Chromatic Index Critical Graphs and Multigraphs

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Chapter 1

Introduction

The theory of graph coloring has its roots in the year 1852 when the four color conjecture arose which states that the countries of every planar map can be colored with four colors such that neighboring countries are colored differently. This conjecture had been unsolved until 1976, when Appel and Haken [2] proved it and it became the four color theorem. Tait [28] proved in 1880 that the four color theorem is equivalent to the statement that the edges of every cubic map can be 3-colored. Although Tait's proof of the theorem with the help of that equivalency was not correct, this was the origin of edge coloring of graphs and multigraphs. The chromatic index $\chi'(M)$ of a multigraph M is defined to be the smallest number k such that the edges of M can be colored with k colors where adjacent edges are colored differently. A trivial lower bound for the chromatic index of M is the maximum degree $\Delta(M)$ of M. König [21] proved that this lower bound is attained for bipartite multigraphs. This was one of the first results determining the chromatic index for a class of multigraphs.

The first upper bound for the chromatic index involving all multigraphs was obtained by Shannon [27] who proved that $\chi'(M) \leq \frac{3}{2}\Delta(M)$ for every multigraph M, and that there are some multigraphs for which this bound is attained. The breakthrough in coloring multigraphs came in 1964 when Vizing [30] proved that $\chi'(M) \leq \Delta(M) + \mu(M)$ for every multigraph M where $\mu(M)$ is the maximum edge multiplicity of M. This result gives a natural partition of the set of simple graphs into two classes, and a graph G is defined to be in class one, if $\chi'(G) = \Delta(G)$, and in class two, if $\chi'(G) = \Delta(G) + 1$.

A multigraph M is called critical, if $\chi'(M) > \Delta(M)$, and the deletion of any edge of M lowers the chromatic index. Let a multigraph with an odd number n of vertices be colored such that adjacent edges are colored differently. Then there are at most $(n-1)/2$ edges colored equally. This yields a lower bound for the chromatic index of a multigraph M with n vertices and m edges: $\chi'(M) \geq \lceil m/\lfloor n/2 \rfloor \rceil$. A conjecture of Goldberg [11], Andersen [1], and Seymour [27] states that for every critical multigraph M with $\chi'(M) > \Delta(M)+1$, this lower bound is attained. On the other hand, there are critical graphs and multigraphs with $\chi'(M) = \Delta(M) + 1$ where the bound is not attained. We define such multigraphs to be non-elementary.

In contrast to elementary critical multigraphs where just the number of edges is responsible for the high chromatic index, there have to be structural properties why a non-elementary critical multigraph is not Δ-colorable.

Another reason to be interested in non- elementary critical graph is the computational complexity of finding the chromatic index of a given graph. As shown by Holyer [17], this problem is NP-complete. On the other hand, Hoffman, Mitchem, and Schmeichel [16] gave an algorithm which decides in polynomial time, if a graph with maximum degree Δ contains a subgraph with n vertices and at least $\lfloor n/2 \rfloor \Delta + 1$ edges. Hence, the chromatic index of a graph G with fixed maximum degree Δ can be determined in polynomial time, if we know that G does not contain a non-elementary critical graph with maximum degree Δ.

There are some famous conjectures in graph theory for which a minimal counterexample would be a k-regular not k-colorable graph without a small edge cut. For example, a minimal counterexample of Tutte's 5-flow conjecture must be a 3-regular graph with chromatic index 4 and without edge-cut of at most four edges. Hence, such a graph contains a non-elementary 3-critical graph.

These examples give an impression why non-elementary critical graphs are important for edge coloring and other graph theoretic fields. Since there are some constructions which produce non-elementary critical graphs from multigraphs, we are also interested in multigraphs, and it is the aim of this thesis to construct new non-elementary critical multigraphs and to derive properties of such multigraphs.

A sufficient condition for a critical multigraph to be non-elementary is to have an even number of vertices. Until 1977 all known critical graphs had an odd number of vertices, and the critical graph conjecture stating that there are no critical graphs of even order was very popular. After this conjecture had been disproved by constructing some counterexamples with maximum degree 3 or 4, Chetwynd and Wilson [6] published the weak critical graph conjecture which states that there is a constant c such that every critical multigraph M with at most $c \cdot \Delta(M)$ vertices has an odd number of vertices. In Chapter 3 we give a construction of Δ-critical multigraphs with exactly twenty vertices for every $\Delta \geq 5$ which disproves even the weak critical graph conjecture. This result has already been published in [14].

A bridgeless 3-regular graph with chromatic index 4 is called a snark. Hence, the four color theorem is equivalent to the statement that there is no planar snark, and a counterexample to Tutte's 5-flow conjecture has to be a snark. Many research has been done to determine snarks, and since every snark contains at least one non-elementary critical subgraph, there are many non-elementary 3-critical graphs known. In Chapter 4 we give a construction which produces several 4-critical multigraphs from given 3-critical graphs. We show that the most famous 4-critical graphs and multigraphs can be obtained from 3-critical graphs by applying this construction, and we construct 4-critical graphs where the number of edges is relatively small.

Seymour [26] conjectured that there is no planar non-elementary critical graph. This is a very strong conjecture, and it implies the four color theorem, the existence of an algorithm determining the chromatic index of a planar graph in polynomial time, and the non-existence of planar class two graphs with maximum degree at least 6. The latter implication of Seymour's conjecture is known as Vizing's planar graph conjecture [31] from 1965. Vizing also proved the case $\Delta(G) \geq 8$. There are (elementary) planar class two graphs known with maximum degree 2, 3, 4, and 5, respectively. The two cases $\Delta = 6$ and $\Delta = 7$ have remained open for the last 35 years. In this thesis we will prove the case $\Delta = 7$.

There are many papers giving necessary conditions for a possible counterexample to the planar graph conjecture. Fiorini [9] proved that there is no outerplanar class two graph with maximum degree at least 3, and Yap ([32], p. 51) derived that $n_3 + n_4 + n_5 \geq 6$ and $n_7 \geq 6 + 2n_2$ for every 7-critical planar graph, where n_i denotes the number of vertices of degree i. For 6-critical graphs, Yap proved that $2n_2 + 3n_3 + 2n_4 + n_5 \geq 12$ and $n_6 \geq 4 + \frac{4}{3}n_2 + \frac{1}{6}n_5$. According to Fiorini and Wilson ([10], p. 109), Kronk, Radlowski, and Franen proved that a planar class two graph with maximum degree at least 5 contains a triangle.

The main tool which is used to derive the above mentioned results is Vizing's adjacency lemma stating that for every k-critical graph with adjacent vertices u, v, the vertex v is adjacent to at least $k - d_G(u) + 1$ vertices of maximum degree. Since there are planar graphs with maximum degree 6 and 7, respectively, which fulfill the adjacency lemma, there are some more structural properties of critical graphs needed for a complete proof of one of these cases.

In the first section of Chapter 5 we consider critical graphs where the number of neighbors of maximum degree of a vertex v attains the lower bound of Vizing's adjacency lemma. In this case we prove several lower bounds for the degree of a vertex u such that the distance between u and v is 2 or 3. These refinements of the adjacency lemma are not restricted to planar graphs and may be useful to settle other questions concerning the structure of critical graphs, for example to derive new lower bounds for the number of edges.

In the second section of Chapter 5 we prove Vizing's planar graph conjecture for $\Delta = 7$. We assume that there is a 7-critical planar graph, and then we use the properties proven in the first section and the discharging method to construct a plane multigraph with average degree at least 6 and without faces containing exactly two vertices. The existence of such a multigraph contradicts Euler's polyhedra formula. Hence, there is no 7-critical planar graph, and every planar graph with maximum degree 7 is in class one.

Chapter 2

Basic definitions and results

In this section we give the definition of the most important terms in edge coloring and some standard results which will be used frequently. For the definition of basic graph theoretic terms, we refer to [4].

We consider connected multigraphs without loops. Throughout the thesis, we denote the edge set and the vertex set of a multigraph M by $E(M)$ and $V(M)$, the degree of a vertex v of M by $d_M(v)$, and the maximum degree of M by $\Delta(M)$. If $m \geq 1$ edges join the same pair v, w of vertices of M, then vw is called a *multiple edge* of *multiplicity* m or an m*-edge*. A *proper multiple edge* is a multiple edge of multiplicity at least 2 and a *graph* is a multigraph without proper multiple edges. An r*-coloring* of M is a function $\pi : E(M) \to \{1, ..., r\}$ so that $\pi(e) \neq \pi(f)$ for adjacent edges e and f. The *chromatic index* $\chi'(M)$ of a multigraph M is the smallest number r so that there is an r-coloring of M. A multigraph M is called critical, if $\chi'(M) > \Delta(M)$ and $\chi'(M - e) = \chi'(M) - 1$ for each edge e of M. We define M to be k*-critical*, if it is critical and $k = \Delta(M) = \chi'(M) - 1$. A k-critical multigraph with n vertices is called *non-elementary*, if it has at most $k\lfloor n/2 \rfloor$ edges, and it is *elementary*, else. Let M be a multigraph with maximum degree k, and let H be a submultigraph on vertices $v_1, ..., v_n$ of M. Then $s_k(H) = kn - \sum_{i=1}^{n} d_M(v_i)$ is the k*-deficiency* of H. We call $s_k(M)$ the *total deficiency* of M. Hence, a k-critical multigraph of odd order is elementary, if its total deficiency equals $k - 2$.

For an r-coloring of a multigraph M, we say that a color i is *missing* in a vertex v of M, if none of the edges incident to v is colored with i. The set of colors missing in v is denoted by $\overline{v}$. Let i be missing in v, and let j be an other color. Then a connected component in the spanning submultigraph of M which contains exactly the edges colored with i or j is an (i, j)*-chain*. If i is missing in v then the (i, j)-chain Q containing v is called the (i, j)-chain *beginning* or *starting* in v.

The Petersen graph will be denoted by P, and the graph obtained from P by deleting a vertex is denoted by P^-. The complete graph with n vertices is denoted by K_n, and $K_{m,n}$ is the complete bipartite graph with m vertices in one partition and n vertices in the other partition.

The most famous statement about the structure of k-critical graphs is Vizing's adjacency lemma [31] which is abbreviated as VAL.

Theorem 2.1 (VAL) *Let G be a k-critical graph with adjacent vertices u, v. Then v is adjacent to at least $k - d_G(u) + 1$ vertices of maximum degree different from u.*

$\square$

One further property of k-critical graphs is given by the following trivial statement a proof of which can be found in [32].

Lemma 2.1 *A k-critical graph is connected and contains no cut-vertex.*

$\square$

A *planar embedding* of a multigraph M is a function ϕ that assigns to each vertex a point in the plane and to each edge a simple curve in the plane, such that the curves $\phi(e_1)$ and $\phi(e_2)$ meet at a point p, if and only if $p = \phi(v)$ for some vertex v which is incident in M to edges e_1 and e_2. A multigraph is *planar*, if it admits a plane embedding, and a *plane* multigraph is a a pair (M, ϕ) of a planar multigraph and ϕ is a planar embedding of M. We will use just M for a plane multigraph, if there is no danger of confusion which embedding of M is meant. The topological closures of the connected components of the complement of the image of ϕ applied to M are the *faces* of M. An edge and a face of M are *incident*, if the edge is contained in the face. The *boundary* of a face F consists of the edges incident to F and is denoted by $E(F)$. Two vertices u, v are said to be *adjacent in* F, if there is an edge $uv \in E(F)$. In a 2-connected plane multigraph, the boundary of each face is a circuit. If the boundary of a face F is a circuit of length k, we say that k is the length of F, and F is a k-face.

A graph H is obtained from G by *subdividing* the edge uw of G, if the edge uw is replaced by a path (uvw) where v is a new vertex. A graph H is called a *subdivision* of G, if H can be derived from G by recursively subdividing edges.

Kuratowski [23] gave a theorem which characterizes planar graphs.

Theorem 2.2 *A graph is planar, if and only if it contains no subdivision of $K_{3,3}$ or K_5.*

$\square$

Euler's formula gives a relation between the numbers of faces, edges, and vertices of a plane multigraph.

Theorem 2.3 *If M is a connected plane multigraph with n vertices, e edges, and f faces, then $n - e + f = 2$.*

□

For 2-connected planar multigraphs without 2-faces, Euler's formula implies a bound for the number of edges.

Lemma 2.2 *If M is a connected plane multigraph with n vertices, $e \geq 2$ edges, and without 2-faces, then $e \leq 3n - 6$.*

□

Chapter 3

Critical multigraphs of even order

Jakobsen [19] formulated the famous critical graph conjecture, which claims that every critical multigraph M has odd order. This conjecture was disproved by Goldberg [13] for $\Delta(M) = 3$, and Chetwynd and Fiol independently disproved it for $\Delta(M) = 4$ (see [6]). The author and Steffen [15] constructed k-critical graphs and multigraphs of even order for all $k \geq 4$, thus the critical graph conjecture is disproved for every maximum degree at least 3. A new conjecture appeared in a paper by Chetwynd and Wilson [6].

Conjecture 3.1 (Weak Critical Graph Conjecture) *There exists a constant $c > 0$ such that every critical multigraph M with at most $c \cdot \Delta(M)$ vertices has odd order.*

According to Chetwynd and Wilson [6], Yap [32], and Jensen and Toft [20], this conjecture was formulated by Goldberg [12], but in fact it does not appear in Goldberg's paper.

We disprove Conjecture 3.1 by proving the following theorem.

Theorem 3.1 *For all $k \geq 5$, there exists a k-critical multigraph of order 20.*

3.1 Basic Theorems

In this section we present several tools for constructing k-critical multigraphs.

Let M be a multigraph and S a set of edges of M. We say that N is obtained from M by *adding S* if the multiplicity of each edge of S increases by 1.

Lemma 3.1 ([15]) *Let $k \geq 0$ and M be a multigraph obtained from P by adding k 1-factors of P. Then $M - v$ is a $(k+3)$-critical multigraph for any $v \in V(M)$.*

$\square$

A multigraph N is said to be obtained from M and M' by *Hajós-union* where v and v' are identified, if $v \in V(M)$, $v' \in V(M')$, and N is constructed from M and M' as follows:

1. One edge $uv \in E(M)$ and one edge $u'v' \in E(M')$ are removed.
2. The vertices v and v' are identified.
3. A new edge uu' is inserted.

Lemma 3.2 ([18]) *Let M and M' be two k-critical multigraphs and $v \in V(M)$, $v' \in V(M')$ two vertices such that $d_M(v) + d_{M'}(v') \leq k+2$. Then any multigraph obtained from M and M' by Hajós-union where v and v' are identified is k-critical.*

□

The next lemma allows the contraction of a proper submultigraph with k-deficiency k of a k-critical multigraph.

Lemma 3.3 ([15]) *Let M be a k-critical multigraph, $k \geq 2$, and N an induced proper submultigraph of odd order. If $s_k(N) = k$, then the multigraph M^- obtained from M by contracting N to a single vertex is k-critical.*

□

Let M be a multigraph. We say that M^+ is obtained from M by *planting* an m-edge on the edge $uv \in E(M)$ if M^+ is obtained from M by removing uv and inserting vertices u^+, v^+ and the edges uu^+, vv^+ and an m-edge u^+v^+.

Lemma 3.4 *Let M be a k-critical multigraph, $k \geq 2$, and let M^+ be a multigraph obtained from M by planting a $(k-1)$-edge on an edge $uv \in E(M)$. Then M^+ is a k-critical multigraph.*

Proof. Let T be the multigraph with three vertices t, u^+, v^+, edges tu^+, tv^+, and a $(k-1)$-edge u^+v^+. Clearly, T is a k-critical multigraph. The multigraph M^+ is obtained from M and T by Hajós-union where t and u are identified, the edges uv and tv^+ are removed, and a new edge vv^+ is inserted. By Lemma 3.2, M^+ is a k-critical multigraph.

□

3.2 Construction

Let the vertices and edges of P be labeled as in Figure 3.1.

Then the edge sets $F_1 = \{v_2v_3, v_1v_6, v_4v_7, v_5x, zy\}$ and $F_2 = \{v_2v_3, v_4v_5, v_1v_7, v_6y, zx\}$ are 1-factors of P. For $k \geq 5$ and $1 \leq j \leq k-2$, let P_k^j be the multigraph obtained from P by adding $j-1$ copies of F_1 and $k-j-2$ copies of F_2. By Lemma 3.1, the

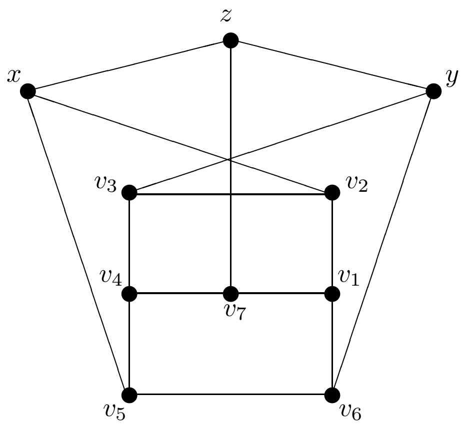

Figure 3.1: The Petersen Graph

multigraph $H_k^j = P_k^j - z$ is k-critical. Furthermore, $d_{H_k^j}(v_7) = k-1$, $d_{H_k^j}(x) = j+1$, and $d_{H_k^j}(y) = k-j$.

Let $1 \leq l \leq k-4$. The multigraphs H_k^l, H_k^{l+1} and H_k^{l+2} are joined by Hajós-union to obtain a new k-critical multigraph. To avoid confusion we indicate the vertices of H_k^j with an upper index j. Firstly, H_k^l and H_k^{l+1} are joined by Hajós-union where the vertices y^l and x^{l+1} are identified, the edges $v_3^l y^l$ and $x^{l+1}v_2^{l+1}$ are removed, and a new edge $v_3^l v_2^{l+1}$ is inserted. Secondly, this multigraph and H_k^{l+2} are joined by Hajós-union where the vertices y^{l+1} and x^{l+2} are identified, the edges $v_3^{l+1}y^{l+1}$ and $x^{l+2}v_2^{l+2}$ are removed, and a new edge $v_3^{l+1}v_2^{l+2}$ is inserted. The obtained multigraph is called M_k^l. By Lemma 3.2, M_k^l is k-critical. The submultigraphs T^l induced by y^l,v_6^l,v_5^{l+1} and T^{l+1} induced by y^{l+1},v_6^{l+1},v_5^{l+2} have k-deficiency k. Contracting T^l to a single vertex t^l and T^{l+1} to a single vertex t^{l+1} yields a new multigraph N_k^l of order 21, where $d_{N_k^l}(x^l) = l+1$ and $d_{N_k^l}(y^{l+2}) = k-l-2$. Lemma 3.3 implies that N_k^l is k-critical. Let Z_k^l be the multigraph obtained from N_k^l by identifying the vertices x^l and y^{l+2} to a single vertex v^l. Clearly, Z_k^l has order 20, maximum degree k, and it has no k-coloring.

Theorem 3.2 *For each $k \geq 5$ and $1 \leq l \leq k-4$, the multigraph Z_k^l is k-critical.*

Proof. The edge sets
$F_3 = \{v^1v_5^1, v_4^1v_7^1, v_1^1t^1, v_4^2v_7^2, v_1^2t^2, v_4^3v_7^3, v_1^3v_6^3, v_2^1v_3^1, v_2^2v_3^2, v_2^3v_3^3\}$ and
$F_4 = \{v_5^1v_4^1, v_7^1v_1^1, t^1v_4^2, v_7^2v_1^2, t^2v_4^3, v_7^3v_1^3, v_6^3v^1, v_2^1v_3^1, v_2^2v_3^2, v_2^3v_3^3\}$ are
1-factors of Z_5^1. It is easy to see that Z_k^l is isomorphic to the multigraph obtained from Z_5^1 by adding $l-1$ copies of F_3 and $k-4-l$ copies of F_4. Since every edge of Z_k^l can be considered as an edge of Z_5^1, it is sufficient to prove that Z_5^1 is a 5-critical multigraph.

Let $V_1 = \{v_1^1, v_2^1, v_7^1, v^1, v_3^3, v_4^3, v_7^3\}$, $V_2 = \{v_1^2, v_7^2, t^2, v_1^3, v_2^3, v_3^3, v_4^3, v_6^3, v_7^3\}$, and
$V_3 = \{v_4^2, v_7^2, t^1, v_1^1, v_2^1, v_3^1, v_4^1, v_5^1, v_7^1\}$. Then for any edge e of Z_5^1 there exists $i \in \{1,2,3\}$ such that $e \in E(Z_5^1 - V_i)$.

