diagram does no longer describe a presheaf since $Z^{\{0,1\}}$ has no $M^{\{0,1\}}$-action.)

The next step is applying e_3 which replaces the space $Y^{\{0,1,2\}}$ by the mapping cylinder $Z^{\{0,1,2\}}$ of the canonical map from the colimit L of the outer six spaces in the diagram to the middle (terminal) space. The upshot is that the map $L \longrightarrow Z^{\{0,1,2\}}$ becomes a cofibration in G-$k\,Top_*$ if Y is weakly cofibrant. In this case, the resulting diagram, considered as a cubical diagram in G-$k\,Top_*$ (with a point as initial vertex) is c-cofibrant, *i.e.*, "cofibrant as a cube" (1.3.4 (1)).

To obtain $\Gamma(Y)$, we finally have to apply γ, *i.e.*, we have to compute the (strict) cofibre of the map $L \longrightarrow Z^{\{0,1,2\}}$.

In what follows, we develop the elementary properties of the global sections functor. As a beginning, we note that suspension commutes with global sections, *i.e.*,

$$\Gamma(\Sigma Y) = \Gamma(S^1 \wedge Y) \cong S^1 \wedge \Gamma(Y) = \Sigma\Gamma(Y)$$

for all $Y \in \mathbf{P}^N(G)$ (this is a special case of 1.3.2 (2)).

Next we consider global sections of a sheaf Y and its extension $\zeta_j(Y)$. In algebraic geometry, extension by zero does not change the cohomology groups of a sheaf. The analogous statement in the present context says that Y and $\zeta_j(Y)$ have *stably* the same global sections:

Lemma 2.7.2. *For an object $Y \in \mathbf{pP}^{N\setminus\{j\}}(G)$, the spaces $\Sigma\Gamma^{N\setminus\{j\}}(Y)$ and $\Gamma^N \circ \zeta_j(Y)$ are naturally isomorphic.*

Proof. By definition $\zeta_j(Y)^A = *$ if $j \in A$. Computing the homotopy cofibre of $\zeta_j(Y)$ in j-direction (cf. 1.3.2 (4)) results in an $N\setminus\{j\}$-cube Z. Its A-component is the mapping cone of $Y^A \longrightarrow *$. But this is one possible model for the suspension of Y^A. Hence Z is isomorphic to $\Sigma(Y)$, and since the global sections functor commutes with suspension, we infer that $\Gamma \circ \zeta_j(Y) \cong \Gamma(Z) \cong \Gamma(\Sigma Y) \cong \Sigma\Gamma(Y)$. □

Now we want to compute global sections of twisted canonical sheaves. For $j \in \mathbf{Z}$ define an N-cube of spaces

$$\mathcal{W}_N(j) \colon \langle N \rangle \longrightarrow k\,Top_*, \quad A \mapsto \widetilde{M}_N^A\big(j \cdot (1 - \#A)\big)$$

where $\widetilde{M}_N^A(k)$ has the discrete topology. Structure maps are $\mathcal{W}_N^A \xrightarrow{t_{B\setminus A}^{-j}} \mathcal{W}_N^B$ for non-empty subsets $A \subseteq B \subseteq N$. As before, N denotes a non-empty finite set with $n+1$ elements.

Lemma 2.7.3. *If $j > -n-1$, the cube $\mathcal{W}_N(j)$ has contractible iterated homotopy cofibre, i.e., $\Gamma\big(\mathcal{W}_N(j)\big) \simeq *$.*

Proof. We give a proof for $N = [2]$ first. The following diagram represents the cube $\mathcal{W}_{[2]}(j)$:

$$\begin{tikzcd}
& (\mathbf{N}\times\mathbf{N}\times\mathbf{Z})^0_+ \arrow[rr, "{(-j,0,0)}"] \arrow[dd, "{(0,-j,0)}" near start] & & (\mathbf{Z}\times\mathbf{N}\times\mathbf{Z})^{-j}_+ \arrow[dd, "{(0,-j,0)}"] \\
(\mathbf{N}\times\mathbf{N}\times\mathbf{N})^j_+ \arrow[ur, "{(0,0,-j)}"] \arrow[rr, "{(-j,0,0)}" near end] \arrow[dd, "{(0,-j,0)}"'] & & (\mathbf{Z}\times\mathbf{N}\times\mathbf{N})^0_+ \arrow[ur, "{(0,0,-j)}"] \arrow[dd, "{(0,-j,0)}" near start] & \\
& (\mathbf{N}\times\mathbf{Z}\times\mathbf{Z})^{-j}_+ \arrow[rr, "{(-j,0,0)}"' near start] & & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z})^{-2j}_+ \\
(\mathbf{N}\times\mathbf{Z}\times\mathbf{N})^0_+ \arrow[ur, "{(0,0,-j)}"] \arrow[rr, "{(-j,0,0)}"'] & & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{N})^{-j}_+ \arrow[ur, "{(0,0,-j)}"'] &
\end{tikzcd}$$

(The lower index "+" means a disjoint basepoint, and an upper index a denotes the subset of tuples with sum a. The symbol $\mathbf{Z}_{<-j}$ denotes all integers less than $-j$.)

Now compute the homotopy cofibre in 2-direction (from front to back). All spaces are discrete and all maps are injective; hence the natural map from the homotopy cofibre into the strict cofibre is a weak equivalence. Thus we can replace "homotopy cofibre" by "strict cofibre" (since we are only interested in the weak homotopy type of the result) and we obtain the following [1]-cube:

$$\begin{tikzcd}
(\mathbf{N}\times\mathbf{N}\times\mathbf{Z}_{<-j})^0_+ \arrow[r, "{(-j,0,0)}"] \arrow[d, "{(0,-j,0)}"'] & (\mathbf{Z}\times\mathbf{N}\times\mathbf{Z}_{<-j})^{-j}_+ \arrow[d, "{(0,-j,0)}"] \\
(\mathbf{N}\times\mathbf{Z}\times\mathbf{Z}_{<-j})^{-j}_+ \arrow[r, "{(-j,0,0)}"] & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}_{<-j})^{-2j}_+
\end{tikzcd}$$

Now we proceed by taking homotopy cofibres in 1-direction (from top to bottom). As above, up to weak equivalence this is the same as strict cofibres which yields the following [0]-cube:

$$(\mathbf{N} \times \mathbf{Z}_{<-j} \times \mathbf{Z}_{<-j})_+^{-j} \xrightarrow{(-j,0,0)} (\mathbf{Z} \times \mathbf{Z}_{<-j} \times \mathbf{Z}_{<-j})_+^{-2j}$$

Finally, we compute the homotopy cofibre in 0-direction (the only remaining direction), which is weakly equivalent to the strict cofibre. This gives the $\emptyset$-cube (*i.e.*, the space) $(\mathbf{Z}_{<-j} \times \mathbf{Z}_{<-j} \times \mathbf{Z}_{<-j})_+^{-2j}$. But this space is a single point (for suppose (a, b, c) is a non-basepoint; then $-2j = a + b + c \leq -3(j+1)$ and consequently $j \leq -3$ which contradicts the assumption on j). This proves $\Gamma\big(\mathcal{W}_{[2]}(j)\big) \simeq *$.

The general case works along the same lines.—It suffices to consider ordered indexing sets $N = [n]$. For $k = -1,\ 0,\ \ldots,\ n$ define an $(n-k)$-cube

$$\mathcal{V}_k(j) \colon \langle [n-k-1] \rangle \longrightarrow k\,Top_*$$

$$A \mapsto \Big\{ (a_0,\ a_1, \ldots,\ a_n) \in \widetilde{M}_n^{A \amalg \{n-k,\ldots,n\}}\big(-j \cdot (i + \#A)\big) \ \Big|\ \forall \ell \geq n-k \colon a_\ell < -j \Big\}$$

with structure maps $\mathcal{V}_k(j)^A \xrightarrow{t_{B \setminus A}^{-j}} \mathcal{V}_k(j)^B$ for $A \subseteq B \subseteq N$. Note that $\mathcal{V}_{-1}(j) = \mathcal{W}_N(j)$, and $\mathcal{V}_n(j)$ is a single space consisting of the basepoint only (the argument is the same as before: if $(a_0,\ a_1,\ \ldots,\ a_n)$ is a non-basepoint in $\mathcal{V}_n(j)$, we have $-nj = \sum a_i \leq -(n+1)(j+1)$ and consequently $j \leq -n-1$ which is impossible by assumption on j). Now $\mathcal{V}_k(j)$ is weakly equivalent to the pointwise homotopy cofibre of the cube $\mathcal{V}_{k-1}(j)$ in $(n-k)$-direction since all spaces are discrete, all maps are injective, and $\mathcal{V}_k(j)$ is the strict cofibre of $\mathcal{V}_{k-1}(j)$ in $(n-k)$-direction. Hence $\Gamma\big(\mathcal{W}_N(j)\big) \simeq \mathcal{V}_n(j) = *$. □

Define, for $j \geq 0$, the number $h(N,\ j) := \#\widetilde{M}_N^{\emptyset}(j) - 1$. Since $\widetilde{M}_N^{\emptyset}(0) = M_N^{\emptyset} = S^0$, we have $h(N, 0) = 1$. (This number does not hide any deep meaning. It is defined exactly as it comes out of the proof of the next corollary.)

Corollary 2.7.4. *(Global sections of canonical sheaves with non-negative twist.)*

Suppose G is a topological monoid with zero which is cofibrant as an object of $k\,Top_*$. For $K \in \mathbf{C}(G)$ and $j \geq 0$, there is a natural chain of weak equivalences connecting $\Sigma \circ \Gamma \circ \psi_j(K)$ and $\bigvee_{h(N,\ j)} \Sigma^{n+1} K$. In particular, we have a natural weak equivalence $\Sigma \circ \Gamma \circ \psi_0(K) \simeq \Sigma^{n+1} K$.

Proof. It suffices to give a proof for $N = [n]$.—First note that we have $\Gamma\big(\psi_j(K)\big) \cong \Gamma\big(\psi_j(K \wedge S^0)\big) \cong K \wedge \Gamma\big(\psi_j(S^0)\big)$. Since K is cofibrant in $k\,Top_*$ by 1.2.1 (3) (at this point we need the hypothesis that G is cofibrant in $k\,Top_*$), the functor $K \wedge \cdot$ is homotopy invariant and it suffices to prove the claim for the special case $K = S^0$ (twisted structure sheaves).

Recall the definition of $\mathcal{W}_N(j)$ from lemma 2.7.3. There is an obvious map of cubes $\psi_j(S^0) \longrightarrow \mathcal{W}_N(j)$ (given by the identity map for $A \neq \emptyset$). For $N = [1]$, we have the following picture:

$$\begin{tikzcd}
 & (\mathbf{N}\times\mathbf{N})^j_+ \arrow[rr, "{(-j,0)}"] \arrow[dd, "{(0,-j)}"] & & (\mathbf{Z}\times\mathbf{N})^0_+ \arrow[dd, "{(0,-j)}"] \\
* \arrow[ur] \arrow[rr] \arrow[dd] & & (\mathbf{Z}\times\mathbf{N})^0_+ \arrow[ur] \arrow[dd, "{(0,-j)}"] & \\
 & (\mathbf{N}\times\mathbf{Z})^0_+ \arrow[rr, "{(-j,0)}"'] & & (\mathbf{Z}\times\mathbf{Z})^{-j}_+ \\
(\mathbf{N}\times\mathbf{Z})^0_+ \arrow[ur] \arrow[rr, "{(-j,0)}"'] & & (\mathbf{Z}\times\mathbf{Z})^{-j}_+ \arrow[ur] &
\end{tikzcd}$$

(The lower index "+" means adding a disjoint basepoint, and an upper index a denotes the subset of tuples with sum a.) The front face of this cube is $\mathcal{O}_{\mathbf{P}^1}(j) = \psi_j(S^0)$, the back face is $\mathcal{W}_{[1]}(j)$.

In the general case, note that all the components of the map $\psi_j(S^0) \longrightarrow \mathcal{W}_N(j)$ are injective, and all components of the cubes are discrete spaces. Hence the pointwise homotopy cofibre is weakly equivalent to the pointwise (strict) cofibre. Computation of the pointwise cofibre yields an $(n+1)$-cube $\mathcal{D}$ with $\mathcal{D}^\emptyset = \mathcal{W}_N^\emptyset(j) = \widetilde{M}_n^\emptyset(j)$ (since $\psi_j(S^0)^\emptyset = *$), and $\mathcal{D}^A = *$ for $A \neq \emptyset$ (since $\psi_j(S^0)^A = \mathcal{W}_N^A(j)$ in this case). The discrete space $\mathcal{D}^\emptyset$ has $h(N,\ j)$ non-basepoint elements (by definition of that number), hence can be written as an $h(N,\ j)$-fold one-point union of zero spheres. Application of Γ to the cofibration sequence $\psi_j(S^0) \longrightarrow \mathcal{W}_N(j) \longrightarrow \mathcal{D}$ yields a sequence of maps

$$\Gamma\big(\psi_j(S^0)\big) \longrightarrow \Gamma\big(\mathcal{W}_N(j)\big) \longrightarrow \Gamma(\mathcal{D}) \cong \Sigma^{n+1}\mathcal{D}^\emptyset \cong \bigvee_{h(n,\ j)} \Sigma^{n+1} S^0 . \qquad (*)$$

Since Γ commutes with pushouts, the last space of $(*)$ is the cofibre of the map on the left. But this map is a cofibration by 1.3.5 (1) since $\psi_j(S^0) \longrightarrow \mathcal{W}_N(j)$ is an f-cofibration in $k\mathit{Top}_*$. Hence its cofibre is weakly equivalent to its homotopy cofibre. By 2.7.3, the space in the middle is contractible, hence the homotopy cofibre is weakly equivalent to the suspension of $\Gamma(\psi_j(S^0))$ which finishes the proof. $\square$

Now we treat negative twists. Define, for $j < 0$, the number

$$a(N,\ j) := \#\left\{(a_i)_{i\in N} \in \mathbf{Z}^N \;\middle|\; \sum_{i\in N} a_i = -nj \text{ and } \forall i: a_i < -j\right\}.$$

(This number does not hide any deep meaning. It is defined exactly as it comes out of the proof of the next corollary.) We have

$$\begin{aligned} a(N,\ j) &= 0 \quad (\text{for } -n-1 < j < 0) \\ a(N,\ -n-1) &= 1 \end{aligned}$$

which can be seen in the following way: suppose $(a_i)_{i\in N}$ is an element of the set occuring in the definition of $a(N,\ j)$. Then $-nj = \sum a_i \leq -(n+1)(j+1)$ and consequently $j \leq -n-1$.—Using the notation of 2.7.3, we have $a(N,\ j) = \#\mathcal{V}_n(j) - 1$, the -1 being due to the basepoint.

Corollary 2.7.5. *(Global sections of canonical sheaves with negative twist.)*

Suppose G is a topological monoid with zero which is cofibrant as an object of $k\,Top_*$. For $K \in \mathbf{C}(G)$ and $j < 0$, there is a natural chain of weak equivalences connecting $\Gamma \circ \psi_j(K)$ and $\bigvee_{a(N,\ j)} K$. In particular, we have a natural weak equivalence $\Gamma \circ \psi_{-n-1}(K) \simeq K$, and for $-n-1 < j < 0$, the space $\Gamma \circ \psi_j(K)$ is contractible.

Proof. It suffices to give a proof for $N = [n]$.—First note that we have $\Gamma\bigl(\psi_j(K)\bigr) \cong \Gamma\bigl(\psi_j(K \wedge S^0)\bigr) \cong K \wedge \Gamma\bigl(\psi_j(S^0)\bigr)$. Since K is cofibrant in $k\,Top_*$ by 1.2.1 (3) (at this point we need the hypothesis that G is cofibrant in $k\,Top_*$), the functor $K \wedge \cdot$ is homotopy invariant and it suffices to prove the claim for the special case $K = S^0$ (twisted structure sheaves).

Recall the definition of $\mathcal{W}_N(j)$ from lemma 2.7.3. For $j < 0$, comparing the definitions shows $\mathcal{O}_{\mathbf{P}^N}(j) = \mathcal{W}_N(j)$. Using the notation from the proof of 2.7.3,

$$\Gamma\bigl(\mathcal{O}_{\mathbf{P}^N}(j)\bigr) = \Gamma\bigl(\mathcal{W}_N(j)\bigr) \simeq \mathcal{V}_n(j)\ .$$

As we have seen, this space is contractible for $j > -n-1$. In general, this space has $a(N,\ j)$ non-basepoints (this is just the definition of $a(N,\ j)$), hence can be written as an $a(N,\ j)$-fold one-point union of copies of S^0. If in particular $j = -n-1$, we have $\Gamma\bigl(\mathcal{O}_{\mathbf{P}^N}(-n-1)\bigr) \simeq S^0$. □

2.8. Spread Sheaves

The purpose of this subsection is to establish a chain of $h_{\{0\}}$-equivalences (see below) in $\mathbf{P}^N(G)$ connecting $\Sigma^2 \circ \psi_0 \circ \Gamma(Y)$ and $\Sigma^{n+2} Y$ for weakly cofibrant Y. (The main tool, spread sheaves, still are somewhat mysterious to the author.)

Definition 2.8.1. Given $S \subseteq \mathbf{Z}$, a map f in $\mathbf{pP}^N(G)$ is called an *h_S-equivalence* if for all $s \in S$ the map $\Gamma \circ \theta_s(f)$ is a weak equivalence of spaces. A map of cubes $g \in G\text{-}k\,Top_*^{\langle N\rangle}$ is called an $h_{\{0\}}$-equivalence if $\Gamma(g)$ is a weak equivalence of G-spaces.

The interplay of h-equivalences and h_S-equivalences (for various choices of S) will be an important tool for calculating the K-theory of non-linear projective space.

Remark 2.8.2. All maps are $h_\emptyset$-equivalences. In [KVWW], $h_{\{0\}}$-equivalences are called h_Γ-equivalences, and $h_{[k]}$-equivalences are called $h_{\leq k}$-equivalences.

Definition 2.8.3. Suppose Y is an object of $\mathbf{pP}^N(G)$, and C is a (possibly empty) subset of N. We define the *C-spreading of Y*, denoted $\operatorname{spr}^C(Y)$, to be the presheaf

$$A \mapsto \operatorname{spr}^C(Y)^A := Y^{C \cup A}$$

($\emptyset \neq A \subseteq N$) with ($\flat$-type) structure maps induced by those of Y. Here $Y^{C\cup A}$ is considered as an $G \wedge M^A$-equivariant space. Define $\widetilde{\operatorname{spr}}^C(Y) := \operatorname{spr}^C(Y)$ if $C \neq \emptyset$ and $\widetilde{\operatorname{spr}}^{\emptyset}(Y) := *$.

For $C \subseteq D \subseteq N$ there is a map of presheaves $\operatorname{spr}^C(Y) \longrightarrow \operatorname{spr}^D(Y)$ with components induced by the structure maps of Y. The assignment $C \mapsto \operatorname{spr}^C(Y)$ for fixed Y is itself functorial, hence:

Lemma 2.8.4. *The cubical diagram $A \mapsto \Gamma \circ \operatorname{spr}^A(Y)$ in G-$k\,Top_*^{\langle N\rangle}$ is commutative.*

□

Example 2.8.5. We draw the diagram of 2.8.4 for the case $N = \langle 1 \rangle$:

$$\begin{array}{ccc}
\Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0\}} \\ \downarrow & & \downarrow \\ Y^{\{1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right) & \longrightarrow & \Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0\}} \\ \downarrow & & \downarrow \\ Y^{\{0,1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right) \\
\downarrow & & \downarrow \\
\Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0,1\}} \\ \downarrow & & \downarrow \\ Y^{\{1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right) & \longrightarrow & \Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0,1\}} \\ \downarrow & & \downarrow \\ Y^{\{0,1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right)
\end{array}$$

As usual, the 0-direction is left to right, *i.e.*, the small square in the upper right corner represents $\operatorname{spr}^{\{0\}}(Y)$, the small squares in the bottom row represent $\operatorname{spr}^{\{1\}}(Y)$ and $\operatorname{spr}^{\{0,1\}}(Y)$, respectively.

As each component of $\operatorname{spr}^C(Y)$ has an M^C-action, the space $\Gamma \circ \operatorname{spr}^C(Y)$ is an object of $G \wedge M^C$-$k\,Top_*$. Because of this and the previous lemma, it is possible to define a new presheaf $\sigma^N(Y) = \sigma(Y)$ by

$$A \mapsto \sigma^N(Y)^A := \Sigma^2 \circ \Gamma \circ \operatorname{spr}^A(Y) \qquad (A \neq \emptyset)\ .$$

In this way we obtain a functor $\sigma^N = \sigma \colon \mathbf{pP}^N(G) \longrightarrow \mathbf{pP}^N(G)$. (For $N = \langle 1 \rangle$ this is the double suspension of Σ' as defined in [KVWW]. For a picture, replace the left upper square in the above example by a point, and suspend twice.)

Lemma 2.8.6.

(1) If Y is a weakly cofibrant object (2.4.7) of $\mathbf{pP}^N(G)$, there is a natural chain of h-equivalences connecting $\sigma(Y)$ and $\Sigma^{n+2}Y$. In particular, if Y is a weakly cofibrant sheaf, then $\sigma(Y) \in \mathbf{P}^N(G)$.

(2) The functor σ^N commutes with pushouts. It preserves h-equivalences and weak cofibrations (2.4.6) between weakly cofibrant objects.

Proof. For a space $K \in k\,Top_*$, let $\mathrm{con}(K)$ denote the N-cube with a point as initial vertex and K everywhere else; structure maps are identity maps (away from the initial vertex). For $Y \in \mathbf{pP}^N(G)$ define a presheaf ϵY by $(\epsilon Y)^C := \Gamma \circ \mathrm{con}(Y^C)$. It is h-equivalent to the n-fold suspension of Y (compute global sections in any direction and observe that the resulting cube has Y^C as initial vertex, and all other vertices are contractible). Thus it is sufficient to show that $\Sigma^2 \circ \epsilon$ is h-equivalent to σ.

For $C \subseteq N$ and $Y \in \mathbf{P}^N(G)$ there is a map of cubes $f^C : \mathrm{con}(Y^C) \longrightarrow \mathrm{spr}^C(Y)$, natural in C and Y, induced by the structure maps of Y.

In the case $N = \langle 1 \rangle$, this map has the following pictorial representation:

$$\begin{tikzcd}
* \arrow[r] \arrow[dd] & Y^C \arrow[dd] & & & * \arrow[r] \arrow[dd] & Y^{\{0\}\cup C} \arrow[dd] \\
 & & {} \arrow[r, "f^C"] & {} & & \\
Y^C \arrow[r] & Y^C & & & Y^{\{1\}\cup C} \arrow[r] & Y^{\{0,1\}\cup C}
\end{tikzcd}$$

The square on the left depicts $\mathrm{con}(Y^C)$, the square on the right represents $\mathrm{spr}^C(Y)$.

Back to the general case, the maps f^C, in turn, induce upon application of $\Sigma^2 \circ \Gamma$ a natural transformation $g : \Sigma^2 \circ \epsilon \longrightarrow \sigma$. We want to prove that g is an h-equivalence of functors, *i.e.*, that all components of g are weak equivalences. Fix a non-empty subset $C \subseteq N$. By definition, the map g^C is given by $\Sigma^2 \circ \Gamma(f^C)$. Thus it is enough to show that $\Gamma(f^C)$ has a contractible mapping cone for all $C \neq \emptyset$ (then $g^C = \Sigma^2 \circ \Gamma(f^C)$ is a homology equivalence by the long exact sequence of homology groups, and hence a homotopy equivalence since source and target are simply connected). But Γ commutes with taking homotopy cofibres (since Γ, applied to a 1-cube, is the mapping cone functor, and since Γ admits iterated calculation in the sense of 1.3.2 (4)). Hence this is equivalent to computing the pointwise mapping cone of f^C and showing that the resulting cube Z has contractible iterated homotopy cofibre.

So define $Z := \text{hocofibre } (f^C)$. Choose $i \in C$. By definition of the C-spreading we have $Z^\emptyset = *$ and $Z^A = \text{hocofibre } (Y^C \longrightarrow Y^{C\cup A})$ for $A \neq \emptyset$ (where "hocofibre" means the mapping cone). This implies $Z^A \simeq Z^{A\amalg\{i\}}$ for all $A \subseteq N$ (even $Z^A = Z^{A\amalg\{i\}}$ if A is not the empty set). Hence computing $\Gamma(Z)$ in i-direction yields a cube with contractible vertices.—This proves that $\sigma(Y)$ and $\Sigma^{n+2}(Y)$ are weakly equivalent (with respect to h-equivalences).

Since the sheaf condition is homotopy invariant (2.3.5), and since the suspension of a weakly cofibrant sheaf is a sheaf, we infer that σ maps weakly cofibrant sheaves to sheaves. This completes the proof of (1).

For (2) note that the functors spr^C commute with pushouts (since colimits are calculated pointwise). Since Γ is also compatible with pushouts (1.3.5 (3)), so is σ.

