Algebraic Computations in Derived Categories

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1 Introduction

Recently, the study of derived categories and related algebraic structures has received much attention. Nowadays they form an important part of the standard toolkit in algebra, algebraic geometry, topology and other fields. Although much is known about the structure of derived categories, in general it remains a difficult problem to carry out specific calculations in a given derived category.

For representation theorists derived categories have gained importance since the discovery of their relation to tilting theory. It is natural to classify algebras up to some equivalence relation; for example, the notion of MORITA equivalence characterizes algebras by their module categories. Derived categories allow one to introduce another kind of equivalence relation on algebras over a fixed field: we call two such algebras derived equivalent if they have equivalent derived categories (as triangulated categories).

In [Ha] D. HAPPEL realized that the derived category of an algebra of finite global dimension is invariant under tilting. J. RICKARD was able to generalize this result in [Rc] and to give a necessary and sufficient condition for derived equivalences to occur. He introduced the notion of a tilting complex, a bounded chain complex of projective modules satisfying various conditions, and showed that if T is a tilting complex over an algebra A then the endomorphism algebra of T is derived equivalent to A.

These results emphasize structural aspects of the theory. Thus we felt a growing need to provide tools for concrete "numerical" calculations. The aim of this thesis (and the attached MuPAD[1] library pathalg[2]) is to describe the theoretical background and the implementation of such tools.

The computer provides a fast and reliable way to check examples in a short time. This helps to develop a good intuition and allows the researcher to support or contradict conjectures. The library pathalg was developed with regard to the study of tilting complexes in a derived category and their endomorphism algebras. Of course, it also can be used as a tool for computations in the module category itself.

The algebras we deal with are assumed to be finite dimensional factors of a path algebra. Based on the concept of noncommutative Gröbner bases we present an algorithm for the construction of a projective resolution of a module. Such resolutions allow us to treat modules as objects of the homotopy category of complexes of projectives which is a full subcategory of the derived category.

If the endomorphism algebra of a tilting complex is again a factor of a path algebra modulo a so-called admissible ideal we are interested in the construction of the quiver with relations associated to it. We develop a general algorithm for finding the quiver with relations of a finite spectroid given by its radical spaces and show how it can be applied to tilting complexes or, more generally, to a finite subspectroid of the homotopy category. In order to do so, we provide various methods for dealing with chain complexes and their morphism spaces in the homotopy category.

The base fields we consider are finite fields or the field of rational numbers because these fields allow for exact computations without rounding errors. Unfortunately, this means we cannot compute examples with an algebraically

[1] "The Open Computer Algebra System", cf [Mu1], [Mu2] and `www.mupad.de`

[2] The library is available at `www.mathematik.uni-bielefeld.de/~akrause/PATHALG`.

closed base field. However, this does not need to be a restriction since we will prove that it is possible to transfer results found by pathalg over a certain ground field to any field extension. In particular, our result states that if we have found a quiver with relations of the endomorphism algebra of a tilting complex, extension of the base field neither changes the quiver nor affects the relations.

We will fix the notations and basic definitions used throughout this paper in chapter 2 and recall some important facts about module categories and their derived categories.

A brief introduction to the theory of noncommutative Gröbner bases is given in chapter 3. Chapter 4 is devoted to the applications of Gröbner bases, the most important in this context being the construction of projective resolutions.

Then we turn to algebraic computations in the category of chain complexes and in the corresponding homotopy category. In chapter 5 it is described how we can determine the morphism space of two complexes in those categories.

An algorithm for finding the radical of a matrix algebra due to Cohen, Ivanyos and Wales in [CIW] is presented in chapter 6. We show how this method can be adapted for our purpose.

By then we have collected all necessary tools to complete in chapter 7 the algorithm for finding the quiver with relations of the endomorphism algebra of a bounded complex in the homotopy category.

In chapter 8 we are concerned with the effect the extension of the base field of an algebra may have on the quiver with relations.

In the last chapter we give an account of the implementation of the presented methods. The author's library package pathalg for MuPAD is used to give a new proof of results in the representation theory of groups due to Holm [Ho] and Asashiba [As].

For further documentation of the library we refer the reader to the pathalg manual [Kr].

2 Preliminaries

2.1 Modules and chain complexes

Let k be an arbitrary field. In the following an algebra A is always a finite dimensional factor of a path algebra over k, i.e., A is isomorphic to kQ/I for some finite quiver Q and an admissible ideal I of kQ. If $I = \{0\}$ then A is called *hereditary*. We call (Q, I) the *quiver with relations of A*. The set of points of a quiver Q will be denoted by Q_0 and the set of arrows by Q_1.

We can view A also as finite k-spectroid (here, we denote the set of objects in A as A_0). By definition, a k-spectroid is a k-category such that no two distinct objects are isomorphic, the morphism spaces are finite dimensional k-vector spaces and the endomorphism rings of all objects are local.

Given a finite quiver Q, the k-category kQ has as objects the points of Q; a basis of a morphism space $kQ(x, y)$ is given by the set of all paths from x to y in Q (including the paths of length zero). If I is an ideal of kQ, the k-category kQ/I has the same objects as kQ and, for $x, y \in Q_0$, the space of morphisms $x \to y$ is the factor space $kQ(x, y)/I(x, y)$. If I is admissible, then kQ/I is a spectroid.

The space of non-invertible morphisms $x \to y$, where x and y are objects of A, is called the (Jacobson) *radical* of $A(x, y)$ and will be denoted by $\operatorname{Rad} A(x, y)$.

In the categorical context, left A-modules are covariant functors from A to k-mod, the category of finite dimensional k-vector spaces. Here and in the following, an A-module will always be a finite dimensional left module over A. The category of finite dimensional left A-modules is denoted by A-mod.

The indecomposable projective modules over $A = kQ/I$ are the representable functors. Hence they are in bijective correspondence with the objects of A (as a spectroid). We denote an indecomposable projective by $P_x := A(x, -)$ for $x \in A_0$. Analogously we write $I_x := DA(-, x)$ for an indecomposable injective module corresponding to $x \in A_0$ (where $D := \operatorname{Hom}_k(-, k)$ is the usual duality functor mod-$A \to A$-mod).

Throughout this paper a *chain complex* C of A-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of A-modules together with module morphisms $d_n^C = d_n : C_n \to C_{n+1}$ such that $d_n d_{n-1} = 0$. This is usually written as

$$C = \cdots \longrightarrow C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \longrightarrow \cdots.$$

A chain complex C is called *bounded* if almost all C_n are zero. In this case, the *bounds* of $C \neq 0$ are defined as the pair (i, j) where i is the smallest integer such that C_i is non-zero and j is the largest integer such that C_j is non-zero. For convenience, the bounds of the zero complex are defined to be $(0, 0)$.

For an interval $[i, j] \subset \mathbb{Z}$ and a complex C we define the *truncated complex* $C_{[i,j]}$ to be the complex with $(C_{[i,j]})_n := C_n$ if $n \in [i, j]$ and $(C_{[i,j]})_n := 0$ otherwise; the differential in degree n is given by d_n^C for $n \in [i, j-1]$ and zero otherwise.

A morphism of chain complexes or, shorter, a *chain map* $f : C \longrightarrow D$ is a sequence of module homomorphisms $(f_n)_{n\in\mathbb{Z}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"'] & C_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & \cdots \\
\cdots \arrow[r] & D_{n-1} \arrow[r, "d^D_{n-1}"] & D_n \arrow[r, "d^D_n"] & D_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Denote the category of bounded chain complexes of A-modules by $\mathcal{C}^b(A\text{-mod})$. The *shift functor* or *translation functor* is an automorphism of $\mathcal{C}^b(A\text{-mod})$ and maps a chain complex C to the shifted complex $C[1]$ with entries $C[1]_n := C_{n-1}$ and differential $d^{C[1]} := -d^C$.

A chain map $f : C \longrightarrow D$ is called *null homotopic* if there is a sequence of morphisms $s_n : C_n \longrightarrow D_{n+1}$ such that $f = ds + sd$. Two chain maps f and g are *homotopic* if $f - g$ is null homotopic. The factor category of $\mathcal{C}^b(A\text{-mod})$ modulo chain homotopy is denoted by $\mathcal{K}^b(A\text{-mod})$. For details we refer the reader to [We, 10.1].

For a chain complex C we define the homology modules of C as the subquotients $H_n(C) := \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$ for $n \in \mathbb{Z}$. A morphism of chain complexes is called a *quasi-isomorphism* if it induces an isomorphism on the homology modules.

2.2 Derived categories

The *bounded derived category* $\mathcal{D}^b(A)$ of chain complexes of A-modules is given by the localization of $\mathcal{K}^b(A\text{-mod})$ with respect to the set of quasi-isomorphisms. Together with the shift functor it is equipped with the structure of a triangulated category.

Note that A-mod is a full subcategory of $\mathcal{D}^b(A)$ if we consider an A-module as a complex concentrated in degree zero.

Consider chain complexes over A-proj, the category of finite dimensional projective modules over A. These chain complexes will be called *complexes of projectives*. They form the full subcategory $\mathcal{C}^b(A\text{-proj})$ of $\mathcal{C}^b(A\text{-mod})$. Its homotopy category is denoted by $\mathcal{K}^b(A\text{-proj})$ which is a full subcategory of $\mathcal{D}^b(A)$ (cf [We, 10.4]). In a special case the inclusion becomes an equivalence:

2.2.1 Theorem. *[We, 10.4.8] Let A be an algebra of finite global dimension. Then the composition of functors*

$$\mathcal{K}^b(A\text{-proj}) \longrightarrow \mathcal{K}^b(A\text{-mod}) \longrightarrow \mathcal{D}^b(A)$$

is an equivalence.