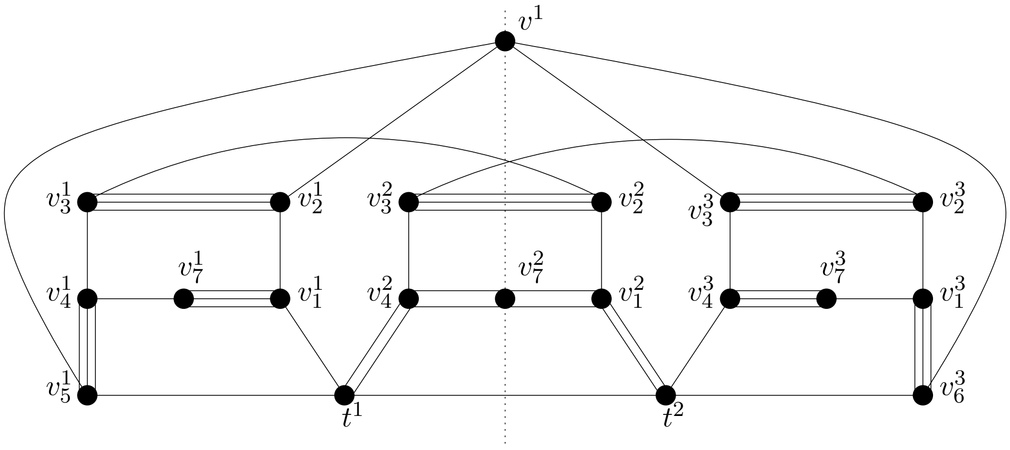

Figure 3.2: The multigraph Z_5^1 with a dotted drawn axis of symmetry

Case 1: $e \in E(Z_5^1 - V_1)$.
For $l = 1$ the above defined edge sets F_3 and F_4 are 1-factors of Z_5^1 where each edge which is contained in both, F_3 and F_4, is a proper multiple edge in Z_5^1. Hence, the multigraph $Z_1 = (Z_5^1 - F_3) - F_4$ (see Figure 3.3) is well defined and has no 3-coloring. Two vertices

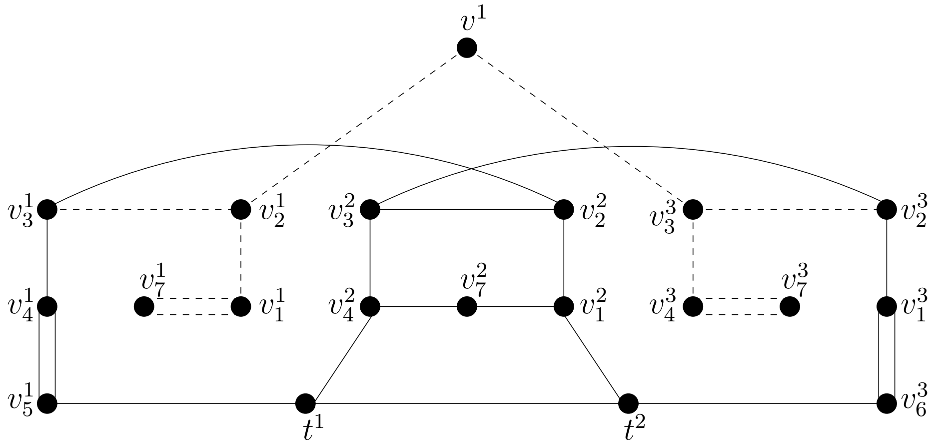

Figure 3.3: The multigraph Z_1 where the edges incident with a vertex of V_1 are drawn dashed

of $Z_1 - V_1$ are adjacent, if and only if they are adjacent in $Z_5^1 - V_1$. Therefore, e can be considered as an edge of $Z_1 - V_1$. The multigraph $Z_1 - V_1$ is 3-critical, since it is obtained from $P - v$ by planting a 2-edge on two edges of $P - v$, respectively. Obviously, any 3-coloring of $(Z_1 - V_1) - e$ can be extended to a 3-coloring of $Z_1 - e$. By coloring F_3 and F_4 with a new color, respectively, we obtain a 5-coloring of $Z_5^1 - e$.

Case 2: $e \in E(Z_5^1 - V_2)$.
The edge sets $F_5 = \{v^1v_3^3, v_2^3v_3^2, v_2^2v_1^2, v_7^2v_4^2, t^1t^2, v_4^3v_7^3, v_1^3v_6^3, v_2^1v_3^1, v_4^1v_5^1, v_1^1v_7^1\}$ and $F_6 = \{v_3^3v_2^3, v_3^2v_2^2, v_1^2v_7^2, v_4^2t^1, t^2v_4^3, v_7^3v_1^3, v_6^3v^1, v_2^1v_3^1, v_4^1v_5^1, v_1^1v_7^1\}$ are 1-factors of Z_5^1 where each edge which is contained in both, F_5 and F_6, is a proper multiple edge in Z_5^1. Hence, the multigraph $Z_2 = (Z_5^1 - F_5) - F_6$ is well defined and has no 3-coloring.

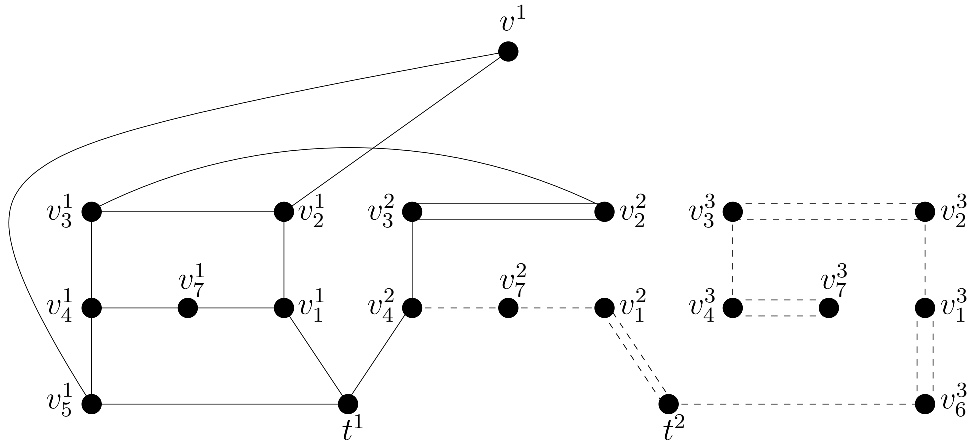

Figure 3.4: The multigraph Z_2 where the edges incident with a vertex of V_2 are drawn dashed

Two vertices of $Z_2 - V_2$ are adjacent, if and only if they are adjacent in $Z_5^1 - V_2$. Therefore, e can be considered as an edge of $Z_2 - V_2$. The multigraph $Z_2 - V_2$ is 3-critical, since it is obtained from $P - v$ by planting a 2-edge on one edge of $P - v$. Obviously, any 3-coloring of $(Z_2 - V_2) - e$ can be extended to a 3-coloring of $Z_2 - e$. By coloring F_5 and F_6 with a new color, respectively, we obtain a 5-coloring of $Z_5^1 - e$.

Case 3: $e \in E(Z_5^1 - V_3)$.
By the symmetry of Z_5^1 (see Figure 3.2), there is an isomorphism of Z_5^1 where the image of each edge of $Z_5^1 - V_3$ is an edge of $Z_5^1 - V_2$. Hence, Case 3 can be reduced to Case 2.

Since there is a 5-coloring of $Z_5^1 - e$ for any edge $e \in E(Z_5^1)$, the multigraph Z_5^1 is 5-critical.

□

3.3 Simple critical graphs of even order

The 3-critical graph of order 22 by Goldberg and the 4-critical graphs of order 18 by Chetwynd and Fiol are the smallest Δ-critical graphs of even order for $\Delta = 3, 4$, respectively, as shown in [5]. We will now construct simple Δ-critical graphs of small even order for $\Delta \geq 5$. We use the following lemmas which are proved in [15].

Lemma 3.5 ([15]) *For $k \geq 2$, let M be a multigraph with $\Delta(M) = k$, and $v \in V(M)$ with $d_M(v) = d \leq k$. Let $v_1, ..., v_d$ be the not necessarily distinct neighbors of v, and $u_1, ..., u_k$ the vertices of degree $k-1$ in the complete bipartite graph $K_{k,k-1}$. Let M' be the multigraph obtained from $M - v$ and $K_{k,k-1}$ by adding edges $v_i u_i$ for $i = 1, ..., d$. Then M is k-critical, if and only if M' is k-critical.*

□

Lemma 3.6 ([15]) *Let $k \geq 3$ be odd, M a multigraph and $v \in V(M)$ with $d_M(v) = \Delta(M) = k$. Let $v_1, ..., v_k$ be the not necessarily distinct neighbors of v, and $u_1, ..., u_k$ the vertices of the complete graph K_k. Let M' be the multigraph obtained from $M - v$ and K_k by adding edges $v_i u_i$ for $i = 1, ..., k$. Then M is k-critical, if and only if M' is k-critical.*

□

Let $S_1 = \{v_4^1, v_1^1, v_4^2, v_1^2, v_4^3, v_1^3, v_2^1, v_2^2, v_2^3\}$. Then every proper multiple edge of Z_5^1 is incident to a vertex of S_1. Hence, the graph obtained from Z_5^1 by replacing each vertex of S_1 by the graph K_5 is simple, and by Lemma 3.6, it is 5-critical. Let $S_2 = \{v_5^1, v_7^1, t^1, v_7^2, t^2, v_7^3, v_6^3, v_2^1, v_2^2, v_2^3\}$. Then every proper multiple edge of Z_k^l with $k \geq 6$ and $1 \leq l \leq k-4$ is incident to a vertex of S_2. By replacing each vertex of S_2 by the graph K_k (for k odd) or by the graph $K_{k,k-1}$ (for k even), we obtain a simple k-critical graph by Lemma 3.6 or Lemma 3.5.

Corollary 3.1 *There is a k-critical simple graph of order*

(a) *56 for $\Delta = 5$,*

(b) $10\Delta + 10$ *for every odd $\Delta \geq 7$,*

(c) 20Δ *for every even $\Delta \geq 6$.*

□

For every $k \geq 5$, the constructed graphs are the smallest known simple k-critical graphs of even order. This improves the result of [15] where k-critical graphs of higher even order are constructed. For even k, it might be possible to construct smaller k-critical graphs of even order by replacing the vertices of S_2 by a graph of odd order at most $2k-3$ and with maximum degree and total deficiency k, but therefore a theorem similar to Lemma 3.6 or 3.5 had to be proved.

Chapter 4

Constructing 4-critical graphs and multigraphs from 3-critical graphs

We give a construction which produces several 4-critical multigraphs from a given 3-critical graph with more than one vertex of degree 2. Especially, this construction yields known 4-critical subgraphs of the line graphs of some snarks and two 4-critical multigraphs of order 16. One of them is new. Further, for every natural number k, we construct 4-critical graphs with $13k+2$ vertices and total deficiency $4k+2$. There is no infinite sequence of Δ-critical simple graphs with constant Δ known where the total deficiency grows faster with the number of vertices.

4.1 Construction

Lemma 4.1 *Let G be a 3-critical graph and G' be a graph obtained from G by replacing a vertex v of G by a triangle. Then G' is 3-critical, too.*

Proof. If v has degree 3, then the statement follows from Lemma 3.6. Let $d_G(v) = 2$, and let w_1, w_2 be the neighbors of v in G. Let (v_1, v_2, v_3) be the triangle replacing v in G' such that v_1w_1 and v_2w_2 are edges of G'. Then G' is not 3-colorable, since otherwise the edges of $G - \{v_1, v_2, v_3\}$ can be colored as before, and the edges vw_1, vw_2 can be colored as v_1w_1, v_2w_2, respectively to obtain a 3-coloring of G. On the other hand, a 3-coloring of $G - e$ for an edge e of $G - v$ yields a 3-coloring of $G' - e$, and a 3-coloring of $G - v$ can be extended to a 3-coloring of $G' - f$ for every edge f which is incident to at least one of the vertices v_1, v_2, v_3.

□

We define A to be the multigraph with five vertices u, v, w, x, y, simple edges uv, uw, vw, and double edges vx, wy. Let G be a multigraph which contains a triangle $T = (u, v, w)$. Then the multigraph H obtained from G by (u, v)-replacing T by A is the multigraph obtained from G by inserting vertices x, y, inserting double edges ux and vy, and replacing every edge uz or vz with $z \notin \{u, v, w\}$ by an edge xz or yz, respectively. The vertex x is called the split vertex of u, and y is the split vertex of v.

Lemma 4.2 *Let G be a multigraph which contains an induced triangle $T = (u, v, w)$. The multigraph H obtained from G by (u, v)-replacing T by A is 4-colorable, if and only if G is 4-colorable.*

Proof. Let π be a 4-coloring of G with $\pi(uv) = 1$, $\pi(uw) = 2$, $\pi(vw) = 3$. Then we obtain a 4-coloring of H by coloring the edges uv with 4, uw with 3, vw with 2, the double edges ux and vy with $(1, 2)$ and $(1, 3)$, respectively, every edge xz or yz with $z \notin \{u, v\}$ with $\pi(uz)$ or $\pi(vz)$, respectively, and leaving the color of the other edges unchanged. Analogously, we obtain a 4-coloring of G from a 4-coloring of H.

□

The next statement is a special case of Lemma 3.3.

Lemma 4.3 *Let G be a multigraph with $\Delta(G) = 4$ containing an induced proper subgraph H of odd order and with 4-deficiency 4. If G is 4-colorable, then the multigraph obtained from G by contracting H to a single vertex is also 4-colorable.*

□

For any 3-critical graph G, let $T(G)$ be the graph obtained from G by replacing each vertex v of G by a triangle T_v. For every edge uw of G, the vertex of T_u which is adjacent to a vertex of T_w is denoted by u_w. If x is a vertex of degree 2 of G, then x_2 denotes the vertex of T_x with $d_{T(G)}(x_2) = 2$. Let $U(G)$ be the multigraph obtained from $T(G)$ by replacing the edge $u_v v_u$ by a double edge for any edge uv of G.

Lemma 4.4 *For every 3-critical graph G, the multigraph $U(G)$ is 4-critical .*

Proof. Assume that there is a 4-coloring of $U(G)$ with colors 1,2,3,4. Then we get a 3-coloring of G by coloring each edge uv of G with 1, if the double edge $u_v v_u$ is colored with $(1, 4)$ or $(2, 3)$, with 2, if $u_v v_u$ is colored with $(2, 4)$ or $(1, 3)$, and with 3, else. This is a 3-coloring of G, since for each 4-coloring of a triangle one color is missing at all three vertices and the other colors are missing at exactly one vertex. Since G is not 3-colorable, $U(G)$ is not 4-colorable.

By Lemma 4.1, $T(G)$ is 3-critical. Since $U(G)$ is obtained from $T(G)$ by adding a set of independent edges, an edge e of $U(G)$ can be interpreted as an edge of $T(G)$. Now we can color the edges of $T(G) - e$ with three colors and the other edges of $U(G) - e$ with a fourth color to obtain a 4-coloring of $U(G) - e$. Hence, $U(G)$ is 4-critical.

□

Let G be a 3-critical simple graph, and let S be a set of edges of G. For each edge uv of S, we identify the vertices u_v and v_u of $U(G)$. The resulting multigraph is denoted by $U_S(G)$.

Theorem 4.1 *If G has at least three vertices of degree 2, then the multigraph $U_S(G)$ is 4-critical for any subset S of $E(G)$.*

Proof. By Lemma 4.4, the statement is true for $S = \emptyset$. Let $e \in S$, and let $U_{S-e}(G)$ be 4-critical. We have to show that $U_S(G)$ is 4-critical to prove the theorem.

Assume that $U_S(G)$ is 4-colorable. Let $u(0), u(1), ..., u(n)$ be a path in G where $e = u(0)u(1)$, and $u(n)$ is a vertex of degree 2. For $i = 1, ..., n-1$, the triangle $T_{u(i)}$ is $(u(i)_{u(i-1)}, u(i)_{u(i+1)})$-replaced by A, and $T_{u(n)}$ is $(u(n)_{u(n-1)}, u(n)_2)$-replaced by A. By assumption and Lemma 4.2, the obtained multigraph $A_S(G)$ is 4-colorable. For $i = 1, ..., n-1$ the vertex $u(i)_{u(i+1)}$ and the split vertices of $u(i)_{u(i+1)}$ and $u(i+1)_{u(i)}$ induce a subgraph of $A_S(G)$ of order 3 and with 4-deficiency 4. We contract each of these induced subgraphs to a single vertex. By Lemma 4.3, the obtained multigraph is 4-colorable. Since it contains the multigraph $U_{S-e}(G)$ as a subgraph, we get a 4-coloring of $U_{S-e}(G)$ which contradicts the induction hypothesis. Hence, $U_S(G)$ is not 4-colorable.

Let f be an edge of $U_S(G)$. Then f is also an edge of $U_{S-e}(G)$. To prove that $U_S(G)-f$ is 4-colorable, we have to consider two cases.
Case 1: f is an edge of a triangle T_v of $U_S(G)$.
Since critical multigraphs contain no cut vertices and G contains more than one vertex of degree 2, there is a path $v(0), v(1), ..., v(k)$ of G such that $v(i) \neq n$ for $i \geq 1$, $e = v(0)v(1)$, and $d_G(v_k) = 2$.
Case 2: f is one edge of a double edge $u_v v_u$.
There is a path $v(0), v(1), ..., v(k)$ of $G - uv$ such that $e = v(0)v(1)$ and $d_G(v_k) = 2$.

In both cases we can $(v(i)_{v(i-1)}, v(i)_{v(i+1)})$-replace the triangle $T_{v(i)}$ of $U_{S-e}(G)-f$ by A for $i = 1, ..., k-1$ and $(v(k)_{v(k-1)}, v(k)_2)$-replace the triangle $T_{v(k)}$ by A. For $i = 1, ..., k-1$, we contract subgraphs of order 3 and with 4-deficiency 4 induced by $v(i)_{v(i+1)}$ and the split vertices of $v(i)_{v(i+1)}$ and $v(i+1)_{v(i)}$, and the subgraph induced by $v(0)_{v(1)}$, $v(1)_{v(0)}$, and the split vertex of $v(1)_{v(0)}$. The obtained graph is 4-colorable and contains $U_S(G) - f$ as a subgraph. Hence, $U_S(G)$ is a 4-critical graph.

□

4.2 Examples and remarks

The proof of Theorem 4.1 almost holds for 3-critical graphs with exactly one vertex v of degree 2. In this case we obtain that $U_S(G)$ is not 4-colorable, and $U_S(G)-e$ is 4-colorable for every edge e which is not contained in the triangle T_v. Since $G-v$ is 3-colorable, the multigraph obtained from $U_S(G) - e$ by removing all edges of T_v is 4-colorable. Hence,

$U_S(G)$ is 4-critical, if it has odd order, and if $U_S(G)$ has even order, then $U_S(G) - v_2$ is 4-critical.

By applying Theorem 4.1, we can construct many non-elementary 4-critical graphs. Let P^- be the graph obtained from the Petersen graph by removing one vertex. Then the graph $U_{E(P^-)}(P^-)$ is a graph with 15 vertices three of which have degree 2. The multigraphs $U_{E(P^-)-e}(P^-)$ for an edge e of P^- have 16 vertices. These two multigraphs (there are two, because e can be incident to a vertex of degree 2 or not) are the smallest known multigraphs of even order. Let P_u be the graph obtained from P^- by replacing a vertex $u \in V(P^-)$ by a triangle. Then the graphs $U_{E(P_v)}(P_v)$ and $U_{E(P_w)}(P_w)$ for vertices v and w of P^- of degree 2 and 3, respectively, have 18 vertices, and they are the smallest 4-critical even order graphs, as shown in [5]. The graph with 15 vertices and one of the multigraphs with 16 and one of the graphs with 18 vertices were found by Chetwynd (see [6]), and Fiol [8] obtained the other graph of order 18.

As an other application, we construct an infinite sequence of 4-critical graphs with fast growing total deficiency. Let $H_1 = P^-$, and for $i \geq 2$, let H_i be a graph obtained from H_{i-1} and P^- by Hajos union where at least one of the identified vertices has degree 2 (see Lemma 3.2). Then H_i is a 3-critical graph of order $6i + 3$ and with total deficiency $2i + 1$, and the graph $U_i = U_{E(H_i)}(H_i)$ is a 4-critical graph of order $13i + 2$ and with total deficiency $4i + 2$.

Corollary 4.1 *For every natural number n, there is a 4-critical graph G with more than n vertices and $s_4(G)/|V(G)| \geq 4/13$.*

□

Chapter 5

A proof of the planar graph conjecture for $\Delta = 7$

By proving the four color theorem, Appel and Haken [2] solved the most famous open problem in edge coloring planar graphs. The theorem implies that every bridgeless cubic map is 3-colorable. However, this is far away from determining the chromatic index for all planar graphs what could be done by proving the following conjecture by Seymour.

Conjecture 5.1 ([26]) *There is no planar non-elementary critical graph.*

A proof of Seymour's conjecture would imply not only the four color theorem, but also the existence of a polynomial algorithm determining the chromatic index for any planar graph (see [16]). One more consequence would be that every planar graph with maximum degree at lest 6 is in class one. This weaker conjecture known as the planar graph conjecture was formulated by Vizing [31] in 1965 who also proved the case $\Delta(G) \geq 8$. Hence, the two cases $\Delta = 6$ and $\Delta = 7$ remain open. In this chapter we prove the case $\Delta = 7$. Since there are planar graphs with maximum degree 7 which fulfill VAL, we need some additional structural properties of critical graphs. In the first section of this chapter we derive those properties that we will need to show that there is no planar graph for which all conditions for a 7-critical graph hold. This is the main theorem of this chapter, and we will prove it in the second section.

5.1 Vizing-type bounds for critical graphs

Let M be a multigraph with a critical edge xy, and let $M - xy$ be colored with $\chi'(M) - 1$ colors. We define a *fan sequence* $F = [xy_0, xy_1, ..., xy_n]$ *at* x to be a sequence of distinct edges $xy_0, ..., xy_n$ with $y_0 = y$ such that for each $i \geq 1$, the edge xy_i is colored with a color missing in y_{i-1}. The set of the end vertices different from x of all edges which are contained in at least one fan sequence at x is denoted by A_x. The set of neighbors of x which are not contained in A_x is denoted by B_x. For every vertex w of G, we define $C'(w)$

to be the set of colors which are missing in w, and for adjacent vertices w_1, w_2, we denote the set of colors used to color the edges between w_1 and w_2 by $C(w_1w_2)$.