Let $f : Y \longrightarrow Z$ denote a weak cofibration between weakly cofibrant objects. This means by definition that all components of f are cofibrations in $G\text{-}kTop_*$. Hence the same is true for $\mathrm{spr}^C(f)$, *i.e.*, $\mathrm{spr}^C(f)$ is an f-cofibration in $G\text{-}kTop_*^{\langle N\rangle}$ between f-cofibrant objects of $G\text{-}kTop_*^{\langle N\rangle}$. Since Γ preserves these cofibrations by 1.3.5 (1), we know that $\Gamma \circ \mathrm{spr}^C(f)$ is a cofibration in $G\text{-}kTop_*$, hence (by suspending twice) so is the C-component of $\sigma(f)$. This means by definition that $\sigma(f)$ is a weak cofibration.

Finally, σ preserves weak equivalences between weakly cofibrant objects since Σ^{n+2} does and both functors are connected by a chain of h-equivalences as shown in (1).

□

We proceed with a technical lemma. Let $Y \in \mathbf{P}^N(G)$, and suppose B is a non-empty subset of N. Fix $i \in B$. There is a natural map

$$\gamma : \mathrm{spr}^B(Y) \longrightarrow \delta_{\{i\}}(Y^B)$$

in $G\text{-}kTop_*^{\langle N\rangle}$ given by the identity id_{Y^B} on $\{i\}$-components (the functor $\delta_{\{i\}}$ has been defined in 1.3.1). For $N = \langle 1\rangle$ and $i = 1 \in B$ this map has the following representation:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\}\cup B} \arrow[d] \arrow[r, phantom, "\xrightarrow{\gamma}"] & * \arrow[r] \arrow[d] & * \arrow[d] \\ Y^{\{1\}\cup B} \arrow[r] & Y^{\{0,1\}\cup B} & Y^B \arrow[r] & * \end{tikzcd}$$

Lemma 2.8.7. *The map γ is an $h_{\{0\}}$-equivalence in the sense that $\Gamma(\gamma)$ is a weak homotopy equivalence.*

Proof. Consider the cube Z which is the same as $\mathrm{spr}^B(Y)$ with $\{i\}$-component replaced by a single point, and let $o : Z \longrightarrow \mathrm{spr}^B(Y)$ denote the inclusion map; it is the identity on all components except the $\{i\}$-component. Picture:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\}\cup B} \arrow[d] \arrow[r, phantom, "\xrightarrow{o}"] & * \arrow[r] \arrow[d] & Y^{\{0\}\cup B} \arrow[d] \\ * \arrow[r] & Y^{\{0,1\}\cup B} & Y^{\{1\}\cup B} \arrow[r] & Y^{\{0,1\}\cup B} \end{tikzcd}$$

There is an obvious map from the pointwise homotopy cofibre of o to $\delta_{\{i\}}(Y^B)$ (given by the identity on $\{i\}$-components), and this map is an h-equivalence since all components of the homotopy cofibre, except the $\{i\}$-component, are contractible. Moreover, the composite $\mathrm{spr}^B(Y) \longrightarrow \text{hocofibre }(o) \longrightarrow \delta_{\{i\}}(Y^B)$ is the map γ.

On application of Γ, we obtain the following diagram:

$$\begin{tikzcd} \Gamma(Z) \arrow[r, "\Gamma(o)"] & \Gamma\big(\mathrm{spr}^B(Y)\big) \arrow[r, "p"] \arrow[d, equal] & \Gamma\big(\text{hocofibre }(o)\big) \arrow[r, "\sim"] & \Gamma\big(\delta_{\{i\}}(Y^B)\big) \arrow[d, equal] \\ & \Gamma\big(\mathrm{spr}^B(Y)\big) \arrow[rr, "\Gamma(\gamma)"] & & \Gamma\big(\delta_{\{i\}}(Y^B)\big) \end{tikzcd}$$

Hence to show that $\Gamma(\gamma)$ is a weak equivalence, it suffices to prove that the map p is a weak equivalence. But the target of p is the homotopy cofibre of $\Gamma(o)$ (since Γ commutes with pushouts and with $\cdot \wedge I$), hence it suffices to show $\Gamma(Z) \simeq *$. But this is clear since computing the homotopy cofibre of Z in i-direction results in a cube with contractible components. $\square$

Let $\Gamma \circ \widetilde{\mathrm{spr}}(Y)$ denote the presheaf $C \mapsto \Gamma \circ \widetilde{\mathrm{spr}}^C(Y)$, and recall the forgetful functor $V : \mathbf{pP}^N(G) \longrightarrow G\text{-}k\mathit{Top}_*^{\langle N \rangle_0}$. Lemma 2.8.4 asserts that there is a canonical map $\Gamma(Y) \longrightarrow \varprojlim \circ V\big(\Gamma \circ \widetilde{\mathrm{spr}}(Y)\big)$. By passing to the adjoint map (2.6.10) we obtain a natural map $\tau : \psi_0 \circ \Gamma(Y) \longrightarrow \Gamma \circ \widetilde{\mathrm{spr}}(Y)$ and, by suspending twice, a natural transformation $\Sigma^2 \circ \psi_0 \circ \Gamma \longrightarrow \sigma$.

We give a graphical representation of τ in the case $N = \langle 1 \rangle$:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Gamma(Y)\wedge M^{\{0\}} \arrow[d] & & * \arrow[r] \arrow[d] & \Gamma(\mathrm{spr}^{\{0\}}(Y)) \arrow[d] \\
\Gamma(Y)\wedge M^{\{1\}} \arrow[r] & \Gamma(Y)\wedge M^{\{0,1\}} \arrow[r, phantom, "\xrightarrow{\tau}"] & & \Gamma(\mathrm{spr}^{\{1\}}(Y)) \arrow[r] & \Gamma(\mathrm{spr}^{\{0,1\}}(Y))
\end{tikzcd}$$

The left square is the picture of $\psi_0 \circ \Gamma(Y)$, the right square symbolizes $\Gamma \circ \widetilde{\mathrm{spr}}(Y)$.

Lemma 2.8.8. *Suppose G is cofibrant as an object of $k\mathit{Top}_*$. Then the natural map $\Sigma^2 \circ \psi_0 \circ \Gamma(Y) \longrightarrow \sigma(Y)$ constructed above is an $h_{\{0\}}$-equivalence for weakly cofibrant objects $Y \in \mathbf{P}^N(G)$.*

As an immediate consequence of this lemma and 2.8.6 (1), we have:

Corollary 2.8.9. *Suppose G is cofibrant as an object of $k\mathit{Top}_*$. Then the two functors $\Sigma^2 \circ \psi_0 \circ \Gamma$ and Σ^{n+2}, restricted to weakly cofibrant objects, are connected by a chain of $h_{\{0\}}$-equivalences.* $\square$

Proof of 2.8.8. There is a canonical map $Y \wedge M^A \longrightarrow \widetilde{\mathrm{spr}}^A(Y)$; its B-component is given by the composite of the structure maps $Y^B \longrightarrow Y^{B\cup A}$ and the M^A-action on $Y^{B\cup A}$. The A-component of τ is the image of this map $Y \wedge M^A \longrightarrow \widetilde{\mathrm{spr}}^A(Y)$ under Γ since $\psi_0 \circ \Gamma(Y)^A = \Gamma(Y) \wedge M^A \cong \Gamma(Y \wedge M^A)$. Let us agree to use the letter B as indexing set for the presheaves Y and $\widetilde{\mathrm{spr}}^A(Y)$, and we will write Γ_b to indicate that Γ belongs to this indexing set B. Similarly, let A denote the indexing set for the presheaves $\psi_0 \circ \Gamma_b(Y)$ and $\Gamma_b \circ \widetilde{\mathrm{spr}}(Y)$ with corresponding global sections functor Γ_a. Then the lemma asserts that

$$\Gamma_a \circ \Sigma^2(\tau) : \quad \Gamma_a \circ \Sigma^2 \circ \psi_0 \circ \Gamma_b(Y) \longrightarrow \Gamma_a \circ \sigma^N = \Gamma_a \circ \Sigma^2 \circ \Gamma_b \circ \widetilde{\mathrm{spr}}(Y)$$

is a weak homotopy equivalence.

Using the identification of τ^A from the beginning of the proof, we can combine both directions (A-direction and B-direction) into a "hypercube": the lemma is equivalent to the claim that the map of $N \amalg N$-cubes

$$\mathcal{O}_{\mathbf{P}^N} \widetilde{\wedge} Y \longrightarrow \widetilde{\mathrm{spr}}(Y) \qquad (*)$$

becomes a weak equivalence after application of $\Sigma^2 \circ \Gamma$, where the source is the "external tensor product" of $\mathcal{O}_{\mathbf{P}^N}$ and Y, *i.e.*, the $N \amalg N$-cube $A \amalg B \mapsto \mathcal{O}^A_{\mathbf{P}^N} \wedge Y^B$ (this is a diagram in G-$k\,Top_*$ because of the G-action of Y^B), and the target is the functor

$$\widetilde{\mathrm{spr}}(Y) \colon N \amalg N \longrightarrow G\text{-}k\,Top_*, \quad A \amalg B \mapsto \left(\widetilde{\mathrm{spr}}^A(Y)\right)^B .$$

Now even for $N = \langle 1 \rangle$, the pictures get huge. The source of the map $(*)$ is a 4-dimensional cube. It has the following graphical representation:

$$\begin{array}{ccc}
\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} & \longrightarrow & \begin{tikzcd} * \arrow[r] \arrow[d] & M^{\{0\}} \wedge Y^{\{0\}} \arrow[d] \\ M^{\{0\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{0\}} \wedge Y^{\{0,1\}} \end{tikzcd} \\
\Big\downarrow & & \Big\downarrow \\
\begin{tikzcd} * \arrow[r] \arrow[d] & M^{\{1\}} \wedge Y^{\{0\}} \arrow[d] \\ M^{\{1\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{1\}} \wedge Y^{\{0,1\}} \end{tikzcd} & \longrightarrow & \begin{tikzcd} * \arrow[r] \arrow[d] & M^{\{0,1\}} \wedge Y^{\{0\}} \arrow[d] \\ M^{\{0,1\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{0,1\}} \wedge Y^{\{0,1\}} \end{tikzcd}
\end{array}$$

Here the "B-direction" is encoded into the small squares, the A-direction is the large square. Similarly, the target of $(*)$ has the following picture:

$$\begin{array}{ccc}
\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} & \longrightarrow & \begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\} \cup \{0\}} \arrow[d] \\ Y^{\{1\} \cup \{0\}} \arrow[r] & Y^{\{0,1\} \cup \{0\}} \end{tikzcd} \\
\Big\downarrow & & \Big\downarrow \\
\begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\} \cup \{1\}} \arrow[d] \\ Y^{\{1\} \cup \{1\}} \arrow[r] & Y^{\{0,1\} \cup \{1\}} \end{tikzcd} & \longrightarrow & \begin{tikzcd} * \arrow[r] \arrow[d] & Y^{\{0\} \cup \{0,1\}} \arrow[d] \\ Y^{\{1\} \cup \{0,1\}} \arrow[r] & Y^{\{0,1\} \cup \{0,1\}} \end{tikzcd}
\end{array}$$

In informal language, the result of computing Γ "in B-direction" (*i.e.*, evaluating Γ_b) is the map τ^A. But computing Γ_b first and then applying Γ_a is the same (up to natural isomorphism) as applying Γ to the "hypercube", which in turn is the same as first computing Γ_a, then applying Γ_b. Symbolically, this change of preference is depicted as follows for the source of $(*)$:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d] & & * \arrow[r] \arrow[d] & M^{\{0\}}\wedge Y^{\{0\}} \arrow[d] \\
* \arrow[r] & * \arrow[rr] \arrow[dd] & & M^{\{1\}}\wedge Y^{\{0\}} \arrow[r] \arrow[dd] & M^{\{0,1\}}\wedge Y^{\{0\}} \\
& & & & \\
* \arrow[r] \arrow[d] & M^{\{0\}}\wedge Y^{\{1\}} \arrow[d] \arrow[rr] & & * \arrow[r] \arrow[d] & M^{\{0\}}\wedge Y^{\{0,1\}} \arrow[d] \\
M^{\{1\}}\wedge Y^{\{1\}} \arrow[r] & M^{\{0,1\}}\wedge Y^{\{1\}} & & M^{\{1\}}\wedge Y^{\{0,1\}} \arrow[r] & M^{\{0,1\}}\wedge Y^{\{0,1\}}
\end{tikzcd}$$

This is really the same cubical diagram as before. The right upper small square is $\mathcal{O}_{\mathbf{P}^1} \wedge Y^{\{0\}}$, and similarly for the other small squares.—There is a similar picture for the target of $(*)$:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d] & & * \arrow[r] \arrow[d] & Y^{\{0\}\cup\{0\}} \arrow[d] \\
* \arrow[r] & * \arrow[rr] \arrow[dd] & & Y^{\{0\}\cup\{1\}} \arrow[r] \arrow[dd] & Y^{\{0\}\cup\{0,1\}} \\
& & & & \\
* \arrow[r] \arrow[d] & Y^{\{1\}\cup\{0\}} \arrow[d] \arrow[rr] & & * \arrow[r] \arrow[d] & Y^{\{0,1\}\cup\{0\}} \arrow[d] \\
Y^{\{1\}\cup\{1\}} \arrow[r] & Y^{\{1\}\cup\{0,1\}} & & Y^{\{0,1\}\cup\{1\}} \arrow[r] & Y^{\{0,1\}\cup\{0,1\}}
\end{tikzcd}$$

Computing "in A-direction", *i.e.*, evaluating Γ_a, results in a map of N-cubes. Its B-component is given by

$$\beta\colon \Gamma_a(\mathcal{O}_{\mathbf{P}^N} \wedge Y^B) \longrightarrow \Gamma_a\big(\widetilde{\mathrm{spr}}(Y)^B\big) \ .$$

We want to show that this map is a weak homotopy equivalence after two suspensions (for all non-empty $B \subseteq N$). Then application of $\Sigma^2 \circ \Gamma_b$ gives a weak homotopy equivalence (1.3.5 (2)), hence the double suspension of the map $(*)$ is an $h_{\{0\}}$-equivalence as claimed.

Now $\widetilde{\mathrm{spr}}(Y)^B$ is the same as $\mathrm{spr}^B(Y)$ which can be seen by tracing the definitions. The source of β is isomorphic to $\Gamma(\mathcal{O}_{\mathbf{P}^N}) \wedge Y^B$. Choose some $i \in B$. The map γ from 2.8.7 is an $h_{\{0\}}$-equivalence. Since weak homotopy equivalences satisfy the saturation axiom, it suffices to show that $\Gamma(\gamma) \circ \beta$ is a weak homotopy equivalence.

Define the map $\alpha\colon \mathcal{O}_{\mathbf{P}^N} \longrightarrow \delta_{\{i\}}(S^0)$ by mapping all non-basepoints of $M^{\{i\}}$ into the non-basepoint of S^0. Picture for $N = \langle 1\rangle$ and $i = 1$:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & M^{\{0\}} \arrow[d] & & * \arrow[r] \arrow[d] & * \arrow[d] \\ M^{\{1\}} \arrow[r] & M^{\{0,1\}} \arrow[r, phantom, "\xrightarrow{\alpha}"] & & S^0 \arrow[r] & * \end{tikzcd}$$

Tracing the definitions shows that $\Gamma(\alpha \wedge \mathrm{id}_{Y^B}) \cong \Gamma(\gamma) \circ \beta$. Since Y^B is cofibrant as a G-space and hence as an object of $k\mathit{Top}_*$ (1.2.1 (3)), and since $\Gamma(\alpha \wedge \mathrm{id}) \cong \Gamma(\alpha) \wedge \mathrm{id}$, it suffices to show that $\Sigma^2(\alpha)$ is an $h_{\{0\}}$-equivalence.

Recall the cube $\mathcal{W} := \mathcal{W}_N(0)$ from 2.7.3. There is a map

$$\omega\colon \mathcal{W} \longrightarrow \delta_\emptyset(S^0) \vee \delta_{\{i\}}(S^0)$$

defined similar to α and fitting into the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{\mathbf{P}^N} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{W} \arrow[r] \arrow[d, "\omega"'] & \delta_\emptyset(S^0) \arrow[d, "\mathrm{id}"] \\ \delta_{\{i\}}(S^0) \arrow[r] & \delta_\emptyset(S^0) \vee \delta_{\{i\}}(S^0) \arrow[r] & \delta_\emptyset(S^0) \end{tikzcd}$$

Note that the two horizontal maps on the left are f-cofibrations of objects in $k\mathit{Top}_*^{\langle N\rangle}$, and the two cubes on the right are the cofibres of these (strict cofibres). Hence by 1.3.5 (1) application of $\Sigma^2 \circ \Gamma$ yields a map of two cofibre sequences:

$$\begin{tikzcd} \Sigma^2 \circ \Gamma(\mathcal{O}_{\mathbf{P}^N}) \arrow[r] \arrow[d, "\Sigma^2\circ\Gamma(\alpha)"'] & \Sigma^2 \circ \Gamma(\mathcal{W}) \arrow[r] \arrow[d, "\Sigma^2\circ\Gamma(\omega)"'] & \Sigma^2 \circ \Gamma\big(\delta_\emptyset(S^0)\big) \arrow[d, "\mathrm{id}"] \\ \Sigma^2 \circ \Gamma\big(\delta_{\{i\}}(S^0)\big) \arrow[r] & \Sigma^2 \circ \Gamma\big(\delta_\emptyset(S^0) \vee \delta_{\{i\}}(S^0)\big) \arrow[r] & \Sigma^2 \circ \Gamma\big(\delta_\emptyset(S^0)\big) \end{tikzcd}$$

The two spaces in the middle are contractible (by 2.7.3 for the upper space, by computing Γ in i-direction for the lower space). Consequently the vertical map in the middle induces an isomorphisms on homology groups. By the five lemma (applied to the long exact sequence of homology groups) this is true also for $\Sigma^2 \circ \Gamma(\alpha)$ which proves that $\Sigma^2(\alpha)$ is an $h_{\{0\}}$-equivalence. $\square$

3. The K-Theory of Projective Space

Throughout this chapter, we assume that N is a non-empty finite set with $n+1$ elements, and that G is a topological monoid with zero which is cofibrant as an object of $kTop_*$. This cofibrancy condition assures properness of the used model structures (1.2.1 (4), 2.4.1, 2.4.4) and allows control about the behaviour of global sections and canonical sheaves (2.7.4, 2.7.5, 2.8.9).

3.1. Finiteness in Projective Space

Now we treat finiteness notions for objects of $\mathbf{pP}^N(G)$. We define different finiteness conditions and show that they are well-behaved in the sense that weak equivalences and formation of pushouts preserve finiteness (for exact statments see below). The main tool is the existence of the two model structures on $\mathbf{pP}^N(G)$ (cf. 2.4).

Definition 3.1.1. An object Y of $\mathbf{P}^N(G)$ is called *locally finite* if all its components Y^A are finite (and hence cofibrant) in their respective categories, *i.e.*, if $Y^A \in \mathbf{C}_f(G \wedge M^A)$ for all non-empty $A \subseteq N$. The full subcategory of these objects is denoted by $\mathbf{P}^N_f(G)$. An object $Y \in \mathbf{P}^N(G)$ is called *homotopy finite* if all its components are homotopy finite in their respective categories. We say that Y is *finitely dominated* if Y is a retract of a homotopy finite sheaf. Finally we call Y *stably finitely dominated* if $\Sigma^k Y$ is finitely dominated for some $k \geq 0$. The full subcategories of locally cofibrant objects which are homotopy finite, finitely dominated and stably finitely dominated will respectively be denoted by $\mathbf{P}^N_{hf}(G)$, $\mathbf{P}^N_{fd}(G)$ and $\mathbf{P}^N_{sfd}(G)$.

We will also need the full subcategory $\check{\mathbf{P}}^N_f(G)$ of objects which are locally finite and strongly cofibrant. Finally let $\overline{\mathbf{P}}^N_{sfd}(G)$ denote the full subcategory of stably finitely dominated, weakly cofibrant objects.

We have defined a sequence

$$\check{\mathbf{P}}^N_f(G) \subseteq \mathbf{P}^N_f(G) \subseteq \mathbf{P}^N_{hf}(G) \subseteq \mathbf{P}^N_{fd}(G) \subseteq \mathbf{P}^N_{sfd}(G) \subseteq \overline{\mathbf{P}}^N_{sfd}(G)$$

of full subcategories of $\mathbf{pP}^N(G)$. It is the purpose of this chapter to define and study the algebraic K-theory of $\mathbf{P}^N_{sfd}(G)$. We will indicate how it compares to alternative definitions of the K-theory of projective space.

Lemma 3.1.2. *All the following conditions are equivalent for a locally cofibrant object $Y \in \mathbf{P}^N(G)$:*

(1) For all $i \in N$, the component $Y^{\{i\}}$ is an object of $\mathbf{C}_{hf}(G \wedge M^{\{i\}})$.

(2) There is a locally finite, strongly cofibrant object $Z \in \mathbf{P}^N(G)$ and a weak equivalence $Z \longrightarrow Y$.

(3) There is a locally finite $Z \in \mathbf{P}^N(G)$ and a chain of weak equivalences in $\mathbf{pP}^N(G)$ connecting Y and Z.

(4) Y is homotopy finite.

Proof. Condition (1) is a special case of (4). Conversely, assume (1) holds. Fix a non-empty subset $A \subseteq N$, and choose $i \in A$. By hypothesis $Y^{\{i\}} \in \mathbf{C}_{hf}(G\wedge M^{\{i\}})$. Since Y is a sheaf, we know that Y^A and $Y^{\{i\}}[t_A^{-1}]$ are weakly equivalent (2.3.4). But the latter space is homotopy finite since inverting the action of t_A preserves homotopy finitenes by lemma 1.2.2 (3). This holds for all A, hence Y is homotopy finite.

Condition (3) is a special case of (2). Moreover, (3) implies (4) since an object of $\mathbf{C}(G\wedge M^A)$ is homotopy finite if it is connected through a chain of weak equivalences in $(G\wedge M^A)$-$k\mathit{Top}_*$ to a homotopy finite object.

It remains to prove (4) $\Longrightarrow$ (2). So assume (4) holds. By the WHITEHEAD theorem ([DS], 4.24), there are spaces $\bar{Z}^A \in \mathbf{C}_f(G\wedge M^A)$ and mutually inverse (M^A-equivariant) homotopy equivalences (in the strong sense) $\alpha^A \colon Y^A \longrightarrow \bar{Z}^A$ and $\bar{\beta}^A \colon \bar{Z}^A \longrightarrow Y^A$. Define $Z^{\{j\}} := \bar{Z}^{\{j\}}$ and $\beta^{\{j\}} := \bar{\beta}^{\{j\}}$ for all $j \in N$.

Let $A \subseteq N$, and suppose that by induction Z^B and β^B have already been constructed for all proper subsets $B \subset A$. Define a map $L_A Z \longrightarrow \bar{Z}^A$ as the composite

$$L_A Z \longrightarrow L_A Y \longrightarrow Y^A \xrightarrow{\alpha^A} \bar{Z}^A$$

where L_A denotes the A-th latching space functor (section 2.4—this makes sense because the definition of $L_A Z$ involves only those spaces Z^B for proper non-empty subsets $B \subset A$). By choice of α^A and $\bar{\beta}^A$, the two maps $L_A Z \longrightarrow L_A Y \longrightarrow Y^A$ and $L_A Z \longrightarrow \bar{Z}^A \xrightarrow{\bar{\beta}^A} Y^A$ are homotopic. Let Z^A denote the mapping cylinder of $L_A Z \longrightarrow \bar{Z}$. Then a choice of a homotopy determines a weak equivalence $\beta^A \colon Z^A \longrightarrow Y^A$, homotopic to $\bar{\beta}^A$, making the following diagram commute:

$$\begin{tikzcd} L_A Z \arrow[r] \arrow[d] & Z^A \arrow[d, "\beta^A"] \\ L_A Y \arrow[r] & Y^A \end{tikzcd} \qquad (*)$$

(Another example of the process of "swallowing a homotopy" is described in more detail below in the proof of lemma 3.1.3.)

We claim that $L_A Z$ (and consequently Z^A) is a finite cofibrant $(G\wedge M^A)$-space. Choose $j \in A$. By 2.4.10 (3), we have a pushout square

$$\begin{tikzcd} L_{A\backslash\{j\}} Z[t_j^{-1}] \arrow[r] \arrow[d] & Z^{A\backslash\{j\}}[t_j^{-1}] \arrow[d] \\ L_A^{+\{j\}} Z \arrow[r] & L_A Z \end{tikzcd}$$

where the upper horizontal map is, by induction, a cofibration between finite spaces. Hence it suffices to show that the restricted latching space $L_A^{+\{j\}} Z$ is a finite cofibrant $(G\wedge M^A)$-equivariant space. This can be shown by induction (note that the restricted latching space is the latching space of the j-th face of Z, *i.e.*, of a lower-dimensional cube). A rigorous induction is similar to the one presented in lemma 3.2.5 below (step 1 of the proof); we omit the details.

