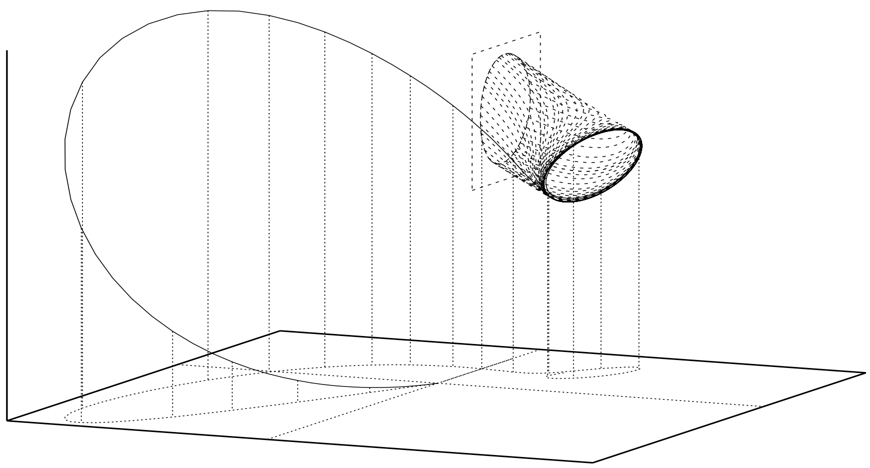
Thorsten Pampel

Numerical approximation of generalized connecting orbits

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned}$$



Numerical approximation of generalized connecting orbits

Dissertation

zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von

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Bielefeld, im Dezember 1999

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Zweiter Gutachter:

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Datum der mündlichen Prüfung: 14. April 2000

Pampel, Thorsten

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generalized connecting orbits

Archivierungsexemplare gedruckt auf
alterungsbeständigem Papier ∞ ISO 9706

Acknowledgments

I am very grateful to Prof. Dr. W.–J. Beyn for the interesting subject, helpful comments and exciting discussions.

Furthermore, I feel bound to thank Prof. Dr. W. Semmler for interesting discussions about applications in economics.

I benefitted a lot from the stimulating environment of the numerical group at the Department of Mathematics and I am grateful to all my colleagues.

This thesis are worked out within the scope of the project "Connecting orbits in highdimensional dynamical systems", supported by the DFG–Schwerpunktprogramm "Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme".

Im Zuge dieser Veröffentlichung wurde die vorliegende Dissertation redaktionell überarbeitet.
Bielefeld, im Mai 2000

Contents

Notation

$\mathbb{R}$, $\mathbb{C}$	real and complex numbers
$\mathbb{R}_+$	$= [0, \infty)$, nonnegative real numbers
$\mathbb{R}^m$	vector space of m–tuples
$\mathbb{R}^{m,n}$	vector space of $m \times n$–matrices
$\mathrm{I}_m \in \mathbb{R}^{m,m}$	identity matrix
$\mathrm{Re}(c)$, $\mathrm{Im}(c)$	real and imaginary part of $c \in \mathbb{C}$
$\mathrm{Re}(M)$	$= \{\mathrm{Re}(c) \mid c \in M\}$ for $M \subset \mathbb{C}$
$\dim(V)$	dimension of a vector space V
$\mathcal{R}(V)$, $\mathcal{N}(V)$	range and kernel of a linear map V
$\mathrm{span}(A)$	linear subspace spanned by the columns of the matrix A
$\sigma(A)$	$= \{\mu \in \mathbb{C} \mid \mu$ is eigenvalue of $A\}$, spectrum of a linear map A
$\mathrm{rank}(A)$	$= \dim\big(\mathcal{R}(A)\big)$, rank of a matrix $A \in \mathbb{R}^{m,n}$
Id_X	identity on a set X
$\dot{x}$	$= \frac{\mathrm{d}}{\mathrm{dt}}x$, derivative with respect to time t
$\mathrm{im}(f)$	image of a function f
$\mathcal{B}_\epsilon(\bar{x})$	$= \big\{x \in \mathcal{X} \;\big\vert\; \lvert\lvert x - \bar{x} \rvert\rvert_{\mathcal{X}} < \epsilon\big\}$, ϵ–ball at $\bar{x}$ in a Banach space $(\mathcal{X}, \lvert\lvert \cdot \rvert\rvert_{\mathcal{X}})$
$T_x\mathcal{W}$	tangent space of a manifold $\mathcal{W}$ at x
$BC^k_\gamma(J, \mathcal{X})$	Banach space of k times differentiable exponentially weighted functions
$BC^k(J, \mathcal{X})$	Banach space of k times differentiable and bounded functions

Introduction

The occurrence of connecting orbits in continuous dynamical systems is often related to structural changes. We mention several examples which illustrate this point. For example, a branch of periodic solutions may terminate in a homoclinic connecting orbit if the period tends to infinity (see [Guckenheimer and Holmes, 1993, Theorem 6.1.1]). Moreover, in a neighborhood of a homoclinic connecting orbit corresponding to a saddle focus there might be chaotic behavior (see [Shilnikov, 1965],[Kuznetsov, 1995]). Complicated behavior may also arise at an orbit-flip bifurcation (for a detailed analysis of this bifurcation see [Sandstede, 1993]). At such a bifurcation a homoclinic orbit exists, which converges to an equilibrium via its strong-stable or strong-unstable manifold. Another example is the Lorenz system [Sparrow, 1982], where the strange invariant set becomes an attractor at those parameter value, where a point to periodic connecting orbit exists (see Figure 0.1).

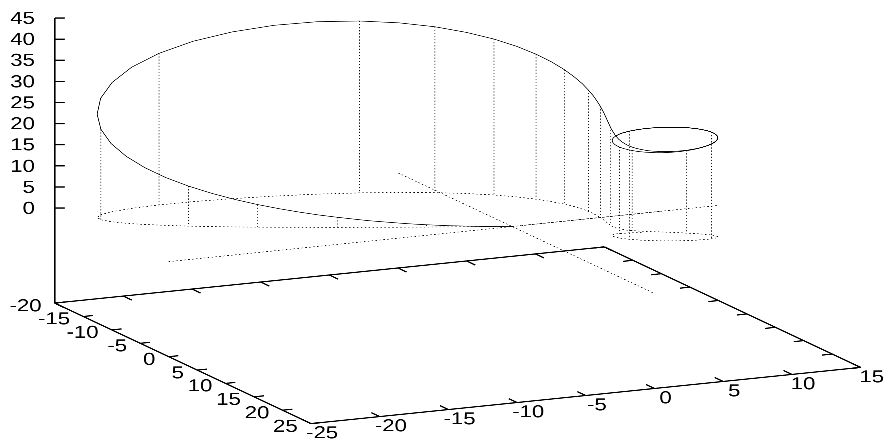

Figure 0.1: Approximation of a point to periodic connecting orbit in the Lorenz system.

In the literature there are several approximation methods for these different types of connecting orbits. All of them base on the truncation of the infinite interval and the use of asymptotic boundary conditions. We mention some of these results.

For orbits connecting hyperbolic equilibria an approximation method and its error estimates are well known [Beyn, 1990a, 1990b]. Results on the persistence and the non-degeneracy of point to periodic and periodic to periodic connecting orbits are derived in [Hale and Lin, 1986],[Beyn, 1994b]. Moreover a numerical method for a periodic to periodic connecting orbit is developed and applied to a reduced water-wave problem in [Champneys and Lord, 1997].

In [Champneys and Kuznetsov, 1994],[Champneys et al., 1996] a numerical approach for the detection and continuation of codimensional-two homoclinic bifurcations is presented. In particular, this approach is used to detect homoclinic orbits which converge on one side to an equilibrium via its strong-stable or strong-unstable manifold, respectively. Moreover, the case, where the leading eigenvalue is 0 is mentioned.

We present a brief summary of the results of this thesis.

We set up and analyze an approximation method for a generalized form of connecting orbits, which includes the connecting orbits mentioned above, except those with leading eigenvalue 0. Moreover, there are other applications which have the structure of "generalized connecting orbits", e. g. we apply these results to optimization problems.

A main result of this thesis are error estimates for our method in the generalized setup. In particular, we show that the error decays exponentially with the length of the approximation interval, even in the strong-stable case and even for periodic solutions. This is in agreement with the known results for orbits connecting hyperbolic equilibria.

For the approximation method we compute appropriate asymptotic boundary conditions. These are typically parameter-dependent. We develop the iterative so called "boundary corrector method" which allows to apply our method with constant asymptotic boundary conditions.

The structure of a generalized connecting orbit implies, that the boundary value problem can be split into two subproblems, which are coupled only by the initial values. Therefore, we also derive an approximation method and its error estimates for such a one sided subproblem, i. e. we approximate a solution with initial value in a given manifold and which converges with an exponential rate to a given solution.

To derive the error estimates we define a stable or strong-stable manifold of such a solution and its smoothness, in particular with respect to parameters and time.

Now we expand on some details.

We develop the notion of a **generalized connecting orbit** which contains two solutions and associated parameters of (possibly different) dynamical systems. The solutions have their initial values in a given manifold and they converge possibly along the strong-stable manifolds to given solutions.

More precisely, a **generalized connecting orbit** consists of a triple (x_+, x_-, λ), where x_+ and x_- are solutions of the parameterized dynamical systems

$$\begin{array}{rclr} \dot{x}_+ & = & f_+(x_+, \lambda), & x_+(t) \in \mathbb{R}^m, \\ \dot{x}_- & = & f_-(x_-, \lambda), & x_-(t) \in \mathbb{R}^m, \end{array} \lambda \in \Lambda \subset \mathbb{R}^p \tag{0.1}$$

defined on $\mathbb{R}_+$, such that both, x_+ and x_- converge with an exponential rate $\gamma_+ < 0$ or $\gamma_- < 0$ to given solutions $y_+(\lambda)$ and $y_-(\lambda)$ of (0.1), respectively.

Moreover, the initial values $(x_+(0), x_-(0), \lambda)$ are restricted to lie in a manifold $\mathcal{W} \subset \mathbb{R}^{2m+p}$.

Connecting orbits in a dynamical system $\dot{x} = f(x, \lambda)$ are special cases of generalized connecting orbits. This can be seen as follows:

We set $f_+ := f$ in case $y_+(\lambda)$ is a family of equilibria and we define $f_+(x, \lambda) := T_+(\lambda) f(x, \lambda)$, if $y_+(\lambda)$ is a family of $T_+(\lambda)$–periodic orbits. Analogously, we define $f_- := -f$ or $f_-(x, \lambda) := -T_-(\lambda) f(x, \lambda)$ in the stationary or periodic case, respectively. Moreover, we define the manifold $\mathcal{W}$, such that it includes the conditions $x_+(0) = x_-(0)$ and possibly an additional phase condition.

Thus we can apply our approximation method and the corresponding error estimates in particular to connecting orbits with the following properties:

1. The connecting orbit converges in each direction with an exponential rate to an equilibrium or to a periodic orbit.

2. The convergence might occur on one side or on both sides through the strong-stable or strong-unstable manifold, respectively.

Moreover, there are other applications which have the structure of "generalized connecting orbits", e. g. we apply these results to optimization problems. In these applications we approximate so called "Skiba" points, which are the initial values of a control problem, where more than one optimal solution exist. At such initial values there can be a global change in optimal control strategies (see [Skiba, 1978]).

The boundary value problem which defines a generalized connecting orbit can be split into two subproblems, which are coupled only by the initial values. Therefore, we consider at first these one sided subproblems. In Figure 0.2 we show the phase portrait of an example for such a subproblem, which is analyzed in Section 3.4.1. Here one wants to compute an orbit that connects a given hyperplane in $\mathbb{R}^3$ to a periodic solution started at a specific point y_0. The dotted line $\mathcal{M}^0(y)$ is the stable set of the point y_0. It is a fibre in the stable manifold of the whole periodic orbit.

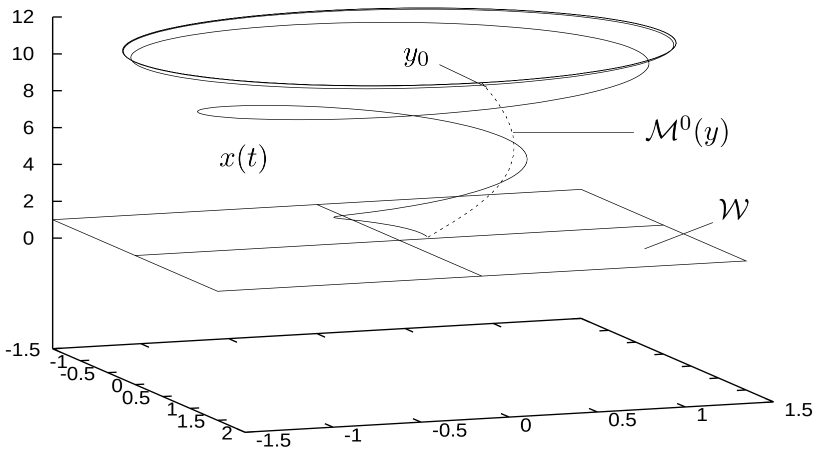

Figure 0.2: Phase portrait of example 3.4.1 where y is a periodic and $\mathcal{W}$ is a plane in $\mathbb{R}^3$.

The setup for the subproblem is as follows:

We consider a parameterized dynamical system

$$\dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \Lambda \subset \mathbb{R}^p \tag{0.2}$$

and a family $\{y(\lambda)(\cdot)\}_{\lambda \in \mathbf{\Lambda}}$ of solutions of (0.2) on $\mathbb{R}_+$. Our aim is to approximate so called **connecting orbits with asymptotic rate $\boldsymbol{\gamma}$**. These are pairs $(\bar{x}, \bar{\lambda})$ where $\bar{x}$ is a solution of (0.2) on $\mathbb{R}_+$ that converges asymptotically to $y(\bar{\lambda})$ with an exponential rate $\gamma < 0$ and that has an initial value in a given manifold $\mathcal{W} \subset \mathbb{R}^{m+p}$, i. e. we require $\big(x(0), \lambda\big) \in \mathcal{W}$ and

$$||x(t) - y(\lambda)(t)|| \leq C_x e^{\gamma t} \quad \forall \quad t \geq 0 \tag{0.3}$$

for some constant $C_x > 0$. More precisely, we say that $x(t)$ is γ–asymptotic with $y(\lambda)(t)$ if (0.3) holds. In other words, we are interested in that part of the stable or strong-stable manifold of an orbit $y(\lambda)$, which intersects the given manifold.

We can approximate parts of the stable or strong-stable manifold of a given solution by fixing an appropriate number of initial values and by using these as continuation parameters. To detect an initial approximation for a connecting orbit we approximate the stable manifold of one solution and calculate a connecting orbit from this approximation to the second solution (see 3.4.2). Problems of the one sided type also occur in solving optimal control problems with infinite horizon [Feichtinger and Novak, 1991], [Semmler and Sieveking, 2000, 1998b].

Both, in the case of connecting orbits with asymptotic rate and generalized connecting orbits the approximation technique is to truncate $\mathbb{R}_+$ to a finite interval and to compute suitable asymptotic boundary condition. For an early reference on the steady state case see [Lentini and Keller, 1980].

To obtain appropriate asymptotic boundary conditions we typically have to solve spectral problems for a linear boundary value problem several times during the approximation process. For example in the periodic case we have to solve the variational equation. Since these boundary conditions usually depend on parameters which are part of the problem it can be cumbersome to compute them within the approximation. Therefore, we develop the **boundary corrector method**. The idea of this method is to compute a first approximation to get a predictor for the unknown parameters and then to use these parameters to update the asymptotic boundary condition. It will be shown that this procedure leads to error estimates of the same quality as if the correct parameter had been known.

As a preparation for the error estimates we first deal with the existence and smoothness of stable or strong-stable manifolds. To obtain this, we use the concept of **shifted exponential dichotomy** as defined in [Hale and Lin, 1986],[Sandstede, 1993] and we employ the implicit function theorem in spaces of exponentially decaying functions.

We show how to define a smooth stable or strong-stable manifold along given solutions which have "special properties" (defined in **A1**–**A3** on page 27). In a rough sense these properties are:

P The family $\{y(\lambda)(\cdot)\}_{\lambda\in\mathbf{\Lambda}}$ is a branch of solutions of (0.2) in the Banach space of bounded functions. This branch is C^k with respect to the parameters. Moreover, its linearization has a shifted exponential dichotomy with exponents α, β, where $\alpha < 0$ and $\beta > \alpha$.

We show that the properties summarized under **P** are satisfied in the case of equilibria and scaled periodic orbits. In particular, we prove parameter dependent results even in the strong stable case. This includes the smoothness of the manifold in the parameter and in the time direction.

In the parameter independent case a general theory of foliations of stable and unstable manifolds of possibly higher dimensional "overflowing invariant sets" has been developed by Fenichel ([Fenichel, 1971, 1974, 1977],[Wiggins, 1994]). There the technique of the Hadamard graph transform is used to get results even for the dynamics on these overflowing invariant sets.

For discrete dynamical systems there are results on the existence and smoothness of stable and strong-stable manifolds of invariant hyperbolic sets in [Hirsch et al., 1977],[Shub, 1987]. Likewise, those results are without parameter-dependence.

The main differences of those approaches to ours are:

We do not determine (strong-) stable manifolds of general "overflowing invariant sets" and hence do not require smoothness with respect to points on these sets. Therefore we do not need additional assumptions on "generalized Liapunov numbers". Instead we determine (strong-) stable manifolds along solutions of parameterized dynamical systems with the properties defined by **P**. This allows us to derive smoothness of type C^{k-1}, even with respect to the parameter and in time direction, which is the direction of the solution.

A related result for foliations of generalized stable and unstable manifolds on infinite dimensional spaces and its smoothness is proved in [Chow et al., 1991]. They assume a global Lipschitz condition and an additional gap condition in order to obtain general foliations. Smoothness in the parameter only appears in the case of generalized center unstable manifolds.

We summarize the main results of this thesis:

1. We define a parameter-dependent (strong-) stable manifold at solutions which satisfy the property **P**. In particular, we obtain smoothness results including the parameter and the time direction.

2. We set up approximation methods for connecting orbits with asymptotic rate and generalized connecting orbits and derive error estimates which decay exponentially with the length of the truncation interval.

3. We develop and analyze the boundary corrector method which allows for an efficient and accurate implementation of the boundary conditions.

Overview of the chapters

To analyze solutions which decay exponentially, we work on function spaces with exponentially weighted norms and we use shifted exponential dichotomies. Thus, in Chapter 1, we collect results on shifted exponential dichotomies, mainly by transferring them from the non-shifted case. Moreover, we show that equilibria and scaled periodic orbits of system (0.2) satisfy the property **P**.

If a family $\{y(\lambda)(\cdot)\}$ of solutions of (0.2) satisfies the property **P** and the k-th derivative of f is Lipschitz, then we prove in Chapter 2 that there exists a γ–stable manifold of this family. The γ–stable manifold contains all points which converge with an asymptotic rate to some $y(\lambda)$. We show that this manifold is of type C^{k-1} even with respect to the parameter. The main tool is to apply the implicit function theorem to an appropriate operator in the space of exponentially decaying functions.

For the approximation results we only use the local γ–stable manifold evaluated at 0 as it is derived in 2.1.1. However, we derive in 2.1.2 a local γ–stable manifold along a solution, which includes the time direction and we prove in Section 2.2 the existence of a global γ–stable manifold.

In Section 2.3 we define non-degeneracy of a connecting orbit with asymptotic rate. We use a transversality condition which turns out to be equivalent to the regularity of a solution of an operator equation.

In Chapter 3 we derive an approximation theorem for connecting orbits with asymptotic rate γ which yields error estimates of the kind

$$\begin{aligned} |\nu_J - \bar{\lambda}| &\leq Ce^{(d\gamma-\beta)T_+} & & \\ ||z_J - \bar{x}|_J||_\infty &\leq Ce^{d\gamma T_+} & &: \beta > 0 \\ ||z_J - \bar{x}|_J||_\infty &\leq Ce^{(d\gamma-\beta)T_+} & &: \beta \leq 0, \end{aligned} \tag{0.4}$$

where $d = 1$ in general and $d = 2$ if we use an appropriate asymptotic boundary condition. Here $(\bar{x}, \bar{\lambda})$ denotes the exact connecting orbit and (z_J, ν_J) is an approximation on a finite interval $J = [0, T_+]$.

As stated before, the computation of the asymptotic boundary condition can be expensive. Thus, in Section 3.3, we develop the boundary corrector method which yields error estimates of the same kind as in (0.4) with $d = 2$. We will apply our general results to orbits that converge to equilibria as well as periodic orbits. The special setup for these cases is defined in Section 3.1. If the $y(\lambda)$ are hyperbolic equilibria, i. e. $\beta > 0$, then we obtain super-convergence with respect to the parameter, similar to [Beyn, 1990a, 1990b], [Sandstede, 1997]. In Section 3.4 we discuss the errors for an example, where the solutions $y(\lambda)$ are known explicitly.

In Chapter 4 and Chapter 5 we transfer these results to the case of generalized connecting orbits. The techniques for proving the approximation theorm are quite similar and we indicate the main differences. In particular, we develop a boundary corrector method for generalized connecting orbits in Section 5.2 and in Section 5.3 we apply the results to approximate the point to periodic connecting orbit in the Lorenz system, plotted in Figure 0.1.

In Chapter 6 we apply our results to optimization problems with infinite horizon. In this application we obtain a dynamical system by using Pontryagin's maximum principle. In this context the "candidates for optimal solutions" are defined by the connecting orbits with asymptotic rate. By duplicating the system we compute two different candidates simultaneously. To identify "Skiba" points we use one coordinate of the initial state value as "free" parameter and we add the condition, that the values of the objective functions are equal. This setup defines a generalized connecting orbit.

Chapter 1

Preliminaries on linear systems

We want to approximate solutions of continuous dynamical systems converging to given ones. The technique is to linearize along these given solutions and to apply results about shifted exponential dichotomies. In this chapter we summarize notions and results used throughout the thesis.

At first, in Section 1.1, we recall existence, uniqueness and smoothness results for continuous dynamical systems and define the solution operator of the linearization along a solution. For proofs we refer to the literature on ordinary differential equations, e. g. [Amann, 1990], [Hale, 1969], [Hartman, 1964].

To analyze linear dynamical systems and solutions which decay exponentially, we work on function spaces with exponentially weighted norms and we use **shifted exponential dichotomies** as defined in [Hale and Lin, 1986], [Sandstede, 1993]. A couple of results known from exponential dichotomies (see [Beyn and Lorenz, 1999], [Coppel, 1978], [Palmer, 1984]) will be transfered to the shifted case. Results on this topic are collected in Section 1.2. In addition we state a transformation lemma for shifted exponential dichotomies (see Lemma 1.8) and results for perturbed projectors (see Lemma 1.10).

Special cases are solutions converging to equilibria or periodic orbits. In Section 1.3 we show that there are branches of equilibria and periodic solutions, respectively, which are smooth with respect to the parameters and we prove that their linearizations have shifted exponential dichotomies.

1.1 Dynamical systems

In this section we summarize results about **autonomous parameter dependent ordinary differential equation**

$$\dot{x}(t) = f\bigl(x(t), \lambda\bigr) \tag{1.1}$$

where $f \in C^k(X \times \mathbf{\Lambda}, \mathbb{R}^m)$, $k \in N$ and $X \subset \mathbb{R}^m$, $\mathbf{\Lambda} \subset \mathbb{R}^p$ are open subsets.

Results on the existence, the uniqueness and the smoothness of solutions of the non-autonomous **initial value problem**

$$\dot{x}(t) = f\bigl(t, x(t), \lambda\bigr), \quad x(\tau) = x_0 \tag{1.2}$$

are well known (see e. g. [Amann, 1990]). We recapitulate them in the following theorem:

THEOREM 1.1 *Let $I \subset \mathbb{R}$, $X \subset \mathbb{R}^m$, $\mathbf{\Lambda} \subset \mathbb{R}^p$ be open subsets and $f \in C^k(I \times X \times \mathbf{\Lambda}, \mathbb{R}^m)$. Then for each $\tau \in I$, $x_0 \in X$, $\lambda \in \mathbf{\Lambda}$ there exists a unique non-extendible solution*

$$u(\cdot, \tau, x_0, \lambda) : J(\tau, x_0, \lambda) \to X$$

of (1.2), where $J\bigl(\tau, x, \lambda\bigr)$ is the maximal interval of existence. The domain of u, given by $\mathcal{D}(\mathbf{\Lambda}) := \bigl\{(t, \tau, x_0, \lambda) \in \mathbb{R}^2 \times X \times \mathbf{\Lambda} \big| t \in J(\tau, x, \lambda)\bigr\}$, is open and $u \in C^k(\mathcal{D}(\mathbf{\Lambda}), \mathbb{R}^m)$. Moreover, the k-th derivative of u is once differentiable with respect to t.

In the autonomous case and with fixed $\lambda \in \mathbf{\Lambda}$ it is sufficient to consider the solution operator $\varphi(t, x_0) := u(t, 0, x_0, \lambda)$ of (1.1) defined on $\mathcal{D}_\varphi := \bigl\{(t, x_0) \in \mathbb{R} \times X \big| t \in J(0, x_0, \lambda)\bigr\}$. The pair $(\varphi, \mathcal{D}_\varphi)$ has the properties of a **flow** or **dynamical system**, i. e.

1. $\mathcal{D}_\varphi$ is open in $\mathbb{R} \times X$,
2. $\varphi : \mathcal{D}_\varphi \mapsto X$ is continuous,
3. $\varphi(0, \cdot) = \mathrm{Id}_X$,
4. For each $x \in X$, $s \in J(0, x, \lambda)$ and $t \in J(0, \varphi(s, x), \lambda)$ we have $s + t \in J(0, x, \lambda)$ and $\varphi\bigl(t, \varphi(s, x)\bigr) = \varphi(s + t, x)$.

Often the ordinary differential equation (1.1) itself is called a dynamical system.

DEFINITION 1.2 *The solution $u(\cdot, \tau, x_0, \lambda)$ of (1.2) is called a* **trajectory** *and its image $\Gamma(\tau, x_0, \lambda) := \bigl\{u(t, \tau, x_0, \lambda) \big| t \in J(\tau, x_0, \lambda)\bigr\}$ is called an* **orbit**.

For the approximation results we linearize about a solution (x, λ) of (1.1). Thus we have to solve the **variational equation** along (x, λ), i. e.

$$\dot{z}(t) = \frac{\partial}{\partial x} f\bigl(x(t), \lambda\bigr) z(t). \tag{1.3}$$

Therefore we deal with linear operators $L(\cdot, \lambda) := \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot, \lambda)$ defined on an interval $J \subset \mathbb{R}$, where $A \in C^l(J \times \mathbf{\Lambda}, \mathbb{R}^{m,m})$, $l \geq 1$. We denote for each $s \in \mathbb{R}$, $\lambda \in \mathbf{\Lambda}$ the unique **solution operator** of

$$L(\cdot, \lambda)U = 0, \quad U(s) = \mathrm{I}_m \tag{1.4}$$

by $S(\cdot, s, \lambda)$. For the existence of a global solution see for example [Amann, 1990, Theorem 7.9]. Theorem 1.1 implies that $S \in C^l(J^2 \times \mathbf{\Lambda}, \mathbb{R}^{m,m})$. The matrix valued function $S(t, \cdot, \lambda)$ satisfies

$$\frac{\partial}{\partial s} S(t, s, \lambda) = -S(t, s, \lambda) A(s, \lambda), \quad S(t, t, \lambda) = \mathrm{I}_m. \tag{1.5}$$

for each $t \in J$. We call $S(t, \cdot, \lambda)$ a solution of the **adjoint variational equation** along (x, λ), if $A(\cdot, \lambda) = \frac{\partial}{\partial x} f\bigl(x(\cdot), \lambda\bigr)$.

1.2 Shifted exponential dichotomies

An important tool for describing linearizations along solutions which converge with an exponential rate to another trajectory is the notion of shifted exponential dichotomy as defined in [Hale and Lin, 1986]. In particular, we apply different forms of roughness theorems to obtain "similar" behavior in a neighborhood of such trajectories.

We define **Banach spaces of *k* times differentiable exponentially weighted functions** from an interval J into a Banach space $(\mathcal{X}, |\cdot|)$ by

$$BC^k_\gamma(J, \mathcal{X}) := \left\{ x \in C^k(J, \mathcal{X}) \middle| \sup_{t \in J}(|x^{(j)}(t)| e^{-\gamma t}) < \infty, \quad j = 0, \ldots, k \right\},$$

where $\gamma \in \mathbb{R}$. The norm on $BC^k_\gamma(J, \mathcal{X})$ is

$$||x||_{\gamma,k} = \sum_{j=0}^{k} ||x^{(j)}||_\gamma, \quad \text{where } ||x||_\gamma = \sup_{t \in J}(|x(t)| e^{-\gamma t}).$$

In case $\gamma = 0$ we obtain the **Banach space of *k* times differentiable and bounded functions** on J

$$BC^k(J, \mathcal{X}) := BC^k_0(J, \mathcal{X}) \text{ with } ||x||_{0,k} = \sum_{j=0}^{k} ||x^{(j)}||_\infty, \; ||x||_\infty := \sup_{t \in J}(|x(t)|).$$

Consider a linear differential operator

$$Lx = \dot{x} - A(\cdot)x$$

where $A \in BC(J, \mathbb{R}^{m,m})$, $x \in C^1(J, \mathbb{R}^m)$ and $J \subset \mathbb{R}$ is some interval. Denote the solution operator of $Lx = 0$ by $S(\cdot, \cdot)$ as in (1.4), i. e. $LS(\cdot, s) = 0$ and $S(s, s) = \mathrm{I}_m$ for all $s \in J$.

We define a shifted exponential dichotomy as in [Hale and Lin, 1986] and its smoothness in the parameter as in [Sandstede, 1993], [Beyn and Lorenz, 1999].

Definition 1.3 (Shifted exponential dichotomy) $L := \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)$ *has a* **shifted exponential dichotomy** *on J with exponents $\alpha < \beta$ if there exist a constant K and projectors $P_a(t)$, $P_b(t)$, $t \in J$ with $P_b(t) = \mathrm{Id}_{\mathbb{R}^m} - P_a(t)$ such that*

$$S(t,s)P_\kappa(s) = P_\kappa(t)S(t,s), \qquad \textit{for } \kappa = a, b \textit{ and all } t, s \in J \tag{1.6}$$

and

$$||P_a(t)S(t,s)|| \le Ke^{\alpha(t-s)}, \quad ||P_b(s)S(s,t)|| \le Ke^{-\beta(t-s)} \quad \textit{for } t \ge s. \tag{1.7}$$

We call $(K, \alpha, \beta, P_a, P_b)$ the **dichotomy data**. *The dichotomy data are* **of type C^l**, $l \ge 0$ *with respect to $\lambda \in \mathbf{\Lambda}$ if $A \in BC^l\big(\Lambda, BC(J, \mathbb{R}^{m,m})\big)$ and $L(\lambda) = \frac{\mathrm{d}}{\mathrm{dt}} - A(\lambda)(\cdot)$ has a shifted exponential dichotomy with dichotomy data $\big(K, \alpha, \beta, P_a(\lambda), P_b(\lambda)\big)$ for all $\lambda \in \Lambda$, where $P_a(\cdot)(t), P_b(\cdot)(t) \in C^l(\Lambda, \mathbb{R}^{m,m})$, and the data K, α, β are independent of λ.*

Remark: The projectors P_κ, $\kappa = a, b$ are as smooth with respect to t as $S(\cdot, \cdot)$, because (1.6) implies $P_\kappa(t) = S(t, \tau)P_\kappa(\tau)S(\tau, t)$ for all $t \in J$ and some fixed $\tau \in J$.

If $A(\cdot) = \frac{\partial}{\partial x} f(\bar{x}(\cdot), \bar{\lambda})$ is the linearization about a bounded C^k–solution $\bar{x}$ of (1.1) on J, then the projectors are in $C^{k-1}(J, \mathbb{R}^{m,m})$.

Using this notion we define Green's function on some interval J by

$$G(t,s) := \begin{cases} S(t,s)P_a(s), & s \le t \\ -S(t,s)P_b(s), & t < s \end{cases} \qquad \text{for } s, t \in J \tag{1.8}$$

and see that

$$s_J(v)(t) := \int_J G(t,s)v(s)\mathrm{ds} \tag{1.9}$$

solves $Lx = v$ for $v \in C(J, \mathbb{R}^m)$ if the integral is well defined for each $t \in J$.

Thus, the basic solution theorem in [Beyn and Lorenz, 1999, Theorem A1] can easily be extended to shifted exponential dichotomies as follows:

Theorem 1.4 *Let* $L = \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)$ *have a shifted exponential dichotomy on* $J = [t_-, \infty)$ *with data* $(K, \alpha, \beta, P_a, P_b)$ *and assume* $\gamma \in (\alpha, \beta)$.

Then any solution $z \in BC^1_\gamma\big([t_-, \infty), \mathbb{R}^m\big)$ *of* $Lz = 0$ *satisfies* $z(t_-) \in \mathcal{R}\big(P_a(t_-)\big)$.

For any $v \in BC^0_\gamma(J, \mathbb{R}^m)$ *and* $\xi_a \in \mathcal{R}\big(P_a(t_-)\big)$ *the boundary value problem*

$$Lz = v, \quad t \in J, \tag{1.10}$$

$$P_a(t_-)z(t_-) = \xi_a \tag{1.11}$$

has a unique solution $z \in BC^1_\gamma(J, \mathbb{R}^m)$, *namely* $z = s_J(v) + S(\cdot, t_-)\xi_a$. *Furthermore, the following estimates hold with* $C_\gamma = \frac{1}{\gamma - \alpha} + \frac{1}{\beta - \gamma}$

$$\begin{aligned} ||s_J(v)||_\gamma &\leq KC_\gamma ||v||_\gamma, &(1.12)\\ ||S(\cdot, t_-)\xi_a||_\gamma &\leq Ke^{-\gamma t_-}|\xi_a|, &(1.13)\\ ||s_J(v)||_{\gamma,1} &\leq \big(K(1 + ||A||_\infty)C_\gamma + 1\big)||v||_\gamma, &(1.14)\\ ||S(\cdot, t_-)\xi_a||_{\gamma,1} &\leq K(1 + ||A||_\infty)e^{-\gamma t_-}|\xi_a|. &(1.15) \end{aligned}$$

Proof: Let $z \in BC^1_\gamma\big([t_-, \infty), \mathbb{R}^m\big)$ be a solution of the homogeneous equation $Lz = 0$ on J, then $z(t) = S(t, t_-)z(t_-)$, $z(t_-) \in \mathbb{R}^m$ and $|z(t)|e^{-\gamma t} \leq ||z||_\gamma$. Thus, for any $t \in [t_-, \infty)$ we see

$$\begin{aligned} |P_b(t_-)z(t_-)| &= |S(t_-, t)P_b(t)S(t, t_-)z(t_-)| \leq ||S(t_-, t)P_b(t)|| \cdot |z(t)| \\ &\leq Ke^{-\beta(t - t_-)} \cdot e^{\gamma t}||z||_\gamma \leq Ke^{\beta t_-}||z||_\gamma e^{(\gamma - \beta)t} \overset{t \to \infty}{\longrightarrow} 0. \end{aligned}$$

Therefore, we obtain $z(t_-) \in \mathcal{R}\big(P_a(t_-)\big)$.

Any solution z of $Lz = 0$ on J with $z(t_-) \in \mathcal{R}\big(P_a(t_-)\big)$ satisfies $z = S(\cdot, t_-)\xi_a$ with $\xi_a = z(t_-)$ and $\dot{z} = A(\cdot)z$. Hence

$$\begin{aligned} ||z||_\gamma &\leq |z(t)|e^{-\gamma t} \leq Ke^{\alpha(t - t_-)}|\xi_a|e^{-\gamma t} \leq Ke^{-\gamma t_-}|\xi_a| \ \text{ for all } t \in J, \\ ||z||_{\gamma,1} &\leq K(1 + ||A(\cdot)||_\infty)e^{-\gamma t_-}|\xi_a|. \end{aligned}$$

Thus, $z \in BC^1_\gamma\big([t_-, \infty), \mathbb{R}^m\big)$ and (1.13), (1.15) hold.

To obtain $s_J(v) \in BC^1_\gamma\big([t_-, \infty), \mathbb{R}^m\big)$ for $v \in BC^0_\gamma\big([t_-, \infty), \mathbb{R}^m\big)$ we prove

$$\begin{aligned} |P_a(t)s_J(v)(t)| &\leq \int_{t_-}^t |S(t, s)P_a(s)v(s)|\mathrm{ds} \\ &\leq K\int_{t_-}^t e^{\alpha(t - s)} \cdot e^{\gamma s}|v(s)|e^{-\gamma s}\mathrm{ds} &(1.16)\\ &\leq Ke^{\alpha t}\frac{1}{\gamma - \alpha}\big(e^{(\gamma - \alpha)t} - e^{(\gamma - \alpha)t_-}\big)||v||_\gamma \\ &\leq K\frac{1}{\gamma - \alpha}e^{\gamma t}||v||_\gamma \end{aligned}$$

and analogously

$$
\begin{aligned}
|P_b(t)s_J(v)(t)| &\leq \int_t^\infty |S(t,s)P_b(s)v(s)|\mathrm{ds} \\
&\leq K\int_t^\infty e^{\beta(t-s)}e^{\gamma s}|v(s)|e^{-\gamma s}\mathrm{ds} \qquad (1.17)\\
&\leq Ke^{\beta t}\frac{1}{\beta-\gamma}e^{(\gamma-\beta)t}||v||_\gamma \\
&\leq K\frac{1}{\beta-\gamma}e^{\gamma t}||v||_\gamma
\end{aligned}
$$

and hence (1.12). By (1.10) we obtain $|\dot{z}(t)| \leq ||A(t)||\ |z(t)| + |v(t)|$ and hence we see $||z||_{\gamma,1} \leq (1+||A(\cdot)||_\infty)\ ||z||_\gamma + ||v||_\gamma$. This implies (1.14).

Therefore $\{s_J(v)+S(\cdot,t_-)\xi|\xi \in \mathcal{R}(P_a(t_-))\}$ is the set of all solutions of (1.10) which are in $BC^1_\gamma([t_-,\infty),\mathbb{R}^m)$ and z is unique by (1.11). ■

We want to derive approximations of solutions by truncating the interval $[t_-,\infty)$. Hence, we need estimates, corresponding to those of Theorem 1.4, on a finite interval $[t_-,t_+]$.

THEOREM 1.5 *Let $L=\frac{\mathrm{d}}{\mathrm{dt}}-A(\cdot)$ have a shifted exponential dichotomy on $J=[t_-,t_+]$ with data (K,α,β,P_a,P_b).*

Then for any $v \in C(J,\mathbb{R}^m)$, $\xi_a \in \mathcal{R}(P_a(t_-))$ and $\xi_b \in \mathcal{R}(P_b(t_+))$ the boundary value problem

$$Lz = v, \quad t \in J \qquad (1.18)$$

$$P_a(t_-)z(t_-) = \xi_a, \qquad P_b(t_+)z(t_+) = \xi_b \qquad (1.19)$$

has a unique solution $z \in C^1(J,\mathbb{R}^m)$ namely $z = s_J(v)+S(t,t_-)\xi_a+S(t,t_+)\xi_b$.

For any $\gamma \in (\alpha,\beta)$ the estimate

$$||s_J(v)||_\gamma \leq KC_\gamma||v||_\gamma \qquad (1.20)$$

holds with $C_\gamma = \frac{1}{\gamma-\alpha}+\frac{1}{\beta-\gamma}$ and for $\gamma \in [\alpha,\beta]$ we estimate

$$
\begin{aligned}
||s_J(v)||_\gamma &\leq K\int_J |v(s)|e^{-\gamma s}\mathrm{ds} \leq K|t_+-t_-|\ ||v||_\gamma, \qquad (1.21)\\
||S(t,t_-)\xi_a+S(t,t_+)\xi_b||_\gamma &\leq K\left(e^{-\gamma t_-}|\xi_a|+e^{-\gamma t_+}|\xi_b|\right). \qquad (1.22)
\end{aligned}
$$

Proof: Defining $s_J(v)$ as in (1.8) and (1.9) implies, on a finite interval, that $s_J(v)$ solves (1.18) and that $P_a(t_-)s_J(v)(t_-)=0$ and $P_b(t_+)s_J(v)(t_+)=0$. Let x be any solution of the homogeneous equation $Lz = 0$ with (1.19), i. e. $x(t) = S(t, t_-)x(t_-)$, then

$$\begin{aligned} x(t) &= P_a(t)x(t) + P_b(t)x(t) \\ &= S(t, t_-)P_a(t_-)x(t_-) + S(t, t_+)P_b(t_+)S(t_+, t_-)x(t_-) \\ &= S(t, t_-)\xi_a + S(t, t_+)\xi_b. \end{aligned}$$

Thus, $z = s_J(v) + S(t, t_-)\xi_a + S(t, t_+)\xi_b$ is the unique solution of (1.18), (1.19). The estimates (1.20) and (1.22) are proved analogously to (1.12), (1.13) in Theorem 1.4, and (1.21) is obtained from

$$\begin{aligned} |s_J(v)(t)|e^{-\gamma t} &\leq K\left(\int_{t_-}^{t} \underbrace{e^{\alpha(t-s)}}_{\leq e^{\gamma(t-s)}} \cdot |v(s)|\mathrm{ds} + \int_{t}^{t_+} \underbrace{e^{\beta(t-s)}}_{\leq e^{\gamma(t-s)}} \cdot |v(s)|\mathrm{ds}\right) e^{-\gamma t} \\ &\leq K\int_{t_-}^{t_+} |v(s)|e^{-\gamma s}\mathrm{ds} \leq K|t_+ - t_-|\ ||v||_\gamma. \end{aligned}$$

■

Another important tool is a quantitative version of the roughness theorem on finite and infinite intervals J, see for example [Coppel, 1978, Palmer, 1984, Sandstede, 1993, Beyn, 1994b, Beyn and Lorenz, 1999]. Here we extend those results to the case of shifted exponential dichotomies.

Proposition 1.6 (Roughness theorem) *Let $\bar{L}$ have a shifted exponential dichotomy on J with dichotomy data $(\bar{K}, \bar{\alpha}, \bar{\beta}, \bar{P}_a, \bar{P}_b)$ and let $B \in C(J, \mathbb{R}^{m,m})$ satisfy*

$$6\bar{K}^2 b_\infty < \bar{\beta} - \bar{\alpha} \text{ where } b_\infty := \sup_{t\in J} |B(t)|. \tag{1.23}$$

Then the perturbed operator $L = \bar{L} - B(\cdot)$ has a shifted exponential dichotomy on J with data $(K, \alpha, \beta, P_a, P_b)$, where

$$\begin{gathered} K = \bar{K}\left(1 + \frac{8b_\infty \bar{K}}{\bar{\beta} - \bar{\alpha} - 6b_\infty \bar{K}}\right), \\ \alpha = \bar{\alpha} + 2b_\infty \bar{K} < \beta = \bar{\beta} - 2b_\infty \bar{K}, \end{gathered}$$

and the projectors $P_a(t)$, $P_b(t)$, $t \in J$ satisfy with $\kappa = a, b$

$$||P_\kappa(t) - \bar{P}_\kappa(t)|| \leq \bar{K}K\int_J e^{-(\beta-\bar{\alpha})|t-s|}||B(s)||\mathrm{ds} \leq 2\bar{K}K\frac{b_\infty}{\beta - \bar{\alpha}}, \tag{1.24}$$

$$||\bar{P}_a(t)P_b(t)|| \leq \bar{K}K\int_{t_-}^{t} e^{-(\beta-\bar{\alpha})|t-s|}||B(s)||\mathrm{ds} \leq \bar{K}K\frac{b_\infty}{\beta - \bar{\alpha}}, \tag{1.25}$$

$$||\bar{P}_b(t)P_a(t)|| \leq \bar{K}K\int_{t}^{t_+} e^{-(\beta-\bar{\alpha})|t-s|}||B(s)||\mathrm{ds} \leq \bar{K}K\frac{b_\infty}{\beta - \bar{\alpha}}. \tag{1.26}$$

If $B \in BC^l\big(\mathbf{\Lambda}, BC(J, \mathbb{R}^{m,m})\big)$ and (1.23) holds uniformly in $\mathbf{\Lambda}$, then the dichotomy data are of type C^l and the estimates (1.24)–(1.26) hold uniformly in $\mathbf{\Lambda}$.

Proof: In the case of an exponential dichotomy, i. e. $\alpha = -\beta$ ([Coppel, 1978]), this is the same result as [Beyn and Lorenz, 1999, Theorem A.3]. Therefore, we shift the exponents to $\frac{1}{2}(\alpha-\beta) < 0 < \frac{1}{2}(\beta-\alpha)$ by using the Transformation lemma 1.8 below with shift $\frac{1}{2}(\alpha+\beta)$. Then we apply [Beyn and Lorenz, 1999, Theorem A.3] and shift the exponents back with $-\frac{1}{2}(\alpha+\beta)$. The estimates (1.24)–(1.26) are obtained as in the proof of [Beyn, 1994b, Proposition 2.3]. ■

We also need a similar result for exponentially decaying functions.

Proposition 1.7 (Asymptotic roughness theorem) *Let $\bar{L}$ have a shifted exponential dichotomy on $J = [0,\infty)$ with dichotomy data $(\bar{K},\bar{\alpha},\bar{\beta},\bar{P}_a,\bar{P}_b)$ and let $B \in C(J,\mathbb{R}^{m,m})$ satisfy $|B(t)| \le C_B e^{\epsilon t}$ for all $t \ge 0$ and some $\epsilon \in (\bar{\alpha}-\bar{\beta},0)$.*

Then the perturbed operator $L = \bar{L} - B(\cdot)$ has a shifted exponential dichotomy on J with exponents $\bar{\alpha}$, $\bar{\beta}$ from the unperturbed operator, a constant K and projectors P_a, P_b satisfying for some $C > 0$ and with $\kappa = a, b$

$$||P_\kappa(t) - \bar{P}_\kappa(t)|| \le Ce^{\epsilon t}, \tag{1.27}$$

$$||\bar{P}_a(t)P_b(t)|| \le Ce^{\epsilon t}, \tag{1.28}$$

$$||\bar{P}_b(t)P_a(t)|| \le Ce^{\epsilon t}. \tag{1.29}$$

If $B \in BC^l_\gamma\big(\mathbf{\Lambda}, BC(J,\mathbb{R}^{m,m})\big)$ and the constants C_B and ϵ are uniform in $\mathbf{\Lambda}$, then the dichotomy data are of type C^l and the estimates (1.27)–(1.29) hold uniformly in $\mathbf{\Lambda}$.

Remark: This is the same result as [Beyn, 1994b, Proposition 2.4], except that we can choose the original exponents $\bar{\alpha}$, $\bar{\beta}$ even for the perturbed operator. For the remainder it is sufficient, that we can choose exponents $\alpha < \beta$ arbitrarily close to $\bar{\alpha}$, $\bar{\beta}$, which is the result of [Beyn, 1994b, Proposition 2.4]. Thus we only remark that the result of Proposition 1.7 can be obtained by following the proof of the non-asymptotic roughness theorem [Beyn and Lorenz, 1999, Theorem A3] and using the exponential behavior of B.

The first part (1. to 3.) of the following lemma is stated in [Beyn, 1994b]. It is used in the proof of Proposition 1.6, where we shift the exponents of a shifted exponential dichotomy. The second transformation (4. to 6.) is used later on in our applications, where we have to scale systems or invert time.

Lemma 1.8 (Transformation Lemma) *Let* $L = \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)$ *have a shifted exponential dichotomy on* $J = [t_-, t_+]$ *(*$t_\pm = \pm\infty$ *allowed) with data* $(K, \alpha, \beta, P_a, P_b)$.

Define $L_\gamma := \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot) + \gamma \mathrm{I}_m$, $\gamma \in \mathbb{R}$ *and let* $S_\gamma(t,s)$ *be its solution operator. Then the following holds:*

1. $e^{\gamma t}\big(L_\gamma x\big)(t) = L\big(e^{\gamma\cdot} x\big)(t)$, *for* $t \in J, x \in C^1(J, \mathbb{R}^m)$,
2. $S_\gamma(t,s) = e^{\gamma(s-t)} S(t,s)$, *for* $t, s \in J$,
3. L_γ *has a shifted exponential dichotomy on* J *with data* $(K, \alpha - \gamma, \beta - \gamma, P_a, P_b)$.

Define $L_c := \frac{\mathrm{d}}{\mathrm{dt}} - cA(c\cdot)$, $c \neq 0$ *on* J_c, *where* $J_c := [\frac{1}{c}t_-, \frac{1}{c}t_+]$ *for* $c > 0$ *and* $J_c := [\frac{1}{c}t_+, \frac{1}{c}t_-]$ *for* $c < 0$. *Moreover, let* $S_c(t,s)$ *be the solution operator of* L_c. *Then the following holds:*

4. $\big(L_c x(c\cdot)\big)(t) = \big(cLx\big)(ct)$, *for* $t \in J_c, x \in C^1(J, \mathbb{R}^m)$,
5. $S_c(t,s) = S(ct, cs)$ *for* $t, s \in J_c$
6. L_c *has a shifted exponential dichotomy on* J_c *with data* $\big(K, c\alpha, c\beta, P_a(c\cdot), P_b(c\cdot)\big)$ *if* $c > 0$ *and* $\big(K, c\beta, c\alpha, P_b(c\cdot), P_a(c\cdot)\big)$ *if* $c < 0$, *respectively.*

Proof: This follows in a straightforward way from the following equations:

1. $L\big(e^{\gamma\cdot}x\big)(t) = \gamma e^{\gamma t}x(t) + e^{\gamma t}\dot{x}(t) - A(t)e^{\gamma t}x(t) = e^{\gamma t}\big(L_\gamma x\big)(t)$,
2. $\frac{\partial}{\partial t}\big(e^{\gamma(s-\cdot)}S(\cdot,s)\big)(t) = e^{\gamma(s-t)}\big(-\gamma + A(t)\big)S(t,s)$, $e^{\gamma(s-s)}S(s,s) = \mathrm{I_m}$,
3. $||P_a(t)S_\gamma(t,s)|| = e^{\gamma(s-t)}||P_a(t)S(t,s)|| \leq Ke^{(\alpha-\gamma)(t-s)}$, $t \geq s$,
4. $\big(cLx\big)(ct) = c\big(\frac{\mathrm{d}}{\mathrm{dt}}x - A(\cdot)x\big)(ct) = \big(\frac{\mathrm{d}}{\mathrm{dt}} - cA(c\cdot)\big)x(c\cdot)(t)$,
5. $\frac{\partial}{\partial t}\big(S(ct,cs)\big) = \big(cA(ct)\big)S(ct,cs)$, $S(cs,cs) = \mathrm{I_m}$,
6. $||P_b(ct)S_c(t,s)|| = ||P_b(ct)S(ct,cs)|| \leq Ke^{c\beta(t-s)}$, $c < 0$, $t \geq s$.

The other estimates of (1.7) and the case $c > 0$ in 6. follow analogously. ∎

A basic tool for the proof is the forthcoming Lemma 1.10 is the following linear perturbation lemma (see [Golub and Van Loan, 1996]):

Lemma 1.9 *Let* $(\mathcal{X}, ||\cdot||)$ *and* $(\mathcal{Y}, ||\cdot||)$ *be Banach spaces and let* $B : \mathcal{X} \to \mathcal{Y}$ *be a linear homeomorphism. Furthermore, let* $H : \mathcal{X} \to \mathcal{Y}$ *be linear with* $||H|| < b := \frac{1}{||B^{-1}||}$. *Then* $B + H : \mathcal{X} \to \mathcal{Y}$ *is a homeomorphism and* $||(B+H)^{-1}|| \leq \frac{1}{b - ||H||}$.

In the approximation theory we can calculate the projectors corresponding to a given solution, but we also need estimates for projectors of solutions nearby. Thus we state the following lemma:

LEMMA 1.10 *Let $P_a, \bar{P}_a \in \mathbb{R}^{m,m}$ be projectors with* $\operatorname{rank}(P_a) = \operatorname{rank}(\bar{P}_a) < m$ *and define* $P_b := \mathrm{Id}_{\mathbb{R}^{m,m}} - P_a$ *and* $\bar{P}_b := \mathrm{Id}_{\mathbb{R}^{m,m}} - \bar{P}_a$. *Furthermore, assume that*

$$\max(||\bar{P}_b P_a||, ||\bar{P}_a P_b||) \le \frac{r}{||P_a|| + ||P_b||} \tag{1.30}$$

holds for some $0 < r < 1$. Then the operators $\bar{P}_\kappa|_{\mathcal{R}(P_\kappa)} : \mathcal{R}(P_\kappa) \to \mathcal{R}(\bar{P}_\kappa)$, $\kappa = a, b$ are homeomorphisms and the inverses $(\bar{P}_\kappa|_{\mathcal{R}(P_\kappa)})^{-1}$ are bounded by $\frac{1}{1-r}(||P_a|| + ||P_b||)$.

