Hall Polynomials for Dynkin Quivers

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CHAPTER 1

Preliminaries

1. Introduction and Main Results

A *quiver* $\vec{Q}$ is a tuple (Q_0, Q_1) where Q_0 is a set of vertices and Q_1 is a set of arrows.

By *finite* we always mean to be of finite cardinality. Let k be a finite field of cardinality q.

Let $\mathcal{A}$ be the path algebra $k\vec{Q}$ of a quiver $\vec{Q}$ with the underlying graph Q of *Dynkin type*, i.e. $Q \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$. (Thus $\mathcal{A}$ is hereditary, connected and representation finite). Such algebras are called *Dynkin algebras*.

All *modules* are finite left $\mathcal{A}$-modules, except where otherwise specified.

Let $[V]$ be the isomorphism class of a module V and let V^t be the direct sum of t copies of V.

The Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is defined as the free abelian group with basis the class F of all modules modulo the ideal generated by the class of short exact sequences. Then $K_0(\mathcal{A})$ is a free abelian group with basis the images of the simple modules. In this way we can identify $K_0(\mathcal{A})$ with $\mathbb{Z}^{|Q_0|}$. We will write $\underline{\dim}$ for the canonical map $F \to \mathbb{Z}^{|Q_0|}$ and the image $\underline{\dim}V$ of a module V is called its dimension vector.

For a Dynkin diagram Q there is the corresponding semisimple complex Lie algebra $\mathbf{g}$. Let Φ^+ be the set of positive roots of $\mathbf{g}$.

According to [**Ga**] $\underline{\dim}$ is a bijection of the set of the isomorphism classes of the indecomposable modules onto the set of the positive roots of $\mathbf{g}$.

For every $\varrho \in \Phi^+$ choose an indecomposable module $X_\varrho(k)$ with $\underline{\dim}X_\varrho(k) = \varrho$.

The theorem of Krull, Remak and Schmidt shows that we can identify the maps $\alpha : \Phi^+ \to \mathbb{N}_0$ with the isomorphism classes of modules via

$$\alpha \mapsto [V_\alpha], \text{ where } V_\alpha := V_\alpha(k) := \bigoplus_{\varrho\in\Phi^+} X_\varrho(k)^{\alpha(\varrho))}.$$

Let us denote the free abelian group generated by the set of the isomorphism classes of finite $\mathcal{A}$-modules by $\mathcal{H}(\mathcal{A})$.

We can define a multiplication $\diamond : \mathcal{H}(\mathcal{A}) \times \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ as follows:

Let $\alpha, \beta, \gamma : \Phi^+ \to \mathbb{N}_0$. Then

$$[V_\gamma] \diamond [V_\alpha] = \sum_{[V_\beta]} \langle V_\gamma V_\alpha \phi\, V_\beta \rangle [V_\beta],$$

where $\langle V_\gamma V_\alpha \phi\, V_\beta\rangle$ is the number of submodules L of V_β such that $L \cong V_\alpha$ and $V/L \cong V_\gamma$.

The analysis of this multiplication shows that $\mathcal{H}(\mathcal{A})$ becomes an associative algebra with 1 (cf [**R2**]), the so-called *integral Hall algebra* or for short *Hall algebra*.

C. M. Ringel showed in [**R2**] that there exists a polynomial $\phi^\beta_{\gamma,\alpha}$ such that $\phi^\beta_{\gamma,\alpha}(q) = \langle V_\gamma(k) V_\alpha(k) \phi\, V_\beta(k)\rangle$ (recall that $q = |k|$). These polynomials are called *Hall polynomials*. By abuse of language it will be convenient to write also $\phi^{V_\beta}_{V_\gamma V_\alpha}$ for $\phi^\beta_{\gamma,\alpha}$.

Since the nature of Hall polynomials is to count something, we are confronted with the question whether the coefficients of the Hall polynomials are also counting something. In particular, whether they are at least non-negative.

But already for $Q = \mathbb{A}_3$ the Hall polynomial $q-1$ occurs as a Hall polynomial and for $Q = \mathbb{D}_4$ there is the polynomial $q-2$.

We describe an algorithm for the calculation of Hall polynomials, which shows that the expansion of the Hall polynomials at specific points has non-negative coefficients.

THEOREM 1.1. *Let $Q = \mathbb{A}_n$ and let U, V and W be modules. Then for $\phi^V_{WU}(q) = \sum_{r\geq 0} c_r(q-1)^r$ all coefficients c_r are non-negative.*

THEOREM 1.2. *Let $Q = \mathbb{D}_n$ and let U, V and W be modules. Then for $\phi^V_{WU}(q) = \sum_{r\geq 0} c_r(q-2)^r$ all coefficients c_r are non-negative.*

A surprising application of Hall polynomials is the following:

Suppose X and Y are indecomposable modules. Let $\mathcal{A}be(X,Y)$ be the abelian extension-closed exact subcategory generated by X and Y. Then $\mathcal{A}be(X,Y)$ is equivalent to the module category of the path algebra of a Dynkin quiver.

Then the natural question arises by which properties of X and Y the category $\mathcal{A}be(X,Y)$ can be classified.

THEOREM 1.3. *Let X and Y be indecomposable modules with $\operatorname{Hom}_{\mathcal{A}}(Y,X) = 0$ and $\operatorname{Ext}^1_{\mathcal{A}}(X,Y) = 0$.*

1. *If $\dim_k \operatorname{Hom}_{\mathcal{A}}(X,Y) + \dim_k \operatorname{Ext}^1_{\mathcal{A}}(Y,X) = 0$, then $\mathcal{A}be(X,Y) \approx k\vec{Q}\text{-}\mathrm{mod}$, where $\vec{Q}$ is of type $\mathbb{A}_1 \times \mathbb{A}_1$.*
2. *If $\dim_k \operatorname{Hom}_{\mathcal{A}}(X,Y) + \dim_k \operatorname{Ext}^1_{\mathcal{A}}(Y,X) = 1$, then $\mathcal{A}be(X,Y) \approx k\vec{Q}\text{-}\mathrm{mod}$, where $\vec{Q}$ is of type $\mathbb{A}_2$.*
3. *If $\dim_k \operatorname{Hom}_{\mathcal{A}}(X,Y) = \dim_k \operatorname{Ext}^1_{\mathcal{A}}(Y,X) = 1$, then $\mathcal{A}be(X,Y) \approx k\vec{Q}\text{-}\mathrm{mod}$, where $\vec{Q}$ is of type $\mathbb{A}_3$.*
4. *If $\dim_k \operatorname{Hom}_{\mathcal{A}}(X,Y) = 1$ and $\dim_k \operatorname{Ext}^1_{\mathcal{A}}(Y,X) = 2$, then $\mathcal{A}be(X,Y) \approx k\vec{Q}\text{-}\mathrm{mod}$, where $\vec{Q}$ is of type $\mathbb{D}_4$.*

It seems to be true that the type of $\mathcal{A}be(X,Y)$ only depends on the sum of the dimensions of $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ and $\operatorname{Ext}^1_{\mathcal{A}}(Y,X)$.

We conjecture that this theorem is also true for infinite fields, but the presented proof requires the knowledge of the Hall polynomials for Dynkin quivers.

Let us now describe the structure of this thesis.

In section 2 of this chapter we will give an brief outline of the historical background of Hall polynomials.

Section 3 of this chapter will introduce the main definitions and notation.

The Hall polynomials $\phi^{V(\alpha)}_{X_\varrho(k)^r X_{\varrho'}(k)^s}$ for an arbitrary module $V(\alpha)$, if the underlying graph of $\vec{Q}$ is of type $\mathbb{A}_n$ and $r, s \in \mathbb{N}_0$, will be calculated in chapter 2.

Then we will give an explicit formula of the polynomials $\phi^{V(\alpha)}_{X_\varrho(k) X_{\varrho'}(k)}$, if $\vec{Q}$ is of type $\mathbb{D}_n$. In addition, theorem 1.3 will be proved in chapter 2.

Chapter 3 is concerned with the algorithm for the calculation of Hall polynomials for arbitrary modules. To apply the algorithm we need to know the Hall polynomials of chapter 2. Also the theorems 1.1 and 1.2 will be proved there.

Finally we will describe the implementation of a computer program to calculate Hall polynomials in the case $\vec{Q}$ is of type $\mathbb{A}_n$ which is available via the Internet[1].

[1] `http://www.mathematik.uni-bielefeld.de/birep/hall`

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2. Historical Overview and Background

Already in 1901 E. Steinitz discussed in [**St**] filtrations of finite abelian p-groups, where p is a prime number. We will call this the *classical case* in contrast to the *Dynkin case* discussed in this thesis. Let us recall the main ideas and definitions.

Let G be a finite abelian p-group. It is well known that G is the direct sum of cyclic groups with orders $(p^{\vartheta_r})_r$. By ordering $\vartheta = (\vartheta_r)_r$ we obtain a partition which is called the *type* of G.

Given two other types ι and κ one may ask how many subgroups $H \subseteq G$ of type ι appear in G such that G/H is of type κ. This number is denoted by $g^{\vartheta}_{\kappa\iota}(p)$. (We distinguish between the classical case and the Dynkin case by using the symbols g and ϕ for the Hall polynomials, respectively.)

E. Steinitz discovered that $g^{\vartheta}_{\kappa\iota}(p)$ can be used to define a product of finite abelian p-groups. He conjectured that this number is given by a polynomial in p and mentioned the following "strange" relationship.

For partitions of finite abelian p-groups he defined the corresponding simple symmetric functions which are nowadays called *Schur functions* (cf [**Mc**]). E. Steinitz realized that the structure constants for the multiplication of Schur functions are given as the leading coefficients of the polynomials $g^{\vartheta}_{\kappa\iota}(p)$. In this way he proved that the structure constants of the multiplication of the Schur functions are always non-negative since otherwise the polynomials would be negative evaluated for large prime numbers p which contradicts their nature as being a number of filtrations.

In 1957 P. Hall stated in [**Ha**] that the number of filtrations of finite abelian p-groups is in fact given by polynomials, the *Hall polynomials*. He defines a scalar product on the space of symmetric functions such that the family of Schur function is a self dual basis.

P. Hall showed also that a Hall polynomial vanishes if and only if the corresponding multiplication constant for the Schur function is zero and additionally he showed that the leading coefficient of the Hall polynomials is given by the scalar product of the corresponding Schur functions.

Furthermore P. Hall remarked that for a partition there is a corresponding symmetric function and the structure constants of the multiplication of these functions are given by the evaluation of the corresponding Hall polynomial at 1.

Then a wide exploration of Hall polynomials and relationships to other parts of mathematics started. We will not try to give a complete list of all papers but let us single out some important results of the last decade.

In [**R2**] C. M. Ringel managed to transfer P. Hall's and E. Steinitz' approach to the representation theory of *finitary* algebras, i.e. algebras where the extension groups of two simple modules have finite cardinality (cf [**R1**]).

Let V be a module with dimension vector $\underline{\dim} V = (v_i)_{i \in Q_0}$. Note that v_i is the Jordan-Hölder multiplicity $[V : S_i]$ of the simple module S_i in the module V (cf [**R2**]).

In our case, where $\mathcal{A}$ is hereditary, the *Euler quadratic form*

$$\chi(\underline{\dim} V) = \dim \operatorname{Hom}_{\mathcal{A}}(V, V) - \dim \operatorname{Ext}^1_{\mathcal{A}}(V, V)$$

coincides with the *Tits form*

$$\mathbf{q}((v_i)_i) = \sum_{i\in Q_0} v_i^2 - \sum_{i\to j\in Q_1} v_i v_j$$

of Q (cf [**Bo**]). Obviously $\mathbf{q}$ is independent of the orientation of $\vec{Q}$.

A *positive root* of $\mathbf{q}$ is a vector $(v_i)_i$ with $\mathbf{q}((v_i)_i) = 1$, where $v_i \geq 0$ for all $i \in Q_0$.

As mentioned above, P. Gabriel has proved already in 1972 that $\underline{\dim}$ yields a bijection between the set of isomorphism classes of indecomposable $\mathcal{A}$-modules and the set Φ^+ of positive roots for Q, if the base field is algebraically closed (cf [**Ga**]). This result was generalized by V. Dlab and C. M. Ringel (cf [**DR**]).

By evaluating the Hall polynomials in the Dynkin case at $q = 1$ we obtain the algebra $\mathcal{H}(\mathcal{A})_1$, with the structure constants $\phi^V_{WU}(1)$. It turns out that $\mathcal{H}(\mathcal{A})_1 \otimes \mathbb{Q}$ is the universal enveloping algebra of $K(\mathcal{A}\text{-}\operatorname{mod}) \otimes \mathbb{Q}$, where $K(\mathcal{A}\text{-}\operatorname{mod})$ is the free abelian group generated by the set of isomorphism classes of indecomposable finite $\mathcal{A}$-modules. Actually, $K(\mathcal{A}\text{-}\operatorname{mod})$ is a Lie subalgebra of $\mathcal{H}(\mathcal{A})_1$ and $K(\mathcal{A}\text{-}\operatorname{mod}) \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbf{n}_+$ are isomorphic as Lie algebras, where $\mathbf{n}_+$ is defined by a triangular decomposition $\mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ of $\mathbf{g}$ with a Cartan subalgebra $\mathbf{h}$.

So the Lie structure of $\mathbf{n}_+$ can be expressed by evaluating the Hall polynomials at 1 but for explicit calculations we need the Hall polynomials for indecomposable modules (the basis elements of $K(\mathcal{A}\text{-}\operatorname{mod})$).

We are supplied with these polynomials by C. M. Ringel's article [**R4**]. With these polynomials it can be shown that there is an isomorphism of $K(\mathcal{A}\text{-}\operatorname{mod}) \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbf{n}_+$ (cf [**R4**]).

In 1993 there was the following result of Butler and Hales dealing with the classical case (cf [**BH**]):

Let G be an abelian p-group of type $(\vartheta_1, \ldots, \vartheta_t)$. The coefficients of $g^{\vartheta}_{\kappa\iota}(p)$ are non-negative for all κ, ι if and only if $\vartheta_1 - \vartheta_t \leq 1$

Inspired by this fact, F. Miller Maley proved in 1996 that $g^{\vartheta}_{\kappa\iota}(p)$ is a positive integral linear combination of powers of $p - 1$ (cf [**Mi**]). This is very similar to the theorem 1.1 and 1.2 but we can not obtain an analogous result in the Dynkin case since already for $Q = \mathbb{D}_4$ the Hall polynomial $q - 2$ occurs

Recently C. M. Ringel has calculated (cf his private notes) the Hall polynomials ϕ^V_{WU} for indecomposable modules U and W in the cases $\mathbb{E}_6$ and $\mathbb{E}_7$. R. Nörenberg did this for $\mathbb{E}_8$ with a computer program.

At long last we refer to a result of P. Hall: If the leading coefficient of $g^{\vartheta}_{\kappa\iota}(p)$ is non-zero, then the degree of $g^{\vartheta}_{\kappa\iota}(p)$ is $n(\vartheta) - n(\kappa) - n(\iota)$, where $n(\vartheta) = \sum_r (r-1)\vartheta_r$ (cf [**Ha**]). This is similar to the degree formula in chapter 3.

3. Main Definitions and Notation

By $\mathcal{A}\text{-}\operatorname{mod}$ we denote the category of finitely generated modules.

For modules V and W the k-vector space of $\mathcal{A}$-linear maps (or for short *homomorphisms*) $f : V \to W$ is denoted by $\operatorname{Hom}_{\mathcal{A}}(V, W)$ or, if no misunderstanding is possible, $\operatorname{Hom}(V, W)$ and its k-dimension by $[V, W]$.

Henceforth we call the set of non-invertible homomorphisms of $\operatorname{Hom}(V, W)$ simply the *radical* and write $\operatorname{rad}(V, W)$ (cf [**R3**]).

Dimension will always mean k-dimension and let $\mathbb{N}_0$ be the set of natural numbers including zero.

Two short exact sequences

$$E: \quad 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

and

$$E' \quad 0 \to U \xrightarrow{f'} V' \xrightarrow{g'} W \to 0$$

are called *equivalent*, if there exists a map $h : V \to V'$ with $hf = f'$ and $g'h = g$. It follows by the five-lemma that h is an isomorphism. For the equivalence class of a short exact sequences E we will write $[E]$.

By writing $\mathrm{Ext}(W, U)$ for modules U and W we always mean $\mathrm{Ext}^1_{\mathcal{A}}(W, U)$, the first homology group, and we identify this with the set of short exact sequences starting at U and ending in W modulo the defined equivalence relation. The dimension of $\mathrm{Ext}(W, U)$ will be abbreviated by $[W, U]^1$.

Let $D = \mathrm{Hom}_k(-, k)$ be the *duality functor* from $\mathcal{A}$-mod to $\mathcal{A}^{\mathrm{op}}$-mod. This is an equivalence of categories.

We emphasize that the dimension of a homomorphism space is easily obtained. We only need to count paths in the mesh-category (cf [**R3**]). The calculation of the dimension of an extension space $\mathrm{Ext}(U, W)$ can be reduced to the calculation of the dimension of homomorphism spaces via the Auslander-Reiten formula

$$\mathrm{Ext}(U, W) \cong D\,\mathrm{Hom}(W, \tau U),$$

where τ is the Auslander-Reiten translation (cf [**R3**] and note that $\mathcal{A}$ is hereditary).