Lemma 5.1 (Yap-Lemma) *Let M be a critical multigraph, let uy be an edge of G, and let $G-uy$ be colored with $\chi'(M)-1$ colors. Let $A_u = \{y, v_1, ..., v_n\}$ and $B_u = \{z_1, ..., z_m\}$. Then all sets $C'(u), C'(y), C'(v_1), ..., C'(v_n), C(uz_1), ..., C(uz_m)$ are pairwise disjoint.*

Proof. Let π be the coloring of $G - uy$ with $\chi'(M) - 1$ colors. Since π is a coloring, the sets $C(uz_1), ..., C(uz_m)$ are pairwise disjoint. For every $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$, the sets $C'(v_j) \cap C(uz_i)$ and $C'(y) \cap C(uz_i)$ are empty, since otherwise $z_i \in A_u$, and $C(uz_i) \cap C'(u) = \emptyset$ by definition. The sets $C'(y)$ and $C'(u)$ are disjoint, because otherwise we can color the edge uy with a color of $C'(u) \cap C'(y)$ and obtain a coloring of M with $\chi'(M) - 1$ colors. Hence, we have to prove that for $i, j \in \{1, ..., n\}$ and $i \neq j$ the sets $C'(y) \cap C'(v_i)$, $C'(u) \cap C'(v_i)$, and $C'(v_i) \cap C'(v_j)$ are pairwise disjoint. If $a \in C'(v_i) \cap C'(u)$, then let $[uy, uv_{i_1}, ..., uv_{i_k}]$ with $i_k = i$ be a fan sequence at u. We color the edges uy with $\pi(uv_{i_1})$, uv_{i_l} with $\pi(uv_{i_{l+1}})$ for $1 \leq l \leq k-1$, and uv_{i_k} with a, and we obtain a coloring of M with $\chi'(M) - 1$ colors. If $a \in C'(v_i) \cap C'(y)$, let $b \in C'(u)$. We recolor the (a, b)-chain starting in v_i. If this chain ends in u, we have $C'(u) \cap C'(y) \neq \emptyset$, else $C'(v_i) \cap C'(u) \neq \emptyset$. If $a \in C'(v_i) \cap C'(v_j)$ with $i \neq j$, let $b \in C'(u)$. The (a, b)-chains starting in v_i and v_j can not both end in u. Hence, we can assume that the (a, b)-chain starting in v_i does not end in u, and by recoloring this chain, we obtain a coloring of $G - uy$ where $C'(v) \cap C'(u) \neq \emptyset$ for a vertex $v \in A_u$.

□

Corollary 5.1 *Let G be a Δ-critical graph with adjacent vertices u and y where $d_G(y) = k + 1$, and u has exactly $\Delta - k$ neighbors $v_{k+1}, ..., v_\Delta$ of degree Δ such that $v_s \neq y$ for $k+1 \leq s \leq \Delta$. Let $v_1, ..., v_l$ be the distinct minor vertices other than y which are adjacent to u. Let $d_G(v_i) \leq \Delta - 2$ for $1 \leq i \leq m$ and $d_G(v_i) = \Delta - 1$ for $m + 1 \leq i \leq l$. Then for every Δ-coloring of $G - uy$, $v_i \notin A_u$ for $1 \leq i \leq m$.*

Proof. Let $n \leq m$ be the number of vertices in $\{v_1, ..., v_m\} \cap A_u$. Then there are $\Delta - k$ colors missing in y, there are $k - l$ colors missing in u, at least $2n$ colors are missing in a vertex of $\{v_1, ..., v_m\} \cap A_u$, and there are $m - n$ colors which the edges uv_i with $1 \leq i \leq m$ and $v_i \in B_u$ are colored with. Additionally, for $m + 1 \leq i \leq l$, we consider the color missing in v_i, if $v_i \in A_u$, or the color of the edge uv_i, if $v_i \in B_u$. In both cases all considered colors have to be different by Lemma 1.1. Hence, we have $\Delta \geq \Delta - k + k - l + 2n + m - n + l - m = \Delta + n$, thus $n = 0$.

□

We will now present several results about critical graphs for which the lower bound for the number of major vertices adjacent to a vertex u given in VAL is attained. In all of the Lemmas 5.2 to 5.8 we suppose that G is a critical graph with adjacent vertices u and y such that $d_G(y) = k+1$, and u has exactly $\Delta - k$ neighbors $v_{k+1}, ..., v_\Delta$ of maximum degree with $v_s \neq y$ for $k+1 \leq s \leq \Delta$. Further, we suppose that $v_1, ..., v_l$ are the distinct minor vertices adjacent to u with $v_s \neq y$ for $1 \leq s \leq l$.

Lemma 5.2 *If $d_G(u) \leq \Delta - 1$ and $d_G(v_s) \leq \Delta - 2$ for $1 \leq s \leq l$, then $d_G(w) = \Delta$ for any vertex w of G for which $w \notin \{u, y, v_1, ..., v_l\}$ and w is adjacent to one of the vertices $v_{k+1}, ..., v_\Delta$.*

Proof. We assume that there is a counterexample. Let w_Δ be a neighbor of v_Δ with $d_G(w_\Delta) < \Delta$ and $w_\Delta \notin \{y, v_1, ..., v_l\}$. Let $G - uy$ be colored with $\{1, ..., \Delta\}$ such that the edges uv_s are colored with s for $s \in \{1, ..., l\} \cup \{k+1, ..., \Delta\}$ and the edges incident with y are colored with $1, ..., k$. Let the edge $v_\Delta w_\Delta$ be colored with j, and let i be a color missing in w_Δ.

Case 1: $l+1 \leq j \leq k$ and $i = \Delta$.
By recoloring the (j, Δ)-chain starting in w_Δ and coloring uy with Δ, we obtain a Δ-coloring of G.

Case 2: $j \leq l$ and $i = \Delta$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{l+1, ..., \Delta\} \cup \{j_0, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ for $0 \leq s \leq r$; $j_{r+1} \in \{l+1, ..., \Delta\} \cup \{j_0, ..., j_{r-1}\}$.
Subcase 2.1: $l+1 \leq j_{r+1} \leq k$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$ and obtain Case 1.
Subcase 2.2: $k+1 \leq j_{r+1} \leq \Delta - 1$.
By recoloring the $(j_{r+1}, l+1)$-chain starting in v_j^r, we obtain Subcase 2.1.
Subcase2.3: $j_{r+1} = \Delta$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$, the edge uv_Δ with j, and the edge $v_\Delta w_\Delta$ with Δ. The obtained coloring of $G - uy$ contradicts the Yap-Lemma.
Subcase 2.4: $j_{r+1} = j$.
We recolor the (j, Δ)-chain starting in v_j^s. If this chain ends in w_Δ, the obtained coloring contradicts the Yap-Lemma, else we are in Subcase 2.3.
Subcase 2.5: There is $s \in \{1, ..., r-1\}$ with $j_{r+1} = j_s$.
We recolor the $(j_s, l+1)$-chain beginning in v_j^r. If this chain ends in u, we color the edges uv_j^t with j_{t+1} for $0 \leq t \leq s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with $l+1$ and uv_j^t with j_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with $l+1$ and uv_j^t with j_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 1.

Case 3: $j \leq k$ and $l+1 \leq i \leq k$.
By recoloring the (i, Δ)-chain starting in w_Δ, we obtain Case 1 or Case 2.

Case 4: $j \leq k$ and $k+1 \leq i \leq \Delta - 1$.
Let $a \in \{l+1, ..., k\}$ and $a \neq j$. We recolor the (i, a)-chain starting in w_Δ and obtain

Case 3.

Case 5: $j \leq k$ and $i \leq l$.

We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ for $0 \leq s \leq r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.

Subcase 5.1: $l+1 \leq i_{r+1} \leq k$.

We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 3.

Subcase 5.2: $k+1 \leq i_{r+1} \leq \Delta$.

Let $a \in \{l+1, ..., k\}$ with $a \neq j$. By recoloring the (i_{r+1}, a)-chain starting in v_i^r, we obtain Subcase 5.1.

Subcase 5.3: $i_{r+1} = i$. Let $a \in \{l+1, ..., k\}$ with $a \neq j$.

We recolor the (i, a)-chain starting in v_i^r. If this chain ends in w_Δ or u, then we obtain Case 3, else we are in Subcase 5.1.

Subcase 5.4: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$.

Let $a \in \{l+1, ..., k\}$ with $a \neq j$. We recolor the (i_s, a)-chain beginning in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with a and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_i^{s-1}, we color the edges uv_i^r with a and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 3.

Case 6: $j \geq k+1$ and $l+1 \leq i \leq k$.

We recolor the (i, j)-chain starting in w_Δ and obtain Case 4.

Case 7: $j \geq k+1$ and $i \geq k+1$.

We recolor the $(i, l+1)$-chain starting in w_Δ and obtain Case 6.

Case 8: $j \geq k+1$ and $i \leq l$.

We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ for $0 \leq s \leq r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.

Subcase 8.1: $l+1 \leq i_{r+1} \leq k$.

We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 6.

Subcase 8.2: $k+1 \leq i_{r+1} \leq \Delta$.

By recoloring the $(i_{r+1}, l+1)$-chain starting in v_i^r, we obtain Case 5 (if $i_{r+1} = j$ and the chain contains the edge $v_\Delta w_\Delta$) or Subsubcase 8.1 (else).

Subcase 8.3: $i_{r+1} = i$.

We recolor the $(i, l+1)$-chain starting in v_i^r. If this chain ends in w_Δ or u, then we obtain Case 6, else we are in Subcase 8.1.

Subsubcase 8.4: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$.

We recolor the $(i_s, l+1)$-chain beginning in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with $l+1$ and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_i^{s-1}, we color the edges

uv_i^r with $l+1$ and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 6.

□

Lemma 5.3 *If $d_G(u) = \Delta$, $d_G(y) \geq 3$, and $d_G(v_s) \leq \Delta - 2$ for $1 \leq s \leq l$, then $d_G(w) = \Delta$ for any vertex w of G for which $w \notin \{y, v_1, ..., v_l\}$ and w is adjacent to one of the vertices $v_{k+1}, ..., v_\Delta$.*

Proof. We assume that there is a counterexample. By definition, we have that $l = k-1 \geq 1$. Let w_Δ be a neighbor of v_Δ with $d_G(w_\Delta) < \Delta$ and $w_\Delta \notin \{y, v_1, ..., v_{k-1}\}$. Let $G - uy$ be colored with $\{1, ..., \Delta\}$ such that the edges uv_s are colored with s for $s \in \{1, ..., k-1\} \cup \{k+1, ..., \Delta\}$ and the edges incident with y are colored with $1, ..., k$. Let the edge $v_\Delta w_\Delta$ be colored with j, and let i be a color missing in w_Δ.

Case 1: $j = k$ and $i = \Delta$.
By recoloring the (j, Δ)-chain starting in w_Δ and coloring uy with Δ, we obtain a Δ-coloring of G.

Case 2: $j \leq k-1$ and $i = \Delta$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{k, ..., \Delta\} \cup \{j_0, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ for $0 \leq s \leq r$; $j_{r+1} \in \{k, ..., \Delta\} \cup \{j_0, ..., j_{r-1}\}$.
Subcase 2.1: $j_{r+1} = k$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$ and obtain Case 1.
Subcase 2.2: $k+1 \leq j_{r+1} \leq \Delta - 1$.
By recoloring the (j_{r+1}, k)-chain starting in v_j^r, we obtain Subcase 2.1.
Subcase 2.3: $j_{r+1} = \Delta$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$, the edge uv_Δ with j, and the edge $v_\Delta w_\Delta$ with Δ. The obtained coloring of $G - uy$ contradicts the Yap-Lemma.
Subcase 2.4: $j_{r+1} = j$.
We recolor the (j, Δ)-chain starting in v_j^r. If this chain ends in w_Δ, the obtained coloring contradicts the Yap-Lemma, else we are in Subcase 2.3.
Subcase 2.5: There is $s \in \{1, ..., r-1\}$ with $j_{r+1} = j_s$.
We recolor the (j_s, k)-chain beginning in v_j^r. If this chain ends in u, we color the edges uv_j^t with j_{t+1} for $0 \leq t \leq s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 1.

Case 3: $j \leq k-1$ and $i = k$.
By recoloring the (i, Δ)-chain starting in w_Δ, we obtain Case 2.

Case 4: $j \leq k-1$ and $k+1 \leq i \leq \Delta - 1$.
We recolor the (i, k)-chain starting in w_Δ and obtain Case 3.

Case 5: $j \leq k-1$ and $i \leq k-1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ for $0 \leq s \leq r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 5.1: $i_{r+1} = k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 3.
Subcase 5.2: $k+1 \leq i_{r+1} \leq \Delta$.
By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Subcase 5.1.
Subcase 5.3: $i_{r+1} = i$.
We recolor the (i, k)-chain starting in v_i^r. If this chain ends in w_Δ or u, then we obtain Case 3, else we are in Subcase 5.1.
Subcase 5.4: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$.
We recolor the (i_s, k)-chain beginning in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 3.

Case 6: $j = k$ and $k+1 \leq i \leq \Delta - 1$.
Subcase 6.1: There is $s \in \{1, ..., k-1\}$ such that $k \in \overline{v_s}$.
By coloring uv_s with k, we obtain Case 4.
Subcase 6.2: There is $s \in \{1, ..., k-1\}$ such that $\Delta \in \overline{v_s}$.
By recoloring the (Δ, k)-chain starting in v_s, we obtain Subcase 6.1.
Subcase 6.3: There is a color $a \neq i$ with $k+1 \leq a \leq \Delta - 1$ and an index $s \in \{1, ..., k-1\}$ such that $a \in \overline{v_s}$.
We recolor the (a, k)-chain starting in v_s. If this chain does not contain the edge $v_\Delta w_\Delta$, we can color the edge uv_s with k and obtain Case 4. Hence, we can assume that the chain contains the edge $v_\Delta w_\Delta$, and we obtain a new coloring of $G - uv$ where $v_\Delta w_\Delta$ is colored with a, the color i is missing in w_Δ, and the color k is missing in v_s. Now we recolor the (k, Δ)-chain beginning in v_s, then we recolor the (i, k)-chain starting in w_Δ, and at last we recolor the (k, a)-chain starting in w_Δ. The obtained coloring suffices the conditions of Subcase 6.2.
Subcase 6.4: There are $s, t \in \{1, ..., k-1\}$ with $s \neq t$ and $i \in \overline{v_s} \cap \overline{v_t}$.
Since the (i, k)-chain starting in w_Δ can not end in v_s and v_t, we can assume that the (i, k)-chain starting in v_s does not contain the edge $v_\Delta w_\Delta$. By recoloring this chain, we obtain Subcase 6.1.
Subcase 6.5: The color i is missing in at most one of the vertices $v_1, ..., v_{k-1}$.
We assume that $\overline{v_1} \subseteq \{1, ..., k-1\} \cup \{i\}$ and $\overline{v_s} \subseteq \{1, ..., k-1\}$ for $s \in \{2, ..., k-1\}$. Let $a_0, ..., a_{r+1}$ and $v_a^0, ..., v_a^r$ be colors and vertices, respectively, such that the following conditions hold: $v_a^0 = v_1$ and $a_0 = 1$; the edges uv_a^s are colored with a_s for $0 \leq s \leq r$; $\overline{v_a^s} \cap \{a_0, ..., a_{s-1}\} = \emptyset$ for $0 \leq s \leq r-1$; $a_{s+1} \in \overline{v_a^s}$, $a_1 \neq i$, and $a_{s+1} \neq 1$ for $0 \leq s \leq r$; $a_{r+1} \in \{a_1, ..., a_{r-1}\}$.
Let $s \in \{1, ..., r-1\}$ with $a_s = a_{r+1}$. Now we recolor the (a_{r+1}, k)-chain starting in v_a^r

and we denote this chain by P.
Subsubcase 6.5.1: P contains the edge $v_\Delta w_\Delta$.
If P ends in u, we color the edge uv_a^{s-1} with a_s and obtain Case 4, else we are already in Case 4.
Subsubcase 6.5.2: P does not contain the edge $v_\Delta w_\Delta$.
If P ends in u, we are in Case 4, else we color the edge uv_a^r with k and obtain Case 4.

Case 7: $j = k$ and $1 \leq i \leq k-1$.
Subcase 7.1: There is $s \in \{1, ..., k-1\}$ such that $k \in \overline{v_s}$.
By coloring uv_s with k, we obtain Case 3 or Case 5.
Subcase 7.2: There is $s \in \{1, ..., k-1\}$ such that $\Delta \in \overline{v_s}$.
By recoloring the (Δ, k)-chain starting in v_s, we obtain Subcase 7.1.
Subcase 7.3: $\{k, \Delta\} \cap \overline{v_s} = \emptyset$ for $1 \leq s \leq k-1$, and there is $a \in \overline{v_i}$ with $k+1 \leq a \leq \Delta-1$.
We recolor the (i, a)-chain beginning in w_Δ. If this chain ends in v_i, the new coloring contradicts the Yap-Lemma, else we obtain Case 6.
Subcase 7.4: $\{k, \Delta\} \cap \overline{v_s} = \emptyset$ for $1 \leq s \leq k-1$ and $\overline{v_i} \subseteq \{1, ..., k-1\}$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k+1, ..., \Delta-1\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq i$ for $0 \leq s \leq r$; $i_{r+1} \in \{k+1, ..., \Delta-1\} \cup \{i_1, ..., i_{r-1}\}$.
Subcase 7.1: $k+1 \leq i_{r+1} \leq \Delta-1$. Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain does not contain the edge $v_\Delta w_\Delta$, we can color the edge uv_i^r with k and obtain Case 5. Hence, we can assume that the chain contains the edge $v_\Delta w_\Delta$, and we obtain a new coloring of $G - uv$ where $v_\Delta w_\Delta$ is colored with i_{r+1}, the color i is missing in w_Δ, and the color k is missing in v_i^r. Now we color the edge uv_i^r with k and the edges uv_i^s with i_{s+1} for $0 \leq s \leq r-1$, and by recoloring the (i, i_{r+1})-chain starting in w_Δ, we obtain Case 6.
Subsubcase 7.4.2: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$, and the (i_{r+1}, k)-chain starting in v_i^r contains the edge $v_\Delta w_\Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in u, we color the edge uv_i^{s-1} with i_s and obtain Case 5, else we are already in Case 5.
Subsubcase 7.4.3: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$, and the (i_{r+1}, k)-chain starting in v_i^r does not contain the edge $v_\Delta w_\Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in u, we are in Case 5, else we color the edge uv_i^r with k and obtain Case 5.

Case 8: $j \geq k+1$ and $i = k$.
We recolor the (i, j)-chain starting in w_Δ and obtain Case 6.

Case 9: $j \geq k+1$ and $i \geq k+1$.
We recolor the (i, k)-chain starting in w_Δ and obtain Case 8.

Case 10: $j \geq k+1$ and $i \leq k-1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ for $0 \leq s \leq r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.

Subcase 10.1: $i_{r+1} = k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 8.
Subcase 10.2: $k+1 \leq i_{r+1} \leq \Delta$.
By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Case 7 (if $i_{r+1} = j$ and the chain contains the edge $v_\Delta w_\Delta$) or Subcase 10.1 (else).
Subcase 10.3: $i_{r+1} = i$.
We recolor the (i, k)-chain starting in v_i^r. If this chain ends in w_Δ or u, then we obtain Case 8, else we are in Subcase 10.1.
Subsubcase 10.4: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$.
We recolor the (i_s, k)-chain beginning in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 8.

□

Lemma 5.4 *If $d_G(u) \leq \Delta - 1$, $d_G(v_s) \leq \Delta - 2$ for $1 \leq s \leq l-1$, and $d_G(v_l) = \Delta - 1$, then $d_G(w) = \Delta$ for any vertex w of G for which $w \notin \{u, y, v_1, ..., v_l\}$ and w is adjacent to one of the vertices $v_{k+1}, ..., v_\Delta$.*

Proof. We assume that there is a counterexample. Let w_Δ be a neighbor of v_Δ with $d_G(w_\Delta) < \Delta$ and $w_\Delta \notin \{y, v_1, ..., v_l\}$.

Case A: There is a coloring π of $G - uy$ such that $v_l \notin A_y$.

Let $G - uy$ be colored with $\{1, ..., \Delta\}$ such that the edges uv_s are colored with s for $s \in \{1, ..., k-1\} \cup \{k+1, ..., \Delta\}$ and the edges incident with y are colored with $1, ..., k$. Let the edge $v_\Delta w_\Delta$ be colored with j, and let i be a color missing in w_Δ.

Case 1: $l+1 \leq j \leq k$ and $i = \Delta$.

By recoloring the (j, Δ)-chain starting in w_Δ and coloring uy with Δ, we obtain a Δ-coloring of G.

Case 2: $l+1 \leq j \leq k$ and $l+1 \leq i \leq k$.

We recolor the (i, Δ)-chain starting in w_Δ and obtain Case 1.

Case 3: $l+1 \leq j \leq k$ and $k+1 \leq i \leq \Delta - 1$.

Let $a \in \{l+1, ..., k\}$ with $a \neq j$. By recoloring the (i, a)-chain starting in w_Δ, we obtain Case 2.

Case 4: $l+1 \leq j \leq k$, $1 \leq i \leq l$, and $j \notin \overline{v_l}$.

We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq j$ for $0 \leq s \leq r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 4.1: $l+1 \leq i_{r+1} \leq k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 2.
Subcase 4.2: $k+1 \leq i_{r+1} \leq \Delta$.

Let $a \in \{l+1, ..., k\}$ with $a \neq j$. By recoloring the (i_{r+1}, a)-chain starting in v_i^r, we obtain Subcase 4.1.
Subcase 4.3: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Let $a \in \{l+1, ..., k\}$ with $a \neq j$. We recolor the (i_{r+1}, a)-chain starting in v_i^r. If this path ends in w_Δ, we are in Case 2, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with a and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with a and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 2.

Case 5: $l+1 \leq j \leq k$, $1 \leq i \leq l$, and $j \in \overline{v_l}$.
By recoloring the (j, Δ)-chain beginning in v_l, we obtain Case 4.

Case 6: $k+1 \leq j \leq \Delta - 1$ and $l+1 \leq i \leq k$.
We recolor the (i, j)-chain starting in w_Δ and obtain Case 3.

Case 7: $k+1 \leq j \leq \Delta - 1$ and $k+1 \leq i \leq \Delta$.
We recolor the (i, k)-chain beginning in w_Δ and obtain Case 5.