Let $\kappa \in \{a, b\}$, $n = \operatorname{rank}(\bar{P}_\kappa)$ and $M \in \mathbb{R}^{n,m}$, such that $M|_{\mathcal{R}(\bar{P}_\kappa)}$ is a homeomorphism and (1.30) holds with

$$r < \frac{1}{2(1 + ||M|| \cdot ||(M|_{\mathcal{R}(\bar{P}_\kappa)})^{-1}||)}. \tag{1.31}$$

Then $M|_{\mathcal{R}(P_\kappa)}$ is a homeomorphism and $\left\|\left(M|_{\mathcal{R}(P_\kappa)}\right)^{-1}\right\| \le 2\left\|\left(M|_{\mathcal{R}(\bar{P}_\kappa)}\right)^{-1}\right\|(||P_a||+||P_b||)$.

Proof: Let us define a linear homeomorphism

$$B : \begin{array}{ccc} \mathcal{R}(P_a) \times \mathcal{R}(P_b) & \longrightarrow & \mathbb{R}^m \\ (\xi_a, \xi_b) & \longmapsto & \xi_a + \xi_b \end{array}.$$

For a given norm $|\cdot|$ in $\mathbb{R}^m$ use the norm $|(\xi_a, \xi_b)|_{\mathcal{R}} := |\xi_a| + |\xi_b|$ in $\mathcal{R}(P_a) \times \mathcal{R}(P_b)$. Then $||B|| \le 1$ and its inverse is $B^{-1}x = \big(P_a x, P_b x\big)$ with $||B^{-1}|| \le ||P_a|| + ||P_b||$. Analogously, we define $\bar{B}$ on $\mathcal{R}(\bar{P}_a) \times \mathcal{R}(\bar{P}_b)$ with the norm $|(\bar{\xi}_a, \bar{\xi}_b)|_{\bar{\mathcal{R}}} := |\bar{\xi}_a| + |\bar{\xi}_b|$. Then the linear function

$$H : \begin{array}{ccc} \mathcal{R}(P_a) \times \mathcal{R}(P_b) & \longrightarrow & \mathbb{R}^m \\ (\xi_a, \xi_b) & \longmapsto & -\big(\bar{P}_b \xi_a + \bar{P}_a \xi_b\big) \end{array}$$

is bounded by $\frac{r}{||B^{-1}||} < \frac{1}{||B^{-1}||}$, because

$$\begin{aligned} |H(\xi_a, \xi_b)| &\le ||\bar{P}_b P_a||\,|\xi_a| + ||\bar{P}_a P_b||\,|\xi_b| \le \max(||\bar{P}_b P_a||, ||\bar{P}_a P_b||)(|\xi_a| + |\xi_b|) \\ &\le \frac{r}{||P_a|| + ||P_b||}(|\xi_a| + |\xi_b|) \le \frac{r}{||B^{-1}||}(|\xi_a| + |\xi_b|). \end{aligned}$$

Thus Lemma 1.9 implies that

$$B + H : \begin{array}{ccc} \mathcal{R}(P_a) \times \mathcal{R}(P_b) & \longrightarrow & \mathbb{R}^m \\ (\xi_a, \xi_b) & \longmapsto & \bar{P}_a|_{\mathcal{R}(P_a)}\xi_a + \bar{P}_b|_{\mathcal{R}(P_b)}\xi_b \end{array}$$

is a homeomorphism with $||(B + H)^{-1}|| \le \frac{||B^{-1}||}{1-r} \le \frac{1}{1-r}(||P_a|| + ||P_b||)$.

Therefore, we obtain $\left(\bar{P}_a|_{\mathcal{R}(P_a)}, \bar{P}_b|_{\mathcal{R}(P_b)}\right) = \bar{B}^{-1} \circ (B+H)$ and for $\kappa = a, b$

$$||(\bar{P}_\kappa|_{\mathcal{R}(P_\kappa)})^{-1}|| \leq ||(B+H)^{-1}|| \cdot ||\bar{B}|| \leq \frac{1}{1-r}(||P_a|| + ||P_b||).$$

We prove the second part of Lemma 1.10 for $\kappa = b$.

Define $B = M \circ \bar{P}_b|_{\mathcal{R}(P_b)}$ and $H = M \circ \bar{P}_a|_{\mathcal{R}(P_b)}$ and therefore $(B+H) = M|_{\mathcal{R}(P_b)}$. With $\bar{N} := ||(M|_{\mathcal{R}(\bar{P}_b)})^{-1}||$ we estimate

$$\begin{aligned} ||B^{-1}|| &\leq ||(\bar{P}_b|_{\mathcal{R}(P_b)})^{-1}|| \cdot \bar{N} \leq \frac{1}{1-r}(||P_a|| + ||P_b||)\bar{N}, \\ ||H|| &\leq ||M|| \cdot ||\bar{P}_a\ P_b|| \leq ||M|| \frac{r}{||P_a|| + ||P_b||} \leq \frac{r\ ||M||\ \bar{N}}{(1-r)||B^{-1}||} < \frac{1}{2||B^{-1}||} \end{aligned}$$

and Lemma 1.9 yields that $(M|_{\mathcal{R}(P_b)})^{-1}$ is a homeomorphism and

$$||(M|_{\mathcal{R}(P_b)})^{-1}|| \leq \frac{\bar{N} \cdot (||P_a|| + ||P_b||)}{1 - r(1 + ||M|| \cdot \bar{N})} \leq 2\bar{N} \cdot (||P_a|| + ||P_b||)$$

The case $\kappa = a$ follows analogously. ∎

1.3 Equilibria and periodic solutions

Later on, we define solutions converging to equilibria and periodic solutions, thus we collect in this section some useful results, always assuming $f \in C^k(\mathbb{R}^m \times \mathbf{\Lambda}, \mathbb{R}^m)$. In particular, in 1.3.1 and 1.3.2 we prove the existence of C^k–branches of equilibria and periodic orbits, respectively. Moreover we show that the linearizations along such solutions have shifted exponential dichotomies.

DEFINITION 1.11 *A point $x \in \mathbb{R}^m$ is called an* **equilibrium** *of (1.1), if $u(t, 0, x, \lambda) = x$ holds for all $t \in J(0, x, \lambda)$.*

A point $x \in \mathbb{R}^m$ is called a **periodic point** *of (1.1), if there exists some $T > 0$ such that $u(t+T, 0, x, \lambda) = u(t, 0, x, \lambda)$ holds for all $t \in J(0, x, \lambda)$ and x is not an equilibrium. If x is a periodic point of (1.1), then $u(\cdot, 0, x, \lambda)$ is called a* **periodic solution** *and its image $\Gamma(x, \lambda)$ is called a* **periodic orbit**.

Remark: In the literature an equilibrium is also called an **equilibrium point**, a **critical point** or a **stationary point**.

The equilibria and periodic orbits have the following properties:

(a) $J(0, x, \lambda) = \mathbb{R}$ for equilibria and periodic points x.

(b) Equilibria are the zeros of f.

(c) If x is a periodic point then there exists a smallest positive period T and x is called $\boldsymbol{T}$**-periodic**.

1.3.1 Equilibria

Lemma 1.12 *Let $(\bar{y}_0, \bar{\lambda})$ be an equilibrium of $\dot{x} = f(x, \bar{\lambda})$ with* $\det\big(\frac{\partial}{\partial x} f(\bar{y}_0, \bar{\lambda})\big) \neq 0$. *Then there exists a C^k–branch $y(\lambda)$ of equilibria in a neighborhood $\boldsymbol{\Lambda}$ of $\bar{\lambda}$.*

Proof: Apply the implicit function theorem to $f(y, \lambda) = 0$. ∎

The linearization at an equilibrium solution yields a linear homogeneous differential equation

$$\dot{z}(t) = \frac{\partial}{\partial x} f(\bar{y}_0, \bar{\lambda})\ z(t)$$

and in the following Lemma we show that it has a shifted exponential dichotomy.

Lemma 1.13 *Let $S(t, s)$ be the solution operator of $L = \frac{\mathrm{d}}{\mathrm{dt}} - A$, $A \in \mathbb{R}^{m,m}$ and $\alpha < \beta$, such that $(\alpha, \beta) \cap \mathrm{Re}\big(\sigma(A)\big) = \emptyset$.*

Then L has a shifted exponential dichotomy with exponents α, β.

Proof: Define for $\gamma \in (\alpha, \beta)$ the hyperbolic matrix $A_\gamma := (A - \gamma\, \mathrm{I}_m)$. Then there exists a direct sum decomposition

$$\mathbb{R}^m = E_a \oplus E_b \ \text{ which decomposes } A_\gamma = A_a \oplus A_b$$

with $\mathrm{Re}\big(\sigma(A_a)\big) < \alpha - \gamma$ and $\mathrm{Re}\big(\sigma(A_b)\big) > \beta - \gamma$ (see [Amann, 1990, Theorem 13.4]).

Let P_a and P_b be projectors on E_a and E_b and let $S_\gamma(\cdot, \cdot)$ be the solution operator of $L_\gamma = \frac{\mathrm{d}}{\mathrm{dt}} - A_\gamma$. Then there exists some constant K such that $||P_a S_\gamma(t, s)|| \leq K e^{(\alpha-\gamma)(t-s)}$ for $t \geq s$ and $||P_b S_\gamma(t, s)|| \leq K e^{-(\beta-\gamma)(s-t)}$ for $t \leq s$. Therefore, L_γ has a shifted exponential dichotomy with dichotomy data $(K, \alpha-\gamma, \beta-\gamma, P_a, P_b)$ and the Transformation Lemma 1.8 implies that $L = L_\gamma - \gamma\, \mathrm{I}_m$ has a shifted exponential dichotomy with dichotomy data $(K, \alpha, \beta, P_a, P_b)$. ∎

1.3.2 Periodic Solutions

In the periodic case let $(\hat{y}(\cdot), \bar{\lambda})$ be a $\bar{T}$–periodic orbit of $\dot{x} = \hat{f}(x, \bar{\lambda})$, $\hat{f} \in C^k(\mathbb{R}^m \times \mathbf{\Lambda}, \mathbb{R}^m)$ with a simple Floquet multiplier 1. Furthermore, let $\bar{y}(t) := \hat{y}(t \cdot \bar{T})$.

Then$(\bar{y}, \bar{T}, \bar{\lambda})$ is a regular solution of

$$F_p(y, T; \lambda) = \begin{pmatrix} \dot{y} - T\hat{f}(y, \lambda) \\ \chi(y, \lambda) \\ y(0) - y(1) \end{pmatrix} = 0, \tag{1.32}$$

where $\chi \in C^k(BC^1(\mathbb{R}_+, \mathbb{R}^m) \times \mathbf{\Lambda}, \mathbb{R})$ satisfies

$$\chi(\bar{y}(\cdot), \bar{\lambda}) = 0, \quad \frac{\partial}{\partial x}\chi(\bar{y}(\cdot), \bar{\lambda})\dot{\bar{y}} \neq 0. \tag{1.33}$$

The system (1.32) is set up and analyzed in [Jepson and Keller, 1984]. A phase fixing functions with the properties (1.33) is used in [Jepson and Keller, 1984] to define a well posed boundary value problem. In particular, there exists a branch $y(\lambda)$ of 1–periodic solutions of

$$\dot{y} = f(y, \lambda), \qquad f(y, \lambda) := T(\lambda)\hat{f}(y, \lambda). \tag{1.34}$$

In the following theorem we see that y satisfies $y \in C^k\big(\mathbf{\Lambda}_0, BC^1(\mathbb{R}_+, \mathbb{R}^m)\big)$.

Theorem 1.14 *Let $\hat{y}(\cdot)$ be a $\bar{T}$ periodic solution of $\dot{x} = \hat{f}(x, \bar{\lambda})$, $\bar{\lambda} \in \mathbf{\Lambda}$ with a simple Floquet multiplier* 1*. Furthermore assume (1.33) is satisfied for $\bar{y}(t) := \hat{y}(t \cdot \bar{T})$.*

Then there exist neighborhoods $\mathbf{\Lambda}_0$ of $\bar{\lambda}$ and $V_0 \subset \mathbb{R}^{m+p}$ of the periodic orbit $\Gamma(\hat{y}, \lambda)$, such that for each $\lambda \in \mathbf{\Lambda}_0$ there exist $T \in C^1(\mathbf{\Lambda}_0, \mathbb{R}_+)$ and a unique $T(\lambda)$–periodic orbit $\tilde{y}(\lambda)(\cdot)$ of $\dot{x} = \hat{f}(x, \lambda)$ with

$$\chi\big(\tilde{y}(\lambda)(\cdot T(\lambda)), \lambda\big) = 0. \tag{1.35}$$

Moreover, $y(\lambda) := \tilde{y}(\lambda)\big(\cdot\ T(\lambda)\big)$ satisfies $y \in C^k\big(\mathbf{\Lambda}_0, BC^k(\mathbb{R}_+, \mathbb{R}^m)\big)$.

Proof: From [Amann, 1990, Theorem 25.15, Remark 25.16] we obtain the existence of functions $T \in C^k(\mathbf{\Lambda}_0, \mathbb{R}_+)$ and $h \in C^k(\mathbf{\Lambda}_0, W_0)$, where W_0 is a neighborhood of $\hat{y}(0)$ in a locally transversal section $H_{\hat{y}(0)}$ with $h(\bar{\lambda}) = \hat{y}(0)$. Moreover, $u(t, 0, h(\lambda), \lambda)$ has the properties claimed in this theorem, except the phase fixing condition (1.35). The latter condition can be achieved by defining a suitable C^k–phase shift $\tau(h(\lambda), \lambda)$.

Thus, $\tilde{y}(\lambda) := u\big(\cdot + \tau(h(\lambda), \lambda), 0, h(\lambda), \lambda\big)$, $\lambda \in \mathbf{\Lambda}_0$ is a $T(\lambda)$–periodic solution which satisfies (1.35) and hence $y(\lambda) := \tilde{y}(\lambda)\big(\cdot\ T(\lambda)\big)$ satisfies $y \in C^k\big(\mathbf{\Lambda}_0, BC^k(\mathbb{R}_+, \mathbb{R}^m)\big)$.

■

Similar to [Beyn, 1994b] we apply classical Floquet theory to see that the linearization of a 1–periodic solution y of the scaled system (1.34) has a shifted exponential dichotomy. Let us assume that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(y, \bar{\lambda})$ has $m_a \geq 1$ Floquet multipliers μ with $|\mu| < e^{\alpha}$, $\alpha < 0$ and $m_b = m - m_a \geq 1$ Floquet multipliers with $|\mu| \geq e^{\beta}$, where $\alpha < \beta \leq 0$.

Then there exists a fundamental matrix with the Floquet representation

$$U(t) = \big(U_a(t)\ U_b(t)\big) = Q(t)e^{tB} = \big(Q_a(t) \quad Q_b(t)\big)e^{\left(t\left(\begin{smallmatrix} B_a & 0 \\ 0 & B_b \end{smallmatrix}\right)\right)},$$

where $Q_a(t) \in \mathbb{C}^{m,m_a}$ and $Q_b(t) \in \mathbb{C}^{m,m_b}$ are 1–periodic and the Floquet exponents are given by the spectra of $B_a \in \mathbb{C}^{m_a,m_a}$, $B_b \in \mathbb{C}^{m_b,m_b}$ and satisfy

$$\mathrm{Re}\big(\sigma(B_a)\big) < \alpha < \beta \leq \mathrm{Re}\big(\sigma(B_b)\big).$$

Thus $U_\kappa(t)$, $\kappa = a, b$ solve the **variational equation**

$$\dot{U} = \frac{\partial}{\partial x} f(y(\cdot), \lambda)U, \tag{1.36}$$

$$U(1) = U(0)e^{B_\kappa}. \tag{1.37}$$

Setting $R(t) = \big(R_a(t)\ R_b(t)\big) := \Big(\big(Q_a(t)\ Q_b(t)\big)^\top\Big)^{-1}$ and $V(t) = \left(\begin{smallmatrix} V_a(t) \\ V_b(t) \end{smallmatrix}\right) := e^{-tB}R(t)^\top$ we find that $V_\kappa(t)$, $\kappa = a, b$ solve the **adjoint variational equation**

$$\dot{V} = -V\frac{\partial}{\partial x} f(y(\cdot), \lambda), \tag{1.38}$$

$$V(1) = e^{-B_\kappa}V(0). \tag{1.39}$$

The solution operator of L then has the form

$$S(t, s) = Q_a(t)e^{(t-s)B_a}R_a(s)^\top + Q_b(t)e^{(t-s)B_b}R_b(s)^\top \tag{1.40}$$

and we see that L has a shifted exponential dichotomy with exponents $\alpha < \beta \leq 0$ and projectors $P_a(t) = Q_a(t)R_a(t)^\top$, $P_b(t) = Q_b(t)R_b(t)^\top$.

Remark: The matrix valued functions Q and R are complex and 1–periodic, nevertheless the projectors P_a and P_b are **real** valued and 1–periodic. In particular, we have for $\kappa = a, b$ that

$$\mathcal{R}\big(P_\kappa(t)\big) = \mathcal{R}\big(Q_\kappa(t)\big)|_{\mathbb{R}^m} \text{ and } \mathcal{N}\big(P_\kappa(t)\big) = \mathcal{N}\big(R_\kappa(t)^\top\big)|_{\mathbb{R}^m}. \tag{1.41}$$

Our analysis contains as a special case the results from [Beyn, 1994b]. There the periodic orbits $(\bar{y}(\cdot), \bar{\lambda})$ are hyperbolic, i. e. there is no Floquet multiplier on the unit circle, except the simple Floquet multiplier 1. In that case L has an “exponential trichotomy” (see [Beyn, 1994b]) with center exponent 0. In particular, L has a shifted exponential dichotomy with $\alpha < 0 = \beta$.

Chapter 2

Connecting orbits with asymptotic rate

In this chapter we define "**connecting orbits with asymptotic rate $\boldsymbol{\gamma}$**". These are solutions of the dynamical system which converge with an asymptotic rate to a given solution $y(\lambda)$ and which have initial values in a given manifold $\mathcal{W}$.

In Section 2.1 we define a local parametrization of "γ–asymptotic" solutions. These are solutions of

$$\dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \mathbf{\Lambda} \subset \mathbb{R}^p \tag{2.1}$$

which converge with asymptotic rate $\gamma < 0$ to $y(\lambda)$, i. e.

$$||x(t) - y(\lambda)(t)|| \leq C_x e^{\gamma t} \quad \forall \quad t \geq 0 \tag{2.2}$$

holds for some constant $C_x > 0$. Here $\{y(\lambda)\}_{\lambda \in \mathbf{\Lambda}}$ is a given family of solutions of (2.1).

Using these parametrizations we show that the set of γ–asymptotic solutions forms a smooth manifold in the crossproduct of the space of bounded functions and the parameter space. We call it the **local $\boldsymbol{\gamma}$–stable manifold**.

We prove smoothness with respect to the parameter and in time direction, even in case of a strong stable manifold. This is not straightforward, because adding parameters λ by defining the additional differential equations $\dot{\lambda} = 0$ would imply that the parameter direction is in the "center part" of the enlarged system. Typically, for example in the periodic case, the time direction belongs to the "center-unstable part". In fact, differentiation with respect to parameters or in time direction causes the loss of one degree of differentiability.

In Section 2.2 we define the **global γ–stable manifold**. The results of Section 2.2 are not used in the remainder, but they illustrate the global structure.

In Section 2.3 we define a **connecting orbit with asymptotic rate** and its **non-degeneracy** and relate this to the non-singularity of a linear map and the regularity of an operator equation.

Throughout this thesis we make the following assumptions for the function f and the family $\{y(\lambda)\}_{\lambda\in\mathbf{\Lambda}}$:

A1 (System properties) Let $f \in C^k(\mathbb{R}^{m+p}, \mathbb{R}^m)$, $k \geq 2$ and let $f^{(k)}$ be locally Lipschitz with respect to x.

A2 (Solution properties) Let $y \in C^k\big(\mathbf{\Lambda}, BC^1(\mathbb{R}_+, \mathbb{R}^m)\big)$, $\mathbf{\Lambda} \subset \mathbb{R}^p$ be an open, bounded set, where $y(\lambda)$ is a solution of $\dot{x} = f(x, \lambda)$ for each $\lambda \in \mathbf{\Lambda}$. Moreover, let $y(\lambda)(t)$ be bounded in $\lambda \in \mathbf{\Lambda}$ and $t \in \mathbb{R}_+$. Let $J_{\mathbf{\Lambda}} := \bigcap_{\lambda\in\mathbf{\Lambda}} J\big(0, y(\lambda)(0), \lambda\big)$, the domain of existence of $y(\lambda)$ for all $\lambda \in \mathbf{\Lambda}$, be an open neighborhood of $\mathbb{R}_+$.

A3 (Dichotomy condition) The linear operator $\bar{L}(\lambda) := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(y(\lambda), \lambda\big)$, $\lambda \in \mathbf{\Lambda}$ has a shifted exponential dichotomy with data $\big(\bar{K}, \bar{\alpha}, \bar{\beta}, \bar{P}_a(\lambda), \bar{P}_b(\lambda)\big)$, $\bar{\alpha} < 0$ which are of type C^{k-1} with respect to λ. The ranks of the projectors $\bar{P}_a(\lambda)$ and $\bar{P}_b(\lambda)$ are independent of λ and given by $m_a := \dim \mathcal{R}\big(\bar{P}_a(t)(\lambda)\big)$ and $m_b := \dim \mathcal{R}\big(\bar{P}_b(t)(\lambda)\big)$.

Remark: **A1** and **A2** imply that $y(\lambda) \in C^{k+1}(\mathbb{R}_+, \mathbb{R}^m)$ and that $f^{(j)}\big(y(\lambda)(t), \lambda\big)$ and $y^{(j)}(\lambda)(t)$ are bounded in $\lambda \in \mathbf{\Lambda}$ and $t \in \mathbb{R}_+$ for $0 \leq j \leq k$. In addition, **A2** implies that there exists some $\epsilon > 0$, such that all $y(\lambda)$ solutions are defined on $[-\epsilon, \infty)$.

The dichotomy condition **A3** can be weakened by the assumption that **A3** holds for some $\bar{\lambda} \in \mathbf{\Lambda}$. Then Proposition 1.6 implies the existence of some open neighborhood $\mathbf{\Lambda}_0 \subset \mathbf{\Lambda}$ such that **A3** holds on $\mathbf{\Lambda}_0$.

As we have seen in Section 1.3, these assumptions hold, if the $y(\lambda)$ are either equilibria or scaled periodic orbits.

2.1 The local γ–stable manifold

In this section we define "γ–asymptotic" solutions and derive a local parametrization of these solutions. To do so, we apply the implicit function theorem in the space of exponentially decaying functions. Using this parametrization, we introduce the notion of a "local γ–stable manifold".

DEFINITION 2.1 *Two solutions x, y of $\dot{x} = f(x, \lambda)$ on $\mathbb{R}_+$ are called* **asymptotic** *if*

$$\lim_{t\to\infty} ||x(t) - y(t)|| = 0 \tag{2.3}$$

and **γ–asymptotic** ($\gamma < 0$) *if in addition there exists some $C_x > 0$ such that*

$$||x(t) - y(t)|| \leq C_x e^{\gamma t} \quad \forall\, t \geq 0. \tag{2.4}$$

We call two points x and $y \in \mathbb{R}^m$ **γ–asymptotic** *if the corresponding solutions $u(\cdot, 0, x, \lambda)$ and $u(\cdot, 0, y, \lambda)$ of $\dot{x} = f(x, \lambda)$ with $x(0) = x$, $y(0) = y$ are γ–asymptotic.*

We establish a local parametrization of γ–asymptotic solution $\bar{x}$. This parametrization defines a manifold structure on the set of points which are γ–asymptotic with $\bar{x}(0)$ (see 2.1.1). We call it the local γ–stable manifold. In 2.1.2 we proceed analogously for points which are γ–asymptotic with some $\bar{x}(\psi)$, where ψ is in a neighborhood of 0.

2.1.1 The local γ–stable manifold evaluated at a point

We establish a local parametrization of γ–asymptotic solutions in Proposition 2.2.

The main idea is to apply the implicit function theorem to an appropriate operator F, such that the zeros of F define the γ–asymptotic solutions. To get smoothness results, in particular with respect to the parameter, we have to take some care and need the Lipschitz continuity of the k–th derivative of f.

PROPOSITION 2.2 (γ–asymptotic solutions) *Let* **A1**–**A2** *hold and let $\bar{x}$ and $\bar{y} = y(\bar{\lambda})$ be $\bar{\gamma}$–asymptotic solutions ($\bar{\gamma} < 0$) of $\dot{x} = f(x, \bar{\lambda})$ for some $\bar{\lambda} \in \mathbf{\Lambda}$.*

Furthermore assume that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{x}(t), \bar{\lambda})$ has a shifted exponential dichotomy on $[0, \infty)$ with dichotomy data $(K, \alpha, \beta, P_a, P_b)$, $\alpha < 0$, $\beta > \alpha$. Then for $\gamma \in \big(\max(\alpha, \bar{\gamma}), \beta\big)$ there exist open neighborhoods $V \times \Omega \subset \mathcal{R}\big(P_a(0)\big) \times \mathbf{\Lambda}$ of $(0, \bar{\lambda})$ and $U \subset BC^1(\mathbb{R}_+, \mathbb{R}^m)$ of $\bar{x}$ and a function $x \in C^{k-1}(V \times \Omega, U)$, such that

P1 $x(0, \bar{\lambda}) = \bar{x}$

P2 $x(\xi, \lambda) \in U$ *for all* $(\xi, \lambda) \in V \times \Omega$

P3 *For $(\xi, \lambda) \in V \times \Omega$ the function $x(\xi, \lambda)$ is the unique solution in U of the problem*

$$\left\{ \begin{array}{l} x \text{ and } y(\lambda) \text{ are } \gamma\text{–asymptotic solutions of } \dot{x} = f(x, \lambda), \\ P_a(0)\big(x(0) - \bar{x}(0)\big) = \xi. \end{array} \right. \tag{$A_\gamma(\bar{x}, \bar{\lambda})$}$$

P4 *The partial derivatives of x at $(0,\bar\lambda)$ are*

$$\frac{\partial}{\partial\xi}x(0,\bar\lambda) \quad = \quad S(\cdot,0)P_a(0) \tag{2.5}$$

$$\frac{\partial}{\partial\lambda}x(0,\bar\lambda) \quad = \quad s_{[0,\infty)}\big(\Delta(\cdot)\big) - S(\cdot,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0) + \frac{\partial}{\partial\lambda}y(\bar\lambda)(\cdot) \tag{2.6}$$

with $s_{[0,\infty)}(w)(t) = \int_0^t S(t,s)P_a(s)w(s)\mathrm{ds} - \int_t^\infty S(t,s)P_b(s)w(s)\mathrm{ds}$ *and*

$$\begin{aligned}\Delta(s) \quad := \quad & \left(\frac{\partial}{\partial x}f\big(\bar x(s),\bar\lambda\big) - \frac{\partial}{\partial x}f\big(\bar y(s),\bar\lambda\big)\right)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s) \\ & + \left(\frac{\partial}{\partial\lambda}f\big(\bar x(s),\bar\lambda\big) - \frac{\partial}{\partial\lambda}f\big(\bar y(s),\bar\lambda\big)\right).\end{aligned} \tag{2.7}$$

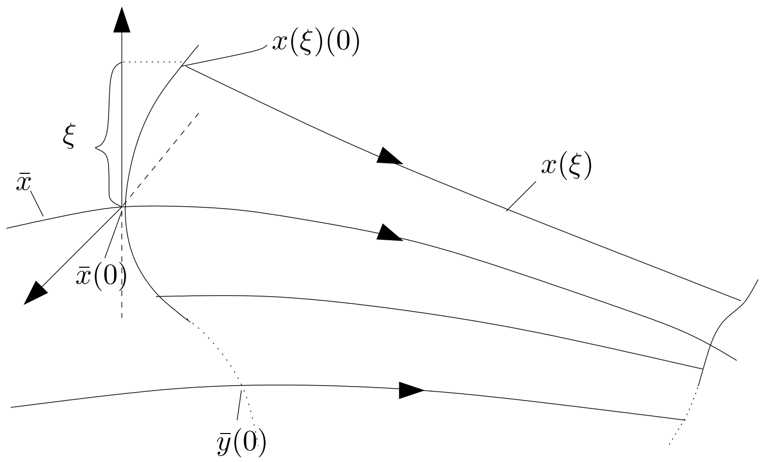

Figure 2.1: Local parametrization of $\bar x$ at 0 (without parameter)

Remark: This proposition asserts a local parametrization of the set of γ–asymptotic solutions. It is parameterized by the (strong) stable subspace $\mathcal{R}\big(P_a(0)\big)$ and the parameter space $\mathbf{\Lambda}$. To derive the approximation theory in Chapter 3 and Chapter 5 we use these results. In particular, we need the derivative with respect to the parameter (2.6).

Notations: As abbreviation let $BC^\kappa_\gamma := BC^\kappa_\gamma(\mathbb{R}_+,\mathbb{R}^m)$ and $\mathcal{X}^\kappa_\gamma := BC^\kappa_\gamma \times \mathcal{R}\big(P_a(0)\big)$ for $\kappa = 0,1$.

As norm on $\mathcal{R}\big(P_a(0)\big)$ we choose the norm on $\mathbb{R}^m$ and for Banach spaces $(\mathcal{X},||\cdot||_\mathcal{X})$, $(\mathcal{Y},||\cdot||_\mathcal{Y})$ we define a norm on $\mathcal{X}\times\mathcal{Y}$ by $(||\cdot||_\mathcal{X} + ||\cdot||_\mathcal{Y})$.

We denote the Lipschitz constant of some function g by L_g and for the j–th derivative by L^j_g. Analogously, we denote the upper bounds by M_g and M^j_g, respectively.

Proof: Any function x which is γ–asymptotic with $y(\lambda)$ can be written as

$$x = \bar{x} + y(\lambda) - \bar{y} + z \text{ for some } z \in BC^1_\gamma(\mathbb{R}_+, \mathbb{R}^m). \tag{2.8}$$

We find an appropriate $z \in BC^1_\gamma(\mathbb{R}_+, \mathbb{R}^m)$, neighborhoods V, Ω, U and a function $x(\xi, \lambda)$ through the following steps:

Step: 1 We define suitable neighborhoods and an operator F such that any zero z of F defines a solution of $(A_\gamma(\bar{x}, \bar{\lambda}))$ by (2.8) (see Lemma 2.3).

Step: 2 In Lemma 2.4 we prove smoothness properties of the operator F. Applying the implicit function theorem yields the existence of $z(\xi, \lambda)$ with $F(z(\xi, \lambda), \xi, \lambda) = 0$.

Step: 3 We use (2.8) and Lemma 2.3 to define sets V, Ω, U and the function $x(\xi, \lambda)$ with the properties of Proposition 2.2. To prove uniqueness in **P3** we apply the contraction mapping theorem.

Step 1: Let (x, ξ, λ) be a solution of $(A_\gamma(\bar{x}, \bar{\lambda}))$ and define

$$z = x - y(\lambda) - \bar{x} + \bar{y}. \tag{2.9}$$

Then (z, ξ, λ) is a solution of the operator equation

$$F(z; \xi, \lambda) := \begin{pmatrix} \dot{z} - h(\cdot, z(\cdot), \lambda) \\ P_a(0)\big(z(0) + y(\lambda)(0) - \bar{y}(0)\big) - \xi \end{pmatrix} = 0, \tag{2.10}$$

where $h \in C^k(\mathbb{R}_+ \times \mathbb{R}^m \times \mathbf{\Lambda}, \mathbb{R}^m)$ is defined by

$$h(t, v, \lambda) := \tag{2.11}$$
$$f\big(v + \bar{x}(t) + y(\lambda)(t) - \bar{y}(t), \lambda\big) - f\big(y(\lambda)(t), \lambda\big) + f\big(\bar{y}(t), \bar{\lambda}\big) - f\big(\bar{x}(t), \bar{\lambda}\big).$$

We relate γ–asymptotic solutions to regular solutions of the operator equation $F = 0$ as follows:

LEMMA 2.3 *Suppose that the assumptions of Proposition 2.2 hold, let $(\xi, \lambda) \in V \times \Omega$ and define for $\delta > 0$*

$$U_\delta := \big\{x \in BC^1(\mathbb{R}_+, \mathbb{R}^m) \big| ||x - \bar{x}||_\infty < \delta\big\}. \tag{2.12}$$

Then $(x, \xi, \lambda) \in U_\delta \times V \times \Omega$ solves $(A_\gamma(\bar{x}, \bar{\lambda}))$ if and only if $(z, \xi, \lambda) \in W^{V,\Omega}_\delta$,

$$W^{V,\Omega}_\delta := \big\{(z, \xi, \lambda) \in BC^1_\gamma \times V \times \Omega \quad \big| \, ||z + y(\lambda) - \bar{y}||_\infty < \delta\big\} \tag{2.13}$$

and (z, ξ, λ) solves the operator equation

$$F(z; \xi, \lambda) = 0,$$

where z is defined as in (2.9).

Proof: "$\Longrightarrow$" Let $(x,\xi,\lambda)\in U_\delta\times V\times\Omega$ be a solution of $(A_\gamma(\bar{x},\bar{\lambda}))$. ¿From (2.9) and $x\in U_\delta$ we see $z\in BC^1(\mathbb{R}_+,\mathbb{R}^m)$ and $||z+y(\lambda)-\bar{y}||_\infty<\delta$. Moreover

$$\begin{aligned}
|z(t)|e^{-\gamma t} &\leq |x(t)-y(\lambda)(t)|e^{-\gamma t}+|\bar{x}(t)-\bar{y}(t)|e^{-\gamma t}\leq C_x+C_{\bar{x}} && (2.14)\\
|\dot{z}(t)|e^{-\gamma t} &= |h(t,z(t),\lambda)|e^{-\gamma t} && (2.15)\\
&\leq |f(x(t),\lambda)-f(y(\lambda)(t),\lambda)|e^{-\gamma t}+|f(\bar{x}(t),\bar{\lambda})-f(\bar{y}(t),\bar{\lambda})|e^{-\gamma t}\\
&\leq L_f|x(t)-y(\lambda)(t)|e^{-\gamma t}+L_f|\bar{x}(t)-\bar{y}(t)|e^{-\gamma t}\leq L_f(C_x+C_{\bar{x}}).
\end{aligned}$$

Therefore $z\in BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)$ and $(z,\xi,\lambda)\in W_\delta^{V,\Omega}\times V$ follow. Clearly, $F(z;\xi,\lambda)=0$ by definition of z and h.

"$\Longleftarrow$" Let $(z,\xi,\lambda)\in W_\delta^{V,\Omega}$ be a solution of (2.10). Then x defined by (2.8) is in U_δ and solves $\dot{x}=f(x,\lambda)$, $P_a(0)(x(0)-\bar{x}(0))=\xi$. The solutions $\bar{x}$, $\bar{y}$ are γ–asymptotic and $z\in BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)$, thus we find

$$|x(t)-y(\lambda)(t)|=|z(t)+\bar{x}(t)-\bar{y}(t)|\leq|z(t)|+|\bar{x}(t)-\bar{y}(t)|\leq(||z||_\gamma+C_{\bar{x}})e^{\gamma t}.$$

This implies that x and $y(\lambda)$ are γ–asymptotic. ■

Step 2: In order to apply the implicit function theorem to F we prove

LEMMA 2.4 *The operator F has the following properties:*

1. $F\in C^{k-1}(\mathcal{X}^1_\gamma\times\Lambda,\mathcal{X}^0_\gamma)$
2. $F(0;0,\bar{\lambda})=0$
3. $Q:=\frac{\partial}{\partial z}F(0;0,\bar{\lambda})\in\mathcal{L}\big(BC^1_\gamma,\mathcal{X}^0_\gamma\big)$ *is nonsingular with inverse*

$$Q^{-1}(y,\xi)=S(\cdot,0)P_a(0)\xi+s_{[0,\infty)}(y)=S(\cdot,0)P_a(0)\xi+\int_0^\infty G(\cdot,s)y(s)\mathrm{d}s,$$

where G is Green's function defined in (1.8).

Proof: Some care is necessary when dealing with differentiability with respect to exponentially weighted norms, for that reason we elaborate the details.

By **A2** we see that the second component F is in $C^k\big(\mathcal{X}^1_\gamma\times\Lambda,\mathcal{R}(P_a(0))\big)$.

The first component of F is independent of ξ, therefore we neglect ξ and define $F_1(z,\lambda)=\dot{z}-h(\cdot,z(\cdot),\lambda)$. To see $F_1(z,\lambda)\in BC^0_\gamma$ we notice (similar to (2.14)) that

$$||\dot{z}-h(\cdot,z,\lambda)||_\gamma\leq||\dot{z}||_\gamma+L_f(||z||_\gamma+2C_{\bar{x}}).$$

We define a linear operator which will be shown to be the derivative of F_1 at (z,λ) by

$$DF_1(z,\lambda):\begin{array}{ccl} BC^1_\gamma\times\boldsymbol{\Lambda} & \to & BC^0_\gamma\\ (v,\mu) & \mapsto & \dot{v}-\frac{\partial}{\partial x}f\big(z+\bar{x}+y(\lambda)-\bar{y},\lambda\big)\,v-\frac{\partial}{\partial\lambda}h\big(\cdot,z,\lambda\big)\,\mu\end{array}$$

To see $DF_1(z,\lambda)(v,\mu) \in BC^0_\gamma$ for any $(v,\mu) \in BC^1_\gamma \times \mathbb{R}^p$ we estimate at $t \geq 0$

$$\begin{aligned}
|\tfrac{\partial}{\partial\lambda} h(t, z(t), \lambda)\mu| e^{-\gamma t} &\leq L^1_f\big(||z||_\gamma + C_{\bar{x}}\big)(1 + M_y)\ |\mu| \\
|\tfrac{\partial}{\partial x} f\big(z(t) + \bar{x}(t) + y(\lambda)(t) - \bar{y}(t), \lambda\big)\ v(t)| e^{-\gamma t} &\leq M^1_f ||v||_\gamma
\end{aligned}$$

and hence we find

$$\begin{aligned}
||DF_1(z,\lambda)||_\gamma &\leq \sup_{\substack{(v,\mu)\in BC^1_\gamma\times\mathbf{\Lambda} \\ ||(v,\mu)||\leq 1}} ||DF_1(z,\lambda)[v,\mu]||_\gamma \\
&\leq \sup_{\substack{(v,\mu)\in BC^1_\gamma\times\mathbf{\Lambda} \\ ||(v,\mu)||\leq 1}} \big(||\dot{v}||_\gamma + M^1_f ||v||_\gamma + L^1_f\big(||z||_\gamma + C_{\bar{x}}\big)(1 + M_y)\ |\mu|\big) \\
&\leq \max\big(1, M^1_f, L^1_f\big(||z||_\gamma + C_{\bar{x}}\big)(1 + M_y)\big).
\end{aligned}$$

In fact, $DF_1(z,\lambda)$ is the derivative of F_1 at (z,λ). This is seen by the following estimate where we use $\tilde{x} := z + \bar{x} + y(\lambda) - \bar{y}$ as abbreviation:

$$\begin{aligned}
&|F_1(z+v, \lambda+\mu)(t) - F_1(z,\lambda)(t) - \big(DF_1(z,\lambda)[v,\mu]\big)(t)| e^{-\gamma t} \qquad (2.16)\\
= \;& \Big|F_1(z+v, \lambda+\mu)(t) - F_1(z+v,\lambda)(t) - \frac{\partial}{\partial\lambda} h(t, z(t)+v(t), \lambda)\ \mu \\
& + f\big(v(t) + \tilde{x}(t), \lambda\big) - f\big(\tilde{x}(t), \lambda\big) - \frac{\partial}{\partial x} f\big(\tilde{x}(t), \lambda\big)\ v(t) \\
& + \frac{\partial}{\partial\lambda} h(t, z(t)+v(t), \lambda)\ \mu - \frac{\partial}{\partial\lambda} h(t, z(t), \lambda)\ \mu\Big| e^{-\gamma t} \\
\leq \;& |\mu|^2 L^2_f\big(1 + 2M^1_y + (M^1_y)^2\big)|\bar{x}(t) - \bar{y}(t)| e^{-\gamma t} \\
& + |v(t)|^2 L^1_f e^{-\gamma t} + |\mu| L^1_f\big(1 + M^1_y\big)|v(t)| e^{-\gamma t} \\
\leq \;& \max\big(L^2_f\big(1 + 2M^1_y + (M^1_y)^2\big)C_{\bar{x}}, L^1_f\big(1 + M^1_y\big)\big)\big(|\mu| + ||v||_\gamma\big)^2
\end{aligned}$$

Similarly, we define multi-linear operators $D^i F_1(z,\lambda)$, $i = 2, \ldots, k$, prove that its norm $||D^i F_1(z,\lambda)||_\gamma$ exists and show that $D^i F_1(z,\lambda)$ is the derivative of $D^{i-1}F_1$ at (z,λ) for $i = 2, \ldots, k-1$. For that purpose we fix $t \in \mathbb{R}_+$ and define

$$\tilde{g}_t(\bar{z},\lambda) = \big(\bar{z} + \bar{x}(t) - \bar{y}(t) + y(\lambda)(t), \lambda\big) \text{ and } g_t(\bar{z},\lambda) = \big(y(\lambda)(t), \lambda\big) \qquad (2.17)$$

for $(\bar{z},\lambda) \in \mathbb{R}^m \times \mathbf{\Lambda}$. Hence

$$D^i F_1(\bar{z},\lambda)(t) = \Big(\big(f \circ g_t\big)^{(i)} - \big(f \circ \tilde{g}_t\big)^{(i)}\Big)(\bar{z},\lambda), \quad i = 2, \ldots, k \qquad (2.18)$$

is the i–th derivatives of $F_1(\bar{z},\lambda)(t)$ for fixed t.

To apply the higher order chain rule (see e. g. [Abraham et al., 1988], [Kleß, 1997], [Beyn and Kleß, 1998]) we introduce the following notations:

For $\eta_j := (v_j, \mu_j) \in \mathbb{R}^{m+p}$ we define $[\eta_j]_{j=1}^{\kappa} := [\eta_1, \ldots, \eta_\kappa]$ and for a permutation $\pi_\kappa = \big(\pi_\kappa(1), \ldots, \pi_\kappa(\rho_\kappa)\big)$ we define $\eta_{\pi_\kappa} := [\eta_{\pi_\kappa(j)}]_{j=1}^{\rho_\kappa}$. Furthermore, a partition of the set of permutations of $\omega = \{1, \ldots, i\}$ is defined by $\mathcal{Q}(i) = \bigcup_{l=1}^{i} \mathcal{Q}_l(i)$, where

$$\begin{aligned}\mathcal{Q}_l(i) := \mathcal{Q}_l(\omega) := \Big\{\pi = (\pi_1, \ldots, \pi_l)\Big|\pi_j \subset \omega, \pi_j \neq \emptyset,\ 1 \leq j \leq l,\ \bigcup_{j=1}^{l} \pi_j = \omega \\ \text{and } \pi_j \cap \pi_\kappa = \emptyset,\ \min(\pi_j) < \min(\pi_\kappa) \text{ for all } 1 \leq j < \kappa \leq l\Big\}.\end{aligned}$$

Moreover we define $\iota(\pi) := (\#\pi_1, \cdots, \#\pi_l)$ for $\pi \in \mathcal{Q}_l(i)$, where $\#\pi_j$ is the number of elements of π_j.

We calculate $D^i F_1(\bar{z}, \lambda)(t)$ by the chain rule as in Lemma A.2 (see Appendix), which is similar to [Kleinkauf, 1998, Lemma A.0.4].

$$\begin{aligned}
D^i F_1(\bar{z}, \lambda)(t)[\eta_j]_{j=1}^{i} &= (f \circ g_t)^{(i)}(\bar{z}, \lambda)[\eta_j]_{j=1}^{i} - (f \circ \tilde{g}_t)^{(i)}(\bar{z}, \lambda)[\eta_j]_{j=1}^{i} \\
= \sum_{l=1}^{i} \sum_{\substack{\pi \in \mathcal{Q}_l(i) \\ \rho := \iota(\pi)}} &\Big\{\Big(f^{(l)}\big(g_t(\bar{z}, \lambda)\big) - f^{(l)}\big(\tilde{g}_t(\bar{z}, \lambda)\big)\Big)\big[g_t^{(\rho_j)}(\bar{z}, \lambda)\eta_{\pi_j}\big]_{j=1}^{l} \\
&+ \sum_{\kappa=1}^{l} f^{(l)}\big(\tilde{g}_t(\bar{z}, \lambda)\big)\Big[\big[\tilde{g}_t^{(\rho_j)}(\bar{z}, \lambda)\eta_{\pi_j}\big]_{j=1}^{\kappa-1}, \\
&\qquad g_t^{(\rho_\kappa)}(\bar{z}, \lambda)\eta_{\pi_\kappa} - \tilde{g}_t^{(\rho_\kappa)}(\bar{z}, \lambda)\eta_{\pi_\kappa}, \big[g_t^{(\rho_j)}(\bar{z}, \lambda)\eta_{\pi_j}\big]_{j=\kappa+1}^{l}\Big]\Big\}.
\end{aligned}$$

The definitions of g_t and $\tilde{g}_t$ (see (2.17)) imply $||g_t^{(j)}(\bar{z}, \lambda)|| \leq M^j$, $||\tilde{g}_t^{(j)}(\bar{z}, \lambda)|| \leq M^j$ for $j \geq 1$, where $M^1 := (1 + M_y^1)$ and $M^j := M_y^j$, $j = 2, \ldots, k$. Moreover,

$$||\big(g_t^{(1)}(\bar{z}, \lambda) - \tilde{g}_t^{(1)}(\bar{z}, \lambda)\big)(v, \mu)|| = |v| \text{ and } ||g_t^{(j)}(\bar{z}, \lambda) - \tilde{g}_t^{(j)}(\bar{z}, \lambda)|| = 0 \tag{2.19}$$

for $j \geq 2$. Furthermore, the Lipschitz continuity of f yields

$$|f^{(l)}\big(g_t(\bar{z}, \lambda)\big) - f^{(l)}\big(\tilde{g}_t(\bar{z}, \lambda)\big)| \leq L_f^l(|\bar{z}| + |\bar{x}(t) - \bar{y}(t)|). \tag{2.20}$$

Now we see for $z, v_j \in BC_\gamma^1(\mathbb{R}_+, \mathbb{R}^m)$ and by using the definitions $\eta_j := (v_j(t), \mu_j)$ that

$$\begin{aligned}
&|D^i F_1(\bar{z}(t), \lambda)(t)[\eta_j]_{j=1}^{i}|e^{-\gamma t} \\
&= \big|(f \circ g_t)^{(i)}(z(t), \lambda)[\eta_j]_{j=1}^{i} - (f \circ \tilde{g})^{(i)}(z(t), \lambda)[\eta_j]_{j=1}^{i}\big|e^{-\gamma t} \\
&\leq \sum_{l=1}^{i} \sum_{\pi \in \mathcal{Q}_l(i)} \Big\{\Big(L_f^l \prod_{j=1}^{l} M^j\Big) \cdot (||z||_\gamma + C_{\bar{x}})\big\|[\eta_j]_{j=1}^{i}\big\| \\
&\qquad + \sum_{\kappa=1}^{l} M_f^l \prod_{\substack{j=1 \\ j \neq \kappa}}^{l} M^j \cdot |v_\kappa(t)|e^{-\gamma t} \cdot \big\|[\eta_j]_{\substack{j=1 \\ j\neq\kappa}}^{i}\big\|\Big\}.
\end{aligned} \tag{2.21}$$

Thus, $|\eta_j| \leq ||v_j||_\gamma + |\mu_j|$ and $|v_\kappa(t)|e^{-\gamma t} \leq ||v_\kappa||_\gamma + |\mu_\kappa|$ imply that the estimates $|D^i F_1(z(\cdot),\lambda)(t)|e^{-\gamma t} \leq C_{D^iF_1}(||z||_\gamma + c_{D^iF_1})$ for $i = 1,\ldots,k$ hold with some constants $C_{D^iF_1} > 0$, $c_{D^iF_1} > 0$. Thus, $D^iF_1(z(\cdot),\lambda)(\cdot)$ is a continuous function defined on $\mathbb{R}_+$ and $D^iF_1(z(\cdot),\lambda)(\cdot)[\eta_j]_{j=1}^i \in BC^0_\gamma$ for any $[\eta_j]_{j=1}^i \in \big(BC^1_\gamma \times \mathbb{R}^p\big)^i$.

Similarly, it can be seen that D^iF_1 is Lipschitz with respect to z. To do so, we apply Lemma A.2 with $g_t = \tilde{g}_t$ ($\tilde{g}_t$ defined as in (2.17)) and $x = \hat{z}$, $\tilde{x} = \bar{z}$. Then we see $||g_t^{(j)}(\hat{z},\lambda) - \tilde{g}_t^{(j)}(\bar{z},\lambda)|| = 0$ and $f^{(l)}\big(g(\hat{z},\lambda)\big) - f^{(l)}\big(\tilde{g}(\bar{z},\lambda)\big) \leq L_f^l(|\bar{z}-\tilde{z}|)$ for $j,l \in \{1,\ldots,k\}$. This implies $||D^iF_1(\hat{z},\lambda) - D^iF_1(\bar{z},\lambda)||_\gamma \leq L_{D^iF_1}||\hat{z}-\bar{z}||_\gamma$ with $L_{D^iF_1} := \sum_{l=1}^i \sum_{\pi\in\mathcal{Q}_l(i)} \Big(L_f^l \prod_{j=1}^l M^j\Big)$ for $i = 1,\ldots,k$.

Finally, we have to prove that $D^{i+1}F_1$ is the derivative of D^iF_1 in the space of exponentially decaying functions. Here we estimate along the line of (2.16) and see for $i \leq k-2$ and $(v,\mu) \in BC^1_\gamma \times \mathbb{R}^p$ with $||(v,\mu)|| < \delta$, $\delta > 0$ that

$$
\begin{aligned}
&\big|D^iF_1(z+v,\lambda+\mu)(t) - D^iF_1(z,\lambda)(t) - \big(DF_1^{i+1}(z,\lambda)[v,\mu]\big)(t)\big|e^{-\gamma t} \qquad (2.22)\\
= \;& \big|D^iF_1(z+v,\lambda+\mu)(t) - D^iF_1(z+v,\lambda)(t) - \big(DF_1^{i+1}(z+v,\lambda)[0,\mu]\big)(t)\\
&+D^iF_1(z+v,\lambda)(t) - D^iF_1(z,\lambda)(t) - \big(DF_1^{i+1}(z,\lambda)[v(t),0]\big)(t)\\
&+\big(D^{i+1}F_1(z+v,\lambda)[0,\mu]\big)(t) - \big(D^{i+1}F_1(z,\lambda)[0,\mu]\big)(t)\big|e^{-\gamma t}\\
\leq \;& C_{D^{i+2}F_1}(||z||_\gamma + \delta + c_{D^{i+2}F_1})|\mu|^2\\
&+L_{D^{i+1}F_1}\,|v(t)|\,|v(t)|e^{-\gamma t} + L_{D^{i+1}F_1}\,|\mu|\,|v(t)|e^{-\gamma t}.\\
\leq \;& C(||v||_\gamma + |\mu|)^2 \qquad (2.23)
\end{aligned}
$$

Remark: The main argument for the exponential decay of the derivatives with respect to λ is the Lipschitz continuity of $f^{(k)}$ which implies

$$
\left|\frac{\partial^l}{\partial\lambda^l} h(t,z,\lambda)\right| \leq C|z(t) + \bar{x}(t) - \bar{y}(t)| \leq C(||z||_\gamma + C_{\bar{x}})e^{\gamma t}.
$$

To differentiate $D^{k-2}F_1$ with respect to λ we use the Lipschitz constant L_f^k of $f^{(k)}$. On the other hand, $D^{k-1}F_1$ is once differentiable with respect to z. This follows in a straightforward way from the previous estimates with $\mu_j = 0$. The continuity of $\frac{\partial}{\partial z}D^{k-1}F_1$ follows from the fact that D^kF_1 is Lipschitz with respect to z.

Now we have proved *1.* in Lemma 2.4.