An algebra $\mathcal{B}$ is called *representation directed*, if there is a linear ordering of the indecomposable $\mathcal{B}$-modules $(X_n)_n$ such that $\mathrm{Hom}_{\mathcal{B}}(X_n, X_m) = 0$ for $n > m$. Equivalently one can say that every indecomposable $\mathcal{B}$-module X is *directing*, i.e. X does not belong to a cycle in $\mathcal{B}$-mod. It is well known that a finite dimensional representation directed algebra is *representation finite*, i.e. there are only finitely many isomorphism classes of indecomposable $\mathcal{B}$-modules (cf [**R3**]).

Note that $\mathcal{A}$ (the path algebra of a Dynkin quiver) is representation directed and that the isomorphism class of a directing module X is uniquely determined by the dimension vector of X (cf [**R3**]).

Let $([X_n])_n$ be the family of the isomorphism classes of the indecomposable modules. Recall that the set $\{\underline{\dim} X_n\}$ is identical to Φ^+, the set of the positive roots of $\mathbf{q}$.

For a module V we denote by $[V]^{\diamond t}$ the t-fold diamond product of $[V]$ with itself.

In the following we will not distinguish any more between modules and isomorphism classes of modules. This will lead to no difficulties, as the meaning is obvious in any case.

Let us first state a reformulation of how to calculate Hall polynomials.

Definition 1.4. *Let*

$$E: \quad 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

and

$$E': \quad 0 \to U \xrightarrow{f'} V \xrightarrow{g'} W \to 0$$

be short exact sequences in $\mathcal{A}$-mod.

E and E' are called middle-term equivalent, *if there is an isomorphism $h : U \to U$ such that the diagram*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "h"'] & V \arrow[d, equal] \\ U \arrow[r, "f'"] & V \end{tikzcd}$$

is commutative. For the middle-term-equivalence class of E we will write $\langle E \rangle$.

Obviously middle-term-equivalence could have been also defined on the right-hand side of a short exact sequence which gives us an equivalent definition.

Whenever we talk about *counting short exact sequences* we mean that we count short exact sequences with fixed end and middle terms up to middle-term equivalence.

The next lemma will state the relationship between middle-term-equivalence and hall polynomials.

LEMMA 1.5. *Let U, V and W be $\mathcal{A}$-modules. The number of short exact sequences*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

up to middle-term equivalence is $\phi^V_{WU}(q)$.

PROOF. We will give a bijective map σ between the middle-term-equivalence classes of short exact sequences with terms U, V and W and the elements of the set $S := \{L \subseteq V \,|\, L \cong U \text{ and } V/L \cong W\}$. Let E be a short exact sequence $0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$. Let σ be the map which sends $\langle E\rangle$ to $\operatorname{im} f$. It is easy to check that σ is well defined.

If L is in S, we have the short exact sequence $0 \longrightarrow L \longrightarrow V \longrightarrow V/L \longrightarrow 0$, where $L \cong U$ and $V/L \cong W$. Thus σ is surjective.

Let $E : 0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$ and $E' : 0 \longrightarrow U \xrightarrow{f'} V \xrightarrow{g'} W \longrightarrow 0$ be short exact sequences with $\sigma(\langle E\rangle) = \sigma(\langle E'\rangle)$. Then we get $\operatorname{im} f = \operatorname{im} f'$ and on $\operatorname{im} f'$ we can invert f'. Via $h := f'^{-1} f$ we get that E and E' are middle-term equivalent. Therefore σ is also injective. □

Let us pay attention to the article [**Ri**] of C. Riedtmann, where a very similar statement is proved, if the base field is the field of complex numbers.

By 1_V we denote the identity map of a module V.

LEMMA 1.6. *Let*

$$E : 0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

and

$$aE : 0 \longrightarrow U \xrightarrow{\frac{1}{a} f} V \xrightarrow{g} W \longrightarrow 0.$$

be short exact sequences in $\mathcal{A}$-mod. For $a \in k \setminus \{0\}$ it holds $a[E] = [aE]$ in $\operatorname{Ext}(W, U)$.

PROOF. By definition $a[E]$ is given by the Pullback along $a1_W : W \to W$. So we get the following commutative diagram

$$\begin{tikzcd}
aE: & 0 \arrow[r] & U \arrow[r, "\frac{1}{a} f"] \arrow[d, equal] & V \arrow[r, "g"] \arrow[d, "h"] & W \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & U \arrow[r] \arrow[d, equal] & \{(v,w) \,|\, g(v) = aw\} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "a1_W"] & 0 \\
E: & 0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W \arrow[r] & 0,
\end{tikzcd}$$

where $h(w) = (aw, g(w))$. The middle row is by definition a representative of $a[E]$ and the top row is an equivalent short exact sequence, since h is an isomorphism. □

LEMMA 1.7. *Let*

$$E : 0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

be a short exact sequence. Then for $a, b \in k \setminus \{0\}$

$$E' : 0 \longrightarrow U \xrightarrow{af} V \xrightarrow{bg} W \longrightarrow 0$$

is also a short exact sequence and E and E' are middle-term equivalent. Additionally, if $b = \frac{1}{a}$, the sequences E and E' are equivalent.

PROOF. Obviously E' is a short exact sequence and E and E' are middle-term equivalent via $\frac{1}{a}1_U : U \to U, u \mapsto \frac{1}{a}u$. If $b = \frac{1}{a}$, the sequences are equivalent via $a1_V$. $\square$

Two remarks concerning the relationship between middle-term equivalence and equivalence of short exact sequences should be added.

REMARK 1.8. *Equivalence does not imply middle-term equivalence.*

Take for instance the path algebra of the quiver $\vec{\mathbb{A}}_2$. Let X be the indecomposable projective injective module and let Y be the simple injective module. Then every short exact sequence starting in X and ending in Y is split. Thus there is only one equivalence class in $\mathrm{Ext}(Y, X)$. But there are q middle-term equivalence classes (see lemma 2.4).

REMARK 1.9. *Middle-term equivalence does not imply equivalence.*

Take again the path-algebra of the quiver $\vec{\mathbb{A}}_2$. Let X be the simple projective module and let Y be the simple injective module. Then all non-split short exact sequences starting in X and ending in Y are middle-term equivalent but in $\mathrm{Ext}(Y, X)$ there are $q - 1$ equivalence classes of non-split sequences (cf theorem 2.17).

CHAPTER 2

The Dynkin Case

Recall some notions and equations which can be found eg in [**CX**] and [**Mc**].

$$
\begin{aligned}
|s] &= \frac{q^s-1}{q-1} = q^{s-1}+\dots+q+1 \\
|s]! &= \prod_{r=1}^{s} |r] \\
\begin{vmatrix} s \\ r \end{bmatrix} &= \frac{|s]!}{|r]!|s-r]!}
\end{aligned}
$$

First we will calculate $X^n \diamond X^m$. This is like counting n-dimensional subspaces of an $n+m$-dimensional vector space.

Lemma 2.1. *Let X be an indecomposable module. Then*

$$X^n \diamond X^m = \phi^{X^{n+m}}_{X^n X^m} X^{n+m},$$

where

$$\phi^{X^{n+m}}_{X^n X^m} = \prod_{r=0}^{m-1} \frac{q^{m+n-r}-1}{q^{m-r}-1}$$

and the degree of $\phi^{X^{n+m}}_{X^n X^m}$ equals nm.

Proof. We will count short exact sequences with fixed end and middle terms up to middle-term equivalence. Let

$$E: \quad 0 \longrightarrow X^m \xrightarrow{f} X^{n+m} \longrightarrow X^n \longrightarrow 0$$

be a short exact sequence. We use the facts that E is split and that $\mathrm{End}(X) = k$. The number of injective maps $f : X^m \to X^{n+m}$ is $\prod_{r=0}^{m-1}(q^{m+n}-q^r)$ and the number of isomorphisms $h : X^m \to X^m$ is $\prod_{r=0}^{m-1}(q^m-q^r)$. This proves the lemma. $\square$

Lemma 2.2. *If X is an indecomposable module X, then*

$$X^{\diamond n} = \prod_{r=1}^{n} \frac{q^r-1}{q-1} X^n = |n]! X^n$$

and

$$X^n \diamond X^m = \begin{vmatrix} n+m \\ n \end{bmatrix} X^{n+m}.$$

Proof. Cf [**Mc**]. $\square$

1. The case $\mathrm{Ext}(Y, X) = 0$

Recall that two modules $V = \bigoplus X_r$ and $W = \bigoplus Y_s$ with indecomposable modules X_r, Y_s are called *disjoint* if X_r and Y_s are non-isomorphic for all r and s. We quote a lemma from [**R3**].

LEMMA 2.3. *Let U, V and W be modules. If there is a non-split short exact sequence*

$$0 \to U \to V \to W \to 0,$$

then $\dim \operatorname{End}(V) < \dim \operatorname{End}(U \oplus W)$.

We will use this lemma to bound the number of summands of the middle-term of a short exact sequences.

For disjoint modules U, W we can calculate the product $W \diamond U$ if there is no non-trivial extension of W by U.

LEMMA 2.4. *Let W and U be disjoint modules. Then $\phi_{WU}^{W\oplus U} = q^{[U,W]}$.*

PROOF. Again we will count short exact sequences with fixed end and middle terms up to middle-term equivalence.

Let

$$E: \quad 0 \to U \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} W \oplus U \xrightarrow{[f' g']} W \to 0$$

be a short exact sequence. If it is not split, we get $\dim \operatorname{End}(W \oplus U) < \dim \operatorname{End}(W \oplus U)$ with lemma 2.3, which is a contradiction. Therefore E is split and the map $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ is a split monomorphism. Thus there is a homomorphism $[\tilde{f}, \tilde{g}] : W \oplus U \to U$ with $[\tilde{f}, \tilde{g}] \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \tilde{f} f + \tilde{g} g = 1_U$. Of course f is in $\operatorname{rad}(U, W)$, because all direct summands of U are not isomorphic to the direct summands of W. It follows that $\tilde{g} g = 1_U - \tilde{f} f$ is invertible and therefore g is also invertible, since g is an endomorphism of a finite dimensional vector space.

Via g the sequence E is middle-term-equivalent to the sequence

$$E': \quad 0 \to U \xrightarrow{\left[\begin{smallmatrix} fg^{-1} \\ 1_U \end{smallmatrix}\right]} W \oplus U \xrightarrow{[1_W, -fg^{-1}]} W \to 0.$$

Observe that the two short exact sequences

$$E: \quad 0 \to U \xrightarrow{\left[\begin{smallmatrix} f \\ 1_U \end{smallmatrix}\right]} W \oplus U \xrightarrow{[1_W, -f]} W \to 0$$

and

$$E': \quad 0 \to U \xrightarrow{\left[\begin{smallmatrix} f' \\ 1_U \end{smallmatrix}\right]} W \oplus U \xrightarrow{[1_Y, -f']} W \to 0$$

are middle-term-equivalent if and only if $f = f'$.

All together we have proven a one to one correspondence between the maps $f : U \to W$ and the middle-term-equivalence classes of short exact sequences. The analysis of this point shows that the number of such sequences is $q^{[U,W]}$. $\square$

Now we can state some frequently applied corollaries.

COROLLARY 2.5. *Let U and W be disjoint modules. If* $\operatorname{Ext}(W, U) = 0$, *then* $W \diamond U = q^{[U,W]} W \oplus U$.

COROLLARY 2.6. *Let X and Y be indecomposable and non-isomorphic modules. If* $\operatorname{Ext}(Y, X) = 0$, *then* $Y^m \diamond X^n = q^{[X^n, Y^m]} Y^m \oplus X^n$.

PROOF. This follows immediately, since Ext is an additive functor. $\square$

Now we want to look at extension-closed abelian subcategories and recall the definition

DEFINITION 2.7. *Let $\mathcal{M}$ be a set of modules. We use $\mathcal{A}be(\mathcal{M})$ as an abbreviation for the abelian extension-closed exact subcategory generated by $\mathcal{M}$.*

THEOREM 2.8. *Let X and Y be indecomposable modules with $[X,Y] = [Y,X] = [X,Y]^1 = [Y,X]^1 = 0$. Then the categories $\mathcal{A}be(X,Y)$ and $\mathbb{A}_1 \times \mathbb{A}_1$-mod are equivalent.*

PROOF. Since there is no homomorphism between X and Y, no extension of X by Y and vice versa, the generated abelian subcategory consists exactly of sums of copies of X and Y. $\square$

If $[Y, X]^1 = 0$ and $[X, Y] = 1$ there is a "hidden" extension. Therefore we leave this case for the next section. This solves the case $\operatorname{Ext}(Y, X) = 0$.

2. The case $\dim \operatorname{Ext}(Y, X) = 1$

Let X and Y be indecomposable modules. We do not know how to determine the Hall polynomial $\phi^Z_{Y^n, X^m}$ in general if $[Y, X]^1 = 1$. But we will calculate $Y^n \diamond X^m$ if $\langle Y X \diamondsuit Z \rangle$ is known. First we need some properties of short exact sequences and abelian extension-closed sub-categories.

LEMMA 2.9. *Let X, Y and Z_r for $1 \le r \le t$ be indecomposable modules. Let*

$$E: \quad 0 \longrightarrow X \xrightarrow{[f_r]} \bigoplus Z_r \xrightarrow{[g_r]} Y \longrightarrow 0$$

be a non-split short exact sequence. Then f_r and g_s are non-zero for $1 \le r, s \le t$. Additionally, if $t > 1$, f_r's injectivity implies that g_s is injective for $r \ne s$ and $[f_r]_{r \ne s}$'s surjectivity implies that g_s is surjective.

PROOF. Without loss of generality we shall only prove the first claim for $r = s = 1$. If $f_1 = 0$, we obtain $Y \cong \bigoplus Z_r / \operatorname{im}[f_r] \cong Z_1 \oplus \bigoplus_{r>1} Z_r / \operatorname{im}[f_r]$. Since Y is indecomposable, we would get $\bigoplus_{r>1} Z_r = \operatorname{im}[f_r]$, but then E is split, which is a contradiction. The proof for $g_1 \ne 0$ is dual.

For the second claim we show that g_s is injective for all $s \ne 1$ if f_1 is injective. Let $z_s \in Z_s$ with $g_s(z_s) = 0$. Then also $[g_1, \dots, g_t](0, \dots, 0, z_s, 0, \dots, 0)^t = 0$. Thus there is an $x \in X$ with $f_1(x) = 0$ and $f_s(x) = z_s$ and therefore $z_s = 0$.

Let us assume $[f_r]_{r \ge 2}$ as being surjective. Let y be an element of Y. Then $[g_1, \dots, g_t](z_1, \dots, z_t)^t = y$. Since $[f_r]_{r \ge 2}$ is surjective, there exists $x \in X$ with $[f_r]_{r \ge 2}(x) = (z_2, \dots, z_t)^t$. Then

$$g_1(z_1 - f_1(x)) = [g_1, \dots g_t]((z_1 - f_1(x), 0, \dots, 0)^t + (f_1(x), \dots, f_t(x))^t) = y.$$

Hence g_1 is surjective. $\square$

LEMMA 2.10. *Let X, Y and Z_r for $1 \le r \le t$ be indecomposable modules. Let*

$$E: \quad 0 \longrightarrow X \xrightarrow{[f_r]} \bigoplus Z_r \xrightarrow{[g_r]} Y \longrightarrow 0$$

be a non-split short exact sequence. Then $g_r f_r$ do not vanish for $1 \le r \le t$.

PROOF. We will only prove $g_1 f_1 \ne 0$. Suppose $g_1 f_1 = 0$. The universal property of the cokernel would yield a map $h : Y \to Y$ with $h[g_1, \dots, g_t] = [g_1, 0, \dots, 0]$. Since Y is indecomposable, h is a multiple of the identity and a contradiction to the first part of lemma 2.9 arises. $\square$

LEMMA 2.11. *Let X, Y and Z be indecomposable modules. If $[Y, X] = 1$ and*

$$E: \quad 0 \longrightarrow X \xrightarrow{f} Z \xrightarrow{g} Y \longrightarrow 0$$

is a non-split short exact sequence, then for every non-split short exact sequence

$$0 \longrightarrow X \longrightarrow Z' \longrightarrow Y \longrightarrow 0$$

the modules Z and Z' are isomorphic.