The spaces Z^A assemble to a presheaf Z; $\flat$-type structure maps are the composites $Z^A \xrightarrow{s_0} Z^A[t_B^{-1}] \longrightarrow L_B Z \longrightarrow Z^B$ for the inclusion $A \subset B$, where s_0 denotes the unit of the adjunction of $\cdot \wedge_{M^A} M^B$ and the restriction functor from $(G \wedge M^B)$-$k\,Top_*$ to $(G \wedge M^A)$-$k\,Top_*$ (1.2.2 (1)). Since Z maps to Y via the weak equivalence β (with components β^A constructed above) it is a sheaf (corollary 2.3.5). It is locally finite since its components Z^A are mapping cylinders of maps between finite spaces. Finally Z is strongly cofibrant since the maps $L_A Z \longrightarrow Z^A$ are inclusions into a mapping cylinder by construction of Z^A, hence are cofibrations. □

Lemma 3.1.3. *Suppose $\bar{Y}$ is a finitely dominated sheaf, and suppose that there is a strongly cofibrant sheaf Y and a weak equivalence $\pi_Y : Y \longrightarrow \bar{Y}$. Then there exists a strongly cofibrant, homotopy finite sheaf Q such that Y is a retract of Q.*

Proof. By hypothesis, $\bar{Y}$ is a retract of a homotopy finite object $\bar{Q}$. Call retraction and section $\bar{r}$ and $\bar{s}$, respectively. We replace $\bar{Q}$ by a strongly cofibrant object: factoring the unique map from the zero sheaf into $\bar{Q}$ as a c-cofibration followed by an acyclic c-fibration yields a strongly cofibrant $\widetilde{Q}$ and a weak equivalence $\pi_Q : \widetilde{Q} \longrightarrow \bar{Q}$. The object $\tilde{Q}$ is homotopy finite since it maps to $\bar{Q}$ via a weak equivalence.

By 5.8 of [DS] we can form $\pi_Y^{-1} \circ \bar{r} \circ \pi_Q$ in $Ho\,\mathbf{pP}^N(G)$, and since Y is c-fibrant we can represent this composite by a map $\widetilde{r} \colon \widetilde{Q} \longrightarrow Y$ in $\mathbf{pP}^N(G)$ ([DS], 5.11). Similarly we can represent $\pi_Q^{-1} \circ \bar{s} \circ \pi_Y$ by a morphism $\widetilde{s} \colon Y \longrightarrow \widetilde{Q}$. Since $\bar{r} \circ \bar{s} = \mathrm{id}_{\bar{Y}}$, we know that $\widetilde{r} \circ \widetilde{s} = \mathrm{id}_Y$ in $Ho\,\mathbf{pP}^N(G)$, *i.e.*, the maps $\widetilde{r} \circ \widetilde{s}$ and id_Y are homotopic.

We show how to swallow this homotopy with a cylinder object. Recall that $Y \wedge I$ is a good cylinder object in the c-structure (lemma 2.4.8). Hence there is a homotopy $H \colon Y \wedge I \longrightarrow Y$ such that $H \circ i_0 = \mathrm{id}_Y$ and $H \circ i_1 = \widetilde{r} \circ \widetilde{s}$ where i_0 and i_1 are the two canonical maps $Y \longrightarrow Y \wedge I$ ([DS], 4.23). Consider the following commutative diagram:

$$\begin{tikzcd}
& Y \arrow[d, "i_0"'] \arrow[dr, "id"] & \\
Y \arrow[r, "i_1"] \arrow[d, "\tilde{s}"'] & Y \wedge I \arrow[r, "H"] \arrow[d, "h"] & Y \\
\widetilde{Q} \arrow[r] & Z_{\tilde{s}} \arrow[ur, "r"'] &
\end{tikzcd}$$

The square on the left is a pushout square, *i.e.*, $Q := Z_{\tilde{s}}$ denotes the mapping cylinder of $\widetilde{s}$. The map r is induced by H and $\widetilde{r} \colon \widetilde{Q} \longrightarrow Y$. Now the map $\widetilde{Q} \longrightarrow Q = Z_{\tilde{s}}$ is the pushout of an acyclic c-cofibration, hence is itself an acyclic c-cofibration. Thus Q is homotopy finite and strongly cofibrant. Define $s := h \circ i_0$. Then Y is a retract of Q with section and retraction given by s and r. □

Lemma 3.1.4. *The classes of homotopy finite, finitely dominated and weakly cofibrant stably finitely dominated objects of* $\mathbf{P}^N(G)$ *are closed under weak equivalences.*

Proof. For homotopy finite objects this holds by definition of homotopy finiteness.— Suppose Z is connected by a chain of weak equivalences to the finitely dominated sheaf $\bar{Y}$. By factoring the unique map $* \longrightarrow \bar{Y}$ we obtain a strongly cofibrant sheaf Y and a weak equivalence $\pi_Y : Y \longrightarrow \bar{Y}$. According to 3.1.3 there exists a strongly cofibrant, homotopy finite sheaf Q such that Y is a retract of Q. Call section and retraction s and r, respectively.

Since Z and $\bar{Y}$ are assumed to be weakly equivalent, the object Z is connected to Y by a chain of weak equivalences. Consider the image of this chain in the homotopy category of $\mathbf{pP}^N(G)$. Since weak equivalences become invertible there, we obtain an isomorphism $Y \longrightarrow Z$ in $Ho\,\mathbf{pP}^N(G)$. This map can be represented by a morphism $\mu : Y \longrightarrow Z$ in $\mathbf{pP}^N(G)$ since Y is strongly cofibrant and Z is fibrant with respect to the c-structure. This morphism is in fact a weak equivalence since it maps to an isomorphism in the homotopy category ([DS], proposition 5.8).

Factor $s : Y \longrightarrow Q$ as a c-cofibration $Y \longrightarrow P$ followed by a weak equivalence $P \longrightarrow Q$ (see diagram below), and observe that since Q is homotopy finite so

$$\begin{tikzcd}
Z & & & Y \arrow[lll, "\sim", "\mu"'] \arrow[dll] \arrow[dl, "s"] \arrow[dd, "\mathrm{id}_Y"] \\
Z \cup_Y P \arrow[d, dashed] & P \arrow[l, "\sim"] \arrow[r, "\sim"'] & Q \arrow[dr, "r"] & \\
Z & & & Y \arrow[lll, "\sim", "\mu"']
\end{tikzcd}$$

is P. There is a map $Z \cup_Y P \longrightarrow Z$ induced by $\mathrm{id} : Z \longrightarrow Z$ and the composite $P \longrightarrow Q \xrightarrow{r} Y \xrightarrow{\mu} Z$; this map has a section given by the canonical map $Z \longrightarrow Z \cup_Y P$. Hence Z is a retract of $Z \cup_Y P$. But by properness (2.4.1), the map $P \longrightarrow Z \cup_Y P$ is a weak equivalence whence the pushout is homotopy finite. This shows that Z is finitely dominated as claimed.

Now assume Z is weakly cofibrant and weakly equivalent to a weakly cofibrant, stably finitely dominated object Y through a chain of weak equivalences. By the model category axioms, we can find a c-cofibrant $\bar{Y}$ together with a weak equivalence $\bar{Y} \xrightarrow{\sim} Y$. By inverting weak equivalences, we obtain an isomorphism $\bar{Y} \longrightarrow Z$ in $Ho\,\mathbf{pP}^N(G)$. Since $\bar{Y}$ is strongly cofibrant, we can lift this isomorphism to a weak equivalence $\bar{Y} \longrightarrow Z$ in $\mathbf{P}^N(G)$.

We have constructed a chain of weak equivalences $Y \longleftarrow \bar{Y} \longrightarrow Z$. All three spaces are weakly cofibrant, hence $\Sigma^k Y \longleftarrow \Sigma^k \bar{Y} \longrightarrow \Sigma^k Z$ is a chain of weak equivalences. But if k is large, the object on the left is finitely dominated by hypothesis, and the previous case asserts that $\Sigma^k(Z)$ is finitely dominated. □

Lemma 3.1.5.

(1) Let ? denote any of the subscripts f, hf, fd or sfd. Suppose $Z \xleftarrow{\alpha} Y \xrightarrow{\gamma} P$ is a diagram in $\mathbf{P}^N_?(G)$, and the map $Y \longrightarrow P$ is an f-cofibration (2.4.4). Then the pushout exists in $\mathbf{P}^N_?(G)$, and the map $Z \longrightarrow Z \cup_Y P$ is an f-cofibration.

(2) If $Z \longleftarrow Y \longrightarrow P$ is a diagram in $\overline{\mathbf{P}}^n_{sfd}(G)$ and the map $Y \longrightarrow P$ is a weak cofibration (2.4.6), then the pushout exists in $\overline{\mathbf{P}}^n_{sfd}(G)$, and the map $Z \longrightarrow Z \cup_Y P$ is a weak cofibration.

Proof. *Ad* (1). In all cases we can form the pushout in the ambient category $\mathbf{pP}^N(G)$. Since the class of cofibrations in $(G \wedge M^A)$-$k\,Top_*$ is closed under cobase changes, the map $Z \longrightarrow Z \cup_Y P$ is an f-cofibration; in particular, the pushout is locally cofibrant.

We have to show that the result satisfies the sheaf condition. By 2.3.4 it suffices to show that for all inclusions $\sigma : A \longrightarrow B$ of non-empty subsets of N, the map

$$(Z \cup_Y P)^\sharp_\sigma : \ (Z \cup_Y P)^A[t_B^{-1}] \longrightarrow (Z \cup_Y P)^B \tag{$*$}$$

is a weak equivalence. But since pushouts are calculated pointwise in $\mathbf{pP}^N(G)$ and $\cdot[t_B^{-1}]$ commutes with pushouts (this functor has a right adjoint), it is isomorphic to the map induced on pushouts of top and bottom row

$$\begin{tikzcd}
Z^A[t_B^{-1}] \arrow[d, "Z^\sharp_\sigma"'] & Y^A[t_B^{-1}] \arrow[l] \arrow[r] \arrow[d, "Y^\sharp_\sigma"'] & P^A[t_B^{-1}] \arrow[d, "P^\sharp_\sigma"] \\
Z^B & Y^B \arrow[l] \arrow[r] & P^B
\end{tikzcd}$$

where the two horizontal maps on the right are cofibrations since $\cdot[t_A^{-1}]$ preserves cofibrations by 2.2.8 (2). All vertical maps are weak equivalences since Z, Y and P are sheaves. Hence the gluing lemma (in $(G \wedge M^B)$-$k\,Top_*$) asserts that $(*)$ is indeed a weak equivalence.

It remains to show that $Z \cup_Y P$ satisfies the correct finiteness condition.

Locally finite objects. For each $A \subseteq N$, we have $(Z \cup_Y P)^A = Z^A \cup_{Y^A} P^A$, and the map $Y^A \longrightarrow P^A$ is a cofibration in $\mathbf{C}_f(M^A)$. But this category is closed under pushouts of this kind. Hence $Z \cup_Y P$ is locally finite.

Homotopy finite objects. This is similar to the previous case.

Finitely dominated objects. We show first that we may assume, without loss of generality, that Z, Y and P are strongly cofibrant. Choose a strongly cofibrant sheaf $\widetilde{Y}$ and a weak equivalence $\widetilde{Y} \longrightarrow Y$ (by factoring the unique map $* \longrightarrow Y$ as a c-cofibration followed by an acyclic c-fibration). Then by factoring the composite maps $\widetilde{Y} \longrightarrow Y \longrightarrow P$ and $\widetilde{Y} \longrightarrow Y \longrightarrow Z$, we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{Z} \arrow[d] & \widetilde{Y} \arrow[l] \arrow[r] \arrow[d] & \widetilde{P} \arrow[d] \\
Z & Y \arrow[l] \arrow[r] & P
\end{tikzcd}$$

where the upper horizontal maps are c-cofibrations (and, in particular, the objects $\widetilde{Z}$ and $\widetilde{P}$ are strongly cofibrant), and all vertical maps are weak equivalences. By the previous lemma, all objects in the upper row are finitely dominated. The category $\mathbf{pP}^N(G)$, equipped with the f-structure, is a proper model category (2.4.4). Hence the gluing lemma asserts that the induced map on pushouts is a weak equivalence. Hence it suffices, by 3.1.4 again, to show that the pushout of the upper row is finitely dominated.

So assume that Z, Y and P are strongly cofibrant. In this case, Y is the retract of a strongly cofibrant, homotopy finite sheaf $\bar{Y}$ by 3.1.3, with section s_Y and retraction r_Y; and similarly for Z and P. We have a commutative diagram

$$\begin{tikzcd}
Z \arrow[d, "s_Z"'] & Y \arrow[l, "\alpha"'] \arrow[r, "\gamma"] \arrow[d, "s_Y"] & P \arrow[d, "s_P"] \\
\bar{Z} \arrow[d, "r_Z"'] & \bar{Y} \arrow[l, "s_Z \circ \alpha \circ r_Y"'] \arrow[r, "s_P \circ \gamma \circ r_Y"] \arrow[d, "r_Y"] & \bar{P} \arrow[d, "r_P"] \\
Z & Y \arrow[l, "\alpha"] \arrow[r, "\gamma"'] & P
\end{tikzcd}$$

with all vertical composites identity maps. The canonical map $i : \bar{Y} \longrightarrow \bar{Y} \cup_Y P$ is an f-cofibration since it is a cobase change of the f-cofibration γ. Factor the induced map $\pi : \bar{Y} \cup_Y P \longrightarrow \bar{P}$ as $\bar{Y} \cup_Y P \xrightarrow{\delta} Q \xrightarrow{\beta} \bar{P}$ through its mapping cylinder $Q := Z_\pi$. Then $\bar{Y} \cup_Y P \longrightarrow Q$ and hence the composite map $\bar{Y} \xrightarrow{i} \bar{Y} \cup_Y P \xrightarrow{\delta} Q$ are f-cofibrations. The object Q is homotopy finite since β is a weak equivalence. Now let $\bar{s}_P$ denote the composite $P \longrightarrow \bar{Y} \cup_Y P \xrightarrow{\delta} Q$, and define $\bar{r}_P$ as the composite $Q \xrightarrow{\beta} \bar{P} \xrightarrow{r_P} P$. We obtain a commutative diagram

$$\begin{tikzcd}
Z \arrow[d, "s_Z"'] & Y \arrow[l, "\alpha"'] \arrow[r, "\gamma"] \arrow[d, "s_Y"] & P \arrow[d, "\bar{s}_P"] \\
\bar{Z} \arrow[d, "r_Z"'] & \bar{Y} \arrow[l, "s_Z \circ \alpha \circ r_Y"'] \arrow[r, "\delta \circ i"] \arrow[d, "r_Y"] & Q \arrow[d, "\bar{r}_P"] \\
Z & Y \arrow[l, "\alpha"] \arrow[r, "\gamma"'] & P
\end{tikzcd}$$

of the same kind as above but with all horizontal maps on the right f-cofibrations. By functoriality of pushouts, it follows that $Z \cup_Y P$ is a retract of $\bar{Z} \cup_{\bar{Y}} Q$. But the latter object is homotopy finite by the above which proves the assertion.

Stably finitely dominated objects. Choose a number k such that $\Sigma^k Z$, $\Sigma^k Y$ and $\Sigma^k P$ are finitely dominated objects. Since suspension preserves f-cofibrations and commutes with pushouts, we know by the previous case that $\Sigma^k(Z \cup_Y P) \cong \Sigma^k Z \cup_{\Sigma^k Y} \Sigma^k P$ is finitely dominated.

Ad (2). Again the colimit can be computed in $\mathbf{pP}^N(G)$. Since the class of cofibrations in G-$k\,Top_*$ is closed under cobase changes, the canonical map $Z \longrightarrow Z \cup_Y P$ is a weak cofibration; in particular, the pushout is weakly cofibrant.

We have to show that $Z \cup_Y P$ is a sheaf. Let $\sigma\colon A \longrightarrow B$ denote an inclusion of non-empty subsets of N. Choose a cofibrant replacement $y\colon \bar{Y}^A \longrightarrow Y^A$ of Y^A, *i.e.*, an object $\bar{Y}^A \in \mathbf{C}(G\wedge M^A)$ and a weak equivalence y. Factor the composite $\bar{Y}^A \xrightarrow{y} Y^A \longrightarrow Z^A$ in $(G\wedge M^A)$-$k\,Top_*$ as a cofibration $\bar{Y}^A \longrightarrow \bar{Z}^A$ followed by a weak equivalence $\bar{Z}^A \xrightarrow{z} Z^A$. Similarly, factor the composite $\bar{Y}^A \xrightarrow{y} Y^A \longrightarrow P^A$ in $(G\wedge M^A)$-$k\,Top_*$ as a cofibration $\bar{Y}^A \longrightarrow \bar{P}^A$ followed by a weak equivalence $\bar{P}^A \xrightarrow{p} P^A$. We have constructed a commutative diagram

$$\begin{tikzcd}
\bar{Z}^A \arrow[d, "z"'] & \bar{Y}^A \arrow[l] \arrow[r] \arrow[d, "y"'] & \bar{P}^A \arrow[d, "p"] \\
Z^A & Y^A \arrow[l] \arrow[r] & P^A
\end{tikzcd}$$

where both upper horizontal maps are cofibrations in $(G\wedge M^A)$-$k\,Top_*$ (and consequently in G-$k\,Top_*$ by 1.2.1 (2)), the lower right horizontal map is a cofibration in G-$k\,Top_*$, and all vertical maps are weak equivalences. Since $\bar{Y}^A$ is cofibrant as a $(G\wedge M^A)$-space, so are $\bar{Z}^A$ and $\bar{P}^A$ as well as their pushout. By the gluing lemma (in G-$k\,Top_*$), the induced map $\bar{Z}^A \cup_{\bar{Y}^A} \bar{P}^A \longrightarrow Z^A \cup_{Y^A} P^A$ is a weak equivalence, *i.e.*, we have constructed a cofibrant replacement of the target of this map.

To show that $Z \cup_Y P$ is a sheaf, it suffices to prove that the map

$$r\colon \Big(\bar{Z}^A \cup_{\bar{Y}^A} \bar{P}^A\Big)[t_B^{-1}] \longrightarrow Z^B \cup_{Y^B} P^B$$

(adjoint to the composite $\bar{Z}^A \cup_{\bar{Y}^A} \bar{P}^A \longrightarrow Z^A \cup_{Y^A} P^A \longrightarrow Z^B \cup_{Y^B} P^B$) is a weak equivalence. For this, we will use the gluing lemma again. By composing the maps y, z and p constructed above with structure maps of Y, Z and P and passing to adjoints, we obtain a commutative diagram

$$\begin{tikzcd}
\bar{Z}^A[t_B^{-1}] \arrow[d] & \bar{Y}^A[t_B^{-1}] \arrow[l] \arrow[r] \arrow[d] & \bar{P}^A[t_B^{-1}] \arrow[d] \\
Z^B & Y^B \arrow[l] \arrow[r] & P^B
\end{tikzcd}$$

where all vertical arrows are weak equivalences since Y, Z and P are sheaves. The horizontal maps on the right are cofibrations in G-$k\,Top_*$: for the lower map this holds by assumption, for the upper map recall that the functor $\cdot[t_B^{-1}]$ preserves cofibrations by 2.2.8 (2), and every cofibration in $(G\wedge M^B)$-$k\,Top_*$ is a cofibration in G-$k\,Top_*$. Hence the gluing lemma (in G-$k\,Top_*$) asserts that the induced map on pushouts is a weak equivalence. Since the functor $\cdot[t_B^{-1}]$ commutes with colimits, this map is isomorphic to the map r. Hence r is a weak equivalence as required.

It remains to show that $Z \cup_Y P$ satisfies the correct finiteness condition. Choose a locally cofibrant sheaf $\bar{Y}$ and a weak equivalence $\bar{Y} \longrightarrow Y$. Factor the composite maps $\bar{Y} \longrightarrow Y \longrightarrow P$ and $\bar{Y} \longrightarrow Y \longrightarrow Z$ as f-cofibrations followed by weak equivalences. Then we have a commutative diagram

$$\begin{tikzcd} \bar{Z} \arrow[d] & \bar{Y} \arrow[l] \arrow[r] \arrow[d] & \bar{P} \arrow[d] \\ Z & Y \arrow[l] \arrow[r] & P \end{tikzcd}$$

with all vertical maps weak equivalences and all upper horizontal maps f-cofibrations. We can consider this as a diagram in $G\text{-}k\,Top_*^{\langle N\rangle}$ which is a proper model category (we use the f-structure in the sense of 1.3, *i.e.*, we have pointwise cofibrations of G-spaces and pointwise weak equivalences). By the gluing lemma the pushouts of top and bottom row are weakly equivalent. But by 3.1.4, all object in the upper row are stably finitely dominated. Hence so is their pushout by case (1), and another application of 3.1.4 shows that $Z \cup_Y P \in \overline{\mathbf{P}}^n_{sfd}(G)$. □

3.2. Finiteness of Global Sections

Our next step is to prove that the global sections functor preserves finiteness in a suitable sense. This is our program:

- Define globally finite sheaves and show that they have stably finitely dominated global sections.
- Show that each finite $(G \wedge M^{\{n\}})$-equivariant space can be used for building a globally finite sheaf (with $\{n\}$-component the given space).
- Show that each locally finite sheaf in $\mathbf{P}^n(G)$ maps into a globally finite sheaf such that the $\{n\}$-component is the identity map.
- Show that the homotopy cofibre of this map of sheaves "comes from one dimension below" and use this for induction.

Definition 3.2.1. For $j \in \mathbf{Z}$ and $k \in \mathbf{N}$, a *j-twisted k-cell* is the sheaf $\psi_j(\Delta^k_+ \wedge G)$; its boundary is given by $\psi_j(\partial\Delta^k_+ \wedge G)$ with the obvious inclusion into the cell. A sheaf $Y \in \mathbf{P}^N(G)$ is *globally finite* if it can be obtained from the zero sheaf by attaching finitely many twisted cells (not necessarily in order of increasing dimension).

Remark 3.2.2. Suppose $Y \in \mathbf{P}^N(G)$ is globally finite, and let $\sigma : A \longrightarrow B$ denote an inclusion of non-empty subsets of N.

(1) The sheaf Y is locally finite and locally cofibrant.
(2) The structure maps $Y^\flat_\sigma : Y^A \longrightarrow Y^B$ are injective.
(3) The structure maps $Y^\sharp_\sigma : Y^A[t_B^{-1}] \longrightarrow Y^B$ are isomorphisms.
(4) If Y is globally finite, so are $\theta_k(Y)$ and $\vartheta_k(Y)$.

Proof. *Ad (1).* This is true since attaching a twisted cell, considered on A-components, amounts to attaching a free $(G{\wedge}M^A)$-equivariant cell. Hence the A-component is an object of $\mathbf{C}_f(G{\wedge}M^A)$.

Ad (2). By induction on the number of cells, it suffices to show that injectivity of the structure maps is preserved by attachment of a single cell. Suppose Z is obtained from Y by attaching one single cell. The structure map $Z^\flat_\sigma : Z^A \longrightarrow Z^B$ is then the map induced on pushouts of top and bottom row of the following diagram:

$$\begin{tikzcd} Y^A \arrow[d, "Y^\flat_\sigma"'] & \psi_j(\partial\Delta^k_+ \wedge G)^A \arrow[l] \arrow[r] \arrow[d] & \psi_j(\Delta^k_+ \wedge G)^A \arrow[d] \\ Y^B & \psi_j(\partial\Delta^k_+ \wedge G)^B \arrow[l] \arrow[r] & \psi_j(\Delta^k_+ \wedge G)^B \end{tikzcd}$$

The left vertical map is injective by induction hypothesis, and the right square has the property that the induced map from the pushout into the terminal vertex is injective. The induced map on pushouts can then be factored as

$$\begin{aligned} & Y^A \cup_{\psi_j(\partial\Delta^k_+\wedge G)^A} \psi_j(\Delta^k_+ \wedge G)^A \\ & \longrightarrow Y^B \cup_{\psi_j(\partial\Delta^k_+\wedge G)^A} \psi_j(\Delta^k_+ \wedge G)^A \\ & \xrightarrow{\cong} Y^B \cup_{\psi_j(\partial\Delta^k_+\wedge G)^B} \psi_j(\partial\Delta^k_+ \wedge G)^B \cup_{\psi_j(\partial\Delta^k_+\wedge G)^A} \psi_j(\Delta^k_+ \wedge G)^A \\ & \longrightarrow Y^B \cup_{\psi_j(\partial\Delta^k_+\wedge G)^B} \psi_j(\Delta^k_+ \wedge G)^B \end{aligned}$$

and all maps are injective since they are cobase changes of injective maps (this is a set-theoretic version of REEDY's patching lemma).