2. By definition of F it is obvious that $F(0;0,\bar{\lambda}) = 0$.

3. Differentiating F at $(0,0,\bar{\lambda})$ yields

$$
F'(0,0,\bar{\lambda}) = \begin{pmatrix} L & 0 & -\Delta(\cdot) \\ P_a(0)E_0 & -\mathrm{Id}_{\mathcal{R}(P_a(0))} & P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0) \end{pmatrix}, \qquad (2.24)
$$

where E_0 is the evaluation operator $E_0 x := x(0)$ and Δ defined as in (2.7). Note that

$$|\Delta(t)| \leq L_f(1+M_y)C_{\bar{x}}e^{\bar{\gamma}t} \leq L_f(1+M_y)C_{\bar{x}}e^{\gamma t} \tag{2.25}$$

implies $\Delta(\cdot) \in BC^0_\gamma(\mathbb{R}_+, \mathbb{R}^m)$.

Theorem 1.4 implies that the inverse of $Q := \frac{\partial}{\partial z}F(0;0,\bar{\lambda})$ is defined by

$$Q^{-1}(y,\xi) = S(\cdot,0)P_a(0)\xi + s_{[0,\infty)}(y) = S(\cdot,0)P_a(0)\xi + \int_0^\infty G(\cdot,s)y(s)\mathrm{d}s.$$

and hence *3.* in Lemma 2.4 holds. ■

Thus we have proved Lemma 2.4 and we proceed with the proof of Proposition 2.2.

Choose $\delta > 0$ and a small neighborhood $\tilde{\Omega} \subset \mathbf{\Lambda}$ of $\bar{\lambda}$ such that $||y(\lambda) - y(\bar{\lambda})||_\infty < \delta$ for all $\lambda \in \tilde{\Omega}$. For the proof of uniqueness later on, we choose δ and $\tilde{\Omega}$ sufficiently small, such that $C_\delta\delta + C_\lambda|\lambda - \bar{\lambda}| < 1$ for all $\lambda \in \tilde{\Omega}$. The constants $C_\delta > 0$ and $C_\lambda > 0$ will be defined in the proof of uniqueness.

We see that $\tilde{W} = \{(z,\xi,\lambda) \in \mathcal{X}^1_\gamma \times \tilde{\Omega} \mid ||z - y(\lambda) + \bar{y}||_\infty < \delta\}$ is an open neighborhood of $(0,0,\bar{\lambda}) \in \mathcal{X}^1_\gamma \times \mathbf{\Lambda}$. We apply the implicit function theorem (see Theorem A.6 in the Appendix) to F defined on $\tilde{W}$. Thus, there exist some open neighborhood $V \times \Omega \subset \mathcal{R}(P_a(0)) \times \tilde{\Omega}$ of $(0,\bar{\lambda})$ and a function $z \in C^{k-1}\big(V \times \Omega, BC^1_\gamma(\mathbb{R}_+, \mathbb{R}^m)\big)$ such that $\big(z(\xi,\lambda),\xi,\lambda\big) \in \tilde{W}$ and $F(z(\xi,\lambda),\xi,\lambda) = 0$ for all $(\xi,\lambda) \in V \times \Omega$ and $z(0,\bar{\lambda}) = 0$. In particular, $\big(z(\xi,\lambda),\xi,\lambda\big)$ is in $W^{V,\Omega}_\delta$ as defined in (2.13). Furthermore, the derivatives of z at $(0,\bar{\lambda})$ are

$$\begin{aligned}
\frac{\partial}{\partial\xi}z(0,\bar{\lambda}) = -Q^{-1}\frac{\partial}{\partial\xi}F(0,0,\bar{\lambda}) &= S(\cdot,0)P_a(0),\\
\frac{\partial}{\partial\lambda}z(0,\bar{\lambda}) = -Q^{-1}\frac{\partial}{\partial\lambda}F(0,0,\bar{\lambda}) &= s_{[0,\infty)}(\Delta(\cdot)) - S(\cdot,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0).
\end{aligned}$$

Step 3: Define $U = U_\delta$ as in (2.12) and set $x(\xi,\lambda) := z(\xi,\lambda) + \bar{x} + y(\lambda) - \bar{y}$, then Lemma 2.3 implies that $x \in C^{k-1}(V \times \Omega, U)$ has the properties **P1**–**P4** claimed in Proposition 2.2 except for the uniqueness in **P3**.

Uniqueness will be proved now by applying the contraction mapping theorem.

Let $(\tilde{x},\xi,\lambda) \in U \times V \times \Omega$ be a solution of $(A_\gamma(\bar{x},\bar{\lambda}))$. Then $\tilde{x}, x(\xi,\lambda) \in \mathcal{B}_\delta(\bar{x})$, where $\mathcal{B}_\delta(\bar{x})$ is the open δ–ball of $\bar{x}$ in BC^0. Moreover $z = \tilde{x} - x(\xi,\lambda)$ is in the 2δ–ball $\mathcal{B}_{2\delta}(0)$ of 0 and $z \in BC^0_\gamma$. Hence $z \in \mathcal{K}_{2\delta}(0) := \{z \in BC^0_\gamma \mid ||z||_\infty < 2\delta\}$. For δ and $\tilde{\Omega}$ sufficiently small we can apply Proposition 1.6, because

$$|\frac{\partial}{\partial x}f\big(\bar{x}(t),\bar{\lambda}\big) - \frac{\partial}{\partial x}f\big(x(\xi,\lambda)(t),\lambda\big)| \leq L^1_f\delta + L^1_f|\lambda - \bar{\lambda}| =: b_\infty. \tag{2.26}$$

This implies that $\tilde{L} := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(x(\xi,\lambda),\lambda\big)$ has a shifted exponential dichotomy with data $(\tilde{K},\tilde{\alpha},\tilde{\beta},\tilde{P}_a,\tilde{P}_b)$, where $\tilde{\alpha} < \gamma < \tilde{\beta}$. We denote the solution operator of $\tilde{L}$ by $\tilde{S}(\cdot,\cdot)$ and define $\tilde{s}_{[0,\infty)}(w)(t) = \int_0^t \tilde{S}(t,s)\tilde{P}_a(s)w(s)\mathrm{d}s - \int_t^\infty \tilde{S}(t,s)\tilde{P}_b(s)w(s)\mathrm{d}s$. In addition (1.25) implies $\big|P_a(0)\tilde{P}_b(0)\big| \le \tilde{K}K\frac{b_\infty}{\beta-\tilde{\alpha}}$.

By Lemma 1.10 with $r=\frac{1}{2}$ we conclude that $\big(P_a(0)|_{\mathcal{R}(\tilde{P}_a(0))}\big)$ is a homeomorphism and its inverse is bounded by $4K$. Now we define for $v \in \mathcal{K}_{2\delta}(0)$

$$\begin{aligned} g(t,v,\lambda) &:= f\big(v(t)+x(\xi,\lambda)(t),\bar{\lambda}\big) - f\big(x(\xi,\lambda)(t),\lambda\big) - \frac{\partial}{\partial x} f\big(x(\xi,\lambda)(t),\lambda\big)v(t) \\ \tilde{\xi}(v) &:= -\big(P_a(0)|_{\mathcal{R}(\tilde{P}_a(0))}\big)^{-1}\big(P_a(0)\tilde{P}_b(0)v(0)\big) \end{aligned}$$

and see that z solves $\tilde{L}z = g(\cdot,z,\lambda)$, $\tilde{P}_a(0)z(0) = \tilde{\xi}(z)$ (notice $P_a(0)z(0) = \xi - \xi = 0$ by definition of z). Thus z is a fixed point of the operator equation

$$z = \tilde{s}_J\big(g(\cdot,z(\cdot),\lambda)\big) + \tilde{S}(\cdot,0)\tilde{\xi}(z) =: \mathcal{G}(z)$$

where $\mathcal{G} : \mathcal{K}_{2\delta}(0) \to BC^0_\gamma$.

We will prove that the operator $\mathcal{G}$ has a Lipschitz constant of the form $C_\delta\delta + C_\lambda|\lambda-\bar{\lambda}|$, thus for sufficiently small δ and Ω it is a contraction and has at most one fixed point in this ball (see Theorem A.3 in the Appendix). In particular, $z = \tilde{x} - x(\xi,\lambda)$ is in this ball and 0 is a fixed point, thus we conclude $\tilde{x} = x(\xi,\lambda)$.

For $t \in \mathbb{R}_+$ and $v \in \mathbb{R}^m$ fixed we see that g is Lipschitz with respect to v by

$$\begin{aligned} &||g(t,v_1,\lambda) - g(t,v_2,\lambda)|| \\ &\le \Big\| \sup_{s\in[0,1]} \Big[\frac{\partial}{\partial x} f\big(sv_1 + (1-s)v_2 + x(\xi,\lambda),\lambda\big) - \frac{\partial}{\partial x} f\big(x(\xi,\lambda),\lambda\big)\Big](v_1-v_2)\Big\| \\ &\le L^1_f \Big\| \sup_{s\in[0,1]} \big(sv_1 + (1-s)v_2\big)(v_1-v_2)\Big\| \end{aligned}$$

and for $z_1, z_2 \in \mathcal{K}_{2\delta}(0)$ this implies $||g(\cdot,z_1(\cdot),\lambda) - g(\cdot,z_2(\cdot),\lambda)||_\gamma \le 2L^1_f\delta||z_1-z_2||_\gamma$ and $\big\|\tilde{s}_J\big[g(\cdot,z_1(\cdot),\lambda) - g(\cdot,z_2(\cdot),\lambda)\big]\big\|_\gamma \le \tilde{K}\tilde{C}_\gamma 2L^1_f\delta||z_1-z_2||_\gamma$.

Finally, we estimate

$$\begin{aligned} |\tilde{S}(t,0)\tilde{\xi}(z)|e^{-\gamma t} &\le |\tilde{S}(t,0)\tilde{P}_a(0)|\ \big\|\big(P_a(0)|_{\mathcal{R}(\tilde{P}_a(0))}\big)^{-1}\big\|\ \big\|P_a(0)\tilde{P}_b(0)\big\|\ |z(0)|e^{-\gamma t} \\ &\le \tilde{K}e^{(\tilde{\alpha}-\gamma)t}4\tilde{K}K^2\frac{b_\infty}{\beta-\tilde{\alpha}}|z(0)|. \end{aligned}$$

Summarizing this implies $||\mathcal{G}(z_1) - \mathcal{G}(z_2)||_\gamma \le (C_\delta\delta + C_\lambda|\lambda-\bar{\lambda}|)||z_1-z_2||_\gamma$. ■

In the later approximation part we only use the evaluation of γ–asymptotic solutions at 0. Hence, we define

$$b(\xi,\lambda) := \big(x(\xi,\lambda)(0),\lambda\big), \quad b \in C^{k-1}(V\times\Omega,\mathbb{R}^{m+p}) \tag{2.27}$$

and its image

$$W^0 := \{b(\xi,\lambda) \mid (\xi,\lambda) \in V \times \Omega\} \subset \mathcal{B}_\delta(\bar{x}(0)) \times \Omega \subset \mathbb{R}^{m+p}, \tag{2.28}$$

where $\delta > 0$ is as in Proposition 2.2. The derivative $B(\xi,\lambda)$ of b is

$$B(\xi,\lambda): \begin{array}{ccc} \mathcal{R}\big(P_a(0)\big) \times \mathbb{R}^p & \to & \mathbb{R}^{m+p} \\ (\eta,\mu) & \mapsto & \begin{pmatrix} \frac{\partial}{\partial\xi} x(\xi,\lambda)(0)\eta + \frac{\partial}{\partial\lambda} x(\xi,\lambda)(0)\mu \\ \mu \end{pmatrix}. \end{array} \tag{2.29}$$

This implies for $(x,\lambda) = \big(x(\xi,\lambda)(0),\lambda\big) \in W^0$ that the tangent space at (x,λ) is

$$T_{(x,\lambda)} W^0 = \mathcal{R}\big(B(\xi,\lambda)\big). \tag{2.30}$$

In particular, with $B := B(0,\bar{\lambda})$ we obtain

$$B(\eta,\mu) = \begin{pmatrix} \eta + \frac{\partial}{\partial\lambda} x(0,\bar{\lambda})(0)\mu \\ \mu \end{pmatrix}. \tag{2.31}$$

Hence b is immersive at $(0,\bar{\lambda})$ and by Theorem A.5 (see Appendix) there exists an open neighborhood $\tilde{V} \times \tilde{\Omega} \subset V \times \Omega$ of $(0,\bar{\lambda})$ such that $\mathrm{im}(b|_{\tilde{V}\times\tilde{\Omega}})$ is a C^{k-1} submanifold of $\mathbb{R}^{m+p}$.

The results of Proposition 2.2 also hold on this (perhaps smaller) neighborhood, thus with $V \times \Omega := \tilde{V} \times \tilde{\Omega}$ and $x := x|_{\tilde{V}\times\tilde{\Omega}}$ we see that $(U \times V \times \Omega, x)$ satisfy **P1**–**P4** and

P5 $W^0 = \mathrm{im}(b|_{V\times\Omega})$ is a C^{k-1} submanifold of $\mathbb{R}^{m+p}$.

This leads to the definition of a "γ–parametrization" and a "local γ–stable manifold" of $\bar{x}$ evaluated at 0.

Definition 2.5 *Suppose that the assumptions of Proposition 2.2 hold, then we call a pair* $(U \times V \times \Omega, x)$ *with* **P1**–**P5** *a* **local $\boldsymbol{\gamma}$–parametrization of $(\bar{\boldsymbol{x}},\bar{\boldsymbol{\lambda}})$ at 0** *and*

$$\begin{aligned} \mathcal{M}_0^{loc}(\bar{x},\bar{\lambda}) \;\; := \;\; \big\{(z,\lambda) \in \mathbb{R}^{m+p} \;\big|\; & z \text{ and } y(\lambda)(0) \text{ are } \gamma\text{–asymptotic at } \lambda \in \Omega, \\ & P_a(0)\big(z - \bar{x}(0)\big) \in V \text{ and } u(\cdot,0,z,\lambda) \in U\big\}. \end{aligned}$$

is called a **local $\boldsymbol{\gamma}$–stable manifold of $(\bar{\boldsymbol{x}},\bar{\boldsymbol{\lambda}})$ evaluated at 0**.

Remark: By Definition 2.1 two points z, $y(\lambda)(0) \in \mathbb{R}^m$ are γ–asymptotic if the solutions $u(\cdot,0,z,\lambda)$ and $y(\lambda)$ are γ–asymptotic. Furthermore, notice that Proposition 2.2 implies that $\mathcal{M}_0^{loc}(\bar{x},\bar{\lambda}) = W^0$ and $\mathcal{M}_0^{loc} \subset \mathcal{B}_\delta\big(\bar{x}(0)\big) \times \Omega$, where $\mathcal{B}_\delta\big(\bar{x}(0)\big)$ is a δ-ball in $\mathbb{R}^m$, where δ is defined as in Proposition 2.2.

Proposition 2.2 yields the existence of a local γ–stable manifold for the $y(\lambda)$ as well as for any x converging to it.

COROLLARY 2.6 *Suppose that* **A1**–**A3** *hold and let* $\gamma \in (\bar{\alpha}, \bar{\beta})$. *Then there exists a local* γ*–stable manifold of* $\big(y(\lambda), \lambda\big)$ *evaluated at* 0 *for all* $\lambda \in \mathbf{\Lambda}$.

For any $(\tilde{x}, \tilde{\lambda}) \in BC^1(\mathbb{R}_+, \mathbb{R}^m) \times \mathbf{\Lambda}$, *where* $\tilde{x}$ *and* $y(\tilde{\lambda})$ *are* γ*–asymptotic, there exists a local* γ*–stable manifold of* $\big(\tilde{x}, \tilde{\lambda}\big)$ *evaluated at* 0.

Proof: Each $y(\lambda)$ is $\bar{\alpha}$–asymptotic with itself and **A3** implies that we can apply Proposition 2.2 with $\bar{x} = y(\lambda)$ and $\bar{\lambda} = \lambda$ to get a local γ–stable manifold of $\big(y(\lambda), \lambda\big)$.

For the general case Proposition 1.7 implies that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\tilde{x}(t), \tilde{\lambda})$ has a shifted exponential dichotomy on $[0, \infty)$ with exponents $\alpha < \gamma < \beta$, α, β near $\bar{\alpha}$, $\bar{\beta}$, a constant K and projectors P_a, P_b. Hence Proposition 2.2 yields the existence of a local γ–stable manifold. ■

For the approximation we have to evaluate the local γ–stable manifold at some $\tau \in \mathbb{R}_+$. Therefore, we state the following corollary:

COROLLARY 2.7 *Suppose that* **A1**–**A3** *hold and let* $\gamma \in (\bar{\alpha}, \bar{\beta})$, $\lambda \in \mathbf{\Lambda}$ *and* $\tau \in \mathbb{R}_+$. *Then there exists a local* γ*–parametrization* $(U_\tau \times V_\tau \times \Omega_\tau, x_\tau)$ *and hence a local* γ*–stable manifold of* $\big(y(\lambda)(\tau + \cdot), \lambda\big)$ *evaluated at* 0. *Moreover the partial derivatives of* x_τ *satisfy*

$$\frac{\partial}{\partial \xi} x_\tau(0, \lambda)(0) = \bar{P}_a(\lambda)(\tau) \tag{2.32}$$

$$\frac{\partial}{\partial \lambda} x_\tau(0, \lambda)(0) = \bar{P}_b(\lambda)(\tau) \frac{\partial}{\partial \lambda} y(\lambda)(\tau) \tag{2.33}$$

and the tangent space of the local γ*–stable manifold* $\mathcal{M}_0^{loc}\big(y(\lambda)(\tau + \cdot), \lambda\big)$ *is*

$$\Big\{ \big(\eta + \bar{P}_b(\lambda)(\tau) \frac{\partial}{\partial \lambda} y(\lambda)(\tau)\mu, \mu\big) \Big| (\eta, \mu) \in \mathcal{R}\big(\bar{P}_a(\lambda)(\tau)\big) \times \mathbb{R}^p \Big\}. \tag{2.34}$$

Proof: Obviously, $y(\lambda)(\tau + \cdot, \lambda\big)$ satisfies **A2** and $\bar{L}_\tau(\lambda) := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(y(\lambda)(\tau + \cdot, \lambda\big)$ has a solution operator $S_\tau(\cdot, \cdot) := S(\tau + \cdot, \tau + \cdot)$ and a shifted exponential dichotomy with dichotomy data $\big(\bar{K}, \bar{\alpha}, \bar{\beta}, \bar{P_a}^\tau(\lambda), \bar{P_b}^\tau(\lambda)\big)$, where $\bar{P}_\kappa^\tau(\lambda) = \bar{P}_\kappa(\lambda)\big)(\tau + \cdot)$, $\kappa = a, b$. Hence **A1**–**A3** is satisfied and we apply Proposition 2.2 with $\bar{x} = \bar{y} = y(\lambda)(\tau + \cdot)$. We notice that (2.7) and $\bar{x} = \bar{y}$ imply $\Delta(\cdot) = 0$. Hence the definitions of the solution operator and the projectors yield by (2.5) and (2.6)

$$\begin{aligned} \frac{\partial}{\partial \xi} x_\tau(0, \lambda)(0) &= \bar{P}_a^\tau(\lambda)(0) = \bar{P}_a(\lambda)(\tau), \\ \frac{\partial}{\partial \lambda} x_\tau(0, \lambda)(0) &= \bar{P}_b^\tau(\lambda)(0) \frac{\partial}{\partial \lambda} y(\lambda)(\tau) = \bar{P}_b(\lambda)(\tau) \frac{\partial}{\partial \lambda} y(\lambda)(\tau). \end{aligned}$$

Thus, (2.29) and (2.30) implies (2.34). ■

DEFINITION 2.8 *We call the manifold defined by Corollary 2.7 a* **local $\boldsymbol{\gamma}$–stable manifold of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$ evaluated at** τ.

To derive quadratic behavior of approximation errors for appropriate asymptotic boundary conditions we state the following two corollaries:

COROLLARY 2.9 *Let $k \geq 3$ and $(U \times V \times \Omega, x)$ be a γ–parametrization of $(\bar{x}, \bar{\lambda})$ at 0, then there exists some $\tilde{V} \subset V$, such that*

$$\big|P_b(0)\big(x(\xi, \bar{\lambda})(0) - \bar{x}(0)\big)\big| \leq C\big|x(\xi, \bar{\lambda})(0) - \bar{x}(0)\big|^2 \tag{2.35}$$

holds for a constant $C > 0$ and for all $\xi \in \tilde{V}$.

Proof: Let us define $h_b(\xi) := P_b(0)\big(x(\xi, \bar{\lambda})(0) - \bar{x}(0)\big)$, then Proposition 2.2 implies $h_b(0) = 0$, $h_b'(0) = P_b(0)P_a(0) = 0$, hence by Taylor expansion $h_b(\xi) = O(|\xi|^2)$ (here we use $x \in C^2$ and thus we have to assume $k \geq 3$). In addition $\xi = P_a(0)\big(x(\xi, \bar{\lambda})(0) - \bar{x}(0)\big)$ implies $|\xi| \leq K|x(\xi, \bar{\lambda})(0) - \bar{x}(0)|$ and hence (2.35) holds for some constant $C > 0$. ■

COROLLARY 2.10 *Let $k \geq 3$ and let $\bar{y}$ be an equilibrium, such that $L = \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{y}, \bar{\lambda})$ has a shifted exponential dichotomy with dichotomy data $(K, \alpha, \beta, P_a, P_b)$ and with projectors P_a, P_b constant in t. Moreover, let $\bar{x}$ be γ–asymptotic with $\bar{y}$. Then there exist constants $\bar{T} > 0$, $C > 0$, such that for all $t \geq \bar{T}$ the following holds*

$$\big|P_b\big(\bar{x}(t) - \bar{y}\big)\big| \leq C\big|\bar{x}(t) - \bar{y}\big|^2. \tag{2.36}$$

Proof: By Corollary 2.6 there exists a local γ–parametrization $(U \times V \times \Omega, x)$ at $\bar{y}$ and we define $\tilde{V} \subset V$ such that (2.35) holds for all $\xi \in \tilde{V}$. By convergence, there exists some $\bar{T} \in \mathbb{R}_+$, such that $\bar{x}(t + \cdot) \in U$ and $\xi(t) := P_a\big(\bar{x}(t) - \bar{y}\big) \in \tilde{V}$ for all $t \geq \bar{T}$. By Proposition 2.2 we have $\bar{x}(t) = x(\xi(t), \bar{\lambda})(0)$ and Corollary 2.9 implies (2.36). ■

2.1.2 The local γ–stable manifold along a solution

In this section we also include time as a parameter and derive similar results as in 2.1.1. We do not use this for the approximation, except to obtain quadratic estimates in the periodic case (analogous to Corollary 2.10 in the equilibrium case). On the other hand, it allows us to define a "global γ–stable" manifold along the whole solutions and to foliate them. This will be the topic of Section 2.2.

PROPOSITION 2.11 *Let* **A1**–**A2** *hold and let* $\bar{x}$ *and* $\bar{y} = y(\bar{\lambda})$ *be* $\bar{\gamma}$*–asymptotic solutions of* $\dot{x} = f(x, \bar{\lambda})$ $(\bar{\gamma} < 0)$ *for some* $\bar{\lambda} \in \mathbf{\Lambda}$*. Moreover, let* $\epsilon > 0$*, such that all* $y(\lambda), \lambda \in \mathbf{\Lambda}$ *and* $\bar{x}$ *are defined on* $[-\epsilon, \infty)$*, i. e.* $[-\epsilon, \infty) \subset J(0, \bar{x}(0), \bar{\lambda}) \cap J_{\mathbf{\Lambda}}$*.*

Furthermore, assume that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{x}(t), \bar{\lambda})$ *has a shifted exponential dichotomy on* $[-\epsilon, \infty)$ *with dichotomy data* $(K, \alpha, \beta, P_a, P_b)$*,* $\alpha < 0$*,* $\beta > \alpha$*. Then for any* $\gamma \in \big(\max(\alpha, \bar{\gamma}), \beta\big)$ *there exist open neighborhoods* $V \times \Omega \times \Psi \subset \mathcal{R}\big(P_a(0)\big) \times \mathbf{\Lambda} \times (-\epsilon, \infty)$ *of* $(0, \bar{\lambda}, 0)$ *and* $U \subset BC^1(\mathbb{R}_+, \mathbb{R}^m)$ *of* $\bar{x}$ *and a function* $x \in C^{k-1}(V \times \Omega \times \Psi, U)$*, such that*

Q1 $x(0, \bar{\lambda}, \psi) = \bar{x}(\cdot + \psi)$ *for all* $\psi \in \Psi$*,*

Q2 $x(\xi, \lambda, \psi) \in U$ *for all* $(\xi, \lambda, \psi) \in V \times \Omega \times \Psi$*,*

Q3 *For* $(\xi, \lambda, \psi) \in V \times \Omega \times \Psi$ *the function* $x(\xi, \lambda, \psi)$ *is the unique solution in* U *of*

$$\begin{cases} x \text{ and } y(\lambda)(\cdot + \psi) \text{ are } \gamma\text{–asymptotic solutions of } \dot{x} = f(x, \lambda), \\ P_a(0)\big(x(0) - \bar{x}(\psi)\big) = \xi. \end{cases} \qquad (B_\gamma(\bar{x}, \bar{\lambda}))$$

Q4 *The partial derivatives of* x *at* $(0, \bar{\lambda}, \psi)$ *w. r. t.* ξ *and* ψ *satisfy for all* $\psi \in \Psi$

$$\frac{\partial}{\partial \psi} x(0, \bar{\lambda}, \psi) = \dot{\bar{x}}(\cdot + \psi) = f\big(\bar{x}(\cdot + \psi), \bar{\lambda}\big), \qquad (2.37)$$

$$P_b(\psi) \frac{\partial}{\partial \xi} x(0, \bar{\lambda}, \psi)(0) = 0. \qquad (2.38)$$

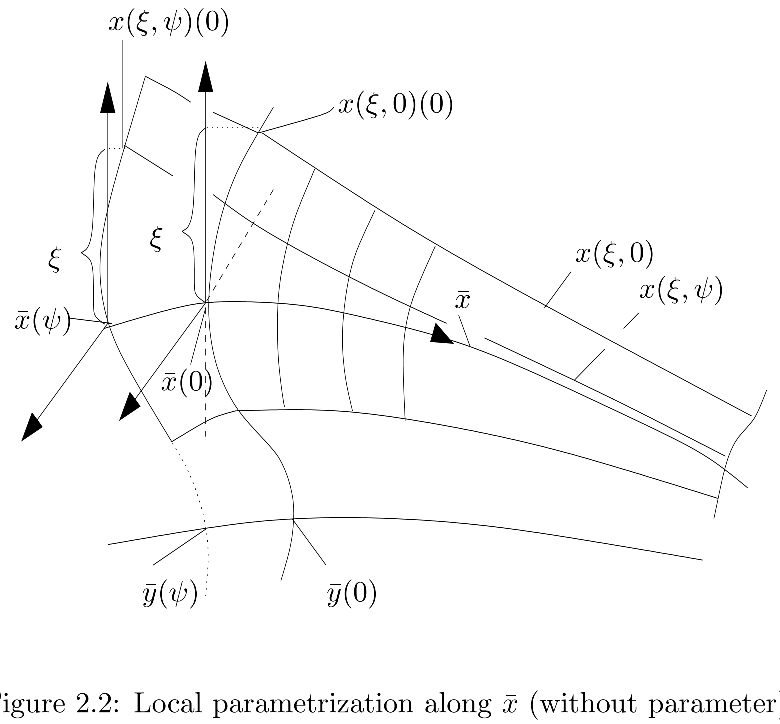

Figure 2.2: Local parametrization along $\bar{x}$ (without parameter)

Remark: The main idea for extending Proposition 2.2 to the case where time is included is to interprete ψ as additional parameter.

The assumption that L has a shifted exponential dichotomy on $[-\epsilon,\infty)$ ensures, that $L_\psi := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(\bar{x}(\cdot+\psi),\bar{\lambda}\big)$ has a shifted exponential dichotomy on $[0,\infty)$ with dichotomy data $\big(K,\alpha,\beta,P_a(\cdot+\psi)P_b(\cdot+\psi)\big)$ for all $\psi \in (-\epsilon,\infty)$ (see the proof of Corollary 2.7).

Proof: In the proof we only demonstrate the differences to the proof of Proposition 2.2.

Let (x,ξ,λ,ψ) be a solution of $(B_\gamma(\bar{x},\bar{\lambda}))$. As in (2.9) we define

$$z = x - y(\lambda)(\cdot+\psi) - \bar{x}(\cdot+\psi) + \bar{y}(\cdot+\psi). \tag{2.39}$$

Then we replace the operator equation (2.10) by

$$F(z;\xi,\lambda,\psi) := \begin{pmatrix} \dot{z} - h(\cdot,z(\cdot),\lambda,\psi) \\ P_a(0)\big(z(0)+y(\lambda)(\psi)-\bar{y}(\psi)\big)-\xi \end{pmatrix} = 0,$$

where $h \in C^k(\mathbb{R}_+ \times \mathbb{R}^m \times \mathbf{\Lambda} \times (\epsilon,\infty),\mathbb{R}^m)$ is defined by

$$\begin{aligned} h(t,v,\lambda,\psi) \quad := \quad & f\big(v+\bar{x}(t+\psi)+y(\lambda)(t+\psi)-\bar{y}(t+\psi),\lambda\big) \\ & -f\big(y(\lambda)(t+\psi),\lambda\big) + f\big(\bar{y}(t+\psi),\bar{\lambda}\big) - f\big(\bar{x}(t+\psi),\bar{\lambda}\big). \end{aligned}$$

Now we see for all $\psi \in (\epsilon,\infty)$ that $F(0;0,\bar{\lambda},\psi)=0$ and

$$F'(0;0,\bar{\lambda},\psi) = \begin{pmatrix} L_\psi & 0 & -\Delta(\cdot+\psi) & 0 \\ P_a(0)E_0 & -\mathrm{Id}_{\mathcal{R}(P_a(0))} & P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(\psi) & 0 \end{pmatrix}, \tag{2.40}$$

where $L_\psi := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(\bar{x}(\cdot+\psi),\bar{\lambda}\big)$. This operator L_ψ has the solution operator $S_\psi(t,s) = S(t+\psi,s+\psi)$ and a shifted exponential dichotomy on $[0,\infty)$ with data $\big(K,\alpha,\beta,P_a^\psi,P_b^\psi\big)$, where $P_\kappa^\psi := P_\kappa(\cdot+\psi)$, $\kappa = a,b$.

Before we proceed with the proof, let us assume, tentatively, that we have applied the implicit function theorem to F and that $z \in C^{k-1}(V\times\Omega\times\Psi, BC^1_\gamma)$ is the result on a suitable neighborhood $V\times\Omega\times\Psi$. Then $F\big(z(\xi,\lambda,\psi);\xi,\lambda,\psi\big)=0$ implies $z(0,\bar{\lambda},\psi)=0$ by uniqueness and $\frac{\partial}{\partial\psi}z(0,\bar{\lambda},\psi)=0$. Furthermore, we get $\frac{\partial}{\partial\xi}z(0,\bar{\lambda},\psi)\eta \in BC^1_\gamma$ for all $\eta \in \mathcal{R}\big(P_a(0)\big)$ and $L_\psi \frac{\partial}{\partial\xi}z(0,\bar{\lambda},\psi)\eta = 0$ by implicit differentiation. Hence, by Theorem 1.4, $\frac{\partial}{\partial\xi}z(0,\bar{\lambda},\psi)(0)\eta \in \mathcal{R}\big(P_a^\psi(0)\big) = \mathcal{R}\big(P_a(\psi)\big)$ and $P_b(\psi)\frac{\partial}{\partial\xi}z(0,\bar{\lambda},\psi)(0)=0$.

We modify the remainder of the proof of Proposition 2.2 as follows:

Lemma 2.3 and its proof remains nearly unchanged. We only add ψ and replace (2.12) and (2.13) by

$$\begin{aligned} U_\delta^\psi \quad &:= \quad \big\{x \in BC^1(\mathbb{R}_+,\mathbb{R}^m) \big| ||x-\bar{x}(\cdot+\psi)||_\infty < \delta\big\}, \\ W_{\delta,\psi}^{V,\Omega} \quad &:= \quad \big\{(z,\xi,\lambda) \in BC^1_\gamma \times V \times \Omega \quad \big|\ ||z+y(\lambda)(\cdot+\psi)-\bar{y}(\cdot+\psi)||_\infty < \delta\big\} \end{aligned}$$

for $\delta > 0$. To obtain differentiability of F with respect to ψ, in exponentially weighted norms, we notice that

$$\Big|\frac{\partial}{\partial\psi}h(t,z,\lambda,\psi)\Big| \leq (M_f^1 L_f + L_f^1 M_f)\cdot\big(|z| + 2|\bar{x}(t+\psi) - \bar{y}(t+\psi)|\big)$$

and get analogous results for higher derivatives up to degree k.

¿From $z \in BC_\gamma^1$ and $|\bar{x}(t+\psi)-\bar{y}(t+\psi)| \leq (C_{\bar{x}}e^{\gamma\psi})e^{\gamma t}$ we conclude that the derivatives are exponentially decaying with rate γ. Similar to (2.17) we define

$$\begin{aligned}\tilde{g}_t(\bar{z},\lambda,\psi) &= \big(\bar{z} + \bar{x}(t+\psi) - \bar{y}(t+\psi) + y(\lambda)(t+\psi), \lambda\big),\\ g_t(\bar{z},\lambda,\psi) &= \big(y(\lambda)(t+\psi),\lambda\big)\end{aligned}$$

and obtain for $(v,\mu,\phi) \in BC_\gamma^1 \times \mathbb{R}^{p+1}$ that

$$\begin{aligned}||\big(g_t^{(1)}(\bar{z},\lambda,\psi) - \tilde{g}_t^{(1)}(\bar{z},\lambda,\psi)\big)(v(t),\mu,\phi)||e^{-\gamma t} &\leq ||v||_\gamma + L_f C_{\bar{x}} e^{\gamma\psi}\,|\phi|,\\ ||g_t^{(j)}(\bar{z},\lambda,\psi) - \tilde{g}_t^{(j)}(\bar{z},\lambda,\psi)||e^{-\gamma t} &\leq CC_{\bar{x}}e^{\gamma\psi} \quad \text{for } j = 2,\dots,k,\end{aligned}$$

Hence $D^iF_1\big(z(\cdot),\lambda,\psi\big)$ is bounded in the exponentially weighted norms (see (2.18) and (2.21)). With the interpretation of (λ,ψ) as parameter (2.22) remains unchanged. Therefore $F \in C^{k-1}$ in exponentially weighted norms.

Applying the implicit function theorem implies that there exist $V\times\Omega\times\Psi$ sufficiently small and $z \in C^{k-1}\big(V\times\Omega\times\Psi, BC_\gamma^1(\mathbb{R}_+,\mathbb{R}^m)\big)$ such that $V\times\Omega\times\Psi$,

$$U := \big\{x \in BC^1(\mathbb{R}_+,\mathbb{R}^m)\big|\,||x - \bar{x}(\cdot+\psi)||_\infty < \delta \text{ for some } \psi\in\Psi\big\} \tag{2.41}$$

and $x(\xi,\lambda,\psi) := z(\xi,\lambda,\psi) + \bar{x}(\cdot+\psi) + y(\lambda)(\cdot+\psi) - \bar{y}(\cdot+\psi)$, satisfy the properties **Q1**–**Q4** claimed in Proposition 2.11 except for the uniqueness in **Q3**. In particular, $F(0;0,\bar{\lambda},\psi) = 0$ and $\frac{\partial}{\partial\psi}F(0;0,\bar{\lambda},\psi) = 0$ implies by uniqueness $z(0,\bar{\lambda},\psi) = 0$ and $\frac{\partial}{\partial\psi}z(0,\bar{\lambda},\psi) = 0$ hence $\frac{\partial}{\partial\psi}x(0,\bar{\lambda},\psi) = \dot{\bar{x}}(\cdot+\psi)$. Moreover, $\frac{\partial}{\partial\xi}z(0,\bar{\lambda},\psi) \in \mathcal{R}\big(P_a(\psi)\big)$ implies $P_b(\psi)\frac{\partial}{\partial\xi}x(0,\bar{\lambda},\psi) = 0$.

For the proof of uniqueness let $(\tilde{x},\xi,\lambda,\psi) \in U\times V\times\Omega\times\Psi$ be a solution of $(B_\gamma(\bar{x},\bar{\lambda}))$. Then there exists a $\tilde{\psi}\in\Psi$, such that $||\tilde{x} - \bar{x}(\cdot+\tilde{\psi})||_\infty < \delta$, hence $z := \tilde{x} - x(\xi,\lambda,\psi)$ satisfies $||z||_\infty = ||\tilde{x} - x(\xi,\lambda,\psi)||_\infty < 2\delta + M_f|\tilde{\psi} - \psi| < 3\delta$ for Ψ sufficiently small. The remainder of the uniqueness proof follows analogously to those of Proposition 2.2 on a 3δ–ball. Thus we obtain $\tilde{x} = x(\xi,\lambda,\psi)$. ■

As in 2.1.1 we define

$$b(\xi,\lambda,\psi) := \big(x(\xi,\lambda,\psi)(0),\lambda\big), \quad b \in C^{k-1}(V\times\Omega\times\Psi,\mathbb{R}^{m+p}) \tag{2.42}$$

and its image

$$W := \big\{ b(\xi, \lambda, \psi) \bigm| (\xi, \lambda, \psi) \in V \times \Omega \times \Psi \big\} \subset \mathbb{R}^{m+p}.$$

Thus, the derivative of b at $(0, \bar{\lambda}, 0)$ is defined by

$$B = \begin{pmatrix} \mathrm{Id}_{\mathcal{R}(P_a(0))} & \frac{\partial}{\partial \lambda} x(0, \bar{\lambda})(0) & \dot{\bar{x}}(0) \\ 0 & \mathrm{Id}_{\mathbb{R}^p} & 0 \end{pmatrix}. \tag{2.43}$$

Hence b is immersive at $(0, \bar{\lambda}, 0)$, if

$$\dot{\bar{x}}(0) \notin \mathcal{R}\big(P_a(0)\big). \tag{2.44}$$

This holds true, if we assume

$$\liminf_{t \to \infty} \big| f\big(\bar{y}(t), \bar{\lambda}\big) \big| > \epsilon \qquad \text{for some } \epsilon > 0,$$

because otherwise $\dot{\bar{x}}(0) \in \mathcal{R}\big(P_a(0)\big)$ would imply $|\dot{\bar{x}}(t)| = |S(t,0)\dot{\bar{x}}(0)| \le Ke^{\alpha t}|\dot{\bar{x}}(0)|$ and from the asymptotic behavior of $\bar{x}$ follows the convergence of $\dot{\bar{y}}(t) = f\big(\bar{y}(t), \bar{\lambda}\big)$ to 0.

Remark: (2.44) is satisfied if $\bar{y}$ is periodic, but it is not true if $\bar{y}$ is an equilibrium or if $\bar{y}$ itself converges with an exponential rate γ to an equilibrium.

As before we assume that we have chosen $V \times \Omega \times \Psi$ small enough, such that

Q5 $W = \mathrm{im}(b|_{U \times \Omega \times \Psi})$ is a C^{k-1} submanifold of $\mathbb{R}^{m+p}$.

This allows us to define a "γ–parametrization" and a local "γ–stable manifold" of $\bar{x}$.

Definition 2.12 *Let the assumptions of Proposition 2.11 and (2.44) hold. Then we call a pair $(U \times V \times \Omega \times \Psi, x)$ with* **Q1**–**Q5** *a* **local γ–parametrization of $(\bar{x}, \bar{\lambda})$** *and*

$$\begin{aligned} \mathcal{M}^{loc}(\bar{x}, \bar{\lambda}) \quad := \quad & \big\{ (z, \lambda) \in \mathbb{R}^{m+p} \bigm| z \text{ and } y(\lambda)(\psi) \text{ are } \gamma\text{–asymptotic at } \lambda \in \Omega, \\ & P_a(0)\big(z - \bar{x}(\psi)\big) \in V \text{ and } u(\cdot, 0, z, \lambda) \in U \text{ for some } \psi \in \Psi \big\} \end{aligned}$$

is called a **local γ-stable manifold along $(\bar{x}, \bar{\lambda})$**.

Remark: Proposition 2.11 implies that $\mathcal{M}^{loc}(\bar{x}, \bar{\lambda}) = W$.

Proposition 2.11 yields the existence of a local γ–stable manifold as follows:

Corollary 2.13 *Suppose that* **A1**–**A3** *and (2.44) hold and $\gamma \in (\bar{\alpha}, \bar{\beta})$. Then there exists a local γ–stable manifold along $\big(y(\lambda), \lambda\big)$ for all $\lambda \in \mathbf{\Lambda}$. For any $(\tilde{x}, \tilde{\lambda})$ where $\tilde{x}$ and $y(\tilde{\lambda})$ are γ–asymptotic there exists a local γ–stable manifold along $\big(\tilde{x}, \tilde{\lambda}\big)$.*

Proof: The arguments for the proof are the same as in the proof of Corollary 2.6. ■

The following two corollaries imply the existence of asymptotic boundary conditions with quadratic error estimates for the approximation of solutions which are γ–asymptotic with periodic orbits.

COROLLARY 2.14 *Let $k \geq 3$ and let $(U \times V \times \Omega \times \Psi, x)$ be a γ–parametrization of $(\bar{x}, \bar{\lambda})$, then there exist open neighborhoods $\tilde{V} \subset V$ of 0 and $\tilde{\Psi} \subset \Psi$ of 0, such that*

$$\left|P_b(\psi)\bigl[x(\xi, \bar{\lambda}, \psi)(0) - \bar{x}(\psi)\bigr]\right| \leq C\left|x(\xi, \bar{\lambda}, \psi)(0) - \bar{x}(\psi)\right|^2 \tag{2.45}$$

holds for a constant $C > 0$ and for all $\xi \in \tilde{V}$, $\psi \in \tilde{\Psi}$.

Proof: Let us define $h_b(\xi, \psi) := P_b(\psi)\bigl[x(\xi, \bar{\lambda}, \psi)(0) - \bar{x}(\psi)\bigr]$, then Proposition 2.11 implies that $h_b(0, \psi) = 0$ and $\frac{\partial}{\partial \xi} h_b(0, \psi) = P_b(\psi)\frac{\partial}{\partial \xi} x(0, \bar{\lambda}, \psi)(0) = 0$ (see (2.38)), hence by Taylor expansion on a sufficiently small neighborhood $\tilde{V} \times \tilde{\Psi} \subset V \times \Psi$ of $(0, 0)$ we get

$$h_b(\xi, \psi) = h_b(0, \psi) + \frac{\partial}{\partial \xi} h_b(0, \psi)\xi + \int_0^1 (1 - s)\frac{\partial^2}{\partial \xi^2} h_b(s\xi, \psi)[\xi]^2 \,\mathrm{ds}.$$

By smoothness of $h_b(\xi, \psi)$ ($x \in C^{k-1}$, $k - 1 \geq 2$) there exists some constant $C > 0$, such that $|h_b(\xi, \psi)| \leq C|\xi|^2$ for all $(\xi, \psi) \in \tilde{V} \times \tilde{\Psi}$.

In addition $\xi = P_a(0)\bigl[x(\xi, \bar{\lambda}, \psi)(0) - \bar{x}(\psi)\bigr]$ implies $|\xi| \leq K|x(\xi, \bar{\lambda}, \psi)(0) - \bar{x}(\psi)|$ and hence (2.45) holds. ■

COROLLARY 2.15 *Let $k \geq 3$ and $\bar{y}$ be a T–periodic orbit, such that $L = \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{y}(\cdot), \bar{\lambda})$ has a shifted exponential dichotomy with dichotomy data $(K, \alpha, \beta, P_a, P_b)$, where P_a, P_b are T–periodic. Moreover, let $\bar{x}$ be γ–asymptotic with $\bar{y}$. Then there exist constants $\bar{T} > 0$, $C > 0$, such that for all $t \geq \bar{T}$ the following holds*

$$\left|P_b(t)\bigl(\bar{x}(t) - \bar{y}(t)\bigr)\right| \leq C \left|\bar{x}(t) - \bar{y}(t)\right|^2. \tag{2.46}$$

Proof: Let use define for each $\tau \in [0, T]$ a local γ–parametrization of $\bigl(\bar{y}(\tau + \cdot), \bar{\lambda}\bigr)$ and $\tilde{V} \times \tilde{\Psi}$ as in Corollary 2.14. Then we cover $[0, T]$ by $\{\tau_i + \tilde{\Psi}_i\}_{i \in I}$, where I is a finite index set and $\tau_i \in [0, T]$ and we define $P_\kappa^i(0) = P_\kappa(\tau_i)$, $\kappa = a, b$ and $\bar{y}^i := \bar{y}(\tau_i + \cdot)$. Moreover let $\delta = \min_{i \in I}(\delta_i)$, δ_i defined in Proposition 2.11 and $C = \max_{i \in I}(C_i)$, C_i defined in Corollary 2.14.

Then we find for each $t \in \mathbb{R}_+$ a number $n \in \mathbb{N}$ with $t - nT \in [0, T]$ and an index $i_t \in I$ with $\psi(t) := (t - nT) - \tau_{i_t} \in \tilde{\Psi}_{i_t}$.

By convergence, there exists some $N \in \mathbb{N}$ and $\bar{T} := NT \in \mathbb{R}_+$, such that $\bar{x}(t + \cdot) \in U_{i_t}$ and $\xi(t) := P_a(t_{i_t})\bigl[\bar{x}(t_{i_t}) - \bar{y}(t)\bigr] = P_a^{i_t}(0)\bigl[\bar{x}(t_{i_t}) - \bar{y}^{i_t}\bigl(\psi(t)\bigr)\bigr] \in \tilde{V}_{i_t}$ for each $t \geq \bar{T}$.

By Proposition 2.11 we see $\bar{x}(t) = x_{i_t}\bigl(\xi(t), \bar{\lambda}, \psi(t)\bigr)(0)$ and Corollary 2.14 implies (2.46), because $P_b(t)\bigl(\bar{x}(t) - \bar{y}(t)\bigr) = P_b^{i_t}\bigl(\psi(t)\bigr)\left[x_{i_t}\bigl(\xi(t), \bar{\lambda}, \psi(t)\bigr)(0) - \bar{y}^{i_t}\bigl(\psi(t)\bigr)\right]$. ■

2.2 The global γ–stable manifold

In this section we will describe the set of solutions which are γ–asymptotic with some $y(\lambda)$, $\lambda \in \mathbf{\Lambda}$ and define a C^{k-1}-structure on it. This will be called the "global γ–stable manifold".

In the parameter independent case results on stable and unstable manifolds have been developed by Fenichel [Fenichel, 1971, 1974, 1977]. Those results hold for overflowing invariant sets (and not only for the manifold of solutions converging to a **single** trajectory, discussed here). The proofs are based on the Hadamard graph transform and allow under additional assumptions to describe the dynamics on the (possibly higher-dimensional) overflowing invariant set.

A foliation of generalized stable and unstable manifolds of a steady state and its smoothness is also proved by Chow, Lin and Lu [Chow et al., 1991]. Their results are valid in infinite dimensional spaces but they need an additional gap condition on α, β to achieve smoothness. Moreover they need a global Lipschitz constant. Smoothness in the parameter is only achieved for the center–unstable manifold (this can also be treated by adding the trivial equation $\dot{\lambda} = 0$). We restrict to single bounded orbits. This allows to achieve smoothness with respect to parameters and with respect to time, even in case of a strong stable manifold without additional global assumptions or conditions on α, β.

In the first part of this section we treat the set of initial values which are γ–asymptotic with $y(\lambda)(0)$, $\lambda \in \mathbf{\Lambda}$, which is

$$\mathcal{M}^0_\gamma := \big\{(x,\lambda) \in \mathbb{R}^m \times \mathbf{\Lambda} \big|\ x \text{ and } y(\lambda)(0) \text{ are } \gamma\text{–asymptotic}\big\}. \tag{2.47}$$

This is a fiber with respect to the base point $y(\lambda)(0)$ of the γ–stable manifold along $y(\lambda)$ which is defined in the second part of this section. There we include time, to define a manifold structure on the set of points which are γ–asymptotic with some $y(\lambda)(\tau)$ for $\tau \in J_{\mathbf{\Lambda}}$. To define an atlas of class C^{k-1} on $\mathcal{M}^0_\gamma$ we follow [Lang, 1972].

Definition 2.16 *Let $\mathcal{M}$ be a set. An* **atlas of class C^l** *($l \geq 0$) on $\mathcal{M}$ is a collection of pairs $\mathcal{A} = \{(W_\iota, \vartheta_\iota)\}_{\iota \in I}$, where I is an index set, satisfying the conditions*

M1 *Each W_ι is a subset of $\mathcal{M}$ and $\{W_\iota\}_{\iota \in I}$ cover $\mathcal{M}$.*

M2 *Each ϑ_ι is a bijection of W_ι onto an open subset $\vartheta_\iota W_\iota$ of some Banach space $\mathcal{Y}_\iota$ and for any ι_1, $\iota_2 \in I$, $\vartheta_{\iota_1}(W_{\iota_1} \cap W_{\iota_2})$ is open in $\mathcal{Y}_{\iota_1}$.*

M3 *The map $\vartheta_{\iota_1} \circ \vartheta_{\iota_2}^{-1} : \ \vartheta_{\iota_2}\big(W_{\iota_1} \cap W_{\iota_2}\big) \longrightarrow \vartheta_{\iota_1}\big(W_{\iota_1} \cap W_{\iota_2}\big)$ is a C^l–isomorphism for each ι_1, $\iota_2 \in I$.*

Now we define an atlas of class C^{k-1} on $\mathcal{M}^0_\gamma$. We take $I = \mathcal{M}^0_\gamma$ itself as index set and define by Corollary 2.6 for each $\iota = (\bar{x}(0), \bar{\lambda})$ a local γ–parametrization $(U_\iota \times V_\iota, x_\iota)$ at $(\bar{x}, \bar{\lambda})$, where $\bar{x} = u(\cdot, 0, \bar{x}(0), \bar{\lambda})$. Furthermore we define $W^0_\iota := \{b_\iota(\xi,\lambda) \mid (\xi,\lambda) \in V_\iota \times \Omega_\iota\}$ as in (2.28) and a function

$$\vartheta_\iota : \begin{array}{ccc} W^0_\iota & \to & V_\iota \times \Omega_\iota \\ (z,\lambda) & \mapsto & \left[P^\iota_a(0)\big(z - \bar{x}(0)\big), \lambda\right]. \end{array} \tag{2.48}$$

We conclude that ϑ_ι is a homeomorphism onto the open subset $V_\iota \times \Omega_\iota \subset \mathcal{R}(P^\iota_a(0)) \times \mathbb{R}^p$ with inverse $b_\iota(\xi,\lambda) = \big(x_\iota(\xi,\lambda)(0), \lambda\big)$ defined as in (2.27) as follows:

Let $(\xi,\lambda) \in V_\iota \times \Omega_\iota$, then $\vartheta_\iota \circ b_\iota(\xi,\lambda) = \big(P^\iota_a(0)\big(x_\iota(\xi,\lambda)(0) - \bar{x}(0)\big), \lambda\big) = (\xi,\lambda)$. Conversely, let $(z,\lambda) \in W^0_\iota$, then the solution $x := u(\cdot, 0, z, \lambda) \in U_\iota$ is γ–asymptotic with $\bar{y}$ and $\big(P^\iota_a(0)\big(x(0) - \bar{x}(0)\big) = \big(P^\iota_a(0)\big(z - \bar{x}(0)\big) \in V_\iota$ and hence by the uniqueness in U_ι we obtain $b_\iota \circ \vartheta_\iota(z,\lambda) = \big(x_\iota\big(P^\iota_a(0)(z - \bar{x}(0)), \lambda\big)(0), \lambda\big) = \big(x(0), \lambda\big) = (z,\lambda)$.