Case 8: $k+1 \leq j \leq \Delta - 1$, $1 \leq i \leq l$, and $j \notin \overline{v_l}$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq j$ for $0 \leq s \leq r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 8.1: $l+1 \leq i_{r+1} \leq k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 6.
Subcase 8.2: $k+1 \leq i_{r+1} \leq \Delta$.
By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Subcase 8.1.
Subcase 8.3: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this path ends in w_Δ, we are in Case 6, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 6.

Case 9: $k+1 \leq j \leq \Delta - 1$, $1 \leq i \leq l$, and $j \in \overline{v_l}$.
We recolor the (j, k)-chain starting in v_l. If this chain contains the edge $v_\Delta w_\Delta$, we are in Case 5, else we obtain Case 8.

Case 10: $1 \leq j \leq l$ and $j \notin \overline{v_l}$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{l+1, ..., \Delta\} \cup \{j_1, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \neq j$ for $0 \leq s \leq r$; $j_{r+1} \in \{l+1, ..., \Delta\} \cup \{j_1, ..., j_{r-1}\}$.
Subcase10.1: $l+1 \leq j_{r+1} \leq k$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$ and obtain one of the Cases 1-5.
Subcase 10.2: $k+1 \leq j_{r+1} \leq \Delta$.
By recoloring the (j_{r+1}, k)-chain starting in v_j^r, we obtain Subcase 10.1.

Subcase 10.3: $j_{r+1} = j_s$ with $1 \le s \le r-1$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r. If this path ends in u, we color the edges uv_j^t with j_{t+1} for $0 \le t \le s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \le t \le s-2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \le t \le r-1$. In all cases we obtain one of the Cases 1-5.

Case 11: $1 \le j \le l$ and $j \in \overline{v_l}$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \le s \le r$; $\overline{v_j^s} \cap (\{l, ..., \Delta\} \cup \{j_1, ..., j_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \ne j$ for $0 \le s \le r$; $j_{r+1} \in \{l, ..., \Delta\} \cup \{j_1, ..., j_{r-1}\}$.
Subcase 11.1: $l+1 \le j_{r+1} \le k$.
We color the edges uv_j^s with j_{s+1} for $0 \le s \le r$ and obtain one of the Cases 1-5.
Subcase 11.2: $k+1 \le j_{r+1} \le \Delta$.
By recoloring the (j_{r+1}, k)-chain starting in v_j^r, we obtain Subcase 11.1.
Subcase 11.3: $j_{r+1} = l$.
We color the edges uv_l with j and uv_j^s with j_{s+1} for $0 \le s \le r$ and obtain Case 10.
Subcase 11.4: $j_{r+1} = j_s$ with $1 \le s \le r-1$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r. If this path ends in u, we color the edges uv_j^t with j_{t+1} for $0 \le t \le s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \le t \le s-2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \le t \le r-1$. In all cases we obtain one of the Cases 1-5.

Case B: There is a coloring π of $G - uy$ where the edges incident with y are colored with $1, ..., k$, $\pi(uv_s) = s$ for $s \in \{1, ..., l-1\} \cup \{k+1, ..., \Delta-1\}$, $\pi(uv_l) = \Delta$, $\overline{v_l} = \{l\}$, and $\pi(uv_\Delta) = l$.
Let the edge $v_\Delta w_\Delta$ be colored with j, and let i be a color missing in w_Δ.

Case 1: $l+1 \le j \le k$ and $i = l$.
We recolor the (l, j)-chain starting in w_Δ, we color the edge uv_l with l and the edge uy with Δ, and we obtain a Δ-coloring of G.

Case 2: $l+1 \le j \le k$ and $l+1 \le i \le k$.
We recolor the (i, l)-chain starting in w_Δ. If this chain ends in u or v_l, the new coloring contradicts the Yap-Lemma, else we are in Case 1.

Case 3: $l+1 \le j \le k$ and $k+1 \le i \le \Delta$.
Let $a \in \{l+1, ..., k\}$ with $a \ne j$. By recoloring the (i, a)-chain starting in w_Δ, we obtain Case 2.

Case 4: $l+1 \le j \le k$ and $1 \le i \le l-1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{l, ..., j-1\} \cup \{j+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne j$ for $0 \le s \le r$; $i_{r+1} \in \{l+1, ..., j-1\} \cup \{j+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 4.1: $l+1 \le i_{r+1} \le k$.
We color the edges uv_i^s with i_{s+1} for $0 \le s \le r$ and obtain Case 2.

Subcase 4.2: $k+1 \leq i_{r+1} \leq \Delta$.
Let $a \in \{l+1, ..., k\}$ with $a \neq j$. By recoloring the (i_{r+1}, a)-chain starting in v_i^r, we obtain Subcase 4.1.
Subcase 4.3: $i_{r+1} = l$.
We recolor the (l, Δ)-chain starting in v_i^r. If this chain ends in y or v_l, we obtain a coloring of $G - uy$ for which $v_l \notin A_y$, else we are in Subcase 4.2.
Subcase 4.4: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Let $a \in \{l+1, ..., k\}$ with $a \neq j$. We recolor the (i_{r+1}, a)-chain starting in v_i^r. If this path ends in w_Δ, we are in Case 2, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with a and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with a and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 2.

Case 5: $k+1 \leq j \leq \Delta$ and $l+1 \leq i \leq k$.
By recoloring the (i, j)-chain starting in w_Δ, we obtain Case 3.

Case 6: $k+1 \leq j \leq \Delta$ and $k+1 \leq i \leq \Delta$.
We recolor the (i, k)-chain beginning in w_Δ and obtain Case 5.

Case 7: $k+1 \leq j \leq \Delta$ and $i = l$.
We recolor the (l, k)-chain starting in w_Δ. If this chain ends in u or v_l, the new coloring contradicts the Yap-Lemma, else we obtain Case 5.

Case 8: $k+1 \leq j \leq \Delta$ and $1 \leq i \leq l-1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$, such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq j$ for $0 \leq s \leq r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 8.1: $l+1 \leq i_{r+1} \leq k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 5.
Subcase 8.2: $k+1 \leq i_{r+1} \leq \Delta$.
By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Subcase 8.1.
Subcase 8.3: $i_{r+1} = l$.
We recolor the (l, Δ)-chain starting in v_i^r. If this chain ends in y or v_l, we obtain a coloring of $G - uy$ for which $v_l \notin A_y$, else we are in Subcase 8.2.
Subcase 8.4: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this path ends in w_Δ, we are in Case 5, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 5.

Case 9: $1 \leq j \leq l-1$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{l, ..., \Delta\} \cup \{j_1, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \neq j$ for $0 \leq s \leq r$; $j_{r+1} \in$

$\{l, ..., \Delta\} \cup \{j_1, ..., j_{r-1}\}$.
Subcase 9.1: $l + 1 \leq j_{r+1} \leq k$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$ and obtain one of the Cases 1-4.
Subcase 9.2: $k + 1 \leq j_{r+1} \leq \Delta$.
By recoloring the (j_{r+1}, k)-chain starting in v_j^r, we obtain Subcase 9.1.
Subcase 9.3: $j_{r+1} = l$.
We recolor the (l, k)-chain starting in v_j^r. If this chain ends in u or v_l, the new coloring contradicts the Yap-Lemma, else we obtain Subcase 9.1.
Subcase 9.4: $j_{r+1} = j_s$ with $1 \leq s \leq r - 1$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r. If this path ends in u, we color the edges uv_j^t with j_{t+1} for $0 \leq t \leq s - 1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \leq t \leq s - 2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \leq t \leq r - 1$. In all cases we obtain one of the Cases 1-4.

Case C: There is a coloring π of $G - uy$ where the edges incident with y are colored with $1, ..., k$, $\pi(uv_s) = s$ for $s \in \{1, ..., l - 1\} \cup \{k + 2, ..., \Delta\}$, $\pi(uv_l) = k + 1$, $\overline{v_l} = \{l\}$, and $\pi(uv_{k+1}) = l$.
Let $v_\Delta w_\Delta$ be colored with j, and let i be a color missing in w_Δ.

Case 1: $j = l$ and $i = \Delta$.
Let P be the (Δ, l)-chain beginning in w_Δ. If P ends in v_l, we recolor the (Δ, l)-chain starting in y. The new coloring contradicts the Yap-Lemma, since both colors l and $k+1$ are missing in y, but the edge uv_l is colored with $k + 1$ and the color l is missing in v_l. If P ends in y, then recolor the (l, Δ)-chain starting in v_l. Again, the new coloring of $G - uy$ contradicts the Yap-Lemma. If P ends neither in v_l nor in y, we recolor P and obtain Case B.

Case 2: $j = l$ and $l + 1 \leq i \leq k$.
We recolor the (i, Δ)-chain starting in w_Δ and obtain Case 1.

Case 3: $j = l$, and $k + 1 \leq i \leq \Delta - 1$.
By recoloring the (i, k)-chain starting in w_Δ, we obtain Case 2.

Case 4: $j = l$ and $1 \leq i \leq l - 1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l + 1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r - 1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq l$ for $0 \leq s \leq r$; $i_{r+1} \in \{l + 1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 4.1: $l + 1 \leq i_{r+1} \leq k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 2.
Subcase 4.2: $k + 1 \leq i_{r+1} \leq \Delta$.
By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Subcase 4.1.
Subcase 4.3: $i_{r+1} = i$.
We recolor the (i, k)-chain starting in v_i^r. If this chain ends in w_Δ or in u, we obtain Case 2, else we are in Subcase 4.1.
Subcase 4.4: $i_{r+1} = i_s$ with $1 \leq s \leq r - 1$.

We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this path ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 2.

Case 5: $l+1 \leq j \leq k$ and $i = l$.
The (l, j)-chain starting in w_Δ cannot end in v_l or u, because by coloring uv_l with j or l and uy with $k+1$, we would obtain a Δ-coloring of G. We recolor this chain and obtain Case 2.

Case 6: $l+1 \leq j \leq k$ and $l+1 \leq i \leq k$.
As in Case 5, the (i, l)-chain starting in w_Δ cannot end in v_l or u. By recoloring this chain, we obtain Case 5.

Case 7: $l+1 \leq j \leq k$ and $k+1 \leq i \leq \Delta$.
Let $a \in \{l+1, ..., k\}$ with $a \neq j$. By recoloring the (i, a)-chain starting in w_Δ, we obtain Case 6.

Case 8: $l+1 \leq j \leq k$ and $1 \leq i \leq l-1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq l$ for $0 \leq s \leq r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 8.1: $l+1 \leq i_{r+1} \leq k$ and $i_{r+1} \neq j$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 6.
Subcase 8.2: $k+1 \leq i_{r+1} \leq \Delta$.
Let $a \in \{l+1, ..., k\}$ with $a \neq j$. By recoloring the (i_{r+1}, a)-chain starting in v_i^r, we obtain Subcase 8.1.
Subcase 8.3: $i_{r+1} = j$.
We recolor the (j, Δ)-chain starting in v_i^r and obtain Subcase 8.2.
Subcase 8.4: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Let $a \in \{l+1, ..., k\}$ with $a \neq j$. We recolor the (i_{r+1}, a)-chain starting in v_i^r. If this path ends in w_Δ, we are in Case 6, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with a and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with a and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 6.

Case 9: $k+1 \leq j \leq \Delta - 1$ and $l+1 \leq i \leq k$.
By recoloring the (i, j)-chain starting in w_Δ, we obtain Case 7.

Case 10: $k+1 \leq j \leq \Delta - 1$ and $k+1 \leq i \leq \Delta$.
We recolor the (i, k)-chain beginning in w_Δ and obtain Case 9.

Case 11: $k+1 \leq j \leq \Delta - 1$ and $i = l$.
Recolor the (l, j)-chain starting in w_Δ. If this chain ends in v_l or y, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we obtain Case 3.

Case 12: $k+1 \leq j \leq \Delta - 1$ and $1 \leq i \leq l-1$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold:

$v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{l+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne l$ for $0 \le s \le r$; $i_{r+1} \in \{l+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 12.1: $l+1 \le i_{r+1} \le k$.
We color the edges uv_i^s with i_{s+1} for $0 \le s \le r$ and obtain Case 9.
Subcase 12.2: $k+1 \le i_{r+1} \le \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we obtain Case 8, else we are in Subcase 12.1.
Subcase 12.3: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this path ends in w_Δ, we are in Case 9, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if it ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 9.

Case 13: $1 \le j \le l-1$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \le s \le r$; $\overline{v_j^s} \cap (\{l+1, ..., \Delta\} \cup \{j_0, ..., j_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \ne j$ for $0 \le s \le r$; $j_{r+1} \in \{l+1, ..., \Delta\} \cup \{j_0, ..., j_{r-1}\}$.
Subcase 13.1: $l+1 \le j_{r+1} \le k$.
We color the edges uv_j^s with j_{s+1} for $0 \le s \le r$ and obtain one of the Cases 5-8.
Subcase 13.2: $k+1 \le j_{r+1} \le \Delta$.
By recoloring the (j_{r+1}, k)-chain starting in v_j^r, we obtain Subcase 13.1.
Subcase 13.3: $j_{r+1} = l$.
We recolor the (l, k)-chain starting in v_j^r. If this chain ends in u or v_l, the new coloring contradicts the Yap-Lemma, else we obtain Subcase 13.1.
Subcase 13.4: $j_{r+1} = j_s$ with $1 \le s \le r-1$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r. If this path ends in u, we color the edges uv_j^t with j_{t+1} for $0 \le t \le s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \le t \le s-2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \le t \le r-1$. In all cases we obtain one of the Cases 5-8.

$\square$

Lemma 5.5 *If $d_G(u) = \Delta$, $d_G(y) \ge 4$, $d_G(v_s) \le \Delta - 2$ for $1 \le s \le l-1$, and $d_G(v_l) = \Delta - 1$, then $d_G(w) \ge \Delta - 1$ for any vertex w of G for which $w \notin \{y, v_1, ..., v_l\}$ and w is adjacent to one of the vertices $v_{k+1}, ..., v_\Delta$.*

Proof. By definition, we have that $l = k-1 \ge 2$. We assume that there is a counterexample. Let w_Δ be a neighbor of v_Δ with $d_G(w_\Delta) < \Delta - 1$ and $w_\Delta \notin \{y, v_1, ..., v_{k-2}\}$.

Case A: There is a coloring π of $G - uy$ such that $v_{k-1} \notin A_y$.
Let $G - uy$ be colored with $\{1, ..., \Delta\}$ such that the edges uv_s are colored with s for

$s \in \{1, ..., k-1\} \cup \{k+1, ..., \Delta\}$ and the edges incident with y are colored with $1, ..., k$. Let the edge $v_\Delta w_\Delta$ be colored with j.

Case 1: $j = k$ and $\Delta \in \overline{w_\Delta}$.

By recoloring the (j, Δ)-chain starting in w_Δ and coloring uy with Δ, we obtain a Δ-coloring of G.

Case 2: $j \leq k-2$ and $\Delta \in \overline{w_\Delta}$.

We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{k, ..., \Delta\} \cup \{j_0, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \neq k-1$ for $0 \leq s \leq r$; $j_{r+1} \in \{k, ..., \Delta\} \cup \{j_0, ..., j_{r-1}\}$.

Subcase 2.1: $j_{r+1} = k$.

We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$ and obtain Case 1.

Subcase 2.2: $k+1 \leq j_{r+1} \leq \Delta - 1$.

By recoloring the (j_{r+1}, k)-chain starting in v_j^r, we obtain Subcase 2.1.

Subcase2.3: $j_{r+1} = \Delta$.

We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$, the edge uv_Δ with j, and the edge $v_\Delta w_\Delta$ with Δ. The obtained coloring of $G - uy$ contradicts the Yap-Lemma.

Subcase 2.4: $j_{r+1} = j$.

We recolor the (j, Δ)-chain starting in v_j^r. If this chain ends in w_Δ, the obtained coloring contradicts the Yap-Lemma, else we are in Subcase 2.3.

Subcase 2.5: There is $s \in \{1, ..., r-1\}$ with $j_{r+1} = j_s$.

We recolor the (j_s, k)-chain beginning in v_j^r. If this chain ends in u, we color the edges uv_j^t with j_{t+1} for $0 \leq t \leq s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 1.

Case 3: $j \leq k-2$ and $k \in \overline{w_\Delta}$.

By recoloring the (k, Δ)-chain starting in w_Δ, we obtain Case 2.

Case 4: $j \leq k-2$ and $i \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta - 1$.

We recolor the (i, k)-chain starting in w_Δ and obtain Case 3.

Case 5: $j \leq k-2$, $i \in \overline{w_\Delta}$ with $i \leq k-1$, and $j \notin \overline{v_{k-1}}$.

We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq j$ for $0 \leq s \leq r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.

Subcase 5.1: $i_{r+1} = k$.

We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 3.

Subcase 5.2: $k+1 \leq i_{r+1} \leq \Delta$.

By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Subcase 5.1.

Subcase 5.3: $i_{r+1} = i$.

We recolor the (i, k)-chain starting in v_i^r. If this chain ends in w_Δ or u, then we obtain Case 3, else we are in Subsubcase 5.1.

Subcase 5.4: There is $s \in \{1, ..., r-1\}$ with $i_{r+1} = i_s$.
We recolor the (i_{r+1}, k)-chain beginning in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if it ends neither in u nor in v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all of the three cases we obtain Case 3.

Case 6: $j \leq k-2$, $i \in \overline{w_\Delta}$ with $i \leq k-2$, and $j \in \overline{v_{k-1}}$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{k, ..., \Delta\} \cup \{i, j_0, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \neq k-1$ for $0 \leq s \leq r$; $j_{r+1} \in \{k, ..., \Delta\} \cup \{i, j_0, ..., j_{r-1}\}$.
Subcase 6.1: $j_{r+1} = k$.
If $r = 0$, we recolor the (k, Δ)-chain starting in v_j, and then we recolor the (j, Δ)-chain starting in v_{k-1}. If that chain ends in v_j, the new coloring contradicts the Yap-Lemma, else we obtain Case 5. If $r \geq 1$, we color the edges uv_j^s with j_{s+1} for $1 \leq s \leq r$ and obtain Subcase 6.1 with $r = 0$.
Subcase 6.2: $k+1 \leq j_{r+1} \leq \Delta$.
By recoloring the (j_{r+1}, k)-chain beginning in v_j^r, we obtain Subcase 6.1.
Subcase 6.3: $j_{r+1} = i$.
Recolor the (i, k)-chain starting in v_j^r. If this chain ends in u or w_Δ, we obtain Case 3, else we are in Subcase 6.1.
Subcase 6.4: $j_{r+1} = j_s$ with $1 \leq s \leq r-1$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r. If this path ends in u, we color the edges uv_j^t with j_{t+1} for $1 \leq t \leq s-1$, if it ends in v_j^{s-1}, we obtain Subcase 6.1 (if s=1) or we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $1 \leq t \leq s-2$ (if $s \geq 2$), and if it ends in none of the vertices u and v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $1 \leq t \leq r-1$. In all cases we obtain Subcase 6.1.
Subcase 6.5: $j_{r+1} = j$.
If the (j, Δ)-chain P starting in v_{k-1} does not end in y, recolor P and obtain a contradiction to the Yap-Lemma (if P contains the edge uv_j) or Case 5 (else). Hence, P ends in y, the (j, Δ)-chain Q starting in v_j^r does not end in y or v_{k-1}, and recoloring Q yields a contradiction to the Yap-Lemma (if Q contains uv_j) or Subcase 6.2.

Case 7: $k+1 \leq j \leq \Delta - 1$ and $k, \Delta \in \overline{w_\Delta}$.
By recoloring the (k, j)-chain starting in w_Δ, we obtain Case 1.

Case 8: $k+1 \leq j \leq \Delta - 1$ and $i, \Delta \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta - 1$.
We recolor the (i, k)-chain beginning in w_Δ and obtain Case 7.

Case 9: $k+1 \leq j \leq \Delta - 1$ and $i, k \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta - 1$.
Recolor the (k, Δ)-chain starting in w_Δ and obtain Case 8.

Case 10: $k+1 \leq j \leq \Delta - 1$ and $h, i \in \overline{w_\Delta}$ with $k+1 \leq h < i \leq \Delta - 1$.
By recoloring the (h, k)-chain starting in w_Δ, we obtain Case 9.

Case 11: $k+1 \leq j \leq \Delta - 1$, $i, \Delta \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$, and $\Delta \notin \overline{v_{k-1}}$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold:

$v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k, ..., \Delta - 1\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne \Delta$ for $0 \le s \le r$; $i_{r+1} \in \{k, ..., \Delta - 1\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 11.1: $i_{r+1} = k$.
We color the edges uv_i^s with i_{s+1} for $0 \le s \le r$ and obtain Case 7.
Subcase 11.2: $k+1 \le i_{r+1} \le \Delta - 1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we are in Case 1, else we obtain Subcase 11.1.
Subcase 11.3: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 7, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 7.

Case 12: $k+1 \le j \le \Delta - 1$, $i, \Delta \in \overline{w_\Delta}$ with $1 \le i \le k-2$, and $\Delta \in \overline{v_{k-1}}$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne k-1$ for $0 \le s \le r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 12.1: $i_{r+1} = k$.
We color the edges uv_i^s with i_{s+1} for $0 \le s \le r$ and obtain Case 7.
Subcase 12.2: $k+1 \le i_{r+1} \le \Delta - 1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we are in Case 1, else we obtain Subcase 12.1.
Subcase 12.3: $i_{r+1} = \Delta$.
The (Δ, k)-chain starting in v_{k-1} ends in w_Δ, since otherwise we can recolor that chain and obtain Case 11. Hence, the (Δ, k)-chain starting in v_i^r ends neither in v_{k-1} nor in w_Δ. By recoloring this chain, we obtain Subcase 12.1.
Subcase 12.4: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 7, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 7.

Case 13: $k+1 \le j \le \Delta - 1$ and $i, k \in \overline{w_\Delta}$ with $1 \le i \le k-2$.
We recolor the (k, Δ)-chain starting in w_Δ and obtain Case 11 or Case 12.