PROOF. Since $\operatorname{Ext}(Y, X) = k$, there are q equivalence classes of short exact sequences starting at X and ending in Y. Lemma 1.6 shows that all the non-split short exact sequences have isomorphic middle-terms. Of course, we get a split sequence if we multiply E with 0. $\square$

Lemma 2.12. *Let X, Y and Z be modules and let X and Y be indecomposable. If*

$$E: \quad 0 \to X \to Z \to Y \to 0$$

is a non-split short exact sequence, then

$$[X,Z] = 1 + [X,Y], \tag{1}$$

$$[Y,X]^1 - 1 = [Y,Z]^1 = [Z,X]^1. \tag{2}$$

If furthermore $[Y,X]^1 = 1$, then

$$\mathrm{Ext}(Z,Z) = 0 \tag{3}$$

and

$$[X,Z] = [Z,Z] = [Z,Y]. \tag{4}$$

Proof. First we apply $\mathrm{Hom}(X,-)$ to E and get

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(X,X) \longrightarrow \mathrm{Hom}(X,Z) \longrightarrow \mathrm{Hom}(X,Y) \longrightarrow \\ \longrightarrow \mathrm{Ext}(X,X) \longrightarrow \cdots \end{array}$$

Now we use that $\mathcal{A}$ is representation-directed and that X is indecomposable. By counting dimensions we arrive at the first claim. Notice that this statement is also true if E is split.

For the second assertion we apply $\mathrm{Hom}(-,X)$ to E and get

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(Y,X) \longrightarrow \mathrm{Hom}(Z,X) \longrightarrow \mathrm{Hom}(X,X) \longrightarrow \\ \longrightarrow \mathrm{Ext}(Y,X) \longrightarrow \mathrm{Ext}(Z,X) \longrightarrow \mathrm{Ext}(X,X) \longrightarrow 0 \quad . \end{array}$$

Lemma 2.9 shows that $\mathrm{Hom}(Z,X) = 0$. Since X is indecomposable we have $\mathrm{Ext}(X,X) = 0$ and $\mathrm{Hom}(X,X) = k$. By counting dimensions we get the second assertion. The proof of the remaining equality is dual.

For the third and fourth assertion we apply $\mathrm{Hom}(Z,-)$ to E and get

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(Z,X) \longrightarrow \mathrm{Hom}(Z,Z) \longrightarrow \mathrm{Hom}(Z,Y) \longrightarrow \\ \longrightarrow \mathrm{Ext}(Z,X) \longrightarrow \mathrm{Ext}(Z,Z) \longrightarrow \mathrm{Ext}(Z,Y) \longrightarrow 0 \quad . \end{array}$$

Because of (1) we know that $\mathrm{Ext}(Z,X) = 0$ and $\mathrm{Ext}(Z,Y) = 0$. Thus (3) is true. From lemma 2.9 we first obtain $\mathrm{Hom}(Z,X) = 0$ and then $\mathrm{Hom}(Z,Z) \cong \mathrm{Hom}(Z,Y)$. The proof for $\mathrm{Hom}(Z,Z) \cong \mathrm{Hom}(X,Z)$ is dual. ☐

Lemma 2.13. *Let X and Z be indecomposable modules. If $\mathrm{Hom}(X,Z) = k$ and $\mathrm{Ext}(Z,X) = 0$ and if there is a short exact sequence*

$$E: \quad 0 \to X \to Z \to Z/X \to 0,$$

then Z/X is indecomposable, $\mathrm{Hom}(Z,Z/X) \cong k$ and $\mathrm{Hom}(X,Z/X) = 0$.

Proof. We want to show that Z/X is indecomposable. Therefore we apply $\mathrm{Hom}(Z,-)$ to E and get

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(Z,X) \longrightarrow \mathrm{Hom}(Z,Z) \longrightarrow \mathrm{Hom}(Z,Z/X) \longrightarrow \\ \longrightarrow \mathrm{Ext}(Z,X) \longrightarrow \cdots \quad . \end{array}$$

Since $\operatorname{Ext}(Z, X) = \operatorname{Hom}(Z, X) = 0$ and Z is indecomposable, $\operatorname{Hom}(Z, Z/X) \cong k$. By applying $\operatorname{Hom}(X, -)$ to E we get

$$0 \longrightarrow \operatorname{Hom}(X, X) \longrightarrow \operatorname{Hom}(X, Z) \longrightarrow \operatorname{Hom}(X, Z/X) \longrightarrow \operatorname{Ext}(X, X) \longrightarrow \cdots .$$

Because $\operatorname{Hom}(X, X) = k \cong \operatorname{Hom}(X, Z)$ and $\operatorname{Ext}(X, X) = 0$ we can conclude that $\operatorname{Hom}(X, Z/X) = 0$. Finally we apply $\operatorname{Hom}(-, Z/X)$ and get

$$0 \longrightarrow \operatorname{End}(Z/X) \longrightarrow \operatorname{Hom}(Z, Z/X) \longrightarrow \cdots .$$

It follows that $\operatorname{Hom}(Z/X, Z/X) = k$ and Z/X is indecomposable. ☐

In [**HR**] we find the following lemma which is true for any hereditary algebra.

LEMMA 2.14. *If X and Y are indecomposable with $\operatorname{Ext}(Y, X) = 0$, then any non-zero map $X \to Y$ is an epimorphism or a monomorphism.*

Now we have all the ingredients to prove

THEOREM 2.15. *Let X and Y be indecomposable modules with $\operatorname{Hom}(X, Y) = 0$ and $\operatorname{Ext}(Y, X) \cong k$. Then there is an equivalence of the categories $\mathcal{A}be(X, Y)$ and $k\vec{\mathbb{A}}_2$-mod. The same is true if $\operatorname{Hom}(X, Y) \cong k$ and $\operatorname{Ext}(Y, X) = 0$*

PROOF. Of course, there is only one orientation of $\mathbb{A}_2$. The Auslander-Reiten quiver of $\vec{\mathbb{A}}_2$ is given by

$$\begin{tikzcd} & Z \arrow[dr] & \\ X \arrow[ur] & & Y. \end{tikzcd}$$

The picture does not only serve the purpose of illustration, but it is also meant to define the desired equivalence on the objects.[1] Lemma 2.12 shows that the middle term of a non-split short exact sequence

$$0 \to X \to Z \to Y \to 0$$

is indecomposable. By using lemma 2.12 we can identify all kernels, cokernels and extensions in the first case. Furthermore all injective maps are sent to injective maps, all surjective maps are sent to surjective maps and the functor defined by this is obviously full, dense and faithful.

If $\operatorname{Hom}(X, Y) = k$ and $\operatorname{Ext}(Y, X) = 0$, we may assume that there is an injective map $f : X \to Y$ by lemma 2.14, otherwise we look at the dual situation. The Auslander-Reiten quiver will now be

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] & & Z. \end{tikzcd}$$

According to lemma 2.13 the module Z is indecomposable, $\operatorname{Hom}(Y, Z) = k$ and $\operatorname{Hom}(X, Z) = 0$. Thus we have a non-split short exact sequence

$$E : \quad 0 \to X \xrightarrow{f} Y \to Z \to 0.$$

Again all injective maps are sent to injective maps, all surjective maps are sent to surjective maps and the functor defined by this is obviously full, dense and faithful.

In both cases we have identified all extensions, kernels, cokernels and images. This completes the proof. ☐

[1]This procedure will also be used in some of the following proofs. But we will not mention this explicitly.

Theorem 2.16. *Let X and Y be indecomposable modules with $[X,Y]=1$ and $[Y,X]^1=1$. Then there is a quiver $\vec{Q}$ with underlying graph $\mathbb{A}_3$ such that there is an equivalence of the categories $\mathcal{A}be(X,Y)$ and $k\vec{\mathbb{Q}}$-*mod.

Proof. Let

$$E_1: \quad 0 \to X \xrightarrow{[f_r]} \bigoplus Z_r \xrightarrow{[g_r]} Y \to 0$$

be a non-split short exact sequence and let $Z=\bigoplus Z_r$. Lemma 2.12 shows that $[Z,Z]=1+[X,Y]=2$, which means that $Z=Z_1\oplus Z_2$, where Z_1 and Z_2 are indecomposable and $\mathrm{Hom}(Z_r,Z_s)=0$ for $r\neq s$.

Case 1: f_1 and f_2 are injective. In this situation we will see that the orientation of $\vec{\mathbb{A}}_3$ is given by

$$\begin{tikzcd} \circ \arrow[dr] & \\ & \circ \\ \circ \arrow[ur] & . \end{tikzcd}$$

The Auslander-Reiten quiver of this $k\vec{\mathbb{A}}_3$ is given by

$$\begin{tikzcd} & Z_1 \arrow[dr] & & I_2 \\ X \arrow[ur] \arrow[dr] & & Y \arrow[ur] \arrow[dr] & \\ & Z_2 \arrow[ur] & & I_1 \end{tikzcd}$$

and we have to identify the modules I_1 and I_2 with modules in $\mathcal{A}be(X,Y)$.

First we look at

$$E_2: \quad 0 \to X \xrightarrow{f_1} Z_1 \to Z_1/X \to 0,$$

which is a non-split short exact sequence. By applying lemma 2.12 to E_1 and lemma 2.13 to E_2 we can conclude that Z_1/X is indecomposable. Let us introduce I_1 as an abbreviation for Z_1/X.

In the same way we get I_2 by looking at the cokernel of $X\to Z_2$. Additionally we have $\mathrm{Hom}(X,I_1)=\mathrm{Hom}(X,I_2)=0$ and $\mathrm{Hom}(Z_1,I_1)\cong\mathrm{Hom}(Z_2,I_2)\cong k$.

Lemma 2.9 ensures that g_1 and g_2 are injective.

Now we look at

$$E_3: \quad 0 \to Z_2 \xrightarrow{g_2} Y \xrightarrow{h_2} Y/Z_2 \to 0.$$

Again we apply lemmas 2.12 and 2.13 and conclude that Y/Z_2 is indecomposable. Counting dimension vectors yields $\underline{\dim} Y/Z_2 = \underline{\dim} Y - \underline{\dim} Z_2 = \underline{\dim} Z_1\oplus Z_2 - \underline{\dim} X - \underline{\dim} Z_2 = \underline{\dim} Z_1 - \underline{\dim} X = \underline{\dim} I_1$. Therefore $Y/Z_2\cong I_1$.

If $h_2g_1=0$, by the universal property of the kernel in E_3 we get a map $Z_1\to Z_2$, which is a contradiction. Thus $h_2g_1: Z_1\to I_1$ is a basis of $\mathrm{Hom}(Z_1,I_1)\cong k$ and we have proven that every map from Z_1 to I_1 factors through Y. Of course, we have the same situation for Z_1 and I_1.

If there is a non-zero map $l: I_1\to I_2$, we have $lh_2g_2=0$ (cf E_3). We obtain a map $lh_2: Y\to I_2$, but $\mathrm{Hom}(Y,I_2)$ is one-dimensional. Thus lh_2 is a multiple of h_1. However, $h_1g_2\neq 0$, so $lh_2=0$ and $l=0$, because h_2 is surjective. This shows that $\mathrm{Hom}(I_1,I_2)=\mathrm{Hom}(I_2,I_1)=0$.

The last homomorphism we have to check is $f: X\to Y$. This is also injective, since $\mathrm{Hom}(X,Y)=k$ and f_1 and g_1 are both injective. We have the short exact sequence

$$E_4: \quad 0 \to X \xrightarrow{f} Y \xrightarrow{g} Y/X \to 0.$$

Look at the diagram

$$\begin{tikzcd} & & Z_1 \oplus Z_2 \arrow[dr, "{[g_1,g_2]}"] & & & \\ 0 \arrow[r] & X \arrow[rr, "f"] & & Y \arrow[r, "g"] \arrow[dr, "\left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right]"'] & Y/X \arrow[r] & 0. \\ & & & & I_1 \oplus I_2 & \end{tikzcd}$$

Since $\begin{bmatrix} 0 & h_1 g_2 \\ h_2 g_1 & 0 \end{bmatrix} = \left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right] [g_1, g_2] : Z_1 \oplus Z_2 \to I_1 \oplus I_2$ and $h_1 g_2$ and $h_2 g_1$ are surjective (cf E_2), $\left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right]$ is surjective. Since there is no map $X \to I_1 \oplus I_2$, we know that $\left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right] f = 0$. All in all we have a surjective map $Y/X \to I_1 \oplus I_2$ by the universal property of the cokernel. The dimension vector of Y/X is $\underline{\dim} Y - \underline{\dim} X = \underline{\dim} Z_1 + \underline{\dim} Z_2 - 2\underline{\dim} X = \underline{\dim} I_1 + \underline{\dim} I_2$. Thus the cokernel of f is isomorphic to $I_1 \oplus I_2$. Now we have identified all kernels and cokernels and all factorizations of maps. Clearly the functor defined this way is full, dense and faithful and also exact.

Case 2: Neither f_1 nor f_2 are injective. In this situation the orientation of $\mathbb{A}_3$ is given by

$$\begin{tikzcd} & \circ \\ \circ \arrow[ur] \arrow[dr] & \\ & \circ \end{tikzcd}$$

and the Auslander-Reiten quiver by

$$\begin{tikzcd} P_1 \arrow[dr] & & Z_1 \arrow[dr] & \\ & X \arrow[ur] \arrow[dr] & & Y. \\ P_2 \arrow[ur] & & Z_2 \arrow[ur] & \end{tikzcd}$$

Lemma 2.14 implies that f_1 and f_2 are surjective.

Lemma 2.9 shows that g_1 and g_2 are surjective. Now we apply the duality functor $D = \operatorname{Hom}(-, k)$. In the dual situation Dg_1 and Dg_2 are both injective, and we are again in the first case.

Case 3: f_1 is injective and f_2 is not injective. In this situation the orientation of $\mathbb{A}_3$ is given by

$$\circ \longrightarrow \circ \longrightarrow \circ$$

and the Auslander-Reiten quiver by

$$\begin{tikzcd} & & Z_1 \arrow[dr] & & \\ & X \arrow[ur] \arrow[dr] & & Y \arrow[dr] & \\ P_2 \arrow[ur] & & Z_2 \arrow[ur] & & I_1. \end{tikzcd}$$

Again we have the sequences E_2 and E_3 as in case 1. I_1 is also indecomposable, $\operatorname{Hom}(X, I_1) = \operatorname{Hom}(Z_2, I_1) = 0$, $\operatorname{Hom}(Z_1, I_1) \cong k$, $Y/Z_2 \cong I_1$ and every map $Z_1 \to I_1$ factors through Y.

Since f_2 is surjective, we get a short exact sequence

$$E_5 : \quad 0 \longrightarrow P_2 \xrightarrow{e_1} X \xrightarrow{g_2} Z_2 \longrightarrow 0.$$

The dual of lemma 2.13 shows that P_2 is indecomposable, $\operatorname{Hom}(P_2, X) \cong k$ and $\operatorname{Hom}(P_2, Z_2) = 0$.

By applying $\operatorname{Hom}(-, Z_1)$ to E_5 we get

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(Z_2, Z_1) \arrow[r] & \operatorname{Hom}(X, Z_1) \arrow[r] & \operatorname{Hom}(P_2, Z_1) \arrow[dll, out=0, in=180] \\ & \operatorname{Ext}(Z_2, Z_1) \arrow[r] & \cdots & . \end{tikzcd}$$

This implies $\operatorname{Hom}(P_2, Z_1) \cong k$, and — as in case 1 — every map $P_2 \to Z_1$ factors through X.

By applying $\operatorname{Hom}(P_2, -)$ to E_1 we get

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(P_2, X) \arrow[r] & \operatorname{Hom}(P_2, Z_1 \oplus Z_2) \arrow[r] & \operatorname{Hom}(P_2, Y) \arrow[dll, out=0, in=180] \\ & \operatorname{Ext}(P_2, X) \arrow[r] & \cdots \end{tikzcd}$$

and $\operatorname{Hom}(P_2, Y) = 0$. Through $\operatorname{Hom}(Z_1, -)$ on E_5 we get

$$\begin{tikzcd} & & \cdots \arrow[r] & \operatorname{Hom}(Z_1, Z_2) \arrow[dll, out=0, in=180] \\ & \operatorname{Ext}(Z_1, P_2) \arrow[r] & \operatorname{Ext}(Z_1, X) \arrow[r] & \cdots \end{tikzcd}$$

and therefore $\operatorname{Ext}(Z_1, P_2) = 0$. Of course, $f_1 e_1$ is injective and lemma 2.13 shows that Z_1/P_2 is indecomposable. Counting dimension vectors supplies us with $Y \cong Z_1/P_2$. This means that we have a non-split short exact sequence

$$E_6 : \quad 0 \longrightarrow P_2 \xrightarrow{f_1 e_1} Z_1 \xrightarrow{g_1} Y \longrightarrow 0.$$

We now have to check that there are no homomorphisms from P_2 to I_1. Apply $\operatorname{Hom}(P_2, -)$ to E_3. The diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(P_2, Z_2) \arrow[r] & \operatorname{Hom}(P_2, Y) \arrow[r] & \operatorname{Hom}(P_2, I_1) \arrow[dll, out=0, in=180] \\ & \operatorname{Ext}(P_2, Z_2) \arrow[r] & \cdots \end{tikzcd}$$

yields $\operatorname{Hom}(P_2, I_1) = 0$. The last homomorphism to consider is $f : X \to Y$. We claim that $\ker f = P_2$ and $\operatorname{im} f = Z_2$. Since $\operatorname{Hom}(X, Y) = k$, every such f is a multiple of $g_2 f_2$. If $g_2 f_2(x) = 0$, it follows that $[g_1, g_2] \begin{pmatrix} 0 \\ f_2(x) \end{pmatrix}$. So we get $x' \in X$ with $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} (x') = \begin{pmatrix} 0 \\ f_2(x) \end{pmatrix}$. Since f_1 is injective, $x \in \ker f_2 = P_2$. On the other hand $g_2 f_2 e_1 = 0$ and therefore $P_2 \subseteq \ker f$.