To see that $\vartheta_{\iota_1}(W_{\iota_1} \cap W_{\iota_2})$ is open in $\mathcal{R}\big(P^{\iota_1}_a(0)\big) \times \mathbb{R}^p$, we define for $(\tilde{\xi}, \tilde{\lambda}) \in \vartheta_{\iota_1}(W_{\iota_1} \cap W_{\iota_2})$ an ϵ–ball $\mathcal{B}_\epsilon(\tilde{\xi},\tilde{\lambda}) \subset V_{\iota_1} \times \Omega_{\iota_1}$, such that for all $(\xi,\lambda) \in \mathcal{B}_\epsilon(\tilde{\xi},\tilde{\lambda})$ the following hold: $||x_{\iota_1}(\xi,\lambda) - \tilde{x}||_\infty < \frac{1}{2}(\delta_{\iota_2} - ||\tilde{x} - \bar{x}_{\iota_2}||_\infty)$, where $\tilde{x} = x_{\iota_1}(\tilde{\xi},\tilde{\lambda}) \in U_{\iota_1} \cap U_{\iota_2}$ and $(\eta,\lambda) := \vartheta^{glob}_{\iota_2} x_{\iota_1}(\xi,\lambda) \in V_{\iota_2} \times \Omega_{\iota_2}$, where $\vartheta^{glob}_{\iota_2}$ is defined as in (2.48), but on $\mathbb{R}^{m+p}$. This can be done, because the $V_\iota \times \Omega_\iota$ are open and x_{ι_1}, b_{ι_1} and $\vartheta^{glob}_{\iota_2}$ are continuous functions. Thus $(x,\lambda) := \big(x_{\iota_1}(\xi,\lambda), \lambda\big) \in W_{\iota_1}$ and x is γ–asymptotic with $\bar{y}$. Moreover, $||x - \bar{x}_{\iota_2}||_\infty < \delta_{\iota_2}$ and $P^{\iota_2}_a(0)\big(x(0) - \bar{x}_{\iota_2}(0)\big) \in V_{\iota_2}$. Thus, the uniqueness implies $(x,\lambda) = \big(x_{\iota_2}(\xi,\lambda),\lambda\big) \in W_{\iota_2}$ and hence $(x,\lambda) \in W_{\iota_1} \cap W_{\iota_2}$.

For any $\iota_1, \iota_2 \in I$ the function

$$\vartheta_{\iota_1} \circ b_{\iota_2} : \begin{array}{ccc} \vartheta_{\iota_2}\big(W_{\iota_1} \cap W_{\iota_2}\big) & \longrightarrow & \vartheta_{\iota_1}\big(W_{\iota_1} \cap W_{\iota_2}\big) \\ (\xi,\mu) & \mapsto & \big(P^{\iota_1}_a(0)\big[x_{\iota_2}(\xi,\mu)(0) - \bar{x}_{\iota_1}(0)\big], \mu\big) \end{array} \tag{2.49}$$

is C^{k-1}. Therefore, $\mathcal{A}^0 = \big\{(W_\iota, \vartheta_\iota)\big\}_{\iota \in \mathcal{M}^0_\gamma}$ is an atlas of type C^{k-1}.

As usual, we add all charts which are C^{k-1}–compatible with this atlas to define a maximal atlas of type C^{k-1}. We denote it by $\mathcal{D}^0$ and hence $(\mathcal{M}^0_\gamma, \mathcal{D}^0)$ is a C^{k-1}–manifold of dimension $m_a + p$.

Remark: Definition 2.16 allows us to use the Banach spaces $\mathcal{Y}_\iota = \mathcal{R}\big(P^\iota_a(0)\big) \times \mathbb{R}^p$, which are in general not the same for different indices, nevertheless they are all isomorphic to $\mathbb{R}^{m_a+p}$.

The inclusion $\mathcal{M}^0_\gamma \to \mathbb{R}^{m+p}$ is an immersion, as we have seen in the previous section by defining local γ–stable manifolds. Thus $\mathcal{M}^0_\gamma$ is a C^{k-1} "immersed submanifold". In general, it is not a submanifold of $\mathbb{R}^{m+p}$.

In the same way we define a for each $\psi \in J_{\mathbf{\Lambda}}$ manifolds $(\mathcal{M}_\gamma^\psi, \mathcal{D}^\psi)$, where

$$\mathcal{M}_\gamma^\psi := \big\{(x,\lambda) \in \mathbb{R}^m \times \mathbf{\Lambda} \big| \ x \text{ and } y(\lambda)(\psi) \text{ are } \gamma\text{–asymptotic}\big\}. \tag{2.50}$$

Notice here, that the $y(\lambda)(\cdot)$ are all defined on $J_{\mathbf{\Lambda}}$ (see **A2** on page 27) and hence the $y(\lambda)(\psi + \cdot)$ are defined on $\mathbb{R}_+$.

Generalizing the notion in Irwin [Irwin, 1980] for periodic orbits we define

Definition 2.17 *The pair $(\mathcal{M}_\gamma^\psi, \mathcal{D}^\psi)$ is called the* **(global) $\boldsymbol{\gamma}$–stable manifold evaluated at $\boldsymbol{\psi}$** *of the family $y(\lambda)$, $\lambda \in \mathbf{\Lambda}$.*

Abbreviating, we call $\mathcal{M}_\gamma^\psi$ the γ–stable manifold.

Now we will define an atlas of type C^{k-1} and hence a manifold on the set of points which are γ–asymptotic with some $y(\lambda)(\psi)$, $\lambda \in \mathbf{\Lambda}$, $\psi \in J_{\mathbf{\Lambda}}$. This is

$$\begin{aligned} \mathcal{M}_\gamma &:= \bigcup_{\psi \in J_{\mathbf{\Lambda}}} \mathcal{M}_\gamma^\psi \qquad (2.51)\\ &= \big\{(x,\lambda) \in \mathbb{R}^m \times \mathbf{\Lambda} \ \big| \ x \text{ and } y(\lambda)(\psi) \text{ are } \gamma\text{–asymptotic at } \lambda \text{ for some } \psi \in J_{\mathbf{\Lambda}}\big\}. \end{aligned}$$

As before, we use the γ–parametrizations at $(\bar{x}, \bar{\lambda})$, now including the time direction (see Definition 2.12). Therefore we always assume (2.44), i. e.

$$\dot{\bar{x}}(0) \notin \mathcal{R}\big(P_a(0)\big) \ \text{ and hence } P_b(0)\dot{\bar{x}}(0) \neq 0. \tag{2.52}$$

To define an atlas of class C^{k-1} on $\mathcal{M}_\gamma$, we cover $\mathcal{M}_\gamma$ by sets

$$W := \big\{\big(x(\xi,\lambda,\psi)(0),\lambda\big) \ \big| \ (\xi,\lambda,\psi) \in V \times \Omega \times \Psi\big\} \subset \mathcal{M}_\gamma, \tag{2.53}$$

where the $(U \times V \times \Psi, x)$ are local γ–parametrizations of $(\bar{x}, \bar{\lambda}) \in \mathcal{M}_\gamma$ (see Definition 2.12 and Corollary 2.13).

Defining a function

$$\vartheta : \begin{array}{ccc} W & \rightarrow & V \times \Omega \times \Psi \\ (z,\lambda) & \mapsto & \big(\xi(z,\lambda), \lambda, \psi(z,\lambda)\big) \end{array} \tag{2.54}$$

which is the inverse of $b = \big(x(\xi,\lambda,\psi)(0),\lambda\big)$, defined in (2.42), and which satisfies **M2** and **M3** requires some additional effort.

We have to define functions $\psi \in C^{k-1}(\Theta,\mathbb{R})$ and $\xi \in C^{k-1}(\Theta, \mathcal{R}\big(P_a(0)\big)$, where Θ is an open neighborhood of $(\bar{x}(0), \bar{\lambda})$ in $\mathbb{R}^{m+p}$, such that $z = x\big(\xi(z,\lambda), \lambda, \psi(z,\lambda)\big)$ for each $(z,\lambda) \in W \cap \Theta$ and $(z,\lambda) \in \mathcal{M}_\gamma^\psi(z,\lambda)$, in particular.

We will define $\psi(z,\lambda)$ by the implicit function theorem applied to

$$H(\psi;z,\lambda) = \big(P_b(0)\dot{\bar{x}}(0)\big)^\top \Big(P_b(\psi)\big[x\big(P_a(0)[z-\bar{x}(\psi)],\lambda,\psi\big)(0)-z\big]\Big). \tag{2.55}$$

The definition of x implies by the uniqueness, that $x\big(P_a(0)(z-\bar{x}(\psi)),\lambda,\psi\big)(0) = z$ for each $(z,\lambda) = \big(x(\xi,\lambda,\psi),\lambda\big) \in W$ and hence $H(\psi;x(\xi,\lambda,\psi),\lambda) = 0$. Furthermore, on a sufficiently small neighborhood $\tilde{\Psi}\times\tilde{U}\times\tilde{\Omega}$ we see $H \in C^{k-1}(\tilde{\Psi}\times\tilde{U}\times\tilde{\Omega},\mathbb{R})$ and the derivative of H at $\big(\psi,\bar{x}(\psi),\bar{\lambda}\big)$ with respect to ψ is (using (2.37) and (2.38))

$$\begin{aligned}
\frac{\partial}{\partial\psi}H\big(\psi,\bar{x}(\psi),\bar{\lambda}\big) &= \big(P_b(0)\dot{\bar{x}}(0)\big)^\top\Big(P_b(\psi)\frac{\partial}{\partial\psi}x(0,\bar{\lambda},\psi)(0) \\
&\quad + P_b'(\psi)\big[x(0,\bar{\lambda},\psi)(0)-\bar{x}(\psi)\big] - P_b(\psi)\frac{\partial}{\partial\xi}x(0,\bar{\lambda},\psi)(0)P_a(0)\dot{\bar{x}}(\psi)\Big) \\
&= \big(P_b(0)\dot{\bar{x}}(0)\big)^\top\big(P_b(\psi)\dot{\bar{x}}(\psi)\big).
\end{aligned}$$

Therefore the implicit function theorem yields a proper $\psi(z,\lambda)$. This function, together with $\xi(z,\lambda) := P_a(0)\big[z-\bar{x}\big(\psi(z,\lambda)\big)\big]$ defines by (2.54) a suitable inverse ϑ of b.

The remainder of the proof of **M2** and **M3** follows analogously to the first part, though the formulas are more complicated.

As before, we take $I = \mathcal{M}_\gamma$ itself a index set, define an atlas $\mathcal{A} = \{(W_\iota,\vartheta_\iota)\}_{\iota\in I}$ by (2.53) and (2.54) and add all charts which are C^{k-1}–compatible with this atlas to define a maximal atlas $\mathcal{D}$ of type C^{k-1}.

Definition 2.18 *The pair $(\mathcal{M}_\gamma,\mathcal{D})$ is called the* **(global) $\boldsymbol{\gamma}$–stable manifold** *of the family $y(\lambda)$, $\lambda\in\boldsymbol{\Lambda}$.*

Remark: This construction allows to define a C^{k-1} **foliation**. $\mathcal{M}_\gamma = \bigcup_{\psi\in J_{\boldsymbol{\Lambda}}}\mathcal{M}_\gamma^\psi$ is a disjoint decomposition of $\mathcal{M}_\gamma$ and each $\mathcal{M}_\gamma^\psi$ is an immersed submanifold of $\mathcal{M}_\gamma$, called a **leaf**. For each chart (W,ϑ), defined as above, we see that $\vartheta^{-1}(U\times V\times\{\psi\})\subset\mathcal{M}_\gamma^\psi$.

2.3 Nondegenerate connecting orbits with asymptotic rate

In this section we define "connecting orbits with asymptotic rate γ" or in shortened form "γ–connections". These are solutions of the dynamical system which are γ–asymptotic with y and have initial values in a given manifold $\mathcal{W}$. They are called "nondegenerate" if a certain transversality condition holds. We relate the non-degeneracy to the non-singularity of a linear function and to the regularity of an operator equation.

In addition to **A1**–**A3** we assume that

A4 (dimension condition) $\mathcal{W}$ is an m_b–dimensional manifold in $\mathbb{R}^{m+p}$ with representation $\mathcal{W} = \{(x, \mu) \in \mathbb{R}^{m+p} | g(x, \mu) = 0\}$ where $g \in C^{k-1}(\mathbb{R}^{m+p}, \mathbb{R}^{m_a+p})$ and $g'(x, \mu)$ has full rank for all $(x, \mu) \in \mathcal{W}$.

A5 (transversality condition) $T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} + T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \mathbb{R}^{m+p}$.

The coupling of the dimension p of the parameter space and the dimension m_b of the manifold $\mathcal{W}$ is reflected by **A4** ($\mathcal{W}$ is $\boldsymbol{m_b}$–dimensional in $\mathbb{R}^{m+\boldsymbol{p}}$). Moreover, $m = m_a + m_b$ (see **A3** on page 27) implies $m + p = (m_a + p) + m_b = \dim(T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0) + \dim(T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W})$ and hence **A5** is equivalent to

$$T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \{0\}. \tag{2.56}$$

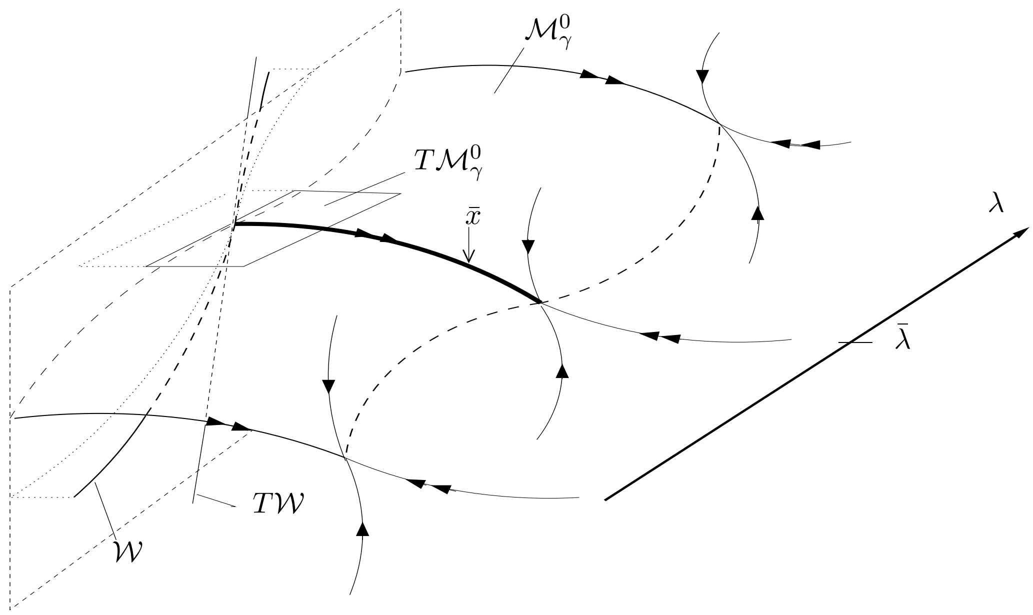

Figure 2.3: Nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to an equilibrium, converging along the strong-stable manifold.

Remark: An arbitrary manifold $\mathcal{W}$ must satisfy the dimension condition **A4**. If **A4** is not satisfied we can proceed as follows:

If **A5** holds but $\mathcal{W}$ is $(m_b + q)$–dimensional, then we may add q extra conditions $h(x, \mu) = 0$ to $g(x, \mu) = 0$ where $\dim\left(\mathcal{N}\left(\frac{\partial}{\partial x}h(x, \lambda), \frac{\partial}{\partial \lambda}h(x, \lambda)\right)\right) = m + p - q$ and

$$\mathcal{N}\left(\frac{\partial}{\partial x}h(x, \lambda), \frac{\partial}{\partial \lambda}h(x, \lambda)\right) \cap \left(T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0\right) = \{0\}. \tag{2.57}$$

The new manifold is m_b–dimensional and the assumptions **A4**, **A5** hold.

To approximate the set of γ–connections in this case we may fix q initial values by using extra conditions $x_{i_j}(0) - \bar{x}_{i_j}$, $j = 1, \ldots, q$ (such that (2.57) holds). Then we use the $\bar{x}_{i_j}$ as continuation parameters.

If $\mathcal{W}$ is only $(m_b - q)$–dimensional and (2.56) holds, then we may use q additional parameters $\mu \in \mathbb{R}^q$ which have no influence on $\mathcal{W}$. If

$$\operatorname{span}\begin{pmatrix} \frac{\partial}{\partial\mu} x\big(0, (\bar{\lambda}, \bar{\mu})\big)(0) \\ \mathrm{Id}_{\mathbb{R}^q} \end{pmatrix} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} = \{0\}$$

and the dimension of the span is q then $\mathcal{W} \times \mathbb{R}^q$ is an m_b–dimensional manifold.

In the periodic case typically the appropriate phase of the periodic orbit $y(\lambda)$ is unknown and the dimension of $\mathcal{W}$ is $m_b - 1$. Thus, we introduce the phase τ as additional parameter and define $y(\lambda, \tau) := y(\lambda)(\cdot + \tau)$. This method presumes that $f(\bar{x}(0), \bar{\lambda}) \notin T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W}$.

Definition 2.19 (connecting orbit with asymptotic rate) *Let* **A1**–**A4** *hold.*

We call $(\bar{x}, \bar{\lambda})$ *a* **connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y** *if* $\bar{x}$ *and* $y(\bar{\lambda})$ *are* γ*–asymptotic at* $\bar{\lambda}$ *and* $(\bar{x}(0), \lambda) \in \mathcal{W}$. *If in addition the transversality condition* $T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}^0_\gamma = \{0\}$ *holds, then* $(\bar{x}, \bar{\lambda})$ *is called a* **nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y**.

In a shortened form we also speak of a **(nondegenerate) γ–connection**.

Let us define for each $(x, \lambda) \in \mathcal{W}$ the matrix

$$D(x, \lambda) := \left(\frac{\partial g(x, \lambda)}{\partial x}, \frac{\partial g(x, \lambda)}{\partial \lambda}\right) \in \mathbb{R}^{m_a+p, m+p}, \tag{2.58}$$

then $T_{(x,\lambda)}\mathcal{W} = \mathcal{N}\big(D(x, \lambda)\big)$ follows from the definition of $\mathcal{W}$.

For any $(x, \lambda) = \big(x(\xi, \lambda)(0), \lambda\big) \in \mathcal{M}^{loc}_0(\bar{x}, \bar{\lambda}) \subset \mathcal{M}^0_\gamma$ we obtain $T_{(x,\lambda)}\mathcal{M}^0_\gamma = \mathcal{R}\big(B(\xi, \lambda)\big)$, where $B(\xi, \lambda)$ is defined as in (2.29) (see also (2.30)).

Thus $T_{(x,\lambda)}\mathcal{W} \cap T_{(x,\lambda)}\mathcal{M}^0_\gamma = \mathcal{N}\big(D(x, \lambda)\big) \cap \mathcal{R}\big(B(\xi, \lambda)\big)$ for $(x, \lambda) = \big(x(\xi, \lambda)(0), \lambda\big) \in \mathcal{W} \cap \mathcal{M}^0_\gamma$ and hence we obtain the following lemma:

Lemma 2.20 *Suppose that* **A1**–**A4** *hold and let* $\bar{x}$ *be* γ*–asymptotic with* $y(\bar{\lambda})$ *at* $\bar{\lambda}$. *Furthermore, let* $(x, \lambda) := (x(\xi, \lambda), \lambda)$ *be a connecting orbit with asymptotic rate* γ *from* $\mathcal{W}$ *to* y *with* $\big(x(0), \lambda\big) \in \mathcal{M}^{loc}_0(\bar{x}, \bar{\lambda})$. *Then* (x, λ) *is nondegenerate if and only if* $D\big(x(0), \lambda\big) \circ B(\xi, \lambda)$ *is nonsingular.*

Proof: Let (x,λ) be nondegenerate and let (η,μ) be a solution of the linear equation $D(x(0),\lambda)\circ B(\xi,\lambda)(\eta,\mu)=0$. We see $B(\xi,\lambda)(\eta,\mu)\in\mathcal{N}\big(D(x(0),\lambda)\big)\cap\mathcal{R}\big(B(\xi,\lambda)\big)=\{0\}$ and hence $(\eta,\mu)=0$ follows from the definition of B (see (2.29)).

Let $D\big(x(0),\lambda\big)\circ B(\xi,\lambda)$ be nonsingular and let $\binom{y}{\mu}\in\mathcal{N}\big(D(x(0),\lambda)\big)\cap\mathcal{R}\big(B(\xi,\lambda)\big)$. Then there exists $\eta\in\mathcal{R}\big(P_a(0)\big)$, such that $\binom{y}{\mu}=B(\xi,\lambda)(\eta,\mu)$. For this (η,μ) we obtain $0=D\big(x(0),\lambda\big)\binom{y}{\mu}=D\big(x(0),\lambda\big)\circ B(\xi,\lambda)(\eta,\mu)$. Hence $(\eta,\mu)=0$ and $\binom{y}{\mu}=0$. Therefore (x,λ) is nondegenerate. ■

For a γ–connection $(\bar{x},\bar{\lambda})$ from $\mathcal{W}$ to y we define $D:=D(\bar{x}(0),\bar{\lambda})$. Similar to Lemma 2.3 we relate γ–connections (x,λ) from $\mathcal{W}$ to y to solutions of an operator equation and show that regular solutions of this operator correspond to nondegenerate γ–connections.

Theorem 2.21 (Nondegenerate γ–connection) *Suppose that* **A1**–**A4** *hold and let $(\bar{x},\bar{\lambda})$ be a γ–connection from $\mathcal{W}$ to y. Then the following conditions are equivalent:*

1. *$(\bar{x},\bar{\lambda})$ is a nondegenerate γ–connection from $\mathcal{W}$ to y.*
2. *$D\circ B\in\mathcal{L}(\mathcal{R}(P_a(0))\times\mathbb{R}^p,\mathbb{R}^{m_a+p})$ is nonsingular.*
3. *$(0,\bar{\lambda})$ is a regular solution of the operator equation*

$$G_\infty(z,\lambda):=\begin{pmatrix}\dot{z}-h(\cdot,z,\lambda)\\ g(z(0)+\bar{x}(0)+y(\lambda)(0)-\bar{y}(0),\lambda)\end{pmatrix}=0, \tag{2.59}$$

where $G_\infty: BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbf{\Lambda}\to BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m_a+p}$.

Recall the definition of h in (2.11).

Proof: "1. $\iff$ 2." This follows from Lemma 2.20.

"2. $\Rightarrow$ 3." By regular we mean that the operator G_∞ has a total derivative $G'_\infty(0,\bar{\lambda})$ at $(0,\bar{\lambda})$ which is a homeomorphism from $BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^p$ into $BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m_a+p}$.

Lemma 2.4 shows that $G_\infty\in C^{k-1}\big(BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\Omega, BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m_a+p}\big)$ and

$$\begin{aligned}G'_\infty(0,\bar{\lambda}) &= \begin{pmatrix}L & -\Delta(\cdot)\\ \frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})E_0 & \frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)+\frac{\partial}{\partial\lambda}g(\bar{x}(0)\bar{\lambda})\end{pmatrix}\\ &= \begin{pmatrix}L & -\Delta(\cdot)\\ R_0E_0 & Q_0+R_0\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\end{pmatrix},\end{aligned}$$

where $(R_0,Q_0):=D=\big(\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\ \frac{\partial}{\partial\lambda}g(\bar{x}(0)\bar{\lambda})\big)$ and E_0 is the evaluation operator at 0. Moreover, L is Fredholm of index m_a, Δ (defined in (2.7)) is in $\mathcal{L}(\mathbb{R}^p,BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m))$,

$\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})E_0 \in \mathcal{L}(BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m),\mathbb{R}^{m_a+p})$ and $\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)+\frac{\partial}{\partial\lambda}g(\bar{x}(0)\bar{\lambda}) \in \mathcal{L}(\mathbb{R}^p,\mathbb{R}^{m_a+p})$. By the Bordering lemma [Beyn, 1990a, Lemma 2.3] we see that $G'_\infty(0,\bar{\lambda})$ is Fredholm of index 0.

Hence it is sufficient to show that $G'_\infty(0,\bar{\lambda})(y,\mu)=0$ has only the trivial solution.

The equations $Ly=\Delta(\cdot)\mu$ and $\mathcal{N}(L)=\{S(\cdot,0)\xi \big| \xi\in\mathcal{R}\big(P_a(0)\big)\}$ imply by Theorem 1.4 that $y=s_{[0,\infty)}(\Delta(\cdot)\mu)+S(\cdot,0)\xi$ for some $\xi\in\mathcal{R}\big(P_a(0)\big)$. Thus, by (2.6), (2.31) and the second component of $G'_\infty(0,\bar{\lambda})(y,\mu)=0$ we obtain

$$\begin{aligned}
0 &= R_0E_0y+Q_0\mu+R_0\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu \\
&= Q_0\mu+R_0\left(\xi+\frac{\partial}{\partial\lambda}x(0,\bar{\lambda})(0)\mu+P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu\right) \\
&= (R_0\ Q_0)\begin{pmatrix}\xi+\frac{\partial}{\partial\lambda}x(0,\bar{\lambda})(0)\mu+P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu \\ \mu\end{pmatrix} \\
&= (D\circ B)\left(\xi+P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu,\mu\right).
\end{aligned}$$

The non-singularity of $D\circ B$ implies $(\xi+P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu,\mu)=0$. Hence $\mu=0$, $\xi=0$ and $y=s_{[0,\infty)}(\Delta(\cdot)\mu)+S(\cdot,0)\xi=0$.

"3. $\Rightarrow$ 2." Let $(\xi,\mu)\in\mathcal{R}\big(P_a(0)\big)\times\mathbb{R}^p$ be a solution of $(D\circ B)(\xi,\mu)=0$. Define the function $y:=s_{[0,\infty)}\big(\Delta(\cdot)\mu\big)+S(\cdot,0)\big(\xi-P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu\big)$, then (y,μ) satisfies $Ly-\Delta(\cdot)\mu=0$ and (using (2.31) and (2.6))

$$\begin{aligned}
0 &= (D\circ B)(\xi,\mu) \\
&= Q_0\mu+R_0\left(\xi+\frac{\partial}{\partial\lambda}x(0,\bar{\lambda})(0)\mu\right) \\
&= Q_0\mu+R_0\left(s_{[0,\infty)}\big(\Delta(\cdot)\mu\big)(0)+S(0,0)\xi+P_b(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\mu\right) \\
&= R_0E_0y+\left(Q_0+R_0\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\right)\mu.
\end{aligned}$$

Thus (y,μ) solves $G'_\infty(0,\bar{\lambda})(y,\mu)=0$ hence by regularity $(y,\mu)=0$. This implies $\mu=0$ and $\xi=y(0)=0$ and hence non-singularity of $D\circ B$. ∎

Chapter 3

The approximation of connecting orbits with asymptotic rate

In this chapter we approximate nondegenerate γ–connections on a finite interval $[0, T_+]$. A nondegenerate γ–connection $(\bar{x}, \bar{\lambda})$ satisfies

$$\dot{\bar{x}} = f(\bar{x}, \bar{\lambda}),\ (\bar{x}(0), \bar{\lambda}) \in \mathcal{W},\ (\bar{x}(T_+), \bar{\lambda}) \in \mathcal{M}_\gamma^{T_+} \text{ for some } T_+ \in \mathbb{R}_+, \tag{3.1}$$

i. e. it solves the differential equation, it has initial values in $\mathcal{W}$ and $\bar{x}(T_+)$ and $y(\bar{\lambda})(T_+)$ are γ–asymptotic at $\bar{\lambda}$ (see (2.50) and Definition 2.17).

Thus we approximate $(\bar{x}, \bar{\lambda})$ by some (x_J, λ_J), where x_J is a solution of $\dot{x} = f(x, \lambda_J)$ on a finite interval $J = [0, T_+]$ with initial value $(x_J(0), \lambda_J) \in \mathcal{W}$ and an asymptotic boundary condition at T_+, i. e. we assume that $(x_J(T_+), \lambda_J)$ is in the tangent space of $\mathcal{M}_\gamma^{T_+}$ at $\big(y(\lambda_J)(T_+), \lambda_J\big)$. Therefore (x_J, λ_J) solve the system

$$\begin{aligned} \dot{x}(t) - f(x(t), \lambda) &= 0, \quad t \in J, && (3.2)\\ g(x(0), \lambda) &= 0, && (3.3)\\ (x(T_+), \lambda) &\in \big(y(\lambda)(T_+), \lambda\big) + T_{\hat{y}}\mathcal{M}_\gamma^{T_+}, && (3.4) \end{aligned}$$

where $\hat{y} := \big(y(\lambda)(T_+), \lambda\big)$. As we have seen in (2.34), the tangent space $T_{\hat{y}}\mathcal{M}_\gamma^{T_+}$ of $\mathcal{M}_\gamma^{T_+}$ at $\hat{y}$ is

$$T_{\hat{y}}\mathcal{M}_\gamma^{T_+} = \Big\{ \big(\eta + \bar{P}_b(\lambda)(T_+) \frac{\partial}{\partial \lambda} y(\lambda)(T_+)\mu, \mu\big) \Big| (\eta, \mu) \in \mathcal{R}\big(\bar{P}_a(\lambda)(T_+)\big) \times \mathbb{R}^p \Big\}.$$

Thus the asymptotic boundary condition (3.4) is equivalent to

$$\big(x(T_+) - y(\lambda)(T_+)\big) \in \mathcal{R}\big(\bar{P}_a(\lambda)(T_+)\big) = \mathcal{N}\big(\bar{P}_b(\lambda)(T_+)\big). \tag{3.5}$$

The space $\mathcal{N}\big(\bar{P}_b(\lambda)(T_+)\big)$ is m_a–dimensional and therefore (3.4) is equivalent to

$$\bar{M}(T_+,\lambda)\big(x(T_+) - y(\lambda)(T_+)\big) = 0 \tag{3.6}$$

where $\bar{M}(T_+,\lambda) \in \mathbb{R}^{m_b,m}$ is a matrix with $\mathcal{N}\big(\bar{M}(T_+,\lambda)\big) = \mathcal{N}\big(\bar{P}_b(\lambda)(T_+)\big)$ for each λ. In the equilibrium case and the periodic case there exists for each $\big(x(0),\lambda\big) \in \mathcal{M}^0_\gamma$ some $\bar{T}_+$ sufficiently large and $C_{\bar{M}} > 0$, such that for all $T_+ \geq \bar{T}_+$

$$|\bar{M}(T_+,\lambda)\big(x(T_+) - y(\lambda)(T_+)\big)| \leq C_{\bar{M}}|x(T_+) - y(\lambda)(T_+)|^2 \tag{3.7}$$

holds if $k \geq 3$. This follows from Corollary 2.10 and Corollary 2.15.

In general (e. g. the periodic case) it is expensive to compute $\bar{M}(T_+,\lambda)$ depending on λ. Thus for the approximation result in Section 3.2 we only assume $M(T_+,\lambda) \in \mathbb{R}^{m_b,m}$ to be a matrix with $\mathcal{N}\big(M(T_+,\lambda)\big) \cap \mathcal{R}\big(\bar{P}_b(\lambda)(T_+)\big) = \big\{0\big\}$ (see **A6** on page 57). In particular, we can choose $M(T_+,\lambda)$ independent of λ.

We apply our theory to the cases where the $y(\lambda)$ are either equilibria or periodic orbits. In Section 3.1 we show how to compute $\bar{M}(T_+,\lambda)$ in these cases.

In Section 3.2 we set up and analyze a numerical method to approximate nondegenerate γ–connections. In particular, in Section 3.3 we develop the boundary corrector method which is based on the error estimates proved in Section 3.2. The boundary corrector method allows us to use a constant matrix $M(T_+,\lambda)$ and we get estimates of the same quality as in the case with $\bar{M}(T_+,\lambda)$, which satisfies (3.7).

In Section 3.4 we provide some numerical examples.

3.1 Equilibria and periodic solutions

For the numerical applications we restrict ourself to the cases where the $y(\lambda)$ are equilibria or periodic orbits. In general it is difficult compute $y(\lambda)$ and to prove that the assumptions **A2** and **A3** hold. We always assume that $f \in C^k(\mathbb{R}^{m+p},\mathbb{R}^m)$ as in **A1**.

3.1.1 Equilibria

In this case we compute the intersection of $\mathcal{W}$ with the (strong) stable manifold of an equilibrium.

Consider a C^k–branch $y(\lambda)$ of equilibria on $\mathbf{\Lambda}$, i. e. $f\big(y(\lambda),\lambda\big) = 0$ for all $\lambda \in \mathbf{\Lambda}$. Choose $\alpha < 0$, $\beta > \alpha$ such that $\frac{\partial}{\partial x}f\big(y(\lambda),\lambda\big)$ has $m_a \geq 1$ eigenvalues μ with $\mathrm{Re}(\mu) < \alpha < 0$ and $m_b = m - m_a \geq 1$ eigenvalues μ with $\mathrm{Re}(\mu) > \beta$ for each $\lambda \in \mathbf{\Lambda}$.

Then $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(y(\lambda), \lambda\big)$ has a shifted exponential dichotomy with exponents $\alpha < \beta$, $\alpha < 0$ and **A1**–**A3** hold (see Section 1.3.1).

Asymptotic boundary conditions for equilibria

To get a smooth matrix valued function $\bar{M}(\lambda)$ (independent of T_+) in a neighbourhood of $\bar{\lambda}$ we choose some $\gamma \in (\alpha, \beta)$ and solve by the implicit function theorem

$$\begin{aligned} V \frac{\partial}{\partial x} f(y(\lambda), \lambda) &= U(\lambda) V, \quad \mathrm{Re}\big(\sigma(U)\big) > \gamma, \quad U \in \mathbb{R}^{m_b, m_b},\ V \in \mathbb{R}^{m_b, m}, && (3.8) \\ \hat{V} V^\top &= \hat{V} \hat{V}^\top, && (3.9) \end{aligned}$$

where $\hat{V}$, $\hat{U}$ solves (3.8) at $\bar{\lambda}$. Using (3.2), (3.3), (3.6), (3.8) and (3.9) we solve

$$G_J^e(z, \nu, y, V, U) = \begin{pmatrix} f(y, \nu) \\ \dot{z} - f(z, \nu) \\ g\big(z(0), \nu\big) \\ V\big(z(T_+) - y\big) \\ V \frac{\partial}{\partial x} f(y, \nu) - UV \\ \hat{V} V^\top - \hat{V} \hat{V}^\top \end{pmatrix} = 0. \qquad (3.10)$$

Instead of adding (3.8) and (3.9) to the boundary value problem we may determine $\bar{M}(\lambda)$ during each evaluation of (3.10) by an eigenvalue solver combined with a normalization procedure ([Beyn, 1990b],[Champneys and Kuznetsov, 1994]) applied to the hyperbolic matrices $\frac{\partial}{\partial x} f(y(\lambda), \lambda) - \gamma \mathrm{I}_m$.

3.1.2 Periodic Solutions

In the periodic case let $(\hat{y}(\cdot), \bar{\lambda})$ be a $\bar{T}$–periodic orbit of $\dot{x} = \hat{f}(x, \bar{\lambda})$ where 1 is a simple Floquet multiplier. Then we consider a C^k–branch $y(\lambda)$ of scaled periodic solutions, i. e. $y(\lambda)$ is a 1–periodic solution of $\dot{y} = T(\lambda) \hat{f}(y, \lambda) =: f(y, \lambda)$ (see Theorem 1.14 and (1.34)).

Furthermore assume that all $y(\lambda)$ have $m_a \geq 1$ Floquet multipliers μ with $|\mu| < e^{\bar{\alpha}}$, $\bar{\alpha} < 0$ and $m_b = m - m_a \geq 1$ Floquet multipliers with $|\mu| \geq e^{\bar{\beta}}$, $\bar{\alpha} < \bar{\beta} \leq 0$. Then each $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(y(\lambda)(\cdot), \lambda)$ has a shifted exponential dichotomy and **A3** (see page 27) holds (see Section 1.3.2).

Asymptotic boundary conditions for periodic orbits

We choose $M(T_+, \lambda)$ to be 1–periodic in T_+ on $\mathcal{D} = \cup_{N \in \mathbb{N}} [N - \tau_-, N + \tau_+]$ for some $\tau_-, \tau_+ \in [0, 1]$, such that $\mathcal{N}\big(M(T_+, \bar{\lambda})\big) \cap \mathcal{R}\big(\bar{P}_b(T_+)\big) = \{0\}$ holds on the interval $[-\tau_-, \tau_+]$.

A good choice for $M(T_+,\lambda) \in \mathbb{R}^{m_b,m}$ is a matrix valued function $\bar{M}(T_+,\lambda)$ which satisfies $\mathcal{N}\big(\bar{M}(T_+,\lambda)\big) = \mathcal{N}\big(\bar{P}_b(\lambda)(T_+)\big)$ (see (3.5), (3.6) and (3.7)) and as we have seen in (1.41) it is appropriate to choose $M(T_+,\lambda) = V(0,\lambda)$ where V is a solution of the adjoint variational equation

$$\begin{aligned} \dot{V} &= -V\frac{\partial}{\partial x} f(y(\lambda)(T_+ + \cdot),\lambda), && (3.11)\\ V(1,\lambda) &= U(\lambda)V(0,\lambda) && (3.12) \end{aligned}$$

for some $U(\lambda) \in \mathbb{R}^{m_b,m_b}$ with $|\mu| \le e^{-\gamma}$ for all eigenvalues μ of $U(\lambda)$ (see [Beyn, 1994b]).

To define the function $M(T_+,\lambda) \in \mathbb{R}^{m_b,m}$, such that (3.11) and (3.12) are smooth in λ, we may apply (as in the equilibrium case) the implicit function theorem to

$$F_M(V(\cdot),U;\lambda) = \begin{pmatrix} \dot{V} + V\frac{\partial}{\partial x} f(y(\lambda),\lambda) \\ V(1) - UV(0) \\ \hat{V}(0)V(0)^\top - \hat{V}(0)\hat{V}(0)^\top \end{pmatrix} = 0, \qquad (3.13)$$

where $\hat{V}$, $\hat{U}$ solves (3.11),(3.12) at $\bar{\lambda}$. Adding these equations to (3.2), (3.3), (3.6) and (1.32) would blow up the number of differential equations in the boundary value problem from $2m$ to $(2+m_b)m$.

As an alternative we may determine $\bar{M}(T_+,\lambda)$ at each evaluation of (3.6) as in [Beyn, 1990b, Appendix C]. To do so, we get $V(\cdot), U$ (without continuity assumptions) from (3.11), (3.12). Then we solve the linear system

$$\left(\hat{V}\; V(\lambda)(0)^\top\right)\Lambda(\lambda) = \hat{V}\hat{V}^\top \qquad (3.14)$$

and the columns of $V(\lambda)(0)^\top\Lambda(\lambda)$ form a C^{k-1}–smooth basis. This implies, that we have to solve an $(m_b \cdot m)$–dimensional boundary value problem whenever we evaluate (3.6).

To avoid this we develop the **boundary corrector method** (for detail see Section 3.3):

1. Compute $M(T_+,\mu_0)$ as the solution of (3.11), (3.12) at μ_0

2. Calculate the solution $\big(x_J,\mu_J,y(\mu_J),T(\mu_J)\big)$ of

$$G_J^p(z,\nu,y,T) = \begin{pmatrix} \dot{y} - T\hat{f}(y,\nu) \\ \chi(y,\nu) \\ y(0)-y(1) \\ \dot{z} - T\hat{f}(z,\nu) \\ g\big(z(0),\nu\big) \\ M(T_+,\mu_0)\big(z(T_+)-y(T_+)\big) \end{pmatrix} = 0 \qquad (3.15)$$

3. Repeat with **1.** once if $\beta = 0$ or twice if $\beta < 0$ with $\mu_0 = \mu_J$.

In Section 3.3 we show in a general set up that this method yields error estimates of the same quality as if we use $\bar{M}(T_+, \lambda)$.

Suppose we have an error estimate for the solutions of the scaled system. Then we will show how this transfers to the unscaled system.

Let $\hat{x}$ and $\hat{y}$ be γ–asymptotic at $\bar{\lambda}$ for the unscaled system. Then $\bar{x}(t) := \hat{x}(t \cdot \bar{T})$ and $\bar{y}(t) := \hat{y}(t \cdot \bar{T})$ are γ–asymptotic at $\bar{\lambda}$ for the scaled system. Furthermore, let (x_J, λ_J) be a solution of (3.15), i. e. (x_J, λ_J) is an approximation of $(\bar{x}, \bar{\lambda})$ in the scaled system. Then $(\hat{x}_J, \lambda_J)$, $\hat{x}_J(t) := x_J\big(\frac{t}{T(\lambda_J)}\big)$ is an approximation of $(\hat{x}, \bar{\lambda})$ in the unscaled system defined on $[0, T_+ \cdot T(\lambda_J)]$ and for any $t \in [0, T_+ \cdot T(\lambda_J)]$ we estimate

$$
\begin{aligned}
\Big|\hat{x}_J(t) - \hat{x}(t)\Big| &= \Big|x_J\Big(\frac{t}{T(\lambda_J)}\Big) - \bar{x}\Big(\frac{t}{\bar{T}}\Big)\Big| \\
&\leq \Big|x_J\Big(\frac{t}{T(\lambda_J)}\Big) - \bar{x}\Big(\frac{t}{T(\lambda_J)}\Big)\Big| + \Big|\bar{x}\Big(\frac{t}{T(\lambda_J)}\Big) - \bar{x}\Big(\frac{t}{\bar{T}}\Big)\Big| \\
&\leq \sup_{s \in [0,T_+]} \Big|x_J(s) - \bar{x}(s)\Big| + \sup_{\mu \in \Gamma} \Big| - f\Big(\bar{x}\Big(\frac{t}{T(\mu)}\Big), \mu\Big) \cdot \frac{T'(\mu)}{T(\mu)^2} \cdot t \Big| \cdot |\lambda_J - \bar{\lambda}| \\
&\leq ||x_J - \bar{x}||_\infty + C\frac{T'(\bar{\lambda})}{T(\bar{\lambda})} \cdot T_+ \cdot \Big|\lambda_J - \bar{\lambda}\Big|
\end{aligned}
$$

where $\Gamma = \{a\lambda_J + (1-a)\bar{\lambda} \mid a \in [0,1]\}$ and $C \approx M_f$.

Thus the error estimates in the scaled system carry over to the unscaled system except for the factor T_+. This is unavoidable since we get similar estimates for the periodic solutions themselves, i. e.

$$
|\hat{y}(\lambda_J)(t) - \hat{y}(\bar{\lambda})(t)| \leq \Big(L_y + C\frac{T'(\bar{\lambda})}{T(\bar{\lambda})} \cdot T_+\Big) \cdot |\lambda_J - \bar{\lambda}|
$$

3.2 The approximation theorem

In this section we assume that $y(\lambda)$ is known and that **A1**–**A4** hold with $k = 2$. Furthermore, let $(\bar{x}, \bar{\lambda})$ be a nondegenerate γ–connection from $\mathcal{W}$ to y, where $\bar{\alpha} < \gamma < \min(0, 2\bar{\beta})$ (see **A3**). Proposition 1.7 implies that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{x}, \bar{\lambda})$ has a shifted exponential dichotomy with projectors P_a and P_b and exponents α, β, $(\alpha < \gamma < 2\beta)$ near $\bar{\alpha}$, $\bar{\beta}$.

As abbreviation we define $\mathcal{R}_\kappa := \mathcal{R}\big(P_\kappa(T_+)\big)$, $\bar{\mathcal{R}}_\kappa := \mathcal{R}\big(\bar{P}_\kappa(\bar{\lambda})(T_+)\big)$ for $\kappa = a, b$.

To set up an appropriate approximation operator G_J (see (3.19)), the boundary condition at T_+ has to be regular in the following sense:

A6 (Regular boundary condition) Let $M(T_+, \cdot) \in C^1\big(\mathbf{\Lambda}, \mathbb{R}^{m_b,m}\big)$ be a matrix valued function with $\mathcal{N}\big(M(T_+, \bar{\lambda})\big) \cap \bar{\mathcal{R}}_b = \{0\}$ for all $T_+ \in \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}_+$ is a set and there exists a sequence $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{D}$ with $\lim_{i\to\infty} T_i = \infty$. Furthermore, let $M(T_+,\lambda)$, $\frac{\partial}{\partial\lambda}M(T_+,\lambda)$ be uniformly bounded by some constants M_M, M_M^1 w. r. t. $T_+ \in \mathcal{D}$ and $\lambda \in \mathbf{\Lambda}$ and Lipschitz continuous w. r. t. $\lambda \in \mathbf{\Lambda}$ with constants L_M, L_M^1. Let $\bar{N} := \big(M(T_+,\bar{\lambda})|_{\bar{\mathcal{R}}_b}\big)^{-1}$ be uniformly bounded by some constant $M_{\bar{N}} > 0$ for all $T_+ \in \mathcal{D}$.

Note that $\mathcal{N}\big(M(T_+,\bar{\lambda})\big) \cap \bar{\mathcal{R}}_b = \{0\}$ implies the non-singularity of $M(T_+,\bar{\lambda})|_{\bar{\mathcal{R}}_b}$. Furthermore, $M(T_+,\bar{\lambda})|_{\mathcal{R}_b}$ is a homeomorphism and its inverse N is uniformly bounded by some constant $M_N := 4KM_{\bar{N}} > 0$ for sufficiently large $T_+ \in \mathcal{D}$. We choose T_+ such that the projectors obtained from Proposition 1.7 satisfy (1.30) with $r = \frac{1}{2}$. Thus we apply Lemma 1.10 to get the upper bound of $\big(M(T_+,\bar{\lambda})|_{\mathcal{R}_b}\big)^{-1}$.

Remark: Uniformity of upper bounds and Lipschitz constants M_M, M_M^1, L_M, L_M^1 and $\bar{N}$ in $T_+ \in \mathbb{R}_+$ might be too strong an assumption. Thus we define $\mathcal{D}$ as in **A6** and only require uniformity on $\mathcal{D}$. In particular, in the case where the $y(\lambda)$ are 1–periodic we choose $\mathcal{D} = \{n + [-a,b] \mid n \in \mathbb{N}\}, a, b \in [0,1]$ and in the case where the $y(\lambda)$ are equilibria we choose $\mathcal{D} = \mathbb{R}_+$.

To get approximation results even for $\beta < 0$, we have to choose functions spaces where the solution properties will be anticipated. Thus, the main trick is to use, even on finite intervals, exponentially weighted norms. In addition the integral-term in (3.17) avoids estimates of type $(1+T_+)||v||_\beta$.

For $T_+ \in \mathcal{D}$, $J := [0,T_+]$ we define Banach spaces $\mathcal{Y} = C^1(J,\mathbb{R}^m) \times \mathbb{R}^p$ with

$$||(x,\lambda)||_{\mathcal{Y}} = ||x||_{\mathcal{Y}}^* + |\lambda|, \quad ||x||_{\mathcal{Y}}^* = \sup_{t\in J}\big(|x(t)|q(t)\big), \quad q(t) = \frac{2}{1+e^{\beta t}} \tag{3.16}$$

and $\mathcal{Z} = C^0(J,\mathbb{R}^m) \times \mathbb{R}^{m_a+p} \times \mathbb{R}^{m_b}$ with

$$\begin{aligned} ||(w,r_0,r_+)||_{\mathcal{Z}} &= C_{\mathcal{Z}}^J||w||_{\mathcal{Z}}^* + |r_0| + C_{\mathcal{Z}}^+|r_+|e^{-\beta T_+}, \\ C_{\mathcal{Z}}^J &:= ||R_0||(K + M_M M_N K^2), \quad C_{\mathcal{Z}}^+ := ||R_0||\ K\ M_N, \\ ||w||_{\mathcal{Z}}^* &= ||w||_\beta + \int_0^{T_+} |w(s)|e^{-\beta s}\mathrm{ds},\ ||w||_\beta = \sup_{t\in J}(|w(t)|e^{-\beta t}). \end{aligned} \tag{3.17}$$

Here we define $(R_0\ Q_0) := D(\bar{x}(0),\bar{\lambda}) := \left(\frac{\partial g(\bar{x}(0),\bar{\lambda})}{\partial x}, \frac{\partial g(\bar{x}(0),\bar{\lambda})}{\partial \lambda}\right) \in \mathbb{R}^{m_a+p,m+p}$ as in (2.58). Defining B as in (2.31) we obtain from Theorem 2.21 that

$$D \circ B(\xi,\mu) = R_0\xi + R_0\frac{\partial}{\partial\lambda}x(0,\bar{\lambda})(0)\mu + Q_0\mu \tag{3.18}$$

is nonsingular. Hence $D \circ B(\xi,0) = R_0\xi$ implies $\mathrm{rank}(R_0) > 0$ and $||R_0|| > 0$.

THEOREM 3.1 *Suppose that* **A1**–**A6** *holds and let* $(\bar{x}, \bar{\lambda})$ *be a nondegenerate connecting orbit with asymptotic rate from* $\mathcal{W}$ *to* y *where* $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{x}, \bar{\lambda})$ *has a shifted exponential dichotomy with dichotomy data* $(K, \alpha, \beta, P_a, P_b)$ *and* $\alpha < \gamma < \min(0, 2\beta)$.

Then there exist constants $\delta > 0$, $C > 0$ *and* $\bar{T}_+$ *sufficiently large, such that for* $T_+ \in \mathcal{D}$ *with* $T_+ \geq \bar{T}_+$ *the operator equation*

$$G_J(z, \nu) = \begin{pmatrix} \dot{z} - f(z, \nu) \\ g\big(z(0), \nu\big) \\ M(T_+, \nu)\big(z(T_+) - y(\nu)(T_+)\big) \end{pmatrix} = 0 \tag{3.19}$$

has a unique solution (z_J, ν_J) *in the ball* $\mathcal{B}_{\frac{\delta}{r(\beta, T_+)}}(\bar{x}|_J, \bar{\lambda})$ *in* $\mathcal{Y}$, *where*

$$r(\beta, T_+) := \begin{cases} e^{|\beta| T_+} + \frac{1}{|\beta|}\left(e^{|\beta| T_+} - 1\right) & : \beta \neq 0 \\ 1 + T_+ & : \beta = 0. \end{cases} \tag{3.20}$$

Furthermore, the following estimates hold with $d = 1$

$$|\nu_J - \bar{\lambda}| \leq Ce^{(d\gamma - \beta)T_+}, \tag{3.21}$$

$$|z_J(t) - \bar{x}|_J(t)| \leq Ce^{(d\gamma - \beta)T_+} \cdot \begin{cases} e^{\beta t} & : \beta > 0 \\ 1 & : \beta \leq 0. \end{cases} . \tag{3.22}$$

If $M(T_+, \lambda)$ *satisfies*

$$|M(T_+, \bar{\lambda})\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big)| \leq C_{\bar{M}} |\bar{x}(T_+) - y(\bar{\lambda})(T_+)|^2 \tag{3.23}$$

for a constant $C_{\bar{M}} > 0$ *and all* $T_+ \in \mathcal{D}$ *with* $T_+ \geq \bar{T}_+$, *then the estimates (3.21), (3.22) hold for* $d = 2$. *In this case the assumption* $\alpha < \gamma < \min(0, 2\beta)$ *can be weakened to* $\alpha < \gamma < \min(0, \beta)$.

Proof: For the proof of this theorem we apply the following lemma (see [Beyn, 1990b], [Vainikko, 1976]):

LEMMA 3.2 (Perturbation lemma) *Let* $F : \mathcal{B}_\delta(y_0) \to \mathcal{Z}$ *be a* C^1*–function from a ball in* $\mathcal{Y}$ *into* $\mathcal{Z}$ *(Banach spaces). Assume that* $F'(y_0)$ *is a homeomorphism and there exist constants* κ *and* σ, *such that*

$$||F'(y) - F'(y_0)|| \leq \kappa < \sigma \leq ||F'(y_0)^{-1}||^{-1}, \tag{3.24}$$

$$||F(y_0)|| \leq (\sigma - \kappa)\delta \tag{3.25}$$

holds for all $y \in \mathcal{B}_\delta(y_0)$. *Then* F *has a unique zero* $\bar{y}$ *in* $\mathcal{B}_\delta(y_0)$ *and*

$$||\bar{y} - y_0|| \leq (\sigma - \kappa)^{-1} ||F(y_0)||, \tag{3.26}$$

$$||y_1 - y_2|| \leq (\sigma - \kappa)^{-1} ||F(y_1) - F(y_2)|| \tag{3.27}$$

for all $y_1, y_2 \in \mathcal{B}_\delta(y_0)$.

Sketch of the proof of Theorem 3.1

Apply Lemma 3.2 with $F = G_J$ and $y_0 = (\bar{x}|_J, \bar{\lambda})$.

S1 There exists some $C_{lin} > 0$ such that $||(v,\mu)||_{\mathcal{Y}} \le C_{lin}||(w, r_0, r_+)||_{\mathcal{Z}}$ holds for each solution (v,μ) of the inhomogeneous equation $G'_J(\bar{x}|_J, \bar{\lambda})(v,\mu) = (w, r_0, r_+)$.

S2 $G'_J(\bar{x}|_J, \bar{\lambda})$ is a homeomorphism and $\sigma := \frac{1}{C_{lin}} \le ||G'_J(\bar{x}|_J, \bar{\lambda})^{-1}||^{-1}$.

S3 There exists some constant $C_{Lip} > 0$ such that for $\delta := \frac{\sigma}{2C_{Lip}} > 0$ and $\tilde{\delta} := \frac{\delta}{r(\beta, T_+)}$ the following holds: $||G'_J(z,\nu) - G'_J(\bar{x}|_J, \bar{\lambda})|| \le \kappa := \frac{\sigma}{2}\ \forall (z,\nu) \in \mathcal{B}_{\tilde{\delta}}(\bar{x}|_J, \bar{\lambda})$.