Case 14. $k+1 \le j \le \Delta - 1$ and $h, i \in \overline{w_\Delta}$ with $1 \le i \le k-2$ and $k+1 \le h \le \Delta - 1$.
By recoloring the (i, k)-chain starting in w_Δ, we obtain Case 13.

Case 15: $k+1 \le j \le \Delta - 1$ and $h, i \in \overline{w_\Delta}$ with $1 \le h < i \le k-2$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup$

$\{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq k-1$ for $0 \leq s \leq r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 15.1: $i_{r+1} = k$.
We color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r$ and obtain Case 13.
Subcase 15.2: $k+1 \leq i_{r+1} \leq \Delta$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we color the edge uv_i^r with k and obtain Case 5 or Case 6, else we are in Subcase 15.1.
Subcase 15.3: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 13, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 13.

Case 16: $k+1 \leq j \leq \Delta-1$, $i, k-1 \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$, and $i, j \notin \overline{v_{k-1}}$.
We define colors $a_0, ..., a_{r+1}$ and vertices $v_a^0, ..., v_a^r$ such that the following conditions hold: $v_a^0 = v_{k-1}$ and $a_0 = k-1$; the edges uv_a^s are colored with a_s for $0 \leq s \leq r$; $\overline{v_a^s} \cap (\{k, ..., \Delta\} \cup \{a_0, ..., a_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $a_{s+1} \in \overline{v_a^s}$ and $a_{s+1} \neq i$ for $0 \leq s \leq r$; $a_{r+1} \in \{k, ..., \Delta\} \cup \{a_0, ..., a_{r-1}\}$.
Subcase 16.1: $a_{r+1} = k$.
We color the edges uv_a^s with a_{s+1} for $0 \leq s \leq r$ and obtain Case 13.
Subcase 16.2: $k+1 \leq a_{r+1} \leq \Delta$.
Recolor the (a_{r+1}, k)-chain starting in v_a^r. If this chain contains the edge $v_\Delta w_\Delta$, we color the edge uv_a^r with k and obtain Case 5, else we are in Subcase 16.1.
Subcase 16.3: $a_{r+1} = a_s$ with $0 \leq s \leq r-1$.
Recolor the (a_{r+1}, k)-chain starting in v_a^r. If this chain ends in w_Δ, we are in Case 13, if it ends in u, we color the edges uv_a^t with a_{t+1} for $0 \leq t \leq s-1$, if it ends in v_a^{s-1}, color the edges uv_a^{s-1} with k and uv_a^t with a_{t+1} for $0 \leq t \leq s-2$, and if the chain ends in none of the vertices w_Δ, u, v_a^{s-1}, we color the edges uv_a^r with k and uv_a^t with a_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 13.

Case 17: $k+1 \leq j \leq \Delta-1$, $i, k-1 \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$, and $j \in \overline{v_{k-1}}$.
Subcase 17.1: $j \in \overline{v_i}$.
Recolor the (i, j)-chain starting in w_Δ. If this chain ends in v_i or y, then the new coloring contradicts the Yap-Lemma, else we obtain Case 4.
Subcase 17.2:$\Delta \in \overline{v_i}$.
We recolor the (j, k)-chain starting in v_{k-1}. If this chain does not contain the edge $v_\Delta w_\Delta$, we are in Case 16, else we recolor the (i, Δ)-chain starting in w_Δ and obtain Case 1 or a contradiction to the Yap-Lemma.
Subcase 17.3: $k \in \overline{v_i}$.
By recoloring the (k, Δ)-chain starting in v_i, we obtain Subcase 17.2.
Subcase 17.4: $k-1 \in \overline{v_i}$.
We recolor the $(k-1, k)$-chain beginning in v_i. If this chain ends in u or w_Δ, we are in

Case 13, else we obtain Subcase 17.3.
Subcase 17.5: $a \in \overline{v_i}$ with $k+1 \leq a \leq \Delta - 1$ and $a \neq j$.
By recoloring the (a, k)-chain starting in v_i, we obtain Subcase 17.3.
Subcase 17.6: $\overline{v_i} \subseteq \{1, ..., k-2\}$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta\} \cup \{i_1, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$, $i_1 \neq j$, and $i_{s+1} \neq i$ for $0 \leq s \leq r$; $i_{r+1} \in \{k-1, ..., \Delta\} \cup \{i_1, ..., i_{r-1}\}$.
Subsubcase 17.6.1: $i_{r+1} = k$.
By coloring the edges uv_i^s with i_{s+1} for $1 \leq s \leq r$, we obtain Subcase 17.2.
Subsubcase 17.6.2: $k+1 \leq i_{r+1} \leq \Delta$ and $i_{r+1} \neq j$.
By recoloring the (i_{r+1}, k)-chain starting in v_i^r, we obtain Subsubcase 17.6.1.
Subsubcase 17.6.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or w_Δ, we obtain Case 13, else we are in Subsubcase 17.6.1.
Subsubcase 17.6.4: $i_{r+1} = i_s$ with $1 \leq s \leq r-1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $1 \leq t \leq s-1$, if it ends in v_i^{s-1}, we are in Subcase 17.3 (if $s = 1$) or we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $1 \leq t \leq s-2$ (if $s \geq 2$), and if the chain ends neither in u nor in v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $1 \leq t \leq r-1$. In all cases we obtain Subcase 17.3.
Subsubcase 17.6.5: $i_{r+1} = j$.
Recolor the (j, k)-chain beginning in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we color the edge uv_i^r with k and obtain Case 5 or Case 6, else we are in Case 10 or Subsubcase 17.6.1.

Case 18: $k+1 \leq j \leq \Delta - 1$, $i, k-1 \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$, and $i \in \overline{v_i}$.
The (i, j)-chains starting in v_{k-1} and w_Δ can not both end in y. If the (i, j)-chain starting in w_Δ does not end in y, we recolor this chain and obtain Case 4 or a contradiction to the Yap-Lemma, else we recolor the (i, j)-chain beginning in v_{k-1} and obtain Case 17 or a contradiction to the Yap-Lemma.

Case 19: $k+1 \leq j \leq \Delta - 1$ and $k-1, \Delta \in \overline{w_\Delta}$.
Subcase 19.1: $k \in \overline{v_{k-1}}$.
Recolor the $(k-1, k)$-chain starting in w_Δ and obtain Case 7.
Subcase 19.2: $a \in \overline{v_{k-1}}$ with $k+1 \leq a \leq \Delta - 1$.
We recolor the (a, k)-chain starting in v_{k-1}. If this chain contains the edge $v_\Delta w_\Delta$, we are in Case 1, else we obtain Subcase 19.1.
Subcase 19.3: $a \in \overline{v_{k-1}}$ with $1 \leq a \leq k-2$.
Let $a_0, ..., a_{r+1}$ and $v_a^0, ..., v_a^r$ be colors and vertices, respectively, such that the following conditions hold: $v_a^0 = v_a$ and $a_0 = a$; the edges uv_a^s are colored with a_s for $0 \leq s \leq r$; $\overline{v_a^s} \cap (\{k-1, ..., \Delta - 1\} \cup \{a_0, ..., a_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $a_{s+1} \in \overline{v_a^s}$ and $a_{s+1} \neq \Delta$ for $0 \leq s \leq r$; $a_{r+1} \in \{k-1, ..., \Delta - 1\} \cup \{a_0, ..., a_{r-1}\}$.

Subsubcase 19.3.1: $a_{r+1} = k$.
We color the edges uv_a^s with a_{s+1} for $0 \le s \le r$ and obtain Subcase 19.1.
Subsubcase 19.3.2: $k+1 \le a_{r+1} \le \Delta - 1$.
Recolor the (a_{r+1}, k)-chain starting in v_a^r and obtain Case 1 (if the chain contains the edge $v_\Delta w_\Delta$) or Subsubcase 19.3.1 (else).
Subsubcase 19.3.3: $a_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_a^r. If this chain ends in u or w_Δ, we are in Case 7, else we obtain Subsubcase 19.3.1.
Subsubcase 19.3.4: $a_{r+1} = a_s$ with $0 \le s \le r-1$.
We recolor the (a_{r+1}, k)-chain starting in v_a^r. If this chain ends in v_{k-1}, we are in Subcase 19.1, if it ends in u, we color the edges uv_a^t with a_{t+1} for $0 \le t \le s-1$, if it ends in v_a^{s-1}, we color the edges uv_a^{s-1} with k and uv_a^t with a_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices v_{k-1}, u, v_a^{s-1}, we color the edges uv_a^r with k and uv_a^t with a_{t+1} for $0 \le t \le r-1$. In all cases we obtain Subcase 19.1.
Subcase 19.4: $\Delta \in \overline{v_{k-1}}$.
The $(\Delta, 1)$-chains starting in v_{k-1} and w_Δ can not both end in y. If the $(\Delta, 1)$-chain starting in w_Δ does not end in y, we recolor it and obtain a contradiction to the Yap-Lemma (if the chain contains uv_1) or one of the Cases 16 and 18 (else). If the $(\Delta, 1)$-chain starting in v_{k-1} does not end in y, we recolor it and obtain a contradiction to the Yap-Lemma (if the chain contains uv_1) or Case 18 or Subcase 19.3 (else).

Case 20: $k+1 \le j \le \Delta - 1$ and $k-1, k \in \overline{w_\Delta}$.
By recoloring the (k, Δ)-chain beginning in w_Δ, we obtain Case 19.

Case 21: $k+1 \le j \le \Delta - 1$ and $i, k-1 \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta - 1$.
We recolor the (i, k)-chain starting in w_Δ and obtain Case 20.

Case 22: $j = k$, $\Delta \notin \overline{w_\Delta}$, and $i \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta - 1$.
By recoloring the (i, k)-chain starting in w_Δ, we obtain one of the Cases 9, 13, and 20.

Case 23: $j = k$ and $\overline{w_\Delta} \subseteq \{1, ..., k-1\}$.
Let $i \in \overline{w_\Delta}$ with $1 \le i \le k-2$. We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne k-1$ for $0 \le s \le r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 23.1: $i_{r+1} = k$.
We color the edges uv_i^s with i_{s+1} for $0 \le s \le r$ and obtain Case 3.
Subcase 23.2: $k+1 \le i_{r+1} \le \Delta$.
By recoloring the (i_{r+1}, k)-chain, we obtain Subcase 23.1 (if the chain does not contain the edge $v_\Delta w_\Delta$) or one of the Cases 15–18 (else).
Subcase 23.3: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 3, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for

$0 \le t \le r-1$. In all cases we obtain Case 3.

Case 24: $j = k-1$.

We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \le s \le r$; $\overline{v_j^s} \cap (\{k, ..., \Delta\} \cup \{j_0, ..., j_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \ne k-1$ for $0 \le s \le r$; $j_{r+1} \in \{k, ..., \Delta\} \cup \{j_0, ..., j_{r-1}\}$.

Subcase 24.1: $j_{r+1} = k$.

We color the edges uv_j^s with j_{s+1} for $0 \le s \le r$ and obtain one of the Cases 1, 22, or 23.

Subcase 24.2: $k+1 \le j_{r+1} \le \Delta$.

By recoloring the (j_{r+1}, k)-chain, we obtain Subcase 24.1.

Subcase 24.3: $j_{r+1} = j_s$ with $1 \le s \le r-1$.

We recolor the (j_{r+1}, k)-chain starting in v_j^r. If this chain ends in u, we color the edges uv_j^t with j_{t+1} for $0 \le t \le s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices u and v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \le t \le r-1$. In all cases we obtain one of the Cases 1, 22, or 23.

Case B: There is a coloring π of $G-uy$ where the edges incident with y are colored with $1, ..., k$, $\pi(uv_s) = s$ for $s \in \{1, ..., k-2\} \cup \{k+1, ..., \Delta-1\}$, $\pi(uv_{k-1}) = \Delta$, $\overline{v_{k-1}} = \{k-1\}$, and $\pi(uv_\Delta) = k-1$.

Let the edge $v_\Delta w_\Delta$ be colored with j.

Case 1: $j = k$ and $k-1 \in \overline{w_\Delta}$.

By recoloring the $(k-1, k)$-chain starting in w_Δ and coloring the edges uv_{k-1} with $k-1$ and uy with Δ, we obtain a Δ-coloring of G.

Case 2: $j = k$ and $\Delta \in \overline{w_\Delta}$.

We recolor the $(\Delta, k-1)$-chain Starting in w_Δ. If this chain ends in v_{k-1} or y, we obtain Case A, else we are in Case 1.

Case 3: $k+1 \le j \le \Delta-1$ and $k-1, k \in \overline{w_\Delta}$.

Recolor the (k, j)-chain beginning in w_Δ and obtain Case 1.

Case 4: $k+1 \le j \le \Delta-1$ and $k, \Delta \in \overline{w_\Delta}$.

By recoloring the (k, j)-chain starting in w_Δ, we obtain Case 2.

Case 5: $k+1 \le j \le \Delta-1$ and $i, \Delta \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta-1$.

We recolor the (i, k)-chain beginning in w_Δ and obtain Case 3.

Case 6: $k+1 \le j \le \Delta-1$ and $k-1, \Delta \in \overline{w_\Delta}$.

Recolor the (Δ, k)-chain starting in w_Δ and obtain Case 3.

Case 7: $k+1 \le j \le \Delta-1$ and $i, \Delta \in \overline{w_\Delta}$ with $1 \le i \le k-2$.

We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta-1\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne \Delta$ for $0 \le s \le r$; $i_{r+1} \in \{k-1, ..., \Delta-1\} \cup \{i_0, ..., i_{r-1}\}$.

Subcase 7.1: $i_{r+1} = k$.

Color the edges uv_i^s with i_{s+1} for $0 \le s \le r-1$ and obtain Case 4.

Subcase 7.2: $k+1 \le i_{r+1} \le \Delta - 1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 7.1.
Subcase 7.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Subcase 7.1.
Subcase 7.4: $i_{r+1} = j$.
We recolor the (j, k)-chain starting in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we obtain Case 2, else we are in Subcase 7.1.
Subcase 7.5: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 4, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 4.

Case 8: $k+1 \le j \le \Delta - 1$, $k \in \overline{w_\Delta}$, and $k-1, \Delta \notin \overline{w_\Delta}$.
Recolor the (k, Δ)-chain starting in w_Δ and obtain one of the Cases 5-7.

Case 9: $k+1 \le j \le \Delta - 1$, $i \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta - 1$, and $k, \Delta \notin \overline{w_\Delta}$.
By recoloring the (i, k)-chain beginning in w_Δ, we obtain Case 3 or Case 8.

Case 10: $k+1 \le j \le \Delta - 1$ and $\overline{w_\Delta} \subseteq \{1, ..., k-1\}$.
Let $i \in \overline{w_\Delta}$ with $1 \le i \le k-2$. We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k-1, ..., j-1\} \cup \{j+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne j$ for $0 \le s \le r$; $i_{r+1} \in \{k-1, ..., j-1\} \cup \{j+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 10.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \le s \le r-1$ and obtain Case 3 or Case 8.
Subcase 10.2: $k+1 \le i_{r+1} \le \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 10.1.
Subcase 10.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, if it ends in w_Δ, we are in Case 8, else in Subcase 10.1.
Subcase 10.4: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 8, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 8.

Case 11: $j = \Delta$ and $k-1 \in \overline{w_\Delta}$.
By recoloring the $(k-1, \Delta)$-chain starting in w_Δ, we obtain Case A.

Case 12: $j = \Delta$ and $k \in \overline{w_\Delta}$.
We recolor the $(k, k-1)$-chain beginning in w_Δ. If this chain ends in u or v_{k-1}, the new

coloring contradicts the Yap-Lemma, else we are in Case 11.

Case 13: $j = \Delta$ and $i \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta - 1$. By recoloring the (i,k)-chain starting in w_Δ, we obtain Case 12.

Case 14: $j = \Delta$ and $i \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta - 1\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq \Delta$ for $0 \leq s \leq r$; $i_{r+1} \in \{k-1, ..., \Delta - 1\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 14.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r-1$ and obtain Case 12.
Subcase 14.2: $k+1 \leq i_{r+1} \leq \Delta - 1$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 14.1.
Subcase 14.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Subcase 14.1.
Subcase 14.4: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 12, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 12.

Case 15: $j = k$ and $i \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta - 1$. By recoloring the (i,k)-chain starting in w_Δ, we obtain one of the Cases 3, 4, or 8.

Case 16: $1 \leq j \leq k-2$ and $k-1 \in \overline{w_\Delta}$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \leq s \leq r$; $\overline{v_j^s} \cap (\{k-1, ..., \Delta\} \cup \{j_1, ..., j_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \neq j$ for $0 \leq s \leq r$; $j_{r+1} \in \{k-1, ..., \Delta\} \cup \{j_1, ..., j_{r-1}\}$.
Subcase 16.1: $j_{r+1} = k$.
Color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r-1$ and obtain Case 1.
Subcase 16.2: $k+1 \leq j_{r+1} \leq \Delta$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r and obtain Subcase 16.1.
Subcase 16.3: $j_{r+1} = k-1$.
We color the edges uv_j^s with j_{s+1} for $0 \leq s \leq r$, uv_Δ with j, and $v_\Delta w_\Delta$ with $k-1$. The new coloring of $G - uy$ contradicts the Yap-Lemma.
Subcase 16.4: $j_{r+1} = j_s$ with $1 \leq s \leq r-1$.
Recolor the (j_{r+1}, k)-chain starting in v_j^r. If this chain ends in u, we color the edges uv_j^t with j_{t+1} for $0 \leq t \leq s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \leq t \leq s-2$, and if the chain ends in none of the vertices u and v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 1.

Case 17: $1 \le j \le k-2$ and $k \in \overline{w_\Delta}$.
We recolor the $(k, k-1)$-chain starting in w_Δ. If this chain ends in u or v_{k-1}, the new coloring contradicts the Yap-Lemma, else we are in Case 16.

Case 18: $1 \le j \le k-2$ and $i \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta$.
By recoloring the (i, k)-chain beginning in w_Δ, we obtain Case 17.

Case 19: $1 \le j \le k-2$ and $i \in \overline{w_\Delta}$ with $1 \le i \le k-2$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne j$ for $0 \le s \le r$; $i_{r+1} \in \{k-1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 19.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \le s \le r-1$ and obtain Case 17.
Subcase 19.2: $k+1 \le i_{r+1} \le \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 19.1.
Subcase 19.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Subcase 19.1.
Subcase 19.4: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 17, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 17.

Case 20: $j = k$ and $\overline{w_\Delta} \subseteq \{1, ..., k-2\}$.
Let $i \in \overline{w_\Delta}$. We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta\} \cup \{i_1, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne i$ for $0 \le s \le r$; $i_{r+1} \in \{k-1, ..., \Delta\} \cup \{i_1, ..., i_{r-1}\}$.
Subcase 20.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \le s \le r-1$ and obtain Case 17.
Subcase 20.2: $k+1 \le i_{r+1} \le \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain one of the Cases 10 and 14 (if the chain contains the edge $v_\Delta w_\Delta$) or Subcase 20.1 (else).
Subcase 20.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Subcase 20.1.
Subcase 20.4: $i_{r+1} = i_s$ with $1 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices u and v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 17.

Case C: There is a coloring π of $G-uy$ where the edges incident with y are colored with $1, ..., k$, $\pi(uv_s) = s$ for $s \in \{1, ..., k-2\} \cup \{k+2, ..., \Delta\}$, $\pi(uv_{k-1}) = k+1$, $\overline{v_{k-1}} = \{k-1\}$, and $\pi(uv_{k+1}) = k-1$.
Let the edge $v_\Delta w_\Delta$ be colored with j.

Case 1: $j = k-1$ and $\Delta \in \overline{w_\Delta}$.
We recolor the $(\Delta, k-1)$-chain starting in w_Δ. If this chain ends in v_{k-1}, the new coloring contradicts the Yap-Lemma, since both colors $k+1$ and Δ are missing in y, the edge uv_{k-1} is colored with $k+1$, and in v_{k-1} the color Δ is missing. If the chain ends in y, we obtain a contradiction to the Yap-Lemma, because $k-1$ and $k+1$ are missing in y, the edge uv_{k-1} is colored with $k+1$, and $k-1$ is missing in v_{k-1}. If the chain does not end in v_{k-1} or y, we obtain Case B.

Case 2: $j = k-1$ and $k \in \overline{w_\Delta}$.
By recoloring the (k, Δ)-chain starting in w_Δ, we obtain Case 1.

Case 3: $j = k-1$ and $i \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta - 1$.
Recolor the (i, k)-chain starting in w_Δ and obtain Case 2.

Case 4: $j = k-1$ and $i \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq k-1$ for $0 \leq s \leq r$; $i_{r+1} \in \{k, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 4.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r-1$ and obtain Case 2.
Subcase 4.2: $k+1 \leq i_{r+1} \leq \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 4.1.
Subcase 4.3: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 2, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 2.

Case 5: $j = k$ and $\Delta \in \overline{w_\Delta}$.
We color the edges $v_\Delta w_\Delta$ with Δ, uv_Δ with k, and uy with Δ, and we obtain a Δ-coloring of G.

Case 6: $j = k$ and $k-1 \in \overline{w_\Delta}$.
We recolor the $(k-1, k)$-chain starting in w_Δ. If this chain ends in v_{k-1} or v_{k+1}, the new coloring contradicts the Yap-Lemma, else we are in Case 2.

Case 7: $k+1 \leq j \leq \Delta - 1$ and $k-1, k \in \overline{w_\Delta}$.
Recolor the (k, j)-chain starting in v_Δ and obtain Case 6.

Case 8: $k+1 \leq j \leq \Delta - 1$ and $k, \Delta \in \overline{w_\Delta}$. By recoloring the (k, j)-chain starting in w_Δ, we obtain Case 5.