Let M be a module in $\mathcal{D}^b(A)$, that is, a complex concentrated in degree zero. The image of M under an inverse of this equivalence is a projective resolution of M considered as a complex of projectives in $\mathcal{K}^b(A\text{-proj})$.

For arbitrary complexes in $\mathcal{D}^b(A)$ there exists an analogous concept to the projective resolution which is called Cartan-Eilenberg *resolution*. It yields a complex of projectives which is quasi-isomorphic to the resolved one. Confer [We, 5.7] for details.

2.3 Tilting modules and tilting complexes

We say that two algebras A and B are *derived equivalent* if the derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

2.3.1 Definition. Let T be an A-module. Then T is called a *tilting module* if it satisfies the following conditions:

(a) $\operatorname{pdim} T \leq 1$,

(b) $\operatorname{Ext}^1(T, T) = 0$,

(c) for any indecomposable projective A-module P there exists an exact sequence $0 \to P \to T' \to T'' \to 0$ where T', T'' are direct summands of finite direct sums of copies of T.

The set of summands of finite direct sums of copies of an object T in A-mod or $\mathcal{D}^b(A)$ will be called $\operatorname{add} T$.

Let $T = \bigoplus_{i=1}^n T_i$ be a tilting module where T_i is indecomposable for all $i \in \{1, \dots, n\}$ and $T_i \ncong T_j$ for $i \neq j$. Then we can also consider T as a *tilting spectroid* with objects $T_1, \dots, T_n$. The set of morphisms $T_i \to T_j$ is given by $\operatorname{Hom}_A(T_i, T_j)$.

2.3.2 Definition. Let C be a complex in $\mathcal{K}^b(A\text{-proj}) \hookrightarrow \mathcal{D}^b(A)$. Then C is called a *tilting complex* or *tilting object* if

(i) $\mathcal{D}^b(A)(C, C[i]) = 0$ for all $i \neq 0$ and

(ii) $\operatorname{add} C$ generates $\mathcal{D}^b(A\text{-mod})$ as a triangulated category.

Note that a tilting module is a special case of a tilting object. If M is a tilting module in A-mod and P is a minimal projective resolution of M, then P is a tilting complex in $\mathcal{K}^b(A\text{-proj})$: condition (a) on a tilting module ensures that P is bounded; conditions (a) and (b) together imply condition (i) in the definition of a tilting complex, and condition (ii) follows from (c) in the definition of a tilting module.

Tilting objects were introduced by Rickard in [Rc] to establish an analogue to the Morita theory of module categories. He showed the following result:

2.3.3 Theorem. *[Rc] Let A be an algebra and $\mathcal{D}^b(A)$ its derived category. If T is a tilting complex in $\mathcal{D}^b(A)$, then the opposite endomorphism algebra $\operatorname{End}(T)^{op}$ of T is derived equivalent to A.*

3 Noncommutative Gröbner Bases

3.1 Multiplicative bases and admissible orders

Let A be a finite dimensional k-algebra and let B be a k-basis of A. We call B a *multiplicative basis* if $b, b' \in B$ implies $bb' \in B$ or $bb' = 0$ under the multiplication in A.

Note that if A is a path algebra, say $A = kQ$ for some quiver Q, then the set of all paths of Q (including the trivial paths, corresponding to the points of Q) form a multiplicative k-basis for A.

Now we assume that B can be equipped with a well order $<$. We will call such an order *admissible* if it is compatible with the product in B, i.e., if it satisfies the following conditions ($p, q, r, s \in B$):

(a) If $p < q$ then $pr < qr$ if both $pr \neq 0$ and $qr \neq 0$.

(b) If $p < q$ then $sp < sq$ if both $sp \neq 0$ and $sq \neq 0$.

(c) If $p = qr$ then $q < p$ and $r < p$.

For example, the *length-lexicographic order* on the set B of all paths of a finite quiver is an admissible order. Let $l(p)$ be the length of a path in B, i.e., the number of arrows in p. An order on the elements of length ≤ 1 then induces an order on longer paths $p, q \in B$ in the following way: We define $p < q$ if $l(p) < l(q)$ or, if $l(p) = l(q)$, then $p < q$ if p is lexicographically less than q.

In the following, let B be a k-basis of A and let $<$ be an admissible order on B. Hence each element in A has a unique *ordered* representation as a linear combination of elements in B.

3.1.1 Definition. Let $0 \neq a \in A$ and $a = \sum_{j=1}^{n} \lambda_j b_j$ ($\lambda_j \in k$, $b_j \in B$). A basis element $b_i \in B$ is called the *tip of a* if $\lambda_i \neq 0$ and $b_j \leq b_i$ for all b_j with non-zero coefficient λ_j. We write $Tip(a) = b_i$ and $CTip(a) = \lambda_i$. If X is a subset of A we define

$$\mathrm{Tip}(X) = \{b \in B \mid b = \mathrm{Tip}(x) \text{ for some non-zero } x \in X\}$$

and

$$\mathrm{NonTip}(X) = B \setminus \mathrm{Tip}(X).$$

With the set of nontips of an ideal I we get a complete description of A/I as a k-vector space.

3.1.2 Theorem. *Let I be an ideal of A. Then*

$$A = I \oplus span(NonTip(I))$$

as a k-vector space.

Proof. cf [Gr, 2.1]. □

In particular, every non-zero element $a \in A$ can be written uniquely as a sum $a = i_r + N(a)$, where $i_r \in I$ and $N(a) \in \mathrm{span}(\mathrm{NonTip}(I))$. We call $N(a)$ the *normal form of a*.

This provides us with a "nice" representative for each element $f + I$ of the quotient A/I.

3.1.3 Proposition. *[Gr, 2.9] Let f, g be elements of A.*

(a) *$f + I = g + I$ if and only if $N(f) = N(g)$.*

(b) *$f + I = N(f) + I$*

(c) *The map $\sigma : A/I \to A$ with $\sigma(f + I) = N(f)$ is a vector space splitting of the canonical surjection $\pi : A \to A/I$.*

(d) *σ is a k-linear isomorphism between A/I and span(NonTip(I)).*

(e) *Identifying A/I with span(NonTip(I)), then NonTip(I) is a k-basis of A/I contained in B.*

Proof. The first two parts and (e) are immediate consequences of theorem 3.1.2 above. The remaining parts can be verified easily. □

3.1.4 Remark. We have $N(f)+N(g) = N(f+g)$, since σ is a linear map and span(NonTip(I)) is a linear subspace. Moreover, the multiplication of elements in A/I is given by $N(f \cdot g) = N(N(f) \cdot N(g))$, since $N(f) \in f + I$ by part (b) of the proposition. In particular, this means that $N(N(f)) = N(f)$.

3.2 The division algorithm

Now we will turn to the question how the normal form of an element of A can be computed. In the following we assume that B is a k-basis of A with admissible order $<$.

We say that $b \in B$ *occurs* in $a = \sum_{i=1}^{n} \lambda_i b_i$ if $b = b_i$ with $\lambda_i \neq 0$ for some i. An element $b \in B$ *divides* $b' \in B$ if there exist $p, q \in B$ such that $pbq = b'$.

Given an ordered subset $X = \{x_1, \ldots, x_n\}$ of A and an element $y \in A$ we want to find nonnegative integers $m_1, \ldots, m_n$ and elements $u_{i,j}, v_{ij}, r \in A$ for $i \leq i \leq n$ and $1 \leq j \leq m_i$ such that

(a) $y = \sum_{i=1}^{n} (\sum_{j=1}^{m_i} u_{ij} x_i v_{ij}) + r$.

(b) $\mathrm{Tip}(u_{ij} x_i v_{ij}) \leq \mathrm{Tip}(y)$ for all i and j.

(c) For $b \in B$ occurring in r, $\mathrm{Tip}(x_i)$ does not divide b for $1 \leq i \leq n$.

Note that condition (c) implies $\mathrm{Tip}(r) \leq \mathrm{Tip}(y)$. We will call r a *remainder of division by $x_1, \ldots, x_n$.*

The division algorithm works as follows (for a presentation in pseudo-code confer fig. 1):

3.2.1 Algorithm. (Division algorithm) The input arguments are non-zero elements $x_1, \ldots, x_n$, y of A. We compute the remainder of the division of y by $X := \{x_1, \ldots, x_n\}$.

Initialize $r := 0$ and $z := y$. We try successively for $i = 1, \ldots, n$ if there exist $u, v \in B$ such that $\mathrm{Tip}(z) = u\mathrm{Tip}(x_i)v$. If not, then z is replaced by $z - \mathrm{Tip}(y)$ and $\mathrm{Tip}(z)$ is added to the remainder r. If $\mathrm{Tip}(z) = u\mathrm{Tip}(x_i)v$ then we replace z by $z - \frac{\mathrm{CTip}(z)}{\mathrm{CTip}(x_i)} \cdot u x_i v$. In both cases, if the replaced z is still non-zero, we continue the division procedure with an element with smaller tip. Since $<$ is a well order and we divide by a finite set, the procedure will stop eventually. Return the remainder r.

```
INPUT:  x[1], ..., x[n], y
OUTPUT: r

m[1]:= 0, ..., m[n]:= 0, r:= 0, z:= y
WHILE z <> 0 DO
  i:= 1
  divisionoccured:= FALSE
  WHILE i <= s AND NOT divisionoccured DO
    IF Tip(z) = u Tip(x[i]) v THEN
      m[i]:= m[i] + 1
      u[i, m[i]]:= (CTip(z)/CTip(x[i])) u
      v[i, m[i]]:= v
      z:= z - (CTip(z)/CTip(x[i])) u x[i] v
      divisionoccured:= TRUE
    ELSE
      i:= i + 1
    END
  END
  IF NOT divisionoccured THEN
    r:= r + CTip(z)Tip(z)
    z:= z - CTip(z)Tip(z)
  END
END
```

Figure 1: A division algorithm

A major drawback is that the remainder of the division does in fact depend on the order of the "divisor" set X, but one can show that this is not the case if X satisfies certain conditions. Thus we are led to introduce a special class of sets called Gröbner bases.