Obviously $\operatorname{im} f = \operatorname{im} g_2 f_2 \subseteq \operatorname{im} g_2 \cong Z_2$ and since f_2 is surjective, we also have equality.

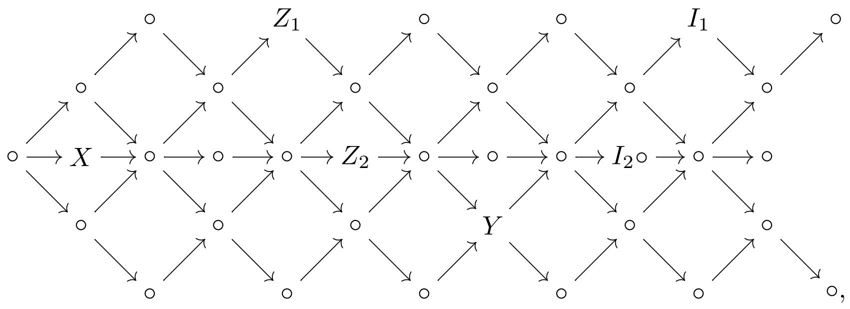
Let us remark that the cokernel of f is isomorphic to Y/Z_2. Again the functor defined this way is full, dense, faithful and exact.

This proves the theorem. $\square$

In order to illustrate the last theorem let us consider $\mathbb{E}_6$ with the orientation

$$\begin{tikzcd} & & \circ \arrow[d] & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \end{tikzcd}$$

The Auslander-Reiten quiver for this algebra is described by the diagram



where the specified modules are those of $k\vec{\mathbb{A}}_3$ in the first case of the proof. The theorems 2.8, 2.15 and 2.16 tell us that Hall polynomials for indecomposable modules X and Y are given by homological data if $[X, Y] \leq 1$ and $[Y, X]^1 \leq 1$. The reason for this is that we only need kernels, cokernels and extensions to calculate $Y \diamond X$. We remark that not all modules in the abelian extension-closed subcategories are needed for the calculation of $Y \diamond X$. But nevertheless they appear $\mathcal{A}be(X, Y)$.

Theorem 2.17. *Suppose X, Y and Z_r, $1 \leq r \leq t$ be indecomposable modules. Let $Z = \bigoplus Z_r$. If*

$$E: \quad 0 \to X \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix}} Z \xrightarrow{[g_1, \ldots, g_t]} Y \to 0$$

is a non-split short exact sequence and $[X, Y] = t - 1$, then

$$\phi^Z_{YX} = (q-1)^{t-1}.$$

Proof. We will count short exact sequences up to middle-term equivalence. Since $\mathcal{A}$ is hereditary, the Auslander-Reiten translation τ is an equivalence from the non-projective to the non-injective modules. Thus we may assume that X is projective. By using the fact that the Hall algebra is independent of the orientation of the quiver (cf [**Mc**]) we may also assume that X is simple and so f_i is injective.

We can take a basis $\bar{f}_1$ of $\operatorname{Hom}(X, Z_1) \cong k$. There is $a_1 \in k$ with $f_1 = a_1 \bar{f}_1$ (cf lemma 2.9). Via $a_1 1_X : X \to X$ the sequence E is middle-term equivalent to

$$E': \quad 0 \to X \xrightarrow{\begin{bmatrix} \bar{f}_1 \\ \frac{1}{a} f_2 \\ \vdots \\ \frac{1}{a} f_t \end{bmatrix}} Z \xrightarrow{[g_1, \ldots, g_t]} Y \to 0.$$

If the short exact sequences

$$E: \quad 0 \to X \xrightarrow{\begin{bmatrix} \bar{f}_1 \\ f_2 \\ \vdots \\ f_t \end{bmatrix}} Z \to Y \to 0$$

and

$$E': \quad 0 \to X \xrightarrow{\begin{bmatrix} \bar{f}_1 \\ f'_2 \\ \vdots \\ f'_t \end{bmatrix}} Z \to Y \to 0$$

are middle-term-equivalent, then there is a map $h : X \to X$ with $\begin{bmatrix} \bar{f}_1 \\ f_2 \\ \vdots \\ f_t \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ f'_2 \\ \vdots \\ f'_t \end{bmatrix} h.$

Since $\bar{f}_1$ is injective, it follows that $h = 1_X$ and $f_r = f'_r$ for $2 \leq r \leq t$.

For all $a_r \in k \setminus \{0\}$, $2 \leq r \leq t$ we obtain a non-split short exact sequence

$$0 \longrightarrow X \xrightarrow{\begin{bmatrix} a_1\bar{f}_1 \\ a_2 f_2 \\ \vdots \\ a_t f_t \end{bmatrix}} Z \xrightarrow{[g_1, \frac{1}{a_2} g_2, \ldots, \frac{1}{a_t} g_t]} Y \longrightarrow 0.$$

So the number of middle-term equivalence classes is identical to the number of possible a_r, $2 \leq r \leq t$, which is $(q-1)^{t-1}$. $\square$

Corollary 2.18. *In the last theorem we can replace the assumption $[X, Y] = t - 1$ by $[Y, X]^1 = 1$.*

Proof. Use lemma 2.12 $\square$.

Lemma 2.19. *Let X and Y be indecomposable modules. Let $[Y, X]^1 = 1$ and Z be a non-trivial extension of Y by X. Then*

$$Y \diamond X = q^{[X,Y]} Y \oplus X + \langle Y X \phi\, Z \rangle Z.$$

Proof. For the right summand we use lemma 2.11 and the left summand results immediately from lemma 2.4. $\square$

Definition 2.20. *Let $h_Y(q)$ and $h_Z(q)$ be different polynomials. We define $\langle s \rangle = {h_Z}^s - {h_Y}^s$ for $i \in \mathbb{N}$. Let*

$$\langle s \rangle ! = \prod_{r=1}^{s} \langle r \rangle$$

and

$$\left\langle {s \atop r} \right\rangle = \frac{\langle s \rangle !}{\langle r \rangle ! \langle s - r \rangle !},$$

where $0 \leq r \leq s$.

The following lemma can be derived by a simple calculation.

Lemma 2.21. *It holds*

$$\left\langle {s+1 \atop r} \right\rangle = {h_Y}^{s-r+1} \left\langle {s \atop r-1} \right\rangle + {h_Z}^r \left\langle {s \atop r} \right\rangle.$$

If the degrees of h_Z and h_Y are different, we can calculate

$$\deg \langle s \rangle ! = \deg \langle 1 \rangle \frac{s(s+1)}{2}$$

and

$$\deg \left\langle {s \atop r} \right\rangle = \deg \langle 1 \rangle (s - r) r.$$

So $\left\langle {s \atop r} \right\rangle$ can be calculated just like the Pascal triangle.

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$
$s=0$	1				
$s=1$	1	1			
$s=2$	1	h_Y+h_Z	1		
$s=3$	1	$h_Y{}^2+h_Yh_Z+h_Z{}^2$	$h_Y{}^2+h_Yh_Z+h_Z{}^2$	1	
$s=4$	1	$h_Y{}^3+h_Y{}^2h_Z+h_Yh_Z{}^2+h_Z{}^3$	$h_Y{}^4+h_Y{}^3h_Z+2h_Y{}^2h_Z{}^2+h_Yh_Z{}^3+h_Z{}^4$	$h_Y{}^3+h_Y{}^2h_Z+h_Yh_Z{}^2+h_Z{}^3$	1

Lemma 2.22. *Let U and W be modules. Then*

$$D(U\diamond W)=DW\diamond DU.$$

Proof. This is true, since $\langle UW \mathbin{\diamond\!\!\!\mid} V\rangle = \langle DWDU \mathbin{\diamond\!\!\!\mid} DV\rangle$. $\square$

Theorem 2.23. *Let X, Y and Z be modules, where X and Y are indecomposable, $[Y,X]^1=1$, $\operatorname{rad}\operatorname{End}(Z)=0$ and*

$$E:\quad 0\longrightarrow X\longrightarrow Z\longrightarrow Y\longrightarrow 0$$

is a non-split short exact sequence. Furthermore let $h_Y=q^{[X,Y]}$, $f(q)=\langle YX \mathbin{\diamond\!\!\!\mid} Z\rangle$ and $h_Z=q^{[X,Z]}$. If we use the notation stated in definition 2.20, then

$$(*)\quad Y^{\diamond n}\diamond X^{\diamond m}=\sum_{r=0}^{l} h_Y^{(m-r)(n-r)}\left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r}\right\rangle \left\langle {m \atop r}\right\rangle X^{\diamond(m-r)}\diamond Z^{\diamond r}\diamond Y^{\diamond(n-r)}$$

for $m,n\geq 0$, where $l=\min\{m,n\}$.

Proof. First we state some previous results by using lemmas 2.12 and 2.19 and corollary 2.6:

$$\begin{aligned} [Z,Z]&=[X,Z] & q^{[Z,Y]}&=h_Z\\ Y\diamond X&=h_YX\diamond Y+fZ & \operatorname{Ext}(Z,Z)&=0\\ Y\diamond Z&=h_ZZ\diamond Y & Z\diamond X&=h_ZX\diamond Z \end{aligned}$$

We claim $(*)$ for $n=1$: This is achieved by induction on m. We start with $m=1$ and obtain

$$Y\diamond X=h_YX\diamond Y+\frac{f}{\langle 1\rangle}\langle 1\rangle! Z,$$

which is shown as true by lemma 2.19. Since $\min\{1,m\}=1$, we can calculate

$$\begin{aligned} Y\diamond X^{\diamond(m+1)} &= \left(\sum_{r=0}^{1} h_Y^{(m-r)(1-r)}\left(\frac{f}{\langle 1\rangle}\right)^r\langle r\rangle!\left\langle {1 \atop r}\right\rangle\left\langle {m \atop r}\right\rangle\right.\\ &\qquad\left. X^{\diamond(m-r)}\diamond Z^{\diamond r}\diamond Y^{\diamond(1-r)}\right)\diamond X\\ &= h_Y^m X^{\diamond m}\diamond Y\diamond X+f\left\langle {m \atop 1}\right\rangle X^{\diamond(m-1)}\diamond Z^{\diamond 1}\diamond X \end{aligned}$$

$$
\begin{aligned}
&= h_Y^m X^{\diamond m} \diamond (h_Y X \diamond Y + fZ) + f \left\langle {m \atop 1} \right\rangle X^{\diamond(m-1)} \diamond (h_Z X \diamond Z) \\
&= h_Y^{m+1} X^{\diamond(m+1)} \diamond Y + f h_Y^m X^{\diamond m} \diamond Z + f \left\langle {m \atop 1} \right\rangle h_Z X^{\diamond m} \diamond Z \\
&= h_Y^{m+1} X^{\diamond m+1} \diamond Y + f\big(h_Y^m + \left\langle {m \atop 1} \right\rangle h_Z\big) X^{\diamond m} \diamond Z.
\end{aligned}
$$

According to lemma 2.21 this equals

$$
h_Y^{m+1} X^{\diamond(m+1)} \diamond Y + f \left\langle {m+1 \atop 1} \right\rangle X^{\diamond m} \diamond Z,
$$

which is

$$
\sum_{r=0}^{1} h_Y^{(m+1-r)(1-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {1 \atop r} \right\rangle \left\langle {m+1 \atop r} \right\rangle X^{\diamond(m+1-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(1-r)}.
$$

Thus $(*)$ is true for $n = 1$ and $m \geq 1$. For $m = 0$ the equation $(*)$ is trivial. This proves the case $n = 1$.

Now we prove $(*)$ for $0 \leq n \leq m$ by induction on n. Obviously $(*)$ is true for $n = 0$. For $0 \leq n < m$ we can calculate:

$$
\begin{aligned}
Y^{\diamond(n+1)} \diamond X^{\diamond m} = &\sum_{r=0}^{n} h_Y^{(m-r)(n-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle \\
&Y \diamond X^{\diamond(m-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n-r)}
\end{aligned}
$$

and with $(*)$ for $n = 1$ this equals

$$
\begin{aligned}
= &\sum_{r=0}^{n} h_Y^{(m-r)(n-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle \\
&\sum_{s=0}^{\min\{1,m-r\}} h_Y^{(m-r-s)(1-s)} \left(\frac{f}{\langle 1\rangle}\right)^s \langle s\rangle! \left\langle {1 \atop s} \right\rangle \left\langle {m-r \atop s} \right\rangle \\
&X^{\diamond(m-r-s)} \diamond Z^{\diamond s} \diamond Y^{\diamond(1-s)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n-r)}.
\end{aligned}
$$

Since $0 \leq r \leq n < m$, this equals:

$$
\begin{aligned}
&\sum_{r=0}^{n} h_Y^{(m-r)(n-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle \sum_{s=0}^{1} h_Y^{(m-r-s)(1-s)} \left(\frac{f}{\langle 1\rangle}\right)^s \langle s\rangle! \\
&\left\langle {1 \atop s} \right\rangle \left\langle {m-r \atop s} \right\rangle X^{\diamond(m-r-s)} \diamond Z^{\diamond s} \diamond Y^{\diamond(1-s)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n-r)} \\
= &\sum_{r=0}^{n} h_Y^{(m-r)(n-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle \\
&\left(h_Y^{m-r} X^{\diamond(m-r)} \diamond Y + f \left\langle {m-r \atop 1} \right\rangle X^{\diamond(m-r-1)} \diamond Z\right) \diamond Z^{\diamond r} \diamond Y^{\diamond(n-r)} \\
= &\sum_{r=0}^{n} h_Y^{(m-r)(n-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle h_Y^{m-r} X^{\diamond(m-r)} \diamond Y \diamond Z^{\diamond r} \diamond Y^{\diamond(n-r)} \\
&+ \sum_{r=0}^{n} h_Y^{(m-r)(n-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle f \left\langle {m-r \atop 1} \right\rangle \\
&X^{\diamond(m-r-1)} \diamond Z^{\diamond(r+1)} \diamond Y^{\diamond(n-r)} \\
= &\sum_{r=0}^{n} h_Y^{(m-r)(n+1-r)} \left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle h_Z^r X^{\diamond(m-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n+1-r)}
\end{aligned}
$$

$$+\sum_{r=1}^{n+1} h_Y^{(m-r+1)(n+1-r)}\left(\frac{f}{\langle 1\rangle}\right)^r \langle r-1\rangle! \left\langle {n \atop r-1} \right\rangle \left\langle {m \atop r-1} \right\rangle \langle 1\rangle \left\langle {m-r+1 \atop 1} \right\rangle$$
$$X^{\diamond(m-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n+1-r)}$$
$$= h_Y^{m(n+1)} X^{\diamond m} \diamond Y^{\diamond n+1} + \sum_{r=1}^{n} h_Y^{(m-r)(n+1-r)}\left(\frac{f}{\langle 1\rangle}\right)^r \langle r-1\rangle! \Big(\left\langle {n \atop r} \right\rangle \left\langle {m \atop r} \right\rangle \langle r\rangle h_Z^r$$
$$+ h_Y^{n+1-r} \left\langle {n \atop r-1} \right\rangle \left\langle {m \atop r-1} \right\rangle \langle 1\rangle \left\langle {m-r+1 \atop 1} \right\rangle \Big) X^{\diamond(m-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n+1-r)}$$
$$+ \left(\frac{f}{\langle 1\rangle}\right)^{n+1} \langle n\rangle! \left\langle {m \atop n} \right\rangle \langle 1\rangle \left\langle {m-n \atop 1} \right\rangle X^{\diamond(m-n-1)} \diamond Z^{\diamond(n+1)} \diamond Y.$$

Since $\left\langle {m \atop r-1} \right\rangle \langle 1\rangle \left\langle {m-r+1 \atop 1} \right\rangle = \left\langle {m \atop r} \right\rangle \langle r\rangle$ for $1 \le r \le n+1$, the preceding sum is identical to

$$h_Y^{m(n+1)} X^{\diamond m} \diamond Y^{\diamond(n+1)} + \sum_{r=1}^{n} h_Y^{(m-r)(n+1-r)}\left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {m \atop r} \right\rangle \Big(\left\langle {n \atop r} \right\rangle h_Z^r$$
$$+ h_Y^{n+1-r} \left\langle {n \atop r-1} \right\rangle \Big) X^{\diamond(m-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n+1-r)}$$
$$+ \left(\frac{f}{\langle 1\rangle}\right)^{n+1} \langle n+1\rangle! \left\langle {n+1 \atop n+1} \right\rangle \left\langle {m \atop n+1} \right\rangle X^{\diamond(m-n-1)} \diamond Z^{\diamond n+1} \diamond Y.$$

Now lemma 2.21 provides the identity with

$$\sum_{r=0}^{n+1} h_Y^{(m-r)(n+1-r)}\left(\frac{f}{\langle 1\rangle}\right)^r \langle r\rangle! \left\langle {m \atop r} \right\rangle \left\langle {n+1 \atop r} \right\rangle X^{\diamond(m-r)} \diamond Z^{\diamond r} \diamond Y^{\diamond(n+1-r)}$$

Thus the cases $0 \le n \le m$ are computed. In the remaining cases $0 \le m \le n$ we look at

$$Y^{\diamond n} \diamond X^{\diamond m} = D(DX^{\diamond m} \diamond DY^{\diamond n})$$

and apply the proven part in $\mathcal{A}^{\mathrm{op}}$-mod. This proves the theorem. □

Corollary 2.24. *Under the same assumptions as in the theorem we obtain that all extensions of Y^n by X^m are of the form $X^{m-r} \diamond Z^{\diamond r} \diamond Y^{n-r}$ for $0 \le r \le \min\{m,n\}$.*

If we want to calculate Hall polynomials more explicitly, we get

Theorem 2.25. *Let $d = [X,Y]$. Then the equations*

$$\langle n\rangle! = q^{d\frac{n(n+1)}{2}}(q-1)^n [n]!,$$
$$\left\langle {n \atop m} \right\rangle = q^{dm(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}$$

and

$$Y^n \diamond X^m = \sum_{r=0}^{\min\{n,m\}} q^{dnm} \prod_{s=1}^{r} (1-q^{-s})^d X^{m-r} \oplus Z^r \oplus Y^{n-r}$$

turn out to be true, if we presuppose the notations of the last theorem.