Case 9: $k+1 \leq j \leq \Delta-1$ and $i, \Delta \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta-1$. We recolor the (i,k)-chain starting in w_Δ and obtain Case 8.

Case 10: $k+1 \leq j \leq \Delta-1$ and $k-1, \Delta \in \overline{w_\Delta}$.
Recolor the (Δ, k)-chain starting in w_Δ and obtain Case 7.

Case 11: $k+1 \leq j \leq \Delta-1$ and $i, \Delta \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta-1\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq \Delta$ for $0 \leq s \leq r$; $i_{r+1} \in \{k-1, ..., \Delta-1\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 11.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r-1$ and obtain Case 8.
Subcase 11.2: $k+1 \leq i_{r+1} \leq \Delta-1$ and $i_{r+1} \neq j$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 11.1.
Subcase 11.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Subcase 11.1.
Subcase 11.4: $i_{r+1} = j$.
We recolor the (j, k)-chain starting in v_i^r. If this chain contains the edge $v_\Delta w_\Delta$, we obtain Case 5, else we are in Subcase 11.1.
Subcase 11.5: $i_{r+1} = i_s$ with $0 \leq s \leq r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 8, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \leq t \leq s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \leq t \leq s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \leq t \leq r-1$. In all cases we obtain Case 8.

Case 12: $k+1 \leq j \leq \Delta-1$, $k \in \overline{w_\Delta}$, and $k-1, \Delta \notin \overline{w_\Delta}$.
Recolor the (k, Δ)-chain starting in w_Δ and obtain Case 9 or Case 11.

Case 13: $k+1 \leq j \leq \Delta-1$, $i \in \overline{w_\Delta}$ with $k+1 \leq i \leq \Delta-1$, and $k, \Delta \notin \overline{w_\Delta}$.
By recoloring the (i,k)-chain starting in w_Δ, we obtain Case 7 or Case 12.

Case 14: $k+1 \leq j \leq \Delta-1$ and $\overline{w_\Delta} \subseteq \{1, ..., k-1\}$.
Let $i \in \overline{w_\Delta}$ with $1 \leq i \leq k-2$. We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \leq s \leq r$; $\overline{v_i^s} \cap (\{k-1, ..., j-1\} \cup \{j+1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \leq s \leq r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq j$ for $0 \leq s \leq r$; $i_{r+1} \in \{k-1, ..., j-1\} \cup \{j+1, ..., \Delta\} \cup \{i_0, ..., i_{r-1}\}$.
Subcase 14.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \leq s \leq r-1$ and obtain Case 7 or Case 12.
Subcase 14.2: $k+1 \leq i_{r+1} \leq \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 14.1.
Subcase 14.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, if it ends in w_Δ, we are in Case 12, else

in Subcase 14.1.
Subcase 14.4: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 12, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 12.

Case 15: $j = k$ and $i \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta - 1$.
By recoloring the (i, k)-chain starting in w_Δ, we obtain one of the Cases 7, 8, and 12.

Case 16: $1 \le j \le k-2$, and $\Delta \in \overline{w_\Delta}$.
We define colors $j_0, ..., j_{r+1}$ and vertices $v_j^0, ..., v_j^r$ such that the following conditions hold: $v_j^0 = v_j$ and $j_0 = j$; the edges uv_j^s are colored with j_s for $0 \le s \le r$; $\overline{v_j^s} \cap (\{k-1, ..., \Delta\} \cup \{j_1, ..., j_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $j_{s+1} \in \overline{v_j^s}$ and $j_{s+1} \neq j$ for $0 \le s \le r$; $j_{r+1} \in \{k-1, ..., \Delta\} \cup \{j_1, ..., j_{r-1}\}$.
Subcase 16.1: $j_{r+1} = k$.
Color the edges uv_j^s with j_{s+1} for $0 \le s \le r-1$ and obtain Case 5.
Subcase 16.2: $k+1 \le j_{r+1} \le \Delta - 1$.
We recolor the (j_{r+1}, k)-chain starting in v_j^r and obtain Subcase 16.1.
Subcase 16.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring contradicts the Yap-Lemma, else we are in Subcase 16.1.
Subcase 16.4: $j_{r+1} = \Delta$.
We color the edges uv_j^s with j_{s+1} for $0 \le s \le r$, uv_Δ with j, and $v_\Delta w_\Delta$ with Δ. The new coloring of $G - uy$ contradicts the Yap-Lemma.
Subcase 16.5: $j_{r+1} = j_s$ with $1 \le s \le r-1$.
Recolor the (j_{r+1}, k)-chain starting in v_j^r. If this chain ends in u, we color the edges uv_j^t with j_{t+1} for $0 \le t \le s-1$, if it ends in v_j^{s-1}, we color the edges uv_j^{s-1} with k and uv_j^t with j_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices u and v_j^{s-1}, we color the edges uv_j^r with k and uv_j^t with j_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 5.

Case 17: $1 \le j \le k-2$ and $k \in \overline{w_\Delta}$.
We recolor the (k, Δ)-chain starting in w_Δ and obtain Case 16.

Case 18: $1 \le j \le k-2$ and $i \in \overline{w_\Delta}$ with $k+1 \le i \le \Delta - 1$.
By recoloring the (i, k)-chain beginning in w_Δ, we obtain Case 17.

Case 19: $1 \le j \le k-2$ and $k-1 \in \overline{w_\Delta}$.
We recolor the $(k-1, k)$-chain starting in w_Δ. If this chain ends in u or v_{k-1}, the new coloring contradicts the Yap-Lemma, else we are in Case 17.

Case 20: $1 \le j \le k-2$ and $i \in \overline{w_\Delta}$ with $1 \le i \le k-2$.
We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta\} \cup \{i_0, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \neq j$ for $0 \le s \le r$; $i_{r+1} \in$

$\{k-1,...,\Delta\} \cup \{i_0,...,i_{r-1}\}$.
Subcase 20.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \le s \le r-1$ and obtain Case 17.
Subcase 20.2: $k+1 \le i_{r+1} \le \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Subcase 20.1.
Subcase 20.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Subcase 20.1.
Subcase 20.4: $i_{r+1} = i_s$ with $0 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in w_Δ, we are in Case 17, if it ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices w_Δ, u, v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 17.

Case 21: $j = k$ and $\overline{w_\Delta} \subseteq \{1, ..., k-2\}$.
Let $i \in \overline{w_\Delta}$. We define colors $i_0, ..., i_{r+1}$ and vertices $v_i^0, ..., v_i^r$ such that the following conditions hold: $v_i^0 = v_i$ and $i_0 = i$; the edges uv_i^s are colored with i_s for $0 \le s \le r$; $\overline{v_i^s} \cap (\{k-1, ..., \Delta\} \cup \{i_1, ..., i_{s-1}\}) = \emptyset$ for $0 \le s \le r-1$; $i_{s+1} \in \overline{v_i^s}$ and $i_{s+1} \ne i$ for $0 \le s \le r$; $i_{r+1} \in \{k-1, ..., \Delta\} \cup \{i_1, ..., i_{r-1}\}$.
Subcase 21.1: $i_{r+1} = k$.
Color the edges uv_i^s with i_{s+1} for $0 \le s \le r-1$ and obtain Case 17.
Subcase 21.2: $k+1 \le i_{r+1} \le \Delta$.
We recolor the (i_{r+1}, k)-chain starting in v_i^r and obtain Case 14 (if the chain contains the edge $v_\Delta w_\Delta$) or Subcase 21.1 (else).
Subcase 21.3: $i_{r+1} = k-1$.
We recolor the $(k-1, k)$-chain starting in v_i^r. If this chain ends in u or v_{k-1}, the new coloring of $G - uy$ contradicts the Yap-Lemma, else we are in Case 4 (if the chain contains the edge $v_\Delta w_\Delta$) or Subcase 21.1 (else).
Subcase 21.4: $i_{r+1} = i_s$ with $1 \le s \le r-1$.
Recolor the (i_{r+1}, k)-chain starting in v_i^r. If this chain ends in u, we color the edges uv_i^t with i_{t+1} for $0 \le t \le s-1$, if it ends in v_i^{s-1}, we color the edges uv_i^{s-1} with k and uv_i^t with i_{t+1} for $0 \le t \le s-2$, and if the chain ends in none of the vertices u and v_i^{s-1}, we color the edges uv_i^r with k and uv_i^t with i_{t+1} for $0 \le t \le r-1$. In all cases we obtain Case 17.

□

Lemma 5.6 *If $d_G(y) = 3$ and $d_G(u) = \Delta - 1$, then $d_G(x) \ge \Delta - 2$ for any vertex x of G for which $x \notin \{u, y\}$ and x is adjacent to a vertex $w \notin \{u, y\}$ which is adjacent to one of the vertices $v_3, ..., v_\Delta$.*

Proof. We assume that there is a counterexample G with w_Δ adjacent to v_Δ and x_Δ adjacent to w_Δ such that $w_\Delta, x_\Delta \notin \{u, y\}$ and $d_G(x_\Delta) \le \Delta - 3$. We further assume that there is a Δ-coloring of $G - uy$ such that the edges incident with y are colored with 1 and 2, and the edges uv_s are colored with s for $3 \leq s \leq \Delta$. We denote the color of the edges $v_\Delta w_\Delta$ and $w_\Delta x_\Delta$ by j and i, respectively.

Claim 1: We can assume that $j = 1$.

Proof of Claim 1.
If $j = 2$, interchange the colors 1 and 2. Hence, we can assume that $3 \leq j \leq \Delta - 1$.
Case 1: $1, j \in \overline{x_\Delta}$.
We recolor the $(1, i)$-chain starting in x_Δ, and then we recolor the $(j, 1)$-chain beginning in x_Δ. The new coloring suffices the conditions of Claim 1.
Case 2: $j \in \overline{x_\Delta}$ and $1 \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta\}$ with $a \neq j$. By recoloring the $(a, 1)$-chain starting in x_Δ, we obtain Case 1.
Case 3: $1 \in \overline{x_\Delta}$ and $j \notin \overline{x_\Delta}$.
We recolor the $(1, j)$-chain beginning in x_Δ. If this chain contains the edge $v_\Delta w_\Delta$, we obtain a coloring as required, else we are in Case 2.
Case 4: $1, j \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta\}$ with $a \neq j$. By recoloring the $(a, 1)$-chain starting in x_Δ, we obtain Case 3.

Claim 2: We can assume that $j = 1$ and $3 \leq i \leq \Delta - 1$.

Proof of Claim 2. Let $j = 1$.
Case 1: $i = 2$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. We recolor the $(a, 2)$-chain starting in x_Δ and obtain a coloring as required.
Case 2: $i = \Delta$ and $2 \in \overline{x_\Delta}$.
Recolor the $(2, \Delta)$-chain starting in x_Δ and obtain Case 1.
Case 3: $i = \Delta$ and $2 \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. We recolor the $(a, 2)$-chain starting in x_Δ and obtain Case 2.

Claim 3: We can assume that $j = 1$, $3 \leq i \leq \Delta - 1$, and $1, 2, \Delta \in \overline{x_\Delta}$.

Proof of Claim 3. Let $j = 1$ and $3 \leq i \leq \Delta - 1$.
Case 1: $1, \Delta \in \overline{x_\Delta}$ and $2 \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. We recolor the $(a, 2)$-chain starting in x_Δ and obtain a coloring as required.
Case 2: $1, 2 \in \overline{x_\Delta}$ and $\Delta \notin \overline{x_\Delta}$.
By recoloring the $(2, \Delta)$-chain beginning in x_Δ, we obtain Case 1.
Case 3: $1 \in \overline{x_\Delta}$ and $2, \Delta \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. We recolor the $(a, 2)$-chain starting in x_Δ and obtain Case 2.
Case 4: $\Delta \in \overline{x_\Delta}$ and $1 \notin \overline{x_\Delta}$.
By recoloring the $(\Delta, 1)$-chain beginning in x_Δ, we obtain Case 2 or Case 3.
Case 5: $2 \in \overline{x_\Delta}$ and $1, \Delta \notin \overline{x_\Delta}$.
Recolor the $(2, \Delta)$-chain starting in x_Δ and obtain Case 4.
Case 6: $1, 2, \Delta \notin \overline{x_\Delta}$. Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$.

We recolor the $(a, 2)$-chain starting in x_Δ and obtain Case 5.

Claim 4: G is Δ-colorable.

Proof of Claim 4. Let $G - uy$ be colored as required in Claim 3. We recolor the $(2, i)$-chain starting in x_Δ, then we recolor the $(\Delta, 2)$-chain starting in x_Δ, and at last we recolor the $(1, \Delta)$-chain beginning in x_Δ. Now we can color the edge uy with Δ and obtain a Δ-coloring of G.

$\square$

Lemma 5.7 *If $d_G(y) = 3$, $d_G(u) = \Delta \geq 6$, and $d_G(v_1) = 3$, then $d_G(x) \geq \Delta - 3$ for any vertex x of G for which $x \notin \{y, v_1\}$ and x is adjacent to a vertex $w \neq u$ which is adjacent to one of the vertices $v_3, ..., v_\Delta$.*

Proof. We assume that there is a counterexample G with vertices w_Δ adjacent to v_Δ and x_Δ adjacent to w_Δ such that $w_\Delta, x_\Delta \notin \{u, v_1, y\}$ and $d_G(x_\Delta) \leq \Delta - 4$. We further assume that there is a Δ-coloring of $G - uy$ such that the edges incident with y are colored with 1 and 2, and the edge uv_1 is colored with 1. We denote the color of the edges $v_\Delta w_\Delta$ and $w_\Delta x_\Delta$ by j and i, respectively.

Claim 1: We can assume that $j = 1$.

Proof of Claim 1.

Case 1: $j = 2$ and $2 \in \overline{v_1}$.

We color the edge uv_1 with 2, and by interchanging the colors 1 and 2, we obtain a coloring as required.

Case 2: $j = 2$ and $\Delta \in \overline{v_1}$.

Recolor the $(\Delta, 2)$-chain starting in v_1 and obtain Case 1.

Case 3: $3 \leq j \leq \Delta - 1$, $3 \leq i \leq \Delta - 1$, and $\Delta \in \overline{v_1}$.

Subcase 3.1: $2, j \in \overline{x_\Delta}$.

We recolor the $(2, i)$-chain starting in x_Δ, then we recolor the $(j, 2)$-chain starting in x_Δ, and we obtain Case 2.

Subcase 3.2: $j \in \overline{x_\Delta}$ and $2 \notin \overline{x_\Delta}$.

Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$ and $a \neq j$. By recoloring the $(a, 2)$-chain starting in x_Δ, we obtain Subcase 3.1.

Subcase 3.3: $2 \in \overline{x_\Delta}$ and $j \notin \overline{x_\Delta}$.

Recolor the $(2, j)$-chain beginning in x_Δ. If this chain contains the edge $v_\Delta w_\Delta$, we are in Case 2, else we obtain Subcase 3.2.

Subcase 3.4: $2, j \notin \overline{x_\Delta}$.

Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. By recoloring the $(a, 2)$-chain starting in x_Δ, we obtain Subcase 3.3.

Case 4: $3 \leq j \leq \Delta - 1$, $3 \leq i \leq \Delta - 1$, and $\Delta \notin \overline{v_1}$.

If $2 \in \overline{v_1}$, then recolor the $(2, \Delta)$-chain starting in v_1, else let $a \in \overline{v_1}$ with $a \neq i, j$, recolor the $(a, 2)$-chain beginning in v_1, and then recolor the $(2, \Delta)$-chain starting in v_1. In both cases we obtain Case 3.

Case 5: $3 \leq j \leq \Delta - 1$ and $i = 2$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$ and $a \neq j$. By recoloring the $(a, 2)$-chain starting in x_Δ, we obtain Case 3 or 4.
Case 6: $3 \leq j \leq \Delta - 1$ and $i = \Delta$.
If $2 \in \overline{x_\Delta}$, then recolor the $(2, \Delta)$-chain starting in x_Δ, else let $a \in \overline{x_\Delta}$ with $a \neq 1, j$, recolor the $(a, 2)$-chain beginning in x_Δ, and then recolor the $(2, \Delta)$-chain starting in x_Δ. In both cases we obtain Case 5.
Case 7: $3 \leq j \leq \Delta - 1$ and $i = 1$.
Let $a \in \overline{v_1} \cap \overline{x_\Delta}$. By Corollary 5.1, the $(a, 1)$-chain starting in x_Δ can not end in v_1 or y. By recoloring this chain, we obtain a coloring as required or one of the Cases 3–6.
Case 8: $j = 2$ and $\overline{v_1} = \{3, ..., \Delta - 1\}$.
We recolor the $(3, 2)$-chain starting in v_1. If this chain contains the edge $v_\Delta w_\Delta$, then we obtain one of the Cases 4–7, else we are in Case 1.

Claim 2: We can assume that $j = 1$ and $3 \leq i \leq \Delta - 1$.
Proof of Claim 2. Let $j = 1$.
Case 1: $i = 2$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. By recoloring the $(a, 2)$-chain beginning in x_Δ, we obtain a coloring as required.
Case 2: $i = \Delta$.
If $2 \in \overline{x_\Delta}$, we recolor the $(2, i)$-chain starting in x_Δ and obtain Case 1, else let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$, and by recoloring the $(a, 2)$-chain beginning in x_Δ and then recoloring the $(2, i)$-chain starting in x_Δ, we obtain Case 1.

Claim 3: We can assume that $j = 1$, $3 \leq i \leq \Delta - 1$, and $1, 2, \Delta \in \overline{x_\Delta}$.
Proof of Claim 3. Let $j = 1$ and $3 \leq i \leq \Delta - 1$.
Case 1: $1, \Delta \in \overline{x_\Delta}$ and $2 \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. Recolor the $(a, 2)$-chain starting in x_Δ and obtain a coloring as required.
Case 2: $1, 2 \in \overline{x_\Delta}$ and $\Delta \notin \overline{x_\Delta}$.
By recoloring the $(2, \Delta)$-chain beginning in x_Δ, we obtain Case 1.
Case 3: $2, \Delta \in \overline{x_\Delta}$ and $1 \notin \overline{x_\Delta}$.
Subcase 3.1: $\Delta \in \overline{v_1}$.
We recolor the $(\Delta, 1)$-chain starting in x_Δ. If this chain ends in v_1 or y, the new coloring contradicts the Yap-Lemma, else we are in Case 2.
Subcase 3.2: $2 \in \overline{v_1}$ and $\Delta \notin \overline{v_1}$. By recoloring the $(2, \Delta)$-chain starting in v_1, we obtain Subcase 3.1.
Subcase 3.3: $\overline{v_1} = \{3, ..., \Delta - 1\}$. Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. We recolor the $(a, 2)$-chain beginning in v_1 and obtain Subcase 3.2.
Case 4: $\Delta \in \overline{x_\Delta}$ and $1, 2 \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. We recolor the $(a, 2)$-chain starting in x_Δ and obtain Case 3.
Case 5: $1 \in \overline{x_\Delta}$ and $2, \Delta \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta} \cap \{3, ..., \Delta - 1\}$. Recolor the $(a, 2)$-chain starting in x_Δ and obtain Case 2.

Case 6: $2 \in \overline{x_\Delta}$ and $1, \Delta \notin \overline{x_\Delta}$.
By recoloring the $(2, \Delta)$-chain beginning in x_Δ, we obtain Case 4.
Case 7: $1, 2, \Delta \notin \overline{x_\Delta}$.
Let $a \in \overline{x_\Delta}$. Recolor the $(a, 2)$-chain starting in x_Δ and obtain Case 6.

Claim 4: The graph G is not Δ-critical.

Proof of Claim 4. Let $G - uy$ be colored as required in Claim 3. We recolor the $(2, i)$-chain starting in x_Δ. Then we recolor the $(\Delta, 2)$-chain beginning in x_Δ. For the new coloring we have that $j = 1$, $i = \Delta$, and $1 \in \overline{x_\Delta}$.
Case 1: $2 \in \overline{v_1}$.
We color the edge uv_1 with 2, and by recoloring the $(1, \Delta)$-chain starting in x_Δ and coloring uy with Δ, we obtain a Δ-coloring of G.
Case 2: $2 \notin \overline{v_1}$ and $\Delta \in \overline{v_1}$.
Recolor the $(1, \Delta)$-chain beginning in x_Δ and obtain a contradiction to the Yap-Lemma.
Case 3: $\overline{v_1} = \{3, ..., \Delta - 1\}$.
Let $a \in \{3, ..., \Delta - 1\}$. Recolor the $(a, 2)$-chain starting in v_1 and obtain Case 1.

□

Lemma 5.8 ([24]) *If $d_G(u) = 2$, then $d_G(x) \geq \Delta - 1$ for any vertex x of G for which $x \neq u$ and x is adjacent to a vertex w which is adjacent to v_Δ.*

□

Now we consider a special case for which the lower bound for the number of major vertices adjacent to a vertex u is not attained, but we can also prove a statement similar to the other lemmas.

Lemma 5.9 *Let G be a 7-critical graph with adjacent vertices u and y with $d_G(y) = 6$ and $d_G(u) = 4$. Let u be adjacent to three major vertices v_1, v_2, v_3. Then there is no vertex $w \neq u$ with $d_G(w) = 4$, such that w is adjacent to v_1, v_2, and v_3.*

Proof. We assume that G is a counterexample, and the edges of $G - uy$ are colored with $1, ..., 7$ such that the edges incident with y are colored with $1, ..., 5$ and the edges uv_1, uv_2 and uv_3 are colored with $1, 6, 7$, respectively. Since not both of v_2w and v_3w can be colored with 1, we can assume that the edge v_3w is not colored with 1.

Case 1: v_3w is colored with $i \in \{2, ..., 5\}$, and the color 7 is missing in w.
By recoloring the $(7, i)$-chain starting in w and coloring uy with 7, we obtain a 7-coloring of G.

Case 2: v_3w is colored with $i \in \{2, ..., 5\}$, and the color 7 is not missing in w.
Let $j \in \{2, ..., 5\}$ be a color missing in w. We recolor the $(j, 7)$-chain beginning in w and obtain Case 1.