Ad (3). As in (2) it suffices to consider the attachment of a single cell. Suppose Z is obtained from Y by attaching one single cell. Since passing to the adjoint map is compatible with pushouts, the map $Z^\sharp_\sigma : Z^A \longrightarrow Z^B$ is the map induced on pushouts of top and bottom row of the following diagram:

$$\begin{tikzcd} Y^A[t_B^{-1}] \arrow[d, "Y^\sharp_\sigma"'] & \psi_j(\partial\Delta^k_+ \wedge G)^A[t_B^{-1}] \arrow[l] \arrow[r] \arrow[d] & \psi_j(\Delta^k_+ \wedge G)^A[t_B^{-1}] \arrow[d] \\ Y^B & \psi_j(\partial\Delta^k_+ \wedge G)^B \arrow[l] \arrow[r] & \psi_j(\Delta^k_+ \wedge G)^B \end{tikzcd}$$

But all vertical maps are isomorphisms (for $Y^\sharp_\sigma$ by induction hypothesis, for the other two by the isomorphism $M^A[t_B^{-1}] \cong M^B$ and the definition of ψ_j), hence the induced map is an isomorphism.

Ad (4). Recall that θ_k and ϑ_k are equivalences of categories (2.6.6 (2)). Hence they commute with cell attachment. □

Lemma 3.2.3. *If $Y \in \mathbf{P}^N(G)$ is a globally finite sheaf, the space $\Gamma(Y)$ is stably finitely dominated as a G-space.*

Proof. For $Y = *$ we have nothing to prove. Now assume Y is obtained from Z by attaching a cell, *i.e.*, $Y = Z \cup_{\psi_j(\partial\Delta^k_+ \wedge G)} \psi_j(\Delta^k_+ \wedge G)$ for some $j \in \mathbf{Z}$ and $k \geq 0$. Assume by induction that $\Gamma(Z)$ is a stably finitely dominated space. Since Γ commutes with pushouts (1.3.5 (3)) we know that the canonical map

$$\Gamma(Z) \cup_{\Gamma(\psi_j(\partial\Delta^k_+ \wedge G))} \Gamma\bigl(\psi_j(\Delta^k_+ \wedge G)\bigr) \longrightarrow \Gamma(Y)$$

is an isomorphism. All spaces on the left are known to be stably finitely dominated (use the induction hypothesis for $\Gamma(Z)$, 2.7.4 and 2.7.5 for the other two spaces), and the map $\Gamma\bigl(\psi_j(\partial\Delta^k_+ \wedge G)\bigr) \longrightarrow \Gamma\bigl(\psi_j(\Delta^k_+ \wedge G)\bigr)$ is a cofibration since the functor $\cdot \wedge G\colon k\mathit{Top}_* \longrightarrow G\text{-}k\mathit{Top}_*$ maps cofibrations in $k\mathit{Top}_*$ to cofibrations in $G\text{-}k\mathit{Top}_*$ (apply 1.2.2 (2)), ψ_j maps cofibrations to f-cofibrations (apply 1.2.2 (2) componentwise), and Γ preserves cofibrations by 1.3.5 (1). Hence the pushout is stably finitely dominated. □

Lemma 3.2.4. *Assume $T \in \mathbf{C}(G \wedge M^{\{n\}})$ is finite. Then there exists a globally finite sheaf $Y \in \mathbf{P}^n(G)$ with $Y^{\{n\}} = T$.*

Proof. We proceed by induction on the number of cells of T. For $T = *$ choose $Y = *$. Now assume $T = S \cup_{\partial C} C$ where $C = \Delta^\ell_+ \wedge G \wedge M^{\{n\}}$ is an equivariant cell. By induction we have a globally finite sheaf $Z \in \mathbf{P}^N$ with $Z^{\{n\}} = S$.

The boundary of C has a compact subspace $\partial C_0 := \partial\Delta^\ell_+ \wedge \{0,1\} \wedge \{0\}_+$. Let $\alpha^{\{n\}} : \partial C_0 \longrightarrow Z^{\{n\}}$ denote the restriction of the attaching map. Fix $i \in [n] \setminus \{n\}$. By 3.2.2 (3) the maps $Z^{\{i\}}[t_n^{-1}] \longrightarrow Z^{\{i,n\}}$ are isomorphisms. Identify the target with the colimit of the (second) telescope construction (2.2.7 (2)) for inverting the action of t_n on $Z^{\{i\}}$. Since all maps in the telescope are cofibrations in $k\mathit{Top}_*$ by 1.2.4 and 1.2.1 (3), the composite

$$\partial C_0 \xrightarrow{\alpha^{\{n\}}} Z^{\{n\}} \longrightarrow Z^{\{i,\, n\}}$$

factors as $\partial C_0 \xrightarrow{\alpha^{\{i\}}} Z^{\{i\}} \xrightarrow{t_i^{k_i} t_n^{-k_i}} Z^{\{i,\, n\}}$ for some $k_i \geq 0$ (smallness of compact spaces; this is similar to 2.2.9). Since all k_i can be enlarged, we may assume that all k_i have the same value $k \geq 0$.

We claim that for all $A \subseteq [n]$ and $i, j \in A$ the diagram

$$\begin{tikzcd} \partial C_0 \arrow[r, "\alpha^{\{i\}}"] \arrow[d, "\alpha^{\{j\}}"'] & Z^{\{i\}} \arrow[d, "t_i^k t_a^{-k}"] \\ Z^{\{j\}} \arrow[r, "t_j^k t_a^{-k}"'] & Z^A \end{tikzcd} \qquad (*)$$

commutes where $a := \max(A)$; lower and right maps are structure maps in $\vartheta_k(Z)$ (cf. 2.6.4). To see this it is sufficient to show that both maps from upper left to lower right are coequalized by the map $Z^A \xrightarrow{t_a^k t_n^{-k}} Z^{[n]}$ since the latter is injective by 3.2.2 (2, 4) (recall that Z and hence $\vartheta_k(Z)$ is globally finite by induction hypothesis). So consider the following diagram:

$$\begin{tikzcd}
\partial C_0 \arrow[r, "\alpha^{\{i\}}"] \arrow[d, "\alpha^{\{n\}}"'] & Z^{\{i\}} \arrow[r, "t_i^k t_a^{-k}"] \arrow[d, "t_i^k t_n^{-k}"'] & Z^A \arrow[d, "t_a^k t_n^{-k}"] \\
Z^{\{n\}} \arrow[r] & Z^{\{i,\, n\}} \arrow[r] & Z^{[n]}
\end{tikzcd}$$

The left square commutes by construction of $\alpha^{\{i\}}$, the square on the right commutes since it consists of structure maps of $\vartheta_k(Z)$. The composition of the lower maps is the structure map $Z^{\{n\}} \longrightarrow Z^{[n]}$. Thus the composite $\partial C_0 \longrightarrow Z^{[n]}$ from upper left to lower right is independent of i which proves commutativity of $(*)$.

Define $\alpha^A : \partial C_0 \longrightarrow Z^A$ as the unique map from upper left to lower right in diagram $(*)$. By virtually the same argument as before (map everything into $Z^{[n]}$) we can show that the α^A are compatible with the structure maps of $\vartheta_k(Z)$, *i.e.*, they assemble to a map $\partial C_0 \longrightarrow \varprojlim V\big(\vartheta_k(Z)\big)$, where V denotes the forgetful functor $\mathbf{pP}^N(G) \longrightarrow G\text{-}k\,Top_*^{\langle N\rangle_0}$. Hence we obtain, by passing to the adjoint map, a morphism $f : \psi_{-k}(\partial C_0) \longrightarrow Z$ (note that 2.6.10 applies since θ_k and ϑ_k are isomorphic functors, cf. 2.6.6 (3)), and by construction $f^{\{n\}}$ coincides with the given attaching map in T. Define Y by attaching an ℓ-cell with twist $-k$ to Z. Then Y is globally finite. On $\{n\}$-components, attaching the twisted cell amounts to attaching a free $(G \wedge M^{\{n\}})$-equivariant cell to $Z^{\{n\}} = S$ along the given attaching map, hence $Y^{\{n\}} = T$. □

Lemma 3.2.5. *Let Y and Z be objects of $\mathbf{P}^n(G)$. Suppose Z is globally finite, and suppose Y is locally finite and strongly cofibrant. Let $g : Y^{\{n\}} \longrightarrow Z^{\{n\}}$ be a given $(G \wedge M^{\{n\}})$-equivariant map. Then there exists an integer $k \geq 0$ and a morphism $f : Y \longrightarrow \vartheta_k(Z)$ with $f^{\{n\}} = g$.*

Proof. The proof consists of three steps: first we construct maps $f^A : Y^A \longrightarrow Z^A$ for $A \subseteq [n]$ with $n \in A$ which are compatible with the structure maps of Y and Z. (In other words: if Y and Z are considered as cubical diagrams, we construct a natural transformation from the n-face of Y to the n-face of Z.) The second step is to extend this partial map to a map of $(n+1)$-cubes (we have to twist Z to do this). In the last step, we check that the components f^A constructed before fit together.

We would like to point out that the lemma might fail if Y is not a sheaf: the induction in step 1 relies on the fact that the sheaf condition is satisfied for the components Y^A with $n \in A$.

Step 1. We proceed by induction on $s := \#A$. Start by defining $f^{\{n\}} := g$.

If $A = \{j,\ n\}$, we claim that the map $Y^{\{n\}}[t_j^{-1}] \longrightarrow Y^{\{j,n\}}$ is an acyclic cofibration. Indeed, by 2.4.10 (4) we know that the map $L^{+\{n\}}_{\{j,n\}} Y \longrightarrow Y^{\{j,n\}}$ is a cofibration.

But $L^{+\{n\}}_{\{j,n\}}Y = Y^{\{n\}}[t_j^{-1}]$ by definition of restricted latching spaces, and the map is a $\sharp$-type structure map of Y. Since Y is a sheaf, it is a weak equivalence by 2.3.4. Hence we can choose a (dotted) lift $f^{\{j,n\}}$ in the following diagram:

$$\begin{tikzcd}
Y^{\{n\}}[t_j^{-1}] \arrow[r, "f^{\{n\}}[t_j^{-1}]"] \arrow[d, "Y^\sharp_\sigma"'] & Z^{\{n\}}[t_j^{-1}] \arrow[r, "Z^\sharp_\sigma"] & Z^{\{j,n\}} \arrow[d] \\
Y^{\{j,n\}} \arrow[urr, dotted, "f^{\{j,n\}}"'] \arrow[rr] & & *
\end{tikzcd}$$

(Here σ denotes the inclusion $\{n\} \longrightarrow \{j,\ n\}$.) This lift is compatible with the structure maps of Y and Z: by passing to adjoint maps, we obtain from the above diagram the following commutative square:

$$\begin{tikzcd}
Y^{\{n\}} \arrow[r, "f^{\{n\}}"] \arrow[d, "Y^\flat_\sigma"'] & Z^{\{n\}} \arrow[d, "Z^\flat_\sigma"] \\
Y^{\{j,n\}} \arrow[r, "f^{\{j,n\}}"] & Z^{\{j,n\}}
\end{tikzcd}$$

Now assume $s = \#A \geq 3$. We give a formulation of the induction hypothesis:

(i)$_s$ We have compatible maps f^B for all proper subsets B of A containing n; *i.e.*, if σ denotes an inclusion $B \longrightarrow C$ of proper subsets of A with $n \in B$, the following diagram commutes:

$$\begin{tikzcd}
Y^B \arrow[r, "f^B"] \arrow[d, "Y^\flat_\sigma"'] & Z^B \arrow[d, "Z^\flat_\sigma"] \\
Y^C \arrow[r, "f^C"'] & Z^C
\end{tikzcd}$$

(ii)$_s$ If B is a proper subset of C with $n \in B$, and C is a proper subset of A with $\#C = \#A - 1 = s-1$, the canonical map $L^{+B}_C Y \longrightarrow Y^C$ is an acyclic cofibration, and the source is cofibrant as an object of $(G \wedge M^C)$-$k\,Top_*$.

To start the induction, we still have to check condition (ii)$_3$, *i.e.*, the case $\#A = 3$. But then necessarily $B = \{n\}$ and $C = \{j,\ n\}$ for some j, and we have checked above that the map $Y^{\{n\}}[t_j^{-1}] \longrightarrow Y^{\{j,n\}}$ is an acyclic cofibration. Moreover, the source of this map is cofibrant in $(G \wedge M^{\{j,n\}})$-$k\,Top_*$ since $Y^{\{n\}}$ is cofibrant in $(G \wedge M^{\{n\}})$-$k\,Top_*$ (apply 2.2.8 (2)).

Assume now that conditions (i)$_s$ and (ii)$_s$ are satisfied. Consider the following diagram:

$$\begin{tikzcd}
L^{+\{n\}}_A Y \arrow[r] \arrow[d] & L^{+\{n\}}_A Z \arrow[r] & Z^A \arrow[d] \\
Y^A \arrow[urr, dotted, "f^A"'] \arrow[rr] & & *
\end{tikzcd} \tag{$**$}$$

The left arrow in the first row is induced by the maps f^B of $(\mathrm{i})_s$. The second map in the first row and the left vertical map are the canonical maps of 2.4.10 (4). In particular, the left vertical map is a cofibration. To construct the dotted lift, it thus suffices to show that the left vertical map is a weak equivalence.

Choose a filtration of A

$$\{n\} = C_1 \subset C_2 \subset \ldots \subset C_{s-1} \subset C_s = A$$

with $\#C_i = i$. By an iterated application of 2.4.10 (2) we realize that the map $L_A^{+C_{s-1}} Y \longrightarrow Y^A$ factors as

$$L_A^{+C_{s-1}} Y \longrightarrow L_A^{+C_{s-2}} Y \longrightarrow \ldots \longrightarrow L_A^{+C_1} Y \longrightarrow Y^A . \qquad (*)$$

By 2.4.10 (4) the last map is a cofibration. All the other maps in this factorization are acyclic cofibrations: fix i with $1 < i \leq s-1$, and write $C_i = C_{i-1} \amalg \{j\}$. By induction hypothesis $(\mathrm{ii})_s$, the map $L_{A\setminus\{j\}}^{+C_{i-1}} Y \longrightarrow Y^{A\setminus\{j\}}$ is an acyclic cofibration, hence so is $L_{A\setminus\{j\}}^{+C_{i-1}} Y\,[t_j^{-1}] \longrightarrow Y^{A\setminus\{j\}}\,[t_j^{-1}]$ (lemma 2.2.8 (2) applies since the source is cofibrant by induction hypothesis). By 2.4.10 (3) we have a pushout square

$$\begin{array}{ccc} L_{A\setminus\{j\}}^{+C_{i-1}} Y\,[t_j^{-1}] & \longrightarrow & Y^{A\setminus\{j\}}\,[t_j^{-1}] \\ \downarrow & & \downarrow \\ L_A^{+C_i} Y & \longrightarrow & L_A^{+C_{i-1}} Y \end{array}$$

which shows that the lower map (appearing also in $(*)$) is an acyclic cofibration.

Write $A = C_{s-1} \amalg \{j\}$. Then the leftmost space in $(*)$ is $L_A^{+C_{s-1}} Y = Y^{A\setminus\{j\}}[t_j^{-1}]$ by definition of restricted latching spaces. Since $Y^{A\setminus\{j\}}$ is cofibrant, this proves that all spaces appearing in $(*)$ are cofibrant. Moreover, the composite map $(*)$ is a $\sharp$-type structure map of Y, hence a weak equivalence. This shows that the rightmost map of $(*)$ is a weak equivalence.

The above arguments apply for all A with s elements, and for all filtrations. This proves $(\mathrm{ii})_{s+1}$.

We also have shown that the left vertical map in $(**)$ is an acyclic cofibration, hence the lift f^A exists. This map is compatible with the f^B constructed earlier since the structure maps of Y and Z are encoded in the restricted latching spaces. Hence $(\mathrm{i})_{s+1}$ holds.

Step 2. We commence now with the construction of maps f^A for $A \subseteq [n-1]$. Given such a set A, consider the composite map $Y^A \longrightarrow Y^{A\amalg\{n\}} \xrightarrow{f^{A\amalg\{n\}}} Z^{A\amalg\{n\}}$ where the second map is as constructed in step 1, and the first map is a $\flat$-type structure map of Y. Since Z is globally finite, we have $Z^{A\amalg\{n\}} \cong Z^A[t_n^{-1}]$, and finiteness of Y^A guarantees that we can factor this map as $Y^A \xrightarrow{f^A} Z^A \xrightarrow{t_m^{k_A}\cdot t_n^{-k_A}} Z^{A\amalg\{n\}}$ with $m = \max(A)$ and some $k_A \geq 0$ (apply 2.2.10). Since all k_A may be enlarged independently, we may assume that all k_A, for the various $A \subseteq [n-1]$, have the same value k.

Step 3. We claim that all the different maps f^A constructed in steps 1 and 2 assemble to a map of diagrams $Y \longrightarrow \vartheta_k(Z)$. So assume $B = A \amalg \{j\} \subseteq [n]$. We have to show that the square

$$\begin{tikzcd} Y^A \arrow[r, "f^A"] \arrow[d] & Z^A \arrow[d, "t_m^k t_\ell^{-k}"] \\ Y^B \arrow[r, "f^B"] & Z^B \end{tikzcd} \tag{$*$}$$

commutes (where $m = \max(A)$, $\ell = \max(B)$, and the right vertical arrow is the structure map of $\vartheta_k(Z)$).

(a) If $n \in A$ this follows from the construction of the f^A in step 1 of this proof since $m = \ell$ in this case, and the right vertical map in $(*)$ coincides with the ($\flat$-type) structure map in Z.

(b) If $j = n$ (and, consequently, $\ell = n$) this follows from the construction of f^A in step 2.

(c) Now assume $n \notin B$, and consider the following cube:

$$\begin{tikzcd} & Y^{A\amalg\{n\}} \arrow[rr, "f^{A\amalg\{n\}}"] \arrow[dd] & & Z^{A\amalg\{n\}} \arrow[dd] \\ Y^A \arrow[ur] \arrow[rr, "f^A"] \arrow[dd] & & Z^A \arrow[ur, "t_m^k t_n^{-k}"] \arrow[dd, "t_m^k t_\ell^{-k}"] & \\ & Y^{B\amalg\{n\}} \arrow[rr, "f^{B\amalg\{n\}}"] & & Z^{B\amalg\{n\}} \\ Y^B \arrow[ur] \arrow[rr, "f^B"] & & Z^B \arrow[ur, "t_\ell^k t_n^{-k}"] & \end{tikzcd}$$

The front face is the diagram $(*)$. The back face of this cube is commutative by (a), for upper and lower face commutativity follows from (b). The left and right face commute by definition of the structure maps of Y and $\vartheta_k(Z)$, respectively. This shows that the two composite maps from initial to terminal vertex, given by $Y^A \xrightarrow{f^A} Z^A \xrightarrow{t_m^k t_\ell^{-k}} Z^B \xrightarrow{t_\ell^k t_n^{-k}} Z^{B\amalg\{n\}}$ and $Y^A \longrightarrow Y^B \xrightarrow{f^B} Z^B \xrightarrow{t_\ell^k t_n^{-k}} Z^{B\amalg\{n\}}$, are equal. But the last map in these compositions is injective since $\vartheta_k(Z)$ is globally finite (3.2.2 (2, 4)). Hence the front face commutes as well.

$\square$

Proposition 3.2.6. *(Finiteness of global sections.)*

The global sections functor Γ *maps objects of* $\overline{\mathbf{P}}^N_{sfd}(G)$ *to stably finitely dominated* G*-spaces.*

The reader may wonder why we focus on stably finitely dominated objects. However, the proof will show that even for a locally finite sheaf Y, the space $\Gamma(Y)$ is, in general, only stably finitely dominated, not finitely dominated.

Proof of 3.2.6. Let P denote a weakly cofibrant object of $\mathbf{P}^N_{sfd}(G)$. By factoring the map $* \longrightarrow Y$ we find a strongly cofibrant sheaf Z and a weak equivalence $Z \longrightarrow P$. Since Z is stably finitely dominated (3.1.4), we know that there exists $k \geq 0$ such that $\Sigma^k(Z)$ is a retract of a strongly cofibrant, homotopy finite sheaf Q (3.1.3). By 3.1.2 (2) we find a strongly cofibrant, locally finite sheaf Y and a weak equivalence $Y \longrightarrow Q$. Now Γ preserves weak equivalences between weakly cofibrant objects (by 1.3.5 (2)) and retractions (by functoriality), and commutes with suspensions. Hence it suffices to show that a strongly cofibrant, locally finite object Y is mapped to a stably finitely dominated space.

Assume $N = [n]$. We proceed by induction on the dimension; for $n = 0$, nothing has to be shown (since $\overline{\mathbf{P}}^0_{sfd}(G) = \mathbf{C}_{sfd}(G)$ and $\Gamma = \mathrm{id}$).

Now assume the statement is true for $n-1$, and let $Y \in \check{\mathbf{P}}^n_f(G)$ be given. Choose a globally finite object Z with $Z^{\{n\}} = Y^{\{n\}}$ (3.2.4). By 3.2.5, we can extend the identity map of $Y^{\{n\}}$ to a morphism $f: Y \longrightarrow \vartheta_k(Z)$ for some $k \geq 0$ such that $f^{\{n\}} = \mathrm{id}_{Y^{\{n\}}}$. For $A \subseteq [n]$ with $n \in A$, let $\sigma : \{n\} \longrightarrow A$ denote the inclusion. Consider the following diagram:

$$\begin{array}{ccccc}
Y^{\{n\}}[t_A^{-1}] & \xrightarrow{\mathrm{id}} & \vartheta_k(Z)^{\{n\}}[t_A^{-1}] & = Y^{\{n\}}[t_A^{-1}] \\
\downarrow{\scriptstyle Y^\sharp_\sigma} & & \downarrow{\scriptstyle \vartheta(Z)^\sharp_\sigma} & \\
Y^A & \xrightarrow{f^A} & \vartheta_k(Z)^A &
\end{array}$$

The vertical maps are the $\sharp$-type structure maps of Y and $\vartheta_k(Z)$, respectively. They are weak equivalences since Y and $\vartheta_k(Z)$ are locally cofibrant sheaves (2.3.4). Hence the components f^A with $n \in A$ are weak equivalences.

Let C_f denote the (pointwise) mapping cone of f (*i.e.*, the space C^A_f is given by the homotopy cofibre of the map f^A). Since Γ commutes with pushouts and preserves cofibrations, we have a cofibration sequence $\Gamma(Y) \xrightarrow{\Gamma(f)} \Gamma\big(\vartheta_k(Z)\big) \longrightarrow \Gamma(C_f)$. By 3.2.3 the space $\Gamma\big(\vartheta_k(Z)\big)$ is stably finitely dominated. Hence it suffices to show that the space on the right is stably finitely dominated. (By using a PUPPE sequence type argument, one can show that if two out of three spaces in a cofibration sequence are stably finitely dominated, so is the third. We omit the details.)

Define $\bar{Y} \in \mathbf{P}^{n-1}(G)$ as the object $A \mapsto C^A_f$ where the spaces C^A_f are equipped with the restricted action (restriction along the inclusions $M^A_{n-1} \subset M^A_n$). (If we consider C_f as a diagram having the shape of an n-simplex, the presheaf $\bar{Y}$ is just the n-th face of C_f.) There is an obvious map $o\colon C_f \longrightarrow \zeta_n(\bar{Y})$ of diagrams in G-$k\mathit{Top}_*$ (not of sheaves) given by the identity and constant maps to the basepoint, respectively.

For $n = 2$, we have the following picture:

$$\begin{tikzcd}[column sep=small]
 & & C_f^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & C_f^{\{0,2\}} \arrow[dr] & & C_f^{\{1,2\}} \arrow[dl] & \\
 & & C_f^{\{0,1,2\}} & & \\
C_f^{\{0\}} \arrow[uur] \arrow[rr] & & C_f^{\{0,1\}} \arrow[u] & & C_f^{\{1\}} \arrow[ll] \arrow[uul]
\end{tikzcd}
\xrightarrow{\ o\ }
\begin{tikzcd}[column sep=small]
 & & * \arrow[dl] \arrow[dr] & & \\
 & * \arrow[dr] & & * \arrow[dl] & \\
 & & * & & \\
C_f^{\{0\}} \arrow[uur] \arrow[rr] & & C_f^{\{0,1\}} \arrow[u] & & C_f^{\{1\}} \arrow[ll] \arrow[uul]
\end{tikzcd}$$

This map is a weak equivalence in $G\text{-}kTop_*^{\langle N\rangle}$ by construction of f: it is given by identity maps (on A-components with $n \notin A$), or maps between contractible spaces (since f^A is a weak equivalence if $n \in A$ by the above). Moreover, $\bar{Y}$ is a sheaf since C_f is.

Caution: o is not a map in $\mathbf{pP}^N(G)$ since it fails to be equivariant with respect to the M_n^A-actions. For example, in the picture we have t_2 acting trivially on the spaces on the lower face in the target (by definition of extension by zero ζ_n), but acting non-trivially in the source; maps are identities on underlying spaces.

Recall that $\Gamma \circ \zeta_n(\bar{Y}) \cong \Sigma \circ \Gamma(\bar{Y})$ by 2.7.2. Hence the weak equivalence o shows that $\Gamma(C_f)$ is stably finitely dominated if $\bar{Y}$ is.