3.3 Gröbner bases

3.3.1 Definition. Let I be an ideal in A. We say that a set $G \subset I$ is a *Gröbner basis for I with respect to* $<$ if

$$\langle \mathrm{Tip}(G) \rangle = \langle \mathrm{Tip}(I) \rangle$$

as two-sided ideals of A.

Equivalently, G is a Gröbner basis if and only if for any $b \in \mathrm{Tip}(I)$ there exist $g \in G$ and $p, q \in B$ such that $p\mathrm{Tip}(g)q = b$, so $\mathrm{Tip}(g)$ divides b. It follows that we can use a Gröbner basis G of I to determine $\mathrm{NonTip}(I)$, since

$$\mathrm{NonTip}(I) = \{b \in B \mid \mathrm{Tip}(g) \text{ does not divide } b \text{ for any } g \in G\}$$

if G is a Gröbner basis of I. This is useful in particular if we can find a finite Gröbner basis.

Note that in general a Gröbner basis is not a k-basis of I. For example, the ideal I itself is always a Gröbner basis, but never a k-basis of I.

In the following we will make use of Gröbner bases to determine the normal form $N(a)$ of an element $a \in A$.

3.3.2 Proposition. *[Gr, 2.7] Let G be a Gröbner basis of an ideal I in A. Let $y \in A$ and let X be the set of $g \in G$ with Tip$(g) \leq$ Tip(y). Then the remainder r of the division by X is independent of the order in X. In fact, $r = N(y)$.*

Proof. Let r be the remainder of y of division by X. By the division algorithm, for each $g \in X$, $\mathrm{Tip}(g)$ does not divide any basis element occurring in r. Moreover, since $\mathrm{Tip}(r) \leq \mathrm{Tip}(y)$, we see that also for each $g \in G \setminus X$, $\mathrm{Tip}(g)$ does not divide any basis element occurring in r. Hence $r \in \mathrm{span}(\mathrm{NonTip}(I))$. Now $y = \sum_i \sum_j u_{ij} g_i v_{ij} + r$ for some $u_{ij}, v_{ij} \in B$, $g_i \in X$. On the other hand, theorem 3.1.2 implies that $y = i_y + N(y)$ with $i_y \in I$ and $N(y) \in \mathrm{span}(\mathrm{NonTip}(I))$ unique. But $\sum_i \sum_j u_{ij} g_i v_{ij} \in I$ and $r \in \mathrm{span}(\mathrm{NonTip}(I))$. Hence $r = N(y)$. □

So this proposition supplies us with an algorithm to compute the normal form of an element in A, provided that G is a Gröbner basis such that for each $b \in B$ there are only finitely many $g \in G$ with $\mathrm{Tip}(g) \leq \mathrm{Tip}(b)$.

3.3.3 Algorithm. (Normal form) Let $a \in A$ and let S be the set of elements $g \in G$ with $g \leq \mathrm{Tip}(a)$. If S is finite then $N(a)$ is the remainder of the division of a by S.

This leads to the question how we can find a Gröbner basis which ensures that the assumption on the divisor is satisfied. We will show that in our case, where we assume that A/I is finite dimensional over k, we can find even a finite Gröbner basis.

3.3.4 Definition. A Gröbner basis G of I is called *minimal* if the the following conditions hold:

(a) If $g \in G$ then $\mathrm{CTip}(g) = 1$.

(b) If $g \in G$ then $\mathrm{Tip}(g) \notin \langle \mathrm{Tip}(G \backslash \{g\}) \rangle$.

If additionally the following condition holds:

(c) if $g \in G$ then $g - \mathrm{Tip}(g) \in \mathrm{span}(\mathrm{NonTip}(I))$,

then G is called *reduced*.

A reduced Gröbner basis has the nice property that it is unique for a given ideal I.

So we aim at finding a finite Gröbner basis, for this gives us an algorithm to compute $\mathrm{NonTip}(I)$, which is a k-basis of A/I.

3.4 Finite Gröbner bases

Let X be a set of elements in A and I_X the ideal generated by X. We use the fact that if $x, x' \in X$ such that $\mathrm{Tip}(x) = u \cdot \mathrm{Tip}(x') \cdot v$ for some $u, v \in B$ then replacing x' by $x - \frac{\mathrm{CTip}(x)}{\mathrm{CTip}(x')} u x' v$ also yields a generating set of I_X where $\mathrm{Tip}(x')$

does not divide $\mathrm{Tip}(x)$. Continuing in this fashion, we obtain a generating set for I_X where no tip of a generator divides the tip of another generator. The process will stop since $<$ is a well order. The obtained set is called *tip reduced*.

So we can safely assume in the following that we always have a tip reduced generating set of an ideal.

3.4.1 Definition. We call an element $\sum_{i=1}^{n} \lambda_i b_i$ with $\lambda_i \in k^*$ and $b_i \in B$ *left uniform* if for each $c \in B$ either $cb_i = 0$ for all $i = 1, \dots, n$ or $cb_i \neq 0$ for all $i = 1, \dots, n$.

Similarly, we call an element $\sum_{i=1}^{n} \lambda_i b_i$ with $\lambda_i \in k^*$ and $b_i \in B$ *right uniform* if for each $c \in B$ either $b_i c = 0$ for all $i = 1, \dots, n$ or $b_i c \neq 0$ for all $i = 1, \dots, n$.

An element is called *uniform* if it is right and left uniform.

An element $a = \sum_{i=1}^{n} \lambda_i b_i$ of a path algebra kQ is uniform if and only if all b_i are paths in Q with a common starting point and a common end point.

3.4.2 Proposition. *Let I be a finitely generated ideal of a path algebra kQ. Then there exists a generating set for I consisting of uniform elements.*

Proof. Let $I = \langle x_1, \dots, x_r \rangle$. Then we can construct a uniform generating set for I in the following way:

Consider x_i for all $i \in \{1, \dots, r\}$. If x_i is uniform we define $S_i := \{x_i\}$. Otherwise x_i is a sum of paths which have different starting point or end point. Let v_s and v_t be two vertices in Q. Then $v_t x_i v_s$ is a uniform element, since all paths occurring must have starting point v_s and end point v_t. We define $S_i := \{v_t x_i v_s \neq 0 \mid v_s, v_t \in Q_0\}$.

We claim that I is generated by $S := \bigcup_{i=1}^{r} S_i$. Obviously S is a subset of I. On the other hand, for each generator x_i of I we have $x_i = \sum_{s \in S_i} s$ and thus I is a subset of $\langle S \rangle$. □

In the following we will always assume that an admissible ideal I of a path algebra kQ is given by a uniform generating set.

3.4.3 Theorem. *[Gr, 2.2] Let A be a finitely generated k-algebra with multiplicative basis and admissible order $<$. Suppose that I is an ideal generated by uniform elements and A/I is finite dimensional. Then I has a finite uniform Gröbner basis with respect to $<$.* □

If $A = kQ$ is a path algebra with finite quiver Q, and I is an admissible ideal of A, the conditions of the above theorem are satisfied and hence we know that a finite Gröbner basis exists.

Consequently, we are concerned with the construction of such a Gröbner basis. However, before we can turn to the description of an algorithm, we need to introduce the concept of overlap relations.

These arise from a fact we want to visualize by the following example: Even if we have a principal ideal $I = \langle g \rangle$ we cannot be sure that $\{g\}$ is a Gröbner basis. For example, let B be the semigroup generated by $\{a, b\}$ with the length-lexicographic order generated by $a < b$, and let I be the ideal generated by $\{aba + b\}$. Then there occurs a *self-overlap* of $\mathrm{Tip}(aba + b) = aba$, i.e., we can choose $p, q \in B$ with $p = ab$ and $q = ba$ such that $paba = abaq$. Since I is an ideal, $p(aba+b)-(aba+b)q = abb - bba$ is also an element of I, which we will call *overlap relation* of aba. But $\mathrm{Tip}(abb-bba) = bba$ is not an element of $\langle \mathrm{Tip}(g)\rangle = \langle aba\rangle$, which shows that $\langle \mathrm{Tip}(\{g\})\rangle \neq \langle \mathrm{Tip}(I)\rangle$ and hence $\{g\}$ is not a Gröbner basis.

One can extend this concept of self-overlaps to overlaps of any two elements of a generating set of I.

3.4.4 Definition. Let $a, a' \in A$ and suppose that there are elements $p, q \in B$ such that

(a) $\mathrm{Tip}(a)q = p\mathrm{Tip}(a')$;

(b) $\mathrm{Tip}(a)$ does not divide p and $\mathrm{Tip}(a')$ does not divide q.

Then the *overlap relation of a and a' by p, q* is defined by

$$o(a, a', p, q) = (1/\mathrm{CTip}(a))aq - (1/\mathrm{CTip}(a'))pa'.$$

The second condition implies that there are only finitely many overlap relations for fixed elements a and a'. In terms of overlap relations one can formulate a condition on a subset X of an ideal I to be a Gröbner basis.

3.4.5 Theorem. *[Gr, 2.3] Let A be a k-algebra with multiplicative basis B and admissible order $<$. Suppose G is a set of uniform, tip reduced elements of A. Suppose for every overlap relation $o(g, g', p, q)$ with $g, g' \in G$, the remainder of division by G is 0. Then G is a Gröbner basis for $\langle G\rangle$.* □

This gives rise to the following procedure for constructing a Gröbner basis.