Proof. For the first two equations are just a matter of notational variants.

Let $M_r = X^{m-r} \oplus Z^r \oplus Y^{n-r}$. Lemma 2.12 shows that $h_Z = qh_Y = q^{d+1}$. Another conclusion of this lemma is $Z^{\diamond r} = (|r]!)^{d+1} Z^r$, since $[Z, Z] = d+1$ and $\operatorname{rad}\operatorname{End}(Z) = 0$. Then one easily calculates

$$\begin{aligned}
Y^n \diamond X^m &= \sum_{r=0}^{\min\{n,m\}} q^{d(n-r)(m-r)-dr}(q-1)^{(d-1)r}\langle r\rangle! \frac{(|r]!)^{d-1}}{\left|\begin{matrix}n\\r\end{matrix}\right]\left|\begin{matrix}m\\r\end{matrix}\right]} \left\langle\begin{matrix}n\\r\end{matrix}\right\rangle \left\langle\begin{matrix}m\\r\end{matrix}\right\rangle M_r \\
&= \sum_{r=0}^{\min\{n,m\}} q^{d((n-r)(m-r)-r+r(n-r)+r(m-r))}(q-1)^{(d-1)r}\langle r\rangle!(|r]!)^{d-1} M_r \\
&= \sum_{r=0}^{\min\{n,m\}} q^{d(nm-r(r+1))}(q-1)^{dr} q^{d\frac{r(r+1)}{2}}(|r]!)^d M_r \\
&= \sum_{r=0}^{\min\{n,m\}} q^{d(nm-\frac{r(r+1)}{2})} \prod_{s=1}^{r}(q^s-1)^d M_r \\
&= \sum_{r=0}^{\min\{n,m\}} q^{dnm} \prod_{s=1}^{r}(1-q^{-s})^d X^{m-r} \oplus Z^r \oplus Y^{n-r}.
\end{aligned}$$

This proves the theorem. $\square$

As an application we will give the Hall polynomials in the following three cases.

Corollary 2.26. *For $\vec{\mathbb{A}}_2 = \circ \to \circ$ with the Auslander-Reiten quiver*

$$\begin{tikzcd}
& Z \arrow[dr] & \\
X \arrow[ur] & & Y
\end{tikzcd}$$

and the modules given by dimension vectors $X = 01$, $Z = 11$ and $Y = 10$ we obtain the Hall polynomials

$$\phi_{Y^n X^m}^{Y^{n-r} \oplus Z^r \oplus X^{m-r}}(q) = 1$$

for all $0 \leq r \leq \min\{n, m\}$.

Corollary 2.27. *For $\vec{\mathbb{A}}_3 = \circ \to \circ \to \circ$ with the Auslander-Reiten quiver*

$$\begin{tikzcd}
& & Z_1 \arrow[dr] & & \\
& X \arrow[ur] \arrow[dr] & & Y \arrow[dr] & \\
\circ \arrow[ur] & & Z_2 \arrow[ur] & & \circ
\end{tikzcd}$$

and the modules given by dimension vectors $X = 011$, $Z = Z_1 \oplus Z_2 = 111 \oplus 010$ and $Y = 110$ we obtain the Hall polynomials

$$\phi_{Y^n X^m}^{Y^{n-r} \oplus Z_1^r \oplus Z_2^r \oplus X^{m-r}}(q) = q^{nm} \prod_{s=1}^{r}(1-q^{-s})$$

for all $0 \leq r \leq \min\{n, m\}$.

COROLLARY 2.28. *For* $\vec{\mathbb{D}}_4 =$ $\begin{tikzcd}[row sep=small, column sep=small] \circ \arrow[dr] & \\ \circ \arrow[r] & \circ \\ \circ \arrow[ur] & \end{tikzcd}$ *with the Auslander-Reiten quiver*

$$\begin{tikzcd}[row sep=small, column sep=small]
 & Z_1 \arrow[dr] & & \circ \arrow[dr] & & \circ \\
X \arrow[ur] \arrow[r] \arrow[dr] & Z_2 \arrow[r] & Y \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[r] & \circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \\
 & Z_3 \arrow[ur] & & \circ \arrow[ur] & & \circ
\end{tikzcd}$$

and the modules given by dimension vectors $X = \begin{matrix}0\\01\\0\end{matrix}$, $Z = Z_1 \oplus Z_2 \oplus Z_3 = \begin{matrix}1\\01\\0\end{matrix} \oplus \begin{matrix}0\\11\\0\end{matrix} \oplus \begin{matrix}0\\01\\1\end{matrix}$ *and* $Y = \begin{matrix}1\\11\\1\end{matrix}$ *we obtain the Hall polynomials*

$$\phi_{Y^n X^m}^{Y^{n-r} \oplus Z_1^r \oplus Z_2^r \oplus Z_3^r \oplus X^{m-r}}(q) = \left(q^{nm} \prod_{s=1}^{r} (1 - q^{-s}) \right)^2$$

for all $0 \leq r \leq \min\{n, m\}$.

COROLLARY 2.29. *For* $\vec{\mathbb{D}}_4 =$ $\begin{tikzcd}[row sep=small, column sep=small] \circ \arrow[dr] & \\ \circ \arrow[r] & \circ \\ \circ \arrow[ur] & \end{tikzcd}$ *with the Auslander-Reiten quiver*

$$\begin{tikzcd}[row sep=small, column sep=small]
 & Z_1 \arrow[dr] & & \circ \arrow[dr] & & \circ \\
X \arrow[ur] \arrow[r] \arrow[dr] & Z_2 \arrow[r] & \circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[r] & \circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \\
 & \circ \arrow[ur] & & Z_3' \arrow[ur] & & \circ
\end{tikzcd}$$

and the modules given by dimension vectors $X = \begin{matrix}0\\01\\0\end{matrix}$, $Z = Z_1 \oplus Z_2 = \begin{matrix}1\\01\\0\end{matrix} \oplus \begin{matrix}0\\11\\0\end{matrix}$ *and* $Z_3' = \begin{matrix}1\\11\\0\end{matrix}$ *we obtain the Hall polynomials*

$$\phi_{Z_3'^n X^m}^{Z_3'^{n-r} \oplus Z_1^r \oplus Z_2^r \oplus X^{m-r}}(q) = q^{nm} \prod_{s=1}^{r} (1 - q^{-s})$$

for all $0 \leq r \leq \min\{n, m\}$.

Note that the upper index for the Hall polynomials in the last four corollaries cover all possible middle-terms. That means that there are no other possible middle-terms of short exact sequences starting at X^m and ending in Y^n.

3. The case $\dim \operatorname{Ext}(Y, X) = 2$

Most of the Hall polynomials for $\mathbb{D}_4$ can be calculated with the results of the last section. There is only one missing. This is the case where $[X, Y] = 1$ and $[Y, X]^1 = 2$.

THEOREM 2.30. *Let* $\vec{\mathbb{D}}_4 = \begin{array}{ccc} & \circ & \\ & & \searrow \\ & \circ \rightarrow & \circ \\ & & \nearrow \\ & \circ & \end{array}$ *and* $\mathcal{A} = k\vec{\mathbb{D}}_4$. *Let* X, Y *and* Z *be indecomposable modules with the dimension vectors* $\underline{\dim}X = \overset{0}{\underset{0}{0}}1$, $\underline{\dim}Z = \overset{1}{\underset{1}{1}}2$ *and* $\underline{\dim}Y = \overset{1}{\underset{1}{1}}1$. *Then the Hall polynomial* ϕ^Z_{YX} *equals* $q-2$.

PROOF. Again we count short exact sequences up to middle-term equivalence. Of course every non-zero map $f : X \longrightarrow Z$ is injective. Therefore we have $q^2 - 1$ possible maps f.

The module Z looks as follows. There are three one-dimensional vector spaces embedded in a two-dimensional vector space. Obviously if and only if f maps onto one of these one-dimensional vector spaces the factor $Z/\operatorname{im} f$ is decomposable. That means we have $q^2-1-3(q-1) = q^2-3q+2 = (q-1)(q-2)$ possible maps f. Since there are always $q-1$ elements in a middle-term equivalence class, the theorem is proved. □

Only one Hall polynomial is missing in the case $\mathbb{D}_5$.

THEOREM 2.31. *Let* $\vec{\mathbb{D}}_5 = \begin{array}{ccc} & \circ & \\ & & \searrow \\ \circ \rightarrow & \circ \rightarrow & \circ \\ & & \nearrow \\ & \circ & \end{array}$ *and* $\mathcal{A} = k\vec{\mathbb{D}}_5$. *Let* X, Y, Z_1 *and* Z_2 *be indecomposable modules with dimension vectors* $\underline{\dim}X = 0\overset{0}{\underset{0}{0}}1$, $\underline{\dim}Z_1 = 0\overset{1}{\underset{1}{1}}2$, $\underline{\dim}Z_2 = 1\overset{0}{\underset{0}{1}}1$ *and* $\underline{\dim}Y = 1\overset{1}{\underset{1}{2}}2$. *Then the Hall polynomial* $\phi^{Z_1 \oplus Z_2}_{YX}$ *equals* $(q-1)(q-2)$.

PROOF. Here Z_1 plays the role of Z in the last proof. Thus we have $(q-2)(q-1)$ maps $f_1 : X \longrightarrow Z_1$ and obviously $q-1$ maps $f_2 : X \longrightarrow Z_2$. Up to middle-term equivalence we obtain $(q-2)(q-1)$ maps $f : X \longrightarrow Z_1 \oplus Z_2$ with indecomposable factor module. □

The last two theorems together with the theorems for the $\mathbb{A}_n$ cases describe all Hall polynomials for $\mathbb{D}_n$ since every $\mathbb{D}_n$ case with $n > 5$ can be reduced to one of the known cases in the following way. Let $\mathbb{D}_n$ have the orientation

$$\begin{array}{c} \circ \\ \downarrow \\ \circ \rightarrow \cdots \rightarrow \circ \longrightarrow \circ \rightarrow \circ. \\ \uparrow \\ \circ \end{array}$$

A module over this quiver has a dimension vector of the form

$$X = \; 00\ldots011\ldots122\ldots\overset{\circ}{\underset{\circ}{2}}\;,$$

where the dimension at the $\circ$s is 0 or 1. The difference between two neighboring dimensions is at most 1 and there need not appear a 2 but if there is a 2 at both $\circ$s there is a 1.

Let us look at a short exact sequence

$$0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$$

of modules over $k\vec{\mathbb{D}}_n$. We want to analyze $\mathrm{Ext}(Y, X)$ and therefore we can without loss of generality assume that X is projective, because the Auslander-Reiten translation is an exact equivalence. In the dimension vector of a projective module does not appear any 2.

Let us illustrate the reduction by an example. We denote only the dimensions in the long branches of X and Y. That means we have for example

$$\begin{array}{lcc|ccc|ccc|ccccc} \text{long branch of } X: & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \text{long branch of } Y: & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \end{array}$$

We have drawn a vertical bar if a map in X or Y on the branch is not an isomorphism. The first reduction is that we can leave out the left zeros in the dimension vectors of X and Y since also in Z there will be zeros at these positions. Then there are two vertical bars remaining. Because of the five lemma the maps between the vertical bars will also be isomorphisms in Z. These isomorphism together with the vector spaces at the points can be reduced to a vector space at a single point. In the above situation we would get

$$\begin{array}{lc|c|c} \text{reduced long branch of } X = & 0 & 1 & 1 \\ \text{reduced long branch of } Y = & 1 & 1 & 2 \end{array}.$$

This procedure applied to the modules of a short exact sequence leads to modules over $k\vec{\mathbb{D}}_4$ or $k\vec{\mathbb{D}}_5$. The Hall polynomial of these modules is the desired Hall polynomial in the case $\mathbb{D}_n$.

The last theorem of this chapter is analogous to theorem 2.16 but for the case $\mathbb{D}_4$. The proof will generalize the arguments of the proof of theorem 2.30.

THEOREM 2.32. *Let X and Y be indecomposable modules with $[X, Y] = 1$ and $[Y, X]^1 = 2$. Then there is a quiver $\vec{Q}$ with underlying graph $\mathbb{D}_4$ such that there is an equivalence of the categories $\mathcal{A}be(X, Y)$ and $k\vec{Q}$-*mod.

To prove this theorem we need two lemmas which analyze the action of the automorphism group of X on $\mathrm{Ext}(Y, X)$ and another to find short exact sequences with decomposable middle-terms.

LEMMA 2.33. *Let X, Y and Z be indecomposable modules. If*

$$E_1: \quad 0 \longrightarrow X \xrightarrow{f_1} Z \xrightarrow{g_1} Y \longrightarrow 0$$

and

$$E_2: \quad 0 \longrightarrow X \xrightarrow{f_2} Z \xrightarrow{g_2} Y \longrightarrow 0$$

are short exact sequences then $\langle E_1 \rangle = \langle E_2 \rangle$ if and only if there exists $a \in k \setminus \{0\}$ with $a[E_1] = [E_2]$.

The lemma states that the elements of a line in $\mathrm{Ext}(Y, X)$ which are corresponding to an indecomposable middle-term are forming a middle-term equivalence class.

PROOF. If E_1 and E_2 are middle-term equivalent we yield an isomorphism $b1_X$. Because of the universal property of the cokernel we obtain an isomorphism $c1_Y$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f_1"] \arrow[d, "b1_X"] & Z \arrow[r, "g_1"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "c1_Y"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f_2"] \arrow[d, equal] & Z \arrow[r, "g_2"] \arrow[d, "b1_Z"] & Y \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "f_1"] & Z \arrow[r, "\frac{c}{b} g_1"] & Y \arrow[r] & 0 \end{tikzcd}$$

is commutative. With $a = \frac{c}{b}$ and lemma 1.6 we proved $a[E_1] = [E_2]$.

Now let $a[E_1] = [E_2]$. Using lemma 1.6 we obtain an isomorphism $b1_Z$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\frac{1}{a}f_1"] \arrow[d, equal] & Z \arrow[r, "g_1"] \arrow[d, "b1_Z"] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "f_2"] & Z \arrow[r, "g_2"] & Y \arrow[r] & 0
\end{tikzcd}$$

is commutative. Then $\frac{a}{b}1_X$ is the desired isomorphism to see that E_1 and E_2 are middle-term equivalent. □

Lemma 2.34. *Let X, Y, Z and Z' be indecomposable modules. If $[X, Y] = 1$ and $[Y, X]^1 = 2$ and if*

$$E_1: \quad 0 \to X \xrightarrow{\left[\begin{smallmatrix} f_1 \\ f_1' \end{smallmatrix}\right]} Z \oplus Z' \xrightarrow{[g_1, g_1']} Y \to 0$$

and

$$E_2: \quad 0 \to X \xrightarrow{\left[\begin{smallmatrix} f_2 \\ f_2' \end{smallmatrix}\right]} Z \oplus Z' \xrightarrow{[g_2, g_2']} Y \to 0$$

are non-split short exact sequences then there is $a \in k$ with $[E_1] = a[E_2]$.

This lemma tells us that the lines in the k-vector space $\operatorname{Ext}(Y, X)$ which elements are corresponding to decomposable middle-terms are classified by there middle-terms (for the other direction of the classification use lemma 1.6).