It was said above that we consider the A-component of $\bar{Y}$ as a $(G \wedge M_{n-1}^A)$-space with the restricted action. However, by its construction as the homotopy cofibre of a map between finite $(G \wedge M_n^A)$-spaces, we know that $\bar{Y}^A = (C_f)^A$ is, in fact, a finite $(G \wedge M_n^A)$-space. Moreover, C_f is a sheaf. Hence if σ denotes the inclusion $A \longrightarrow A \amalg \{n\}$, we have a weak equivalence

$$\bar{Y}^A[t_n^{-1}] = (C_f)^A[t_n^{-1}] \xrightarrow{(C_f)^\sharp_\sigma} (C_f)^{A\amalg\{n\}} \simeq * \,.$$

In this situation, we can apply lemma 2.2.11 to conclude that for each $A \subseteq [n-1]$, the space $\Sigma\bar{Y}^A \vee \bar{Y}^A$ is homotopy finite as a $(G \wedge M_{n-1}^A)$-space, *i.e.*, $\Sigma\bar{Y} \vee \bar{Y}$ is a homotopy finite object of $\mathbf{P}^{n-1}(G)$. This implies that the sheaf $\bar{Y} \in \mathbf{P}^{n-1}(G)$ (which is a retract of $\Sigma\bar{Y} \vee \bar{Y}$) is finitely dominated. In addition, $\bar{Y}$ is certainly weakly cofibrant since C_f is locally cofibrant as an object of $\mathbf{P}^n(G)$ (apply 1.2.1 (2) to the components of $\bar{Y}$). Hence the induction hypothesis applies, asserting that $\Gamma(\bar{Y})$ is stably finitely dominated as claimed. □

3.3. K-Theory Structures

Definition 3.3.1. Let ? denote any of the subscripts f, hf, fd or sfd. A map f in $\mathbf{P}^N_?(G)$ will be called a *cofibration* if it is an f-cofibration (2.4.4) when considered as a map in $\mathbf{pP}^N(G)$. We call f a *weak equivalence* or an *h-equivalence* if it is a weak equivalence in $\mathbf{pP}^N(G)$ (cf. 2.4).

Lemma 3.3.2. *Let ? denote one of the subscripts f, hf, fd or sfd.*

(1) With the above definitions $\mathbf{P}^N_?(G)$ is a category with cofibrations and weak equivalences ([W, 1.2]) satisfying the saturation axiom.

(2) The category $\mathbf{P}^N_?(G)$ has a cylinder functor given by componentwise mapping cylinder construction. The cylinder axiom is satisfied.

Proof. *Ad (1).* Axioms (Cof 1), (Cof 2) and (Weq 1) hold by definition, and (Cof 3) has been checked in 3.1.5 (1). Axiom (Weq 2), the gluing lemma, follows from properness of the f-structure (2.4.4). The saturation axiom holds since $\mathbf{P}^N_?(G)$ is a full subcategory of a model category.

Ad (2). It was checked in 2.4.8 (2) that the mapping cylinder construction provides a cylinder functor for maps between locally cofibrant objects. The projection from the cylinder is a weak equivalence, hence the cylinder axiom holds. It remains to check that the mapping cylinder construction respects finiteness. For $? = f$, this follows from the fact that it preserves finiteness on each component (1.2.3 (2)). In all other cases, the mapping cylinder is weakly equivalent to the target of the map, hence is an object of $\mathbf{P}^N_?(G)$ by 3.1.4. □

Proposition 3.3.3. *The c-cofibrations (2.4.1) and h-equivalences make $\check{\mathbf{P}}^N_f(G)$ into a category with cofibrations and weak equivalences satisfying the saturation axiom. The mapping cylinder construction equips $\check{\mathbf{P}}^N_f(G)$ with a cylinder functor satisfying the cylinder axiom.*

Proof. This is similar to the previous proposition (use 2.4.8 (4) for existence of the cylinder functor). □

We indicate briefly how the different finiteness notions behave in K-theory. However, we will only be concerned with the K-theory of stably finitely dominated objects in the rest of the paper.

Proposition 3.3.4. *(Different models for the K-theory of projective space.)*

(1) The inclusion $\check{\mathbf{P}}^N_f(G) \longrightarrow \mathbf{P}^N_f(G)$ induces an equivalence on $\mathcal{S}_\bullet$-constructions.

(2) The inclusion $\mathbf{P}^N_f(G) \longrightarrow \mathbf{P}^N_{hf}(G)$ induces an equivalence on $\mathcal{S}_\bullet$-constructions.

(3) The inclusion $\mathbf{P}^N_{hf}(G) \longrightarrow \mathbf{P}^N_{fd}(G)$ induces an isomorphism on all K-groups except possibly on K_0.

(4) The inclusion $\mathbf{P}^N_{fd}(G) \longrightarrow \mathbf{P}^N_{sfd}(G)$ induces an equivalence on $\mathcal{S}_\bullet$-constructions.

Proof. To prove (1) and (2) it suffices to check that the inclusions of $\check{\mathbf{P}}^N_f(G)$ into $\mathbf{P}^N_f(G)$ and $\mathbf{P}^N_{hf}(G)$ have the approximation property ([W], 1.6). By definition both inclusions detect weak equivalences, *i.e.*, satisfy (App 1).

Let ? denote the subscript f or hf, and let $f\colon Y \longrightarrow Z$ be a map in $\mathbf{P}^N_?(G)$ where Y is an object of $\in \check{\mathbf{P}}^N_f(G)$. By lemma 3.1.2 there is an object $P \in \check{\mathbf{P}}^N_f(G)$

and a weak equivalence $g\colon P \xrightarrow{\sim} Z$. We can form $g^{-1}\circ f$ in the homotopy category $Ho\,\mathbf{P}^N(G)$, and by [DS], proposition 5.11 we find a map $h\colon Y \longrightarrow P$ in $\mathbf{P}^N(G)$ representing it. Since $g\circ h$ and f have the same image in $Ho\,\mathbf{P}^N(G)$, we infer that $g\circ h$ and f are homotopic. By swallowing this homotopy as in the proof of 3.1.3 we obtain a factorization $Y \longrightarrow Z_h \longrightarrow Z$ of f where Z_h is the mapping cylinder of h. By construction Z_h is locally finite and strongly cofibrant (the mapping cylinder construction is a cylinder functor for $\check{\mathbf{P}}^N_f(G)$ by 2.4.8 (4)) and maps to Z via the weak equivalence $Z_h \xrightarrow{\sim} P \xrightarrow{\sim} Z$. This proves (App 2).

For (3), we follow the argument given in [KVWW, 3.3 (2)]: let Y be an object of $\mathbf{P}^N_{fd}(G)$. Then Y is homotopy finite if and only if for all $i\in N$, the component $Y^{\{i\}}$ is a homotopy finite $(G\wedge M^{\{i\}})$-space (3.1.2). We have maps of abelian groups

$$\pi_1|h\mathcal{S}_\bullet\mathbf{C}_{hf}(G\wedge M^{\{i\}})| \longrightarrow \pi_1|h\mathcal{S}_\bullet\mathbf{C}_{fd}(G\wedge M^{\{i\}})|$$

with cokernel H_i. The space $Y^{\{i\}}$ gives rise to an element of $\pi_1|h\mathcal{S}_\bullet\mathbf{C}_{fd}(M^{\{i\}})|$ by the remarks in [W, p. 329], and $Y^{\{i\}}$ is homotopy finite if and only if the image of this element in H_i is zero. Thus the sheaf Y determines an element of $\prod_{i\in N} H_i$ which is zero if and only if Y is homotopy finite.

By the cofinality theorem of [TT, 1.10.1], we obtain a fibration sequence

$$\Omega|h\mathcal{S}_\bullet\mathbf{P}^N_{hf}(G)| \longrightarrow \Omega|h\mathcal{S}_\bullet\mathbf{P}^N_{fd}(G)| \longrightarrow \prod_{i\in N} H_i$$

where the abelian group on the right has the discrete topology. This proves (3).

For (4), let $\mathcal{C}_k$ denote the full subcategory of $\mathbf{P}^N_{sfd}(G)$ of objects whose k-th suspension is finitely dominated. Then $\mathbf{P}^N_{sfd}(G) = \bigcup_0^\infty \mathcal{C}_k$ and $\mathcal{C}_0 = \mathbf{P}^N_{fd}(G)$. We have inclusion maps $\mathcal{C}_k \longrightarrow \mathcal{C}_{k+1}$ and suspensions $\mathcal{C}_{k+1} \xrightarrow{\Sigma} \mathcal{C}_k$. Since suspension (considered as an endofunctor) induces a self-equivalence of $\mathcal{S}_\bullet$-constructions by [W, 1.6.2], we know that the $\mathcal{S}_\bullet$-constructions of $\bigcup_0^\infty \mathcal{C}_k$ and $\mathcal{C}_0$ are homotopy equivalent. □

We have already introduced the notion of h_S-equivalences (2.8.1). The interplay with h-equivalences allows us to use the fibration theorem of [W]. Fix sets $T\subseteq S\subseteq \mathbf{Z}$.

Definition 3.3.5. We define $\mathbf{P}^{N,T}_{sfd}(G)$ to be the full subcategory of $\mathbf{P}^N_{sfd}(G)$ consisting of the objects Y such that $\Gamma\circ\theta_j(Y)\simeq *$ for all $j\in T$. Similarly we define the category $\overline{\mathbf{P}}^{N,T}_{sfd}(G)$ as the full subcategory of $\overline{\mathbf{P}}^N_{sfd}(G)$ whose objects satisfy $\Gamma\circ\theta_j(Y)\simeq *$ for all $j\in T$.

Lemma 3.3.6. *The axioms for a category with cofibrations and weak equivalences hold for $\mathbf{P}^{N,T}_{sfd}(G)$ with respect to f-cofibrations and h_S-equivalences. The saturation axiom is satisfied. The mapping cylinder construction provides a cylinder functor, and the cylinder axiom holds.*

Proof. Axioms (Cof 1), (Cof 2) and (Weq 1) hold by definition. The saturation axiom follows from the model category axioms. To prove (Cof 3), suppose we have a diagram $Z \longleftarrow Y \longrightarrow P$ with the right map an f-cofibration. Since $\mathbf{P}^{N}_{sfd}(G)$ satisfies axiom (Cof 3) by 3.3.2 (1), the pushout $Z \cup_Y P$ exists in $\mathbf{P}^{N}_{sfd}(G)$, and the map $Z \longrightarrow Z \cup_Y P$ is an f-cofibration. We are left to check that the pushout is an object of $\mathbf{P}^{N,T}_{sfd}(G)$, *i.e.*, that for all $t \in T$, the space $\Gamma \circ \theta_t(Z \cup_Y P)$ is weakly contractible. But Γ and θ_k commute with pushouts (the former by 1.3.5 (3), the latter since it is an equivalence of categories by 2.6.6 (2)). Moreover, every f-cofibration is a weak cofibration, *i.e.*, a pointwise cofibration in $G\text{-}k\mathit{Top}_*^{\langle N\rangle}$ (cf. remark following definition 2.4.6), and Γ maps weak cofibrations to cofibrations (1.3.5 (1)). Hence the right horizontal maps in the diagram

$$\begin{tikzcd} \Gamma\circ\theta_t(Z) \arrow[d, "\sim"'] & \Gamma\circ\theta_t(Y) \arrow[l] \arrow[r] \arrow[d, "\sim"] & \Gamma\circ\theta_t(P) \arrow[d, "\sim"] \\ * & * \arrow[l] \arrow[r] & * \end{tikzcd}$$

are cofibrations of G-spaces, and all three vertical arrows are weak equivalences since Z, Y and P are objects of $\mathbf{P}^{N,T}_{sfd}(G)$. The gluing lemma in $G\text{-}k\mathit{Top}_*$ then asserts

$$\Gamma\circ\theta_t(Z\cup_Y P) \cong \Gamma\circ\theta_t(Z)\cup_{\Gamma\circ\theta_t(Y)}\Gamma\circ\theta_t(P) \simeq * \,.$$

The gluing lemma (axiom (Weq 2)) is proved with a similar argument. Suppose we have a diagram

$$\begin{tikzcd} Z \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & P \arrow[d] \\ \bar{Z} & \bar{Y} \arrow[l] \arrow[r] & \bar{P} \end{tikzcd}$$

where all vertical maps are h_S-equivalences, and the two horizontal maps on the right are f-cofibrations. We have to show that for all $s \in S$, the induced map

$$\Gamma\circ\theta_s(Z\cup_Y P)) \longrightarrow \Gamma\circ\theta_s(\bar{Z}\cup_{\bar{Y}}\bar{P})$$

is a weak equivalence. But since Γ and θ_s commute with pushouts, this map is isomorphic to the morphism induced on pushouts of top and bottom row in the following diagram:

$$\begin{tikzcd} \Gamma\circ\theta_s(Z) \arrow[d, "\sim"'] & \Gamma\circ\theta_s(Y) \arrow[l] \arrow[r] \arrow[d, "\sim"] & \Gamma\circ\theta_s(P) \arrow[d, "\sim"] \\ \Gamma\circ\theta_s(\bar{Z}) & \Gamma\circ\theta_s(\bar{Y}) \arrow[l] \arrow[r] & \Gamma\circ\theta_s(\bar{P}) \end{tikzcd}$$

The two horizontal maps on the right are cofibrations since Γ preserves f-cofibrations, and by hypothesis the three vertical maps are weak equivalences. The gluing lemma in $G\text{-}k\mathit{Top}_*$ then finishes the argument.

The mapping cylinder construction is a cylinder functor by 3.3.2 (2). Since all h-equivalences between weakly cofibrant objects are in particular h_S-equivalences, the projection from the cylinder is an h_S-equivalence. Hence the cylinder axiom holds.

$\square$

Similar arguments can be used to prove the following two lemmas for all $T \subseteq S \subseteq \mathbf{Z}$:

Lemma 3.3.7. *The axioms for a category with cofibrations and weak equivalences hold for $\mathbf{P}^{N,T}_{sfd}(G)$ with respect to f-cofibrations and h-equivalences. The saturation axiom is satisfied. The mapping cylinder construction provides a cylinder functor, and the cylinder axiom holds.* □

Lemma 3.3.8. *The axioms for a category with cofibrations and weak equivalences hold for $\overline{\mathbf{P}}^{N,T}_{sfd}(G)$ with respect to weak cofibrations and h_S-equivalences. The saturation axiom is satisfied. The mapping cylinder construction provides a cylinder functor, and the cylinder axiom holds.* □

Recall the notion of an exact functor ([W, 1.2]). Since we deal with several notions of weak equivalences, we say a functor is h-exact (or exact with respect to h-equivalences) if, in particular, it preserves h-equivalences. Similarly, we have h_S-exact functors. Again, if we have a weak equivalence of exact functors, and we want to specify the notion of weak equivalences, we speak of an h-equivalence of functors, or a weak equivalence with respect to h_S-equivalences.

Proposition 3.3.9. *The h_S-exact inclusion $\mathbf{P}^{N,T}_{sfd}(G) \longrightarrow \overline{\mathbf{P}}^{N,T}_{sfd}(G)$ induces an equivalence on $\mathcal{S}_\bullet$-constructions.*

Proof. Exactness of the inclusion functor is obvious. We check that it has the approximation property. Axiom (App 1) holds by definition of weak equivalences (h_S-equivalences in both categories).

Given a map $f: Y \longrightarrow Z$ with $Y \in \mathbf{P}^{N,T}_{sfd}(G)$ and $Z \in \overline{\mathbf{P}}^{N,T}_{sfd}(G)$, we can find a factorization $f = g \circ h$ with $h: Y \longrightarrow P$ an f-cofibration (making P a locally cofibrant object) and an acyclic f-fibration $g: P \longrightarrow Z$. From 3.1.4 we infer that P is stably finitely dominated, and since Γ preserves weak equivalences of weakly cofibrant objects (1.3.5 (2)) we conclude that $P \in \overline{\mathbf{P}}^{N,T}_{sfd}(G)$. Application of the approximation theorem finishes the proof. □

Lemma 3.3.10. *Let ? denote any of the subscripts f, hf, fd or sfd.*

(1) The functors Σ, ϑ_k and θ_k are exact endofunctors of $\mathbf{P}^N_?(G)$.

(2) The canonical sheaf functors ψ_k are exact functors $\mathbf{C}_?(G) \longrightarrow \mathbf{P}^N_?(G)$.

(3) The functor $\Gamma: \mathbf{P}^N_?(G) \longrightarrow \mathbf{C}_{sfd}(G)$ is exact.

(4) The functor $\sigma: \mathbf{P}^{N,T}_{sfd}(G) \longrightarrow \overline{\mathbf{P}}^{N,T}_{sfd}(G)$ is exact with respect to h_S-equivalences (where $T \subseteq S \subseteq \mathbf{Z}$).

Proof. For the proof we have to gather material from earlier lemmas.

Ad (1). Since cofibrations (f-cofibrations) and weak equivalences (h-equivalences) are defined componentwise, the claim is true for suspension. Since $\vartheta_k(Y)$ and Y have the same components, ϑ_k preserves cofibrations, weak equivalences and all finiteness notions. Compatibility with pushouts follows from the fact that ϑ_k is an equivalence of categories (2.6.6 (2)). Finally the natural isomorphism $\theta_k \cong \vartheta_k$ from 2.6.6 (3) implies exactness of θ_k.

Ad (2). The A-component of $\psi_j(K)$ is given by $K \wedge M^A$. But the functor $\cdot \wedge M^A \colon G\text{-}k\,Top_* \longrightarrow (G \wedge M^A)\text{-}k\,Top_*$ preserves cofibrations, and maps weak equivalences between cofibrant objects to weak equivalences (apply 1.2.2 (2) to the morphism $G \longrightarrow G \wedge M^A$ and note that the functors $\cdot \wedge M^A$ and $\cdot \wedge_G (G \wedge M^A)$ are isomorphic). Hence ψ_j maps cofibrations to f-cofibrations, and ψ_j preserves weak equivalences between cofibrant spaces. Since ψ_j has a right adjoint (2.6.10), it commutes with pushouts.

It remains to prove that ψ_j preserves finiteness. For $? = f$ and $? = hf$ this is true since ψ_j preserves finiteness (*resp.* homotopy finiteness) on each component (1.2.2 (3)). For $? = fd$, let K denote an object of $\mathbf{C}_{fd}(G)$. As in the proof of 1.2.2 (3), we can assume that K is the retract of a cofibrant homotopy finite G-space L. Then $\psi_j(K)$ is a retract of the homotopy finite sheaf $\psi_j(L)$. Finally, ψ_j commutes with suspension, hence maps stably finitely dominated spaces in $\mathbf{C}(G)$ to stably finitely dominated sheafs.

Ad (3). The functor Γ maps stably finitely dominated, locally cofibrant sheaves to stably finitely dominated G-spaces by 3.2.6. The other conditions have been checked in 1.3.5.

Ad (4). This follows immediately from 2.8.6. □

Lemma 3.3.11. *(Shifting lemma.)*

Suppose F, $G \colon \mathbf{P}^{N,T}_{sfd}(G) \longrightarrow \overline{\mathbf{P}}^{N,T}_{sfd}(G)$ are h_S-exact functors, and $\tau \colon F \longrightarrow G$ is a weak equivalence of functors (with respect to h_S-equivalences). Then the induced natural transformation

$$\theta_{-k}(\tau_{\theta_k}) \colon \theta_{-k} \circ F \circ \theta_k \longrightarrow \theta_{-k} \circ G \circ \theta_k$$

is a weak equivalence of h_{k+S}-exact functors where $k+S$ denotes the set $\{k+s \mid s \in S\}$.

Proof. To show that $\bar{F} := \theta_{-k} \circ F \circ \theta_k$ is h_{k+S}-exact whenever F if h_S-exact, it is sufficient to prove that $\bar{F}$ preserves h_{k+S}-equivalences (all the other axioms for exact functors remain valid since θ_j is an equivalence of categories and maps f-cofibrations to weak cofibrations). Suppose f is an h_{k+S}-equivalence. Then $\theta_k(f)$ is an h_S-equivalence since $\Gamma \circ \theta_s \circ \theta_k(f) \cong \Gamma \circ \theta_{k+s}(f)$ is a weak equivalence for all $s \in S$ by hypothesis on f. But F preserves h_S-equivalences, hence for all $m \in k+S$ the map

$$\Gamma \circ \theta_m \circ \bar{F}(f) = \Gamma \circ \theta_m \circ \theta_{-k} \circ F \circ \theta_k(f) \cong \Gamma \circ \theta_{m-k} \circ F \circ \theta_k(f)$$

is a weak equivalence (note that $m - k \in S$, and $F \circ \theta_k(f)$ is an h_S-equivalence since F is h_S-exact).

To prove that $\theta_{-k}(\tau_{\theta_k})$ is an h_{k+S}-equivalence of functors, we have to show that for each object Y and each element $m \in k+S$, the morphism $\theta_{-k}(\tau_{\theta_k(Y)})$ gets mapped to a weak homotopy equivalence by $\Gamma \circ \theta_m$. But $\Gamma \circ \theta_m \circ \theta_{-k}(\tau_{\theta_k(Y)}) \cong \Gamma \circ \theta_{m-k}(\tau_{\theta_k(Y)})$ is a weak equivalence since $m - k \in S$ and τ is an h_S-equivalence. □

3.4. The Splitting of the K-Theory

Definition 3.4.1. Let G denote a topological monoid with zero. The algebraic K-theory of G-equivariant non-linear projective N-space is the algebraic K-theory in the sense of [W, 1.3] of the category $\mathbf{P}^N_{sfd}(G)$ with respect to f-cofibrations and h-equivalences, *i.e.*, the space $\Omega|h\mathcal{S}_\bullet\mathbf{P}^N_{sfd}(G)|$.

The particular choice of cofibrations and finiteness condition is somewhat arbitrary: we have defined c-cofibrations, f-cofibrations, and weak cofibrations; and we have four finiteness notions indicated by f, hf, fd and sfd. Each combination gives a valid definition of the K-theory, and we have checked explicitly for some of them (propositions 3.3.4 and 3.3.9) the expected result that, apart from a change in K_0, all possible definitions lead to the same result. The above definition displays, to the author's taste, the most natural choice of cofibrations. The finiteness condition, however, is forced upon us if we want to use the functor Γ since it maps locally finite sheaves to stably finitely dominated spaces (see the remark following proposition 3.2.6).

From now on we will restrict ourselves to the stably finitely dominated case. We will only consider f-cofibrations unless otherwise stated. Let G denote a topological monoid with zero which is cofibrant as an object of $k\,Top_*$, and let N denote a non-empty finite set with $n+1$ elements.

Lemma 3.4.2. *For each $k \in [n]$, the functors $\psi_{-k}\colon \mathbf{C}_{sfd}(G) \longrightarrow \mathbf{P}^{N,[k-1]}_{sfd}$ and $\Gamma \circ \theta_k \colon \mathbf{P}^{N,[k-1]}_{sfd} \longrightarrow \mathbf{C}_{sfd}(G)$ induce equivalences on $\mathcal{S}_\bullet$-constructions with respect to $h_{[k]}$-equivalences in $\mathbf{P}^{N,[k-1]}_{sfd}$. (We consider $[k-1]$ as a subset of $\mathbf{Z}$.) Explicitly, the map $|h\mathcal{S}_\bullet\mathbf{C}_{sfd}(G)| \longrightarrow |h_{[k]}\mathcal{S}_\bullet\mathbf{P}^{N,[k-1]}_{sfd}|$ induced by ψ_{-k} is a homotopy equivalence, and similarly for $\Gamma \circ \theta_k$.*

Proof. By corollary 2.7.5, the spaces $\Gamma \circ \theta_j \circ \psi_{-k}(K) = \Gamma \circ \psi_{j-k}(K)$ are contractible for all $K \in G\text{-}k\,Top_*$ and $0 \le j < k$. Hence $\psi_{-k}\colon \mathbf{C}_{sfd}(G) \longrightarrow \mathbf{P}^N_{sfd}(G)$ factors through $\mathbf{P}^{N,[k-1]}_{sfd}(G)$, so ψ_{-k} induces a well-defined map in K-theory. Since Σ induces a self-equivalence ([W, 1.6.2]), it is sufficient to show that the map $\Sigma^2 \circ \psi_{-k}$ induces an equivalence on $\mathcal{S}_\bullet$-constructions. Now $(\Gamma\circ\theta_k)\circ(\Sigma^2\circ\psi_{-k}) \cong \Sigma^2\circ\Gamma\circ\psi_0 \simeq \Sigma^{n+2}$ by 2.7.4, and the suspension functor induces an isomorphism on homotopy groups. Thus $\Sigma^2\circ\psi_{-k}$ induces an injection on homotopy groups, and $\Gamma \circ \theta_k$ induces a surjection.