3.5 Buchberger's algorithm

Following [Gr], we can now use the noncommutative analogue of BUCHBERGER's algorithm to construct a Gröbner basis of a given ideal. Unfortunately, in the noncommutative case the termination of this procedure cannot be proved in general.

3.5.1 Algorithm. (Buchberger) Let I be an ideal of A generated by a set $\{f_1, \dots f_n\} \subset A$ of uniform elements. As input we have $\{f_1, \dots, f_n\}$. The output will be a (possibly infinite) sequence of uniform elements $\{g_1, g_2, \dots\}$ where $g_i = f_i$ for $i = 1, \dots, n$ and, for $i > n$, $g_i \in I$ such that

$$\mathrm{Tip}(g_i) \notin \langle \mathrm{Tip}(g_1), \dots, \mathrm{Tip}(g_{i-1})\rangle.$$

Then it can be shown that $\{g_1, g_2, \dots\}$ is a Gröbner basis of I.

The procedure is based on theorem 3.4.5. We start by defining the set G as the set of generators of I. We check for every pair of elements g, g' in G if for any overlap relation $o(g, g', p, q)$ there exists a non-zero remainder r of the division by G. If so, we replace G be the union of G and $\{r\}$ and start checking again. See fig. 2 for a presentation in pseudo-code.

```
INPUT: f[1], ... , f[n]
OUTPUT: g[1], g[2], g[3], ...

FOR i = 1 TO n DO
  g[i]:= f[i]
END
  G:= {g[1], ... , g[n]}
  count:= n
DO
  H:= G
  FOR h in H DO
    FOR k in H DO
      FOR each overlap relation o(h,k,p,q) of h and k DO
        r := remainder of division of o(h,k,p,q) by H
        IF r <> 0 THEN
           count:= count + 1
           g[count]:= r
           G:= G union {g[count]}
        END
      END
    END
  END
WHILE H <> G
```

Figure 2: BUCHBERGER's algorithm

By definition, for any two elements there exist only a finite number of overlap relations. However, as we add new elements to G every time a remainder is non-zero it is not clear that the computations will stop. But in case the procedure terminates then theorem 3.4.5 tells us that we have found a finite Gröbner basis of I.

If A/I is finite dimensional, theorem 3.4.3 asserts that a finite Gröbner basis exists. In this case, it can be shown that BUCHBERGER's algorithm terminates and produces a Gröbner basis.

3.5.2 Theorem. *[Gr, 2.3] Let A/I be finite dimensional with I generated by a finite set of uniform, tip reduced elements. Then* BUCHBERGER*'s algorithm yields a finite Gröbner basis of I.* □

We now return to the special case that A is the path algebra of a quiver Q and I is an ideal of A which must be *admissible* in the following sense:

(a) All paths in I are of length at least 2.

(b) For each point x of the quiver, there exists $l_x \in \mathbb{N}$ such that all paths of length at least l_x starting in x are contained in I.

Since we assume that Q is finite, the second condition implies that there exists $l \in \mathbb{N}$ such that all paths in Q with length at least l are contained in I.

3.5.3 Algorithm. (Gröbner basis and non-tips of an ideal) As input we have a quiver Q and a finite generating set H of an ideal I of kQ. We compute a finite Gröbner basis and $\mathrm{NonTip}(I)$.

We choose the length-lexicographic order on the basis B of kQ, which is the set of all paths in Q. Starting with H as input we can construct a Gröbner basis G of I as described in BUCHBERGER's algorithm 3.5.1. Theorem 3.5.2 states that G is finite.

Thus $\mathrm{Tip}(G)$ is also a finite set. As shown in section 3.3, $\mathrm{NonTip}(I)$ is the set of elements in B which are not divisible by an element of $\mathrm{Tip}(G)$. But for actual computations we may encounter the problem that although Q is finite, B may not be finite if Q has cycles, and hence we would have to consider infinitely many paths. This can be avoided by working inductively on the length of the paths.

We start with the paths of length 2 (since I is admissible and does not contain paths of shorter length) and proceed by induction. There are only finitely many paths of a given length. If a path p is divisible by an element t of $\mathrm{Tip}(G)$ then any longer path containing p is of course also divisible by t. This reduces the number of paths we have to consider in the next step. Even better, since I is admissible we know that in our induction we must reach a bound l such that any path of length at least l is an element of I and thus is divisible by an element of $\mathrm{Tip}(G)$. This shows that the procedure terminates.

An implementation of BUCHBERGER's algorithm is supplied by the function `pathalg::GroebnerBasis`. The function `pathalg::Nontips` computes the nontips of an ideal given by a Gröbner basis.

4 Projective Resolutions

4.1 Indecomposable projective modules

The last chapter supplied us with the necessary tools to compute indecomposable projective modules over a path algebra $A = kQ/I$. These modules are the basic building blocks for projective resolutions.

A k-basis B of kQ is given by the (possibly infinite) set of paths in Q. Hence, $B \cup \{0\}$ is a pointed semigroup with set of generators the arrows of Q and the paths of length 0 which correspond to the points of Q. Choose total orders on the points $\{x_1, \dots, x_n\} = Q_0$ and on the arrows $\{\alpha_1, \dots, \alpha_m\} = Q_1$. Then we equip B with the length-lexicographic order generated by $x_1 < x_2 < \dots < x_n < \alpha_1 < \dots < \alpha_m$.

Assume that a finite Gröbner basis G of I and the set $\mathrm{NonTip}(I)$ have been constructed as described in algorithm 3.5.3. In the following let us view A as a finite k-spectroid. The indecomposable projective modules over A are the representable functors $P_x = A(x, -)$ for $x \in A_0$.

An arbitrary module M is determined up to isomorphism by its dimension vector $\underline{\dim} M = (\dim_k M(x))_{x \in A_0}$ and a list of matrices $(M(\alpha))_{\alpha \in Q_1}$. In the following we will show how one can compute this data for $P_x = A(x, -)$ explicitly.

For any $y \in A_0$, a k-basis of $P_x(y) = A(x, y)$ is given by all b in $\mathrm{NonTip}(I)$ which (as paths in Q) have starting point x and target point y. We will denote this subset of $\mathrm{NonTip}(I)$ by $\mathrm{NonTip}(I)(x, y)$.

Next, we want to determine the matrices which represent the linear maps $P_x(\alpha) = A(x, \alpha) : P_x(z) \to P_x(y)$ for $\alpha : y \to z$ in Q_1. Such a map $P(\alpha)$ is given by left multiplication by α. The image of a basis element b of $P_x(y)$ under left multiplication by $\alpha : y \to z$ is the path $\alpha\, b$. Now we consider $\alpha \cdot b$ as a product of elements of A. Then remark 3.1.4 tells us that the normal form $N(\alpha\, b)$ is the product $\alpha \cdot b$ written as a linear combination of elements in $\mathrm{NonTip}(I)(x, z)$. This yields the image of the basis vector b in $P_x(z)$. Of course the images of the basis elements determine the linear maps $P_x(\alpha)$ for each arrow α.

> The construction of an arbitrary indecomposable projective A-module is implemented in the the function `pathalg::Projective`.

4.2 Top and Rad

The Jacobson radical $\mathrm{rad}\, M$ of an A-module M is the submodule given by the intersection of all maximal submodules of M. It is the minimal submodule such that the quotient $M/\mathrm{rad}\, M$ is semisimple.

The Jacobson radical of the algebra A coincides with the radical of A as a left module over itself. It can be shown that a k-basis of the radical of A is the set of all paths of length ≥ 1 in $\mathrm{NonTip}(I)$. Moreover, it is well known that $\mathrm{rad}\, M = (\mathrm{Rad}\, A)M$. If we view A as a spectroid and consequently M as a functor we have $a \cdot m = M(a)(m)$ for $a \in A(x, y)$, $m \in M(x)$. Then it follows that $\mathrm{rad}\, M(y)$ is the k-space spanned by

$$\{M(n)(m) \mid n \in \mathrm{NonTip}(I)(x, y),\, l(n) \geq 1, m \in M(x)\}.$$

We assume that an A-module M is given by its dimension vector and linear maps represented by matrices. This means that we have fixed a k-basis for M. Therefore, when computing $\operatorname{rad} M(y)$ for $y \in A_0$, we run through all $x \in A_0$ and apply to each basis element b_x of $M(x)$ the image $M(n)$ of each element n of $\operatorname{NonTip}(I)(x, y)$ with length at least 1. In this way we obtain elements $M(n)(b_x) \in M(y)$ which span the vector space $\operatorname{rad} M(y)$. Note that $n \in \operatorname{NonTip}(I)(x, y)$ is a path in the quiver with starting point x and target point y. The image $M(n)$ of n is a linear map $M(x) \to M(y)$, represented as a matrix.

The *top of* M is the factor module $\operatorname{top} M := M/\operatorname{rad} M$. It is a semisimple module, i.e., a direct sum of simple modules. For each $x \in A_0$ we have $\operatorname{top} M(x) = M(x)/\operatorname{rad} M(x)$ and the following split short exact sequence of k-vector spaces

$$0 \longrightarrow \operatorname{rad} M(x) \longrightarrow M(x) \xrightarrow{\pi} \operatorname{top} M(x) \longrightarrow 0.$$

From a theoretical point of view we know the factor module $\operatorname{top} M$. But for computational purposes we need to know the dimension vector and the list of matrices which determine the module. Let $x \in A_0$, then we obtain a complement of $\operatorname{rad} M(x)$ in $M(x)$ by fixing a basis W_1 of $\operatorname{rad} M(x)$ and choosing a subset W_2 of our given basis of $M(x)$ such that $W_1 \cup W_2$ is again a basis of $M(x)$. Then the subspace T spanned by W_2 is isomorphic to $\operatorname{top} M$ by the restriction $\pi|_T : T \to \operatorname{top} M(x)$. Under this isomorphism, a linear map $\operatorname{top} M(\alpha) : \operatorname{top} M(x) \to \operatorname{top} M(y)$ corresponds to the restriction of $M(\alpha)$ to T, i.e., $\operatorname{top} M(\alpha)$ is a submatrix of $M(\alpha)$.