S4 There exist constants $C_{tr} > 0$ and $\bar{T}_+$ sufficiently large, such that $||G_J(\bar{x}|_J, \bar{\lambda})|| \le C^+_{\mathcal{Z}}\, C_{tr} e^{(d\gamma - \beta)T_+} \le (\sigma - \kappa)\frac{\delta}{r(\beta, T_+)}$ for all $T_+ \in \mathcal{D}$, $T_+ \ge \bar{T}_+$.

S5 Lemma 3.2 implies that (3.19) has a unique solution (z_J, ν_J) in $\mathcal{B}_{\tilde{\delta}}(\bar{x}|_J, \bar{\lambda})$ with $C := 2C^+_{\mathcal{Z}} C_{lin} C_{tr}$

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \le C e^{(d\gamma - \beta)T_+}.$$

We consider the steps in detail.

S1 Let $(w, r_0, r_+) \in \mathcal{Z}$ and let $(v,\mu) \in \mathcal{Y}$ be a solution of the inhomogeneous equation

$$G'_J(\bar{x}|_J, \bar{\lambda})(v,\mu) = (w, r_0, r_+). \tag{3.28}$$

This is equivalent to the variational equation

$$\begin{aligned} \dot{v} - A(\cdot)v - V(\cdot)\mu &= w & (3.29)\\ R_0 v(0) + Q_0\mu &= r_0 & (3.30)\\ R_+ v(T_+) + Q_+\mu &= r_+ & (3.31) \end{aligned}$$

where

$$\begin{aligned} A(\cdot) &= \tfrac{\partial}{\partial x} f(\bar{x}(\cdot), \bar{\lambda}), & V(\cdot) &= \tfrac{\partial}{\partial \lambda} f(\bar{x}(\cdot), \bar{\lambda}),\\ R_0 &= \tfrac{\partial}{\partial x} g(\bar{x}(0), \bar{\lambda}), & Q_0 &= \tfrac{\partial}{\partial \lambda} g(\bar{x}(0), \bar{\lambda}),\\ R_+ &= M(T_+, \bar{\lambda}), & Q_+ &= \tfrac{\partial}{\partial \lambda} M(T_+, \bar{\lambda})\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big)\\ & & &\quad - M(T_+, \bar{\lambda})\tfrac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+). \end{aligned}$$

Defining $\xi_a = P_a(0)v(0) \in \mathcal{R}(P_a(0))$, $\xi_b = P_b(T_+)\big(v(T_+) - \frac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+)\mu\big) \in \mathcal{R}\big(P_b(T_+)\big)$, then Theorem 1.5 implies that

$$\begin{gathered} Lv = \big(V(\cdot)\mu + w\big), \quad t \in J\\ P_a(0)v(0) = \xi_a, \quad P_b(T_+)v(T_+) = \xi_b + P_b(T_+)\tfrac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+)\mu \end{gathered}$$

has a unique solution v. This is

$$\begin{aligned} v(t) &= S(t,0)\xi_a + S(t,T_+)\xi_b + s_J(w)(t) + s_J(V(\cdot)\mu)(t) \\ &\quad + S(t,T_+)P_b(T_+)\frac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+)\mu. \end{aligned} \tag{3.32}$$

Now we derive another representation of $s_J(V(\cdot)\mu)(t)$ and hence v.

Using $\frac{\partial}{\partial s}S(t,s) = -S(t,s)A(s)$ (see (1.5)) and (2.7) we get

$$\begin{aligned}
\frac{\partial}{\partial s}\left(S(t,s)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s)\right) &= -S(t,s)\frac{\partial}{\partial x}f\big(\bar x(s),\bar\lambda\big)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s) \\
&\quad +S(t,s)\Big(\frac{\partial}{\partial x}f\big(\bar y(s),\bar\lambda\big)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s)+\frac{\partial}{\partial\lambda}f\big(\bar y(s),\bar\lambda\big)\Big) \\
&= S(t,s)\Big(V(s)-\Delta(s)\Big)
\end{aligned}$$

and hence by integration and using (2.6) we get for $t\in J$

$$\begin{aligned}
& s_J\big(V(\cdot)\big)(t) - s_J\big(\Delta(\cdot)\big)(t) \\
={}& \int_0^t P_a(t)S(t,s)\big(V(s)-\Delta(s)\big)\mathrm{ds} - \int_t^{T_+} P_b(t)S(t,s)\big(V(s)-\Delta(s)\big)\mathrm{ds} \\
={}& \big[P_a(t)S(t,s)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s)\big]_0^t - \big[P_b(t)S(t,s)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s)\big]_t^{T_+} \\
={}& \frac{\partial}{\partial\lambda}y(\bar\lambda)(t) - S(t,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0) - S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) \qquad (3.33)\\
={}& \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(t) - s_{[0,\infty)}(\Delta(\cdot))(t) - S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) \qquad (3.34)\\
={}& \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(t) - S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) \qquad (3.35)\\
& -s_J\big(\Delta(\cdot)\big)(t) + S(t,T_+)\int_{T_+}^\infty S(T_+,s)P_b(s)\Delta(s)\mathrm{ds}.
\end{aligned}$$

This and (3.32) imply further representations of v. Here (3.36) follows from (3.33) and (3.37) follows from (3.35).

$$\begin{aligned}
v(t) &= S(t,0)\xi_a + S(t,T_+)\xi_b + s_J(w)(t) \qquad (3.36)\\
&\quad +\frac{\partial}{\partial\lambda}y(\bar\lambda)(t)\mu - S(t,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu + s_J\big(\Delta(\cdot)\big)(t)\mu \\
&= S(t,0)\xi_a + S(t,T_+)\xi_b + s_J(w)(t) + \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(t)\mu \qquad (3.37)\\
&\quad +S(t,T_+)\int_{T_+}^\infty S(T_+,s)P_b(s)\Delta(s)\mu\mathrm{ds}.
\end{aligned}$$

We use $D\circ B(\xi_a,\mu){=}R_0\xi_a + R_0\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu + Q_0\mu$ (see (3.18)) to get equation (3.39) below. By $R_0 v(0) + Q_0\mu = r_0$ (see (3.30)) and (3.37) evaluated at 0 we get (3.40).

$$\begin{aligned}
&(D\circ B \quad R_0 S(0,T_+)|_{\mathcal{R}_b})\begin{pmatrix}(\xi_a,\mu)\\ \xi_b\end{pmatrix} && (3.38)\\
&= D\circ B(\xi_a,\mu) + R_0 S(0,T_+)\xi_b &&\\
&= Q_0\mu + R_0\xi_a + R_0\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu + R_0 S(0,T_+)\xi_b && (3.39)\\
&= r_0 - R_0 S(0,T_+)\int_{T_+}^{\infty} S(T_+,s)P_b(s)\Delta(s)\mu \mathrm{ds} &&\\
&\quad + R_0\int_0^{T_+} S(0,s)P_b(s)w(s)\mathrm{ds} \quad =: \quad R_a(T_+). && (3.40)
\end{aligned}$$

From $P_a(T_+)v(T_+) = S(T_+,0)\xi_a + s_J(w)(T_+) + P_a(T_+)\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(T_+)\mu$ (see (3.37)) and $P_b(T_+)v(T_+) = \xi_b + P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+)\mu$ (see (3.32) and notice that $P_b(T_+)s_J(w+V(\cdot))(T_+) = P_b(T_+)\int_0^{T_+} S(T_+,s)P_a(s)(w(s)+V(s))\mathrm{ds} = 0$) follows

$$\begin{aligned}
&M(T_+,\bar\lambda)\Big(\xi_b + S(T_+,0)\xi_a + s_J(w)(T_+)\\
&\qquad + P_a(T_+)\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(T_+)\mu + P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+)\mu\Big)\\
&= R_+ v(T_+) \quad = \quad r_+ - Q_+\mu\\
&= r_+ + M(T_+,\bar\lambda)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+)\mu - \frac{\partial}{\partial\lambda}M(T_+,\bar\lambda)\mu\big(\bar x(T_+) - y(\bar\lambda)(T_+)\big)
\end{aligned}$$

by definition of R_+ and Q_+. Thus we see

$$\begin{aligned}
M(T_+,\bar\lambda)\xi_b &= r_+ - \frac{\partial}{\partial\lambda}M(T_+,\bar\lambda)\mu\big(\bar x(T_+) - y(\bar\lambda)(T_+)\big) && (3.41)\\
&\quad + M(T_+,\bar\lambda)P_a(T_+)\big(\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) - \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(T_+)\big)\mu &&\\
&\quad - M(T_+,\bar\lambda)S(T_+,0)\xi_a - M(T_+,\bar\lambda)s_J(w)(T_+) &&\\
&=: R_b(T_+). &&
\end{aligned}$$

Defining $\hat\xi_b := \xi_b e^{-\beta T_+}$ we get the linear system

$$\begin{pmatrix} D\circ B & R_0 S(0,T_+)|_{\mathcal{R}_b}\cdot e^{\beta T_+}\\ 0 & M(T_+,\bar\lambda)|_{\mathcal{R}_b}\end{pmatrix}\begin{pmatrix}(\xi_a,\mu)\\ \hat\xi_b\end{pmatrix} = \begin{pmatrix} R_a(T_+)\\ R_b(T_+)e^{-\beta T_+}\end{pmatrix}. \qquad (3.42)$$

We can solve this linear system, because $D\circ B$ and $M(T_+,\bar\lambda)|_{\mathcal{R}_b}$ are nonsingular and $R_0 S(0,T_+)|_{\mathcal{R}_b}\cdot e^{\beta T_+}$ is uniformly bounded by $||R_0||\ K$. Moreover, $R_a(T_+)$ and $R_b(T_+)e^{-\beta T_+}$ are bounded as we will see later on in in (3.47) and (3.48).

Before we estimate this, we recall some useful results from the preceding parts and define some constants.

$$
\begin{aligned}
|\Delta(t)|e^{-\gamma t} &\le L_f(1+M_y)C_{\bar{x}} \quad \forall t \in \mathbb{R}_+, && (3.43)\\
||s_{[T_+,\infty)}(\Delta(\cdot))(T_+)||e^{-\gamma T_+} &\le K\frac{1}{\beta-\gamma}L_f^1(1+M_y^1)C_{\bar{x}} =: -c_{\Delta,\beta}, && (3.44)\\
||s_J(\Delta(\cdot))(T_+)||e^{-\gamma T_+} &\le K\frac{1}{\gamma-\alpha}L_f^1(1+M_y^1)C_{\bar{x}} =: c_{\Delta,\alpha}, && (3.45)\\
||s_J(\Delta(\cdot))(t)||e^{-\gamma t} &\le KC_\gamma L_f^1(1+M_y^1)C_{\bar{x}} =: c_\Delta. && (3.46)
\end{aligned}
$$

Here (2.25) implies (3.43) and therefore $\Delta \in BC_\gamma^0(\mathbb{R}_+,\mathbb{R}^m)$, and (1.16), (1.17) imply (3.44) and (3.45). Furthermore, we see $c_\Delta = c_{\Delta,\alpha} - c_{\Delta,\beta} = C_\gamma L_f^1(1+M_y^1)C_{\bar{x}}$.

We estimate $R_a(T_+)$, defined in (3.40), by

$$
\begin{aligned}
|R_a(T_+)| &\le |r_0| + ||R_0||\int_0^{T_+} ||S(0,s)P_b(s)||\,|w(s)|\mathrm{ds} \qquad (3.47)\\
&\quad + ||R_0||\cdot||S(0,T_+)P_b(T_+)||\cdot||s_{[T_+,\infty)}(\Delta(\cdot))(T_+)||\,|\mu|\\
&\le |r_0| + ||R_0||K||w||_{\mathcal{Z}}^* + ||R_0||Kc_{\Delta,\beta}e^{(\gamma-\beta)T_+}|\mu|\\
&\le ||(w,r_0,0)||_{\mathcal{Z}} + ||R_0||Kc_{\Delta,\beta}e^{(\gamma-\beta)T_+}|\mu|
\end{aligned}
$$

and by using (2.6) we obtain for $R_b(T_+)$, defined in (3.41), the estimate

$$
\begin{aligned}
&|R_b(T_+)|e^{-\beta T_+} \le |r_+|e^{-\beta T_+} + M_M^1 C_{\bar{x}} e^{(\gamma-\beta)T_+}|\mu| \qquad (3.48)\\
&\quad + M_M||S(T_+,0)P_a(0)||M_y^1|\mu|e^{-\beta T_+} + M_M||s_J(\Delta(\cdot))(T_+)||\,|\mu|e^{-\beta T_+}\\
&\quad + M_M||S(T_+,0)P_a(0)||e^{-\beta T_+}|\xi_a| + M_M e^{-\beta T_+}\int_0^{T_+} \underbrace{||S(T_+,s)P_a(s)||}_{\le Ke^{\alpha(T_+-s)}\le Ke^{\beta(T_+-s)}} |w(s)|\mathrm{ds}\\
&\le |r_+|e^{-\beta T_+} + M_M K||w||_{\mathcal{Z}}^* + M_M K e^{(\alpha-\beta)T_+}|\xi_a|\\
&\quad + \big(M_M^1 C_{\bar{x}} + M_M K M_y^1 e^{(\alpha-\gamma)T_+} + M_M c_{\Delta,\alpha}\big)e^{(\gamma-\beta)T_+}|\mu|\\
&\le \frac{1}{K\ M_N\ ||R_0||}||(w,0,r_+)||_{\mathcal{Z}} + \frac{\tilde{c}}{KM_N C_{\xi,\mu}||R_0||}e^{(\gamma-\beta)T_+}(|\xi_a|+|\mu|)
\end{aligned}
$$

with the constants

$$
\begin{aligned}
C_{\xi,\mu} &:= ||(D\circ B)^{-1}||,\\
\tilde{c} &:= C_{\xi,\mu}\ K\ M_N\ ||R_0||\cdot\big(\max(1,M_y^1)M_M K + M_M^1 C_{\bar{x}} + \max(\frac{1}{M_N},M_M)c_\Delta\big).
\end{aligned}
$$

For some $a > 1$ and for all $T_+ \geq \bar{T} := \frac{\ln(\tilde{c})-\ln(1-\frac{1}{a})}{\beta-\gamma}$, i. e. $\tilde{c}e^{(\gamma-\beta)T_+} \leq 1-\frac{1}{a}$, we estimate

$$\begin{aligned}
|(\xi_a,\mu)| &\leq \left|(D\circ B)^{-1}\Big(R_0 S(0,T_+)\big(M(T_+,\bar{\lambda})|_{\mathcal{R}_b}\big)^{-1}R_b(T_+)+R_a(T_+)\Big)\right| \\
&\leq ||(D\circ B)^{-1}||\,\Big(|r_0|+||R_0||K||w||^*_{\mathcal{Z}}+||R_0||Kc_{\Delta,\beta}e^{(\gamma-\beta)T_+}|\mu| \\
&\quad +||R_0||KM_N\big(|r_+|e^{-\beta T_+}+M_MK||w||^*_{\mathcal{Z}}+M_MK|\xi_a|\,e^{(\alpha-\beta)T_+} \\
&\quad +(M^1_M C_{\bar{x}}+M_MKM^1_y+M_Mc_{\Delta,\alpha})e^{(\gamma-\beta)T_+}|\mu|\big)\Big) \\
&\leq ||(D\circ B)^{-1}||\;||(w,r_0,r_+)||_{\mathcal{Z}}+\tilde{c}(|\xi_a|+|\mu|)e^{(\gamma-\beta)T_+} \\
&\leq a\cdot C_{\xi,\mu}||(w,r_0,r_+)||_{\mathcal{Z}} \qquad (3.49)
\end{aligned}$$

and

$$\begin{aligned}
|\hat{\xi}_b| &\leq M_N|R_b(T_+)|e^{-\beta T_+}\leq \frac{1}{K\;||R_0||}||(w,0,r_+)||_{\mathcal{Z}}+\frac{\tilde{c}}{C_{\xi,\mu}K\;||R_0||}(|\xi_a|+|\mu|)e^{(\gamma-\beta)T_+} \\
&\leq a\cdot\frac{1}{K\;||R_0||}\;||(w,r_0,r_+)||_{\mathcal{Z}} \qquad (3.50)
\end{aligned}$$

We estimate $v(t)$ using (3.36), (3.46) and (1.21)

$$\begin{aligned}
|v(t)| &\leq Ke^{\alpha t}|\xi_a|+Ke^{-\beta(T_+-t)}|\xi_b|+K||w||^*_{\mathcal{Z}}e^{\beta t} \\
&\quad +\big(\big|s_J\big(\Delta(\cdot)\big)(t)\big|+\big|\frac{\partial}{\partial\lambda}y(\bar{\lambda})(t)\big|+\big|S(t,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\big|\big)|\mu| \\
&\leq \max\big(Ke^{\alpha t},c_\Delta e^{\gamma t}+(K^{\alpha t}+1)M^1_y\big)\big(|\xi_a|+|\mu|\big) \\
&\quad +Ke^{-\beta T_+}|\xi_b|e^{\beta t}+K||w||^*_{\mathcal{Z}}e^{\beta t} \\
&\leq \big(\big[\max(1,M^1_y)Ke^{\alpha t}+c_\Delta e^{\gamma t}+M^1_y\big]C_{\xi,\mu}+\frac{1}{||R_0||}(1+\frac{1}{a})e^{\beta t}\big)\;a\;||(w,r_0,r_+)||_{\mathcal{Z}} \\
&\leq \Big(\big(\big[\max(1,M^1_y)K+c_\Delta\big]C_{\xi,\mu}+\frac{2}{||R_0||}\big)e^{\beta t}+M^1_yC_{\xi,\mu}\Big)\;a\;||(w,r_0,r_+)||_{\mathcal{Z}} \\
&\leq \frac{1+e^{\beta t}}{2}C_v\;a\;||(w,r_0,r_+)||_{\mathcal{Z}}. \qquad (3.51)
\end{aligned}$$

Here we define $C_v := 2\big(\max(1,M^1_y)K+c_\Delta\big)C_{\xi,\mu}+\frac{4}{||R_0||}$ and notice that $2M^1_yC_{\xi,\mu}\leq C_v$. This leads to $||v||^*_{\mathcal{Y}}=\sup_{t\in J}\big(||v(t)||q(t)\big)\leq aC_v||w,r_0,r_+||_{\mathcal{Z}}$.
Thus $||(v,\mu)||_{\mathcal{Y}}\leq C_{lin}||(w,r_0,r_+)||_{\mathcal{Z}}$ holds with $C_{lin}=a(C_{\mu,\xi}+C_v)$.

Remark: The factor $a > 1$ can be chosen. Large a implies that the constant C_{lin} becomes worse and small a implies that we have to choose large $\bar{T}$. On the other hand if $\bar{T} > \frac{\ln(\tilde{c})}{\beta-\gamma}$ is given, then we may choose $a=\frac{1}{1-\tilde{c}e^{(\gamma-\beta)\bar{T}}}$.

S2 Assumption **A4** implies that $G'_J(\bar{x}|_J,\bar{\lambda})(v,\mu)=(w,r_0,r_+)$ is a linear boundary value problem of dimension $m_a+p+m_b=m+p$ for which the Fredholm alternative holds. Thus $\sigma := \frac{1}{C_{lin}}\leq ||G'_J(\bar{x}|_J,\bar{\lambda})^{-1}||^{-1}$.

S3 For any $(z,\nu),(v,\mu)\in\mathcal{Y}$ we find

$$\left\|\Big(G_J'(z,\nu)-G_J'(\bar{x}|_J,\bar{\lambda})\Big)\begin{pmatrix} v\\ \mu\end{pmatrix}\right\|_{\mathcal{Z}} \tag{3.52}$$

$$= C_{\mathcal{Z}}^J\Big\|\big(\frac{\partial}{\partial x}f(z,\nu)-\frac{\partial}{\partial x}f(\bar{x}|_J,\bar{\lambda})\big)v+\big(\frac{\partial}{\partial\lambda}f(z,\nu)-\frac{\partial}{\partial\lambda}f(\bar{x}|_J,\bar{\lambda})\big)\mu\Big\|_{\mathcal{Z}}^* \tag{3.53}$$

$$+\Big|\big(\frac{\partial}{\partial x}g(z(0),\nu)-\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\big)v(0) \tag{3.54}$$

$$\quad+\big(\frac{\partial}{\partial\lambda}g(z(0),\nu)-\frac{\partial}{\partial\lambda}g(\bar{x}(0),\bar{\lambda})\big)\mu\Big|$$

$$+C_{\mathcal{Z}}^+\Big|\big(M(T_+,\nu)-M(T_+,\bar{\lambda})\big)v(T_+) \tag{3.55}$$

$$\quad+\frac{\partial}{\partial\lambda}M(T_+,\nu)\mu\big(z(T_+)-y(\nu)(T_+)\big)$$

$$\quad-\frac{\partial}{\partial\lambda}M(T_+,\bar{\lambda})\mu\big(\bar{x}(T_+)-y(\bar{\lambda})(T_+)\big)$$

$$\quad-M(T_+,\nu)\frac{\partial}{\partial\lambda}y(\nu)(T_+)\mu+M(T_+,\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+)\mu\Big|e^{-\beta T_+}$$

In order to estimate (3.53)–(3.55) we define the functions

$$q(t)=\frac{2}{1+e^{\beta t}},\quad p(t)=\frac{1+e^{-\beta t}}{2},\quad \rho(t)=\frac{1+e^{|\beta|t}}{2}$$

and notice $p(t)\le q(t)\cdot\rho(t)$, $p(t)\cdot q(t)=e^{-\beta t}$ and $q(t)\le p(t)\le\rho(t)$.

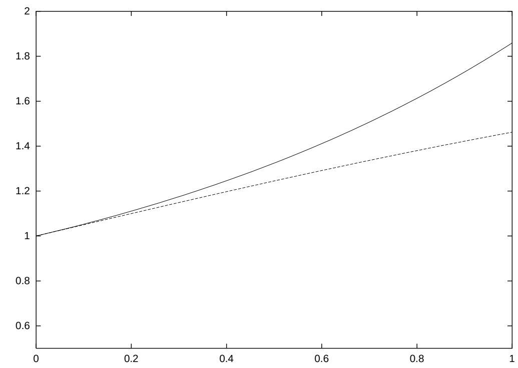

Figure 3.1: For $\beta=-1$: $q(t)$ dashed (- - -) and $p(t)=\rho(t)$ lined (—)

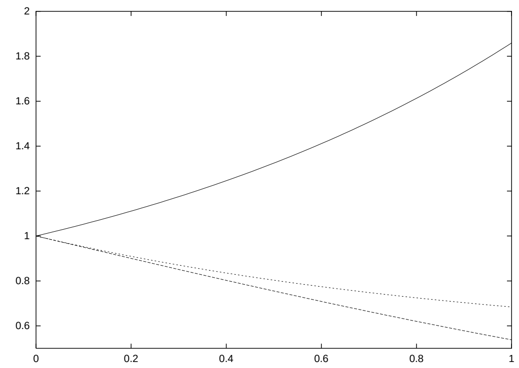

Figure 3.2: For $\beta=1$: $q(t)$ dashed (- - -), $p(t)$ dotted $(\cdots)$ and $\rho(t)$ lined (—)

In addition, we use

$$\sup_{t\in J}(\rho(t))+\int_0^{T_+}\rho(t)\mathrm{dt}\le r(\beta,T_+)=\begin{cases} e^{|\beta|T_+}+\frac{1}{|\beta|}\left(e^{|\beta|T_+}-1\right) & :\beta\neq 0\\ 1+T_+ & :\beta=0.\end{cases}$$

With this we estimate (3.53) as follows:

$$\begin{aligned}
&\Big\|\big(\frac{\partial}{\partial x}f(z,\nu)-\frac{\partial}{\partial x}f(\bar{x}|_J,\bar{\lambda})\big)v+\big(\frac{\partial}{\partial\lambda}f(z,\nu)-\frac{\partial}{\partial\lambda}f(\bar{x}|_J,\bar{\lambda})\big)\mu\Big\|^*_{\mathcal{Z}}\\
\leq\ & \sup_{t\in J}\big(\big|\frac{\partial}{\partial x}f\big(z(t),\nu\big)-\frac{\partial}{\partial x}f\big(\bar{x}(t),\bar{\lambda}\big)\big|\cdot p(t)\cdot|v(t)|q(t)\big)\\
&+\int_0^{T_+}|(\frac{\partial}{\partial x}f(z(t),\nu)-\frac{\partial}{\partial x}f(\bar{x}(t),\bar{\lambda}))v(t)|e^{-\beta t}\mathrm{dt}\\
&+\sup_{t\in J}(|(\frac{\partial}{\partial\lambda}f(z(t),\nu)-\frac{\partial}{\partial\lambda}f(\bar{x}(t),\bar{\lambda}))|e^{-\beta t}|\mu|)\\
&+\int_0^{T_+}|(\frac{\partial}{\partial\lambda}f(z(t),\nu)-\frac{\partial}{\partial\lambda}f(\bar{x}(t),\bar{\lambda}))|e^{-\beta t}|\mu|\mathrm{dt}\\
\leq\ & L_f^1\sup_{t\in J}\big(|z(t)-\bar{x}(t)|q(t)\cdot\rho(t)+|\nu-\bar{\lambda}|\cdot p(t)\big)\cdot||v||^*_{\mathcal{Y}}\\
&+L_f^1\left(\int_0^{T_+}|z(t)-\bar{x}(t)|q(t)\cdot\rho(t)\mathrm{dt}+\int_0^{T_+}p(t)\mathrm{dt}|\nu-\bar{\lambda}|\right)\cdot||v||^*_{\mathcal{Y}}\\
&+L_f^1\sup_{t\in J}\left(|z(t)-\bar{x}(t)|q(t)\cdot p(t)+|\nu-\bar{\lambda}|e^{-\beta t}\right)|\mu|\\
&+L_f^1\left(\int_0^{T_+}|z(t)-\bar{x}(t)|q(t)\cdot p(t)\mathrm{dt}+\int_0^{T_+}e^{-\beta t}\mathrm{dt}|\nu-\bar{\lambda}|\right)|\mu|\\
\leq\ & L_f^1 r(\beta,T_+)(||z-\bar{x}|_J||^*_{\mathcal{Y}}+|\nu-\bar{\lambda}|)\cdot(||v||^*_{\mathcal{Y}}+|\mu|).
\end{aligned}$$

The estimate for (3.54) is

$$\begin{aligned}
&|(\frac{\partial}{\partial x}g(z(0),\nu)-\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda}))v(0)+(\frac{\partial}{\partial\lambda}g(z(0),\nu)-\frac{\partial}{\partial\lambda}g(\bar{x}(0),\bar{\lambda}))\mu|\\
\leq\ & L_g^1(|z(0)-\bar{x}(0)|+|\nu-\bar{\lambda}|)|v(0)|+L_g^1(|z(0)-\bar{x}(0)|+|\nu-\bar{\lambda}|)|\mu|\\
\leq\ & L_g^1(||z-\bar{x}|_J||^*_{\mathcal{Y}}+|\nu-\bar{\lambda}|)\cdot(||v||^*_{\mathcal{Y}}+|\mu|)
\end{aligned}$$

and finally for the term (3.55) we find

$$\begin{aligned}
&|(\bar{M}(T_+,\nu)-\bar{M}(\bar{\lambda})(T_+))v(T_+)|e^{-\beta T_+}\\
&\quad\leq\ L_M|\nu-\bar{\lambda}|\cdot|v(T_+)|e^{-\beta T_+}\leq L_M(|\nu-\bar{\lambda}|)\cdot||v||^*_{\mathcal{Y}}\,p(T_+).
\end{aligned}$$

and

$$
\begin{aligned}
&\Big|\frac{\partial}{\partial\lambda}\bar{M}(T_+,\nu)\mu(z(T_+)-y(\nu)(T_+))-\frac{\partial}{\partial\lambda}\bar{M}(T_+,\bar{\lambda})\mu(\bar{x}(T_+)-y(\bar{\lambda})(T_+))\\
&\qquad -\bar{M}(T_+,\nu)\frac{\partial}{\partial\lambda}y(\nu)(T_+)\mu+\bar{M}(T_+,\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+)\mu\Big|\,e^{-\beta T_+}\\
\le\;& \Big|\frac{\partial}{\partial\lambda}\bar{M}(T_+,\nu)\mu\big(z(T_+)-\bar{x}(T_+)\big)\\
&+\frac{\partial}{\partial\lambda}\bar{M}(T_+,\nu)\mu\big(y(\bar{\lambda})(T_+)-y(\nu)(T_+)\big)\\
&+\big(\frac{\partial}{\partial\lambda}\bar{M}(T_+,\nu)-\frac{\partial}{\partial\lambda}\bar{M}(T_+,\bar{\lambda})\big)\mu\big(\bar{x}(T_+)-y(\bar{\lambda})(T_+)\big)\\
&+\big(\bar{M}(T_+,\bar{\lambda})-\bar{M}(T_+,\nu)\big)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+)\mu\\
&-\bar{M}(T_+,\nu)\big(\frac{\partial}{\partial\lambda}y(\nu)(T_+)\mu-\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+)\mu\big)\Big|\,e^{-\beta T_+}\\
\le\;& M_M^1|z(T_+)-\bar{x}(T_+)|q(T_+)\cdot p(T_+)|\mu|\\
&+M_M^1|\mu|L_y|\bar{\lambda}-\nu|e^{-\beta T_+}\\
&+L_M^1|\mu|\;|\bar{\lambda}-\nu|\;|\bar{x}(T_+)-y(\bar{\lambda})(T_+)|e^{-\beta T_+}\\
&+(L_M M_y^1+M_M L_y^1)|\bar{\lambda}-\nu|\;|\mu|e^{-\beta T_+}\\
\le\;& \max\Big(M_M^1 p(T_+),c_M e^{-\beta T_+}\Big)\cdot||(z-\bar{x}|_J,\nu-\bar{\lambda})||_{\mathcal{Y}}\cdot|\mu|\\
\le\;& \max\big(M_M^1,c_M\big)\cdot||(z-\bar{x}|_J,\nu-\bar{\lambda})||_{\mathcal{Y}}\cdot|\mu|\cdot r(\beta,T_+)
\end{aligned}
$$

where $c_M := L_M^1 C_{\bar{x}} + M_M^1 L_y + L_M M_y^1 + M_M L_y^1$. All this implies that we estimate (3.52) by

$$
\begin{aligned}
&\left\|\Big(G_J'(z,\nu)-G_J'(\bar{x}|_J,\bar{\lambda})\Big)\begin{pmatrix} v\\ \mu\end{pmatrix}\right\|_{\mathcal{Z}}\\
&\quad\le\; C_{Lip}||(z-\bar{x}|_J,\nu-\bar{\lambda})||_{\mathcal{Y}}\cdot||(v,\mu)||_{\mathcal{Y}}\cdot r(\beta,T_+),
\end{aligned}
\tag{3.56}
$$

where $C_{Lip} := C_{\mathcal{Z}}^J\, L_f^1 + L_g^1 + C_{\mathcal{Z}}^+ \max(M_M^1, c_M) > 0$.

Therefore, with $\kappa := \frac{\sigma}{2}$ and $\delta := \frac{\sigma}{2C_{Lip}}$ we see for any $(z,\nu)\in\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J,\bar{\lambda})$

$$
||G_J'(z,\nu)-G_J'(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}\mathcal{Z}}\le\frac{\sigma}{2}=\kappa<\sigma\le||G_J'(x|_J,\bar{\lambda})^{-1}||_{\mathcal{Y}\mathcal{Z}}^{-1}.
$$

S4 The truncation error is

$$\begin{aligned}
||G_J(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Z}} &= \left\| \begin{pmatrix} 0 \\ 0 \\ M(T_+)(\bar{\lambda})(\bar{x}(T_+) - y(\bar{\lambda})(T_+)) \end{pmatrix} \right\|_{\mathcal{Z}} \\
&= C_{\mathcal{Z}}^+ \big| M(T_+,\bar{\lambda})\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big)\big| e^{-\beta T_+} \\
&\leq C_{\mathcal{Z}}^+ \tilde{C}_{tr} ||\bar{x}(T_+) - y(\bar{\lambda})(T_+)||^d e^{-\beta T_+} \\
&\leq C_{\mathcal{Z}}^+ C_{tr} e^{(d\gamma-\beta)T_+}
\end{aligned}$$

where $\tilde{C}_{tr} := \begin{cases} M_M & : d = 1 \\ C_{\bar{M}} & : d = 2, \quad (C_{\bar{M}} \text{ defined in (3.7)}) \end{cases}$ and $C_{tr} := \tilde{C}_{tr} C_{\bar{x}}^d$.

We estimate $r(\beta,T_+)$ defined in (3.20) by $r(\beta,T_+) \leq e^{|\beta|T_+} \cdot \min(1+\frac{1}{|\beta|}, 1+T_+)$, with the interpretation $\min(1+\frac{1}{|\beta|}, 1+T_+) = 1+T_+$ for $\beta = 0$. This implies

$$e^{(d\gamma-\beta)T_+} r(\beta,T_+) \leq e^{(d\gamma-\min(0,2\beta))T_+} \cdot \min(1+\frac{1}{|\beta|}, 1+T_+). \tag{3.57}$$

In the cases ($d=1$, $\gamma < 2\beta$) and ($d=2$, $\gamma < \beta$) we achieve that $d\gamma - \min(0,2\beta) < 0$ and hence there exists $\bar{T}_+ \in \mathcal{D}$ sufficiently large, such that

$$||G_J(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Z}} \leq C_{\mathcal{Z}}^+ C_{tr} e^{(d\gamma-\beta)T_+} \leq (\sigma-\kappa)\frac{\delta}{r(\beta,T_+)} \tag{3.58}$$

holds for all $T_+ \in \mathcal{D}$ and $T_+ \geq \bar{T}_+$.

S5 Now we can apply Lemma 3.2 which implies that there exists a unique solution (z_J,ν_J) in $\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J,\bar{\lambda})$ with

$$||(z_J,\nu_J) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} \leq Ce^{(d\gamma-\beta)T_+},$$

where $C = 2C_{\mathcal{Z}}^+ C_{lin} C_{tr} C_{\bar{x}}$. This implies the estimates in Theorem 3.1 ■

As a direct result of the estimates in Theorem 3.1 a super-convergence phenomenon with respect to the parameter as in the connecting orbit case [Beyn, 1990a], [Beyn, 1990b], [Sandstede, 1997] occurs for $\beta > 0$.

Corollary 3.3 *Suppose that the assumptions of Theorem 3.1 hold. Then the following estimates hold:*

$$\begin{aligned}
|\nu_J - \bar{\lambda}| &\leq Ce^{(d\gamma-\beta)T_+} & & \\
||z_J - \bar{x}|_J||_\infty &\leq Ce^{d\gamma T_+} & &: \beta > 0 \\
||z_J - \bar{x}|_J||_\infty &\leq Ce^{(d\gamma-\beta)T_+} & &: \beta \leq 0.
\end{aligned}$$

Remark: Theorem 3.1 also holds, if we replace β by $\hat{\beta} \in (\frac{\gamma}{2},\beta]$, because all assumptions of the theorem are also satisfied for $\hat{\beta}$. This will be of importance for similar results on generalized connecting orbits in Chapter 5.

3.3 The boundary corrector method

The asymptotic boundary conditions with (3.7) yield "quadratic" error estimates (see Theorem 3.1). On the other hand, e. g. in the periodic case, it can be cumbersome to compute $\bar{M}(T_+, \lambda)$ at $\bar{\lambda}$ since $\bar{\lambda}$ is unknown (see section Section 3.1). But it is realistic to assume

A7 There exists a neighborhood $\mathbf{\Lambda}_0$ of $\bar{\lambda}$ and a matrix valued function $\bar{M}(T_+, \cdot) \in C^1(\mathbf{\Lambda}_0, \mathbb{R}^{m_b,m})$, such that **A6** and (3.7) holds for each $\lambda \in \mathbf{\Lambda}_0$.

In (3.15) we set up the the boundary corrector method for periodic orbits. In this section we define it more generally and analyze the error of the method.

The main idea is to compute $\bar{M}(T_+, \lambda)$ only at the beginning of each step, fix this matrix and approximate a solution with this matrix. Then we take the better approximation of the parameter for the following step. Using the boundary corrector method we have to compute $\bar{M}(T_+, \lambda)$ only twice (case $\beta \geq 0$) or three times (case $\beta < 0$).

Boundary Corrector Method

1. Start with some $\mu_0 \in \mathbf{\Lambda}_0$, $i = 0$

2. Calculate the solution (z_{i+1}, μ_{i+1}) of

$$G_J^{\mu_i}(z, \nu) = \begin{pmatrix} \dot{z} - f(z, \nu) \\ g\big(z(0), \nu\big) \\ \bar{M}(T_+, \mu_i)\big(z(T_+) - y(\nu)(T_+)\big) \end{pmatrix} = 0 \tag{3.59}$$

3. Repeat with **2.** once for $\beta \geq 0$ or twice for $\beta < 0$ with "$i = i + 1$".

The following proposition states that the error of the solution of the boundary corrector method has the same exponential rate $(2\gamma - \beta)$ as the solution of Theorem 3.1 with $d = 2$.

PROPOSITION 3.4 *Suppose that the assumptions of Theorem 3.1 and* **A7** *hold and let* $\bar{T}_+$ *be sufficiently large. Then there exists some* $C_{bcm} > 0$ *such that for* $T_+ \in \mathcal{D}$, $T_+ \geq \bar{T}_+$ *and* $\mu_0 \in \mathbf{\Lambda}_0$ *the result* (z_J, ν_J) *of the* **boundary corrector method** *can be estimated by*

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq C_{bcm} e^{(2\gamma - \beta)T_+}. \tag{3.60}$$

Proof: For each $\mu_i \in \mathbf{\Lambda}_0$ we see that (3.59) is the same as (3.19) with the constant matrix $M(T_+, \nu) := \bar{M}(T_+, \mu_i)$ for all $\nu \in \mathbf{\Lambda}_0$.

By inspection of the proof of Theorem 3.1 and the uniformity assumptions in **A6** we see that we can choose $\delta > 0$ and $\bar{T}_+$ independent of ν such that G_J^{ν} has a unique solution

(z_J^ν, μ_J^ν) in $\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ for each $\nu \in \mathbf{\Lambda}_0$. Furthermore the estimates (3.21) and (3.22) hold with some constant C independent of ν. In particular, the estimates hold with $d = 2$ if $(\bar{z}_J, \bar{\mu}_J) \in \mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ is the (unknown) solution of $G_J^{\bar{\lambda}} = 0$. Using this and (3.27) we estimate for each $\mu \in \mathbf{\Lambda}_0$ the solution $(z_J, \mu_J) \in \mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ of G_J^μ by

$$
\begin{aligned}
&||(z_J,\mu_J) - (\bar{z}_J,\bar{\mu}_J)||_{\mathcal{Y}} \\
\leq\ & \frac{2}{\sigma}||G_J^\mu(z_J,\mu_J) - G_J^\mu(\bar{z}_J,\bar{\mu}_J)||_{\mathcal{Z}} \\
=\ & |\underbrace{\bar{M}(T_+,\mu)(z_J(T_+) - y(\mu_J)(T_+))}_{=0} - \bar{M}(T_+,\mu)(\bar{z}_J(T_+) - y(\bar{\mu}_J)(T_+))|C_{\mathcal{Z}}^+\frac{2}{\sigma}e^{-\beta T_+} \\
=\ & |\underbrace{\bar{M}(T_+,\bar{\lambda})(\bar{z}_J(T_+) - y(\bar{\mu}_J)(T_+))}_{=0} - \bar{M}(T_+,\mu)(\bar{z}_J(T_+) - y(\bar{\mu}_J)(T_+))|C_{\mathcal{Z}}^+\frac{2}{\sigma}e^{-\beta T_+} \\
=\ & |\big(\bar{M}(T_+,\bar{\lambda}) - \bar{M}(T_+,\mu)\big)\big(\bar{z}_J(T_+) - y(\bar{\mu}_J)(T_+)\big)|\ 2\ C_{\mathcal{Z}}^+\ C_{lin}e^{-\beta T_+} \\
\leq\ & ||\bar{M}(T_+,\bar{\lambda}) - \bar{M}(T_+,\mu)|| \cdot \big(\underbrace{|\bar{z}_J(T_+) - \bar{x}|_J(T_+)|}_{\leq Ce^{(2\gamma-\beta)T_+}} \\
& \quad + \underbrace{|\bar{x}|_J(T_+)) - y(\bar{\lambda})(T_+)|}_{\leq C_{\bar{x}}e^{\gamma T_+}} + \underbrace{|y(\bar{\lambda})(T_+) - y(\bar{\mu}_J)(T_+)|}_{\leq L_y Ce^{(2\gamma-\beta)T_+}}\big)\ 2\ C_{\mathcal{Z}}^+\ C_{lin}e^{-\beta T_+} \\
\leq\ & ||\bar{M}(T_+,\bar{\lambda}) - \bar{M}(T_+,\mu)||\ 2\ C_{\mathcal{Z}}^+\ C_{lin}(C + CL_y + C_{\bar{x}})e^{(\gamma-\beta)T_+} \\
\leq\ & |\mu - \bar{\lambda}|C_b e^{(\gamma-\beta)T_+}
\end{aligned}
$$

with $C_b = L_{\bar{M}}\ 2\ C_{\mathcal{Z}}^+\ C_{lin}(C + CL_y + C_{\bar{x}})$.

Applying this estimate to the boundary corrector method we get

$$
\begin{aligned}
||(z_2,\mu_2) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} &\leq ||(z_2,\mu_2) - (\bar{z}_J,\bar{\mu}_J)||_{\mathcal{Y}} + ||(\bar{z}_J,\bar{\mu}_J) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} \\
&\leq |\mu_1 - \bar{\lambda}|C_b e^{(\gamma-\beta)T_+} + Ce^{(2\gamma-\beta)T_+} \\
&\leq CC_b e^{2(\gamma-\beta)T_+} + Ce^{(2\gamma-\beta)T_+}.
\end{aligned}
$$

Here we used $|\mu_1 - \bar{\lambda}| \leq Ce^{(\gamma-\beta)T_+}$ from (3.21). In case $\beta \geq 0$ this implies (3.60) with $C_{bcm} = C(C_b + 1)$. Furthermore we see

$$
\begin{aligned}
||(z_3,\mu_3) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} &\leq ||(z_3,\mu_3) - (\bar{z}_J,\bar{\mu}_J)||_{\mathcal{Y}} + ||(\bar{z}_J,\bar{\mu}_J) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} \\
&\leq |\mu_2 - \bar{\lambda}|C_b e^{(\gamma-\beta)T_+} + Ce^{(2\gamma-\beta)T_+} \\
&\leq CC_b^2 e^{3(\gamma-\beta)T_+} + (C_b e^{(\gamma-\beta)T_+} + 1)Ce^{(2\gamma-\beta)T_+}.
\end{aligned}
$$

Thus $\gamma < 2\beta$ yields (3.60) with $C_{bcm} = CC_b^2 + C(1 + C_b)$ in case $\beta < 0$. ■

3.4 Examples

In the examples we solve the boundary value problems of type (3.10) and (3.15). We use the collocation boundary value solver COLCON and its version COLPAR for parameter continuation by Bader and Kunkel [Bader and Kunkel, 1989].

3.4.1 A first example

This example can be studied analytically and it will be used to test the behavior of the truncation error.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1(\mu - ar^2) - x_2(1+\lambda r^2) \\ x_1(1+\lambda r^2) + x_2(\mu - ar^2) \\ cx_3 + d\mu - \nu r + \frac{\nu}{c} r(\mu - ar^2) \end{pmatrix} =: \hat{f}(x;\lambda,\mu,a,c,d,\nu) \tag{3.61}$$

with $x \in \mathbb{R}^3$, $\mu, a, d, \nu > 0$, $c \neq 0$, $\lambda > -\frac{a}{\mu}$ and $r = \sqrt{x_1^2 + x_2^2}$. In polar coordinates with $\varphi = \arccos(\frac{x_1}{r})$ we get $\dot{r} = r(\mu - ar^2)$, $\dot{\varphi} = 1 + \lambda r^2$ for the first two equations.

We use $\chi(x, \tilde{\varphi}_0) := -\sin(\tilde{\varphi}_0)x_1 + \cos(\tilde{\varphi}_0)x_2 = 0$ as phase condition for periodic orbits (see 1.3.2) and $\tilde{\varphi}_0$ as the extra parameter to fix the phase of the periodic orbit.

For given $\tilde{\varphi}_0$ there exists a periodic solution with $\chi(\tilde{x}(0), \tilde{\varphi}_0) = 0$ namely

$$\tilde{x}(t) = \left(\sqrt{\frac{\mu}{a}} \cos\left(\tilde{\varphi}(t)\right), \sqrt{\frac{\mu}{a}} \sin\left(\tilde{\varphi}(t)\right), -\frac{d\mu}{c} + \frac{\nu}{c}\sqrt{\frac{\mu}{a}}\right)$$

with $\tilde{r}(t) = \sqrt{\frac{\mu}{a}}$, $\tilde{\varphi}(t) = t(1 + \lambda\frac{\mu}{a}) + \tilde{\varphi}_0$. The period is $T = T(\mu, a, \lambda) := \frac{2\pi}{1+\lambda\frac{\mu}{a}}$. Scaling $\hat{f}$ by $f(x;\lambda,\mu,a,c,d,\nu) = T(\lambda,\mu,a)\hat{f}(x;\lambda,\mu,a,c,d,\nu)$ we see that

$$y(t) = \left(\sqrt{\frac{\mu}{a}} \cos(2\pi t + \tilde{\varphi}_0), \sqrt{\frac{\mu}{a}} \sin(2\pi t + \tilde{\varphi}_0), -\frac{d\mu}{c} + \frac{\nu}{c}\sqrt{\frac{\mu}{a}}\right)$$

is a 1–periodic solution of $\dot{x} = f(x;\lambda,\mu,a,c,d,\nu)$ with $\chi\big(y(0), \tilde{\varphi}_0\big) = 0$.

The linearization at the periodic orbit y implies that

$$\zeta(t) = \begin{pmatrix} y_1(t) + \frac{\lambda}{a} y_2(t) & -y_2(t) & 0 \\ y_2(t) - \frac{\lambda}{a} y_1(t) & y_1(t) & 0 \\ \frac{\nu}{c}\sqrt{\frac{\mu}{a}} & 0 & -\frac{\nu}{c}\sqrt{\frac{a}{\mu}} \end{pmatrix} \begin{pmatrix} e^{-2\mu T t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{cTt} \end{pmatrix}$$

solves $\dot{\zeta}(t) = \frac{\partial}{\partial x} f(y(t), \lambda)\zeta(t)$ and

$$\chi^T(t) = \begin{pmatrix} e^{2\mu T t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-cTt} \end{pmatrix} \begin{pmatrix} \frac{a}{\mu} y_1(t) & \frac{a}{\mu} y_2(t) & 0 \\ \frac{1}{\mu}(\lambda y_1(t) - a y_2(t)) & \frac{1}{\mu}(\lambda y_2(t) + a y_1(t)) & 0 \\ y_1(t) & y_2(t) & -\frac{c}{\nu}\sqrt{\frac{\mu}{a}} \end{pmatrix}$$

solves $\dot{\chi}^T(t) = -\chi^T(t)\frac{\partial}{\partial x} f(y(t), \lambda)$ with $\chi^T(t)\zeta(t) = \mathrm{Id}_{\mathbb{R}^3}$.

Thus there are Floquet exponents $-2\mu T < 0 < cT$. Using the periodic solutions y depending on $\tilde{\varphi}_0$ we can define

$$\bar{M}(\tilde{\varphi}_0)(t) = \begin{pmatrix} \frac{1}{\mu}(\lambda y_1(t) - a y_2(t)) & \frac{1}{\mu}(\lambda y_2(t) + a y_1(t)) & 0 \\ y_1(t) & y_2(t) & -\frac{c}{\nu}\sqrt{\frac{\mu}{a}} \end{pmatrix}. \tag{3.62}$$

Then we can apply Theorem 3.1 with $d = 2$. On the other hand the exact solution of $\dot{x} = f(x; \lambda, \mu, a, c, d, \nu)$ with initial value $x_0 \in \mathbb{R}^3$ and $r_0 := \sqrt{x_{0_1}^2 + x_{0_2}^2}$, $\varphi_0 := \arccos(\frac{x_{0_1}}{r_0})$ is

$$\begin{aligned} x(t) &= \Big(r(t)\cos\big(\varphi(t)\big), r(t)\sin\big(\varphi(t)\big), e^{cTt}\big(x_{0_3} + \frac{d\mu}{c} - \frac{\nu}{c}r_0\big) - \frac{d\mu}{c} + \frac{\nu}{c}r(t)\Big), \\ r(t) &= \frac{r_0}{\sqrt{e^{-2\mu T t}(1 - \frac{a r_0^2}{\mu}) + \frac{a r_0^2}{\mu}}}, \\ \varphi(t) &= Tt + \frac{\lambda}{2a}\ln\left(1 + \frac{a r_0^2}{\mu}(e^{2\mu T t} - 1)\right) + \varphi_0. \end{aligned}$$

The solution $\bar{x}$ starting in the plane $\mathcal{W} = \{z \in \mathbb{R}^3 | z_3 = \bar{z}_3\}$, $\bar{z}_3 > 0$ which is $-2\mu T$–asymptotic with y has the initial value

$$\bar{x}_0 = (\bar{r}_0\cos(\bar{\varphi}_0), \bar{r}_0\sin(\bar{\varphi}_0), \bar{z}_3), \quad \text{where } \bar{r}_0 = \frac{c}{\nu}\left(\bar{z}_3 + \frac{d\mu}{c}\right), \ \bar{\varphi}_0 = \tilde{\varphi}_0 - \frac{\lambda}{2a}\ln\left(\frac{a\bar{r}_0^2}{\mu}\right).$$

This yields the phase portrait in Figure 3.3 (which was also shown as Figure 0.2 in the introduction).

In typical applications the phase is not fixed, thus we will use $\tilde{\varphi}_0$ as free parameter. In this case the 2–dimensional manifold $\mathcal{W}$ in $\mathbb{R}^4$ is given by $\{(z, \tilde{\varphi}_0) \in \mathbb{R}^4 | z_1 = \bar{z}_1, z_3 = \bar{z}_3\}$ with $\bar{z}_3 > 0$. The solution has initial values $\bar{x}_0 = (\bar{z}_1, \bar{r}_0\sin(\bar{\varphi}_0), \bar{z}_3)$, where

$$\bar{r}_0 = \frac{c}{\nu}\left(\bar{z}_3 + \frac{d\mu}{c}\right), \bar{\varphi}_0 = \arccos\left(\frac{\bar{z}_1}{\bar{r}_0}\right) \text{ and } \tilde{\varphi}_0 = \bar{\varphi}_0 + \frac{\lambda}{2a}\ln\left(\frac{a\bar{r}_0^2}{\mu}\right) =: \hat{\varphi}.$$

For an approximation (x_J, φ_J) of $(\bar{x}, \hat{\varphi})$ we define the approximation errors by

$$e_{tr}^J(t) := |x_J(t) - \bar{x}(t)|, \ e_{max}(T_+) := \max_{t\in[0,T_+]}\left(e_{tr}^J(t)\right) \text{ and } e_{par}^J(T_+) := |\varphi_J - \hat{\varphi}|.$$

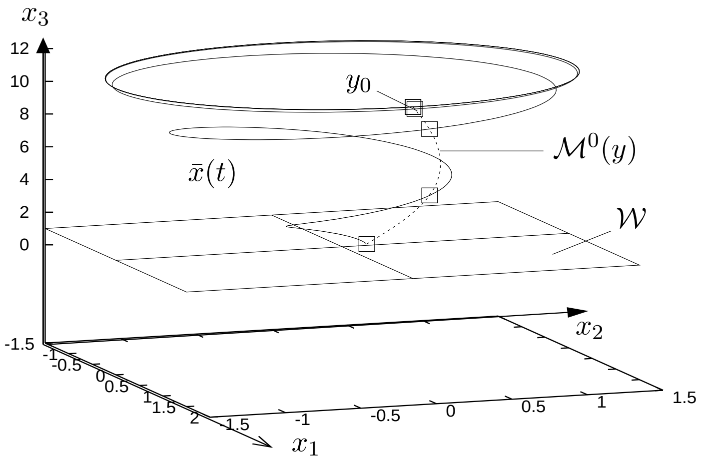

Figure 3.3: The phase portrait of (3.61) with parameters $\mu = 0.25$, $a = \frac{1}{9}$, $\lambda = 0.1$, $c = 0.5$, $d = 1$, $\nu = 4$ and $\tilde{\varphi}_0 = 0$. The connecting orbit $\bar{x}$ starts in a plane $\mathcal{W}$ and converges to the periodic orbit y. It is indicated by (—). The stable set of y_0 (dashed line) is approximated by using $\bar{z}_3$ as continuation parameter in $[1, 11.5]$. The intersection of the connecting orbit $\bar{x}$ with the stable set is $\bar{x}(N), N = 0, 1, \ldots$ marked by the symbol $\square$. The plane $\mathcal{W}$ is defined by $\{z \in \mathbb{R}^3 \mid \bar{z}_3 = 1.0\}$ and the periodic orbit has an initial value $y_0 = (1.5, 0, 11.5)$.