It remains to prove that the functors $\Sigma^2 \circ \psi_{-k} \circ \Gamma \circ \theta_k$ and Σ^{n+2} induce homotopic self-maps on the $\mathcal{S}_\bullet$-construction. By 3.3.9 it is sufficient to prove that they are weakly equivalent as exact functors $\mathbf{P}^{N,[k-1]}_{sfd}(G) \longrightarrow \overline{\mathbf{P}}^{N,[k-1]}_{sfd}(G)$ with respect to $h_{[k]}$-equivalences (cofibrations are f-cofibrations in the source and weak cofibrations in the target). But by 2.8.9 and 3.3.10 (4), the functors $\Sigma^2\circ\psi_0\circ\Gamma$ and Σ^{n+2} are connected by a chain of $h_{\{0\}}$-equivalences of functors. Hence

$$\begin{aligned}\Sigma^2 \circ \psi_{-k} \circ \Gamma \circ \theta_k &\cong \theta_{-k} \circ \Sigma^2 \circ \psi_0 \circ \Gamma \circ \theta_k \\ &\simeq \theta_{-k} \circ \Sigma^{n+2} \circ \theta_k \qquad \text{(by 2.8.9)} \\ &\cong \Sigma^{n+2}\end{aligned}$$

with respect to $h_{\{k\}}$-equivalences by the shifting lemma (3.3.11). We claim that this is an $h_{[k]}$-equivalence of functors, *i.e.*, for all $j \in [k]$, the components of the natural transformations involved in the chain are mapped to weak equivalences upon application of $\Gamma \circ \theta_j$. For $j = k$ this is what we have just checked. For $j < k$, observe that source and target of the components of the natural transformations are objects of $\overline{\mathbf{P}}^{N,[k-1]}_{sfd}(G)$, hence are mapped to contractible spaces, and any map between contractible spaces is a weak homotopy equivalence. □

Lemma 3.4.3. *For $k \in [n]$, the map*

$$\iota \vee \psi_{-k} : \mathbf{P}^{N,[k]}_{sfd} \times \mathbf{C}_{sfd}(G) \longrightarrow \mathbf{P}^{N,[k-1]}_{sfd}, \quad (Y, K) \mapsto Y \vee \psi_{-k}(K)$$

(where $\iota \colon \mathbf{P}^{N,[k]}_{sfd} \longrightarrow \mathbf{P}^{N,[k-1]}_{sfd}$ symbolizes the inclusion functor and $\vee$ denotes one-point union, the coproduct in $\mathbf{P}^{N,[k-1]}_{sfd}$) induces an equivalence on $\mathcal{S}_\bullet$-constructions with respect to h-equivalences.

Proof. For $n = 0$, *i.e.*, if N is a set with one element, we know $k = 0$, $\mathbf{P}^{N,[-1]}_{sfd}(G) = \mathbf{C}_{sfd}(G)$, $\Gamma = \mathrm{id}$, $\psi_{-k} \cong \mathrm{id}$, and $\mathbf{P}^{N,[0]}_{sfd}(G)$ is the subcategory of $\mathbf{C}_{sfd}(G)$ of contractible spaces. Hence the assertion is true.

So assume N has at least two elements ($n \geq 1$), and let $\Sigma^2\mathbf{P}^{N,[k]}_{sfd}(G)$ denote the full subcategory of $\mathbf{P}^{N,[k]}_{sfd}(G)$ consisting of objects with simply connected components. This category inherits the structure of a category with cofibrations and weak equivalences. Similarly we can define all the other categories present in the following square:

$$\begin{array}{ccc} h\mathcal{S}_\bullet\Sigma^2\mathbf{P}^{N,[k]}_{sfd}(G) & \longrightarrow & h_{[k]}\mathcal{S}_\bullet\Sigma^2\mathbf{P}^{N,[k]}_{sfd}(G) \\ \downarrow & & \downarrow \\ h\mathcal{S}_\bullet\Sigma^2\mathbf{P}^{N,[k-1]}_{sfd}(G) & \longrightarrow & h_{[k]}\mathcal{S}_\bullet\Sigma^2\mathbf{P}^{N,[k-1]}_{sfd}(G) \end{array}$$

The $h_{[k]}$-equivalences satisfy the extension axiom in these modified categories (since a map of simply connected cofibrant spaces is a weak homotopy equivalence if and only if it is a homology isomorphism, and cofibration sequences give rise to long exact sequences of homology groups; hence the five lemma is available to prove the extension axiom—we omit the details). Hence we can apply the fibration theorem ([W], 1.6.4) to conclude that the above square is homotopy cartesian with contractible upper right corner. This square has a canonical map into the square

$$\begin{tikzcd} h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G) \arrow[r] \arrow[d] & h_{[k]}\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G) \arrow[d] \\ h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G) \arrow[r] & h_{[k]}\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G) \end{tikzcd}$$

induced by inclusion of categories. There is a map going backwards induced by double suspension. But Σ^2, considered as an endofunctor, induces a self-equivalence on $\mathcal{S}_\bullet$-constructions ([W, 1.6.2]). Hence the maps of squares are weak equivalences. This implies that the second square is homotopy cartesian with contractible upper right corner.

By the previous lemma, $\Sigma^n \circ \Gamma \circ \theta_k$ induces a homotopy equivalence from the lower right space to $h\mathcal{S}_\bullet \mathbf{C}_{sfd}(G)$. Hence we obtain the following homotopy cartesian square:

$$\begin{tikzcd} h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G) \arrow[r] \arrow[d] & h_{[k]}\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G) \arrow[d, "\Sigma^n\circ\Gamma\circ\theta_k"] \simeq * \\ h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G) \arrow[r, "\Sigma^n\circ\Gamma\circ\theta_k"'] & h\mathcal{S}_\bullet \mathbf{C}_{sfd}(G) \end{tikzcd} \tag{$*$}$$

As in the proof of the previous lemma, we see that the canonical sheaf functor ψ_{-k} induces a map $\mathbf{C}_{sfd}(G) \longrightarrow \mathbf{P}^{N,[k-1]}_{sfd}(G)$. But there is a chain of weak equivalences of functors

$$\begin{aligned} \Sigma^n \circ \Gamma \circ \theta_k \circ \psi_{-k} &\cong \Sigma \circ \Gamma \circ \theta_k \circ \psi_{-k} \circ \Sigma^{n-1} \\ &\cong \Sigma \circ \Gamma \circ \psi_0 \circ \Sigma^{n-1} \\ &\simeq \Sigma^{n+1} \circ \Sigma^{n-1} \qquad \text{(by 2.7.4)} \\ &\cong \Sigma^{2n} \quad , \end{aligned}$$

and Σ^{2n} induces, on $\mathcal{S}_\bullet$-construction, a map which is homotopic to the identity. This shows that ψ_{-k} induces a section-up-to-homotopy of the lower horizontal map.

We claim that the composite map

$$\begin{aligned} |h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G)| \times |h\mathcal{S}_\bullet \mathbf{C}_{sfd}(G)| &\xrightarrow{\iota\times\psi_{-k}} |h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G)| \times |h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G)| \\ &\xrightarrow{\vee} |h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G)| \end{aligned}$$

where ι is induced by the forgetful functor and $\vee$ denotes one-point union (the coproduct inducing the H-space structure) is a weak homotopy equivalence. Since source and target are connected, it suffices to check that the induced map $(\iota \vee \psi_{-k})_*$ on j-th homotopy groups is an isomorphism for $j \geq 1$.

To see this, recall the homotopy cartesian square $(*)$ above. Since its upper right corner is contractible, we obtain a long exact sequence of homotopy groups (which are abelian because the $\mathcal{S}_\bullet$-construction has an abelian H-group structure induced by the coproduct)

$$\begin{aligned} \ldots \longrightarrow \pi_j|h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G)| &\xrightarrow{\iota_*} \pi_j|h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G)| \xrightarrow{(\Sigma^n\circ\Gamma\circ\theta_k)_*} \pi_j|h\mathcal{S}_\bullet \mathbf{C}_{sfd}(G)| \\ &\longrightarrow \pi_{j-1}|h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G)| \xrightarrow{\iota_*} \ldots \end{aligned}$$

with maps induced by ι and $\Sigma^n \circ \Gamma \circ \theta_k$ as indicated. But we have shown above that the latter map has a section-up-to-homotopy, hence the long exact sequence gives rise (for $j \geq 1$) to short exact sequences

$$0 \longrightarrow \pi_j|h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G)| \xrightarrow{\iota_*} \pi_j|h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G)| \xrightarrow{(\Sigma^n\circ\Gamma\circ\theta_k)_*} \pi_j|h\mathcal{S}_\bullet \mathbf{C}_{sfd}(G)| \longrightarrow 0$$

which are split by the map induced from ψ_{-k}, *i.e.*, for all $j \geq 1$ we have an isomorphism of abelian groups

$$\iota_*{+}(\psi_{-k})_*: \ \pi_j|h\mathcal{S}_\bullet \mathbf{P}^{N,[k]}_{sfd}(G)| \oplus \pi_j|h\mathcal{S}_\bullet \mathbf{C}_{sfd}(G)| \xrightarrow{\cong} \pi_j|h\mathcal{S}_\bullet \mathbf{P}^{N,[k-1]}_{sfd}(G)|$$

where "+" means the sum of group elements.

But the target is a homotopy group of an H-space. Hence the group structure is induced from the H-space structure ([S], I.6, corollary 10), *i.e.*, from the coproduct in $\mathbf{P}^{N,[k-1]}_{sfd}(G)$. Consequently, the isomorphism $\iota_*{+}(\psi_{-k})_*$ is the same map as $(\iota \vee \psi_{-k})_*$. □

Lemma 3.4.4. *Suppose $S \subseteq \mathbf{Z}$ contains $n+1$ successive integers. The inclusion functor induces a homotopy equivalence $|h\mathcal{S}_\bullet \mathbf{P}^{N}_{sfd}(G)| \longrightarrow |h_S\mathcal{S}_\bullet \mathbf{P}^{N}_{sfd}(G)|$.*

Proof. By passing to objects with simply connected components as in the proof of the previous lemma we see that there is a "fibration sequence"

$$|h\mathcal{S}_\bullet \mathbf{P}^{N,S}_{sfd}(G)| \longrightarrow |h\mathcal{S}_\bullet \mathbf{P}^{N}_{sfd}(G)| \longrightarrow |h_S\mathcal{S}_\bullet \mathbf{P}^{N}_{sfd}(G)| \ .$$

We show that the space on the left is contractible. Since all the spaces are connected, this implies the lemma.

For $k \geq 0$, let $\mathcal{C}^k$ denote the full subcategory of $\mathbf{P}^{N,S}_{sfd}(G)$ generated by the objects Y with $\Sigma^k(Y) \simeq *$. These categories inherit the structure of categories with cofibrations (f-cofibrations) and weak equivalences (h-equivalences).

Now $|h\mathcal{S}_\bullet\mathcal{C}^0| \simeq *$ since the map $Y \longrightarrow *$ is a weak equivalence for all objects $Y \in \mathcal{C}^0$. Furthermore, suspension defines a map $\mathcal{C}^{k+1} \longrightarrow \mathcal{C}^k$ which shows that the inclusion $\mathcal{C}^k \subseteq \mathcal{C}^{k+1}$ induces a homotopy equivalence on $\mathcal{S}_\bullet$-constructions. Hence the category $\mathcal{C}^\infty := \bigcup_0^\infty \mathcal{C}^k$ has contractible $\mathcal{S}_\bullet$-construction. We claim that $\mathbf{P}^{N,S}_{sfd}(G) = \mathcal{C}^\infty$. By definition of $\mathcal{C}^k$, we have $\mathcal{C}^\infty \subseteq \mathbf{P}^{N,S}_{sfd}(G)$. For the reverse inclusion, it suffices to prove the following assertion: if Y is an object of $\mathbf{P}^{N,S}_{sfd}(G)$, some finite suspension of Y is acyclic (*i.e.*, the map $\Sigma^\ell(Y) \longrightarrow *$ is an h-equivalence for $\ell \geq 0$ large).

It is enough to consider ordered indexing sets $N = [n]$. We proceed by induction on the dimension n. The statement is clear for $n = 0$ (in which case $\mathbf{P}^0_{sfd}(G) = \mathbf{C}_{sfd}(G)$ and $\Gamma = \mathrm{id}$). So assume we have shown the result for $n-1$. Let $Y \in \mathbf{P}^{n,S}_{sfd}$. Fix $j \in [n]$, and choose $k \in S$ with $k{+}1 \in S$. The functors θ_k and ϑ_k are naturally isomorphic by 2.6.6 (3), and the natural transformations κ_j and $\bar\kappa_j$ (defined in 2.6.7 and 2.6.8, respectively) correspond under this isomorphism. Hence lemma 2.6.11 applies, asserting the existence of a cofibration sequence

$$\vartheta_k\big(\Sigma^2(Y)\big) \xrightarrow{\bar\kappa_j} \vartheta_{k+1}\big(\Sigma^2(Y)\big) \longrightarrow \zeta_j \circ \rho_j \circ \vartheta_{k+1}\big(\Sigma^2(Y)\big)$$

which induces, by application of Γ, a cofibration sequence of spaces

$$\begin{aligned}\Gamma \circ \vartheta_k\big(\Sigma^2(Y)\big) \longrightarrow \Gamma \circ \vartheta_{k+1}\big(\Sigma^2(Y)\big) \longrightarrow{}& \Gamma\big(\zeta_j \circ \rho_j \circ \vartheta_{k+1}(\Sigma^2(Y))\big)\\ &\cong \Sigma \circ \Gamma\big(\vartheta_{k+1} \circ \rho_j(\Sigma^2(Y))\big)\end{aligned}$$

(we have used lemma 2.7.2 and the fact that restriction commutes with twisting for the last isomorphism). Since suspension commutes with restriction and global sections (1.3.2 (2)), all spaces in this sequence are simply connected. Furthermore, by hypothesis on Y the first two spaces are contractible, hence so is the third (use the long exact sequence in homology), which implies (by varying k) that the global sections of n successive twists of $\rho_j\big(\Sigma^2(Y)\big)$ are contractible. In other words, the induction hypothesis applies to the object $\rho_j\big(\Sigma^2(Y)\big)$. Thus we find that some finite suspension of $\rho_j\big(\Sigma^2(Y)\big)$ is acyclic. Since twisting, suspension, and restriction commute, this means that $\rho_j \circ \vartheta_{k+1}\big(\Sigma^\ell(Y)\big) \simeq *$ for some $\ell \geq 2$.

By lemma 2.6.11, there is a cofibration sequence

$$\vartheta_k\big(\Sigma^\ell(Y)\big) \xrightarrow{\bar\kappa_j} \vartheta_{k+1}\big(\Sigma^\ell(Y)\big) \longrightarrow \zeta_j \circ \rho_j \circ \vartheta_{k+1}\big(\Sigma^\ell(Y)\big)$$

whose last term is acyclic by what we have just shown. From the long exact sequence in homology, applied componentwise, we infer that all components of $\bar\kappa_j$ are weak equivalences.

Now suppose we have a non-empty set $A \subseteq [n]$. Let $m := \max(A)$. The A-component of $\bar\kappa_j$ is given by $\Sigma^\ell(Y^A) \xrightarrow{t_j t_m^{-1}} \Sigma^\ell(Y^A)$ which is a weak homotopy equivalence by what we have shown above. This implies that all maps in the telescope construction 2.2.7 (2) for $\Sigma^\ell(Y^A)[t_j^{-1}]$ are weak homotopy equivalences. Consequently, the $(G\wedge M^A)$-equivariant inclusion of $\Sigma^\ell(Y^A)$ into $\Sigma^\ell(Y^A)[t_j^{-1}]$ (the unit of the adjunction desribed in 2.2.6 (1)), denoted s_0, is a weak homotopy equivalence.

Let σ denote the inclusion $A \longrightarrow A \cup \{j\}$. The $\flat$-type structure map $\vartheta_k\big(\Sigma^\ell(Y)\big)^\flat_\sigma$ is given by the composite

$$\Sigma^\ell(Y^A) \xrightarrow{s_0} \Sigma^\ell(Y^A)[t_j^{-1}] \xrightarrow{\vartheta_k(\Sigma^\ell(Y))^\sharp_\sigma} \Sigma^\ell(Y)^{A\cup\{j\}} .$$

The second map is a weak equivalence since $\vartheta_k\big(\Sigma^\ell(Y)\big)$ is a locally cofibrant sheaf (2.3.4), and s_0 is a weak equivalence by the above argument. Hence $\vartheta_k\big(\Sigma^\ell(Y)\big)^\flat_\sigma$ is a weak homotopy equivalence.

This argument applies for all $j \in [n]$. Thus all $\flat$-type structure maps in $\vartheta_k\big(\Sigma^\ell(Y)\big)$ (away from the initial vertex) are weak homotopy equivalences, *i.e.*, $\vartheta_k\big(\Sigma^\ell(Y)\big)$ is special as a cube of spaces. By lemma 1.3.6, we conclude that $\vartheta_k\big(\Sigma^\ell(Y)\big)$ has contractible components since $\Gamma \circ \vartheta_k\big(\Sigma^\ell(Y)\big) \simeq *$. □

Corollary 3.4.5. *The space $|h\mathcal{S}_\bullet \mathbf{P}^{N,[n]}_{sfd}(G)|$ is weakly contractible.*

Proof. The map $|h\mathcal{S}_\bullet \mathbf{P}^N_{sfd}(G)| \longrightarrow |h_{[n]}\mathcal{S}_\bullet \mathbf{P}^N_{sfd}(G)|$ is a weak homotopy equivalence by lemma 3.4.4. Hence its homotopy fibre $|h\mathcal{S}_\bullet \mathbf{P}^{N,[n]}_{sfd}(G)|$ is weakly contractible. □

Now we can state and prove the main theorem of this paper.

Theorem 3.4.6. *Suppose N is a non-empty finite set with $n+1$ elements, and G is a topological monoid with zero which is cofibrant as an object of $k\mathit{Top}_*$. The map*

$$\bigvee_{k=0}^{n} \psi_{-k} : \underbrace{\mathbf{C}_{sfd}(G) \times \ldots \times \mathbf{C}_{sfd}(G)}_{n+1 \text{ factors}} \longrightarrow \mathbf{P}^N_{sfd}(G)$$

induces an equivalence on $\mathcal{S}_\bullet$-constructions with respect to h-equivalences and hence a weak homotopy equivalence

$$\underbrace{A^{sfd}(*,G) \times \ldots \times A^{sfd}(*,G)}_{n+1 \text{ factors}} \longrightarrow \Omega|h\mathcal{S}_\bullet \mathbf{P}^N_{sfd}(G)|$$

where A^{sfd} denotes the version of Waldhausen*'s algebraic K-theory of spaces functor using stably finitely dominated spaces. In other words: the algebraic K-theory of G-equivariant non-linear projective N-space decomposes naturally as a product of $n+1$ copies of $A^{sfd}(*,G)$.*

Proof. By 3.4.5 and 3.4.3, the composite functor

$$\begin{aligned}
\mathbf{C}_{sfd}(G)^{n+1} &\xrightarrow{*\times \mathrm{id}^{n+1}} \mathbf{P}^{N,[n]}_{sfd}(G) \times \mathbf{C}_{sfd}(G)^{n+1} \\
&\xrightarrow{(\iota\vee\psi_{-n})\times \mathrm{id}^{n}} \mathbf{P}^{N,[n-1]}_{sfd}(G) \times \mathbf{C}_{sfd}(G)^{n} \\
&\xrightarrow{(\iota\vee\psi_{-n+1})\times \mathrm{id}^{n-1}} \ldots \\
&\xrightarrow{(\iota\vee\psi_{-1})\times \mathrm{id}} \mathbf{P}^{N,[0]}_{sfd}(G) \times \mathbf{C}_{sfd}(G) \\
&\xrightarrow{\iota\vee\psi_{0}} \mathbf{P}^{N}_{sfd}(G)
\end{aligned}$$

induces an equivalence on $\mathcal{S}_\bullet$-constructions with respect to h-equivalences. But up to an associativity isomorphism this is the functor mentioned in the theorem. Looping once yields the second assertion. □

It is natural to ask if the specific range of twists used in the theorem is important, and if there is an explicit homotopy equivalence going backwards. In fact, we have the following two corollaries of theorem 3.4.6:

Corollary 3.4.7. *Let $k_0 \in \mathbf{Z}$ be any integer. The map*

$$\bigvee_{k=0}^{n} \psi_{k_0-k} : \underbrace{\mathbf{C}_{sfd}(G) \times \ldots \times \mathbf{C}_{sfd}(G)}_{n+1 \text{ factors}} \longrightarrow \mathbf{P}^N_{sfd}(G)$$

induces an equivalence on $\mathcal{S}_\bullet$-constructions with respect to h-equivalences.

Proof. Since θ_{k_0} is an equivalence of categories (2.6.6 (2)), it induces a homotopy equivalence on $\mathcal{S}_\bullet$-constructions. By the previous theorem, we realize that

$$\theta_{k_0} \circ \bigvee_{k=0}^{n} \psi_{-k} \cong \bigvee_{k=0}^{n} \psi_{k_0-k}$$

induces an equivalence on $\mathcal{S}_\bullet$-constructions as well. □

Corollary 3.4.8. *Let $k_0 \in \mathbf{Z}$ be any integer. The map*

$$(\Gamma \circ \theta_{-k_0},\ \Gamma \circ \theta_{-k_0+1},\ \ldots, \Gamma \circ \theta_{-k_0+n}) : \mathbf{P}^N_{sfd}(G) \longrightarrow \mathbf{C}_{sfd}(G)^{n+1}$$

induces an equivalence on $\mathcal{S}_\bullet$-constructions with respect to h-equivalences.

Proof. The composite map $f := \Sigma \circ (\Gamma \circ \theta_{-k_0},\ \Gamma \circ \theta_{-k_0+1},\ \ldots, \Gamma \circ \theta_{-k_0+n}) \circ \bigvee_{k=0}^{n} \psi_{k_0-k}$ is given, up to homotopy, by the matrix

$$\begin{pmatrix} \Sigma^{n+1} & 0 & 0 & 0 & \ldots & 0 \\ ? & \Sigma^{n+1} & 0 & 0 & \ldots & 0 \\ ? & ? & \Sigma^{n+1} & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ ? & \ldots & & \ldots & ? & \Sigma^{n+1} \end{pmatrix}$$

where ? means "don't care", and 0 denotes null-homotopic maps (we have used 2.7.4 to identify the terms on the diagonal, and 2.7.5 for the zero entries). More precisely, this means that the map induced on homotopy groups is described by the above matrix. But since Σ induces a homotopy equivalence on $\mathcal{S}_\bullet$-constructions, the matrix is invertible (on the level of homotopy groups). Hence the map f induces a weak homotopy equivalence, hence so does the map of the corollary. □

To conclude this chapter, let us briefly comment on the rôle of G. Suppose X is a connected pointed simplicial set, and let $G(X)$ denote its KAN loop group. Then $G := |G(X)|_+$ is a topological monoid with zero which is cofibrant as an object of $k\,Top_*$. By a variant of [W, theorem 2.1.5] we know that $A^{sfd}(X)$ and $A^{sfd}(*, G)$ are homotopy equivalent (with a suitable definition of "stably finitely dominated" objects of $R(X)$). The above theorem asserts that the K-theory of $\mathbf{P}^N_{sfd}(G)$ is homotopy equivalent to $A^{sfd}(X)^{n+1}$, and $\mathbf{P}^N_{sfd}(G)$ may be thought of as the category of quasi-coherent sheaves on relative projective space over X in the sense of [Ha, II.7, p. 162]. Hence theorem 3.4.6 really is the analogue of QUILLEN's result ([Q2], § 8, theorem 2.1).

A. Appendix: Twisted Diagrams

The material in this appendix is joint work with O. Röndigs.

A.1. Introduction

In practice one often encounters a construction which looks like diagrams in some category but isn't quite. A prominent example is the notion of (naïve) spectra, a sequence of pointed spaces $X_0,\ X_1,\ \ldots$ and structure maps $\Sigma X_n \longrightarrow X_{n+1}$. This almost determines a diagram indexed over $\mathbf{N}$ (regarded as a category), and in fact can be described by a "twisted diagram" with "twists" given by iterated suspension functors. Another example is the category $\mathbf{pP}^N$ discussed in this paper. The formalism also applies to diagram categories in the usual sense (*i.e.*, functor categories) like $k\,Top_*^{\langle N\rangle}$.

We show how to construct model structures for categories of twisted diagrams if the indexing category satisfies certain conditions. The weak equivalences will always be pointwise weak equivalences.

This appendix presents in some detail a slight generalization of the construction of model category structures for functor categories as given in [DHK]. The main tool is the theory of adjoint functors.

A.2. Basic Definitions

Let $\mathcal{I}$ be a (fixed, small) category. It will serve as the index category for our diagrams. Later we will restrict ourselves to posets.

For the purpose of this paper, the relevant examples are $\mathcal{I} = \langle N\rangle$, the category of subsets of a finite non-empty set N, and $\mathcal{I} = \langle N\rangle_0$, the category of non-empty subsets.