Proof. The lemmas 2.12 and 2.9 show that $[X, Z] = [X, Z'] = [Z, Y] = [Z', Y] = 1$ and therefore we obtain $b, b', c, c' \in k \setminus \{0\}$ with $f_2 = bf_1$, $f_2' = b'f_1'$, $g_2 = cg_1$ and $g_2' = c'g_1'$. With lemma 2.10 we obtain $g_i f_i \neq 0 \neq g_i' f_i'$ and since E_i is exact $g_i f_i + g_i' f_i' = 0$ for $i = 1, 2$. So we conclude

$$0 = g_2 f_2 + g_2' f_2' = bcg_1 f_1 + b'c' g_1' f_1' = (bc - b'c') g_1 f_1.$$

Thus $c = \frac{c'b'}{b}$ and we get the commutative diagram

$$\begin{tikzcd}
E_2: & 0 \arrow[r] & X \arrow[r, "\left[\begin{smallmatrix} bf_1 \\ b'f_1' \end{smallmatrix}\right]"] \arrow[d, equal] & Z \oplus Z' \arrow[r, "c'\left[\frac{b'}{b}g_1, g_1'\right]"] \arrow[d, "\left[\begin{smallmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1}{b'} \end{smallmatrix}\right]"] & Y \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & X \arrow[r, "\left[\begin{smallmatrix} f_1 \\ f_1' \end{smallmatrix}\right]"] & Z \oplus Z' \arrow[r, "c'b'[g_1, g_1']"] & Y \arrow[r] & 0.
\end{tikzcd}$$

With $a = \frac{1}{c'b'}$ and lemma 1.6 we yield $[E_2] = a[E_1]$. □

Lemma 2.35. *Let X and Y be indecomposable modules. If $[X, Y] = 1$ and $[Y, X]^1 = 2$ then*

1. *there are exactly three elements in the k-vector space* $\operatorname{Ext}(Y, X)$ *which are pairwise linear independent and the middle-terms of the corresponding non-split short exact sequences are decomposable and*
2. *there are exactly $q - 2$ elements in the k-vector space* $\operatorname{Ext}(Y, X)$ *which are pairwise linear independent and the middle-terms of the corresponding non-split short exact sequences are indecomposable.*

Proof. First let us assume that all short exact sequences starting at X and ending in Y have decomposable middle-terms. Because $[X, Y] = 1$ and the lemmas 2.12 and 2.3 we know that each such middle-term has exactly two indecomposable summands. Since $[Y, X]^1 = 2$ there are $q + 1$ lines in $\operatorname{Ext}(Y, X)$. For different lines the middle-term of the corresponding short exact sequences are not isomorphic (cf lemma 2.34). That means we would have $2(q+1)$ non-isomorphic modules, which is a contradiction since $\mathcal{A}$ is representation finite.

Thus for large fields there is an indecomposable module Z with $\underline{\dim}Z = \underline{\dim}X + \underline{\dim}Y$ and a short exact sequence starting at X and ending in Y with middle-term Z. Since the module category is independent of the field this module Z exist also for small k, i.e. also if k has only two elements. Pay attention to the fact that Z is uniquely determined by its dimension vector.

Now let us look at the Hall polynomial ϕ^Z_{YX}. The number $\phi^Z_{YX}(q)$ counts the lines in $\operatorname{Ext}(Y,X)$ which elements have indecomposable middle-terms. Since the entire two-dimensional vector space $\operatorname{Ext}(Y,X)$ has $q+1$ lines the Hall polynomial ϕ^Z_{YX} has degree at most one. Ringel proved in [**R4**] that the Hall polynomials with indecomposable end- and middle-terms in the Dynkin cases are one of the table

1	$\mathbb{A}_2$
$q-2$	$\mathbb{D}_4$
$(q-2)^2$	$\mathbb{E}_6$
$q^3-5q^2+10q-7$	$\mathbb{E}_7$
$(q-2)^3$	$\mathbb{E}_8$
$(q-2)(q^3-4q^2+8q-6)$	$\mathbb{E}_8$
$q^5-6q^4+15q^3-23q^2+25q-13$	$\mathbb{E}_8$,

where we have denoted in the second column the "smallest" algebra for which the Hall polynomial appears first. If $\phi^Z_{YX}(q) = 1$ there would be $2q$ non-isomorphic indecomposable modules which is again a contradiction. Therefore $\phi^Z_{YX} = q-2$ and there are exactly three other lines to exhaust $\operatorname{Ext}(Y,X)$. $\square$

Let us draw attention to the following remark. If k has only two elements there is no short exact sequence with an indecomposable middle-term since $\phi^Z_{YX}(2) = 0$. Thus in this case all four elements of $\operatorname{Ext}(Y,X)$ correspond to sequences with decomposable middle-terms.

Now we come to

Proof of theorem 2.32. The last lemma supplies us with three non-split short exact sequences

$$E_r: \quad 0 \longrightarrow X \xrightarrow{\left[\begin{smallmatrix} f_r \\ f'_r \end{smallmatrix}\right]} Z_r \oplus Z'_r \xrightarrow{[g_r\, g'_r]} Y \longrightarrow 0 \quad r = 1,2,3.$$

and if the field has more than two elements with another sequence

$$E_0: \quad 0 \longrightarrow X \xrightarrow{f_0} Z_0 \xrightarrow{g_0} Y \longrightarrow 0,$$

where Z_r $(r = 0,1,2,3)$ and Z'_r $(r = 1,2,3)$ are indecomposable, $[Z_r \oplus Z'_r, Z_r \oplus Z'_r] = 2$ $(r = 1,2,3)$ and $\{Z_r \,|\, r = 0,1,2,3\} \cup \{Z'_r \,|\, r = 1,2,3\}$ is a set of pairwise non-isomorphic elements. Furthermore from lemma 2.9 it follows that $[X, Z_r] = [X, Z'_r] = [Z_r, Y] = [Z'_r, Y] = 1$ $(r = 1,2,3)$.

By applying $\operatorname{Hom}(Z_r \oplus Z'_r, -)$ to E_r $(r = 1,2,3)$ we get $[Z_r \oplus Z'_r, Z_r \oplus Z'_r]^1 = 1$. Without loss of generality we may assume $[Z_r, Z'_r]^1 = 0$ $(r = 1,2,3)$, thus there are non-split short exact sequences

$$E_{r+3}: \quad 0 \longrightarrow Z_r \longrightarrow Z_0 \longrightarrow Z'_r \longrightarrow 0 \quad (r = 1,2,3),$$

where the middle-term is the module Z_0 since it is indecomposable (cf lemma 2.12) with dimension vector $\underline{\dim}Z_r + \underline{\dim}Z'_r = \underline{\dim}X + \underline{\dim}Y = \underline{\dim}Z_0$.

By applying $\operatorname{Hom}(-, X)$ to E_4 we obtain $[Z, X]^1 = 1$. Let us summarize the up to now known dimensions of the homomorphism and extension spaces in the following table which are computed as in theorem 2.16.

	X	Z_r	Z_0	Z'_r	Y
X		1	2	1	1
Z_r	0		1	0	1
Z_0	1	0		1	2
Z'_r	1	1	0		1
Z'_r	2	1	1	0	

, for $r = 1, 2, 3$

In the upper triangle we have denoted the dimension of the homomorphism spaces and in the lower triangle the dimensions of the extensions spaces. Remark that $\mathcal{A}$ is representation directed. The dimensions of the extension spaces $\operatorname{Ext}(Z'_r, X)$ and $\operatorname{Ext}(Z_0, X)$ are marked, since we have up to now no short exact sequences for these spaces.

Let us first analyze $\operatorname{Ext}(Z_0, X)$. We have a non-split short exact sequence

$$E_7: \quad 0 \to X \to M \to Z_0 \to 0$$

with $\dim \operatorname{End}(M) = 3$ (apply $\operatorname{Hom}(X, -)$ and $\operatorname{Hom}(M, -)$ to E_7). The dimension of the homomorphism and extension spaces we will need in the following are: $[X, M] = [M, Z_0] = 3$, $[Z_r, M] = 1$, $[M, Z_r] = 1$ and $[M, X]^1 = [M, M]^1 = [Z_0, M]^1 = 0$ for $r = 1, 2, 3$ and all these are easy by applying $\operatorname{Hom}(-,-)$-functors or they are immediate by lemma 2.12.

By using $\operatorname{Ext}(Z_r, X) = 0$ we obtain the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & M \arrow[r] & Z_0 \arrow[r] & 0 \\ 0 \arrow[r] & X \arrow[r] \arrow[u, equal] & X \oplus Z_r \arrow[r] \arrow[u, hook] & Z_r \arrow[r] \arrow[u, hook] & 0. \end{tikzcd}$$

This shows that Z_r is a submodule of M for all $r = 1, 2, 3$. Let us look at this situation.

Since $[M, M] = 3$ the module M has two or three indecomposable summands. Let us assume $M = M_1 \oplus M_2$ with indecomposable modules M_1 and M_2 and $[M_1, M_2] = 1$.

Then without loss of generality M_1 maps to Z_1 and Z_1 maps injectively to M_2, where $M_1 \ncong Z_1 \ncong M_2$.

If $M_1 \longrightarrow Z_1$ is surjective then the image of $M_1 \twoheadrightarrow Z_1 \hookrightarrow M_2$ is identical to Z_1 and the image of $M_1 \longrightarrow Z_2 \hookrightarrow M_2$ has also to be equal to Z_1 since $[M_1, M_2] = 1$. So we would achieve $Z_1 \subseteq Z_2$. Using the fact that $M_1 \longrightarrow Z_2$ is surjective or injective (cf lemma 2.14) with $\operatorname{im}(M_1 \longrightarrow Z_2) = Z_1$ we obtain a contradiction to $Z_1 \ncong Z_2$ or $M_1 \ncong Z_1$. This means $M_1 \longrightarrow Z_1$ is injective. Since there is a surjective map $M \longrightarrow Z_0$ the map $M_2 \longrightarrow Z_0$ cannot be injective (in this case the image of M in Z_0 would be only M_2). So we conclude $M_2 \twoheadrightarrow Z_0$ is surjective. Using lemma 2.9 we are supplied with a surjective map $X \twoheadrightarrow M_1$. Thus the image of $X \twoheadrightarrow M_1 \hookrightarrow Z_1 \hookrightarrow M_2$ is identical to M_1. However the map $X \hookrightarrow M_2$ is injective (use the injectivity of $M_1 \hookrightarrow Z_1 \hookrightarrow Z_0$ and lemma 2.9 for E_7). Therefore the number of summands of M cannot be two thus it is three.

If $M = M_1 \oplus M_2 \oplus M_3$ it is immediate that $M = Z_1 \oplus Z_2 \oplus Z_3$. (eg $M_1 \hookrightarrow Z_1 \longrightarrow M_1$ is a cycle and therefore $M_1 \cong Z_1$.) Furthermore we get $[Z_i, Z_j] = 0$ for $i \neq j$. Then E_7 becomes

$$E_7: \quad 0 \to X \to Z_1 \oplus Z_2 \oplus Z_3 \to Z_0 \to 0$$

and dually we obtain

$$E_8: \quad 0 \to Z_0 \to Z'_1 \oplus Z'_2 \oplus Z'_3 \to Y. \to 0$$

Now we analyze $\operatorname{Ext}(Z_1', X)$. There is a non-split short exact sequence

$$E_9: \quad 0 \to X \to N_1 \oplus N_2 \to Z_1' \to 0,$$

where N_1 and N_2 are indecomposable. Now we have the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & X \arrow[r] & N_1 \oplus N_2 \arrow[r] \arrow[u] & Z_1' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[u, equal] & Pb \arrow[r] \arrow[u] & Z_0 \arrow[r] \arrow[u] & 0 \\
 & & Z_1 \arrow[r, equal] \arrow[u] & Z_1 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The pullback Pb is isomorphic to $Z_1 \oplus Z_2 \oplus Z_3$ since the bottom row can not split because there is no map from Z_1 to X and X is indecomposable. Therefore the left column splits and $N_1 \oplus N_2$ is isomorphic to $Z_2 \oplus Z_3$. Thus E_9 becomes

$$E_9: \quad 0 \to X \to Z_2 \oplus Z_3 \to Z_1' \to 0.$$

Analogously we obtain non-split short exact sequences

$$\begin{aligned}
E_{10}: &\quad 0 \to X \to Z_1 \oplus Z_3 \to Z_2' \to 0,\\
E_{11}: &\quad 0 \to X \to Z_1 \oplus Z_2 \to Z_3' \to 0,\\
E_{12}: &\quad 0 \to Z_1 \to Z_2' \oplus Z_3' \to Y \to 0,\\
E_{13}: &\quad 0 \to Z_2 \to Z_1' \oplus Z_3' \to Y \to 0,
\end{aligned}$$

and

$$E_{14}: \quad 0 \to Z_3 \to Z_1' \oplus Z_2' \to Y \to 0.$$

All in all we have the exact equivalence to the following part of $\mathbb{D}_4$.

$$\begin{tikzcd}
 & Z_1 \arrow[dr] & & Z_1' \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[dr] & Z_2 \arrow[r] & Z_0 \arrow[ur] \arrow[r] \arrow[dr] & Z_2' \arrow[r] & Y \\
 & Z_3 \arrow[ur] & & Z_3' \arrow[ur] &
\end{tikzcd}$$

The remaining sequences and modules (only kernels and/or cokernels) are obtained with theorem 2.16, lemma 2.9 and by counting dimension vectors.

Let $f_{rs}: Z_r \longrightarrow Z_s'$ for $1 \le r, s \le 3$ and $r \ne s$. Then f_{rs} is either injective or surjective. Let $P_{rs} = \ker f_{rs}$ and $I_{rs} = Z_s' / \operatorname{im} f_{rs}$. Exactly one of the modules P_{rs} and I_{rs} is in indecomposable and the other one is zero (cf lemma 2.13) and we have the exact sequences

$$E_{rs}: \quad 0 \to P_{rs} \to Z_r \xrightarrow{f_{rs}} Z_s' \to I_{rs} \to 0.$$

Counting dimension vectors yields $\underline{\dim} P_{rs} = \underline{\dim} Z_r + \underline{\dim} I_{rs} - \underline{\dim} Z_s' = \underline{\dim} Z_r \oplus Z_s + \underline{\dim} I_{rs} - \underline{\dim} Z_0 = \underline{\dim} Z_1 \oplus Z_2 \oplus Z_3 - \underline{\dim} Z_{6-r-s} + \underline{\dim} I_{rs} - \underline{\dim} Z_0 = \underline{\dim} X - \underline{\dim} Z_{6-r-s} + \underline{\dim} I_{rs}$. If $\underline{\dim} X - \underline{\dim} Z_{6-r-s}$ is in $\mathbb{N}_0^{|Q_0|}$ the module I_{rs} must be the zero-module and if $\underline{\dim} Z_{6-r-s} - \underline{\dim} X$ is in $\mathbb{N}_0^{|Q_0|}$ the module P_{rs} must be the zero-module. Now it is immediate that $P_{sr} \cong P_{rs}$ and $I_{rs} \cong I_{sr}$. That means if f_{rs} is injective also f_{sr} is injective and if f_{rs} is surjective also f_{sr} is surjective.

All in all we have three short exact sequences E_{rs}, $0 \leq r < s \leq 3$. Now theorem 2.16 for the modules X and Z'_r, $r = 1, 2, 3$ supplies us with the short exact sequences

$$E^{rs}: \quad 0 \longrightarrow P_{rs} \longrightarrow Z'_{6-r-s} \longrightarrow Y \longrightarrow 0$$

if $I_{rs} = 0$ or

$$E^{rs}: \quad 0 \longrightarrow X \longrightarrow Z_{6-r-s} \longrightarrow I_{rs} \longrightarrow 0$$

if $P_{rs} = 0$. Finally we see that if $P_{rs} \neq 0$ we have a short exact sequence

$$0 \longrightarrow P_{rs} \longrightarrow X \longrightarrow Z_{6-rs} \longrightarrow 0.$$

The orientation of $\vec{Q}$ depends now on the surjectivity or injectivity of f_{sr}. This proves the theorem. $\square$

CHAPTER 3

The Algorithm

We will describe the algorithm to calculate Hall polynomials with arbitrary end- and middle-terms.

1. Description and Main Results

Let X be an indecomposable module. A module V is called *isotypical (of type X)* provided all its indecomposable direct summands are isomorphic (to X). An *isotypical component of type* X of a module V is a direct summand X^n of V, where n is maximal.

Since $\mathcal{A}$ is representation directed, we can order the indecomposable modules X_n such that $\mathrm{Hom}(X_n, X_m) = \mathrm{Ext}(X_m, X_n) = 0$ for $n > m$. We will call an indecomposable module X_n *smaller* ($<$) than a module X_m if $n < m$. Obviously this order can be extended to an order of isotypical modules.

Modules U and W can be decomposed to $U = \bigoplus X_n^{u_n}$ and $W = \bigoplus X_n^{w_n}$, where u_n and w_n may be zero.

By ordering the isotypical components of U and W we obtain two sequences $T_U = (X_1^{u_1}, X_2^{u_2}, \dots)$ and $T_W = (X_1^{w_1}, X_2^{w_2}, \dots)$. For a sequence T, the *reduced sequence*, that is, the sequence in which all zero-modules are eliminated is denoted by $\overline{T}$.

By concatenating the two reduced sequences $\overline{T_W}$ and $\overline{T_U}$ we obtain the sequence

$$T = \overline{(X_1^{w_1}, X_2^{w_2}, \dots, X_1^{u_1}, X_2^{u_2}, \dots)}.$$

We say T corresponds to the product $W \diamond U$ and call it a *product sequence*. All sequences we look at are product sequences and for the sake of simplicity we will call them *sequences*. Of course, such sequences have finite length.