This method for choosing a complement to a vector space has been applied frequently in the implementation.

If P_x is an indecomposable projective module corresponding to some $x \in A_0$ then $\operatorname{top} P_x$ is the simple module S_x given by $S_x(x) = k$ and zero otherwise. Moreover, P_x is a projective cover of S_x, hence a projective module is uniquely determined by its top (up to isomorphism).

4.3 Construction of a projective resolution

Once we have constructed the indecomposable projective modules of A and know how to compute the radical and the top of a module M we can start with our task, the construction of a projective resolution of M.

We will determine a minimal projective resolution of an A-module M inductively: First we have to find the projective cover P_0 of M together with an epimorphism $\pi_0 : P_0 \to M$. The kernel of π_0 is the first syzygy module $\Omega_0(M)$ of M. Then we proceed in this fashion, i.e., in step i we construct a projective cover P_i of $\Omega_{i-1}(M)$ together with the epimorphism π_i and we obtain $\Omega_i(M)$ as the kernel of π_i. One may visualize the construction by the following diagram:

$$\begin{tikzcd}
\cdots P_3 \arrow[rr] \arrow[dr, "\pi_3"] & & P_2 \arrow[rr] \arrow[dr, "\pi_2"] & & P_1 \arrow[rr] \arrow[dr, "\pi_1"] & & P_0 \arrow[rr] \arrow[dr, "\pi_0"] & & 0 \\
& \Omega_2(M) \arrow[ur, "\iota_2"] \arrow[dr] & & \Omega_1(M) \arrow[ur, "\iota_1"] \arrow[dr] & & \Omega_0(M) \arrow[ur, "\iota_0"] \arrow[dr] & & M \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & 0
\end{tikzcd}$$

The procedure stops when $\Omega_n(M)$ is the zero module for some $n \in \mathbb{N}_0$ in which case the projective dimension of M is n. If the procedure does not stop we say that M has infinite projective dimension.

Let us consider the construction of a projective cover of a module M. If $P \xrightarrow{\pi} M$ is a projective cover, then π restricts to an isomorphism of $\operatorname{top} P$ and $\operatorname{top} M$. We will use this to determine P. Recall that $\operatorname{top} M$ is a semisimple module, and $\operatorname{top} P_x$ is the simple module S_x corresponding to $x \in A_0$, if P_x is indecomposable projective. On the other hand, a projective module is uniquely determined by its top. Now we decompose $\operatorname{top} M$ as a direct sum of simple modules

$$\operatorname{top} M = \bigoplus_{x \in A_0} S_x^{m_x}$$

with $m_x \in \mathbb{N}_0$. It follows that the projective cover of M is

$$P = \bigoplus_{x \in A_0} P_x^{m_x}.$$

The direct sum of two A-modules X and Y is given by

$$(X \oplus Y)(x) = X(x) \oplus Y(x) \text{ for } x \in A_0$$

and

$$(X \oplus Y)(\alpha) = \begin{pmatrix} X(\alpha) & 0 \\ 0 & Y(\alpha) \end{pmatrix} \text{ for } \alpha \in A(x,y).$$

By YONEDA's lemma, $\operatorname{Hom}(P_x, M)$ is isomorphic to $M(x)$ via $f \mapsto f_x(id_x)$ and the projection $\pi : P \to M$ is determined by its restriction, the isomorphism $\operatorname{top} P \longrightarrow \operatorname{top} M$. Since A_0 is finite, we have

$$\operatorname{Hom}_A(\bigoplus_{x \in A_0} P_x^{m_x}, M) \cong \bigoplus_{x \in A_0} (\operatorname{Hom}_A(P_x, M))^{m_x}.$$

This provides us with a recipe to compute π by its restrictions to each of the direct summands of P. Fix a direct summand P_x of P and denote the restriction of π to P_x by $\tilde{\pi}$. S_x is a direct summand of $\operatorname{top} M$. Let $\{b\}$ be the k-basis of $S_x(x) \cong k$ which is contained in our given basis of $\operatorname{top} M$. Then the isomorphism of $\operatorname{top} P_x$ and $S_x \hookrightarrow \operatorname{top} M$ is given by $P_x(x) \ni id_x \mapsto b \in S_x(x)$ and by YONEDA's lemma this determines the map $P_x \to M$. Explicitly, this means that $\tilde{\pi}_y$ maps an element v of $P_x(y)$ to $M(v)(b) \in M(y)$. This yields the module homomorphism $\tilde{\pi} = (\tilde{\pi}_y)_{y \in A_0} : P_x \to M$.

Once we have computed the projective cover P of M together with the projection π, we can compute the kernel K of π together with an inclusion $\iota : K \hookrightarrow P$ by solving systems of linear equations as follows:

The kernel map $\iota : K \to P$ is a natural transformation and thus for each $\alpha \in A(x,y)$ the following diagram is commutative with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K(x) \arrow[r, "\iota_x"] \arrow[d, dashed, "K(\alpha)"'] & P(x) \arrow[r, "\pi_x"] \arrow[d, "P(\alpha)"'] & M(x) \arrow[r] \arrow[d, "M(\alpha)"'] & 0 \\
0 \arrow[r] & K(y) \arrow[r, "\iota_y"] & P(y) \arrow[r, "\pi_y"] & M(y) \arrow[r] & 0
\end{tikzcd}$$

Each row is a split exact sequence of k-vector spaces and thus we can compute ι_x for all $x \in A_0$ by solving the matrix equation $\pi_x \iota_x = 0$. With this information we can calculate matrices $K(\alpha)$ making the left square of the diagram commutative, by solving another matrix equation $\iota_y K(\alpha) = P(\alpha)\iota_x$.

As we have seen before, $K = \Omega_0(M)$ and we can proceed with the construction of the projective resolution inductively by computing a projective cover of K.

We can view A-mod as a full subcategory of $\mathcal{D}^b(A)$ by considering a module as a chain complex concentrated in degree 0. As shown above, we can produce a projective resolution of an A-module M algorithmically if the projective dimension of M is finite. The sequence of projective modules $(P_i)_{i=0}^n$ together with the compositions $(\iota_i \pi_{i+1})_{i=1}^n$ form a bounded chain complex P_M which is quasi-isomorphic to M. If the global dimension of A is finite then $\mathcal{D}^b(A)$ is equivalent to $\mathcal{K}^b(A\text{-proj})$, and the chain complex P_M is the image of M under the equivalence of categories $\mathcal{D}^b(A) \longrightarrow \mathcal{K}^b(A\text{-proj})$.

An implementation of the procedure for the construction of a projective resolution is realized in the function `pathalg::projectiveResolution`. Also the kernel and cokernel of a module morphism can be explicitly computed with the functions `pathalg::Kernel` and `pathalg::Cokernel`.

5 Homomorphisms and Homotopy

5.1 Computing the homomorphism space of two modules

Throughout this section, let $A = kQ/I$ be a finite k-spectroid with quiver Q and admissible ideal I.

Let M, N be A-modules. Our aim is to compute the homomorphism space of M and N. A homomorphism $f : M \to N$ is a natural transformation (here, we consider M and N as functors $A \to k$-mod). Thus, for each object $x \in Q_0$ we have a map f_x such that for each arrow $\alpha : x \to y$ in Q_1 the following square commutes:

$$\begin{tikzcd}
M(x) \arrow[r, "f_x"] \arrow[d, "M(\alpha)"'] & N(x) \arrow[d, "N(\alpha)"] \\
M(y) \arrow[r, "f_y"'] & N(y)
\end{tikzcd}$$

Since $M(x)$ and $N(x)$ are k-vector spaces for every $x \in A_0$, the linear map f_x can be represented as a matrix and also $M(\alpha)$ and $N(\alpha)$ are given by matrices. Thus we can calculate all homomorphisms $M \to N$ in A-mod as the solutions $(f_x)_{x \in A_0}$ to the system of matrix equations

$$f_y M(\alpha) = N(\alpha) f_x$$

for all arrows $\alpha : x \to y$ in Q_1.

If $M = A(x, -)$ is an indecomposable projective module, we can use a less time consuming method which avoids solving a system of linear equations. By YONEDA's lemma, every morphism $f : M \to N$ is uniquely determined by the image $M(x) \ni id_x \mapsto f_x(id_x) \in N(x)$. So we can find a basis of $\mathrm{Hom}_A(M, N)$ by mapping id_x successively to each basis element of $N(x)$ and extending to morphisms of modules $f : M \to N$ in the following way:

Let $z \in A_0$. The paths in $\mathrm{NonTip}(I)(x, z)$ form a basis of $M(z)$ since $M = A(x, -)$ is an indecomposable projective module. On the other hand, a path $p \in \mathrm{NonTip}(I)(x, z)$ is a morphism $x \to z$ in A, and hence induces a map $M(p) : M(x) \to M(z)$ which is given by left multiplication with p. Thus we have $M(p)(id_x) = p \circ id_x = p$ and by naturality of f it then follows that

$$f_z(p) = f_z M(p)(id_x) = N(p) f_x(id_x).$$

So when the image $b = f_x(id_x)$ of id_x in $N(x)$ is fixed we can find the images of the basis of $M(z)$ by computing $N(p)(b)$ for all paths p in $\mathrm{NonTip}(I)(x, z)$. Obviously, this provides us with the linear map $f_z : M(z) \to N(z)$ and finally with the morphism of modules $f = (f_z)_{z \in A_0}$.