Definition A.2.1. An *adjunction bundle* $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ *over* $\mathcal{I}$ consists of the following data:

- for each object $i \in \mathcal{I}$ a category $\mathcal{C}_i$,
- for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ a pair of adjoint functors $F_\sigma : \mathcal{C}_i \longrightarrow C_j$ and $U_\sigma : \mathcal{C}_j \longrightarrow \mathcal{C}_i$ (with F_σ being the left adjoint),

such that $\mathfrak{B}$ is functorial, *i.e.*, $U_{\mathrm{id}_i} = \mathrm{id}_{\mathcal{C}_i}$, and for each pair of composable arrows $i \xrightarrow{\sigma} j \xrightarrow{\tau} k$, the equality $U_{\tau\circ\sigma} = U_\sigma \circ U_\tau$ holds. (In particular, U is a functor $\mathcal{I}^{\mathrm{op}} \longrightarrow Cat$.) In addition, we require $F_{\mathrm{id}_i} = \mathrm{id}_{\mathcal{C}_i}$. The properties of adjunctions guarantee that there is a canonical isomorphism $F_{\tau\circ\sigma} \cong F_\tau \circ F_\sigma$ (which will sometimes be refered to as "uniqueness isomorphism") since both functors are left adjoint to $U_{\tau\circ\sigma} = U_\sigma \circ U_\tau$ ([ML, IV.1, corollary 1, p. 83]).

We call $\mathfrak{B}$ an *adjunction bundle of model categories* if all the $\mathcal{C}_i$ are pointed model categories, and all the F_σ preserve cofibrations and acyclic cofibrations. If in addition all the $\mathcal{C}_\sigma$ are proper model categories, $\mathfrak{B}$ is called *proper*.

There is an obvious variant of the definition requiring F to be a functor $\mathcal{I} \longrightarrow Cat$. In this case, there are natural isomorphisms $U_{\tau\circ\sigma} \cong U_\sigma \circ U_\tau$. All the material presented in this appendix works in both cases.

Definition A.2.2. Let $\mathfrak{B}$ be an adjunction bundle. A *twisted diagram Y with coefficients in* $\mathfrak{B}$ consists of the following data:

- for each object $i \in \mathcal{I}$ an object $Y_i \in \mathcal{C}_i$,
- for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ a map $Y_\sigma^\flat : Y_i \longrightarrow U_\sigma(Y_j)$ in $\mathcal{C}_i$

such that Y behaves like a functor, *i.e.*, $Y^\flat_{\mathrm{id}_i} = \mathrm{id}_{Y_i}$ and $Y^\flat_{\tau\circ\sigma} = U_\sigma(Y^\flat_\tau) \circ Y^\flat_\sigma$ for each pair $i \xrightarrow{\sigma} j \xrightarrow{\tau} k$ of composable arrows in $\mathcal{I}$. (A reformulation using the left adjoints will be given below.)

A map $f : Y \longrightarrow Z$ of twisted diagrams is a collection of maps $f_i : Y_i \longrightarrow Z_i$ in $\mathcal{C}_i$, one for each object $i \in \mathcal{I}$, such that for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ the equality $U_\sigma(f_j) \circ Y^\flat_\sigma = Z^\flat_\sigma \circ f_i$ holds. (A reformulation using the left adjoints will be given below.) The category of twisted diagrams and their maps is denoted $\mathfrak{B}^{\mathcal{I}}$.

If $\mathfrak{B}$ is a *trivial adjunction bundle*, *i.e.*, if all $\mathcal{C}_i = \mathcal{C}$ and the functors are the identity of $\mathcal{C}$, we recover the usual functor category $Func(\mathcal{I},\, \mathcal{C})$.

For each of the structure maps $Y^\flat_\sigma : Y_i \longrightarrow U_\sigma(Y_j)$ there is a corresponding adjoint map $Y^\sharp_\sigma : F_\sigma(Y_i) \longrightarrow Y_j$. The idea is to think of the symbol Y_σ as a "structure map" having two incarnations as a $\flat$-type map (a morphism in $\mathcal{C}_i$) and a $\sharp$-type map (a morphism in $\mathcal{C}_j$).

Given a collection of maps $f_i : Y_i \longrightarrow Z_i$ in $\mathcal{C}_i$, we can form two squares for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$

$$\begin{tikzcd} Y_i \arrow[r, "f_i"] \arrow[d, "Y^\flat_\sigma"'] & Z_i \arrow[d, "Z^\flat_\sigma"] \\ U_\sigma(Y_j) \arrow[r, "U_\sigma(f_j)"'] & U_\sigma(Z_j) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_\sigma(Y_i) \arrow[r, "F_\sigma(f_i)"] \arrow[d, "Y^\sharp_\sigma"'] & F_\sigma(Z_i) \arrow[d, "Z^\sharp_\sigma"] \\ Y_j \arrow[r, "f_j"'] & Z_j \end{tikzcd}$$

and the definition of adjunctions imply that the left square commutes if and only if the right square commutes. Thus the family $(f_i)_{i\in\mathcal{C}}$ determines a map of twisted diagrams if and only if $Z^\sharp_\sigma \circ F_\sigma(f_i) = f_j \circ Y^\sharp_\sigma$.

For later use, we record the following fact:

Lemma A.2.3. *Suppose we have, for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$, a map $Y^\flat_\sigma : Y_i \longrightarrow U_\sigma(Y_j)$ in $\mathcal{C}_i$, and denote by $Y^\sharp_\sigma$ the adjoint map $F_\sigma(Y_i) \longrightarrow Y_j$. Let $\tau : j \longrightarrow k$ be another morphism in $\mathcal{I}$. Then if one of the squares*

$$\begin{tikzcd} F_\tau \circ F_\sigma(Y_i) \arrow[r, "\cong"] \arrow[d, "F_\tau(Y^\sharp_\sigma)"'] & F_{\tau\circ\sigma}(Y_i) \arrow[d, "Y^\sharp_{\tau\circ\sigma}"] \\ F_\tau(Y_j) \arrow[r, "Y^\sharp_\tau"'] & Y_k \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Y_i \arrow[r, equal] \arrow[d, "Y^\flat_\sigma"'] & Y_i \arrow[d, "Y^\flat_{\tau\circ\sigma}"] \\ U_\sigma(Y_j) \arrow[r, "U_\sigma(Y^\flat_\tau)"'] & U_{\tau\circ\sigma}(Y_k) \end{tikzcd}$$

commutes so does the other (the upper horizontal map in the left square is the uniqueness isomorphism). In other words, if one of the squares commutes, the objects Y_i together with the maps $Y^\flat_\sigma$ form a twisted diagram.

Proof. Assume that the square on the left commutes. We want to show that the square on the right is commutative. The strategy is to devide the square into smaller pieces which are known to commute.

Let $\sigma\colon i \longrightarrow j$ denote a morphism in $\mathcal{I}$. There are natural transformations $\eta^\sigma\colon \mathrm{id} \longrightarrow U_\sigma \circ F_\sigma$ called *unit* of the adjunction of F_σ and U_σ. Given the structure map $Y_\sigma^\sharp\colon F_\sigma(Y_i) \longrightarrow Y_j$, we obtain the corresponding adjoint map $Y_\sigma^\flat\colon Y_i \longrightarrow U_\sigma(Y_j)$ as the composite

$$Y_i \xrightarrow{\eta^\sigma_{Y_i}} U_\sigma \circ F_\sigma(Y_i) \xrightarrow{U_\sigma(Y_\sigma^\sharp)} U_\sigma(Y_j)$$

(cf. [ML], IV.1, p. 80). In particular, the functors $F_{\tau\circ\sigma}$ and $U_{\tau\circ\sigma}$ are adjoint with unit $\eta^{\tau\circ\sigma}\colon \mathrm{id} \longrightarrow U_{\tau\circ\sigma} \circ F_{\tau\circ\sigma}$. But $U_{\tau\circ\sigma} = U_\sigma \circ U_\tau$ has another left adjoint $F_\tau \circ F_\sigma$, and we denote the corresponding unit by $\bar\eta^{\tau\circ\sigma}\colon \mathrm{id} \longrightarrow U_\sigma \circ U_\tau \circ F_\tau \circ F_\sigma$.

Now we redraw the square on the right with some extra data added:

$$\begin{tikzcd}
Y_i \arrow[rr, equal] \arrow[d, "\eta^\sigma_{Y_i}"'] \arrow[drr, phantom, "1"] & & Y_i \arrow[rr, equal] \arrow[d, "\bar\eta^{\tau\circ\sigma}_{Y_i}"'] \arrow[drr, phantom, "2"] & & Y_i \arrow[d, "\eta^{\tau\circ\sigma}_{Y_i}"] \\
U_\sigma \circ F_\sigma(Y_i) \arrow[rr, "U_\sigma(\eta^\tau_{F_\sigma(Y_i)})"] \arrow[d, "U_\sigma(Y_\sigma^\sharp)"'] \arrow[drr, phantom, "3"] & & U_\sigma \circ U_\tau \circ F_\tau \circ F_\sigma(Y_i) \arrow[rr, "\cong"] \arrow[d, "U_\sigma \circ U_\tau \circ F_\tau(Y_\sigma^\sharp)"'] \arrow[drr, phantom, "4"] & & U_{\tau\circ\sigma} \circ F_{\tau\circ\sigma}(Y_i) \arrow[d, "U_{\tau\circ\sigma}(Y^\sharp_{\tau\circ\sigma})"] \\
U_\sigma(Y_j) \arrow[rr, "U_\sigma(\eta^\tau_{Y_j})"'] & & U_\sigma \circ U_\tau \circ F_\tau(Y_j) \arrow[rr, "U_{\tau\circ\sigma}(Y_\tau^\sharp)"'] & & U_{\tau\circ\sigma}(Y_k)
\end{tikzcd}$$

The outer square is the right hand square of the lemma. Square 1 commutes by the composition rules for adjunctions and units ([ML], IV.8.1, p. 101). Square 2 commutes by definition of the uniqueness isomorphism. Square 3 commutes since η^τ is a natural transformation of functors, and since U_σ is a functor. Finally, square 4 commutes by hypothesis (apply $U_\sigma \circ U_\tau = U_{\tau\circ\sigma}$ to the left diagram of the lemma).

The other direction of the lemma is proved using similar techniques. We omit the details. $\square$

If $\mathcal{J}$ is a subcategory of $\mathcal{I}$ and $\mathfrak{B}$ is an adjunction bundle over $\mathcal{I}$, we can restrict $\mathfrak{B}$ in an obvious way to an adjunction bundle over $\mathcal{J}$, denoted $\mathfrak{B}|_{\mathcal{J}}$. Similarly, one can restrict a twisted diagram Y with coefficients in $\mathfrak{B}$ to a diagram $Y|_{\mathcal{J}}$ with coefficients in $\mathfrak{B}|_{\mathcal{J}}$. In this way we obtain the restriction functor $\mathfrak{B}^{\mathcal{I}} \longrightarrow \mathfrak{B}^{\mathcal{J}}$ where we have written $\mathfrak{B}^{\mathcal{J}}$ instead of $(\mathfrak{B}|_{\mathcal{J}})^{\mathcal{J}}$. As a special case (if $\mathcal{J} = \{i\}$ is the category with one object and no non-identity morphism) we obtain the evaluation functors $Ev_i\colon \mathfrak{B}^{\mathcal{I}} \longrightarrow \mathcal{C}_i,\ Y \mapsto Y_i$.

The next lemma says that $\mathfrak{B}^{\mathcal{I}}$ *is as complete and cocomplete as all the* $\mathcal{C}_i$, and limits *resp.* colimits can be computed “pointwise” in the categories $\mathcal{C}_i$.

Lemma A.2.4. *Let $G\colon \mathcal{D} \longrightarrow \mathfrak{B}^{\mathcal{I}}$ be a functor, and suppose that for all i the limit of $Ev_i \circ G$ exists. Then $\varprojlim G$ exists and the canonical map*

$$Ev_i(\varprojlim G) \longrightarrow \varprojlim (Ev_i \circ G)$$

is an isomorphism, and similarly for colimits.

Proof. The proof relies on the compatibility of left (*resp.* right) adjoint functors with colimits (*resp.* limits): if F is a left adjoint, and D is a functor, then there is a unique natural isomorphism $\varinjlim (F \circ D) \longrightarrow F(\varinjlim D)$, and similarly for right adjoints and limits ([ML, V.5, theorem 1, p. 114]).

To prove the lemma, we treat the case of colimits only. (For limits one has to use similar techniques. Since U is supposed to be functorial, this is slightly easier.) Let $G_i := Ev_i \circ G$, and define $C_i := \varinjlim G_i$. We claim that the objects C_i assemble to a twisted diagram C, and it is almost obvious that C is "the" colimit of G.

Let $\sigma : i \longrightarrow j$ denote a morphism in $\mathcal{I}$. The $\sharp$-type structure maps of the twisted diagrams $G(d)$ (for objects $d \in \mathcal{D}$) assemble to a natural transformation

$$G^\sharp_\sigma : F_\sigma \circ G_i \longrightarrow G_j$$

of functors $\mathcal{D} \longrightarrow \mathcal{C}_j$. Hence we can define the $\sharp$-type structure map $C^\sharp_\sigma$ as the composite

$$F_\sigma(C_i) = F_\sigma(\varinjlim G_i) \cong \varinjlim (F_\sigma \circ G_i) \xrightarrow{f} \varinjlim G_j = C_j$$

with f induced by $G^\sharp_\sigma$. By lemma A.2.3 we are left to show that the following square commutes for all composable morphisms σ and τ in $\mathcal{I}$:

$$\begin{tikzcd} F_\tau \circ F_\sigma(C_i) \arrow[r, "\cong"] \arrow[d, "F_\tau(C^\sharp_\sigma)"'] & F_{\tau\circ\sigma}(C_i) \arrow[d, "C^\sharp_{\tau\circ\sigma}"] \\ F_\tau(C_j) \arrow[r, "C^\sharp_\tau"'] & C_k \end{tikzcd} \qquad (*)$$

We replace the symbols C_ℓ and the structure maps by their definition and obtain the following bigger diagram:

$$\begin{tikzcd} F_\tau \circ F_\sigma(\varinjlim G_i) \arrow[r, equal] \arrow[d, "\cong"'] \arrow[dr, phantom, "1"] & F_\tau \circ F_\sigma(\varinjlim G_i) \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[dr, phantom, "2"] & F_{\tau\circ\sigma}(\varinjlim G_i) \arrow[d, "\cong"] \\ F_\tau\big(\varinjlim (F_\sigma \circ G_i)\big) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "3"] & \varinjlim (F_\tau \circ F_\sigma \circ G_i) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "4"] & \varinjlim (F_{\tau\circ\sigma} \circ G_i) \arrow[d] \\ F_\tau\big(\varinjlim (G_j)\big) \arrow[r, "\cong"'] & \varinjlim (F_\tau \circ G_j) \arrow[r] & \varinjlim (G_k) \end{tikzcd} \qquad (**)$$

All the small squares commute: for 1 this is true by uniqeness of the isomorphisms for commuting left adjoints with colimits. The horizontal maps of 2 are induced by the uniqueness isomorphism, the vertical maps are induced by the isomorphism for commuting left adjoints with colimits. By uniqueness, 2 commutes. Both horizontal maps of 3 are induced by the isomorphism for commuting colimits with F_τ, and both vertical maps are induced by the natural transformation $G^\sharp_\sigma : F_\sigma \circ G_i \longrightarrow G_j$. Hence 3 commutes. Finally, square 4 commutes by lemma A.2.3, applied componentwise to the diagrams G_ℓ, and by functoriality of $\varinjlim$.

Hence the diagram $(**)$ commutes. But the square $(*)$ is contained in there as the outer square, thus is commutative as claimed. $\square$

A.3. The Model Structures

Before defining the model structures on twisted diagrams, we make two technical observations which will allow an inductive procedure in the proof of the main result of this appendix.

Remark A.3.1. Suppose $\mathcal{C} = \prod_\nu \mathcal{C}_\nu$ is the product of model categories $\mathcal{C}_\nu$. Then there is a "product" model structure on $\mathcal{C}$ where a map is a weak equivalence (*resp.* fibration, *resp.* cofibration) if its image under the canonical projection is a weak equivalence (*resp.* fibration, *resp.* cofibration) in $\mathcal{C}_\nu$ for all ν. If all the $\mathcal{C}_\nu$ are proper, $\mathcal{C}$ is a proper model category.

Suppose $\mathfrak{B}_\nu = (\mathcal{C}^\nu,\ F^\nu,\ U^\nu)$ is a family of adjunction bundles of model categories indexed by $\mathcal{I}_\nu$. Then we can form an adjunction bundle $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ of model categories $\mathfrak{B} := \prod_\nu \mathfrak{B}_\nu$ indexed by the disjoint union $\mathcal{I} := \amalg_\nu \mathcal{I}_\nu$ given by "putting diagrams side by side": for each $i \in \mathcal{I}$ there is exactly one ν with $i \in \mathcal{I}_\nu$, and we define $\mathcal{C}_i = \mathcal{C}^\nu_i$ (and similarly for the F and U). This description shows that $\mathfrak{B}^\mathcal{I} = \prod \mathfrak{B}_\nu^{\mathcal{I}_\nu}$. Thus if each category $\mathfrak{B}_\nu^{\mathcal{I}_\nu}$ admits a model structure, so does $\mathfrak{B}^\mathcal{I}$.

Remark A.3.2. Consider an inverse system

$$\cdots \xrightarrow{\rho_{n+1}} \mathcal{C}_n \xrightarrow{\rho_n} \mathcal{C}_{n-1} \xrightarrow{\rho_{n-1}} \cdots$$

of model categories, and assume that the model structures are *compatible* in the following sense: the restriction functors ρ_n preserve all the structure, and a factorization (*resp.* lifting) in $\mathcal{C}_n$ can be extended to one in $\mathcal{C}_{n+1}$. (Explicitly: if f is a map in $\mathcal{C}_n$ and $\rho(f) = p \circ i$ is a factorization in $\mathcal{C}_{n-1}$ with i an acyclic cofibration and p a fibration, then there is a factorization $f = \bar{p} \circ \bar{\imath}$ in $\mathcal{C}_n$ with $\bar{\imath}$ an acyclic cofibration, $\bar{p}$ a fibration, $\rho(\bar{p}) = p$ and $\rho(\bar{\imath}) = i$; there are similar conditions for the other axioms.) Then the inverse limit $\mathcal{C} := \varprojlim \mathcal{C}_n$ has a model structure where a map is a weak equivalence (*resp.* fibration, *resp.* cofibration) if and only if its image under the canonical map is a weak equivalence (*resp.* fibration, *resp.* cofibration) in $\mathcal{C}_n$ for all n. If all the $\mathcal{C}_n$ are proper, $\mathcal{C}$ is a proper model category.

Definition A.3.3. A *category with degree function* is a (small) category $\mathcal{I}$ together with a $\mathbf{Z}$-valued function d, defined on the objects, such that whenever there is a non-identity morphism $i \longrightarrow j$ we have $d(i) \neq d(j)$. (We say that all non-identity arrows change the degree. In particular, objects have no non-trivial endomorphisms.) The category is called *bounded* if d is bounded below, and it is called *locally bounded* if each connected component is bounded. If non-identity arrows always increase the degree and the category is (locally) bounded, we say that $\mathcal{I}$ is a *(locally) direct category*.

All finite dimensional categories (*i.e.*, categories with finite dimensional nerve) admit degree functions and can be made into direct categories. A disjoint union of locally direct categories is locally direct. If $\mathcal{I}$ is (locally) direct, so are subcategories, under and over categories formed with $\mathcal{I}$. In particular, the full subcategory $\mathcal{I}_{d<m}$ of objects of degree strictly less than m is (locally) direct, and similarly for $\mathcal{I}_{d\leq m}$. A finite product of direct categories is direct (with degree given by sum of partial degrees).

In what follows we assume that $\mathcal{I}$ is a partially ordered set and $\mathfrak{B}$ is an adjunction bundle of model categories over $\mathcal{I}$. Note that a degree function on a direct poset is nothing but a strongly monotone map $\operatorname{Ob}\mathcal{I} \longrightarrow \mathbf{Z}$ with a lower bound.

Let Y denote a twisted diagram with coefficients in $\mathfrak{B}$. For each object $i \in \mathcal{I}$, let $\mathcal{I}^{\leq i}$ denote the full subcategory of objects equal to or smaller than i with respect to the given partial order. (This is the over category $\mathcal{I} \downarrow i$.) Similarly, we can form $\mathcal{I}^{<i}$.

Definition A.3.4. The *latching object of Y at i* (it is an object in $\mathcal{C}_i$) is defined as $L_iY := \varinjlim_{\mathcal{I}^{<i}} R(Y)$. Here $R(Y)$ denotes the restriction of Y to the subcategory $\mathcal{I}^{<i}$ pushed forward into $\mathcal{C}_i$; explicitly, it is the functor

$$R(Y)\colon \mathcal{I}^{<i} \longrightarrow \mathcal{C}_i, \quad (\sigma : j \longrightarrow i) \mapsto F_\sigma(Y_j) \ .$$

The structure maps are defined as follows: suppose $j \xrightarrow{\alpha} k \xrightarrow{\tau} i$ is a factorization of σ (*i.e.*, a map $\alpha\colon \sigma \longrightarrow \tau$ in $\mathcal{I}^{<i}$). The functors F_σ and $F_\tau \circ F_\alpha$ are canonically isomorphic since both are left adjoint to $U_\sigma = U_\alpha \circ U_\tau$, and we define the structure map $R(Y)(\alpha)$ as the composite

$$R(Y)_j = F_\sigma(Y_j) \cong F_\tau \circ F_\alpha(Y_j) \xrightarrow{F_\tau(Y_\alpha^\#)} F_\tau(Y_k) = R(Y)_k \ . \tag{$*$}$$

For the definition of latching spaces to make sense, we have to show that $R(Y)$ is a commutative diagram in $\mathcal{C}_i$:

Lemma A.3.5. *The construction given above defines a functor $R(Y)\colon \mathcal{I}^{<i} \longrightarrow \mathcal{C}_i$.*

Proof. As a start, note that $R(Y)(\mathrm{id}_j) = \mathrm{id}_{F_\sigma(Y_j)}$ where σ denotes the unique map $j \longrightarrow i$: set $\alpha = \mathrm{id}_j$ in $(*)$, and recall that $F_{\mathrm{id}_j} = \mathrm{id}_{\mathcal{C}_j}$ by definition of adjunction bundles (A.2.1). Moreover, $Y^\flat_{\mathrm{id}_j} = \mathrm{id}_{Y_j}$ by definition of twisted diagrams (A.2.2), hence $Y^\sharp_{\mathrm{id}_j} = \mathrm{id}_{Y_j}$.

Now suppose given a commutative diagram

$$\begin{tikzcd} j \arrow[r, "f"] \arrow[dr, "\sigma"'] & k \arrow[r, "g"] \arrow[d, "\tau"] & \ell \arrow[dl, "\mu"] \\ & i & \end{tikzcd}$$

($i.e.$, two composable arrows f and g in $\mathcal{I}^{<i}$), we prove $R(Y)(g\circ f) = R(Y)(g)\circ R(Y)(f)$. In other words, we prove that the following diagram commutes:

$$\begin{tikzcd}
R(Y)_j \arrow[rr, "R(Y)(f)"] \arrow[d, equal] \arrow[drr, phantom, "1"] & & R(Y)_k \arrow[rr, "R(Y)(g)"] \arrow[d, equal] \arrow[drr, phantom, "2"] & & R(Y)_\ell \arrow[d, equal] \\
F_\sigma(Y_j) \arrow[r, "\cong"] \arrow[d, equal] & F_\tau \circ F_f(Y_j) \arrow[r, "F_\tau(Y_f^\sharp)"] & F_\tau(Y_k) \arrow[r, "\cong"] \arrow[d, phantom, "4"] & F_\mu \circ F_g(Y_k) \arrow[r, "F_\mu(Y_g^\sharp)"] & F_\mu(Y_\ell) \arrow[d, equal] \\
F_\sigma(Y_j) \arrow[r, "\cong"] \arrow[d, equal] & F_\mu \circ F_{g\circ f}(Y_j) \arrow[rrr, "F_\mu(Y_{g\circ f}^\sharp)"] & {} \arrow[d, phantom, "3"] & & F_\mu(Y_\ell) \arrow[d, equal] \\
R(Y)_j \arrow[rrrr, "R(Y)(g\circ f)"'] & & {} & & R(Y)_\ell
\end{tikzcd}$$

The squares 1, 2 and 3 are just the definitions of the structure maps in $R(Y)$. We redraw rectangle 4 with some extra data added:

$$\begin{tikzcd}
F_\sigma(Y_j) \arrow[r, "\cong"] \arrow[d, equal, "\mathrm{id}"'] \arrow[dr, phantom, "1"] & F_\tau \circ F_f(Y_j) \arrow[r, "F_\tau(Y_f^\sharp)"] \arrow[d, "\cong"] \arrow[dr, phantom, "2"] & F_\tau(Y_k) \arrow[d, "\cong"] \\
F_\sigma(Y_j) \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[dr, phantom, "3"] & F_\mu \circ F_g \circ F_f(Y_j) \arrow[r, "F_\mu\circ F_g(Y_f^\sharp)"] \arrow[d, "\cong"] \arrow[dr, phantom, "4"] & F_\mu \circ F_g(Y_k) \arrow[d, "F_\mu(Y_g^\sharp)"] \\
F_\mu \circ F_{g\circ f}(Y_j) \arrow[r, equal] & F_\mu \circ F_{g\circ f}(Y_j) \arrow[r, "F_\mu(Y_{g\circ f}^\sharp)"'] & F_\mu(Y_\ell)
\end{tikzcd}$$

All marked isomorphisms are uniqueness isomorphisms. Hence squares 1 and 3 commute by uniqueness of the isomorphisms themselves. Square 2 commutes by naturality of the uniqueness isomorphism, and 4 is the image of the diagram

$$\begin{tikzcd}
F_g \circ F_f(Y_j) \arrow[r, "F_g(Y_f^\sharp)"] \arrow[d, "\cong"'] & F_g(Y_k) \arrow[d, "Y_g^\sharp"] \\
F_{g\circ f}(Y_j) \arrow[r, "Y_{g\circ f}^\sharp"'] & Y_\ell
\end{tikzcd}$$

under the functor F_μ. This diagram is known to commute by lemma A.2.3. □

We may consider L_i as a functor $\mathfrak{B}^{\mathcal{I}^{<i}} \longrightarrow \mathcal{C}_i$, $i.e.$, the expression L_iY makes sense for objects $Y \in \mathfrak{B}^{\mathcal{I}^{<i}}$ since only the objects Y_j for $j < i$ are involved in the definition.