Each curve in Figure 3.4 shows the natural logarithm of the error $e^J_{tr}(t)$ for $J = [0, T_+]$ with different T_+, when we use the boundary corrector method.

In Figure 3.5 we compare the natural logarithm of the errors in the first and the second step of the boundary corrector method.

- -	first step	$\square$	e^J_{par}
—	second step	$\times$	$e_{max}(T_+)$

In this example the period of the periodic orbit is $T = \frac{2\pi}{1+\lambda\frac{\mu}{a}} = 5.129$ and we apply Theorem 3.1 with $\alpha = -2\mu T$, $\beta = 0$. Hence the theoretical exponential rate for the error estimates is $d\alpha - \beta = -d \cdot 2\mu T = -d \cdot 2.565$ (see Corollary 3.3). The slopes in Figure 3.5 taken from $T_+ = 4$ to $T_+ = 6$ are approximatively 2.563 in the first step and 5.15 in the second step (nearly the same for $e_{max}(T_+)$ and e^J_{par}).

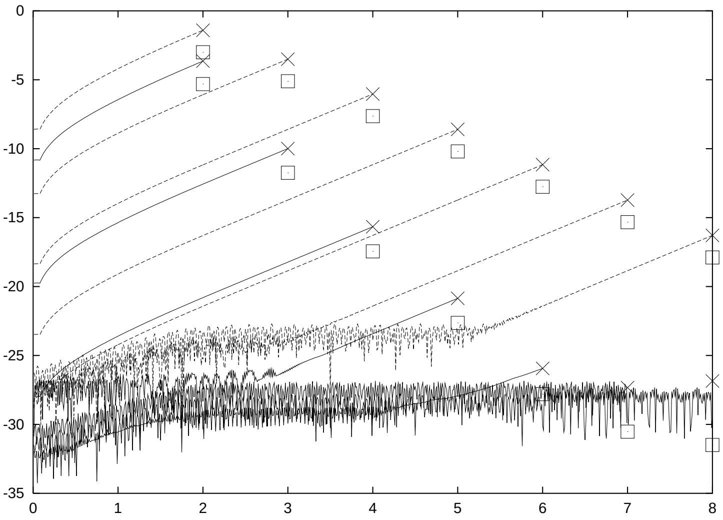

Figure 3.4: The natural logarithm of the error $e^J_{tr}(t)$, $J = [0, T_+]$, $T_+ = 2, \ldots, 8$, for the boundary corrector method. The first approximation with starting parameter $\tilde{\varphi}_0 = 0$ (to compute (3.62)) is represented by (- -). The result of the first step yields the parameter $\tilde{\varphi}_1 = -0.3007$. Using $\bar{M}(\tilde{\varphi}_1)$ from (3.62) we get the second approximation (—).

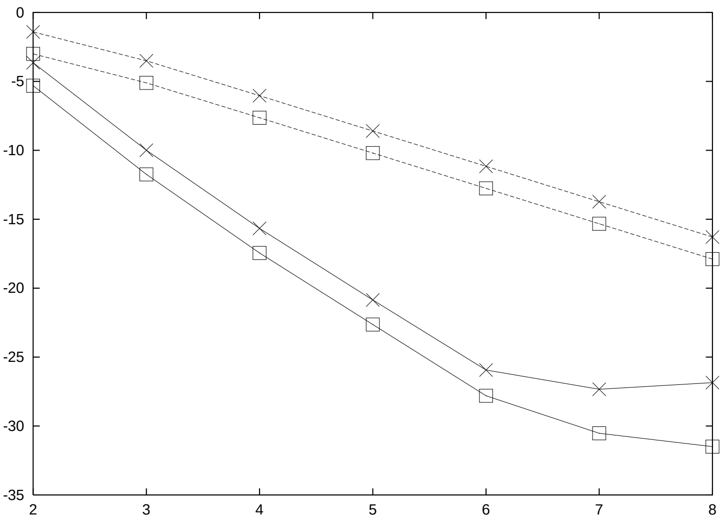

Figure 3.5: The natural logarithm of the errors $e_{max}(T_+)$ ($\times$) and e^J_{par} ($\Box$) for $T_+ = 2, \ldots, 8$ in the first (- - -) and the second (—) step of the boundary corrector method.

To study the error behavior of the method in the **strong-stable** case we use the same example with $c = -0.05 < 0$. The Floquet exponents are $-2\mu T < cT < 0$. The third coordinate of the periodic orbit is $y_3 = -115.0$ and we start in $\mathcal{W} = \{z \in \mathbb{R}^3 \mid \bar{z}_3 = -20.0\}$. This yields the phase portrait in Figure 3.6. Error diagrams for this case are shown in Figure 3.7 and Figure 3.8. The theoretical exponential rate for the error estimates is $-d \cdot (2\mu - c)T = -d \cdot 2.31$. The slopes in Figure 3.8 taken from $T_+ = 4$ to $T_+ = 6$ are approximatively 2.3 and 4.5, respectively (again, nearly the same for $e_{max}(T_+)$ and e^J_{par}).

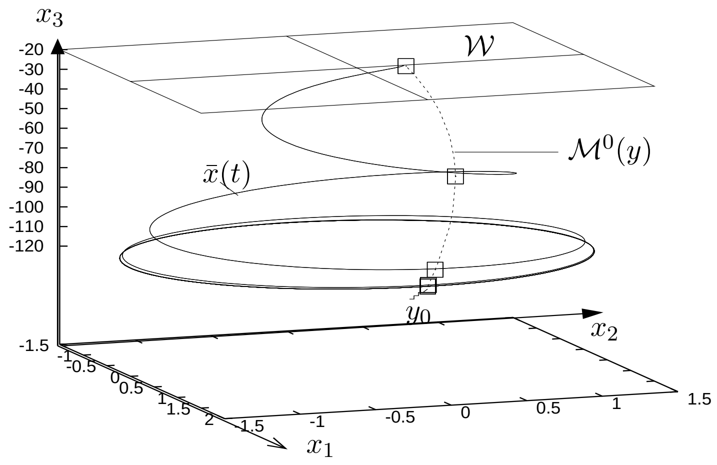

Figure 3.6: The phase portrait of (3.61) with the same parameters as in Figure 3.3, except $c = -0.05$. The connecting orbit is starting in $\mathcal{W} = \{z \in \mathbb{R}^3 \mid \bar{z}_3 = -20.0\}$ and converging in the strong-stable manifold to the periodic orbit y with initial value $y_0 = (1.5, 0, -115)$.

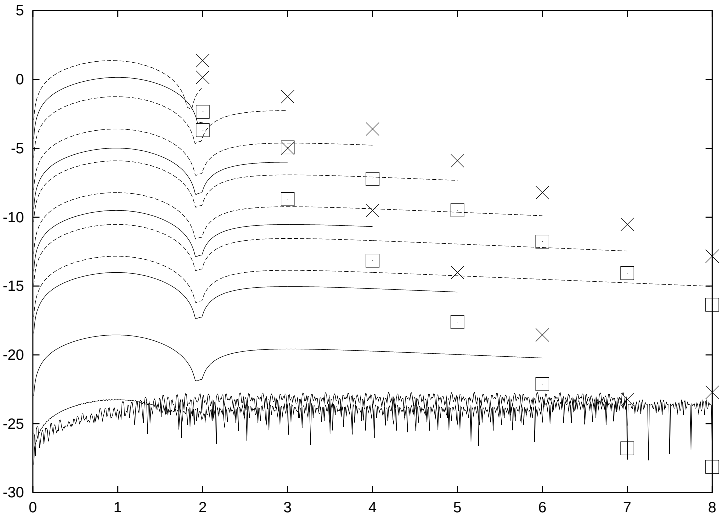

Figure 3.7: The natural logarithm of the error $e_{tr}^{J}(t)$, $J = [0, T_+]$, $T_+ = 2, \ldots, 8$, for the boundary corrector method in the strong-stable case with parameter $c = -0.05$.

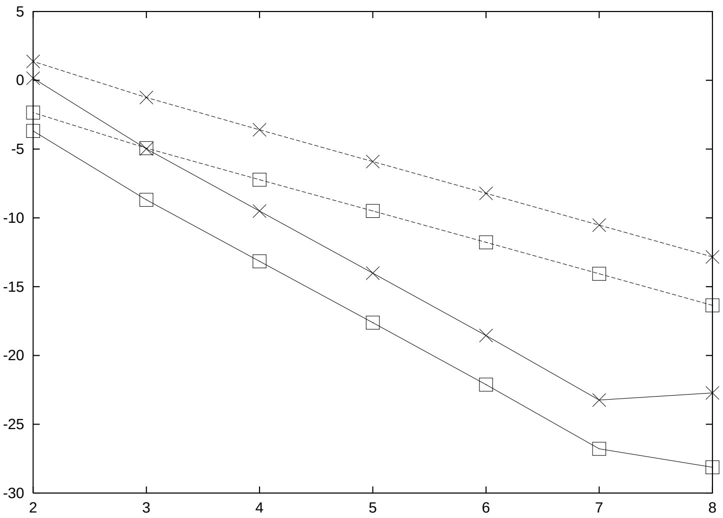

Figure 3.8: The natural logarithm of the errors $e_{max}(T_+)$ and e_{par}^{J} for $T_+ = 2, \ldots, 8$ in the first and the second step of the boundary corrector method in the strong-stable case with parameter $c = -0.05$.

3.4.2 The Lorenz system

For the Lorenz system we use our techniques to approximate the stable manifold of the periodic orbit and the strong stable manifold of the non-trivial equilibrium. Moreover, we set up a procedure to find an initial approximation of a point to periodic connecting orbit (see [Beyn, 1994b]).

We use the result of this procedure later on in Section 5.3 as initial approximation for computing the connecting orbit by the method derived in Chapters 4 and 5.

For further details about the Lorenz system see [Sparrow, 1982].

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1)\\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3\\ \dot{x}_3 &= x_1x_2 - bx_3\end{aligned}$$

We use the parameter values $\sigma = 10$, $b = \frac{8}{3}$ and $r = 24$.

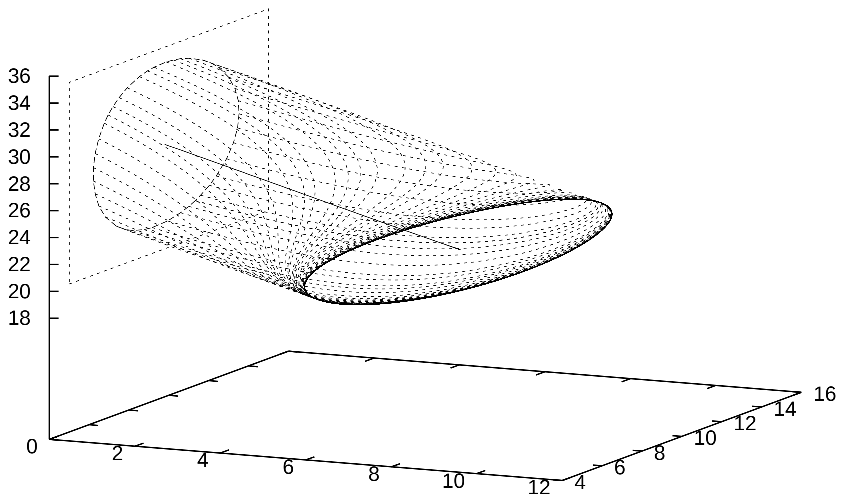

Figure 3.9: Stable manifold of a periodic orbit and strong stable manifold of a non-trivial equilibrium. The parameters are $\sigma = 10$, $b = \frac{8}{3}$ and $r = 24$. All solutions have initial values with $x_1(0) = 0$.

First we approximate connecting orbits starting in the plane with $x_1(0) = 0$, which are either in the stable manifold of a periodic orbit or in the strong-stable manifold of a non-trivial equilibrium. This yields Figure 3.9.

To approximate the strong-stable manifold we apply Theorem 3.1 in an obvious way. The strong-stable manifold has dimension $m_a = 1$, the plane has dimension $m_b = 2$ and $\sigma x_2 > 0$ for $x_2 > 0$ implies non-degeneracy. The approximation is the line (—) in the interior of the cylinder in Figure 3.9.

Remark: Using the parameter r for continuation through the Hopf-bifurcation at $r_H = 24.737$ does not effect the computations. Of course, at this point the strong-stable manifold turns into the stable manifold (which is also 1-dimensional at $r > r_H$).

For the approximation of the stable manifold of a periodic orbit we use the phase as free parameter and fix $x_3(0)$ in addition to $x_1(0)$. As a result we get one of the dashed lines (- - -) in Figure 3.9. Using $x_3(0)$ as continuation parameter yields the other solutions and hence the whole figure.

Remark: To get a "nice picture", we do not use the phase-variable as continuation parameter, because the resulting curves diverge exponentially and do not give a good graphical representation of the surface.

To find an initial approximation of a point to periodic connecting orbit we use the following procedure:

First we approximate the 1–dimensional unstable manifold of the origin for $r = 30$. It intersects the x_2–x_3 plane at $(0, 10.7, 34.3)$. By continuation of the parameter r to $r = 15$ we get a curve $\big(0, x_2(r), x_3(r), r\big)$ which is a 1–dimensional manifold in $\mathbb{R}^4$. Its projection onto $\mathbb{R}^3$ is plotted in Figure 3.10 by (- - -). Then we choose a linear approximation $\big(0, a_2 r, a_3 r, r\big)$ of $\big(0, x_2(r), x_3(r), r\big)$.

The center unstable manifold of the periodic orbit has dimension two, thus the manifold $\mathcal{W}$ must be 2–dimensional to get a solution converging to the periodic orbit. Therefore we use the phase fixing parameter φ as additional parameter and define a 2–dimensional manifold $\mathcal{W}$ by $\big(0, a_2 r, a_3 r, r, \varphi\big)$ in $\mathbb{R}^5$. We approximate a solution with initial value on this plane and converging to the periodic orbit to achieve the orbit plotted by (—) in Figure 3.10. The resulting parameter is $r = 24.05803$. This is in agreement with the values from [Beyn, 1994b].

In Section 5.3 we use this result as initial solution for the approximation method for connecting orbits, which we derive in the forthcoming Chapters 4 and 5.

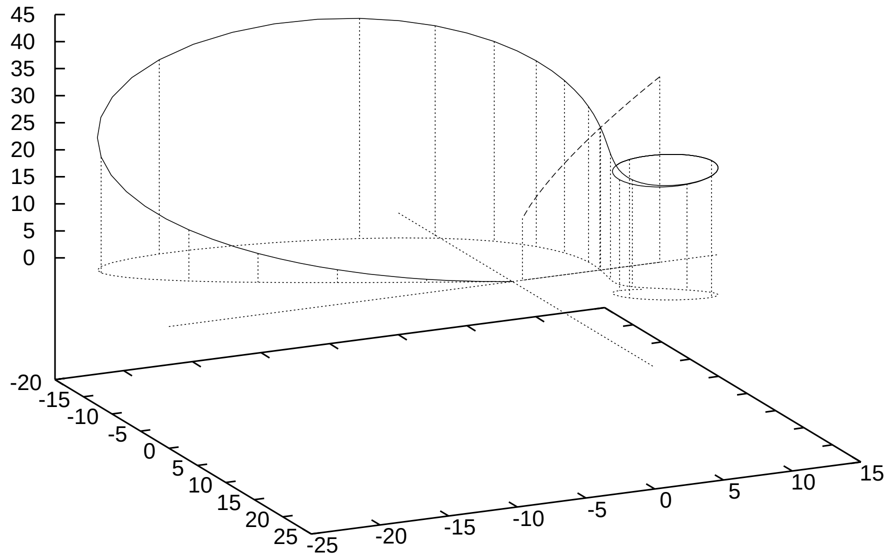

Figure 3.10: Approximation of a point to periodic connecting orbit

Chapter 4

Generalized connecting orbits

Usually in the literature a connecting orbit is a pair (x, λ), where λ is a parameter and x is a solution of $\dot{x} = f(x, \lambda)$ which converges for $t \to \pm\infty$ to given sets. In particular, these sets are either equilibria or periodic orbits. Therefore, the connecting orbit converges to solutions $v_+(\lambda)$ or $v_-(\lambda)$ in these sets, respectively. More precisely, a connecting orbit (x, λ) solves

$$\dot{x} = f(x, \lambda), \qquad \lim_{t \to \pm\infty} |x(t) - v_\pm(\lambda)(t)| = 0. \tag{4.1}$$

We reformulate this and solve for $t \in \mathbb{R}_+$

$$\begin{aligned} \dot{x}_+ &= f_+(x_+, \lambda), \qquad x_+(t) \in \mathbb{R}^m, \\ \dot{x}_- &= f_-(x_-, \lambda), \qquad x_-(t) \in \mathbb{R}^m, \\ & x_+(0) = x_-(0), \\ & \lim_{t \to \infty} |x_\pm(t) - y_\pm(\lambda)(t)| = 0, \end{aligned}$$

where $f_+ = f$, $f_- = -f$ and $y_\pm(\lambda)(t) = v_\pm(\lambda)(\pm t)$. Then $x(t) := x_+(t)$ for $t \geq 0$ and $x(t) := x_-(-t)$ for $t < 0$ solves (4.1) and is smooth in $t = 0$.

Both x_+ and x_- are in the stable manifolds, either of $y_+(\lambda)(t)$ or of the time inverted solution $y_-(\lambda)(-t)$, respectively. Moreover, the initial values $\big(x_+(0), x_-(0), \lambda\big)$ are in a manifold $\mathcal{W}$ of dimension $m + p$ in $\mathbb{R}^{2m+p}$, defined by $x_+(0) = x_-(0)$. Notice that x_+ and x_- are only coupled at $t = 0$ through $\mathcal{W}$.

Though this has the structure of the problems treated in Chapter 3, it is not appropriate to apply those results directly, because the solutions $y_+(\lambda)$ and $y_-(\lambda)$ have, in general, different asymptotic rates. Hence the gap $\alpha < \beta$ might shrink or even vanish.

Therefore, we consider in this chapter a block partitioning of the system where we take the different asymptotic rates into account. Moreover, this allows us to generalize the concept of connecting orbits.

In Section 4.1 we define a **generalized connecting orbit** and its **non-degeneracy** and in Section 4.2 we embed the usual connecting orbits in the concept of generalized connecting orbits and provide some additional applications.

4.1 Nondegenerate generalized connecting orbits

In this section we generalize the concept of connecting orbits and we use a transversality condition to define the **non-degeneracy** of a generalized connecting orbit. Moreover, we relate the non-degeneracy to the non-singularity of a linear operator and to the regularity of an operator equation.

Given two parameterized dynamical systems

$$\begin{aligned} \dot{x}_+ &= f_+(x_+, \lambda), \qquad (4.2) \\ \dot{x}_- &= f_-(x_-, \lambda), \qquad (4.3) \end{aligned}$$

two families of solutions $\{y_+(\lambda)\}_{\mathbf{\Lambda}}$ of (4.2) and $\{y_-(\lambda)\}_{\mathbf{\Lambda}}$ of (4.3) and a manifold $\mathcal{W}$. Then a **generalized connecting orbit** consists of (x_+, x_-, λ) where $(x_+(0), x_-(0), \lambda) \in \mathcal{W}$ and x_+ and x_- are solutions of (4.2) and (4.3), respectively, such that $x_\pm$ converge with an exponential rate $\gamma_\pm < 0$ to a solution $y_\pm(\lambda)$. This means that each pair $(x_+(0), \lambda)$ and $(x_-(0), \lambda)$ is in the γ_+–stable manifold of $y_+(\lambda)$ evaluated at 0 and in the γ_-–stable manifold of $y_-(\lambda)$ evaluated at 0, respectively. In other words, a generalized connecting orbit $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ has the properties

$$(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}) \in \mathcal{W}, \quad (\bar{x}_+(0), \bar{\lambda}) \in \mathcal{M}^0_{\gamma_+}, \quad (\bar{x}_-(0), \bar{\lambda}) \in \mathcal{M}^0_{\gamma_-}, \qquad (4.4)$$

where $\mathcal{M}^0_{\gamma_\pm}$ are the $\gamma_\pm$–stable manifold of $y_\pm$ evaluated at 0 (see Definition 2.17).

Notation: The notion of a generalized connecting orbit is not only applied to connecting orbits where we have a positive and a negative part. Nevertheless we index the first two parts of (x_+, x_-, λ) by "+" and "−". Definitions and assertions which hold for both parts are often indexed by "±", e. g. "$(x_\pm, \lambda)$ solves (4.2), (4.3)" means that (x_+, λ) solves $\dot{x}_+ = f_+(x_+, \lambda)$ and (x_-, λ) solves $\dot{x}_- = f_-(x_-, \lambda)$.

We abbreviate pairs (x_+, x_-) by x and use notions from Chapter 3 by indexing them with "+" or "−", e. g. $\mathcal{M}^0_{\gamma_+}$ is the γ_+–stable manifold of y_+ evaluated at 0 (we set an index "±" only at γ and not in addition at $\mathcal{M}$).

To define a generalized connecting orbit precisely we assume as in Chapter 2 on page 27

A1$_\pm$ (System properties) $f_\pm \in C^k(\mathbb{R}^{m+p}, \mathbb{R}^m)$, $k \geq 2$ and $f_\pm^{(k)}$ is locally Lipschitz with respect to $x_\pm$.

A2$_\pm$ (Solution properties) $y_\pm \in C^k\big(\mathbf{\Lambda}, BC^1(\mathbb{R}_+, \mathbb{R}^m)\big)$ where $\mathbf{\Lambda} \subset \mathbb{R}^p$ is an open, bounded set. Moreover $y_\pm(\lambda)$ is a solution of $\dot{x} = f_\pm(x, \lambda)$, for each $\lambda \in \mathbf{\Lambda}$ and $y_\pm$ is bounded in $\lambda \in \mathbf{\Lambda}$ and $t \in \mathbb{R}_+$.

A3$_\pm$ (Dichotomy condition) $\bar{L}_\pm(\lambda) := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f_\pm\big(y_\pm(\lambda), \lambda\big)$, $\lambda \in \mathbf{\Lambda}$ has a shifted exponential dichotomy with data $\big(\bar{K}_\pm, \bar{\alpha}_\pm, \bar{\beta}_\pm, \bar{P}_a^\pm(\lambda), \bar{P}_b^\pm(\lambda)\big)$, $\bar{\alpha}_\pm < 0$ which are of type C^{k-1} with respect to λ. The ranks of the projectors are independent of λ and given by $m_a^\pm := \dim \mathcal{R}\big(\bar{P}_a^\pm(t)(\lambda)\big)$ and $m_b^\pm := \dim \mathcal{R}\big(\bar{P}_b^\pm(t)(\lambda)\big)$.

We describe the set of points $(x, \lambda) := (x_+, x_-, \lambda)$ where the $x_\pm$ are $\gamma_\pm$–asymptotic with $y_\pm(\lambda)(0)$ by

$$\mathcal{M}_\gamma^0 := \big\{(x_+, x_-, \lambda)\big|\lambda \in \mathbf{\Lambda}, (x_+, \lambda) \in \mathcal{M}_{\gamma_+}^0, (x_-, \lambda) \in \mathcal{M}_{\gamma_-}^0\big\} \tag{4.5}$$

with the interpretation of $\gamma = (\gamma_+, \gamma_-)$ as index.

Later on, we see that $\mathcal{M}_\gamma^0$ is an $(m_a^+ + m_a^- + p)$–dimensional C^{k-1}–manifold.

To define non-degeneracy we assume in addition to **A1**$_\pm$–**A3**$_\pm$, as in Section 2.3 on page 49

A4$_\pm$ (dimension condition) $\mathcal{W}$ is an $(m_b^+ + m_b^-)$–dimensional manifold in $\mathbb{R}^{2m+p}$ with representation $\mathcal{W} = \{(x_+, x_-, \mu) \in \mathbb{R}^{2m+p} | g(x_+, x_-, \mu) = 0\}$, where g is a function with $g \in C^{k-1}(\mathbb{R}^{2m+p}, \mathbb{R}^{m_a^+ + m_a^- + p})$ and $g'(x_+, x_-, \mu)$ has full rank for all $(x_+, x_-, \mu) \in \mathcal{W}$.

A5$_\pm$ (transversality condition) $T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} + T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \mathbb{R}^{2m+p}$.

Assumption **A4**$_\pm$ implies by $2m+p = (m_a^+ + m_a^- + p) + (m_b^+ + m_b^-)$ that **A5**$_\pm$ is equivalent to

$$T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \{0\}.$$

Remark: Arbitrary choices for the number of parameters p and the manifold $\mathcal{W}$ in different applications are defined in Section 4.2

Definition 4.1 (Generalized connecting orbit) *Let* **A1**$_\pm$–**A4**$_\pm$ *hold.*

We call $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ *a* **generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_+, γ_-)** *if* $\bar{x}_\pm$ *and* $y_\pm(\bar{\lambda})$ *are* $\gamma_\pm$*–asymptotic at* $\bar{\lambda}$ *and* $(\bar{x}_+(0), \bar{x}_-(0), \lambda) \in \mathcal{W}$.

If in addition the transversality condition **A5**$_\pm$ *holds then it is called a* **nondegenerate generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_+, γ_-)**.

For each $(x,\lambda)=(x_+,x_-,\lambda)\in\mathcal{W}$ we define a matrix

$$D(x_+,x_-,\lambda):=\left(\frac{\partial}{\partial x_+}g(x,\lambda),\frac{\partial}{\partial x_-}g(x,\lambda),\frac{\partial}{\partial\lambda}g(x,\lambda)\right)\in\mathbb{R}^{m_a^++m_a^-+p,2m+p}. \tag{4.6}$$

Thus, $T_{(x,\lambda)}\mathcal{W}=\mathcal{N}\big(D(x_+,x_-,\lambda)\big)$ follows from the definitions of $\mathcal{W}$ and D.

Now we prove the manifold structure of $\mathcal{M}_\gamma^0$. Similar to Section 2.2 we define a C^{k-1}–atlas on $\mathcal{M}_\gamma^0$ as follows:

As in Section 2.2 we cover $\mathcal{M}_\gamma^0$ by itself and define for each $(\bar{x}_+(0),\bar{x}_-(0),\bar{\lambda})\in\mathcal{M}_\gamma^0$ the indices $\iota_\pm=(\bar{x}_\pm(0),\bar{\lambda})$. Using the notions of Section 2.2 we define a neighborhood of $(0,0,\bar{\lambda})$ by $V\times\Omega:=V_{\iota_+}^+\times V_{\iota_-}^-\times\big(\Omega_{\iota_+}^+\cap\Omega_{\iota_-}^-\big)\subset\mathcal{R}\big(P_a^+(0)\big)\times\mathcal{R}\big(P_a^-(0)\big)\times\mathbb{R}^p$. Moreover, using $x_\pm:=x_{\iota_\pm}$, we define a function $b(\xi_+,\xi_-,\lambda):=\big(x_+(\xi_+,\lambda)(0),x_-(\xi_-,\lambda)(0),\lambda\big)$ on $V\times\Omega$ which maps into

$$W:=\big\{\big(x_+(\xi_+,\lambda)(0),x_-(\xi_-,\lambda)(0),\lambda\big)\;\big|\;(\xi_+,\xi_-,\lambda)\in V\times\Omega\big\}.$$

The inverse of b is

$$\vartheta:\begin{array}{ccc} W & \to & V\times\Omega\\ (z_+,z_-,\lambda) & \mapsto & \big(P_a^+(0)\big(z_+-\bar{x}_+(0)\big),P_a^-(0)\big(z_--\bar{x}_-(0)\big),\lambda\big).\end{array}$$

Thus (W,ϑ) is a chart of $\mathcal{M}_\gamma^0$ at $(\bar{x}_+(0),\bar{x}_-(0),\bar{\lambda})$. Analogously to Section 2.2 we see that all these charts define an atlas $\mathcal{A}_\pm$ of type C^{k-1}. Adding all C^{k-1}–compatible charts we get a maximal atlas $\mathcal{D}_\pm$ of type C^{k-1} and $(\mathcal{M}_\gamma^0,\mathcal{D}_\pm)$ defines a C^{k-1}–manifold of dimension $m_a^++m_a^-+p$.

As in (2.29), we define the tangent map $B(\xi^+,\xi^-,\lambda)$ of b by

$$\begin{array}{ccc}\mathcal{R}(P_a^+(0))\times\mathcal{R}(P_a^-(0))\times\mathbb{R}^p & \to & \mathbb{R}^{2m+p}\\ (\eta_+,\eta_-,\mu) & \to & \begin{pmatrix}\frac{\partial}{\partial\xi^+}x_+(\xi^+,\lambda)(0)\eta_++\frac{\partial}{\partial\lambda}x_+(\xi^+,\lambda)(0)\mu\\ \frac{\partial}{\partial\xi^-}x_-(\xi^-,\lambda)(0)\eta_-+\frac{\partial}{\partial\lambda}x_-(\xi^-,\lambda)(0)\mu\\ \mu\end{pmatrix}.\end{array} \tag{4.7}$$

This implies for the tangent space of $\mathcal{M}_\gamma^0$ at $(x^0,\lambda):=\big(x_+(\xi^+,\lambda)(0),x_-(\xi^-,\lambda)(0),\lambda\big)\in W$ that

$$T_{(x^0,\lambda)}\mathcal{M}_\gamma^0=\mathcal{R}\big(B(\xi^+,\xi^-,\lambda)\big).$$

Using the definitions (4.6) of D and (4.7) of B we obtain:

LEMMA 4.2 *Suppose that* $\mathbf{A1}_{\pm}$–$\mathbf{A4}_{\pm}$ *hold, let* $\bar{x}_{\pm}$ *be* $\gamma_{\pm}$*–asymptotic with* $y_{\pm}(\bar{\lambda})$ *at* $\bar{\lambda}$ *and let* $(x,\lambda) := (x_{+}(\xi^{+},\lambda), x_{-}(\xi^{-},\lambda), \lambda) \in W$ *be a generalized connecting orbit from* $\mathcal{W}$ *to* $y_{\pm}$ *of type* (γ_{+},γ_{-})*. Then* (x,λ) *is nondegenerate if and only if* $D\big(x_{+}(0), x_{-}(0), \lambda\big) \circ B(\xi^{+},\xi^{-},\lambda)$ *is nonsingular.*

Proof: Analogous to the proof of Lemma 2.20. ■

As in Section 2.2 we define for a generalized connecting orbit $(\bar{x}_{+}, \bar{x}_{-}, \bar{\lambda})$ from $\mathcal{W}$ to $y_{\pm}$ a linear operator $D := D(\bar{x}_{+}(0), \bar{x}_{-}(0), \bar{\lambda})$ and for $(\xi^{+},\xi^{-},\lambda) = (0,0,\bar{\lambda})$ we define B by

$$B(\eta_{+},\eta_{-},\mu) := B(0,0,\bar{\lambda})(\eta_{+},\eta_{-},\mu) = \begin{pmatrix} \eta_{+} + \frac{\partial}{\partial\lambda} x_{+}(0,\bar{\lambda})(0)\mu \\ \eta_{-} + \frac{\partial}{\partial\lambda} x_{-}(0,\bar{\lambda})(0)\mu \\ \mu \end{pmatrix}. \tag{4.8}$$

Thus, we obtain a theorem which allows to relate non-degeneracy of generalized connecting orbits (x,λ) from $\mathcal{W}$ to y to both, the non-singularity of $D \circ B$ and the regularity of an operator equation.

THEOREM 4.3 (Nondegenerate generalized connecting orbit) *Suppose that* $\mathbf{A1}_{\pm}$–$\mathbf{A4}_{\pm}$ *hold and let* $(\bar{x},\bar{\lambda})$ *be a generalized connecting orbit from* $\mathcal{W}$ *to* $y_{\pm}$*. Then the following conditions are equivalent:*

1. $(\bar{x},\bar{\lambda})$ *is a nondegenerate generalized connecting orbit from* $\mathcal{W}$ *to* $y_{\pm}$ *of type* (γ_{+},γ_{-})*.*
2. $D \circ B \in \mathcal{L}(\mathcal{R}(P_a^{+}(0)) \times \mathcal{R}(P_a^{-}(0)) \times \mathbb{R}^p, \mathbb{R}^{m_a^{+}+m_a^{-}+p})$ *is nonsingular.*
3. $(0,0,\bar{\lambda})$ *is a regular solution of the operator equation*

$$G_{\infty}(z_{+},z_{-},\lambda) := \begin{pmatrix} \dot{z}_{+} - h_{+}(\cdot, z_{+}, \lambda) \\ \dot{z}_{-} - h_{-}(\cdot, z_{-}, \lambda) \\ g(z(0) + \bar{x}(0) + y(\lambda)(0) - \bar{y}(0), \lambda) \end{pmatrix} = 0 \tag{4.9}$$

where $z + \bar{x} + y(\lambda) - \bar{y} = (z_{+}, z_{-}) + (\bar{x}_{+}, \bar{x}_{-}) + \big(y_{+}(\lambda), y_{-}(\lambda)\big) - (\bar{y}_{+}, \bar{y}_{-})$ *and* $h_{\pm}$ *is defined in terms of* $f_{\pm}$ *as in (2.11).*

Moreover, $G_{\infty} : BC^1_{\gamma_{+}} \times BC^1_{\gamma_{-}} \times \mathbf{\Lambda} \to BC^0_{\gamma_{+}} \times BC^0_{\gamma_{-}} \times \mathbb{R}^{m_a^{+}+m_a^{-}+p}$.

Proof: Similar to the proof of Theorem 2.21. ■

4.2 Connecting orbits and other applications

In the literature ([Beyn, 1994b]) a connecting orbit from one compact invariant sets $\mathcal{V}_-(\lambda)$ to another $\mathcal{V}_+(\lambda)$ is a solution $(\bar{x}, \bar{\lambda})$ on $\mathbb{R}$ of a parameterized dynamical system

$$\dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \Lambda \subset \mathbb{R}^p \tag{4.10}$$

which converges to $\mathcal{V}_\pm(\lambda)$ as $t \to \pm\infty$.

In particular, a connecting orbit is called nondegenerate if a transversality condition holds and the number of parameters is

$$p = m_{+u} - m_{-u} - m_{-c} + 1 = m_{+u} + m_{-s} - m + 1 \tag{4.11}$$

where $m_{\pm s} + m_{\pm c}$, $m_{\pm u} + m_{\pm c}$ are the dimensions of the stable and unstable manifolds of $\mathcal{V}_\pm(\lambda)$ and $m = m_{\pm s} + m_{\pm c} + m_{\pm u}$ holds (see [Beyn, 1994b]).

Connecting orbits where the invariant sets are either equilibria or periodic orbits are analyzed in [Hale and Lin, 1986] and [Beyn, 1994b].

In [Hale and Lin, 1986] "bifurcation functions" are defined and it is proved that connecting orbits exist if and only if the "bifurcation functions" are zero.

In [Beyn, 1994b] the non-degeneracy is related to the regularity of an operator and the non-singularity of a linear map. A similar concept is used here.

An approximation method for periodic to periodic connections, which is similar to ours, is set up in [Champneys and Lord, 1997]. They get numerical results for a specific Hamiltonian system arising from a reduced water-wave problem, but without error estimates for the method.

In our approach we only deal with single orbits on $\mathcal{V}_-(\lambda)$ respectively $\mathcal{V}_+(\lambda)$ depending on λ. In case of periodic solutions we fix the phase of the periodic orbit to get a single solution, whereas equilibria are single orbits. As in Chapter 2 page 49 we may use the phase fixing parameter as additional "free" parameter to find the appropriate phase.

An approximation method for connecting orbits of hyperbolic stationary points is analyzed in [Beyn, 1990a], [Beyn, 1990b].

In our generalization we also approximate connections starting or ending in the strong-unstable respectively strong-stable manifold. For example we can apply this to approximate orbit flip solutions (see [Sandstede, 1993] for details of this codimension-2-bifurcation).

In addition several other applications are also possible. For example we can compute solutions with a discontinuity at time $t = 0$ and we can detect so called "Skiba" points in control theory (see Chapter 6).

Throughout this section we define appropriate frameworks to show that all these applications fit under the notion of "generalized connecting orbits".

4.2.1 Point to point connecting orbits

The case of connecting orbits of hyperbolic stationary points is analyzed in [Beyn, 1990a], [Beyn, 1990b]. As generalization we also consider cases where at one side or at both sides the connecting orbit converges along the strong stable respectively strong unstable manifold, for example the approximation of an orbit flip solution (see [Sandstede, 1993]).

In this section we set up an appropriate reformulation of a nondegenerate point to point connecting orbit as defined in [Beyn, 1990b], such that it is a nondegenerate generalized connecting orbit.

Let $(\bar{x}, \bar{\lambda})$ be a nondegenerate connecting orbit pair in the sense of [Beyn, 1990b], i. e. the following holds:

1. $\bar{x}$ solves $\dot{x} = f(x, \bar{\lambda})$ on $\mathbb{R}$ and $\lim_{t\to\pm\infty} x(t) = z_\pm$.
2. The matrices $A_\pm = \lim_{t\to\pm\infty} \frac{\partial}{\partial x} f(\bar{x}(t), \bar{\lambda}) = \frac{\partial}{\partial x} f(z_\pm, \bar{\lambda})$ are hyperbolic.
3. The number of parameters is $p = -m + 1 + m_{-s} + m_{+u}$, where $m_{\pm s}$, $m_{\pm u}$ are the dimensions of the stable and unstable manifolds of $A_\pm$.
4. For any $(y, \mu) \in BC^1(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^p$ where $\lim_{t\to\pm\infty} y(t)$ exist and which solve
$$\dot{y} = \frac{\partial}{\partial x} f(\bar{x}, \bar{\lambda}) y + \frac{\partial}{\partial \lambda} f(\bar{x}, \bar{\lambda}) \mu \tag{4.12}$$
the following holds: $\mu = 0$, $y = c\dot{\bar{x}}$ for some constant c.

In addition define a phase fixing function $\chi \in C^k(\mathbb{R}^m \times \mathbf{\Lambda}, \mathbb{R})$ satisfying
$$\chi(\bar{x}(0), \bar{\lambda}) = 0, \quad \frac{\partial}{\partial x}\chi(\bar{x}(0), \bar{\lambda})\dot{\bar{x}}(0) \neq 0, \tag{4.13}$$
e. g. $\chi(x, \bar{\lambda}) = \dot{\hat{x}}(0)^\top \big(x - \hat{x}(0)\big)$, where $\hat{x}$ is a given approximation.

In [Beyn, 1990b] a more general phase fixing functions are allowed, but for the setup of generalized connecting orbits we have to restrict to this kind of functions.

Define $f_+ = f$, $f_- = -f$ and $x_+(t) = \bar{x}(t)$, $x_-(t) = \bar{x}(-t)$ for $t \in \mathbb{R}_+$. Then x_+ is in the stable manifold of z_+ and x_- is in the stable manifold of z_- of the time reversed system (this is the unstable manifold of the original system). Thus we have the correspondences $m_b^+ \widehat{=} m_{+u}$ and $m_b^- \widehat{=} m_{-s}$. This implies with $g(x_+, x_-, \mu) = \binom{x_+ - x_-}{\chi(x_+,\mu)}$ that

$$\mathcal{W} = \{(x_+, x_-, \mu) \in \mathbb{R}^{2m+p} | g(x_+, x_-, \mu) = 0\}$$

is a manifold in $\mathbb{R}^{2m+p}$ of dimension $2m + p - (m+1) = m + p - 1 = m_b^+ + m_b^-$.

In addition, any solution $(v_+(0), v_-(0), \mu)$ in

$$T_{(x_+(0), x_-(0), \bar{\lambda})} \mathcal{W} \cap T_{(x_+(0), x_-(0), \bar{\lambda})} \mathcal{M}_\gamma^0$$

satisfies $v_+(0) = v_-(0)$ and the composite function $v(t) = \frac{\partial}{\partial \xi} x_+(0, \bar{\lambda})(t) \xi_a^+ + \frac{\partial}{\partial \lambda} x_+(0, \bar{\lambda})(t) \mu$ for $t \geq 0$ and $v(t) = \frac{\partial}{\partial \xi} x_-(0, \bar{\lambda})(-t) \xi_a^- + \frac{\partial}{\partial \lambda} x_-(0, \bar{\lambda})(-t) \mu$ for $t \leq 0$ is a solution of (4.12). Thus by non-degeneracy and (4.13) we see $(v_+(0), v_-(0), \mu) = 0$ and therefore non-degeneracy for the generalized connecting orbit.

For the strong stable, respectively strong unstable case we use the same setup, only the dimensions m_b^+ and m_b^- and hence the appropriate number p of parameters changes.

4.2.2 Point to periodic and periodic to periodic connecting orbits

This kind of connecting orbit and its non-degeneracy is analyzed in [Beyn, 1994b].

For the point to periodic connecting orbits we assume that $T(\lambda)$–periodic solutions $y(\lambda)$ are given and notice that the phase of each of these orbits is fixed.

To define the dynamical systems we set $f_+(x, \lambda) = T(\lambda) f(x, \lambda)$, $f_-(x, \lambda) = -f(x, \lambda)$ and $x_+(t) = \bar{x}(t)$, $x_-(t) = \bar{x}(-t)$ for $t \in \mathbb{R}_+$. This implies that x_+ is in the stable manifold of a periodic orbit $y(\lambda)(\mathbb{R})$ and x_- is in the stable manifold of z_- of the time reversed system (this is the unstable manifold of the original system). Thus we have the correspondences $m_b^+ \widehat{=} m_{+c} + m_{+u} = m_{+u} + 1$ and $m_b^- \widehat{=} m_{-s}$. This and (4.11) implies that

$$\mathcal{W} = \{(x_+, x_-, \mu) \in \mathbb{R}^{2m+p} | x_+ - x_- = 0\} \tag{4.14}$$

is a manifold in $\mathbb{R}^{2m+p}$ of dimension $m + p = m_{+u} + m_{-s} + 1 = m_b^+ + m_b^-$.

Here the periodic orbit is fixed, thus we do not need an additional phase fixing function.

Similarly, in order to define periodic to periodic connections we assume that the $T_\pm(\lambda)$–periodic solutions $y_\pm(\lambda)$ are given and notice that the phase of each of these orbits is fixed.

Moreover, we define $f_\pm(x,\lambda) = \pm T_\pm(\lambda) f(x,\lambda)$ and $x_\pm(t) = \bar{x}(\pm t)$ for $t \in \mathbb{R}_+$. Then x_+ is in the stable manifold of a periodic orbit $y_+(\lambda)(\mathbb{R})$ and x_- is in the stable manifold of $y_-(\lambda)(\mathbb{R})$ of the time reversed system (this is the unstable manifold of the original system). Thus we have the correspondences $m_b^+ \widehat{=} m_{+c} + m_{+u} = m_{+u} + 1$ and $m_b^- \widehat{=} m_{-s} + m_{-c}$. The relation (4.11) yields $m_b^+ + m_b^- = m + p + 1$. This is not surprising, because we have two phase fixing conditions, one at each periodic orbit. Thus we have to introduce an additional "free" parameter (similar to the remark on page 49). A suitable choice is a phase parameter τ for one of the periodic orbits. This implies that

$$\mathcal{W} = \{(x_+, x_-, \mu, \tau) \in \mathbb{R}^{2m+p+1} | x_+ - x_- = 0\} \tag{4.15}$$

is a manifold in $\mathbb{R}^{2m+p+1}$ of dimension $m + p + 1 = m_{+u} + m_{-s} + 1 = m_b^+ + m_b^-$.

4.2.3 Solutions with discontinuity

The splitting at 0 allows discontinuities, e. g. f_+ and f_- do not necessarily have the form $f_\pm = a_\pm f$ as in the connecting orbit cases before. We also might approximate solutions with "jumps" at 0, in this case the condition at 0 is $x_+(0) - x_-(0) = v$ with $v \in \mathbb{R}^m$. This can be used find initial approximations for connecting orbits as follows: First compute solutions which satisfy all conditions of a connecting orbit except $x_+(0) = x_-(0)$ and define the difference vector v. Then use the components of v as continuation parameters and try to continue v to 0. Of course it is not clear whether such (global) continuation works or not, nevertheless it is a heuristic approach for getting initial approximations.

A similar method for locating connecting orbits is developed in [Doedel et al., 1997]. It is called "successive continuation" and a local convergence analysis for this method is presented.

4.2.4 Skiba points

The control problems discussed in Chapter 6, lead to $m = 2n$–dimensional dynamical systems, where the first n components are fixed. We want to approximate two solutions $x_\pm = (x_\pm^s, x_\pm^c) \in \mathbb{R}^{2n}$ of $\dot{x} = f(x)$ which converge to different solutions $y_\pm$, but which satisfy $x_+^s(0) = x_-^s(0) = v$ for given $v \in \mathbb{R}^n$. Typically, each $y_\pm$ is either an equilibrium with an n–dimensional stable manifold (i. e. $m_b^\pm = n$) or a periodic orbit with an $(n-1)$–dimensional stable manifold (i. e. $m_b^\pm = n + 1$). Thus we set $f_\pm = f$ and

$$g\big(x_+(0), x_-(0)\big) = \begin{pmatrix} x_+(0) - v \\ x_-(0) - v \end{pmatrix} \in \mathbb{R}^{2n}.$$

In the periodic case we add the phase of each periodic orbit as "free" parameter (see remark on page 49), i. e. we have $p = 0$ parameter if the $y_\pm$ are both equilibria, $p = 1$ parameter if one of the $y_\pm$ is a periodic orbit and $p = 2$ parameters if both $y_\pm$ are periodic orbits.

Thus we see that

$$\mathcal{W} = \{(x_+, x_-, \lambda) \in \mathbb{R}^{2m+p} | g(x_+, x_-) = 0\} \tag{4.16}$$

is a manifold in $\mathbb{R}^{2m+p}$ of dimension $2m + p - m = m_b^+ + m_b^-$.

The aim is to compare the values of an objective function $\hat{U}$ for both trajectories. In particular, we want to "free" one component v_i and approximate solutions which satisfy $\hat{U}(x_+(0)) = \hat{U}(x_-(0))$.

Thus we substitute for a given index the side conditions $x_{+_i}(0) - v_i = 0$ and $x_{-_i}(0) - v_i = 0$ by $x_{+_i}(0) - x_{-_i}(0) = 0$ and $\hat{U}(x_+(0)) = \hat{U}(x_-(0))$.

For further details see Chapter 6.

Chapter 5

The approximation of generalized connecting orbits

A numerical method for approximating generalized connecting orbits on finite intervals is presented in this chapter. Moreover, we analyze the error which is shown to decay exponentially with the length of the intervals.

In Section 5.2 we present the boundary corrector method for generalized connecting orbits. The computation of the point to periodic solution in the Lorenz system is shown in Section 5.3 and in Section 5.4 we give some concluding remarks on the theory of generalized connecting orbits.

5.1 The approximation theorem

In this section we set up an approximation theory for generalized connecting orbits. The technique is the same as in Section 3.2. We truncate $\mathbb{R}_+$ to finite intervals $J_+ = [0, T_+]$ and $J_- = [0, T_-]$ and approximate both part $\bar{x}_+$ and $\bar{x}_-$ of a generalized connecting orbit on J_+ and J_-, respectively. At T_+ and T_- we use asymptotic boundary conditions. The proof and a lot of estimates are similar to those in Section 3.2.

Assume **A1**$_\pm$–**A4**$_\pm$ and let $(\bar{x}, \bar{\lambda}) = (\bar{x}_+, \bar{x}_-, \bar{\lambda})$ be a nondegenerate generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_-, γ_+) $(\bar{\alpha}_\pm < \gamma_\pm < \min(0, 2\bar{\beta}_\pm))$.

Moreover, suppose that the linearizations $L_\pm := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f_\pm(\bar{x}_\pm, \bar{\lambda})$ have shifted exponential dichotomies with exponents $\alpha_\pm$, $\beta_\pm$ $(\alpha_\pm < \gamma_\pm < 2\beta_\pm)$ near $\bar{\alpha}_\pm$, $\bar{\beta}_\pm$, projectors $P_a^\pm$, $P_b^\pm$ and constants $K_\pm$.

Furthermore, as in **A6** (see page 57), we assume that the boundary conditions are regular.

A6$_\pm$ (Regular boundary condition) Let $M_\pm(T_\pm,\cdot) \in C^1\big(\mathbf{\Lambda}, \mathbb{R}^{m_b^+ + m_b^-, m}\big)$ be matrix valued functions with $\mathcal{N}\big(M_\pm(T_\pm,\bar\lambda)\big) \cap \bar{\mathcal{R}}_b^\pm = \{0\}$ for all $T_\pm \in \mathcal{D}_\pm$, where $\mathcal{D}_\pm \subset \mathbb{R}_+$ are sets as in **A6**. Furthermore, assume that $M_\pm(T_\pm,\lambda)$ and $\frac{\partial}{\partial\lambda}M_\pm(T_\pm,\lambda)$ are uniformly bounded by $M_{M_\pm}$, $M^1_{M_\pm}$ and Lipschitz continuous with constants $L_{M_\pm}$, $L^1_{M_\pm}$. Moreover, let the inverses $\bar N_\pm := (M_\pm(T_\pm,\bar\lambda)|_{\bar{\mathcal{R}}_b^\pm})^{-1}$ be bounded by $M_{\bar N_\pm}{}^{-1} > 0$ uniformly in $T_\pm \in \mathcal{D}_\pm$.

As in Section 3.2, we see that the $M_\pm(T_\pm,\bar\lambda)|_{\mathcal{R}_b^\pm}$ are nonsingular and that their inverses $N_\pm := \big(M_\pm(T_\pm,\bar\lambda)|_{\mathcal{R}_b^\pm}\big)^{-1}$ are uniformly bounded by $M_{N_\pm} := 4KM_\pm{}^{-1} > 0$ for sufficiently large $T_\pm \in \mathcal{D}_\pm$.

For some $T_\pm \in \mathcal{D}_\pm$, $J_\pm := [0, T_\pm]$ (notice that $T_- > 0$) we define Banach spaces
$\mathcal{Y} = C^1(J_+,\mathbb{R}^m) \times C^1(J_-,\mathbb{R}^m) \times \mathbb{R}^p$ with norm

$$\begin{aligned} ||(x_+,x_-,\lambda)||_{\mathcal{Y}} &= ||x_+||^+_{\mathcal{Y}} + ||x_-||^-_{\mathcal{Y}} + |\lambda|, \\ ||x_\pm||^\pm_{\mathcal{Y}} &= \sup_{t\in J_\pm}\big(|x_\pm(t)|q_\pm(t)\big), \quad q_\pm(t) = \frac{2}{1+e^{\beta_\pm t}} \end{aligned}$$

and $\mathcal{Z} = C^0(J_+,\mathbb{R}^m) \times C^0(J_-,\mathbb{R}^m) \times \mathbb{R}^{m_a^+ + m_a^- + p} \times \mathbb{R}^{m_b^+} \times \mathbb{R}^{m_b^-}$ with norm

$$\begin{aligned} &||(v_+,v_-,r_0,r_+,r_-)||_{\mathcal{Z}} \\ &\quad = C^{J_+}_{\mathcal{Z}}||v_+||^+_{\mathcal{Z}} + C^{J_-}_{\mathcal{Z}}||v_-||^-_{\mathcal{Z}} + |r_0| + C^+_{\mathcal{Z}}|r_+|e^{-\beta_+T_+} + C^-_{\mathcal{Z}}|r_-|e^{-\beta_-T_-}, \end{aligned}$$

where

$$\begin{aligned} C^{J\pm}_{\mathcal{Z}} &= (K_\pm + M_{M_\pm}\, M_{N_\pm}\, K_\pm^2)||R_0^\pm||, \\ C^\pm_{\mathcal{Z}} &= M_{N_\pm}\, K_\pm\, ||R_0^\pm||, \\ ||v_\pm||^\pm_{\mathcal{Z}} &= ||v_\pm||_{\beta_\pm} + \int_0^{T_\pm} |v_\pm(s)|e^{-\beta_\pm s}\mathrm{d}s, \quad ||v_\pm||^\pm_{\beta_\pm} = \sup_{t\in J_\pm}(|v_\pm(t)|e^{-\beta_\pm t}. \end{aligned}$$

Here we define $(R_0^+\ R_0^-\ Q_0) := D(\bar x_+(0), \bar x_-(0), \bar\lambda)$ as in (4.6). Defining B as in (4.8) we obtain from Theorem 4.3 that

$$D \circ B(\xi_a^+,\xi_a^-,\mu) = R_0^+\big(\xi_a^+ + \frac{\partial}{\partial\lambda}x_+(0,\bar\lambda)(0)\mu\big) + R_0^-\big(\xi_a^- + \frac{\partial}{\partial\lambda}x_-(0,\bar\lambda)(0)\mu\big) + Q_0\mu \quad (5.1)$$

is nonsingular. Hence $D \circ B(\xi^+,\xi^-,0) = R_0^+\xi^+ + R_0^-\xi^-$ implies $\mathrm{rank}(R_0^\pm) > 0$ and $||R_0^\pm|| > 0$.