An easier method applies to the case of M and N both being indecomposable projective A-modules, $M := P_x$ and $N := P_y$, say. Then a basis of $\mathrm{Hom}_A(P_x, P_y) \cong P_y(x) = A(y, x)$ (by YONEDA's lemma) is given by all paths in $\mathrm{NonTip}(I)(y, x)$. For the composition of two morphisms $f : P_x \to P_y$ and

$g : P_y \to P_z$, we do not need to compute the matrix product $g_v f_v$ for each $v \in A_0$ but it suffices to compute the product $g_y(id_y) \cdot f_x(id_x)$ as elements in A.

If M is a decomposable projective module, say $M = \bigoplus_{i=1}^{s} M_i$ with M_i indecomposable, we can decompose the homomorphism space accordingly. Thus, $\mathrm{Hom}_A(\bigoplus_{i=1}^{s} M_i, N) = \bigoplus_{i=1}^{s} \mathrm{Hom}_A(M_i, N)$ and we can compute a k-basis for $\mathrm{Hom}_A(M_i, N)$ for each i as shown above. This yields a k-basis for $\mathrm{Hom}_A(M, N)$.

> The function `pathalg::HomSpace` computes a basis of the homomorphism space of two arbitrary modules, while `pathalg::HomSpaceProjective` is an implementation for the special case of projective modules.

5.2 The homomorphism space of two complexes

By definition, a chain map $f : C \to D$ of two complexes C and D in $\mathcal{C}^b(A\text{-mod})$ is given by a sequence of maps $f_n : C_n \to D_n$ compatible with the boundary maps.

We use the fact that C and D are bounded complexes. Let (m_C, n_C) and (m_D, n_D) be the bounds of C and D respectively. Then $\mathcal{C}^b(A\text{-mod})(C, D)$ is zero if $m_C > n_D$ or $m_D > n_C$.

Let $m = \max(m_C, m_D)$ and $n = \min(n_C, n_D)$. If $m > n$ then we have $\mathcal{C}^b(A\text{-mod})(C, D) = 0$. Otherwise it suffices to compute the homomorphism space of the truncated complexes $C_{[m-1,n+1]}$ and $D_{[m-1,n+1]}$.

5.2.1 Algorithm. Let C and D be chain complexes. We compute a k-basis of $\mathcal{C}^b(A\text{-mod})(C, D)$.

Let m and n be defined as above. If $m > n$ we return $\emptyset$. If $m \leq n$, then we will proceed by induction on the degree and start by determining a basis B_m of $\mathrm{Hom}_A(C_m, D_m)$ as described in section 5.1. Since $\mathrm{Hom}_A(C_{m-1}, D_{m-1}) = 0$ by definition of m, compatibility with the boundary maps yields the equation

$$(\sum_{b \in B_m} \lambda_b b) d^C_{m-1} = 0$$

which is in fact a system of matrix equations. Let Λ be a k-basis of the solution space.

In the following we will denote a sequence $(t_i)_{i \in \{m,\ldots,n\}}$ of maps $t_i : C_i \to D_i$ by the vector $(t_m, \ldots, t_n)$. Let

$$T_m := \{(\sum_{b \in B_m} \lambda_b b, 0, \ldots, 0) \mid \lambda \in \Lambda\}.$$

For the induction step $m < i \leq n$ we compute a basis B_i of $\mathrm{Hom}_A(C_i, D_i)$. The compatibility with the boundary maps again yields a system of matrix equations

$$(\sum_{b \in B_i} \lambda_b b) d^C_{i-1} - d^D_{i-1}(\sum_{t \in T_{i-1}} \lambda_t t_{i-1}) = 0$$

where t_{i-1} is the $(i-1)$-th entry of a sequence $t \in T_{i-1}$ and therefore a module morphism $C_{i-1} \longrightarrow D_{i-1}$. Let Λ be a k-basis of the solution space to this equation. Then we define

$$T_i := \{\sum_{b\in B_i}(0,\dots,0,\lambda_b b,0,\dots,0) + \sum_{t\in T_{i-1}} \lambda_t t \mid \lambda \in \Lambda\}$$

where $(0,\dots,0,\lambda_b b,0,\dots,0)$ is the sequence with the only non-zero entry at position i. One may think of T_i as a set of "chain maps" $C \to D$ which are compatible with the boundary maps in the degrees $m-1,\dots,i$.

Finally, we have to compute one last step: Here, we have the equation

$$d_n^D(\sum_{t\in T_n} \lambda_t t_n) = 0$$

where t_n is the n-th entry of the sequence t. Let Λ be a k-basis of the solution space. Then we define

$$T_{n+1} := \{\sum_{t\in T_n} \lambda_t t \mid \lambda \in \Lambda\}.$$

After this step we return T_{n+1} which is a k-basis of $\mathcal{C}^b(A\text{-mod})(C,D)$. An element of T is a sequence $(t_j)_{j=m,\dots,n}$ where $t_j \in \mathrm{Hom}_A(C_j,D_j)$ such that $t=(t_j)_{j\in\mathbb{Z}}$ with $t_j=0$ for $j \notin \{m,\dots,n\}$ is a chain map.

The algorithm is implemented in the function `pathalg::HomComplex` for computing a basis of the morphism space of two arbitrary complexes and in the faster function `pathalg::HomComplexProjective` for finding the morphism space of two complexes of projectives.

5.3 Null homotopic chain maps

For chain complexes C and D we need to compute $\mathcal{K}^b(A\text{-proj})(C,D)$ which, by definition, is $\mathcal{C}^b(A\text{-proj})(C,D)$ modulo null homotopic morphisms. We start by giving a useful characterization of null homotopic maps.

A *projective complex* in $\mathcal{C}^b(A\text{-proj})$ is a complex whose indecomposable direct summands are of the form P^i for an indecomposable projective P defined by $P^i_i := P$, $P^i_{i+1} := P$, and zero in all other degrees; the differential is $d_i := id_P$ and $d_j = 0$ for $j \neq i$. So P^i is the chain complex $0 \to P = P \to 0$ in degrees i and $i+1$.

5.3.1 Lemma. *A morphism f in $\mathcal{C}^b(A\text{-proj})$ is null homotopic if and only if f factors through a projective complex in $\mathcal{C}^b(A\text{-proj})$.*

Proof. Let $f : C \to D$ be null homotopic, so $f = ds + sd$, say. Let P be the projective complex with $P_n := C_n \oplus C_{n+1}$ and $d_n^P := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all $n \in \mathbb{Z}$. We define $g : C \to P$ and $h : P \to D$ by

$$g_n := \begin{pmatrix} 1 \\ d_n^C \end{pmatrix}, \quad h_n := (d_{n-1}^D s_n, s_{n+1}) \text{ for } n \in \mathbb{Z}.$$

One verifies easily that g and h are chain maps. Then $f = hg$ since

$$h_n g_n = (d^D_{n-1} s_n, s_{n+1}) \begin{pmatrix} 1 \\ d^C_n \end{pmatrix} = d^D_{n-1} s_n + s_{n+1} d^C_n = f_n.$$

Conversely, let $f = hg$ where $g : C \to P$, $h : P \to D$ and P is a projective complex. Hence there exist projective modules Q_n (which are non-zero for only finitely many $n \in \mathbb{Z}$) such that $P = (P_n)_{n\in\mathbb{Z}}$ is of the form $P_n = Q_{n-1} \oplus Q_n$ and $d^P_n := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We define $s = (s_n)_{n\in\mathbb{Z}}$ with $s_n := h_{n-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_n$. Then f is null homotopic via s since

$$\begin{aligned}
d^D_{n-1} s_n + s_{n+1} d^C_n &= d^D_{n-1} h_{n-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_n + h_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_{n+1} d^C_n \\
&= h_n d^P_{n-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_n + h_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} d^P_n g_n \\
&= h_n(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) g_n \\
&= h_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_n \\
&= h_n g_n \\
&= f_n
\end{aligned}$$

□

Thus we can compute the homomorphism space of two complexes in $\mathcal{K}^b(A\text{-proj})$ in the following way:

5.3.2 Algorithm. Let C, D be chain complexes in $\mathcal{K}^b(A\text{-proj})$. We compute a k-basis of $\mathcal{K}^b(A\text{-proj})(C, D)$.

As a first step we determine a k-basis of the space $\mathcal{C}^b(A\text{-proj})(C, D)$ using algorithm 5.2.1.

Next, we have to find the set S of all indecomposable projective complexes P such that $\mathcal{C}^b(A\text{-proj})(P, D) \neq 0$ and all indecomposable projective complexes Q with $\mathcal{C}^b(A\text{-proj})(C, Q) \neq 0$. This is a finite set since C and D are bounded and there are only finitely many indecomposable projective complexes in these bounds.

Let (m_C, n_C) and (m_D, n_D) be the bounds of C and D, respectively. As before, we define $m := \max(m_C, m_D)$ and $n := \min(n_C, n_D)$. If $m > n$ then $\mathcal{K}^b(A\text{-proj})(C, D) = 0$ and we are finished. Otherwise we compute for each indecomposable projective complex P^i (where P is an indecomposable projective module and $i \in \{m-1, \ldots, n\}$) a k-basis $V(P^i)$ of $\mathcal{C}^b(A\text{-proj})(C, P^i)$ and a k-basis $W(P^i)$ of $\mathcal{C}^b(A\text{-proj})(P^i, D)$ using algorithm 5.2.1. Thus S is the set of all P^i where $V(P^i)$ or $W(P^i)$ is not empty.