Note that if Y is an object of $\mathfrak{B}^{\mathcal{I}^{\leq i}}$ (if, for example, Y is the restriction of an object of $\mathfrak{B}^{\mathcal{I}}$), L_iY comes equipped with a map to Y_i induced by the structure maps $Y_\sigma^\sharp : F_\sigma(Y_j) \longrightarrow Y_i$.

Fix an object $i \in \mathcal{I}$. For $T \in \mathcal{C}_i$, define a twisted diagram $\gamma^i(T) \in \mathfrak{B}^{\mathcal{I}^{<i}}$ by the assignment

$$(\sigma : j \longrightarrow i) \mapsto U_\sigma(T)$$

with identities as $\flat$-type structure maps. (This yields a twisted diagram because of functoriality of U.) We obtain a functor $\gamma^i : \mathcal{C}_i \longrightarrow \mathfrak{B}^{\mathcal{I}^{<i}}$ called "diagonal" or "constant diagram" functor. (For this definition to make sense we need that $\mathcal{I}$ is a poset.)

Proposition A.3.6. *The functor $L_i : \mathfrak{B}^{\mathcal{I}^{<i}} \longrightarrow \mathcal{C}_i$ is left adjoint to γ^i.*

Proof. Let $\mathcal{C}_i^{\mathcal{I}^{<i}} = Func(\mathcal{I}^{<i}, \mathcal{C}_i)$ denote the functor category. The diagonal functor $\Delta : \mathcal{C}_i \longrightarrow \mathcal{C}_i^{\mathcal{I}^{<i}}$ is right adjoint to the colimit functor, hence there is a natural bijection of sets $\hom_{\mathcal{C}_i}(\varinjlim R(Y), T) \cong \hom_{\mathcal{C}_i^{\mathcal{I}^{<i}}}(R(Y), \Delta(T))$. We claim that the sets

$$\hom_{\mathcal{C}_i^{\mathcal{I}^{<i}}}(R(Y), \Delta(T)) \qquad \text{and} \qquad \hom_{\mathfrak{B}^{I^{<i}}}(Y, \gamma^i(T))$$

are naturally isomorphic. The concatenation of these two bijections gives the desired adjunction since $\varinjlim \circ R = L_i$.

Given $f \in \hom_{\mathcal{C}_i^{\mathcal{I}^{<i}}}(R(Y), \Delta(T))$, define $\bar{f} : Y \longrightarrow \gamma^i(T)$ as the componentwise adjoint: if $\sigma : j \longrightarrow i$ is a non-identity map in $\mathcal{I}$, *i.e.*, an object of $\mathcal{I}^{<i}$, let $\bar{f}_j : Y_j \longrightarrow U_\sigma(T) = \gamma^i(T)_j$ be the map adjoint to $f_j : F_\sigma(Y_j) \longrightarrow T$.

Conversely, given $g \in \hom_{\mathfrak{B}^{I^{<i}}}(Y, \gamma^i(T))$, define $\tilde{g} : R(Y) \longrightarrow \Delta(T)$ by choosing $\tilde{g}_j : R(Y)_j = F_\sigma(Y_j) \longrightarrow T$ as the adjoint of the map $g_j : Y_j \longrightarrow U_\sigma(T) = \gamma^i(T)_j$.

By the properties of adjunctions, we have $\tilde{\bar{f}}_j = f_j$ and hence $\tilde{\bar{f}} = f$. Similarly, $\bar{\tilde{g}} = g$. It remains to check the following things:
(1) The maps $\bar{f}_j$ assemble to a map $\bar{f}$ of twisted diagrams.
(2) The maps $\tilde{g}_j$ assemble to a map $\tilde{g}$ of diagrams.
The techniques are similar to those used in the proof of the previous lemma; we omit the details. □

Definition A.3.7. *(c-structure)*

Let $f : Y \longrightarrow Z$ be a map in $\mathfrak{B}^{\mathcal{I}}$. We call f a *weak equivalence* if f_i is a weak equivalence in $\mathcal{C}_i$ for every object $i \in \mathcal{I}$. We call f a *c-cofibration* if for all objects i of I, the induced map $Y_i \cup_{L_i Y} L_i Z \longrightarrow Z_i$ is a cofibration. We call f a *c-fibration* if all f_i are fibrations in $\mathcal{C}_i$.

Theorem A.3.8. *Suppose $\mathcal{I}$ is a locally direct poset, and all the $\mathcal{C}_i$ are cocomplete.*
(1) The c-structure is a model structure.
(2) If f is a c-cofibration, all its components are cofibrations in their respective categories.
(3) A map f of twisted diagrams is an acyclic c-cofibration if and only if for all objects $i \in \mathcal{I}$, the induced map $Y_i \cup_{L_i Y} L_i Z \longrightarrow Z_i$ is an acyclic cofibration in $\mathcal{C}_i$.
(4) If $\mathfrak{B}$ is a proper bundle, the c-structure is proper.

Proof. First we note that the class of weak equivalences is closed under composition since weak equivalences are defined pointwise. Similarly, the composition of two c-fibrations is a c-fibration again.

Now assume we have two composable c-cofibrations $A \xrightarrow{f} B \xrightarrow{g} C$. To show that $g \circ f$ is a c-cofibration, we have to prove that for all objects $i \in \mathcal{I}$ the induced map

$$A_i \cup_{L_i A} L_i C \longrightarrow C_i$$

is a cofibration in $\mathcal{C}_i$. But we can factor this map as

$$\begin{aligned} A_i \cup_{L_i A} L_i C &\cong A_i \cup_{L_i A} L_i B \cup_{L_i B} L_i C \\ &\xrightarrow{x} B_i \cup_{L_i B} L_i C \\ &\xrightarrow{y} C_i \end{aligned}$$

where x is induced by f, and y is induced by g. But both of these maps are cofibrations (since they are cobase changes of cofibrations), hence so is their composite.

Let $\mathcal{I}_\nu$ denote the family of path components of $\mathcal{I}$. Then $\mathcal{I} = \coprod \mathcal{I}_\nu$, and each of the $\mathcal{I}_\nu$ is a direct poset. Since $\mathfrak{B}^{\mathcal{I}} = \prod \mathfrak{B}^{\mathcal{I}_\nu}$ by A.3.1 in this case, it is enough to show that the c-structure is a model structure for each of the categories $\mathfrak{B}^{\mathcal{I}_\nu}$. By A.3.1 we can equip $\mathfrak{B}^{\mathcal{I}}$ with the product model structure.

So assume that $\mathcal{I}$ is direct. Consider the filtration $\mathcal{I} = \bigcup_n \mathcal{I}_{d \leq n}$ given by the degree function. This induces an inverse system

$$\dots \longrightarrow \mathfrak{B}^{\mathcal{I}_{d\leq n+1}} \longrightarrow \mathfrak{B}^{\mathcal{I}_{d\leq n}} \longrightarrow \dots$$

which is bounded on the right (since $\mathcal{I}$ is direct by hypothesis). The inverse limit is the category $\mathfrak{B}^{\mathcal{I}}$. By induction on n, we will construct compatible model structures on the members of the inverse system. By remark A.3.2 this gives the desired model structure on $\mathfrak{B}^{\mathcal{I}}$.

We use the axioms for model categories as given in [DS].

To start the induction, take $n := \min d$ where d is the degree function of $\mathcal{I}$. Since all non-identity morphisms raise the degree, $\mathcal{I}_{d \leq n}$ is a discrete category with objects all degree-n objects of $\mathcal{I}$ (and no morphisms except the identities). We can write $\mathcal{I}_{d\leq n}$ as a coproduct of trivial categories with one object and one morphism. Thus by remark A.3.1 the category $\mathfrak{B}^{\mathcal{I}_{d\leq n}}$ has a model structure satisfying all the conditions (1)–(4).

Assume that the theorem has been proved for $\mathfrak{B}^{\mathcal{I}_{d\leq m}}$ for all $m < n$. We show that it also holds for $\mathfrak{B}^{\mathcal{I}_{d\leq n}}$.

Axiom **MC1**: existence of finite limits and colimits is guaranteed by A.2.4 since they exist in all $\mathcal{C}_i$.

Axiom **MC2**: the "2-of-3" property for weak equivalences is satisfied since weak equivalences are defined pointwise and **MC2** holds in all the categories $\mathcal{C}_i$.

Axiom **MC3**: the class of weak equivalences is closed under retracts since weak equivalences are defined pointwise, and in each category $\mathcal{C}_i$ a retract of a weak equivalence is a weak equivalence. Similarly, the class of fibrations is closed under retracts.

Suppose $g\colon Y \longrightarrow Z$ is a retract of $f\colon A \longrightarrow B$ and f is a c-cofibration. We have to show that for all objects $i \in \mathcal{I}_{d\leq n}$, the map $L_iZ \cup_{L_iY} Y_i \longrightarrow Z_i$ induced by g is a cofibration in $\mathcal{C}_i$. But by functoriality of pushouts and latching spaces, this map is a retract of $L_iB \cup_{L_iA} A_i \longrightarrow B_i$ induced by f which is a cofibration by hypothesis. Since **MC3** is valid in $\mathcal{C}_i$, the former map is a cofibration. Hence g is a c-cofibration as claimed.

Axiom **MC4**: suppose we have a diagram in $\mathfrak{B}^{\mathcal{I}_{d\leq n}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "f"'] & X \arrow[d, two heads, "\wr" ', "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

with f a c-cofibration and p an acyclic c-fibration. We want to construct a lifting $B \longrightarrow X$.

By induction we obtain a lift of restricted twisted diagrams

$$\begin{tikzcd} A|_{I_{d\leq n-1}} \arrow[r] \arrow[d, tail] & X|_{I_{d\leq n-1}} \arrow[d, two heads, "\wr"'] \\ B|_{I_{d\leq n-1}} \arrow[r] \arrow[ur, dotted, "l"'] & Y|_{I_{d\leq n-1}} \end{tikzcd}$$

(note that restriction preserves the c-structure). Assume that $i \in \mathcal{I}$ has degree n. We claim that there is a diagram in $\mathcal{C}_i$ of the following form:

$$\begin{tikzcd} A_i \cup_{L_iA} L_iB \arrow[r] \arrow[d, tail] & X_i \arrow[d, two heads, "\wr"', "p_i"] \\ B_i \arrow[r] & Y_i \end{tikzcd} \tag{$*$}$$

The left vertical map is a cofibration by definition of c-cofibrations. The upper horizontal arrow is constructed as follows: let j be an object of $\mathcal{I}^{<i}$ and $\sigma : j \longrightarrow i$ the unique morphism. Then $d(j) < d(i)$, hence we have a map $l_j : B_j \longrightarrow X_j$. Composing with the structure maps of X we obtain a map $F_\sigma(B_j) \xrightarrow{F_\sigma(l_j)} F_\sigma(X_j) \xrightarrow{X^\sharp_\sigma} X_a$. All these maps for various σ induce a map $L_iB \longrightarrow X_i$ which is compatible with the given map $A_i \longrightarrow X_i$.

Now define l_i to be a lifting in diagram $(*)$, and observe that this extends l to a map of twisted diagrams indexed by $\mathcal{I}_{d\leq n-1} \cup \{i\}$. Repeat this construction for all objects i with degree n.

To prove (2), note that $\gamma^i : \mathcal{C}_i \longrightarrow \mathfrak{B}^{\mathcal{I}^{<i}}$ preserves fibrations and acyclic fibrations. Hence by adjointness (proposition A.3.6) $L_i : \mathfrak{B}^{\mathcal{I}^{<i}} \longrightarrow \mathcal{C}_i$ preserves cofibrations and acyclic cofibrations. Furthermore, the restriction of a c-cofibration in $\mathfrak{B}^{\mathcal{I}_{d\leq n}}$ to a map in $\mathfrak{B}^{\mathcal{I}^{<i}}$ is a c-cofibration. It follows that the map $L_iA \longrightarrow L_iB$ is a cofibration. Consequently, $A_i \longrightarrow B_i$ is a cofibration since it is given by the composition $A_i \rightarrowtail A_i \cup_{L_iA} L_iB \rightarrowtail B_i$.

Now we prove the second half of **MC4**. Suppose we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "f"', "\wr"] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

where the left vertical map is an acyclic c-cofibration, and the right vertical map is a c-fibration.

Fix $i \in \mathcal{I}$ with $d(i) = n$, and suppose by induction that we have lifts for the restricted diagrams $l\colon B|_{\mathcal{I}_{d<n}} \longrightarrow X|_{\mathcal{I}_{d<n}}$. Consider the following diagram:

$$\begin{tikzcd} A_i \cup_{L_iA} L_iB \arrow[r] \arrow[d, tail] & X_i \arrow[d, "p_i"] \\ B_i \arrow[r] & Y_i \end{tikzcd}$$

The maps are given as in diagram $(*)$ above. By part (2) of the theorem, all components of the map $A|_{\mathcal{I}<i} \longrightarrow B|_{\mathcal{I}<i}$ of restricted diagrams are acyclic cofibrations in their respective categories. By adjointness of L_i and γ^i, the map $L_iA \longrightarrow L_iB$ is thus an acyclic cofibration. Hence $A_i \longrightarrow B_i$ is a cofibration since it is the composite $A_i \longrightarrow A_i \cup_{L_iA} L_iB \longrightarrow B_i$. Here the first map is a cobase change of an acyclic cofibration, hence is itself an acyclic cofibration. The second map is a cofibration since f is a c-cofibration by hypothesis. This also implies that the composite is acyclic if and only if the map $A_i \cup_{L_iA} L_iB \longrightarrow B_i$ is a weak equivalence which proves (3). Moreover, we have shown that the left vertical map in the above square is an acyclic cofibration, hence there is a lift $l_i : B_i \longrightarrow X_i$ compatible with the partial lifts constructed so far.

Axiom **MC5**: we want to factor a map $A \longrightarrow B$ as an acyclic c-cofibration followed by a c-fibration. By induction, we can assume that we have constructed an object $T \in \mathfrak{B}^{\mathcal{I}_{d<d(i)}}$ and a (partial) factorization $A|_{\mathcal{I}_{d<d(i)}} \longrightarrow T \longrightarrow B|_{\mathcal{I}_{d<d(i)}}$ of the restricted diagrams.

The left vertical map in the diagram

$$\begin{tikzcd} L_iA \arrow[r] \arrow[d] & A_i \arrow[d] \\ L_iB \arrow[r] & B_i \end{tikzcd}$$

factors through L_iT where the map $L_iA \longrightarrow L_iT$ is an acyclic cofibration in $\mathcal{C}_a$ by an adjointness argument (use A.3.6). Hence A_i maps into $K := L_iT \cup_{L_iA} A_i$ by an acyclic cofibration. Factor the induced map from K to B_i as $K \overset{\sim}{\rightarrowtail} T_i \twoheadrightarrow B_i$ and observe that T can be extended by T_i to a twisted diagram over $\mathcal{I}_{d<d(i)} \cup \{i\}$ ($\sharp$-type structure maps are given by the composites $F_\sigma(T_j) \longrightarrow K \longrightarrow T_i$). Furthermore, the resulting factorization $A \longrightarrow T \longrightarrow B$ of diagrams over $\mathcal{I}_{d<d(i)} \cup \{i\}$ has the desired form.

The other factorization is constructed similarly, which completes the proof of parts (1), (2) and (3).

To prove (4), recall from lemma A.2.4 that pushouts and pullbacks are calculated pointwise. Since the components of a weak equivalence (fibration, cofibration) are weak equivalences (fibrations, cofibrations) in the respective categories (use part (2) for the cofibrations), the assertion follows from remarks A.3.1 and A.3.2. $\square$

Example A.3.9. *(Spectra)*

In this case, $\mathcal{I} = \mathbf{N}$, all $\mathcal{C}_i = Top_*$ and the U_σ and F_σ are given by (iterated) loop and suspension functors, respectively. Note that $\mathbf{N}$ is a direct poset (with $d(n) = n$). Hence we obtain the model structure with levelwise weak equivalences and fibrations. (This structure can be used as a starting point for the construction of the stable model structure of spectra.)

Everything of the above can be dualized. There is a notion of a (locally) inverse category, and matching spaces allow us to define an f-structure (with pointwise cofibrations and weak equivalences). We omit the proofs.

Definition A.3.10. A category with degree function is called a *(locally) inverse category* if its opposite category (with the same degree function) is (locally) direct. If $\mathcal{I}$ is a poset, denote by $\mathcal{I}^{>i}$ the full subcategory of objects strictly bigger than i. The *matching object of Y at i* (it is an object in $\mathcal{C}_i$) is defined as $M^iY := \lim_{\mathcal{I}^{>i}} S(Y)$, where $S(Y)$ denotes the restricted diagram pulled back to $\mathcal{C}_i$, *i.e.*, the functor

$$S(Y) : \mathcal{I}^{>i} \longrightarrow \mathcal{C}_i, \quad (\sigma : i \longrightarrow j) \mapsto U_\sigma(Y_j) \ .$$

Structure maps are defined by a similar procedure as for latching objects. Note that because the right adjoint functors U_σ commute strictly we obtain a strictly commuting diagram.

We may consider M^i as a functor $\mathfrak{B}^{\mathcal{I}^{>i}} \longrightarrow \mathcal{C}_i$. Note that M^iY comes equipped with a map from Y_i induced by the structure maps $Y^\flat_\sigma : Y_i \longrightarrow U_\sigma(Y_j)$.

Fix an object $i \in \mathcal{I}$. For $T \in \mathcal{C}_i$, define a twisted diagram $\beta_i(T) \in \mathfrak{B}^{\mathcal{I}^{>i}}$ by the assignment

$$(\sigma : i \longrightarrow j) \mapsto F_\sigma(T)$$

with "identities" as $\sharp$-type structure maps; to be precise, if $i \xrightarrow{\tau} j \xrightarrow{\alpha} k$ is a factorization of σ (*i.e.*, a map $\tau \longrightarrow \sigma$ in $\mathcal{I}^{>i}$), the structure map $\beta_i(T)^\sharp_\alpha$ is given by the canonical isomorphism $F_\alpha \circ F_\tau(T) \xrightarrow{\cong} F_\sigma(T)$. Application of lemma A.2.3 shows that this defines a twisted diagram. Hence we obtain a functor $\beta_i : \mathcal{C}_i \longrightarrow \mathfrak{B}^{\mathcal{I}^{>i}}$ called "diagonal" or "constant diagram" functor.

Proposition A.3.11. *The functor M^i is right adjoint to β_i.* $\square$

Definition A.3.12. *(f-structure)*

Let $f: Y \longrightarrow Z$ be a map in $\mathfrak{B}^{\mathcal{I}}$. We call f a *weak equivalence* if f_i is a weak equivalence in $\mathcal{C}_i$ for every object $i \in \mathcal{I}$. We call f an *f-fibration* if for all objects i of I, the induced map $Y_i \longrightarrow Z_i \times_{M^i Z} M^i Y$ is a fibration. We call f an *f-cofibration* if all f_i are cofibrations in $\mathcal{C}_i$.

Theorem A.3.13. *Suppose $\mathcal{I}$ is a locally inverse poset, and all the $\mathcal{C}_i$ are complete.*

(1) The f-structure is a model structure.

(2) If f is an f-fibration, all its components are fibrations in their respective categories.

(3) A map f of twisted diagrams is an acyclic f-fibration if and only if for all objects $i \in \mathcal{I}$, the induced map $Y_i \longrightarrow Z_i \times_{M^i Y} M^i Z$ is an acyclic fibration in $\mathcal{C}_i$.

(4) If $\mathfrak{B}$ is a proper bundle, the f-structure is proper. $\square$

B. List of Symbols and Terminology

I	$[0,1]_+$, the interval with a disjoint basepoint	
G	a topological monoid with zero	1.2
$k\,Top_*$	category of compactly generated spaces	1.2
$G\text{-}k\,Top_*$	category of G-equivariant compactly generated spaces	1.2
$\mathbf{C}(G)$	subcategory of cofibrant objects of $G\text{-}k\,Top_*$	1.2
$\mathbf{C}_?(G)$	cofibrant objects with finiteness condition	1.2
$h,\ hf,$		
$fd,\ sfd$	finiteness conditions for spaces	1.2
$\cdot \wedge_G \cdot$	"tensor product" of G-spaces	1.2
$G\text{-}k\,Top_*^{\langle N\rangle}$	category of N-cubical diagrams in $G\text{-}k\,Top_*$	1.3.1
L_A	A-th latching space functor	1.3, 2.4, A.3
Γ	iterated homotopy cofibre, global sections	1.3.1, 2.7.1
$G\text{-}k\,Top_*^{\langle N\rangle_0}$	category of almost N-cubical diagrams in $G\text{-}k\,Top_*$	
N	a non-empty finite set	
n	$n = \#N - 1$	
$\langle n\rangle$	the set $\{0,\ 1,\ \ldots,\ n\}$	
$[n]$	the ordered set $\{0,\ 1,\ \ldots,\ n\}$	
$\langle A\rangle$	power set of A	
$\langle A\rangle_0$	set of non-empty subsets of A	
$\langle A\rangle_0^1$	set of non-empty proper subsets of A	
$M_N^A, \widetilde{M}_N^A$	certain discrete monoids with zero	2.2.1
$\widetilde{M}_N^A(j)$		2.2.1
t_i	a distinguished element of $\widetilde{M}_N^A$	2.2.1
t_A	multi-index notation: $t_A := \prod_{i\in A} t_i$	
$\cdot[t_B^{-1}]$	inverting the action of t_B	2.2.6
$\mathbf{pP}^N(G)$	category of presheaves	2.3.1
$Y_\sigma^\flat$	$\flat$-type structure maps	2.3.1, A.2.2
$Y_\sigma^\sharp$	$\sharp$-type structure maps	2.3.1, A.2.2
$\mathbf{P}^N(G)$	category of sheaves	2.3.3
$\mathbf{P}_?^N(G)$	locally cofibrant sheaves with finiteness condition	3.1.1
$\overline{\mathbf{P}}_{sfd}^N(G)$	weakly cofibrant, stably finitely dominated sheaves	3.1.1
$\check{\mathbf{P}}_f^N(G)$	strongly cofibrant, locally finite sheaves	3.1.1
$\mathbf{P}_{sfd}^{N,T}(G)$	a certain subcategory of $\mathbf{P}_{sfd}^N$	3.3.5
$\overline{\mathbf{P}}_{sfd}^{N,T}(G)$	a certain subcategory of $\overline{\mathbf{P}}_{sfd}^N$	3.3.5

h, hf,		
fd, sfd	finiteness conditions for sheaves	3.1.1
$\mathcal{O}_{\mathbf{P}^N}$	structure sheaf of $\mathbf{P}^N$	2.3.2
L_A^{+C}	restricted latching space	2.4.9
	c-structure	1.3, 2.4.1, A.3
	f-structure	1.3, 2.4.4, A.3
	weak cofibration	2.4.6
	weakly cofibrant, locally cofibrant, strongly cofibrant	2.4.7
ζ_j	extension by zero	2.5.3
ρ_j	restriction	2.5.1
$\cdot \wedge_{\mathcal{O}} \cdot$	tensor product of sheaves	2.6.1
θ_k	k-th twist functor (symmetric version)	2.6.2
κ	natural transformation $\theta_k \longrightarrow \theta_{k+1}$	2.6.7
ϑ_k	k-th twist functor (asymmetric version)	2.6.4
$\bar{\kappa}$	natural transformation $\vartheta_k \longrightarrow \vartheta_{k+1}$	2.6.8
ψ_k	twisted canonical sheaf functor	2.6.9
σ	a funny model for the suspension functor	2.8.6
	globally finite sheaves	3.2.1
	h-equivalence	2.4, 3.3.1
	h_S-equivalence	2.8.1
	adjunction bundle	A.2.1
	direct and inverse categories	A.3.3, A.3.10

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