Using the definition of $r^\pm$ as in (3.20) and $(\beta,T) := (\beta_+,\beta_-,T_+,T_-)$, we define

$$\begin{aligned} r(\beta,T) &:= r^+(\beta_+,T_+) + r^-(\beta_-,T_-), \\ r^\pm(\beta_\pm,T_\pm) &:= \begin{cases} e^{|\beta_\pm|T_\pm} + \frac{1}{|\beta_\pm|}\big(e^{|\beta_\pm|T_\pm} - 1\big) & : \beta_\pm \neq 0 \\ 1 + T_\pm & : \beta_\pm = 0 \end{cases}. \end{aligned} \quad (5.2)$$

Before setting up the theorem we define a set of pairs (T_-, T_+) for which the estimate

$$e^{\Delta_+ T_+} + e^{\Delta_- T_-} \leq \frac{\epsilon}{r^+(\beta_+, T_+) + r^-(\beta_-, T_-)} \tag{5.3}$$

holds (see Figure 5.1). The set depends on $\epsilon > 0$, $\beta_\pm \in \mathbb{R}$ and $\Delta_\pm < 0$. Moreover we assume $\Delta_\pm < -|\beta_\pm|$.

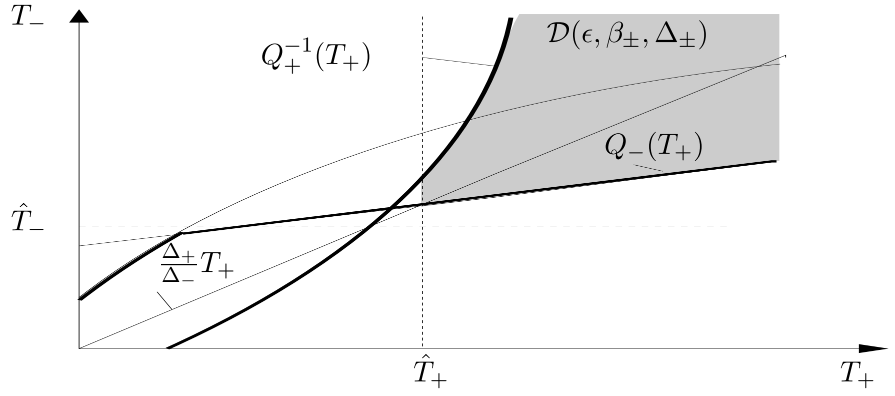

Figure 5.1: Typical diagram for $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$, where $\beta_\pm \neq 0$.

As in (3.57) we see $r^\pm(\beta_\pm, T_\pm) \leq e^{|\beta_\pm| T_\pm} \cdot q(\beta_\pm, T_\pm)$ with $q(\beta_\pm, T_\pm) = \min(1 + \frac{1}{|\beta_\pm|}, 1 + T_\pm)$, for $\beta_\pm \neq 0$ and $q(0, T_\pm) = 1 + T_\pm$. Therefore, (5.3) holds for each pair (T_+, T_-) which satisfies

$$\left(e^{\Delta_+ T_+} + e^{\Delta_- T_-}\right)\left(e^{|\beta_+| T_+} q(\beta_+, T_+) + e^{|\beta_-| T_-} q(\beta_-, T_-)\right) \leq \epsilon. \tag{5.4}$$

The estimate (5.4) is satisfied, if

$$\Delta_+ T_+ + |\beta_-| T_- + \ln\left(q(\beta_-, T_-)\right) \quad \leq \quad \ln\left(\frac{\epsilon}{4}\right), \tag{5.5}$$

$$\Delta_- T_- + |\beta_+| T_+ + \ln\left(q(\beta_+, T_+)\right) \quad \leq \quad \ln\left(\frac{\epsilon}{4}\right), \tag{5.6}$$

$$(\Delta_\pm + |\beta_\pm|) T_\pm + \ln\left(q(\beta_\pm, T_\pm)\right) \quad \leq \quad \ln\left(\frac{\epsilon}{4}\right) \tag{5.7}$$

holds. First we choose the minimal $\hat{T}_\pm \in \mathbb{R}_+$, such that (5.7) holds for all $T_\pm \geq \hat{T}_\pm$. Then we define

$$Q_+(T_-) = \frac{|\beta_-| T_- + \ln\left(q(\beta_-, T_-)\right) - \ln(\frac{\epsilon}{4})}{-\Delta_+}, \; Q_-(T_+) = \frac{|\beta_+| T_+ + \ln(q(\beta_+, T_+)) - \ln(\frac{\epsilon}{4})}{-\Delta_-}$$

and obtain that (5.5) (5.6) holds in the domain

$$\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm) = \left\{(T_-, T_+) \;\middle|\; T_+ \geq \max\left(\hat{T}_+, Q_+(T_-)\right), \;\; \text{and } T_- \geq \max\left(\hat{T}_-, Q_-(T_+)\right)\right\}.$$

If $\beta_\pm \neq 0$ and $T_\pm \geq \frac{1}{|\beta_\pm|}$, then the functions Q_+ and Q_- are linear with slopes $\frac{|\beta_-|}{-\Delta_+}$ and $\frac{|\beta_+|}{-\Delta_-}$, respectively. Therefore the assumption $\Delta_\pm < -|\beta_\pm|$ implies that $|\beta_+|\cdot|\beta_-| < \Delta_+ \cdot \Delta_-$ and hence $\frac{|\beta_+|}{-\Delta_-}$, the slope of Q_- is less than $\frac{-\Delta_+}{|\beta_-|}$, the slope of Q_+^{-1}. Thus, Q_- and Q_+^{-1} intersect and $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm) \neq \emptyset$. In particular, for each $a \in (\frac{|\beta_+|}{-\Delta_-}, \frac{-\Delta_+}{|\beta_-|})$ there exists some $\bar{T}$, such that $(T, aT) \in \mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$ for all $T \geq \bar{T}$. Similar results hold for $\beta_+ = 0$ or $\beta_- = 0$ by using definition of $Q_\pm$.

Thus we obtain the following Lemma:

Lemma 5.1 *Let $\epsilon > 0$, $\beta_\pm \in \mathbb{R}$, and $\Delta_\pm < -|\beta_\pm|$. Then the inequality (5.3) is satisfied for all $(T_+, T_-) \in \mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$ and $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm) \neq \emptyset$*

Typically, ϵ is small. If $\epsilon < 4$, then $\hat{T}_\pm$ satisfy (5.7) with equality and $Q_+(\hat{T}_-) = \frac{\Delta_-}{\Delta_+}\hat{T}_-$, $Q_-(\hat{T}_+) = \frac{\Delta_+}{\Delta_-}\hat{T}_+$.

Remark: Typically $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$ has a large intersection with $\mathcal{D}_+ \times \mathcal{D}_-$, e. g. as in our remark on page 58.

Theorem 5.2 *Suppose that* **A1**$_\pm$–**A6**$_\pm$ *hold, let $d_\pm = 1$ and let $(\bar{x}, \bar{\lambda})$ be a nondegenerate generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_-, γ_+).*

Let the linear operators $L_\pm := \frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial}{\partial x} f_\pm(\bar{x}_\pm, \bar{\lambda})$ have shifted exponential dichotomies with dichotomy data $(K_\pm, \alpha_\pm, \beta_\pm, P_a^\pm, P_b^\pm)$ and $\alpha_\pm < \gamma_\pm < \min(0, 2\beta_\pm)$.

Then there exist constants $\delta > 0$, $C^\pm > 0$, $\epsilon > 0$ and $\bar{T}_\pm$ sufficiently large, such that for all $(T_+, T_-) \in \mathcal{D}(\epsilon, \beta_\pm, d_\pm\gamma_\pm - \beta_\pm) \cap \mathcal{D}_+ \times \mathcal{D}_-$, $T_\pm \geq \bar{T}_\pm$ the operator equation

$$H_J(z_+, z_-, \nu) = \begin{pmatrix} \dot{z}_+ - f_+(z_+, \nu) \\ \dot{z}_- - f_-(z_-, \nu) \\ g\big(z_+(0), z_-(0), \nu\big) \\ M^+(T_+, \nu)(z_+(T_+) - y_+(\nu)(T_+)) \\ M^-(T_-, \nu)(z_-(T_-) - y_-(\nu)(T_-)) \end{pmatrix} = 0 \tag{5.8}$$

has a unique solution $(z_J, \nu_J) = (z_J^+, z_J^+, \nu_J)$ in $\mathcal{B}_{\frac{\delta}{r(\beta,T)}}(\bar{x}_+|_{J_+}, \bar{x}_-|_{J_-}, \bar{\lambda})$ in $\mathcal{Y}$.

Furthermore, the following pointwise estimates with $d_\pm = 1$ hold

$$\begin{aligned} |\nu_J - \bar{\lambda}| &\leq C^+ e^{(d_+\gamma_+ - \beta_+)T_+} + C^- e^{(d_-\gamma_- - \beta_-)T_-}, && (5.9) \\ ||z_J^\pm(t) - \bar{x}_\pm|_{J_\pm}(t)|| &\leq \left(C^+ e^{(d_+\gamma_+ - \beta_+)T_+} + C^- e^{(d_-\gamma_- - \beta_-)T_-}\right) \begin{cases} e^{\beta_\pm t} & : \beta_\pm > 0 \\ 1 & : \beta_\pm \leq 0. \end{cases} && (5.10) \end{aligned}$$

If $M^+(T_+, \lambda)$ satisfies

$$|\bar{M}^+(T_+,\bar{\lambda})(\bar{x}(T_+) - y(\bar{\lambda})(T_+))| \le C^+_{\bar{M}} |\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)|^2 \tag{5.11}$$

for a constant $C^+_{\bar{M}} > 0$ and sufficiently large $T_+ \in \mathcal{D}_+$, then the estimates hold for $d_+ = 2$ and we can replace $\alpha_+ < \gamma_+ < \min(0, 2\beta_+)$ by $\alpha_+ < \gamma_+ < \min(0, \beta_+)$. The analogous result holds in the "$-$" case.

Proof: The proof is similar to that of Theorem 3.1 and a lot of ideas and estimates can be carried over. Again, we apply the Perturbation Lemma 3.2.

Sketch of the proof of Theorem 5.2

Apply Lemma 3.2 with $F = H_J$ and $y_0 = (\bar{x}_+|_{J_+}, \bar{x}_-|_{J_-}, \bar{\lambda}) =: (\bar{x}|_J, \bar{\lambda})$. Here and in the remainder part we abbreviate sometimes for example $(w_+, w_-, r_0, r_+, r_-)$ by (w, r_0, r).

S1 There exists some $C_{lin} > 0$ such that $||(v,\mu)||_{\mathcal{Y}} \le C_{lin}||(w, r_0, r_+)||_{\mathcal{Z}}$ holds for each solution (v, μ) of the inhomogeneous equation $H_J'(\bar{x}|_J, \bar{\lambda})(v, \mu) = (w, r_0, r_+)$.

S2 $H_J'(\bar{x}|_J, \bar{\lambda})$ is a homeomorphism and $\sigma := \frac{1}{C_{lin}} \le ||H_J'(\bar{x}|_J, \bar{\lambda})^{-1}||^{-1}$.

S3 There exists some constant $C_{Lip} > 0$ such that for $\delta := \frac{\sigma}{2C_{Lip}} > 0$ and $\tilde{\delta} := \frac{\delta}{r(\beta, T_+)}$ the following holds: $||H_J'(z,\nu) - H_J'(\bar{x}|_J, \bar{\lambda})|| \le \kappa := \frac{\sigma}{2}\ \forall (z,\nu) \in \mathcal{B}_{\tilde{\delta}}(\bar{x}|_J, \bar{\lambda})$.

S4 There exist constants $C^\pm > 0$ and $\bar{T}_\pm$ sufficiently large, such that $||H_J(\bar{x}|_J, \bar{\lambda})|| \le C^+e^{(d_+\gamma_+ - \beta_+)T_+} + C^-e^{(d_+\gamma_- - \beta_-)T_-} \le (\sigma - \kappa)\frac{\delta}{r(\beta, T_+)}$ for all $T_\pm \in \mathcal{D}_\pm$, $T_\pm \ge \bar{T}_\pm$

S5 Lemma 3.2 implies that (5.8) has a unique solution (z_J^+, z_J^-, ν_J) in $\mathcal{B}_{\frac{\delta}{r(\beta,T)}}(\bar{x}|_J, \bar{\lambda})$ with

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \le 2C_{lin}\big(C^+e^{(d_+\gamma_+ - \beta_+)T_+} + C^-e^{(d_-\gamma_- - \beta_-)T_-}\big)$$

The detailed computations are similar to those in Section 3.2, therefore we often refer to Section 3.2.

S1 Let $(w_+, w_-, r_0, p_+, r_-) \in \mathcal{Z}$ and $(v_+, v_-, \mu) \in \mathcal{Y}$ be a solution of

$$H_J'(\bar{x}_+|_J, \bar{x}_-|_J, \bar{\lambda})(v_+, v_-, \mu) = (w_+, w_-, r_0, r_+, r_-).$$

This is equivalent to the variational equation

$$\begin{aligned}
\dot{v}_+ - A_+(\cdot)v_+ - V_+(\cdot)\mu &= w_+, \\
\dot{v}_- - A_-(\cdot)v_- - V_-(\cdot)\mu &= w_-, \\
R_0^+ v_+(0) + R_0^- v_-(0) + Q_0\mu &= r_0, \\
R_+ v_+(T_+) + Q_+\mu &= r_+, \\
R_- v_-(T_-) + Q_-\mu &= r_-,
\end{aligned}$$

where

$$\begin{array}{rclrcl} A_\pm(\cdot) &=& \frac{\partial}{\partial x} f_\pm(\bar{x}_\pm(\cdot),\bar\lambda), & V_\pm(\cdot) &=& \frac{\partial}{\partial \lambda} f_\pm(\bar{x}_\pm(\cdot),\bar\lambda), \\ R_0^\pm &=& \frac{\partial}{\partial x_\pm} g(\bar{x}_+(0),\bar{x}_-(0),\bar\lambda), & Q_0 &=& \frac{\partial}{\partial \lambda} g(\bar{x}_+(0),\bar{x}_-(0),\bar\lambda), \\ R_\pm &=& M_\pm(T_+,\bar\lambda), & Q_\pm &=& \frac{\partial}{\partial \lambda} M_\pm(T_\pm,\bar\lambda)\big(\bar{x}_\pm(T_\pm) - y_\pm(\bar\lambda)(T_\pm)\big) \\ &&&&& -M_\pm(T_\pm,\bar\lambda)\frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(T_\pm). \end{array}$$

Defining $\xi_a^\pm = P_a^\pm(0)v_\pm(0) \in \mathcal{R}(P_a^\pm(0))$ and $\xi_b^\pm = P_b^\pm(T_+)\big(v_\pm(T_\pm) - \frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(T_\pm)\mu\big) \in \mathcal{R}\big(P_b^\pm(T_+)\big)$ then Theorem 1.5 implies that each $v_\pm$ is the unique solution of

$$\begin{gathered} L_\pm v_\pm = \big(V_\pm(\cdot)\mu + w_\pm\big), \quad t \in J_\pm, \\ P_a^\pm(0)v_\pm(0) = \xi_a^\pm, \quad P_b^\pm(T_\pm)v_\pm(T_\pm) = \xi_b^\pm + P_b^\pm(T_\pm)\tfrac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(T_\pm)\mu. \end{gathered}$$

Analogous to (3.32), (3.36) and (3.37) these are

$$\begin{aligned} v_\pm(t) &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(w_\pm)(t) + s_J^\pm(V_\pm(\cdot)\mu)(t) && (5.12)\\ &\quad + S_\pm(t,T_\pm)P_b^\pm(T_\pm)\frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(T_\pm)\mu \\ &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(w_\pm)(t) && (5.13)\\ &\quad + \frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(t)\mu - S_\pm(t,0)P_a^\pm(0)\frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(0)\mu + s_J^\pm\big(\Delta_\pm(\cdot)\big)(t)\mu \\ &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(w_\pm)(t) + \frac{\partial}{\partial\lambda} x^\pm(0,\bar\lambda)(t)\mu && (5.14)\\ &\quad + S_\pm(t,T_\pm)\int_{T_\pm}^\infty S_\pm(T_\pm,s)P_b^\pm(s)\Delta_\pm(s)\mu \mathrm{d}s. \end{aligned}$$

Using the definitions of D and B and (5.1)

$$D \circ B(\xi_a^+,\xi_a^-,\mu) = Q_0\mu + R_0^+\big(\xi_a^+ + \frac{\partial}{\partial\lambda} x_+(0,\bar\lambda)(0)\mu\big) + R_0^-\big(\xi_a^- + \frac{\partial}{\partial\lambda} x_-(0,\bar\lambda)(0)\mu\big)$$

we obtain analogously to (3.38)

$$\begin{aligned} &\big(D\circ B \quad R_0^+ S_+(0,T_+)|_{\mathcal{R}_b^+} \quad R_0^- S_-(0,T_-)|_{\mathcal{R}_b^-}\big)\begin{pmatrix} (\xi_a^+,\xi_a^-,\mu) \\ \xi_b^+ \\ \xi_b^- \end{pmatrix} && (5.15)\\ &\quad = D\circ B(\xi_a^+,\xi_a^-,\mu) + R_0^+ S_+(0,T_+)\xi_b^+ + R_0^- S_-(0,T_-)\xi_b^- && (5.16)\\ &\quad = r_0 + R_a^+(T_+) + R_a^-(T_-) && (5.17) \end{aligned}$$

with

$$\begin{aligned} R_a^\pm(T_\pm) \;&:=\; -R_0^\pm S_\pm(0,T_\pm)\int_{T_\pm}^\infty S(T_\pm,s)P_b^\pm(s)\Delta_\pm(s)\mu \mathrm{d}s \\ &\qquad + R_0^\pm \int_0^{T_\pm} S_\pm(0,s)P_b^\pm(s)w_\pm(s)\mathrm{d}s \end{aligned}$$

Analogous to (3.41) we see

$$\begin{aligned} M_\pm(T_\pm,\bar\lambda)\xi_b^\pm &= r_\pm - \frac{\partial}{\partial\lambda}M_\pm(T_\pm,\bar\lambda)\mu\big(\bar x(T_\pm) - y_\pm(\bar\lambda)(T_\pm)\big) \\ &\quad + M_\pm(T_\pm,\bar\lambda)P_a^\pm(T_\pm)\big(\frac{\partial}{\partial\lambda}y_\pm(\bar\lambda)(T_\pm) - \frac{\partial}{\partial\lambda}x_\pm(0,\bar\lambda)(T_\pm)\big)\mu \\ &\quad - M_\pm(T_\pm,\bar\lambda)S_\pm(T_\pm,0)\xi_a^\pm - M_\pm(T_\pm,\bar\lambda)s_J^\pm(w_\pm)(T_\pm) \\ &=: R_b^\pm(T_\pm). \end{aligned}$$

Defining $\hat\xi_b^\pm := \xi_b^\pm e^{-\beta_\pm T_\pm}$ and $R_a(T_+,T_-) =: R_a^+(T_+) + R_a^-(T_-)$ we get the linear system

$$\begin{pmatrix} D\circ B & R_0^+S_+(0,T_+)|_{\mathcal{R}_b^+}e^{\beta_+T_+} & R_0^-S_-(0,T_-)|_{\mathcal{R}_b^-}e^{\beta_-T_-} \\ 0 & M_+(T_+,\bar\lambda)|_{\mathcal{R}_b^+} & 0 \\ 0 & 0 & M_-(T_-,\bar\lambda)|_{\mathcal{R}_b^-} \end{pmatrix} \begin{pmatrix} \xi_a^+,\xi_a^-,\mu \\ \hat\xi_b^+ \\ \hat\xi_b^- \end{pmatrix} = \begin{pmatrix} r_0 + R_a(T_+,T_-) \\ R_b^+(T_+)e^{-\beta_+T_+} \\ R_b^-(T_-)e^{-\beta_-T_-} \end{pmatrix}.$$

As in (3.47) and (3.48) we estimate $|R_a^\pm(T_\pm)|$ and $|R_b^\pm(T_\pm)|e^{-\beta T_\pm}$ by

$$\begin{aligned} |R_a^\pm(T_\pm)| &\le ||(\tilde w_\pm,0,0)||_{\mathcal{Z}} + ||R_0^\pm||K_\pm c^\pm_{\Delta,\beta}e^{(\gamma_\pm-\beta_\pm)T_+}|\mu|, \\ |R_b^\pm(T_\pm)|e^{-\beta_\pm T_\pm} &\le \frac{1}{K_\pm\, M_N^\pm\, ||R_0^\pm||}\Big(||(\tilde w_\pm,0,\tilde r_\pm)||_{\mathcal{Z}} + \frac{\tilde c_\pm}{C_{\xi,\mu}}e^{(\gamma_\pm-\beta_\pm)T_\pm}(|\xi_a^\pm| + |\mu|)\Big). \end{aligned}$$

with $\tilde w_+ := (w_+,0)$, $\tilde r_+ := (r_+,0)$, $\tilde w_- := (0,w_-)$, $\tilde r_+ := (0,r_-)$ and, as in Section 3.2, $\tilde c^\pm := C_{\xi,\mu}\, K_\pm\, M_N^\pm\, ||R_0^\pm|| \cdot \big(\max(1,M^1_{y_\pm})M_M^\pm K_\pm + M^1_{M_\pm}C^\pm_{\bar x} + \max(\frac{1}{M_N^\pm},M_M^\pm)c^\pm_\Delta\big)$ and $C_{\xi,\mu} := ||(D\circ B)^{-1}||$.

To estimate $|(\xi_a^+,\xi_a^-,\mu)|$ we notice that μ is involved in both parts of the problem, thus we have to choose $\bar T_\pm$ larger as follows:

Let $a > 1$ and $T_\pm \ge \bar T_\pm := \frac{\ln(\tilde c_\pm) - \ln(1-\frac{1}{a}) + \ln(2)}{\beta_\pm - \gamma_\pm}$, then $\tilde c_\pm e^{(\gamma_\pm-\beta_\pm)T_\pm} \le \frac{1}{2}(1-\frac{1}{a})$ holds.

Now we can estimate as in (3.49) and (3.50)

$$\begin{aligned} |(\xi_a^+,\xi_a^-,\mu)| &\le ||(D\circ B)^{-1}||\; ||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}} \\ &\quad + \tilde c_+(|\xi_a^+| + |\mu|)e^{(\gamma_+-\beta_+)T_+} + \tilde c_-(|\xi_a^-| + |\mu|)e^{(\gamma_--\beta_-)T_-} \\ &\le a\cdot C_{\xi,\mu}||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}} \end{aligned}$$

and

$$\begin{aligned} |\hat\xi_b^\pm| &\le \frac{1}{K_\pm\, ||R_0^\pm||}||(\tilde w_\pm,0,\tilde r_\pm)||_{\mathcal{Z}} + \frac{1}{C_{\xi,\mu}K_\pm\, ||R_0^\pm||}\cdot \tilde c_\pm\, (|\xi_a^\pm| + |\mu|)e^{(\gamma_\pm-\beta_\pm)T_\pm} \\ &\le \frac{1+a}{2}\frac{1}{K_\pm\, ||R_0^\pm||}\, ||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}} \le \frac{a}{K_\pm\, ||R_0^\pm||}||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}} \end{aligned}$$

The estimate of $||v_\pm(t)||$ is analogous to (3.51)

$$|v_\pm(t)| \quad \leq \quad \frac{1+e^{\beta_\pm t}}{2} C_v^\pm \; a \; ||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}}$$

where $C_v^\pm$ is defined as in Section 3.2 with indices $\pm$.

Therefore $||v_\pm||^\pm_{\mathcal{Y}} = \sup_{t\in J_\pm}\big(||v_\pm(t)||q_\pm(t)\big) \leq aC_v^\pm||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}}$.

Thus $||(v_+,v_-,\mu)||_{\mathcal{Y}} \leq C_{lin}||(w_+,w_-,r_0,r_+,r_-)||_{\mathcal{Z}}$ with $C_{lin} = a\,(C_{\xi,\mu}+C_v^+ + C_v^-)$.

S2 Assumption **A4**$_\pm$ implies that $H'_J(\bar{x}|_J,\bar{\lambda})(v_+,v_-,\mu) = (w_+,w_-,r_0,r_+,r_-)$ is a linear boundary value problem of dimension $m_a+p+m_b = m+p$ for which the Fredholm alternative holds. Thus $\sigma := \frac{1}{C_{lin}} \leq ||H'_J(\bar{x}|_J,\bar{\lambda})^{-1}||^{-1}$.

S3 For any $(z,\nu),(v,\mu)\in\mathcal{Y}$ we can estimate as in (3.52)

$$\begin{aligned}
&\left\| H'_J(z,\nu) - H'_J(\bar{x}|_J,\bar{\lambda})\begin{pmatrix} v\\ \mu\end{pmatrix}\right\|_{\mathcal{Z}} \\
&\leq \; C^+_{Lip}||(z_+-\bar{x}_+|_{J_+})||^+_{\mathcal{Y}}\cdot(||v_+||^+_{\mathcal{Y}}+|\mu|)\cdot r^+(\beta_+,T_+) \\
&\quad\; +C^-_{Lip}||(z_--\bar{x}_-|_{J_-})||^-_{\mathcal{Y}}\cdot(||v_-||^-_{\mathcal{Y}}+|\mu|)\cdot r^-(\beta_-,T_-) \\
&\quad\; +C_{Lip}|\nu-\bar{\lambda}|\cdot||(v_+,v_-,\mu)||_{\mathcal{Y}}\cdot r(\beta,T) \\
&\leq \; C_{Lip}||(z-\bar{x}|_J,\nu-\bar{\lambda})||_{\mathcal{Y}}\cdot||(v,\mu)||_{\mathcal{Y}}\cdot r(\beta,T)
\end{aligned}$$

where $c_M^\pm$ is defined as c_M on page 67 with index $\pm$ and

$$\begin{aligned}
C^\pm_{Lip} &= C^{J_\pm}_{\mathcal{Z}}\,L^1_{f_\pm} + L^1_g + C^\pm_{\mathcal{Z}}\max(M^1_{M_\pm},c^\pm_M),\\
C_{Lip} &= C^{J_+}_{\mathcal{Z}}\,L^1_{f_+} + C^{J_-}_{\mathcal{Z}}\,L^1_{f_-} + C^+_{\mathcal{Z}}\max(M^1_{M_+},c^+_M) + C^-_{\mathcal{Z}}\max(M^1_{M_-},c^-_M) + L^1_g.
\end{aligned}$$

Therefore with $\kappa := \frac{\sigma}{2}$ and $\delta := \frac{\sigma}{2C_{Lip}}$ we see for any $(z,\nu)\in\mathcal{B}_{\frac{\delta}{r(\beta,T)}}(\bar{x}|_J,\bar{\lambda})$

$$||H'_J(z,\nu) - H'_J(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}\mathcal{Z}} \leq \frac{\sigma}{2} = \kappa < \sigma \leq ||H'_J(x|_J,\bar{\lambda})^{-1}||^{-1}_{\mathcal{Y}\mathcal{Z}}.$$

S4 On the other hand the truncation error is

$$
\begin{aligned}
||H_J(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Z}} &= \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ M_+(T_+)\big(\bar{\lambda}\big)\big(\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)\big) \\ M_-(T_-)\big(\bar{\lambda}\big)\big(\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)\big) \end{pmatrix} \right\|_{\mathcal{Z}} \\
&= C^+_{\mathcal{Z}}\big|M_+(T_+,\bar{\lambda})\big(\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)\big)\big|e^{-\beta_+T_+} \\
&\quad +C^-_{\mathcal{Z}}\big|M_-(T_-,\bar{\lambda})\big(\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)\big)\big|e^{-\beta_-T_-} \\
&\leq C^+_{\mathcal{Z}}\ \tilde{C}^+_{tr}||\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)||^{d_+}e^{-\beta_+T_+} \\
&\quad +C^-_{\mathcal{Z}}\ \tilde{C}^-_{tr}||\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)||^{d_-}e^{-\beta_-T_-} \\
&\leq C^+_{\mathcal{Z}}C^+_{tr}e^{(d_+\gamma_+-\beta_+)T_+} + C^-_{\mathcal{Z}}C^-_{tr}e^{(d_-\gamma_- -\beta_-)T_-} \\
&\leq \bar{C}\big(e^{(d_+\gamma_+-\beta_+)T_+} + e^{(d_-\gamma_- -\beta_-)T_-}\big),
\end{aligned}
$$

where $\bar{C} = \max(C^+_{\mathcal{Z}}C^+_{tr}, C^-_{\mathcal{Z}}C^-_{tr})$, $C^{\pm}_{tr}$ as on page 68.

Lemma 5.1 implies that

$$\bar{C}\big(e^{(d_+\gamma_+-\beta_+)T_+} + e^{(d_-\gamma_- -\beta_-)T_-}\big) \leq \frac{\delta}{r(\beta,T)}\frac{\sigma}{2}$$

is satisfied for all $(T_+,T_-) \in \mathcal{D}(\epsilon,\beta_\pm,d_\pm\gamma_\pm-\beta_\pm)\cap\mathcal{D}_+\times\mathcal{D}_-$, $T_\pm \geq \bar{T}_\pm$, where $\epsilon = \frac{\sigma\delta}{2\bar{C}}$.

Thus, by Lemma 3.2, there exists a unique solution (z_J,ν_J) in $\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J,\bar{\lambda})$ with

$$||(z_J,\nu_J) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} \leq C^+e^{(d_+\gamma_+-\beta_+)T_+} + C^-e^{(d_-\gamma_- -\beta_-)T_-} \tag{5.18}$$

where $C^\pm = 2C_{lin}C^\pm_{\mathcal{Z}}C^\pm_{tr}$. Moreover, we see

$$
\begin{aligned}
|\nu_J-\bar{\lambda}| &\leq C^+\ e^{(d_+\gamma_+-\beta_+)T_+} + C^-\ e^{(d_-\gamma_- -\beta_-)T_-}, \\
||z^\pm_J(t) - \bar{x}_\pm|_{J_\pm}(t)|| &\leq \big(C^+e^{(d_+\gamma_+-\beta_+)T_+} + C^-e^{(d_-\gamma_- -\beta_-)T_-}\big)\cdot\frac{1+e^{\beta_\pm t}}{2},
\end{aligned}
$$

■

For $\beta_\pm \leq 0$ the following estimate holds

$$||z^\pm_J - \bar{x}_\pm|_{J_\pm}||_\infty \leq C^+e^{(d_+\gamma_+-\beta_+)T_+} + C^-e^{(d_-\gamma_- -\beta_-)T_-}.$$

As in the remark on page 68 we see that we can apply Theorem 5.2 with $\hat{\beta}_\pm \in (\frac{\gamma_\pm}{2},\beta\pm]$. Hence, if $\beta_+>0$ and $\beta_-<0$, then we can apply Theorem 5.2 with $\beta_+=0$. The result is an approximation which satisfies the following estimates:

$$
\begin{aligned}
|\nu_J-\bar{\lambda}| &\leq C^+\ e^{d_+\gamma_+T_+} + C^-\ e^{(d_-\gamma_- -\beta_-)T_-}, \\
||z^\pm_J - \bar{x}_\pm|_{J_\pm}||_\infty &\leq C^+e^{d_+\gamma_+T_+} + C^-e^{(d_-\gamma_- -\beta_-)T_-}.
\end{aligned}
$$

We get analogous results if $\beta_+ < 0$ and $\beta_- > 0$ or if both are positive.

To get a super-convergence in the parameter as in [Beyn, 1990a, 1990b], [Sandstede, 1997] for $\beta_\pm > 0$ we restrict to those pairs (T_+, T_-) satisfying $\beta_+ T_+ = \beta_- T_-$. For these pairs we see

$$\begin{aligned} |\nu_J - \bar{\lambda}| &\leq C^+ \, e^{(d_+\gamma_+ - \beta_+)T_+} + C^- \, e^{(d_-\gamma_- - \beta_-)T_-}, \\ ||z_J^\pm - \bar{x}_\pm|_{J_\pm}||_\infty &\leq C^+ e^{d_+\gamma_+ T_+} + C^- e^{d_-\gamma_- T_-}. \end{aligned}$$

5.2 The boundary corrector method

As in Section 3.3 we set up a boundary corrector method for generalized connecting orbits and assume as in **A7** on page 69

A7$_\pm$ There exists a neighborhood $\mathbf{\Lambda}_0$ of $\bar{\lambda}$ and functions $\bar{M}_\pm(T_\pm, \cdot) \in C^1(\mathbf{\Lambda}_0, \mathbb{R}^{m_b^\pm, m})$, such that **A6$_\pm$** and (5.11) hold for both, "+" and "−" and each $\lambda \in \mathbf{\Lambda}_0$.

The **Boundary Corrector Method** for generalized connecting orbits is defined as follows:

1. Start with some $\mu_0 \in \mathbf{\Lambda}_0$, $i = 0$
2. Calculate $(z_{i+1}, \mu_{i+1}) = (z_{i+1}^+, z_{i+1}^-, \mu_{i+1})$ as the solution of

$$H_J^{\mu_i}(z_+, z_-, \nu) = \begin{pmatrix} \dot{z}_+ - f_+(z_+, \nu) \\ \dot{z}_- - f_-(z_-, \nu) \\ g\big(z_+(0), z_-(0), \nu\big) \\ M^+(T_+, \mu_i)(z_+(T_+) - y_+(\nu)(T_+)) \\ M^-(T_-, \mu_i)(z_-(T_-) - y_-(\nu)(T_-)) \end{pmatrix} = 0$$

3. Repeat with **2.** ("$i = i + 1$") once if $\beta_\pm \geq 0$ or twice if $\beta_+ < 0$ or $\beta_- < 0$.

We state the analogous result to Proposition 3.4 as follows:

PROPOSITION 5.3 *Suppose that the assumptions of Theorem 5.2 and let* **A7$_\pm$** *hold for both, $\bar{M}_+$ and $\bar{M}_-$. Moreover let $\bar{T}_\pm$ be sufficiently large.*

Then there exists some $C_{bcm} > 0$ and $\bar{T}_\pm$ sufficiently large, such that for all $T_\pm \geq \bar{T}_\pm$ with $(T_+, T_-) \in \mathcal{D}(\epsilon, \beta_\pm, d_\pm\gamma_\pm - \beta_\pm)$ for both, $d_\pm = 1$ and $d_\pm = 2$, the result (z_J, ν_J) of the **boundary corrector method** *can be estimated by*

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq C_{bcm}\big(e^{(d_+\gamma_+ - \beta_+)T_+} + e^{(d_-\gamma_- - \beta_-)T_-}\big).$$

Proof: Noticing that $e^{(\gamma_+ - \beta_+)T_+} \cdot e^{(\gamma_- - \beta_-)T_-} \leq \max(e^{2(\gamma_+ - \beta_+)T_+}, e^{2(\gamma_- - \beta_-)T_-})$ we follow the proof of Proposition 3.4. ■

5.3 Example: The Lorenz system

In this section we apply our theoretical results to the Lorenz system, mentioned in 3.4.2. We detect the point to periodic connecting orbit plotted in Figure 0.1. Moreover, we continue this connecting orbit by varying the parameter σ.

As approximation we solve the boundary value problem (5.19) for the "first part" of the connecting orbit $z(t) \in \mathbb{R}^3$, the "second part" $x(t) \in \mathbb{R}^3$ and the periodic orbit $y(t) \in \mathbb{R}^3$. The system is defined by $f(x, r; \sigma, b) = \big(\sigma(x_2 - x_1), rx_1 - x_2 - x_1x_3, x_1x_2 - bx_3\big)^\top$ as in 3.4.2. The parameters are $\sigma = 10$ and $b = \frac{8}{3}$.

An initial approximation for the unknown parameter $r \approx 24.05803$, the unknown period $\bar{T} \approx 0.677167$ and the solutions x, y and z are taken from the approximation computed by the method in 3.4.2.

We truncate the scaled solution on the interval $[0, 3]$, such that we compute three periods of the periodic orbit. The periodic orbit is fixed at $t = 2$ and $t = 3$ by $y(2) = y(3)$ and the phase fixing function $\chi\big(y(2), \phi\big) = y_1(2) - \phi$, where the value $\phi = 6.5043$ as well as the matrices $V_+, V_- \in \mathbb{R}^{2,3}$ for the asymptotic boundary conditions are also defined by the initial approximation in 3.4.2.

The boundary value solver restricts us to compute on the interval $[0, 3]$ even for the first part z of the connecting orbit. Thus we also scale the first part by a constant $\tau = \frac{2}{3}$, which is chosen to get roughly similar exponents $\alpha_+ \approx -7.845$ and $\alpha_- \approx -6.76$ for the scaled system. At $r = 24.0$ the "unstable eigenvalue" of 0 is 10.1365 and the "stable Floquet multiplier" of the periodic orbit is $0.93 \cdot 10^{-5}$. This yields for the systems scaled by $\bar{T} \approx 0.677167$ and $\tau = \frac{2}{3}$ the exponents $\alpha_\pm$ mentioned above.

The boundary value problem on $[0, 3]$ is defined as follows:

$$\begin{pmatrix} \dot{x} - \bar{T} f(x, r; \sigma, b) \\ \dot{y} - \bar{T} f(y, r; \sigma, b) \\ \dot{z} + \tau f(z, r; \sigma, b) \\ x(0) - z(0) \\ \chi\big(y(2), \phi\big) \\ y(2) - y(3) \\ V_+\big(x(3) - y(3)\big) \\ V_- z(3) \end{pmatrix} = 0 \tag{5.19}$$

The phase portrait is plotted in Figure 0.1 and in Figure 5.2 we plot the first and third component of the connecting orbit (—) and the periodic orbit (- -) in the rescaled version, i. e. $\tilde{x} = x(\cdot/\bar{T})$, $\tilde{z} = z(-\cdot/\tau)$ and $\tilde{y} = y(\cdot/\bar{T})$ for $t \geq 0$ and $\tilde{y} = y\big((\cdot + 3)/\bar{T}\big)$ for $t < 0$.

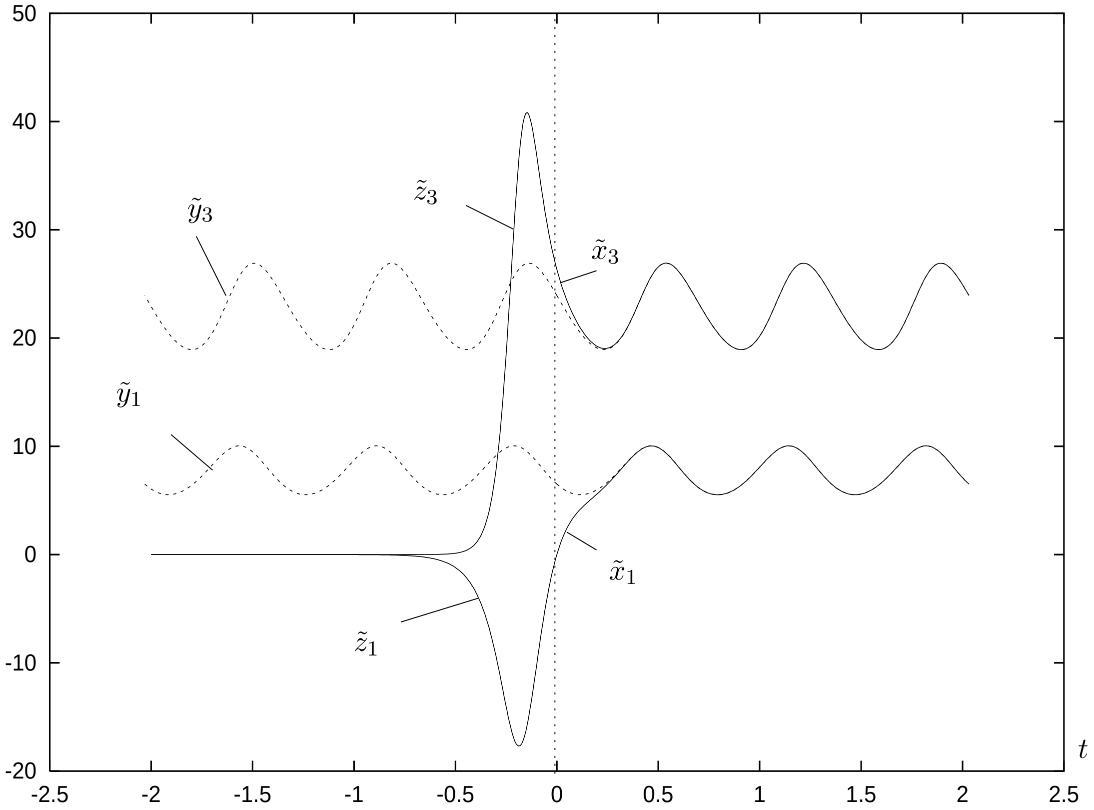

Figure 5.2: The first and third component of the connecting orbit (—) and the periodic orbit (- -) in the rescaled version.

Parameter continuation with respect to σ yields Figure 5.3. During this continuation a new phenomenon occurs, a limit point in the parameter space.

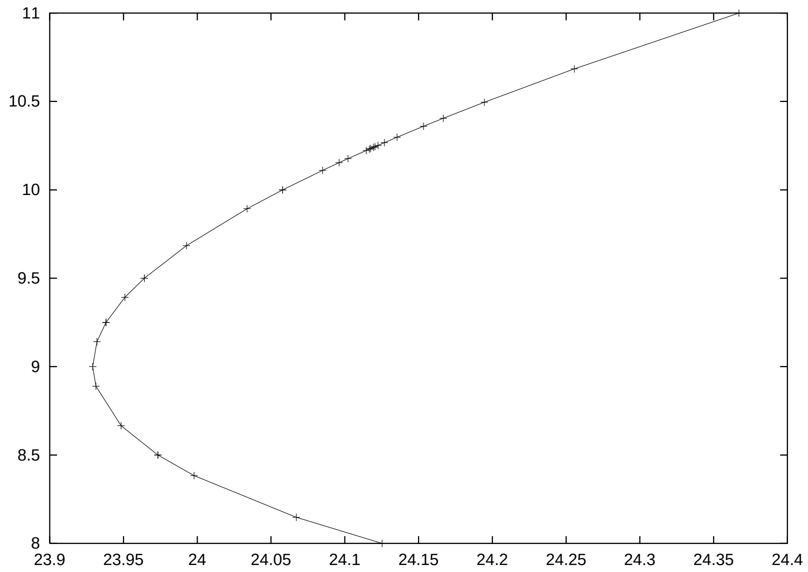

Figure 5.3: Parameterspace r–σ. Branch of point to periodic connecting orbits.

5.4 Concluding remarks on the theory

We have developed an approximation theory for a general type of connecting orbits, which includes most of the common connecting orbits.

However, there is an additional kind of connecting orbit not covered by our theory. These are solutions which converge, but not with an exponential rate (in our notion this means $\alpha = 0$). The case of a homoclinic connecting orbit of a semi-hyperbolic equilibrium is analyzed in [Schecter, 1993, Sandstede, 1997].

Continuation of the stable manifold of an equilibrium to a Hopf-bifurcation yields that α has to tend to 0 and hence the approximation interval must be enlarged and the error estimates become worse. Therefore we cannot apply the theory to approximate a Hopf-Shilnikov bifurcation.

It seems straightforward to transfer the results of Chapter 4 and Chapter 5 from the case with two subproblems to the case with a finite number of subproblems.

Chapter 6

Applications to control problems with infinite horizon

In Section 6.1 we define a class of optimization problems with infinite horizon we want to treat by our approximation results. In 6.1.1 we employ a proper version of Pontryagin's maximum principle to define the set of "candidates for optimal solutions" with "stationary or periodic asymptotic behavior". These solutions are of particular interest, because such candidates are bounded in state space. Connecting orbits now live in the space of state and costate variables and they determine optimal controls as well as the system response under this control, if the optimal solutions are asymptotically stationary or periodic. We apply the results of Chapter 3 to approximate such orbits.

The topic of 6.1.2 is the computation of "Skiba" points with "stationary or periodic asymptotic behavior" by using the methods of Chapter 5.

In Section 6.2 we analyze a model of optimal investment and consumption policy which is presented in [Semmler and Sieveking, 1998]. In particular, we approximate two solutions converging to different equilibria and we show, that there exists a Skiba point.

A model of optimal exploitation of renewable resources and optimal consumption policy, presented in [Semmler and Sieveking, 2000], is analyzed in Section 6.3. Here we detect a periodic solution and approximate a solution converging to it.

6.1 The general setup

In this section we follow [Michel, 1982] to define a class of optimization problems with infinite horizon. In 6.1.1 and 6.1.2 we define candidates for optimal solutions and Skiba points with "stationary or periodic asymptotic behavior".

The optimization problem is: Maximize the objective function

$$\hat{U}(x,c) := \int_0^\infty e^{-\theta t} U(x(t), c(t)) \mathrm{dt} \tag{6.1}$$

subject to

$$\dot{x}(t) = f\big(x(t), c(t)\big),\ t \in \mathbb{R}_+, \qquad x(0) = x_0, \tag{6.2}$$

where $x(t) \in \mathbb{R}^n$ are the state variables and $c(t) \in \mathbb{R}^n$ are the control variables (in [Michel, 1982] $c(t)$ is more generally in a topological space).

We assume that the control function c is piecewise continuous, $x \in C(\mathbb{R}_+, \mathbb{R}^n)$ and piecewise differentiable, $U \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and $f \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n)$.

DEFINITION 6.1 *A pair* $\big(x(\cdot), c(\cdot)\big)$ *is called an* **admissible trajectory**, *if* $x(\cdot)$ *is a solution of (6.2) with control* $c(\cdot)$ *and if the integral (6.1) converges. The set of all admissible trajectories is denoted by* $\mathcal{T}_{ad}(x_0)$.

A pair $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big)$ *is called an* **optimal solution** *of the problem (6.1), (6.2), if it is admissible and if it maximizes (6.1) over all admissible pairs* $\big(x(\cdot), c(\cdot)\big) \in \mathcal{T}_{ad}(x_0)$. *The set of all optimal solutions is denoted by* $\mathcal{T}_{opt}(x_0)$.

We will apply our approximation technique to approximate "candidates for optimal solutions" and to compare the corresponding values of the objective function (6.1). It is not our intention to prove existence of optimal solutions. Therefore we assume henceforth

Assumption 1: There exists an optimal solution of the problem (6.1), (6.2).

This assumption implies $\mathcal{T}_{opt}(x_0) \neq \emptyset$.

The technique used here is to apply Pontryagin's maximum principle to detect candidates for optimal solutions. These are admissible trajectories which satisfy the necessary conditions for optimal solutions stated in the forthcoming Theorem 6.2. The theorem is a special version of Pontryagin's maximum principle from [Michel, 1982, Theorem A].

Theorem 6.2 *A necessary condition for an admissible trajectory $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big) \in \mathcal{T}_{ad}(x_0)$ to be an optimal solution of (6.1), (6.2) is, that there exist $a \in \{0,1\}$ and an initial costate value $y_0 \in \mathbb{R}^n$, such that*

1. $(a, y_0) \neq 0$.

2. *The costate variable $y(\cdot)$ is a solution of the initial value problem*

$$\begin{aligned} \dot{y}(t) &= -ae^{-\theta t}\Big(\tfrac{\partial}{\partial x}U\big(\bar{x}(t), \bar{c}(t)\big)\Big)^{\top} - \Big(\tfrac{\partial}{\partial x}f\big(\bar{x}(t), \bar{c}(t)\big)\Big)^{\top} \cdot y(t), \\ y(0) &= y_0. \end{aligned} \tag{6.3}$$

3. *For each $t \in \mathbb{R}_+$ where $\bar{c}(t)$ is continuous, $(\bar{x}(t), \bar{c}(t), y(t), a)$ satisfies*

$$H(\bar{x}(t), \bar{c}(t), y(t), a, t) = \max_c H(\bar{x}(t), c, y(t), a, t), \tag{6.4}$$

   where H is the Hamiltonian function

$$H(x(t), c, y(t), a, t) := ae^{-\theta t}U\big(x(t), c\big) + f\big(x(t), c\big)^{\top} \cdot y(t).$$

4. *The maximum value of the Hamiltonian satisfies for every t*

$$M(t) := H(\bar{x}(t), \bar{c}(t), y(t), t) = -q(t), \tag{6.5}$$

   where the function $q(t) \in \mathbb{R}$ is the solution of

$$\begin{aligned} &\dot{q} = \theta e^{-\theta t}U\big(\bar{x}(t), \bar{c}(t)\big), \\ &\lim\nolimits_{t\to\infty} q(t) = 0. \end{aligned} \tag{6.6}$$

Proof: See [Michel, 1982, Theorem A]. ∎

Remark: The initial costate value y_0 is unknown. If y_0 and $\bar{c}(\cdot)$ were given, then (6.2) and (6.3) constitute an initial value problem for x and y.

For admissible $(\bar{x}, \bar{c})$ the integral (6.1) converges by definition and hence

$$q(t) = -\theta \int_t^{\infty} e^{-\theta s}U(\bar{x}(s), \bar{c}(s))\mathrm{d}s \tag{6.7}$$

is the solution of (6.6).

Definition 6.3 *An admissible trajectory $\big(x(\cdot), c(\cdot)\big) \in \mathcal{T}_{ad}(x_0)$ is called a* **candidate for an optimal solution***, if it satisfies the necessary conditions of Theorem 6.2. The value y_0 is called the* **corresponding costate value** *and the set of all candidates for optimal solutions is denoted by $\mathcal{T}_{can}(x_0)$.*

As abbreviation we also call a "candidate for an optimal solution" simply a **candidate**.

In typical cases $a = 1$, nevertheless this requires a proof in the applications. In this section we will henceforward assume $a = 1$.

In particular, we deal with systems where we can compute $\bar{c}(t)$ depending on $\bar{x}(t)$ and the scaled costate function $z(t) := e^{\theta t} y(t)$ explicitly from (6.4), i. e. we find $\bar{c}(t) = c\big(\bar{x}(t), z(t)\big)$.

Thus we assume henceforward

Assumption 2: We have $a = 1$ and there exists a function c, such that $c(x, z)$ maximizes $\tilde{H}(x, c, z) := U(x, c) + f(x, c)^\top \cdot z$ for all $(x, z) \in \mathbb{R}^{2n}$.

Using $z(t) = e^{\theta t} y(t)$, $\bar{c}(t) = c\big(\bar{x}(t), z(t)\big)$, then (6.2) and (6.3) imply that $(\bar{x}(t), z(t))$ solves the autonomous dynamical system

$$\begin{aligned} \dot{x} &= f\big(x, c(x, z)\big), & x(0) = x_0, \qquad (6.8) \\ \dot{z} &= \theta z - \Big(\frac{\partial}{\partial x} U\big(x, c(x, z)\big)\Big)^\top - \Big(\frac{\partial}{\partial x} f\big(x, c(x, z)\big)\Big)^\top z, & z(0) = y_0 \qquad (6.9) \end{aligned}$$

if $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big)$ is an optimal solution.

This is an initial value problem depending on $y_0 \in \mathbb{R}^n$. For each $y_0 \in \mathbb{R}^n$ we denote its solution by $x(y_0), z(y_0)$. We abbreviate $c(y_0)(t) := c\big(x(y_0)(t), z(y_0)(t)\big)$.

Suppose that the assumptions 1 and 2 hold, then we reformulate the optimization problem and maximize

$$\bar{U}(y_0) := \int_0^\infty e^{-\theta t} U\big(x(y_0)(t), c(y_0)(t)\big) \mathrm{d}t \tag{6.10}$$

over all $y_0 \in \mathbb{R}^n$, where the integral (6.10) exists.