The *entropy $e_T(T_n)$ of an isotypical module T_n* in a sequence $T = (T_m)$ is

$$e_T(T_n) = \sum_{\{m \mid m<n \text{ and } T_m \geq T_n\}} 1.$$

This means that we count the misplacements which occur on the left-hand side of T_n. The *entropy $e_T(X_n)$ of the isotypical type X_n* is

$$e_T(X_n) = \sum_{T_n \text{ is of type } X_n} e_T(T_n).$$

Now we want to calculate $W \diamond U$ and remark that we will urgently need the associativity of $\diamond$. Assume that we know all products $Y \diamond X = \sum_Z \langle Y X \phi\, Z \rangle Z$ for isotypical Y and X. These are the main ingredients of the algorithm. Remember that we do not distinguish between modules and isomorphism classes. Lemma 2.9 shows that $X \leq Z_n \leq Y$ for all isotypical direct summands Z_n of Z. Equality holds only for $Z = X \oplus Y$.

We shall call a sequence $T = (T_n)$ ordered if $T_n < T_m$ for all $n < m$. Obviously a sequence T is ordered if and only if $e_T(X_n) = 0$ for all isotypical types X_n.

Let $n(X_n)$ be the largest index, whereby $T_{n(X_n)}$ is of type X_n.

If $e_T(X_1) \neq 0$ we look at the positions $n(X_1)$ and $n(X_1)-1$. This corresponds to the product

$$T_{n(X_1)-1} \diamond T_{n(X_1)} = \sum_Z \langle T_{n(X_1)-1} T_{n(X_1)} \diamond Z \rangle Z.$$

By replacing $T_{n(X_1)-1}, T_{n(X_1)}$ in T with $Z_1, \ldots, Z_t$, where Z_r are the isotypical summands of Z, we find a new sequence T' (one for each summand in the above formula). Obviously $e_{T'}(X_1) < e_T(X_1)$ for all these new sequences, since there is one misplacement less. Going on with this procedure simultaneously for all sequences we obtain a sequence T_1 with $e_{T_1}(X_1) = 0$. We proceed with X_2 and so on. Finally we get a list of sorted sequences.

In order to illustrate the procedure we want to calculate the Hall polynomials for $\vec{\mathbb{A}}_3 = \circ \to \circ \to \circ$. For this we name the isotypical types just with the numbers from 1 up to 6 as illustrated in the Auslander-Reiten quiver by

$$\begin{tikzcd}[column sep=small] & & 3 \arrow[dr] & & \\ & 2 \arrow[ur] \arrow[dr] & & 5 \arrow[dr] & \\ 1 \arrow[ur] & & 4 \arrow[ur] & & 6. \end{tikzcd}$$

We want to calculate $(4^a \oplus 5^b \oplus 6^c) \diamond (1^d \oplus 2^e \oplus 4^f) = 4^a \diamond 5^b \diamond 6^c \diamond 1^d \diamond 2^e \diamond 4^f$ (use corollary 2.5), where $a, b, c, d, e, f \in \mathbb{N}$. In the left factor we leave out the projective direct summands and in the right factor the injective direct summands, since $\mathrm{Ext}(Y, -)$ vanishes for Y of type 1, 2 or 3 and analogously for the injective summands.

First we see that $T = (4^a, 5^b, 6^c, 1^d, 2^e, 4^f)$, $e_T(1) = 3$ and $n(1) = 4$. So we flip positions 3 and 4. Since $6^c \diamond 1^d = 1^d \diamond 6^c$ (cf 2.6), we obtain $T' = (4^a, 5^b, 1^d, 6^c, 2^e, 4^f)$. For T' we obtain $e_{T'}(1) = 2$ and due to $n(1) = 3$ we flip positions 2 and 3. Since $\mathrm{Hom}(1,5) = 0$ and $\mathrm{Ext}(5,1) \cong k$, the theorems 2.15 and 2.25 supply us with $5^b \diamond 1^d = \sum_{r=0}^{\min\{b,d\}} 1^{d-r} \oplus 3^r \oplus 5^{b-r}$.

We also want to decrease $e_{T'}(2)$. Thus we look at the positions $n(2) = 5$ and 4. This is the dual situation to the positions 2 and 3. Altogether we yield

$$\begin{aligned} 4^a \diamond \sum_{r=0}^{\min\{b,d\}} (1^{d-r} \oplus 3^r \oplus 5^{b-r}) \diamond \sum_{r'=0}^{\min\{c,e\}} (2^{e-r'} \oplus 3^{r'} \oplus 6^{c-r'}) \diamond 4^f \\ = \sum_{r=0}^{\min\{b,d\}} \sum_{r'=0}^{\min\{c,e\}} 4^a \diamond 1^{d-r} \diamond 3^r \diamond 5^{b-r} \diamond 2^{e-r'} \diamond 3^{r'} \diamond 6^{c-r'} \diamond 4^f. \end{aligned}$$

So we have $(\min\{b,d\}+1)(\min\{c,e\}+1)$ sequences of the form

$$T'' = (4^a, 1^{d-r}, 3^r, 5^{b-r}, 2^{e-r'}, 3^{r'}, 6^{c-r'}, 4^f).$$

If an exponent is zero we have to reduce these sequences. In each of the sequences with $d > r$ we have $e_{T''}(1) = 1$ and $n(1) = 2$. In the sequences with $d = r$ the algorithm goes on with the isotypical modules of type 2. To speed up the process remember that we do not only want to shift the 1 to the left, but also the 6 to the right. So we look now at the position-pairs (1,2), (4,5) and (7,8). The products $4^a \diamond 1^{d-r}$ and $6^{c-r'} \diamond 4^f$ are $\mathbb{A}_2$ cases due to theorem 2.15, but $5^{b-r} \diamond 2^{e-r'}$ is an $\mathbb{A}_3$ case according to theorem 2.16. Thus from theorem 2.25 we deduce that the preceding product is identical to

$$\sum_{r=0}^{\min\{b,d\}} \sum_{r'=0}^{\min\{c,e\}} \left(\sum_{s=0}^{\min\{a,d-r\}} (1^{d-r-s} \oplus 2^s \oplus 4^{a-s}) \right) \diamond 3^r \diamond$$

$$\sum_{t=0}^{\min\{b-r,e-r'\}} \left(q^{(b-r)(e-r')} \prod_{i=1}^{t} (1-q^{-i}) 2^{e-r'-t} \oplus (3 \oplus 4)^t \oplus 5^{b-r-t} \right) \diamond$$
$$3^{r'} \diamond \left(\sum_{s'=0}^{\min\{c-r',f\}} (4^{f-s'} \oplus 5^{s'} \oplus 6^{c-r'-s'}) \right)$$
$$= \sum_{r=0}^{\min\{b,d\}} \sum_{r'=0}^{\min\{c,e\}} \sum_{s=0}^{\min\{a,d-r\}} \sum_{t=0}^{\min\{b-r,e-r'\}} \sum_{s'=0}^{\min\{c-r',f\}} q^{(b-r)(e-r')} \prod_{i=1}^{t} (1-q^{-i}) \cdot$$
$$1^{d-r-s} \diamond 2^s \diamond 4^{a-s} \diamond 3^r \diamond 2^{e-r'-t} \diamond 3^t \diamond 4^t \diamond 5^{b-r-t} \diamond 3^{r'} \diamond 4^{f-s'} \diamond 5^{s'} \diamond 6^{c-r'-s'}.$$

The corresponding sequences are of the form

$$T_1 = (1^{d-r-s}, 2^s, 4^{a-s}, 3^r, 2^{e-r'-t}, 3^t, 4^t, 5^{b-r-t}, 3^{r'}, 4^{f-s'}, 5^{s'}, 6^{c-r'-s'}).$$

In every sequence the equality $e_{T_1}(1) = 0$ is fulfilled and if the first five exponents do not vanish we have $e_{T_1}(2) = 3$ and $n(2) = 5$. This means that we have to flip positions 4 and 5 and for the sake of shortening the procedure we look at the dual situation again, that is, at positions 8 and 9. Since there is no extension in either case, the Hall polynomial is $q^{r(e-r'-t)} q^{(b-r-t)r'}$ and we obtain the sequence

$$S_1' = (1^{d-r-s}, 2^s, 4^{a-s}, 2^{e-r'-t}, 3^r, 3^t, 4^t, 3^{r'}, 5^{b-r-t}, 4^{f-s'}, 5^{s'}, 6^{c-r'-s'}).$$

Now we flip positions 3 and 4 and also positions 9 and 10. Again there are no extensions and the the Hall polynomial is given by the dimension of the homomorphism-space. It is $q^{(a-s)(e-r'-t)} q^{(b-r-t)(f-s')}$ and the corresponding sequence is

$$T_1'' = (1^{d-r-s}, 2^s, 2^{e-r'-t}, 4^{a-s}, 3^r, 3^t, 4^t, 3^{r'}, 4^{f-s'}, 5^{b-r-t}, 5^{s'}, 6^{c-r'-s'}),$$

where $e_{T''}(2) = 1$. Since $3 \diamond 4 = 4 \diamond 3$, we get the sequence

$$T_1'' = (1^{d-r-s}, 2^s, 2^{e-r'-t}, 3^r, 3^t, 3^{r'}, 4^{a-s}, 4^t, 4^{f-s'}, 5^{b-r-t}, 5^{s'}, 6^{c-r'-s'}).$$

Finally we obtain the Hall polynomial

$$\begin{vmatrix} s+e-r'-t \\ s \end{vmatrix} \begin{vmatrix} r+t \\ r \end{vmatrix} \begin{vmatrix} r+t+r' \\ r' \end{vmatrix}$$
$$\begin{vmatrix} a-s+t \\ t \end{vmatrix} \begin{vmatrix} a-s+t+f-s' \\ f-s' \end{vmatrix} \begin{vmatrix} b-r-t+s' \\ s' \end{vmatrix}$$

and the sequence

$$T_2 = (1^{d-r-s}, 2^{s+e-r'-t}, 3^{r+t+r'}, 4^{a-s+t+f-s'}, 5^{b-r-t+s'}, 6^{c-r'-s'}),$$

which is sorted. All in all we deduce:

$$(4^a \oplus 5^b \oplus 6^c) \diamond (1^d \oplus 2^e \oplus 4^f) =$$
$$\sum_{r=0}^{\min\{b,d\}} \sum_{r'=0}^{\min\{c,e\}} \sum_{s=0}^{\min\{a,d-r\}} \sum_{t=0}^{\min\{b-r,e-r'\}} \sum_{s'=0}^{\min\{c-r',f\}} q^{(b-r)(e-r')} \prod_{i=1}^{t} (1-q^{-i})$$
$$q^{r(e-r'-t)} q^{(b-r-t)r'} q^{(a-s)(e-r'-t)} q^{(b-r-t)(f-s')} \begin{vmatrix} s+e-r'-t \\ s \end{vmatrix} \begin{vmatrix} r+t \\ r \end{vmatrix}$$
$$\begin{vmatrix} r+t+r' \\ r' \end{vmatrix} \begin{vmatrix} a-s+t \\ t \end{vmatrix} \begin{vmatrix} a-s+t+f-s' \\ f-s' \end{vmatrix} \begin{vmatrix} b-r-t+s' \\ s' \end{vmatrix}$$
$$1^{d-r-s} \oplus 2^{s+e-r'-t} \oplus 3^{r+t+r'} \oplus 4^{a-s+t+f-s'} \oplus 5^{b-r-t+s'} \oplus 6^{c-r'-s'}.$$

The degree of the polynomial in each of the summand above is

$$(a+b)(e+f) - f(r+s) - a(r'+s') - st - ts' + ss'$$

$$= (a+b)(e+f) - f(r+s) - a(r'+s') + \frac{1}{2}(t-s-s')^2 - \frac{1}{2}t^2 - \frac{1}{2}s^2 - \frac{1}{2}s'^2.$$

Clearly there are several combinations of s, s', r, r' and t with the same module $1^{d-r-s} \oplus 2^{s+e-r'-t} \oplus 3^{r+t+r'} \oplus 4^{a-s+t+f-s'} \oplus 5^{b-r-t+s'} \oplus 6^{c-r'-s'}$. So we introduce new variables to fix the upper indices. Let $c_1 = r+s$, $c_2 = s-r'-t$, $c_3 = r+t+r'$, $c_4 = t-s-s'$, $c_5 = s'-r-t$ and $c_6 = r'+s'$ and note the equalities $c_2 = -c_4 - c_6$, $c_3 = c_4 + c_1 + c_6$ and $c_5 = -c_4 - c_1$.

In this notation the module is

$$1^{d-c_1} \oplus 2^{e-c_4-c_6} \oplus 3^{c_1+c_4+c_6} \oplus 4^{a+f+c_4} \oplus 5^{b-c_4-c_1} \oplus 6^{c-c_6}.$$

and the degree of

$$\phi^{1^{d-c_1}\oplus 2^{e-c_4-c_6}\oplus 3^{c_1+c_4+c_6}\oplus 4^{a+f+c_4}\oplus 5^{b-c_4-c_1}\oplus 6^{c-c_6}}_{4^a\oplus 5^b\oplus 6^c, 1^d\oplus 2^e\oplus 4^f}.$$

is the maximum over s and s' of

$$(a+b)(c+f) - fc_1 - ac_6 + \frac{1}{2}c_4^2 - \frac{1}{2}((c_4+s+s')^2 + s^2 + s'^2).$$

If $c_4 \geq 0$, this is obviously achievable if $s = s' = 0$. In this case $r = c_1$, $r' = c_6$ and $t = c_4$. All in all we have proven

Theorem 3.1. *If $0 \leq c_4 \leq \min\{b-c_1, e-c_6\}$, $0 \leq c_1 \leq \min\{b,d\}$ and $0 \leq c_6 \leq \min\{c,e\}$, then the degree of the Hall polynomial*

$$\phi^{1^{d-c_1}\oplus 2^{e-c_4-c_6}\oplus 3^{c_1+c_4+c_6}\oplus 4^{a+f+c_4}\oplus 5^{b-c_4-c_1}\oplus 6^{c-c_6}}_{4^a\oplus 5^b\oplus 6^c, 1^d\oplus 2^e\oplus 4^f}$$

is

$$(a+b)(c+f) - fc_1 - ac_6.$$

If we want to calculate the degree in homological terms, we deduce

Corollary 3.2. *If $U = 1^d \oplus 2^e \oplus 4^f$, $W = 4^a \oplus 5^b \oplus 6^c$ and $V = 1^{d-c_1} \oplus 2^{e-c_4-c_6} \oplus 3^{c_1+c_4+c_6} \oplus 4^{a+f+c_4} \oplus 5^{b-c_4-c_1} \oplus 6^{c-c_6}$ and the stipulated conditions are the same as in the theorem, then the degree of ϕ^V_{WU} is*

$$[U,W] + [U,V]^1 - [U,U]^1 + [V,W]^1 - [W,W]^1.$$

The analysis of the algorithm leads to

Theorem 3.3. *Let $\vec{Q}$ be a quiver with underlying graph of type $\mathbb{A}_n$ and U, V and W be $k\vec{Q}$-modules. Then for*

$$\phi^V_{WU}(q) = \sum_{r\geq 0} \phi_r (q-1)^r,$$

all coefficients ϕ_r are non-negative.

Proof. This is clear since in every step of the algorithm each polynomial which occurs has this property. □

Theorem 3.4. *Let $\vec{Q}$ be a quiver with underlying graph of type $\mathbb{D}_n$ and U, V and W be $k\vec{Q}$-modules. Then for*

$$\phi^V_{WU}(q) = \sum_{r\geq 0} \phi_r (q-2)^r,$$

all coefficients ϕ_r are non-negative.

Theorem 3.5. *Let $\vec{Q}$ be a quiver with underlying graph of type $\vec{\mathbb{E}}_6, \vec{\mathbb{E}}_7, \vec{\mathbb{E}}_8$ and U, V and W be $k\vec{Q}$-modules. Then for*

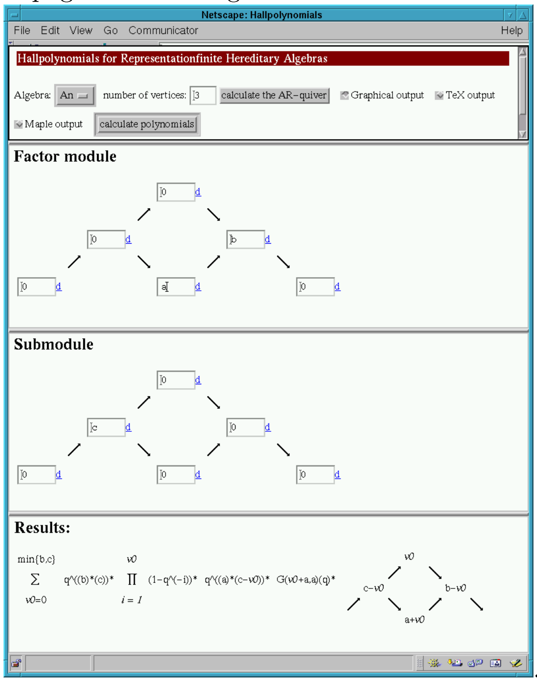
$$\phi^V_{WU}(q) = \sum_{r\geq 0} \phi_r (q-2)^r$$

all coefficients ϕ_r are non-negative.