Then we determine the subspace $\mathcal{P}(C, D)$ of $\mathcal{C}^b(A\text{-proj})(C, D)$ which contains all morphisms which factor through some projective complex. This is the

k-vector space spanned by all compositions gf where $f \in V(P)$ and $g \in W(P)$, and P runs through all elements of S. We compute a basis of $\mathcal{P}(C, D)$.

Finally we find $\mathcal{K}^b(A\text{-proj})(C, D)$ by choosing a vector space complement of $\mathcal{P}(C, D)$ in $\mathcal{C}^b(A\text{-proj})(C, D)$ as described in section 4.2.

6 The Radical of an Algebra

6.1 The algebra of endomorphisms of a module

Let $A = kQ/I$ and let M be an A-module. The set $\mathrm{End}_A(M)$ of endomorphisms of M is a k-vector space and can be equipped with a multiplication given by the usual composition of morphisms. Thus it is again an algebra over k.

In our setup, where M is a functor from A to k-mod, an endomorphism f of M is a natural transformation $M \to M$. Explicitly, f is given by a collection of linear maps $f_{x_i} : M(x_i) \to M(x_i)$ (for $x_i \in A_0$) subject to a certain compatibility condition. If we fix bases of the k-vector spaces $M(x_i)$, the map f_{x_i} can be represented by a matrix f_{x_i}. Since $M \cong \bigoplus_{x_i \in A_0} M(x_i)$ as a k-vector space, f is represented by a block diagonal matrix

$$\begin{pmatrix} f_{x_1} & 0 & 0 & \cdots & 0 \\ 0 & f_{x_2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & f_{x_n} \end{pmatrix}$$

In this way, we can view $\mathrm{End}_A(M)$ as a subalgebra of $\mathcal{M}_d(k)$, the algebra of $d \times d$-matrices over k (where $d = \dim_k M$).

Moreover, an endomorphism of a chain complex can be represented in this fashion. Let C be a chain complex in $\mathcal{C}^b(A\text{-mod})$ and $f = (f^i)_{i \in \mathbb{Z}}$ an endomorphism of C. If (r, s) are the bounds of C then we can write f as a diagonal block matrix with blocks $f^i_{x_j}$ for $i \in \{r, \dots, s\}$ and $j \in \{1, \dots, n\}$. Thus we can view f as a square matrix of size d, where $d = \sum_{i=r}^{s} \dim_k C_i$.

This point of view allows us to apply known methods for computing the radical of a subalgebra of $\mathcal{M}_d(k)$ on which we will concentrate for the rest of this chapter.

If we have found the radical of $\mathrm{End}_A(M)$ for a module M we can also decide if M is indecomposable in A-mod. For, M is indecomposable if and only if $\mathrm{End}_A(M)$ is a local algebra and thus $K = \mathrm{End}_A(M)/\mathrm{Rad}(\mathrm{End}_A(M))$ is a skew field. If k is assumed to be finite, then K must be a field.

This also applies to chain complexes: A complex $C \in \mathcal{C}^b(A\text{-mod})$ is indecomposable if and only if the endomorphism algebra of C is local. We remark that if C is indecomposable as an object of $\mathcal{C}^b(A\text{-mod})$ then C is also indecomposable as an object of $\mathcal{D}^b(A)$. However, the converse is not true in general.

6.2 Computing the radical in characteristic 0

In this section, k will be a field of characteristic 0. We denote the trace of a matrix $m \in \mathcal{M}_d(k)$ by $\mathrm{Tr}(m)$.

6.2.1 Theorem. *Let A be a finite-dimensional algebra of $d \times d$ matrices over a field k. If char $k = 0$ or char $k > d$ then*

$$\mathrm{Rad}(A) = \{x \in A \,|\, \mathrm{Tr}(xy) = 0 \textit{ for all } y \in A\}.$$

Proof. Let $x \notin \mathrm{Rad}(A)$. This is equivalent to the existence of elements $a, b \in A$ such that axb is idempotent. Since axb is a matrix over k, it must be similar to a matrix E of the form

$$E = \begin{pmatrix} 1 & 0 & 0 & & & \dots & & 0 \\ 0 & 1 & 0 & & & \dots & & 0 \\ & & \ddots & & & & & \vdots \\ & & & 1 & 0 & & \dots & 0 \\ \vdots & & & & 0 & & \dots & 0 \\ & & & & & & \ddots & \vdots \\ 0 & & & \dots & & & & 0 \end{pmatrix}.$$

Therefore $\mathrm{Tr}(axb) = \mathrm{Tr}(xab) = \mathrm{Tr}(E) = m \cdot 1_k$ with $m := \mathrm{rank}(E) \leq d$. By assumption on the characteristic of k, we know that $m \cdot 1_k \neq 0$. It follows that $\mathrm{Tr}(xy) \neq 0$ for some $y \in A$ if and only if $x \notin \mathrm{Rad}(A)$. □

Thus, under the hypotheses of the theorem, we can compute $\mathrm{Rad}(A)$ from a k-basis $B = \{b_1, \dots, b_r\}$ of A by solving a system of linear equations. In particular, $\mathrm{Rad}(A)$ is the space of solutions $\sum_{i=1}^{r} \lambda_i b_i$ to the equations

$$\mathrm{Tr}(\sum_{i=1}^{r} \lambda_i b_i b_j) = 0$$

for $j = 1, \dots, r$.

6.3 Computing the radical in non-zero characteristic

Let k be a field of characteristic $p > 0$ and let A be a subalgebra of $\mathcal{M}_d(k)$.

From the proof of theorem 6.2.1 it can be seen that the condition on the characteristic of k cannot be dropped: If char $k = p < d$ then there may be elements $x \notin \mathrm{Rad}(A)$ and $y \in A$ such that $\mathrm{Tr}(xy) = m \cdot 1_k = 0 \bmod p$.

We present here an algorithm due to Cohen, Ivanyos, and Wales [CIW] for finding the radical of an algebra of linear transformations over a field of non-zero characteristic, applied to our situation. The algorithm is based on the following fact (which holds for base fields of arbitrary characteristic):

6.3.1 Proposition. *Let A be a finite dimensional algebra with 1. Then*

$$\mathrm{Rad}(A) = \{x \in A \,|\, (xy)^d = 0 \text{ for all } y \in A\}$$

for some $d \in \mathbb{N}$.

Proof. Let $x \in A$ such that $(xy)^d = 0$ for all $y \in A$. Then

$$(1 - xy)(1 + xy + (xy)^2 + \ldots + (xy)^{d-1}) = 1 + (xy)^d = 1.$$

Hence $1 - xy$ is invertible for all $y \in A$ and thus $x \in \mathrm{Rad}(A)$.

Conversely, let x be an element of $\mathrm{Rad}(A)$. Since $\mathrm{Rad}\,(A)$ is a nilpotent ideal we have $\{0\} = \mathrm{Rad}^d(A)$ for some $d \in \mathbb{N}$. But $(xy)^d \in \mathrm{Rad}^d(A) = \{0\}$. □

Following [CIW], we define an action of the matrix algebra A on k^d by matrix multiplication. Let $x \in A$. We know that $x^d = 0$ if and only if the characteristic polynomial $\chi_x(t)$ of x equals t^d. Now $\chi_x(t) = \sum_{i=0}^d (-1)^i \mathrm{Tr}(i,x) t^{d-i}$, where $\mathrm{Tr}(i,x)$ is the trace of the x action on the i-th exterior product space $\Lambda^i(k^d)$. Thus, $x^d = 0$ if and only if $\mathrm{Tr}(i,x) = 0$ for all $i = 1, \ldots, d$.

We will define a decreasing sequence $A_0, A_1, A_2, \ldots$ of ideals of A which contain $\mathrm{Rad}\,(A)$ by setting $A_0 := A$. For $j > 0$ we define A_j to be the set of all elements $a \in A_{j-1}$ for which $\mathrm{Tr}(p^{j-1}, a) = 0$ and $\mathrm{Tr}(p^{j-1}, ab) = 0$ for all $b \in A_{j-1}$.

Explicitly, we have

$$A_1 = \{x \in A \,|\, \mathrm{Tr}(xy) = 0 \text{ for all } y \in A\}.$$

Moreover, $A_1 = \mathrm{Rad}(A)$ if $d > \mathrm{char}\, k$ by theorem 6.2.1.

6.3.2 Theorem. *[CIW, Thm 7] The following statements hold for all $j \geq 1$:*

(a) *Each A_j is an ideal in A_{j-1} which contains Rad(A).*

(b) *The multiplicities of non-zero eigenvalues for elements of A_j acting on k^d are all multiples of p^j.*

(c) *For a, b in A_j we have $\mathrm{Tr}(p^j, a+b) = \mathrm{Tr}(p^j, a) + \mathrm{Tr}(p^j, b)$ and $Tr(l, a) = 0$ for all $l < p^j$.*

(d) *For $a \in A$, $\lambda \in k$ we have $\mathrm{Tr}(j, \lambda a) = \lambda^j\, \mathrm{Tr}(j, a)$.*

This gives rise to a procedure for the successive computation of the ideals A_j for $j = 1, 2, \ldots$ if we are supplied with a k-basis of A. It suffices to consider basis elements by 6.3.2 (c). Part (b) of the theorem states that all non-zero eigenvalues of elements of A_j occur with multiplicities which are multiples of p^j. Thus, if $p^j > d$, all eigenvalues of elements in A_j must be zero and hence A_j, by (a), is the radical of A. This shows that the process terminates. See fig. 3 for a description in pseudo-code.