Case 3: v_3w is colored with 6.
Let $j \in \{2, ..., 5\}$ be a color missing in w. Recolor the $(j, 6)$-chain starting in w and obtain one of the Cases 1 or 2.

□

5.2 Main Theorem

Theorem 5.1 *There is no 7-critical planar graph.*

Proof. We assume that G is a 7-critical plane graph, and from this assumption we deduce with the discharging method, that then there is a plane multigraph without 2-faces and with average degree at least 6, contradicting Euler's formula. For every vertex $v \in V(G)$ we define the initial charge $ch(v) = d_G(v)$. These charges will be modified according to the following discharging rules, and the obtained charges will be denoted by $ch_1(v)$ for every vertex v of G. Each of the rules sends an amount of 1 from a major vertex v to one or more vertices of degree at most 5.

Let v be a major vertex, and w a vertex of degree 4. Then we say that w is a T_4-*neighbor* of v, if w is contained in four 3-faces, v is adjacent to w and to two other vertices x_1, x_2 of degree at most 5, and both vertices y_1, y_2 for which 3-faces (y_1, v, w) and (y_2, v, w) exist are adjacent to x_1 or x_2, respectively. A vertex of degree 4 is called a T_4-*vertex*, if it is a T_4-neighbor of at least one major vertex. By VAL, Lemma 5.3, and Lemma 5.5, a T_4-vertex is adjacent to four major vertices. By $N_i(v)$ we denote the number of vertices of degree i adjacent to v.

Rule 1: If v is adjacent to two vertices w_1, w_2 of degree 3, v sends $1/2$ to each of w_1 and w_2.

Rule 2: If v is adjacent to one vertex w of degree 3 and one vertex x of degree 4, such that x is adjacent to four major vertices, x is contained in four 3-faces, and w is adjacent to three neighbors of x, then v sends $2/3$ to w and $1/3$ to x.

Rule 3: If v is adjacent to exactly one vertex w of degree 3 and the conditions of Rule 2 do not hold, v sends 1 to w.

Rule 4: If v is adjacent to a major vertex w such that w is adjacent to two vertices x_1, x_2 of degree 3 and v is adjacent to none of x_1, x_2, then v sends $1/2$ to each vertex x_1 and x_2.

Rule 5: If v is adjacent to a vertex w of degree 6 such that w is adjacent to a vertex x of degree 3 and v is not adjacent to x, then v sends 1 to x.

Rule 6: If $N_3(v) = 0$ and v has two T_4-neighbors w_1, w_2, then v sends $1/2$ to each of w_1 and w_2.

Rule 7: If $N_3(v) = 0$ and v has exactly one T_4-neighbor w and two other neighbors x_1, x_2 of degree at most 5, then v sends $1/2$ to w and $1/4$ to each of x_1 and x_2.

Rule 8: If $N_3(v) = 0$ and v has no T_4-neighbor and is adjacent to a vertex w of degree 4 and to two other vertices x_1, x_2 of degree 4 or 5, v sends $1/3$ to each vertex w, x_1, and x_2.

Rule 9: If $N_3(v) = N_5(v) = 0$, and v has exactly two neighbors w_1, w_2 of degree 4, v sends $1/2$ to each of w_1 and w_2.

Rule 10: If $N_3(v) = 0$, v has exactly one neighbor w of degree 4 and one neighbor x of degree 5, and w is adjacent to a vertex of degree at most 6 and to three distinct major vertices adjacent to x, then v sends $2/3$ to w and $1/3$ to x.

Rule 11: If $N_3(v) = 0$, v has exactly one neighbor w of degree 4 and one neighbor x of degree 5, and x is not adjacent to three distinct neighbors of w and to a vertex of degree

at most 6, then v sends $1/2$ to each of w and x.
Rule 12: If $N_3(v) = N_5(v) = 0$ and v is adjacent to exactly one vertex w of degree 4, v sends 1 to w.
Rule 13: If $N_3(v) = N_4(v) = 0$ and v is adjacent to exactly three vertices w, x_1, x_2 of degree 5 such that w is adjacent to exactly two major vertices and is contained in 5 3-faces, then v sends $1/2$ to w and $1/4$ to each of x_1 and x_2.
Rule 14: If $N_3(v) = N_4(v) = 0$, $1 \leq N_5(v) \leq 4$ and the condition of Rule 13 does not hold, then v sends $1/N_5(v)$ to each neighbor of degree 5 of v.

To show that Rule 13 is clear, we have to prove that a vertex v with $N_5(v) = 3$ can not be adjacent to two vertices of degree 5 each of which has exactly two neighbors of degree 7 and is contained in five 3-faces. Let w be a neighbor of degree 5 of v such that w has exactly two neighbors of degree 7 and is contained in five 3-faces. Since all faces containing w are 3-faces, there are two vertices x_1, x_2 such that there are 3-faces (v, w, x_1) and (v, w, x_2), and one of x_1, x_2, say x_1, must have degree 6, because w is adjacent to only two major vertices. Since v has three neighbors of degree 5, it can not have a neighbor of degree 6 other than x_1. If there is a neighbor $y \neq w$ of v of degree 5 such that y has exactly two neighbors of degree 7 and is contained in five 3-faces, then G contains a 3-face (v, y, x_1) by the same argument as (v, w, x_1). Since w and y are both contained in five 3-faces, there must be vertices $z_1, z_2 \neq v$ such that there are 3-faces (w, x_1, z_1) and (y, x_1, z_2). If $z_1 = z_2$, then G contains 3-faces (x_1, v, w), (x_1, w, z_1), (x_1, z_1, y), and (x_1, y, v), a contradiction to $d_G(x_1) = 6$. Hence, x_1 is adjacent to two vertices w, y of degree 5 and to two vertices z_1, z_2 of degree 6. Since $d_G(x_1) = 6$, this is a contradiction to VAL.

By Lemma 5.3 and Lemma 5.2, we have that a vertex v which sends 1 to one or $1/2$ to two vertices by Rule 4 or 5 is not adjacent to a vertex of degree at most 5, and by Lemma 5.6 and Lemma 5.7, we know that v either sends 1 to exactly one vertex of degree 3, or v sends $1/2$ to exactly two vertices of degree 3. Hence, for all major vertices v of G we have $ch_1(v) \geq 6$. Since vertices of degree 6 do not send or receive an amount by any of the Rules 1–14, we have $ch_1(v) = 6$ for every vertex v of degree 6.

Let $ch_1(w) < 6$ for a vertex w of degree 3, 4, or 5. We say that a face F is a *bigface* of w, if F has length at least 4, F contains w, and $d_G(x_1), d_G(x_2) \geq 6$ for both neighbors x_1, x_2 of w in F. In the following four claims we prove that for every vertex w of degree 3, 4, or 5, either $ch_1(w) \geq 6$ holds, or w has at least $\lceil 6 - ch_1(w) \rceil$ bigfaces.

Claim 1 *For every vertex $v \in V(G)$ with $d_G(v) = 3$, either $ch_1(v) \geq 6$, or $ch_1(v) \geq 5$ and there is a bigface F of v.*

Proof of Claim 1. If a vertex v of degree 3 has a neighbor w of degree 6, then there are at least 3 major vertices adjacent to w which are not adjacent to v, and each of them sends 1 to v, thus $ch_1(v) \geq 6$. Let v be adjacent to three major vertices w_1, w_2, w_3.

If there is a vertex x of degree 3 such that each of w_1, w_2, w_3 is adjacent to x, then w_1 has three neighbors y_1, y_2, y_3 of degree 7 such that $y_1, y_2, y_3, w_1, w_2, w_3$ are distinct, and by Lemma 5.3, each of these vertices sends $1/2$ to v, thus $ch_1(v) \geq 6$.

If there is a vertex x of degree 3 such that exactly two vertices of w_1, w_2, w_3, say w_1 and w_2, are adjacent to x, and there is one vertex y of degree 3 adjacent to w_3, then w_1 and w_3 can not be adjacent by Lemma 5.3. Hence, there are three major vertices z_1, z_2, z_3 adjacent to w_1 but not to x such that $z_1, z_2, z_3, w_1, w_2, w_3$ are distinct, and by Lemma 5.3, each of these vertices sends $1/2$ to v, thus $ch_1(v) \geq 6$.

If there is a vertex x of degree 3 such that exactly two vertices of w_1, w_2, w_3, say w_1 and w_2, are adjacent to x and w_3 is not adjacent to a vertex of degree 3 other than v, then there are two vertices y_1, y_2 adjacent to w_1 but not to x such that y_1, y_2, w_1, w_2, w_3 are distinct and by Lemma 5.3, each of w_1, w_2, y_1, y_2 sends $1/2$ to v and w_3 sends 1 to v.

If there are two different vertices x_1, x_2 of degree 3 such that each of them is adjacent to exactly one of w_1, w_2, w_3, say x_1 is adjacent to w_1 and x_2 is adjacent to w_2, then the vertices w_1 and w_2 can not be adjacent by Lemma 5.3, and they have no common neighbor other than v by Lemma 5.7. Hence, one vertex of w_1, w_2, say w_1, is not adjacent to w_3, and there are three major vertices y_1, y_2, y_3 adjacent to w_1 but not to x_1 such that $y_1, y_2, y_3, w_1, w_2, w_3$ are distinct and each of them sends at least $1/2$ to v.

If two vertices of w_1, w_2, w_3, say w_2 and w_3, are adjacent to no vertex of degree 3 other than v, and w_1 is adjacent to a vertex x of degree 3 other than v, then there is a major vertex y adjacent to w_1 such that y is not adjacent to v or x. Each of w_1 and y sends $1/2$ to v and each of w_2, w_3 sends 1 to v, thus $ch_1(v) \geq 6$.

If none of w_1, w_2, w_3 is adjacent to a vertex of degree 3 other than v, and there is no vertex x of degree 4 such that w_1, w_2, w_3 are neighbors of x and x is contained in four 3-faces, then w_1, w_2, and w_3 send 1 to v, respectively.

If there is a vertex x of degree 4 such that w_1, w_2, w_3 are neighbors of x and x is contained in four 3-faces, then each of w_1, w_2, w_3 sends $2/3$ to v. Further, in the last case v is contained in at least one face F with at least four vertices, since otherwise we can assume that there are 3-faces (w_2, w_1, x), (w_2, x, w_3), (w_2, w_3, v), and (w_2, v, w_1), a contradiction to $d_G(w_2) = 7$. Since v has no neighbor of degree at most 5, F is a bigface of v.

Claim 2 *For every vertex $v \in V(G)$ with $d_G(v) = 4$ and $ch_1(v) < 5$, there are two bigfaces F_1, F_2 of v.*

Proof of Claim 2. By Lemma 5.2, v can not be adjacent to a vertex w of degree 5, since otherwise there are three major vertices x_1, x_2, x_3 adjacent to v, and none of them can be adjacent to a vertex of degree at most 5 other than v and w. If one of x_1, x_2, x_3, say x_1, is not adjacent to w, then x_1 sends 1 and each of x_2, x_3 sends at least $1/2$ to v, else each of x_1, x_2, x_3 sends $2/3$ to v by Rule 10. This implies $ch_1(v) \geq 6$. Now we have to consider three possible cases for the number of major vertices adjacent to v.

Case 1: $N_7(v) = 2$.
Let w be a major vertex adjacent to v such that w sends at most $1/3$ to v by the discharging rules. Then w is adjacent to a vertex of degree 3 or to two vertices of degree at most 5

other than v. Since $N_7(v) = 2$, there is a vertex x_1 of degree 6 adjacent to v and a face F_1 such that the edges $vw, vx_1 \in E(F_1)$. By VAL, F_1 can not be a 3-face. Let x_2 be the vertex other than x_1 such that vw, vx_2 are contained in one face F_2. Let x_3 be the neighbor of v other than w, x_1, x_2. If $d_G(x_2) = 6$, then F_2 can not be a 3-face by VAL. If $d_G(x_2) = 7$, then we have $d_G(x_3) = 6$. Let F_3 be the face containing vx_2, vx_3. We now prove that at least one of the faces F_2, F_3 is not a 3-face. Since $ch_1(v) < 5$, there must be a vertex $y \neq v$ adjacent to x_2 with $d_G(y) \leq 5$. If F_3 is a 3-face, then $d_G(y) \geq 4$ and all neighbors of x_2 other than v, x_3, y are major vertices by VAL. Hence, F_2 can not be a 3-face, since otherwise Lemma 5.5 implies that w can not be adjacent to a vertex of degree at most 5 other than v and y contradicting the fact that w is adjacent to a vertex of degree 3 or to two vertices of degree at most 5 other than v.

Case 2: $N_7(v) = 3$.
Let w_1, w_2, w_3 be the major vertices adjacent to v, and let x be the vertex of degree 6 adjacent to v. Since $ch_1(v) < 5$, one of the vertices w_1, w_2, w_3, say w_1, sends at most $1/4$ to v. Then w_1 is adjacent to a vertex of degree 3, or w_1 has a T_4-neighbor.

We first consider the case that w_1 is adjacent to a vertex y of degree 3. If w_2 and w_3 are adjacent to y, then both faces containing the edge vx can not be 3-faces by VAL. If exactly one of the vertices w_2, w_3, say w_2 is adjacent to y, then every face containing exactly one of the edges vw_1, vw_2 is not a 3-face by VAL (if the face contains x) or by Lemma 5.3 (if the face contains w_3), since w_3 is adjacent to at least one vertex of degree at most 5 other than v, because otherwise w_3 sends 1 to v. If none of the vertices w_2, w_3 is adjacent to y, then both faces containing the edge vw_1 can not be 3-faces by VAL or Lemma 5.3.

Now we consider the case that w_1 has a T_4-neighbor y. Then it follows from the definition of a T_4-vertex that there is a vertex z of degree at most 5 other than v, y adjacent to w_1. If both faces containing the edge vw_1 have length at least 4, then the statement of Claim 2 is true. Hence, we can assume that G contains a 3-face (v, w_1, w_2). By Lemma 5.3, w_2 can have no neighbor of degree at most 5 other than v, y, z, and w_2 must be adjacent to at least one of y, z, since otherwise $ch_1(v) \geq 5$. Hence, x can not be adjacent to w_1 (by VAL) or to w_2 (by Lemma 5.5). If G does not contain a 3-face (x, v, w_3), then we have that both faces containing the edge vx have length at least 4. Thus we assume that x is adjacent to w_3. Since w_3 is adjacent to a vertex v of degree 4 and to a vertex x of degree 6, w_3 has no neighbor of degree 3 and at most one neighbor of degree at most 5 other than v, thus w_3 sends at least $1/2$ to v. Hence, w_2 must be adjacent to y and z, since $ch_1(v) < 5$. If w_3 is adjacent to none of w_1, w_2, then both faces containing exactly one of the edges vw_3, vx have length at least 4. If w_3 is adjacent to w_2, then w_2 has two neighbors v, y of degree 4, one neighbor z of degree at most 5, and one neighbor w_3 of degree 7 which is adjacent to a vertex x of degree 6, but that is impossible by Lemma 5.3. If w_3 is adjacent to w_1, then w_1 has two neighbors v, y of degree 4, a neighbor z of degree at most 5 and a neighbor w_3 of degree 7 which is adjacent to a vertex x of degree 6, what is not possible by Lemma 5.3.

Case 3: $N_7(v) = 4$.
Let w_1, w_2, w_3, w_4 be the neighbors of v. Since $ch_1(v) < 5$, each of the neighbors of v is adjacent to at least one vertex of degree at most 5, and one of them, say w_1, does not send any amount to v. Hence, w_1 is adjacent to a vertex of degree 3, or w_1 has two T_4-neighbors.

We first assume that w_1 is adjacent to a vertex x of degree 3. If two of the vertices w_2, w_3, w_4, say w_2 and w_3 are adjacent to x, then both faces containing the edge vw_4 can not be 3-faces by Lemma 5.3. If exactly one of the vertices w_2, w_3, w_4, say w_2, is adjacent to x, then every face containing exactly one of the edges vw_1, vw_2 is not a 3-face by Lemma 5.3. If none of the vertices w_2, w_3, w_4 is adjacent to x, then both faces containing the edge vw_1 can not be 3-faces by Lemma 5.3.

Now we consider the case that w_1 has two T_4-neighbors x_1, x_2. If none of the faces containing the edge vw_1 is a 3-face, then Claim 2 is true. Hence, we can assume that there is a 3-face (v, w_1, w_2). Let the other face containing vw_1 also be a 3-face, say (v, w_1, w_3), then both vertices w_2, w_3 have no neighbor of degree at most 5 other than x_1, x_2 by Lemma 5.3. If none of w_2, w_3 is adjacent to both, x_1 and x_2, then each of w_2, w_3 sends at least 1/2 to v, thus $ch_1(v) \geq 5$. Hence, we can assume that w_2 is adjacent to x_1 and x_2. The vertex w_3 is not adjacent to none of x_1, x_2, since otherwise w_3 sends 1 to v, and w_3 is not adjacent to x_1 and x_2, since otherwise the graph induced by $v, x_1, x_2, w_1, w_2, w_3$ contains $K_{3,3}$. Hence, we can assume that w_3 is adjacent to x_1 but not to x_2.

If G contains a 3-face (v, w_2, w_4), then w_4 has no neighbor of degree at most 5 other than x_1, x_2 by Lemma 5.3, but w_4 can not be adjacent to x_1 and x_2, since otherwise the graph induced by $v, x_1, x_2, w_1, w_2, w_4$ contains $K_{3,3}$. Hence, each of w_3, w_4 sends at least 1/2 to v, thus $ch_1(v) \geq 5$.

If G contains a 3-face (v, w_3, w_4), then w_4 must have two neighbors of degree at most 5 other than v, since otherwise each of w_2, w_4 sends at least 1/2 to v. By Lemma 5.3, one of these vertices must be x_1. This implies that x_1 is adjacent to $w_1, ..., w_4$ and to no other vertex, but this is not possible, since each face containing x_1 is a 3-face, (v, w_1, w_2), (v, w_1, w_3) are 3-faces, and w_1 is adjacent to x_2. Hence, both faces containing the edge vw_4 have length at least 4, and we have proved that Claim 2 is true, if a 3-face (v, w_1, w_3) exists.

Now we assume that the face other than (v, w_1, w_2) containing the edge vw_1 is not a 3-face. Then we only have to consider the case that the face other than (v, w_1, w_2) containing the edge vw_2, say (v, w_2, w_3), and (v, w_3, w_4) must be 3-faces, since otherwise v is contained in two bigfaces. By Lemma 5.3, w_2 has no neighbors of degree at most 5 other than x_1, x_2.

If w_2 is adjacent to both, x_1 and x_2, then w_3 can not be adjacent to a vertex of degree at most 5 other than x_1, x_2, but w_3 can not be adjacent to x_1 and x_2, since otherwise the graph induced by $v, x_1, x_2, w_1, w_2, w_3$ contains $K_{3,3}$. On the other hand, w_3 must be adjacent to one of x_1, x_2, say to x_1, since otherwise w_3 sends 1 to v. Now w_4 must be adjacent to two vertices of degree at most 5, since otherwise w_3 and w_4 send 1/2 to v. By

Lemma 5.3, one of these vertices must be x_1. This implies that x_1 is adjacent to $w_1, ..., w_4$ which is not possible, as shown above.

If w_2 is adjacent to exactly one of x_1, x_2, say to x_1, then w_3 must have two neighbors of degree at most 5 other than v, since otherwise each of w_2, w_3 sends $1/2$ to v. By Lemma 5.3, one of these vertices must be x_1. The vertex w_4 must have two neighbors of degree at most 5 other than v, since otherwise each of w_2, w_4 sends $1/2$ to v. Again by Lemma 5.3, x_1 must be one of these neighbors, thus x_1 is adjacent to all neighbors of v, a contradiction.

Claim 3 *For every vertex $v \in V(G)$ with $d_G(v) = 4$ and $5 \leq ch_1(v) < 6$, there is a bigface F of v.*

Proof of Claim 3. As in Claim 2 we have that v is not adjacent to a vertex of degree 5, since otherwise Lemma 5.2 implies $ch_1(v) \geq 6$. We assume that all faces containing v are 3-faces.

Case 1: $N_7(v) = 2$.
Let w_1, w_2 be the major vertices adjacent to v, and let x_1, x_2 be the neighbors of degree 6 of v. If w_1 and w_2 are adjacent to both, x_1 and x_2, then w_1 and w_2 have no neighbors of degree at most 5 other than v, a contradiction to $ch_1(v) < 6$. Hence, we can assume that G contains the 4-cycle (w_1, x_1, x_2, w_2). Since x_1 is adjacent to v and x_2, Lemma 5.4 implies that w_1 has no neighbors of degree at most 5 other than v. The same holds for w_2 by applying Lemma 5.4 to the edge x_2v. Hence, we have $ch_1(v) = 6$, a contradiction.

Case 2: $N_7(v) = 3$.
Let w be the vertex of degree 6 adjacent to v, let x_1, x_2 be the major vertices such that there are 3-faces (v, w, x_1) and (v, w, x_2), and let y be the major vertex adjacent to v, x_1, and x_2. If x_1 and x_2 have no neighbors of degree at most 5 other than v, then we have $ch_1(v) \geq 6$. Hence, we can assume that x_1 is adjacent to a vertex $z_1 \neq v$ of degree at most 5. Lemma 5.5 implies that y has no neighbor of degree at most 5 other than v, z_1.

If there is a vertex z_2 other than z_1, v of degree at most 5 such that z_2 is adjacent to x_2, then Lemma 5.5 implies that y has no neighbor of degree at most 5 other than v, z_2. Now we have that y is adjacent to no vertex of degree at most 5 other than v, and by the discharging rules, y sends 1 to v and each of x_1, x_2 sends $1/2$ to v, thus $ch_1(v) = 6$.