PROOF. Our task is to check whether $\phi^Z_{Y X}$ has non-negative coefficients if evaluated in powers of $q-2$ and if X and Y are indecomposable. C. M. Ringel has done this in the cases $\mathbb{E}_6$ and $\mathbb{E}_7$. R. Nörenberg wrote a computer program which proves this theorem also in the case $\mathbb{E}_8$. □

2. The Implementation

A program to calculate Hall polynomials in the case $\mathcal{A} = k\vec{\mathbb{A}}_n$ is available via the world wide web on page:
`http://www.mathematik.uni-bielefeld.de/birep/hall`.
By entering the page one should get a window which looks like this:



.

In the upper frame one can choose the number of points of $\mathbb{A}_n$. The orientation of $\mathbb{A}_n$ is always

$$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ\ .$$

There is a button to calculate the Auslander-Reiten quiver. This is done on the client-side with Java Script. In the second frame from the top one can input the factor module W, in the third frame the submodule U. When the button `calculate Hall polynomials` is pressed the data is sent to the server, where $W \diamond U$ is calculated. The result is displayed in the bottom frame. In the following we will describe the program on the server. Therefore we will give the synopsis of all procedures and data-structures. The header files are

(1) `debug.h`,
(2) `list.h`,
(3) `module.h`
(4) `ptmodule.h`,
(5) `character.h`,
(6) `factor.h`,
(7) `html.h`,
(8) `diamanten.h`,
(9) `calculate.h`,

and the program files are

(1) `calculate_An.cpp`,
(2) `character.cpp`,
(3) `diamanten.cpp`,
(4) `html.cpp`,
(5) `ptmodule.cpp`,
(6) `calculate_Dn.cpp`,
(7) `debug.cpp`,
(8) `factor.cpp`,
(9) `module.cpp`.

The program is written in `C++` and is compilable with the `gcc` version 2.95.2. and the `libc5` version 1.9.9. To make things easier there is a `Makefile` with the rules

`dep:` to make the dependencies
`clean:` to delete the object-files, the program-file and the dependencies.
`build:` this is a short form for `make dep, make clean, make all`.
`all:` this is the default. `make all` builds the object-files and links them.

Usually one only types `make`.

2.1. The file `list.h`. This file supplies us with a template for building doubly linked lists.

```
template <class ELEMENT>
   class Element;
```

This is the base class for elements of a doubly linked list.

```
template <class ELEMENT>
   class List;
```

This is a template for a doubly linked list. Each element must be a class with base class `Element`. Of course, there is a constructor and a destructor for a whole list.

```
template <class ELEMENT>
   List<ELEMENT>::List (ELEMENT *elem)
```

Create a new list of elements with first element `elem`.

```
template <class ELEMENT>
   ELEMENT* List<ELEMENT>::element ( int n )
```

Return the `n`-th `ELEMENT` of the list, where the enumeration is starting at zero.

```
template <class ELEMENT>
   void List<ELEMENT>::insert ( int n , ELEMENT *elem)
```

Insert the element at which `elem` is pointing after position `n` in the list. if `n` = -1 the new element will inserted on top of the list.

```
template <class ELEMENT>
   void List<ELEMENT>::insert ( int n , ELEMENT& elem)
```

The same as above, but `elem` is a reference to an element.

```
template <class ELEMENT>
   void List<ELEMENT>::insert( ELEMENT *elem1, ELEMENT *elem2 )
```

This is a very intrinsic function. One should use it only with care. `elem1` should point to an element of the list. `elem2` could be an arbitrary element. This procedure allocates memory space and copies `elem2` to this new space and inserts this new element in the list behind `elem1`.

```
template <class ELEMENT>
   void List<ELEMENT>::swap ( ELEMENT *elem1, ELEMENT *elem2 )
```

The elements `elem1` and `elem2` in the list are interchanged. This is done by changing the pointers inside the list. Be careful that `elem1` and `elem2` belong to the same list, since only the pointers of the doubly linked list are interchanged. Otherwise strange things may happen.

```
template <class ELEMENT>
   void List<ELEMENT>::swap ( int i )
```

The elements at positions `i` and `i+1` are interchanged. The enumeration starts at zero.

```
template <class ELEMENT>
   ELEMENT* List<ELEMENT>::nebe( ELEMENT* elem, int n )
```

If `n` is positive, the `n`-th element behind `*elem` is returned. If `n` is negative, the `n`-th element before `*elem` is returned.

```
template <class ELEMENT>
   ELEMENT* List<ELEMENT>::nebe( ELEMENT& elem, int n )
```

The same as above, but `elem` is a reference to an element.

```
template <class ELEMENT>
   void List<ELEMENT>::wipeout ( int n )
```

Delete the n-th element of the list. The enumeration starts at zero.

```
template <class ELEMENT>
   void List<ELEMENT>::wipeout ( ELEMENT* elem )
```

Delete the element `*elem` from the list.

```
template <class ELEMENT>
   void List<ELEMENT>::wipeout ( ELEMENT& elem )
```

The same as above, but `elem` is a reference.

```
template <class ELEMENT>
   ostream &operator<<(ostream &stream , List<ELEMENT>* list)
```

The whole list is printed. The elements are separated by `char *list->septor`. After the output `list->septor` is set to `EMPTY` which is "".

```
template <class ELEMENT>
   ostream &operator<<(ostream &stream , List<ELEMENT>& list)
```

The same as above, but `list` is a reference.

```
template <class ELEMENT>
   void List<ELEMENT>::operator=(List<ELEMENT> &list)
```

This is the `=` operator for a list. First the left-hand side is deleted and then the right-hand side is copied. Be careful that also pointers are only copied, but not the memory they are pointing to. That means that one gets a new list, but if an element points to some memory space, both lists are pointing to the same memory space.

2.2. The files `ptmodule.h` and `module.h`. Here are the following two types of lists declared. A list of modules and a list of pointers to modules. A module is a class with the following data-structure.

```
class Module : private Element<Module> {

public:

  int x, y;
  int value;
```

```
  char name[MAXLENGTH+1];
  Module *tau, *tauminus;
  PtModuleList succ, pred;
  Module();
  ~Module();
  void operator=(Module &ch);
  friend ostream &operator<<(ostream &stream, Module* mod);
  friend ostream &operator<<(ostream &stream, Module& mod);
  friend class List<Module>;
}
```

x and y are the coordinates of the module in the Auslander-Reiten quiver. value is for storing dimensions of homomorphism spaces. char* name stores the name of the module. This is for the communication with the CGI-interface. tau points to the τ-translation and tau-minus to the τ^{-1}-translation of this module in the Auslander-Reiten quiver. succ is the list of pointers to the direct successors of the module in the Auslander-Reiten quiver and pred is the list of its predecessor. Finally there the = and << operators.

The list of pointers to modules is declared in ptmodule.h. It is

```
class PtModule : private Element<PtModule> {

public:

  Module* module;
  PtModule();
  PtModule(Module* mod);
  ~PtModule();
  void operator=(PtModule &mod);
  friend ostream &operator<<(ostream &stream, PtModule* mod);
  friend ostream &operator<<(ostream &stream, PtModule& mod);
  friend class List<PtModule>;
}
```

The data of this element is module, which is a pointer to a module. In order to get access to the list there are the type definitions

```
typedef List<PtModule> PtModuleList
typedef List<Module> ModuleList
```

2.3. The file character.h. To handle arbitrary strings we build a list of characters. There is

```
class Character : private Element<Character> {

public:

  char c;

  Character();
  Character(char ch);
  ~Character();
  void operator=(Character &ch);
  friend ostream &operator<<(ostream &stream, Character *ch);
  friend ostream &operator<<(ostream &stream, Character& ch);
  friend class List<Character>;
```

```
}
```

and the type definition

```
typedef List<Character> CharacterList;
```

> `char c` contains the character. Here the `=` operator is not very useful to set a CharacterList. This can better be handled by the procedure `insert_string` (see below).

2.4. The file `factor.h`. In a product $Y^n \diamond X^m$ a factor Y^n or X^m is stored in the data-structure

```
class Factor : private Element<Factor> {

 public:

  Module *base;
  CharacterList expo;
  int expoint;

  Factor();
  ~Factor();

  void operator=( Factor &fac );
  int operator<( Factor &fac );

  int operator==( Factor &fac );

  int operator<=( Factor &fac );

  friend ostream &operator<<(ostream &stream, Factor* fac);
  friend ostream &operator<<(ostream &stream, Factor& fac);
  friend class List<Factor>;
}
```

> The base in our example X is a module which is stored in `*base`. The exponent `expo` is a list of characters or an integer `expoint`.

2.5. The file `html.h`. Since the program can make three kinds of output (graphically, LaTeX or Maple), we have the datatype

```
enum OutputShape { graphical , tex , maple };
```

and the type of the algebra is stored in

```
enum Type { An , Dn , E6 , E7 , E8 };
```

In this file the following procedures are defined

```
void read_from_cgi(ModuleList &quiver,
                   FactorList &fac,
                   FactorList &sub,
                   Type &type,
                   OutputShape &output_shape);
```

> This procedure reads from `STDIN`, which is usually the output of a `CGI`. The data is a quiver which is a list of modules. Since we want to calculate $Y^n \diamond X^m$, the factor Y^n is stored in `fac` and X^m is stored in `sub`. The type of the algebra is stored in `type`. Up to now this program calculates only $\mathbb{A}_n$ cases, but any contributions are welcome. The source is available for extensions. The type of the output the user wants is stored in `output_shape`.

```
Module* find_module(ModuleList& quiver,
                    int x,
                    int y);
```

This procedure returns the pointer to a module in a `quiver` with the coordinate `x` and `y`.

```
void error ();
```

If something in the input stream is strange (maybe someone wants to attack the server), only a very short error message is sent. I do not want to give someone hints how to attack my server.

```
void draw_quiver(ModuleList &quiver,
                 FactorList &fac ,
                 OutputShape &output_shape);
```

Print the Auslander-Reiten quiver and `fac` for the modules.

2.6. The files `diamanten.h` and `calculate.h`. These are the main files.

```
int  main (void)
```

In `main` the main variables are initialized. These are `ModuleList quiver` to store the Auslander-Reiten quiver, `FactorList sub` and `FactorList fac` to store the factors of the product, `Type type` to store the type of the algebra, `CharacterList polynom` to store the polynomials and `OutputShape output_shape` to store the type of the output.

Then the data is read by the procedure `read_from_cgi` and the calculation is started via the procedure `calculate`.

```
void hammock (ModuleList &quiver ,
              int x1 , int y1 , int x2 , int y2)
```

This procedure calculates the dimension of the homomorphism spaces starting at the module with the coordinates `(x1,y1)`. If the module with the coordinates `(x2,y2)` is reached, the calculation stops. The dimensions are stored in `module.value` for each `module` in `quiver`.

```
int calculate (ModuleList &quiver , FactorList &fac ,
               Type &type , CharacterList &polynom ,
               OutputShape &output_shape)
```

This procedure calls the different calculating procedures depending on the `type` of the algebra.

```
void calculate_An (ModuleList &quiver ,
                   FactorList &fac ,
                   CharacterList &polynom ,
                   OutputShape &output_shape)
```

This procedure calculates the product with the factor in `fac` and returns the polynomial in `CharacterList polynom`.

```
void calculate_Dn (ModuleList &quiver ,
                   FactorList &fac ,
                   CharacterList &polynom ,
                   OutputShape &output_shape)
```

This is just a template. This procedure is not yet implemented.

```
void insert_string (CharacterList &clist ,
                    char *string ,int position)
```

Insert the string `string` at position `position` in the list of characters `clist`.

`int min (Factor &f1, Factor &f2)`

Calculate the minimum of the exponents of the two factors `f1` and `f2`. If the result is -1, at least one of the factors has an exponent which is not an integer.

List of Symbols and Terminology

	Dynkin algebra	5
	Dynkin case	7
	Dynkin type	5
	Hall polynomials in the classical case	7
	Schur functions	7
	dimension	8
	directing	9
	disjoint	13
	equivalent short exact sequences	9
finite	of finite cardinality	5
	finitary algebra	7
	homomorphism	8
	integral Hall algebra	5
	isotoypical (of type X)	35
	isotoypical component	35
	middle-term equivalent	9
module	a finite left $\mathcal{A}$-module	5
	positive root	8
	product sequences	35
quiver	a tuple (Q_0, Q_1)	5
	radical	8
	representation directed	9
	representation finite	9
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	the classical case	7
	type of a finite abelian group	7
$([X_n])_n$	familiy of isomorphism classes of modules	9
$(\vartheta_i)_i$	a partition	7
1_V	the identity map of a module V	10
$<$	an order for modules	35
D	$\mathrm{Hom}_k(-, k)$	9
E	a short exact sequence	
F	the class of all modules	5
$H \subseteq G$	finite ablelian p-groups	7
$K(\mathcal{A}\text{-}\mathrm{mod})$	a subalgebra of $\mathcal{H}(\mathcal{A})_1$	8
$K_0(\mathcal{A})$	the Grothendiek group of $\mathcal{A}$	5
Q	a Dynkin graph	5
S_i	a simple module	7
V_α	an arbitrary module	5
V^t	direct sum of t copies of V	5
$X_\varrho(k)$	see X_n	5
$[E]$	the equivalence class of E	9

$[V, W]$	the dimension of $\mathrm{Hom}(V, W)$	8
$[V]$	the isomorphism class of a module V	5
$[V]^{\diamond t}$	t-fold diamond product of $[V]$	9
$[W, U]^1$	the dimension of $\mathrm{Ext}(W, U)$	9
$\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8}$	the Dynkin diagrams	5
$\mathcal{A}be(\mathcal{M})$	abelian extension-closed subcategory	6,14
$\mathcal{A}$	the path algebra $k\vec{Q}$	5
$\mathcal{A}$- mod	the category finitely generated modules	8
$\mathcal{B}$	a representation directed algebra	9
$\mathrm{Ext}(W, U)$	the extension group of W by U	9
$\mathrm{Hom}(V, W)$	the vectorspace of homomorphisms $f, g, h : V \to W$	8
$\mathrm{Hom}_{\mathcal{A}}(V, W)$	the vectorspace of $\mathcal{A}$-linear maps $f : V \to W$	8
$\mathbb{N}$	the set $\{1, 2, 3, \dots\}$	
Φ^+	the set of positive roots	8
α, β, γ	maps $\Phi+ \to \mathbb{N}_0$	5
$\mathcal{H}(\mathcal{A})$	the Hall algebra	5
$\mathcal{H}(\mathcal{A})_1$	$\mathcal{H}(\mathcal{A})$ evaluated at 1	8
$\left\lvert {s \atop r} \right]$	$\frac{\lvert s]!}{\lvert r]!\lvert s-r]!}$	13
χ	Euler quatric form	7
$\diamond$	the multiplication sign in $\mathcal{H}(\mathcal{A})$	5
$\underline{\mathrm{dim}}V$	the dimension vector of V	5
$\mathbf{g}$	semisimple complex Lie algebra	5
ι	the type of H	7
κ	the type of G/H	7
$\langle E \rangle$	the middle term equivalence class of E	9
$\langle WU \mathbin{\diamond\!\!\!\mid} V \rangle$	the structure constants for $\mathcal{H}(\mathcal{A})$	5
$\langle s \rangle!$	$\langle 1 \rangle \cdot \langle 2 \rangle \cdot \dots \cdot \langle s \rangle$	22
$\langle s \rangle$	${h_Z}^s - {h_Y}^s$	22
$\left\langle {s \atop r} \right\rangle$	$\frac{\langle s \rangle!}{\langle r \rangle! \langle s-r \rangle!}$	22
$\mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$	a triangular decompositon of $\mathbf{g}$	8
$\overline{T}$	a reduced sequence	35
ϕ^V_{UW}	a Hall polynomial	5
$\phi^\beta_{\alpha\gamma}$	a Hall polynomial	5
$\mathbf{q}$	Tits form	8
$\mathrm{rad}(V, W)$	the non invertible homomorphism of $\mathrm{Hom}(V, W)$	8
τ	the Auslander-Reiten translation	9
$\vec{Q}$	a Dynkin quiver	5
$a, b, \dots \in k$	a finite field with elements $a, b, \dots$	5
$e_T(T_i)$	entropy of an isotypical module T_i	35
$e_T(X_i)$	entropy of the isotypical type X_i	35
$g^\vartheta_{\kappa\iota}(p)$	number of filtrations or Hall polanomial	7
$i, j \in Q_0$	a set of vertices with elements i and j	5
$i \to j \in Q_1$	a set of arrows with an element $i \to j$	5
$n(X_n)$	the largest index, whereby $T_{n(X_n)}$ is of type X_n	35
p	a prime number	7
q	the cardinality of k	5
$r, s, t \in \mathbb{N}_0$	the set $\{0, 1, 2, 3, \dots\}$ with elements r, s, t	8
$\lvert s]!$	$\lvert 1] \cdot \lvert 2] \cdot \dots \cdot \lvert s]$	13
$\lvert s]$	$q^{s-1} + \dots + q + 1$	13

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