Thus, the set of candidates for optimal solutions is

$$\mathcal{T}_{can}(x_0) = \Big\{ \big(x(y_0), c(y_0)\big) \;\Big|\; y_0 \in \mathbb{R}^n \text{ and the integral (6.10) exists and (6.5) holds} \Big\}.$$

Additional information whether a solution is optimal can be obtained from (6.5) and (6.7). If $(\bar{x}, \bar{c})$ is a candidate, then we have

$$\int_t^\infty e^{-\theta s} U(\bar{x}(s), \bar{c}(s)) \mathrm{d}s = \frac{e^{-\theta t}}{\theta} \Big(U\big(\bar{x}(t), \bar{c}(t)\big) + f\big(\bar{x}(t), \bar{c}(t)\big)^\top z(t) \Big). \tag{6.11}$$

Maximizing $\frac{1}{\theta}\big[U\big(x(y_0)(0), c[x(y_0)(0), z(y_0)(0)]\big) + \dot{x}(y_0)(0)^\top y_0\big]$ over all initial values of costate variables $y_0 \in \mathbb{R}^n$ does not solve the original problem, because it is also defined for non-optimal solutions. On the other hand we can test (6.11) for single admissible trajectories by numerical integration to see whether it satisfies (6.5) or not.

Remark: The equation (6.11) always holds true for equilibria $(\bar{x}, z)$ of (6.8), (6.9). If $(\bar{x}(t), z(t))$ is an optimal T–periodic solution, then (6.11) is equivalent to

$$\int_0^T e^{-\theta s} U(\bar{x}(s+t), \bar{c}(s+t))\mathrm{ds} = \frac{1-e^{-\theta T}}{\theta}\Big(U\big(\bar{x}(t), \bar{c}(t)\big) + f\big(\bar{x}(t), \bar{c}(t)\big)^\top z(t)\Big). \quad (6.12)$$

for all $t \in \mathbb{R}_+$.

6.1.1 Asymptotic solutions

Using the techniques of Chapter 3 we approximate candidates for optimal solutions with "special asymptotic behavior", i. e. those solutions which converge to given solutions η. In particular, we restrict to cases where η is either an equilibrium or a periodic solution.

Henceforward, we assume that $x_0 \in \mathbb{R}^n$ is given and that $\mathcal{Y} = \{\eta_i\}_{i \in I}$, $I = \{1, \dots, l\}$, $l \in \mathbb{N}$ is a finite number of solutions $\eta_i(t) \in \mathbb{R}^m$, $m = 2n$ of $\binom{\dot{x}}{\dot{z}} = F(x, z)$, where F is defined by (6.8), (6.9). Moreover we define:

Definition 6.4 *Suppose that the Assumptions 1 and 2 hold.*

A candidate for an optimal solution $\big(x(\cdot), c(\cdot)\big) \in \mathcal{T}_{can}(x_0)$ with corresponding costate value y_0 is called **asymptotic to $\mathcal{Y}$**, *if $\big(x, z(y_0)\big)$, solves (6.8), (6.9) and is asymptotic to some $\eta_i \in \mathcal{Y}$. The set of these solutions is denoted by $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$.*

A candidate $(\bar{x}, \bar{c}) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)$ for an optimal solution which is asymptotic with $\{\eta_i\}$, $i \in I$ is called **nondegenerate** *if there exist neighborhoods V of x_0 and W of $\bar{x}$, such that for each $\tilde{x}_0 \in V$ there is a unique $(x, c) \in \mathcal{T}_{asy}(\mathcal{Y})(\tilde{x}_0)$ converging to η_i with $x \in W$.*

We call a candidate $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)$ which is asymptotic with $\mathcal{Y}$ **optimal with respect to $\mathcal{Y}$**, *if it maximizes the objective function over all $(x, c) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)$, i. e.*

$$\hat{U}\big(\bar{x}(\cdot), \bar{c}(\cdot)\big) = \max_{(x,c) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)} \big(\hat{U}(x, c)\big), \quad \textit{where } \hat{U}(x, c) = \int_0^\infty e^{-\theta t} U(x(t), c(t))\mathrm{dt}.$$

The set of candidates which are optimal with respect to $\mathcal{Y}$ is denoted by $\mathcal{T}_{asy}^{opt}(\mathcal{Y})(x_0)$.

Remark: The Definitions 6.1, 6.3 and 6.4 yield

$$\mathcal{T}_{asy}^{opt}(\mathcal{Y})(x_0) \subset \mathcal{T}_{asy}(\mathcal{Y})(x_0) \subset \mathcal{T}_{can}(x_0) \subset \mathcal{T}_{ad}(x_0).$$

If there is an optimal solution in $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$, then $\mathcal{T}_{asy}^{opt}(\mathcal{Y})(x_0) \subset \mathcal{T}_{opt}(x_0)$.

We apply the techniques of Chapter 3 to approximate $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$ and hence $\mathcal{T}_{asy}^{opt}(\mathcal{Y})(x_0)$ in those cases, where the η_i are either equilibria periodic solutions. Moreover we assume

Assumption 3: The $\eta_i \in \mathcal{Y}$ are either hyperbolic equilibria or hyperbolic periodic orbit of $\binom{\dot{x}}{\dot{z}} = F(x, z)$ and for each $\eta_i \in \mathcal{Y}$ there exists a unique nondegenerate candidate for an optimal solution which is asymptotic with $\{\eta_i\}$.

Remark: An equilibrium is called **hyperbolic**, if its linearization has no eigenvalue μ with $\mathrm{Re}(\mu) = 0$ and a periodic orbit is called **hyperbolic**, if it has a simple Floquet multiplier 1 and no other multiplier on the unit circle. Recall, that a periodic orbit is the image of a periodic solution (Definition 1.11).

If η_i is a hyperbolic periodic orbit with an n–dimensional center-stable manifold, then the stable manifold of η_i evaluated at $t = 0$ is $(n-1)$–dimensional, because the time direction belongs to the center manifold and not to the stable manifold evaluated at 0.

If the dimension of a stable or center–stable manifold of $\eta_i \in \mathcal{Y}$ is less than n, then the non-degeneracy condition is not satisfied, because the tangent spaces of the stable manifold at $t = 0$ and $\{0\} \times \mathbb{R}^n$, the tangent space of the set $\mathcal{W} = \{(x, z) \in \mathbb{R}^{2n} | x = x_0\}$, do not span the $\mathbb{R}^{2n}$.

If the dimension of a stable or center–stable manifold of $\eta_i \in \mathcal{Y}$ is $n + q$, then there exists typically a q–dimensional surface of candidates which are asymptotic with $\{\eta_i\}$. This leads to the problem of maximizing $\bar{U}(y_0)$ on a q–dimensional surface. We can approximate parts of this surface by fixing additional q costate variables, applying the forthcoming procedure and using the additional parameters for continuation.

For the approximation of nondegenerate candidates, we define $f = F$ by (6.8) and (6.9) and $g(x, z) = x - x_0$. Moreover, in the periodic case we define the phase of the periodic solution as unknown parameter and hence $g(x, z, \tau) = x - x_0$ (see the remark on page 49). Therefore, in the equilibrium case $\mathcal{W} = \{(x, z) \in \mathbb{R}^m | x = x_0\}$, $m = 2n$ is a manifold of dimension $m_b = n$ in $\mathbb{R}^{2n} = \mathbb{R}^m$ and in the case of periodic solutions the manifold $\mathcal{W} = \{(x, z, \tau) \in \mathbb{R}^{m+1} | x = x_0\}$ is of dimension $m_b = n + 1$ in $\mathbb{R}^{m+1}$.

Thus, a nondegenerate connecting orbit in the sense of Definition 2.19 defines a nondegenerate candidate for an optimal solution which is asymptotic with $\{\eta_i\}$ (in the sense of Definition 6.4).

Suppose that the assumptions 1–3 hold and let $x_0 \in \mathbb{R}^n$ be an initial value, then we approximate $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$ and $\mathcal{T}_{asy}^{opt}(\mathcal{Y})(x_0)$ by the following procedure:

Detecting asymptotic candidates

1. For each $i \in I$ apply (3.10) or the boundary corrector method defined by (3.15), respectively. Let (x_i, z_i) be the solution and define $c_i = c(x_i, z_i)$.
2. Test (6.11) by numerical integration to decide whether these solutions are candidates or not. This defines an approximation of the set $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$.
3. Choosing those solutions which maximize the objective function (6.1) defines the set of candidates which are optimal with respect to $\mathcal{Y}$.

6.1.2 Skiba points

Now we define "Skiba" points [Skiba, 1978] and derive an approximation procedure based on the results of Chapter 5.

DEFINITION 6.5 *Suppose that assumption 1 holds. We call an initial value $x_0 \in \mathbb{R}^n$ a* **Skiba point***, if there are multiple optimal solutions.*

Suppose that in addition the assumptions 2 and 3 hold. Then we call an initial value $x_0 \in \mathbb{R}^n$ a **Skiba point with respect to $\mathcal{Y}$***, if there are multiple candidates which are optimal with respect to $\mathcal{Y}$. We call a Skiba point with respect to $\mathcal{Y}$* **nondegenerate***, if all candidates are nondegenerate.*

In order to approximate Skiba points with respect to $\mathcal{Y}$ we have to solve two problems of the type in 6.1.1 and detect initial state values where (6.11) is the same for both solutions.

At first we approximate simultaneously two solutions $\big(x_\pm(t), z_\pm(t)\big) \in \mathbb{R}^{2n}$ which converge to different $\eta_+, \eta_- \in \mathcal{Y}$ and have the same initial state value x_0. The corresponding controls are denoted by $c_\pm(t) = c\big(x_\pm(t), z_\pm(t)\big)$. This is done in an obvious way by doubling the system and possibly scaling the differential equations, if the theoretical convergence rates in (3.21) and (3.22) are different. Notice that these systems are not coupled.

To detect a Skiba point we define one coordinate j of the initial state value $x_0 \in \mathbb{R}^n$ as unknown parameter. Thus we substitute the conditions $x_{+_j}(0) = x_{0_j}$, $x_{-_j}(0) = x_{0_j}$ by $x_{+_j}(0) = x_{-_j}(0)$ and the condition that the values of the objective functions are the same. To define the second "new" condition we use (6.11) evaluated at $t = 0$ for both solutions to achieve a condition which only depends on the initial values.

Typically the set of Skiba points with respect to $\mathcal{Y}$ is locally of dimension $n-1$, i. e. it is a point, if there is one state variable and it is a curve if there are two.

This is seen by following arguments:

If $\bar{x}_0$ is a Skiba point and there are two corresponding nondegenerate optimal candidates with respect to $\mathcal{Y}$ which are asymptotic to η_+ and η_-, then there is a neighborhood V of $\bar{x}_0$, such that there exists unique solutions $\big(x_\pm(x_0), c_\pm(x_0)\big)$ which are asymptotic with $\eta_\pm$ for each $x_0 \in V$.

Let $\hat{U}_\pm(x_0)$ be the value of the objective function at $\big(x_\pm(x_0), c_\pm(x_0)\big)$ for each $x_0 \in V$, then we require $\hat{U}_+(\bar{x}_0) = \hat{U}_-(\bar{x}_0)$ for a Skiba point $\bar{x}_0$.

The sets $W_\pm = \big\{\big(x_0, \hat{U}_+(x_0)\big) \,|x_0 \in \mathbb{R}^n\big\}$ are of dimension n in $\mathbb{R}^{n+1}$. Moreover, typically they intersect transversally at $\bar{x}_0$ and hence $T_{\bar{x}_0}W_+ + T_{\bar{x}_0}W_- = \mathbb{R}^{n+1}$. Thus $T_{\bar{x}_0}W_+ \cap T_{\bar{x}_0}W_-$ is of dimension $2n-(n+1) = n-1$.

To approximate a higher dimensional "Skiba set", we can use the initial state variables (except the j above) as continuation parameters.

6.2 Optimal investment

We analyze a model of optimal investment and consumption policy which is presented in [Semmler and Sieveking, 1998]. There it is shown that a non-explosiveness condition yields that there is a region in state space, where the borrower remains creditworthy.

The full problem can be split into the problem of optimal investment for maximizing the wealth of the economy and into the problem of optimal consumption policy to maximize the utility of this wealth.

We restrict to the problem of optimal investment. Using Pontryagin's maximum principle yields a 2-dimensional dynamical system in state-costate space. It will be seen that the stable manifolds of stationary points are candidates for optimal solutions.

The model

There are two control variables, the investment $j(t)$ and the consumption $c(t) \geq 0$. The two state equations for the per capita capital stock $k(t) \geq 0$ and the per capita stock of debt $b(t)$ are:

$$\dot{k} = i(k,j), \qquad k(0) = k_0, \tag{6.13}$$

$$\dot{b} = \theta b + c - f(k,j), \qquad b(0) = b_0 \tag{6.14}$$

with the net investment function $i(k,j) = j - \sigma k$ (depreciation rate σ), net income function $f(k,j) = c_f k^\alpha - (j + \frac{j^2}{k^\gamma})$, $k > 0$, $f(0,0) = 0$ and the rate of interest θ.

We assume $\sigma \geq 0$, $\theta \geq 0$, $c_f \geq 0$, $\gamma \in (0,1)$, $0 < \alpha < 2-\gamma$.

Moreover, the following non-explosiveness condition for the debt is required:

$$\lim_{t\to\infty} e^{-\theta t} b(t) = 0. \tag{6.15}$$

This asymptotic boundary condition implies that the state variables have to stay bounded if the borrower remains creditworthy (see [Semmler and Sieveking, 1998]).

To optimize the utilization $\hat{U}(c) := \int_0^\infty e^{-\theta_1 t} U(c(t))\mathrm{d}t$ we have to solve

$$\begin{cases} \max_{c,j} \int_0^\infty e^{-\theta_1 t} U(c(t))\mathrm{d}t, & \\ \dot{k} = i(k,j); & k(0) = k_0, \\ \dot{b} = \theta b + c - f(k,j); & b(0) = b_0, \\ \lim_{t\to\infty} e^{-\theta t} b(t) = 0, & \end{cases} \tag{$P_F(k_0,b_0)$}$$

where U is a strictly monotone increasing utilization function.

This problem can be seperated into two optimization problems.

a) Solve the investment problem for $k_0 \in \mathbb{R}_+$

$$\begin{cases} \max_j \hat{F}(k,j), \quad \text{where } \hat{F}(k,j) := \int_0^\infty e^{-\theta s} f(k(s),j(s))\mathrm{d}s, & \\ \dot{k} = i(k,j); & k(0) = k_0. \end{cases} \tag{$P_I(k_0)$}$$

By using an optimal solution $(k^*(t), j^*(t))$ of $(P_I(k_0))$ we define the wealth of the economy at time $t=0$ by $w^* := \int_0^\infty e^{-\theta s} f(k^*(s), j^*(s))\mathrm{d}s - b_0$.

b) Solve the problem of optimal consumption for given (k,j,b_0), and $w \in \mathbb{R}_+$

$$\begin{cases} \max_{c,\bar{c}\leq w} \int_0^\infty U(c(s)) e^{-\theta_1 s}\mathrm{d}s, & \\ \dot{b} = \theta b + c - f(k,j), & b(0) = b_0, \\ \lim_{t\to\infty} e^{-\theta t} b(t) = 0, & \end{cases} \tag{$P_C(w,k,j,b_0)$}$$

where $\bar{c} := \int_0^\infty e^{-\theta s} c(s)\mathrm{d}s$. We denote a solution of $(P_C(w,k,j,b_0))$ by $(b^*(t), c^*(t))$.

It can be seen, that $(\tilde{k},\tilde{b},\tilde{j},\tilde{c})$ is an optimal solution of $(P_F(k_0,b_0))$ if and only if $(\tilde{k},\tilde{j})$ is an optimal solution of $(P_I(k_0))$ and $(\tilde{b},\tilde{c})$ is an optimal solution of $(P_C(\tilde{w},k,j,b_0))$, where $\tilde{w} := \int_0^\infty e^{-\theta s} f(\tilde{k}(s),\tilde{j}(s))\mathrm{d}s - b_0$.

An analytical treatment why and under what conditions such problems can be separated is given in [Sieveking and Semmler, 1998].

Henceforward we treat $(P_I(k_0))$. Let use apply Theorem 6.2 to this system.

THEOREM 6.6 *A necessary condition for $(k^*(t), j^*(t))$, $0 \le t < \infty$, to be an optimal solution of $(P_I(k_0))$ is that there exist a real number $a \in \{0,1\}$ and an initial costate value $x_0 \in \mathbb{R}$, such that*

1. $(a, x_0) \neq 0$.
2. *The costate variable $x(\cdot)$ is the continuous and piecewise differentiable solution of*
$$\begin{cases} \dot{x}(t) = -ae^{-\theta t}\frac{\partial}{\partial k}f(k^*(t), j^*(t)) - x(t)\frac{\partial}{\partial k}i(k^*(t), j^*(t)), \\ x(0) = x_0. \end{cases} \tag{6.16}$$
3. *The investment $j^*(t)$ maximizes*
$$H(k^*(t), j, x(t), a, t) = ae^{-\theta t}f(k^*(t), j(t)) + x\, i(k^*(t), j(t)),$$
the Hamiltonian function at each t where $j^*(t)$ is continuous.
4. *The maximum of the Hamiltonian satisfies for every t*
$$M(t) := H(k^*(t), j^*(t), x(t), a, t) = -q(t),$$
where the function $q(t)$ is the solution of
$$\begin{cases} \dot{q} = \theta e^{-\theta t}f(k^*(t), j^*(t)), \\ \lim_{t\to\infty} q(t) = 0. \end{cases} \tag{6.17}$$

Now we consider the specific problem and draw some conclusions from this theorem. Let (k^*, j^*) be an optimal solution of $(P_I(k_0))$ and let a, x_0, x and q be as in Theorem 6.6.

- If $a = 0$, then *1.* and *2.* imply $x_0 \neq 0$ and $\dot{x}(t) = x(t)\sigma$ (notice $i(k, j) = j - \sigma k$) and hence $x(t) = e^{\sigma t}x_0 \neq 0$. Thus $\frac{\partial}{\partial j}H(k^*(t), j, x(t), a, t) = x(t)\frac{\partial}{\partial j}i(k^*(t), j) = x(t) \neq 0$. This leads to a contradiction to the maximization condition *3.* and hence $a = 1$.
- The maximization condition *3.* implies $j^*(t) = j\big(e^{\theta t}x(t), k^*(t)\big)$ for each $0 \le t < \infty$, where $j(y, k) = \frac{y-1}{2}k^\gamma$. Continuity of x and k^* imply continuity of j^*.
- The scaled function $(k(t), y(t)) := (k^*(t), e^{\theta t}x(t))$ is a solution of

$$\dot{k} = \frac{y-1}{2}k^\gamma - \sigma k, \qquad k(0) = k_0 \tag{6.18}$$
$$\dot{y} = (\theta + \sigma)y - \alpha c_f k^{\alpha - 1} - \gamma\frac{(y-1)^2}{4}k^{\gamma - 1}, \qquad y(0) = x_0. \tag{6.19}$$

- By integration (6.17) implies $q(t) = -\theta \int_t^\infty e^{-\theta s} f(k^*(s), j^*(s))\mathrm{ds}$.
- The definition of y and 4. yields for each t:

$$y(t)\dot{k}^*(t) = \theta \int_t^\infty e^{-\theta(s-t)} f(k^*(s), j^*(s))\mathrm{ds} - f(k^*(t), j^*(t)). \tag{6.20}$$

For the phase portrait of the dynamical system (6.18), (6.19) see Figure 6.1 and Figure 6.2. The system has two stationary points, (k_u, y_u) is a sink and (k_s, y_s) is a hyperbolic saddle.

We approximate a solution on the stable manifold of (k_s, y_s) by the techniques from Chapter 3 and get the solution $z_+(t) = \big(k_+(t), y_+(t)\big)$ drawn in Figure 6.1.

The differential equation (6.18), (6.19) is not defined for $k = 0$ if $\gamma < 1$, nevertheless the problem $(P_I(k_0))$ has an admissible stationary solution for $k_0 = 0$ defined by $k = 0$, $j = 0$, since $f(0,0) = 0$. The corresponding point in the dynamical system (6.18), (6.19) is $(0,1)$ (where (6.19) is not defined), because if $\big(k, g(k)\big)$ defines the curve where $\dot{y} = 0$ holds then $\lim_{k\to 0} g(k) = 1$. Nevertheless for sufficiently small k_0 there exist trajectories $\big(k_-(t), y_-(t)\big)$ which converge in finite time $\hat{T}$ to $(0,1)$. Therefore we can define another bounded admissible solution for $(P_I(k_0))$ by

$$\big(k(t), j(t)\big) = \begin{cases} \big(k_-(t), j(k_-(t), y_-(t))\big) & \text{for} \quad t \in [0, \hat{T}), \\ (0,0) & \text{for} \quad t \in [\hat{T}, \infty). \end{cases}$$

We cannot apply our approximation method to find this solution, thus we use a heuristic approach without error estimates. As reference for an analytical approximation near singularities see [Coddington and Levinson, 1955]. Let

$$\begin{aligned} K(t) &:= 0 + b_1 t^{\epsilon_1} + b_2 t^{\epsilon_2}, \\ Y(t) &:= 1 + a_1 t^{\delta_1} + a_2 t^{\delta_2} \end{aligned}$$

and choose coefficients a_1, a_2, b_1, b_2, $0 < \delta_1 < \delta_2$ and $0 < \epsilon_1 < \epsilon_2$, such that $\big(K(t), Y(t)\big)$ satisfies

$$\begin{pmatrix} \dot{K}(t) \\ \dot{Y}(t) \end{pmatrix} + F(K(t), Y(t)) = \begin{pmatrix} o(t^{\epsilon_2}) \\ o(t^{\delta_2}) \end{pmatrix}$$

where F is defined by (6.18) and (6.19) and $p(t) = o(t^\delta)$ means $\frac{p(t)}{t^\delta} \overset{t\to 0}{\to} 0$. In this particular case with $\alpha > 1$, the exponents are $\delta_1 = 1$, $\epsilon_1 = \frac{2}{1-\gamma}$, $\delta_2 = \epsilon_1(\alpha - 1) + 1$ and $\epsilon_2 = \epsilon_1 + 1$. Thus (K, Y) is an approximation for a solution of the time inverted system (6.18), (6.19) with $K(0) = 0$, $Y(0) = 1$. We choose some (small) $\bar{T} > 0$ and solve the following boundary value problem on $[0, 1]$

$$\begin{pmatrix} \dot{k} \\ \dot{y} \end{pmatrix} = T \cdot F(k, y),\ k(0) = k_0,\ k(1) = K(\bar{T}),\ y(1) = Y(\bar{T}),$$

where T is an additional "free" parameter, defining the time, needed to reach $\big(K(\bar{T}), Y(\bar{T})\big)$. Thus we define an approximation for $z_- = (k_-, y_-)$ by

$$\big(k_-(t), y_-(t)\big) = \begin{cases} \big(k(\frac{t}{T}), y(\frac{t}{T})\big) & : \quad t \in [0, T] \\ \big(K(\bar{T} + T - t), Y(\bar{T} + T - t)\big) & : \quad t \in (T, \bar{T} + T] \\ (0, 1) & : \quad t > \bar{T} + T. \end{cases}$$

In particular, we obtain an approximation of the time $\hat{T} = \bar{T} + T$ which is needed to reach $k = 0$.

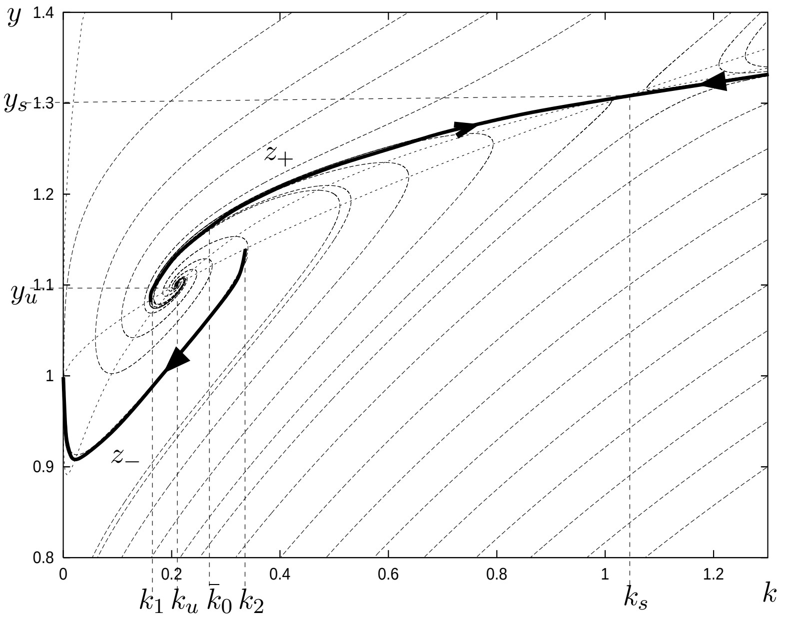

Figure 6.1: k-y phase portrait for (6.18), (6.19) with parameters $\alpha = 1.1$, $c_f = 0.29$, $\sigma = 0.15$, $\theta = 0.1$ and $\gamma = 0.7$.

All solutions which leave the region $\{(k, y) \mid |y| \leq \sqrt{4c_f k^{\alpha-\gamma} + 1}\}$, where $f(k, \frac{y-1}{2} k^\gamma) \geq 0$ (see Figure 6.2 below the bold line) and stay outside do not satisfy (6.20), because its right hand side is negative and its left hand side is positive.

The only solutions which do not leave this region $\{(k, y) \mid f(k, \frac{y-1}{2} k^\gamma) \geq 0\}$ are those converging to $(0, 1)$, (k_u, y_u) or (k_s, y_s).

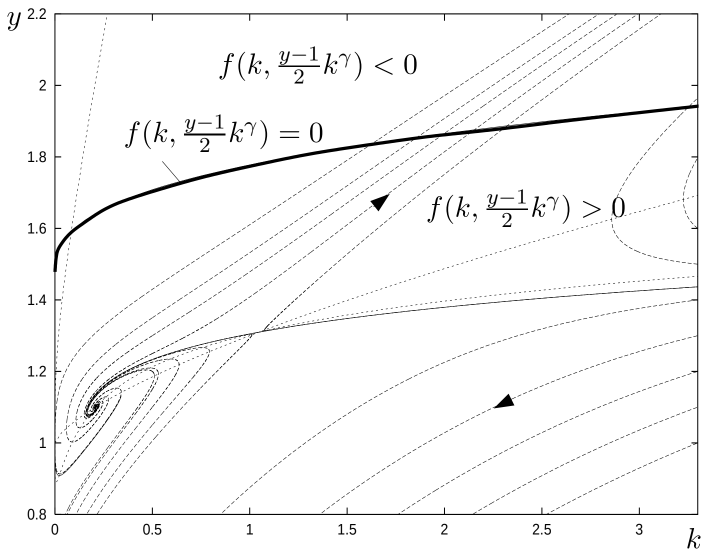

Figure 6.2: k-y phase portrait for (6.18), (6.19) with parameters $\alpha = 1.1$, $c_f = 0.29$, $\sigma = 0.15$, $\theta = 0.1$ and $\gamma = 0.7$.

If there is an optimal trajectory for $(P_I(k_0))$ with $k(\bar{t}) = k_0$, $y(\bar{t}) \neq y_0$ for some $\bar{t} > 0$ then the trajectory $\big(k(\bar{t} + \cdot), y(\bar{t} + \cdot)\big)$ is also optimal for $(P_I(k_0))$.

Hence we can restrict to those solutions which converge monotonically to one of the equilibria.

Let k_1 and k_2 be the values where the z_+ and z_-, respectively, intersect the line with $\dot{y} = 0$ (see Figure 6.1). If $k(0) \leq k_2$, then the solution defined by z_- which converges to $(0, 1)$ is one candidate, if $k(0) \geq k_1$, then the solution defined by z_+ which converges to (k_s, y_s) is another candidate. At $k(0) = k_u$ the stationary solution $(k_u, \sigma k_u)$ is another candidate, but in a neighborhood of k_u there exists no solution converging to k_u and hence k_u is not nondegenerate. Moreover, the value of its objective function is not optimal (see Figure 6.4). If we compute the values of the objective function we get Figure 6.3. In Figure 6.4 we show the difference of the values of the objective function to $\frac{f(k_0, \sigma k_0)}{\theta}$ which is the value of the objective function of the admissible stationary solution of $(P_I(k_0))$ defined by $(k_0, \sigma k_0)$ (notice that this solution does not solve the differential equation (6.18), (6.19)).

Figure 6.4 implies that there is a Skiba point at a value $\bar{k}_0 \approx 0.29$. Notice that this is not k_u, the value of the unstable equilibrium.

At the Skiba point the global optimal strategy changes, for an initial capital stock less than $\bar{k}_0$ the optimal strategy is to "sell" the capital stock and for an initial capital stock larger than $\bar{k}_0$ it is optimal to reach the upper equilibrium.

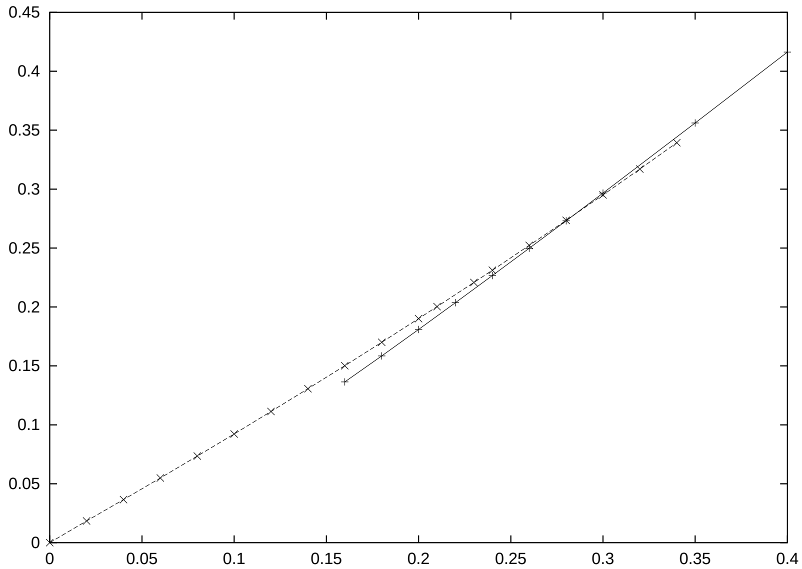

Figure 6.3: Value of the objective function depending on k_0, for z_- plotted by (×- -×) and z_+ plotted by (+—+).

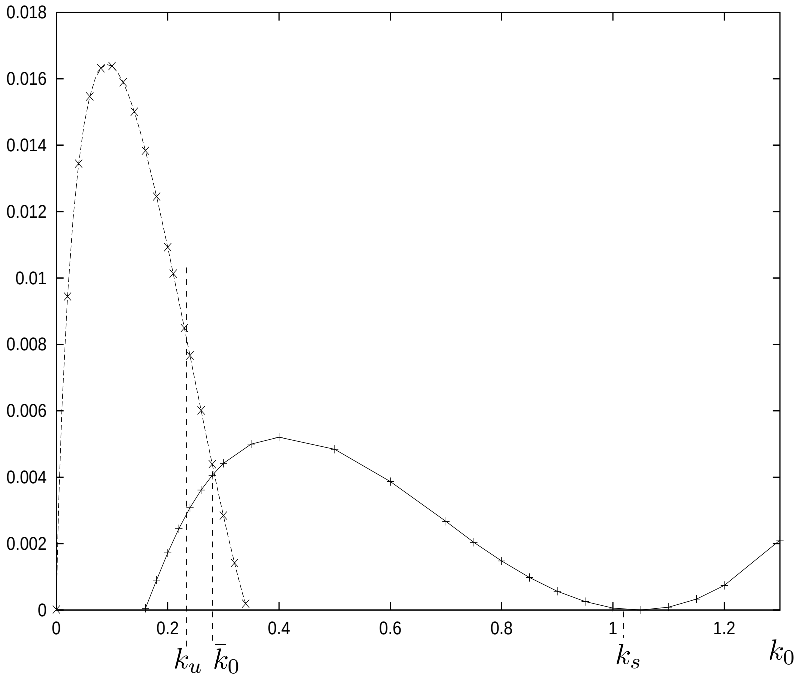

Figure 6.4: Difference of the values from Figure 6.3 to those with constant $k = k_0$, achieved by $j(t) := \sigma k_0$, for different k_0.

6.3 Resource management

A model of optimization in resource management is presented in [Rauscher, 1990] and the solutions near a steady state are analyzed. The existence of periodic orbits in this model, for rather large discount rates is proved in [Feichtinger and Novak, 1991] by detecting a Hopf bifurcation.

We analyze a similar model of optimal exploitation of renewable resources and optimal consumption policy due to [Semmler and Sieveking, 2000].

We show in this section the existence of periodic orbits and how to approximate solutions converging to it by using the method from Chapter 3.

Using Pontryagin's maximum principle we get a 4-dimensional ordinary differential equation and we detect periodic orbits by parameter continuation and bifurcation theory using CONTENT [Kuznetsov and Levitin, 1994].

The model

The control variables are the rate of extraction of resources $q(t) \geq 0$ and the consumption policy $c(t) \geq 0$. The change of the stock of resource $R(t)$ and the level of countries debt $B(t)$ is given by

$$\dot{R} = g(R) - qR, \qquad R(0) = R_0, \tag{6.21}$$

$$\dot{B} = h(B) - f(qR) + c, \qquad B(0) = B_0, \tag{6.22}$$

where $g(R) = eR(1-R)$, $e > 0$ is the natural growth function of resource and the costs of debt are $h(B) = aB^{\sigma}$, $a > 0$, $\sigma > 1$. The production function is defined by $f(D) = p((1+D)^{\gamma} - 1)$, $0 < \gamma < 1$, where D is the amount of used resources.

There exists a critical curve $x^*(R)$, such that it is possible to hold debt bounded if $B(t) \leq x^*(R(t))$. In [Semmler and Sieveking, 2000] it is described how to compute this curve by using vector field analysis.

We want to optimize the utility stream over an infinite time horizon

$$\int_0^{\infty} U(c(t), R(t)) e^{-\delta t} \mathrm{dt}$$

with a discount rate $\delta > 0$ and a utility function $U(c, R) = dc^{\epsilon} R^{\rho}$, $0 < \epsilon, \rho < 1$.

Using Pontryagin's maximum principle (Theorem 6.2) and scaling the costate variables we get necessary condition for an optimal solution. In particular, we obtain that $a = 1$.

Defining scaled costate variables x, y we obtain, that an optimal solution (R, B, x, y) solves the 4-dimensional dynamical system

$$\begin{aligned}
\dot{R} &= eR(1-R) - qR; \qquad R(0) = R_0, & (6.23)\\
\dot{B} &= aB^{\sigma} - p((1+qR)^{\gamma} - 1) + c; \qquad B(0) = B_0, & (6.24)\\
\dot{x} &= x\delta - xe(1-2R) - d\rho c^{\epsilon} R^{\rho-1}, & (6.25)\\
\dot{y} &= y\big(\delta - a\sigma B^{\sigma-1}\big) & (6.26)
\end{aligned}$$

with $q = ((-\frac{yp\gamma}{x})^{\frac{1}{1-\gamma}} - 1)/R$ and $c = (-\frac{dR^{\rho}\epsilon}{y})^{\frac{1}{1-\epsilon}}$.

As in [Semmler and Sieveking, 2000] we start with parameters $\gamma = 0.76$, $\sigma = 1.01$, $d = 0.19$, $\epsilon = 0.21$, $a = 0.1$, $e = 0.08$, $\rho = 0.77$, $\delta = 0.1$, $p = 24.85$ and we detect two equilibria

$$\begin{aligned}
(R_{E1}, B_{E1}, x_{E1}, y_{E1}) &= (0.5762\ ,\ 0.3697\ ,\ 1.1738\ ,\ -0.0624) \quad \text{with}\\
(c_{E1}, q_{E1}) &= (0.3315\ ,\ 0.0339)
\end{aligned}$$

and

$$\begin{aligned}
(R_{E2}, B_{E2}, x_{E2}, y_{E2}) &= (0.0272\ ,\ 0.3697\ ,\ 4.174,\ ,\ -0.2211) \quad \text{with}\\
(c_{E2}, q_{E2}) &= (0.0034\ ,\ 0.0778).
\end{aligned}$$

For further analysis of the system we use the continuation software CONTENT to detect Hopf bifurcations and hence periodic orbits in this system. CONTENT has been developed by Kuznetsov and V. V. Levitin (see [Kuznetsov and Levitin, 1994]).

We follow the branch of equilibria by variation of the parameter δ and reach a limit point. By continuation of the limit point in the 2-dimensional (ϵ, δ)–parameter space we reach a Takens-Bogdanov point at $\epsilon_{TB} = 0.5273669$, $\delta_{TB} = 0.1010684$.

At a Takens-Bogdanov bifurcation a branch of homoclinic orbits and a branch of Hopf bifurcations emenate (see [Kuznetsov, 1995]). For details on the numerical treatment of a Takens-Bogdanov bifurcation and the continuation of the branch of homoclinic orbits see [Beyn, 1994a].

We are interested in periodic orbits and hence follow the branch of Hopf points. This yields the bifurcation diagram of Figure 6.5 and which is shown in detail near $\delta = 0.1$ and $\epsilon = 0.6075$ in Figure 6.6

We concentrate on the equilibrium with the larger value of R. The equilibria in region A have a 4-dimensional unstable manifold. At the branch of Hopf bifurcations unstable

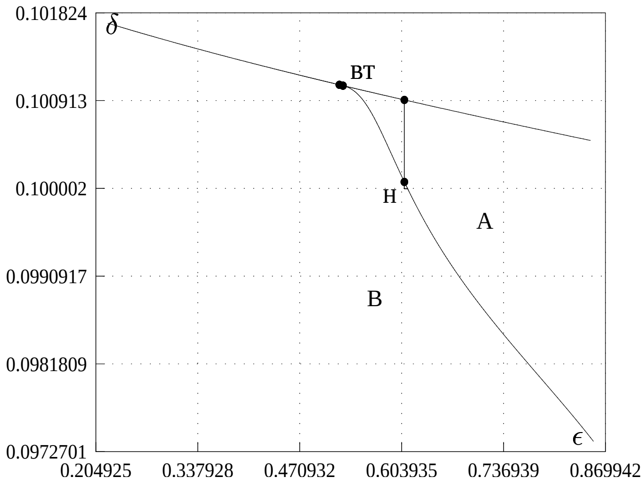

Figure 6.5: Bifurcation diagram in ϵ-δ space. T is the Takens-Bogdanov point an H is on the branch of Hopf points.

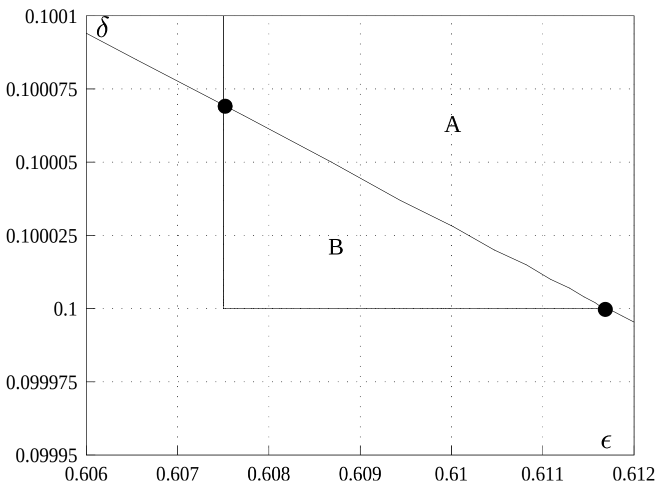

Figure 6.6: Bifurcation diagram in ϵ-δ space near $(0.6075, 0.1)$. It shows the branch of Hopf points and the lines of continuation of equilibria.

periodic orbits arise which are in region B. Moreover, the equilibria in region B have stable and unstable manifolds of dimension 2.

We follow the branch of periodic orbits by continuation with parameter ϵ. This yields Figure 6.7, where we draw ϵ versus the period of the orbit.

Starting at the Hopf point H with unstable periodic orbits we reach a period doubling P, such that one stable direction for the periodic orbit arises. At the turning point for periodic orbits L there is also a period doubling and the periodic orbits which turns back have again one stable direction. Therefore, at $\epsilon = 0.6075$ there are two periodic orbits for the same parameter value plotted by lines (—) in Figure 6.8. Moreover at the same parameter value there is a periodic orbit with "double period", the dashed line (- -) in Figure 6.8.

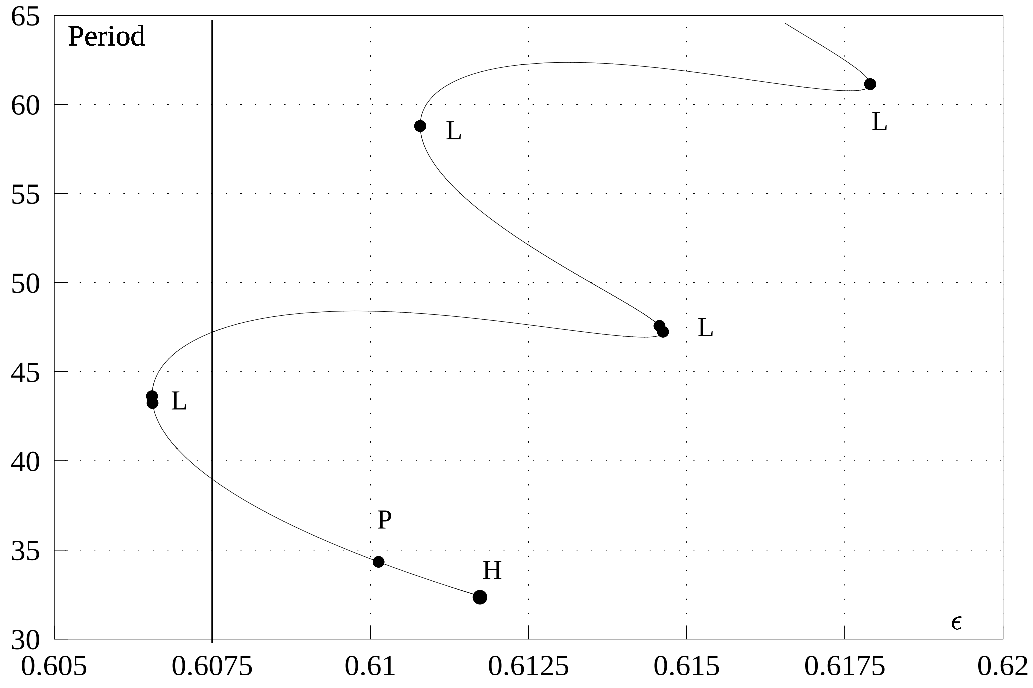

Figure 6.7: Continuation of periodic orbits.

We approximate two solution which converges to the periodic orbits. The phase portrait is drawn in Figure 6.9.

The structure of the dynamics near these periodic orbits is complicated and there are a couple of periodic orbits emenating from period doublings nearby. Thus it seems that the non-degeneracy is "weak". It was not possible to detect an initial value where two candidates exist which are asymptotic with each of the periodic solutions.

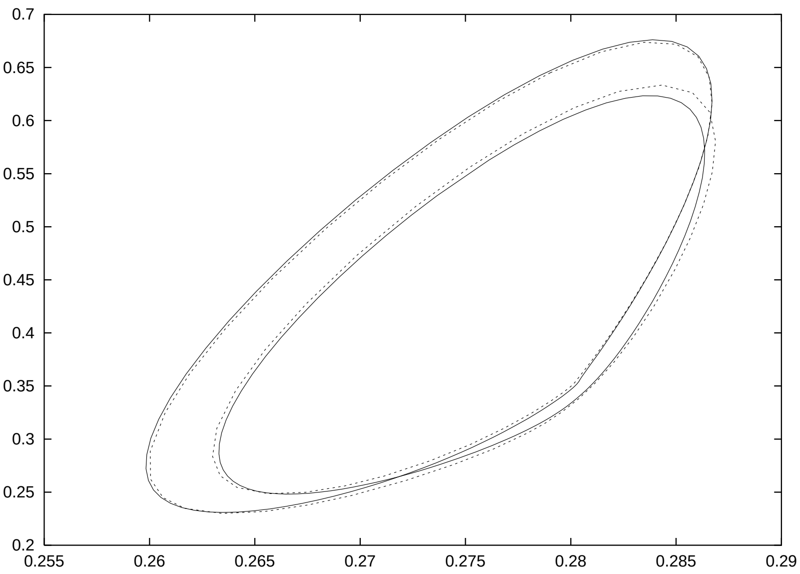

Figure 6.8: Projection of three periodic orbits onto the R–B state space.

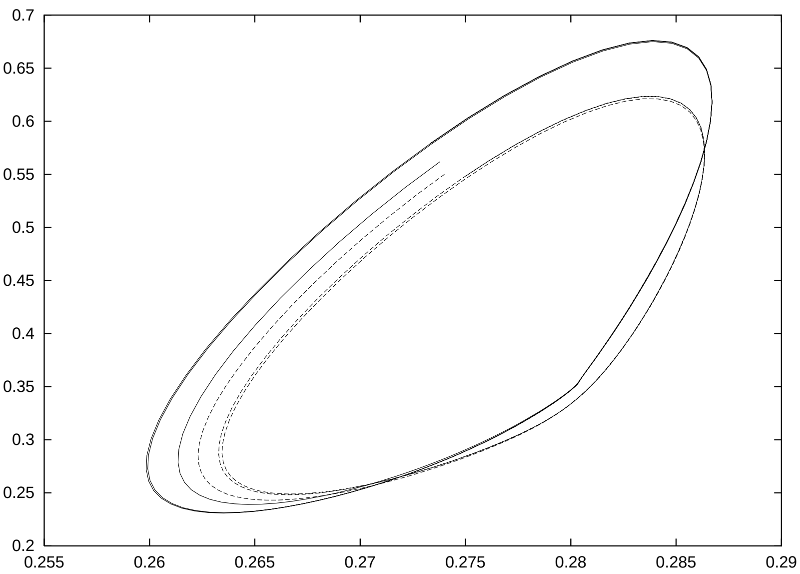

Figure 6.9: Projection of two approximations of solutions converging to different periodic orbits onto the R–B state space.

Appendix A

The higher order chain rule

We use the higher order chain rule as defined in [Kleß, 1997], [Beyn and Kleß, 1998]. There the chain rule is represented by means of a partition of the index set.

Let $i, l \in \mathbb{N}$, $1 \le l \le i$ and $\omega = \{1, \dots, i\}$. We define a set

$$\mathcal{Q}_l(i) := \mathcal{Q}_l(\omega) := \Big\{\pi = (\pi_1, \dots, \pi_l) \Big| \pi_j \subset \omega, \pi_j \neq \emptyset,\ 1 \le j \le l,\ \bigcup_{j=1}^{l} \pi_j = \omega$$
$$\text{and } \pi_j \cap \pi_\kappa = \emptyset,\ \min(\pi_j) < \min(\pi_\kappa) \text{ for all } 1 \le j < \kappa \le l\Big\}$$

and $\iota(\pi) := (\#\pi_1, \cdots, \#\pi_l)$ for $\pi \in \mathcal{Q}_l(i)$, where $\#\pi_j$ is the number of elements of π_j. With this notion we define $\mathcal{Q}(i) := \bigcup_{l=1}^{i} \mathcal{Q}_l(i)$.

Let $[y_j]_{j=1}^{\kappa} := [y_1, \dots, y_\kappa]$ for y_j in some Banach space and $1 \le j \le \kappa \le i$. Moreover, define $\rho_j := \#\pi_j$ and $y_{\pi_j} := [y_{\pi_j(\kappa)}]_{\kappa=1}^{\rho_j}$ for any permutation $\pi_j = (\pi_j(1), \dots, \pi_j(\rho_j))$.

Theorem A.1 *Let $U \subset \mathbb{R}^n$ be open, let $\mathcal{X}$ and $\mathcal{Y}$ Banach spaces and let $g \in C^k(U, \mathcal{X})$, $h \in C^k(\mathcal{X}, \mathcal{Y})$ with $k \ge 1$. Then the following holds for all $x \in U$, $i \in \{1, \dots, k\}$ and $y_1, \dots, y_i \in \mathbb{R}^n$:*

$$(h \circ g)^{(i)}(x)[y_j]_{j=1}^{i} = \sum_{l=1}^{i} \sum_{\substack{\pi \in \mathcal{Q}_l(i) \\ \rho := \iota(\pi)}} h^{(l)}\big(g(x)\big)\left[g^{(\rho_j)}(x) y_{\pi_j}\right]_{j=1}^{l}.$$

The multi-linearity of the higher derivatives of h implies the following lemma which is similar to [Kleinkauf, 1998, Lemma A.0.4].

Lemma A.2 *Let U, $\mathcal{X}$, $\mathcal{Y}$, k, g and h as in Theorem A.1 and $\tilde{g} \in C^k(U,\mathcal{X})$. Then, for all $x, \tilde{x} \in U$, $i \in \{1,\dots,k\}$ and $y_1,\dots,y_i \in \mathbb{R}^n$ the following holds:*

$$
\begin{aligned}
&(h\circ g)^{(i)}(x)[y_j]_{j=1}^{i} - (h\circ\tilde{g})^{(i)}(\tilde{x})[y_j]_{j=1}^{i} \\
&= \sum_{l=1}^{i} \sum_{\substack{\pi\in\mathcal{Q}_l(i)\\ \rho:=\iota(\pi)}} \Big\{ \Big(h^{(l)}\big(g(x)\big) - h^{(l)}\big(\tilde{g}(\tilde{x})\big)\Big) \big[g^{(\rho_j)}(x) y_{\pi_j}\big]_{j=1}^{l} \\
&\qquad + \sum_{\kappa=1}^{l} h^{(l)}\big(\tilde{g}(\tilde{x})\big) \Big[\big[\tilde{g}^{(\rho_j)}(\tilde{x}) y_{\pi_j}\big]_{j=1}^{\kappa-1}, g^{(\rho_\kappa)}(x) y_{\pi_\kappa} - \tilde{g}^{(\rho_\kappa)}(\tilde{x}) y_{\pi_\kappa}, \big[g^{(\rho_j)}(x) y_{\pi_j}\big]_{j=\kappa+1}^{l} \Big] \Big\}
\end{aligned}
$$

The contraction mapping theorem

We use the contraction mapping theorem as in [Irwin, 1980, Theorem C.5].

Theorem A.3 *Let X and Z be subsets of Banach spaces and let $F : X \to Z$ be a contraction, i. e. F is Lipschitz with constant $\kappa < 1$, then F has at most one fixed point. If $X = Z$ and X is complete then F has a fixed point.*

To get smoothness results we apply [Irwin, 1980, Theorem C.7].

Theorem A.4 *Let $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ and $Z \subset \mathcal{X}$ be subsets of Banach spaces and let $F : X \times Y \to Z$ be a uniform contraction on the first factor, i. e. $F(\cdot, y)$ is Lipschitz with constant $\kappa < 1$ for all $y \in Y$. Moreover, let $g : Y \to X$ satisfy $F\big(g(y), y\big) = 0$ for all $y \in Y$. Then $g \in C^k(Y,\mathcal{X})$ if $F \in C^k(X \times Y, \mathcal{X})$.*

The immersive mapping theorem

To define local embeddings we apply the immersive mapping theorem [Irwin, 1980, Exercise C.12].

Theorem A.5 *Let $f : X \to Y$ be a C^l map of manifolds ($l \geq 1$) and let f be immersive at x_0, i. e. $T_{x_0}f$ is injective and $\mathcal{R}(T_{x_0}f)$ splits $T_{f(x_0)}Y$, then there exists a neighborhood U of x_0 such that the image of $f|_U$ is a C^l submanifold of Y.*

Remark: If $\mathcal{R}(T_{x_0}f)$ has finite dimension then splitting is automatic (see [Irwin, 1980, Note A.39]).

The implicit function theorem

The following version of the implicit function theorem is derived from [Dieudonné, 1969, (10.2.1),(10.2.3)].

THEOREM A.6 (Implicit function theorem) *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be three Banach spaces and let $F \in C^k\big(A, \mathcal{Z}\big)$, $k \geq 1$, where A is an open subset of $\mathcal{X} \times \mathcal{Y}$. Let $(x_0, y_0) \in A$, such that $F(x_0, y_0) = 0$ and such that the partial derivative $\frac{\partial}{\partial x} F(x_0, y_0)$ is a linear homeomorphism of $\mathcal{X}$ onto $\mathcal{Z}$. Then, there exist open neighborhoods V of x_0 in $\mathcal{X}$, U of y_0 in $\mathcal{Y}$ and a mapping $u \in C^k\big(U, V\big)$, such that $u(y_0) = x_0$, $\big(u(y), y\big) \in A$ and $F\big(u(y), y\big) = 0$ for any $y \in U$. Furthermore the derivative of u is given by*

$$u'(y) = -\Big(\frac{\partial}{\partial x} F\big(u(y), y\big)\Big)^{-1} \circ \Big(\frac{\partial}{\partial y} F\big(u(y), y\big)\Big).$$

Moreover, for each $(x, y) \in V \times U$ the following equivalence holds true:

$$x = u(y) \Longleftrightarrow F\big(x, y\big) = 0.$$

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