If x_2 has no neighbor of degree at most 5, then x_2 sends 1 to v and each of x_1, y sends at least $1/2$ to v, thus $ch_1(v) \geq 6$. If x_1 and x_2 but not y are adjacent to z_1, then y sends 1 to v and each of x_1, x_2 sends at least $1/2$ to v, thus $ch_1(v) = 6$. If x_1, x_2, y are all adjacent to z_1, then Lemma 5.9 implies that $d_G(z_1) = 5$, and since z_1 is adjacent to three neighbors of v, each of x_1, x_2, y sends $2/3$ to v, thus $ch_1(v) = 6$.

Case 3: $N_7(v) = 4$.
Since $ch_1(v) < 6$, one of the neighbors of v must be adjacent to a vertex of degree 3 or to two vertices other than v of degree at least 5.

We first consider the case that there is a vertex w of degree 3 adjacent to a neighbor of v. If w is adjacent to three neighbors of v, then each of them sends $1/3$ to v by Rule

2 and, by Lemma 5.3, the fourth neighbor of v is not adjacent to a vertex of degree at most 5 other than v and sends 1 to v, thus $ch_1(v) = 6$. If w is adjacent to exactly two neighbors x_1, x_2 of v, then both neighbors of v other than x_1, x_2 send 1 to v by Lemma 5.3. If w is adjacent to exactly one neighbor x of v, then both neighbors of v which are also adjacent to x send 1 to v by Lemma 5.3.

Now we consider the case that a neighbor w_1 of v is adjacent to two vertices x_1, x_2 of degree at most 5. Let w_2, w_3, w_4 be the other vertices adjacent to v, such that G contains a 4-cycle (w_1, w_2, w_4, w_3). By Lemma 5.3, both vertices w_2 and w_3 can not be adjacent to a vertex of degree at most 5 other than x_1, x_2. Not both of w_2, w_3 can be adjacent to both, x_1 and x_2, since otherwise the graph induced by $v, x_1, x_2, w_1, w_2, w_3$ contains $K_{3,3}$.

If one of w_2, w_3, say w_2, is adjacent to x_1 and x_2, and w_3 is adjacent to one of x_1, x_2, say to x_1, then w_4 can not have a neighbor of degree at most 5 other than v, x_1, x_2 by Lemma 5.3, and w_4 can not be adjacent to x_2, since otherwise G contains a subdivision of $K_{3,3}$ with main vertices $v, x_1, x_2, w_1, w_2, w_3$. If w_4 is adjacent to x_1, then v is T_4-neighbor of w_1 and w_2, and w_3 and w_4 have exactly two neighbors of degree at most 5, respectively, thus every neighbor of v sends 1/2 to v, and we have $ch_1(v) = 6$. If w_4 is not adjacent to x_1, then v is T_4-neighbor of w_1, and w_1 sends 1/2 to v, w_2 has exactly two vertices of degree at most 5 and sends 1/2 to v, and w_4 is not adjacent to any vertex of degree at most 5 other than v and sends 1 to v, thus $ch_1(v) \geq 6$. The same argument holds true, if w_4 but not w_3 is adjacent to x_1.

If w_2 is adjacent to x_1 and x_2, and w_3 and w_4 are not adjacent to x_1 or x_2, then both of w_3 and w_4 send 1 to v, thus $ch_1(v) \geq 6$. Hence, we can assume that none of w_2, w_3 is adjacent to x_1 and x_2.

If both, w_2 and w_3, are neighbors of one of x_1, x_2, say of x_1, then w_4 can not be adjacent to x_2, since otherwise G contains a subdivision of K_5 with main vertices v, w_1, w_2, w_3, w_4. By Lemma 5.3, w_4 can not be adjacent to two vertices of degree at most 5 other than x_1. If w_4 is adjacent to x_1 and to one more vertex x_3 of degree at most 5, then v is a T_4-neighbor of w_1 and w_4, and w_2, w_3 have exactly two neighbors v, x_1 of degree at most 5. Hence, each of w_1, w_2, w_3, w_4 sends 1/2 to v and we have $ch_1(v) = 6$. If w_4 has at most one neighbor of degree at most 5 other than v, then w_4 sends at lest 1/2 to v, and again we obtain $ch_1(v) \geq 6$.

If w_2 is adjacent to one of x_1 and x_2, say to x_1, and w_3 is adjacent to x_2, then either w_4 has at most one neighbor of degree at most 5 other than v, or w_4 is adjacent to x_1 and x_2. In the latter case v is T_4-neighbor of w_1 and w_4, and w_2 and w_3 have exactly two neighbors of degree at most 5, thus every neighbor of v sends 1/2 to v and $ch_1(v) = 6$. In the first case w_4 sends at lest 1/2 to v, and again we obtain $ch_1(v) \geq 6$.

If one of w_2, w_3, say w_2, is adjacent to exactly one of x_1, x_2, say to x_1, and w_3 is not adjacent to x_1 or x_2, then w_4 is adjacent to x_1, or w_4 has at most one neighbor of degree at most 5 other than v. In the latter case w_2 and w_4 send at least 1/2 to v and w_3 sends 1 to v, thus $ch_1(v) \geq 6$. If w_4 is adjacent to x_1 and to a vertex of degree at most 5 other than v, x_1, then w_2 sends 1/2 and w_3 sends 1 to v. Hence, $ch_1(v) < 6$ is only possible, if one of w_1, w_4 has two T_4-neighbors other than v, which implies that x_1 is a T_4-neighbor of w_1 or w_4. Therefore, all faces containing x_1 are 3-faces, but G can not contain 3-faces (x_1, w_1, w_2) and (x_1, w_2, w_4), since $d_G(w_2) = 7$. Hence, there must exist a 3-face (x_1, w_1, w_4) which implies that w_1 is adjacent to w_4. By Lemma 5.3, we obtain that w_4 can not have a neighbor of degree at most 5 other than v, x_1, x_2. Now we can assume that w_1 has two T_4-neighbors x_1, x_2, since otherwise we can interchange the labeling of w_1 and w_4. Let y be the neighbor of x_1 other than w_1, w_2, w_4. Since all faces containing x_1 are 3-faces, y must be adjacent to w_2. Since x_2 is a T_4-neighbor of w_1, and x_2 is not adjacent to w_2 or w_3, there must be a neighbor of x_2 other than w_1, w_4 which is also adjacent to x_1, thus x_2 is adjacent to y. Now we obtain a subdivision of $K_{3,3}$ with main vertices $x_1, x_2, v, w_1, w_2, w_4$ as a subgraph of G, a contradiction.

The last case we have to consider is that both vertices w_2, w_3 are not adjacent to x_1 or x_2, but then each of w_2, w_3 sends 1 to v and we have $ch_1(v) \geq 6$.

Claim 4 *For every vertex $v \in V(G)$ with $d_G(v) = 5$ and $ch_1(v) < 6$, there is a bigface F of v.*

Proof of Claim 4. By Lemma 5.2, v can not be adjacent to a vertex w of degree 4, since otherwise there are four vertices of degree 7 adjacent to v, and none of them can be adjacent to a vertex of degree at most 5 other than v, w, thus $ch_1(v) > 6$.

If v is adjacent to two vertices w_1, w_2 of degree 5, then v has three neighbors of degree 7 each of which is adjacent to no vertex of degree at most 6 other than v, w_1, w_2 by Lemma 5.2. Since the major vertices adjacent to v have no neighbor of degree 6, the condition of rule 13 holds for none of them, thus $ch_1(v) \geq 6$.

If v is adjacent to a vertex w of degree 5 and to a vertex x of degree 6, then v has three neighbors of degree 7 each of which is adjacent to no vertex of degree at most 5 other than v, w by Lemma 5.4, thus $ch_1(v) > 6$.

Hence, we only have to prove the cases that v has exactly one neighbor of degree 5 and four neighbors of degree 7, or that all neighbors of v have degree at least 6.

We now consider the case that v has one neighbor w of degree 5 and four neighbors of degree 7. We assume that all faces containing v but not the edge vw are 3-faces. Let x_1, x_2, x_3, x_4 be the major vertices adjacent to v such that 3-faces (v, x_i, x_{i+1}) exist for $1 \leq i \leq 3$. Since $ch_1(v) < 6$, there is one neighbor of v of degree 7 which is adjacent to a vertex of degree 3 or to two T_4-neighbors.

If one neighbor x_i of degree 7 is adjacent to a vertex y of degree 3, then there is one vertex x_j of degree 7 adjacent to v such that x_j is not adjacent to y but a neighbor x_k of x_j is adjacent to v and y. Hence, x_j has no neighbor of degree at most 5 other than v by Lemma 5.3, thus $ch_1(v) \geq 6$.

Now we consider the case that a vertex adjacent to v has two T_4-neighbors y_1, y_2. If one of y_1, y_2, say y_1, is adjacent to all vertices $x_1, ..., x_4$, then G contains 3-faces $(x_2, x_1, y_1), (x_2, y_1, x_3), (x_2, x_3, v), (x_2, v, x_1)$, a contradiction to $d_G(x_2) = 7$.

We first consider the case that one of the vertices x_1, x_4, say x_1, is adjacent to y_1 and y_2. Then x_2 has no neighbors of degree at most 5 other than v, y_1, y_2 by Lemma 5.3.

If x_2 is adjacent to both, y_1 and y_2, then x_3 has no neighbors of degree at most 5 other than v, y_1, y_2 by Lemma 5.3, but x_3 can not be adjacent to y_1 and y_2, since otherwise the graph induced by $x_1, x_2, x_3, v, y_1, y_2$ contains $K_{3,3}$. From $ch_1(v) < 6$ it follows that x_3 is adjacent to one of y_1, y_2, say to y_1. Then x_3 sends $1/2$ to v, since the condition of Rule 10 does not hold, because y_1 is a T_4-vertex and therefore y_1 has four neighbors of degree 7. If x_4 has at most one neighbor of degree at most 5, then x_4 also sends at least $1/2$ to v and we obtain $ch_1(v) \geq 6$. If x_4 has two neighbors of degree at most 5, then y_1 must be one of them by Lemma 5.3, thus y_1 is adjacent to each of $x_1, ..., x_4$. Hence, x_2 is not adjacent to y_1 and y_2.

If x_2 is adjacent to exactly one of y_1, y_2, say to y_1, then x_2 sends $1/2$ to v. Hence, x_3 must be adjacent to at least two vertices of degree at most 5 other than v. By Lemma 5.3, x_3 is not adjacent to three vertices of degree 5 other than v. Further, x_3 is not adjacent to exactly two vertices of degree 5 other than v, since otherwise x_3 sends at least $1/4$ to v. In this case, x_4 can not send any amount to v which implies that x_4 has two T_4-neighbors, but that is impossible by Lemma 5.3.

Hence, x_3 is adjacent to one vertex of degree 4 and to one more vertex of degree at most 5 other than v, and by Lemma 5.3, x_3 is adjacent to y_1. If x_4 is adjacent to at most one vertex of degree at most 5 other than v, then each of x_2, x_4 sends at least $1/2$ to v, else y_1 is adjacent to each of $x_1, ..., x_4$.

If x_2 is adjacent to none of y_1, y_2, then x_2 sends 1 to v, a contradiction. Hence, none of x_1, x_4 has two T_4-neighbors.

Now let one of x_2, x_3, say x_2, be adjacent to y_1 and y_2. Then x_1 and x_3 have no neighbors of degree at most 5 other than v, y_1, y_2. Each of x_1, x_3 must be adjacent to one of y_1, y_2 and one of x_1, x_3 must be adjacent to both, y_1 and y_2, because otherwise $ch_1(v) \geq 6$.

Since we have already considered the case that x_1 is adjacent to x_1 and x_3, we can assume that x_3 is adjacent to y_1 and y_2, and x_1 is adjacent to one of y_1, y_2. In this case x_1 sends $1/2$ to v, and x_4 also sends at least $1/2$ to v (if x_4 has at most one neighbor of degree at most 5 other than v), or x_4 is adjacent to y_1 and y_2, but then the graph induced by $x_2, x_3, x_4, v, y_1, y_2$ contains $K_{3,3}$.

The only remaining case is that v has no neighbors of degree at most 5. We assume that all faces containing v are 3-faces, and we have to consider four cases for the number of major vertices adjacent to v.

Case 1: $N_7(v) = 2$.
Let w_1, w_2 be the major vertices adjacent to v. Since w_1 is adjacent to one vertex of degree 6 and to v, the vertex w_1 has no neighbor of degree at most 3, and w_1 has either one neighbor of degree 4 and no other neighbor of degree at most 5 other than v, or w_1 is adjacent to at most two vertices of degree 5 other than v by VAL. In both cases w_1 sends at least $1/2$ to v by the discharging rules. The same holds for w_2, thus $ch_1(v) \geq 6$.

Case 2: $N_7(v) = 3$.

Let w_1, w_2, w_3 be the major vertices and x_1, x_2 the vertices of degree 6 adjacent to v. If there is one vertex $w_i \in \{w_1, w_2, w_3\}$ such that 3-faces (v, w_i, x_1) and (v, w_i, x_2) exist, then we assume that $i = 1$. By VAL, w_1 has no neighbor of degree at most 4 and at most one neighbor of degree 5 other than v. Hence, w_1 sends at least $1/2$ to v. Again by VAL, w_2 has either exactly one neighbor of degree 4 and no other neighbor of degree at most 5 other than v, or w_2 has no neighbor of degree at most 4 and at most two neighbors of degree 5. In both cases w_2 sends at least $1/4$ to v. The same holds for w_3, thus $ch_1(v) \geq 6$.

Now we assume that there are 3-faces (v, w_1, w_2) and (v, w_2, w_3). Then w_1 and w_3 are adjacent to a vertex of degree 6. We can assume that one of w_1, w_3, say w_1, sends at most $1/3$ to v, since otherwise $ch_1(v) \geq 6$. By VAL, w_1 has either exactly one neighbor of degree 4 and no other neighbor of degree at most 5 other than v, or w_2 has no neighbor of degree at most 4 and at most two neighbors of degree 5.

If w_1 is adjacent to a vertex x of degree 4, then x must be adjacent to w_2 and to w_3, since otherwise the conditions of Rule 10 do not hold, and w_1 sends $1/2$ to v. By Lemma 5.5, w_2 has no neighbor of degree at most 5 other than v, x, thus each of w_1, w_2, w_3 sends $1/3$ to v and $ch_1(v) = 6$.

If w_1 is adjacent to no vertex of degree at most 4 and to at most one vertex of degree 5 other than v, then w_1 sends at least $1/2$ to v, a contradiction.

If w_1 has two neighbors x_1, x_2 of degree 5, then w_2 has no neighbor of degree at most 5 other than v, x_1, x_2 by Lemma 5.5. If w_2 is adjacent to at most one of x_1, x_2, then w_2 sends at least $1/2$ and each of w_1, w_3 sends at least $1/4$ to v, thus $ch_1(v) \geq 6$. Let w_2 be adjacent to x_1 and x_2. Then w_3 can not have a neighbor of degree 4 by Lemma 5.5. If w_3 has at most one neighbor of degree at most 5 other than v, then w_3 sends at least $1/2$ and each of w_1, w_2 sends at least $1/4$ to v, thus $ch_1(v) \geq 6$. On the other hand, w_3 can not have two neighbors of degree 5 other than v, since otherwise x_1, x_2 must be these neighbors by Lemma 5.5, thus G contains a subdivision of K_5 with main vertices w_1, w_2, w_3, x_1, x_2.

Case 3: $N_7(v) = 4$.

Let $w_1, ..., w_4$ be the major vertices adjacent to v such that for $1 \leq i \leq 3$, there are 3-faces (v, w_i, w_{i+1}). Then w_1 and w_4 are adjacent to a vertex of degree 6. We can assume that one of w_1, w_4, say w_1, sends at most $1/3$ to v, since otherwise $ch_1(v) \geq 6$. By VAL, w_1 has either exactly one neighbor of degree 4 and no other neighbor of degree at most 5 other than v, or w_1 has no neighbor of degree at most 4 and at most two neighbors of degree 5.

If w_1 is adjacent to a vertex x of degree 4, then x must be adjacent to two vertices of w_2, w_3, w_4, since otherwise the conditions of Rule 10 do not hold and w_1 sends $1/2$ to v. If w_4 is one of these vertices, then none of $w_1, ..., w_4$ can have a neighbor of degree at most 5 other than v, x by Lemma 5.5, thus $ch_1(v) > 6$. If x is adjacent to w_2, w_3 but not to w_4, then w_4 is adjacent to no vertex of degree at most 4 and to at most one vertex of degree 5 other than v by Lemma 5.5, thus w_4 sends at lest $1/2$ and each of w_1, w_2 sends at least $1/3$ to v. Hence, w_1 and w_4 have no neighbor of degree 4.

If w_1 is adjacent to exactly one vertex of degree 5, then w_1 sends at least 1/2 to v, a contradiction. Hence, we can assume that w_1 is adjacent to two vertices x_1, x_2 of degree 5 other than v. By Lemma 5.5, w_2 has no neighbors of degree at most 5 other than v, x_1, x_2, thus each of w_1, w_2 sends at least 1/4 to v. Therefore, w_4 must send at most 1/3 to v which implies that w_4 must have two neighbors y_1, y_2 of degree 5 other than v. By Lemma 5.5, w_3 can not have a neighbor of degree at most 5 other than v, y_1, y_2, thus each of w_3, w_4 sends at least 1/4 to v and $ch_1(v) \geq 6$.

Case 4: $N_7(v) = 5$.
Let $w_1, ..., w_5$ be the neighbors of v such that for $1 \leq i \leq 4$, there are 3-faces (v, w_i, w_{i+1}). Since $ch_1(v) < 6$, there must be one neighbor of v, say w_1, such that w_1 does not send any amount to v. Hence, w_1 is adjacent to a vertex of degree 3 or w_1 has two T_4-neighbors.

If w_1 is adjacent to a vertex x of degree 3, then there is a vertex $w_i \in \{w_2, ..., w_5\}$ such that w_i is not adjacent to x, but a neighbor of w_i is adjacent to x. Hence, w_i has no neighbor of degree at most 5 other than v by Lemma 5.3, thus w_i sends 1 to v.

Now we consider the case that w_1 is adjacent to two T_4-vertices x_1, x_2. Then Lemma 5.3 implies that w_2 and w_5 have no neighbors of degree at most 5 other than v, x_1, x_2. If one of w_2, w_5 is adjacent to none of x_1, x_2, then this vertex sends 1 to v, thus $ch_1(v) \geq 6$. If both, w_2 and w_5 are adjacent to exactly one of x_1, x_2, then each of w_2, w_5 sends at least 1/2 to v, thus $ch_1(v) \geq 6$. Hence, we can assume that w_2 is adjacent to x_1 and x_2. Now w_5 can not be adjacent to x_1 and x_2, since otherwise the graph induced by $v, x_1, x_2, w_1, w_2, w_5$ contains $K_{3,3}$. By Lemma 5.3, w_3 has no neighbor of degree at most 5 other than v, x_1, x_2, but w_3 can not be adjacent to x_1 and x_2, since otherwise the graph induced by $v, x_1, x_2, w_1, w_2, w_3$ contains $K_{3,3}$. Hence, each of w_3, w_5 sends at least 1/2 to v, thus $ch_1(v) \geq 6$, and the proof of Claim 4 is complete. □

Altogether we have $ch_1(v) \geq 6$ for each vertex v of degree 6 or 7, and by Claims 1-4, either $ch_1(w) \geq 6$, or w has at least $\lceil 6 - ch_1(w) \rceil$ bigfaces for each vertex w of degree 3, 4, or 5. By Lemma 5.8, every vertex v which is adjacent to a vertex of degree 2 does not send any amount to an other vertex, thus $ch_1(v) = 7$.

We construct a planar multigraph without 2-faces from G by inserting additional edges. For each face F which is a bigface of at least one vertex, we assume that $F = (v_F^1, v_F^2, ..., v_F^{l(F)})$ where $l(F)$ is the length of F. If F is a bigface of exactly one vertex v, then we assume that $v = v_F^1$ and we insert an edge $v_F^1 v_F^3$. By VAL, $d_G(v_F^3) \geq 3$. If F is a bigface of each of $v_F^{i_1}, ..., v_F^{i_m}$ with $m \geq 2$ and $i_1 < i_2 < ... < i_m$, then we insert edges $v_F^{i_k} v_F^{i_{k+1}}$ for $1 \leq k \leq m-1$. The obtained planar multigraph is denoted by G_1. Although G_1 may contain multiple edges, it can not contain faces of length 2 by the definition of a bigface. For each vertex v of G we define $n(v)$ to be the number of edges incident with v which are contained in $E(G_1)$ but not in $E(G)$. Further, we define new charges $ch_2(v) = ch_1(v) + n(v)$ for every vertex v of G. It follows that $\sum_{v \in V(G)} ch_2(v) = \sum_{v \in V(G)} d_{G_1}(v)$ and $ch_2(v) \geq 6$ for every vertex v with $d_G(v) \geq 3$. Further, we have $ch_2(v) \geq 7$ for every vertex v adjacent to a vertex of degree 2. We define G_2 to be the planar multigraph obtained from G_1 by removing all vertices of degree 2, and we define new charges $ch_3(v) = ch_2(v) - 1$, if

v is in G adjacent to a vertex of degree 2, and $ch_3(v) = ch_2(v)$ else. Then G_2 is a planar multigraph, and G_2 contains no 2-faces, because G is a simple graph, and G_1 is obtained from G by inserting edges each of which is incident with at least one minor vertex of G. Hence, for every vertex v of degree 2 of G_1, the neighbors of v are joined by at most one edge. By the construction of G_2 and ch_3, we have $\sum_{v \in V(G_2)} ch_3(v) = \sum_{v \in V(G_2)} d_{G_2}(v)$ and $ch_3(v) \geq 6$ for every vertex $v \in V(G_2)$. Hence, G_2 is a planar multigraph without 2-faces and with average degree at least 6, but such a multigraph can not exist by Euler's formula, thus there is no 7-critical planar graph.

□

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