6.3.3 Algorithm. (Radical of a matrix algebra) As input we take a basis of a matrix algebra $A = A_0$ over k. We will compute a k-basis of each A_j inductively for $j = 1, 2, \ldots$ by solving systems of linear equations.

Let $B^{j-1} = \{b_1, \ldots, b_r\}$ be a k-basis of A_{j-1} and let $s := p^j$. Denote by Γ the solution space to both

$$\sum_{i=1}^r \mathrm{Tr}(s, bb')x_i = 0 \text{ for } b, b' \in B^{j-1}$$

and

$$\sum_{i=1}^r \mathrm{Tr}(s, b)x_i = 0 \text{ for } b \in B^{j-1}.$$

By part (d) of theorem 6.3.2, we have the equality $\mathrm{Tr}(s, \lambda bb') = \lambda^s \mathrm{Tr}(s, bb')$, so each solution $(x_1, \ldots, x_r) \in \Gamma$ of the equations above gives rise to an element

```
INPUT:  B = {b[1], ..., b[r]}, basis for A
OUTPUT: a basis for Rad A

s:= 1
WHILE s <= d DO
  B':= B UNION {I}
  r := |B|
  r':= |B'|
  FOR i = 1 TO r DO
    FOR j = 1 TO r' DO
      g[i,j]:= Tr(s, b[i]*b'[j])
    END
  END
  G:= {g[i,j] | i=1,...,r; j=1,...,r'}
  H:= a basis of the space of solutions to
      g[1,j]*x[1] + ... + g[r,j]*x[r], j = 1,..., r'
  IF 1 < s < |k| THEN
    H':= {(a[1]^(1/s), ..., a[r]^(1/s)) | a in H}
  END
  B:= {a[1]*b[1] + ... + a[r]*b[r] | a in H'}
  s:= s * p
END
```

Figure 3: Finding the radical of an algebra over a finite field

$x = x_1^{1/s} b_1 + \ldots + x_r^{1/s} b_r$ of A with $\mathrm{Tr}(s, x) = 0$ and $\mathrm{Tr}(s, xb) = 0$ for all $b \in B^{j-1}$, so x is an element in A_j.

The algorithm terminates if $s > d$ and we return a basis of A_j which coincides with the radical of A.

Note that we can easily compute an s-th root of an element λ of k for $s = p^j$ if k is a finite field. Let $q = p^l$ be the number of elements in k. Thus we have $\lambda^q = \lambda$. Now choose $a, b \in \mathbb{N}_0$ such that $j = al + b$ with $b < l$. It follows that $\lambda^{\frac{1}{s}} = (\lambda^{\frac{1}{p^{al}}})^{\frac{1}{p^b}} = \lambda^{\frac{1}{p^b}}$ and now we can infer from $\lambda = \lambda^q$ that $\lambda^{\frac{1}{p^b}} = \lambda^{\frac{q}{p^b}} = \lambda^{p^{l-b}}$.

The most time-consuming part of the algorithm is the calculation of $\mathrm{Tr}(p^j, f)$ which is given as a coefficient of the characteristic polynomial χ_f of f. As we have seen above, in case A is the endomorphism algebra of a module, an element f of A is a block matrix with blocks f_x ($x \in A_0$) on the diagonal and zero otherwise. This has the advantage that we can calculate χ_f as the product of the characteristic polynomials of the blocks f_x of f.

> An implementation of the algorithm is provided by `pathalg::Radical` which chooses the appropriate algorithm depending on the characteristic of the base field and returns a basis of the radical of the endomorphism algebra of a module. The function `pathalg::RadicalComplex` does the same for a chain complex.

7 The Quiver of an Endomorphism Algebra

7.1 The quiver of a spectroid

Let S be a finite k-spectroid. By a theorem of GABRIEL (cf [Ri, 2.1]), if k is algebraically closed, we know that S is isomorphic to the path spectroid of a finite quiver modulo an admissible ideal.

Our aim is to construct this quiver and a set of generators of the ideal in the case that S is given by its radical spaces $\operatorname{Rad} S(x, y)$ for all objects x and y of S.

If k is not algebraically closed we cannot be sure that S is isomorphic to a path spectroid modulo an admissible ideal. In this case, the method described below will fail. This happens if and only if there exists an object x in S such that the k-dimension of the space $\operatorname{Rad}(\operatorname{End}(x))$ is not equal to $\dim_k \operatorname{End}(x) - 1$. We will test for this criterion and stop the computations if it is not satisfied.

So assume S is isomorphic to kQ_S/I_S. The points of the quiver Q_S are corresponding to the objects of S. Let $\{x_1, \dots, x_n\}$ be the objects of S. The arrows $\alpha_1, \dots, \alpha_r : x_i \to x_j$ of Q_S correspond to a k-basis $b_1, \dots, b_r$ of the space $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$. So the number of arrows $x_i \to x_j$ is given by the dimension of $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$.

This determines the quiver Q_S uniquely. In the following we describe how we find k-bases of the vector spaces $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$ for each pair of objects of S.

Let x_i, x_j be objects of S. Then $\operatorname{Rad}^2 S(x_i, x_j)$ is the space of all elements in $\operatorname{Rad} S(x_i, x_j)$ which are linear combinations of morphisms factoring through another object $x_s \in S_0$. Thus, $\operatorname{Rad}^2 S(x_i, x_j)$ is the vector space generated by all compositions gf where $f \in \operatorname{Hom}(x_i, x_s)$ and $g \in \operatorname{Hom}(x_s, x_j)$ for some $x_s \in S_0$. It suffices to consider only those f and g which are elements of a (fixed) basis of the respective spaces. Their compositions gf span the vector space $\operatorname{Rad}^2 S(x_i, x_j)$. Given a basis of $\operatorname{Rad}^2 S(x_i, x_j)$ we can compute a basis of a vector space complement in $\operatorname{Rad} S(x_i, x_j)$.

For later use, we define a map Φ which sends an arrow $\alpha : x_i \to x_j$ of Q_S to its corresponding basis element in $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$. We can extend Φ to a map from the set of finite paths in Q_S to the set of morphisms in S by defining $\Phi(p) = \Phi(\alpha_m) \cdots \Phi(\alpha_1)$ for a path $p = \alpha_m \cdots \alpha_1$ in Q_S.

Once we have found the quiver of S we want to compute an ideal I_S such that $S \cong kQ_S/I_S$. For this we have to find all linear combinations of paths in Q_S which lie in I_S, meaning that they are zero in S. But if Q_S has cycles there exists an infinite number of paths. On the other hand, I_S is an admissible ideal, so I_S contains all paths of length $\geq l$ for some $l \in \mathbb{N}$.

This allows us to determine I_S even if Q_S is not directed:

7.1.1 Algorithm. (Relations) The input is a quiver Q_S of a spectroid S and the map Φ from the set of paths in Q_S to the set of morphisms in S. A generating set G for the ideal I_S is returned.

First, we check for zero relations, i.e., paths which lie in I_S and thus are zero in S. Initialize $Z := \emptyset$. We proceed by induction on the length of the paths in Q. We start with paths of length 2 since I_S does not contain shorter paths.

Length 2: Initialize $P_2 := \emptyset$. Let $\{p_1, \dots, p_r\}$ be the set of paths with length 2 in Q_S. For each p_i we determine $\Phi(p_i)$ and check if $\Phi(p_i)$ is zero in S. If so, we replace Z by $Z \cup \{p_i\}$. If not we extend P_2 by $\{p_i\}$.

Length i: Initialize $P_i := \emptyset$. By induction, P_{i-1} contains all non-zero paths in (Q_S, I_S) of length $i-1$. We will go through the following procedure for each $p \in P_{i-1}$.

For each arrow α in Q_S which can be appended to the path p we determine $\Phi(\alpha p)$. If $\Phi(\alpha p)$ is zero then we enlarge Z by the path αp. If $\Phi(\alpha p)$ is not zero, then replace P_i by $P_i \cup \{\alpha p\}$.

This procedure must stop because I_S is an admissible ideal and so P_l will be empty for some $l \in \mathbb{N}$. We have constructed a set Z of generators for the zero-relations in S and we obtained $P := \bigcup_{i=2}^{l} P_i$, the finite set of non-zero paths in S.

Now we partition P as $P = \bigcup_{i,j \in S_0} P_{ij}$ where P_{ij} contains all paths in P which start in i and end in j. For each P_{ij} we solve the linear equation $\sum_{p \in P_{ij}} \lambda_p \Phi(p) = 0$ and obtain a k-basis B_{ij} of the solution space. We define

$$R := \{ \sum_{p \in P_{ij}} \lambda_p p \mid (\lambda_p)_{p \in P_{ij}} \in B_{ij} \}.$$

Finally we obtain $G := R \cup Z$ as the required set of generators for I_S.

For a description of this algorithm in pseudo-code see fig. 4.

7.2 Finding the quiver of an endomorphism algebra

We now apply the algorithms described in the previous section to a special case: Given an object in the homotopy category $\mathcal{K}^b(A\text{-proj})$, we want to find the quiver with relations of its endomorphism algebra.

Let T be a complex in $\mathcal{K}^b(A\text{-proj})$ which decomposes as $T = \bigoplus_{i=1}^{n} T_i$ with T_i indecomposable for $i = 1, \dots, n$ and $T_i \ncong T_j$ for $i \neq j$. We will view T in the following as a full subspectroid of $\mathcal{K}^b(A\text{-proj})$ with the objects $T_1, \dots, T_n$ and try to construct a quiver Q and an admissible ideal I of kQ such that $T \cong kQ/I$.

First we compute the quiver Q as explained in the previous section.

7.2.1 Algorithm. (Quiver of a full subspectroid of $\mathcal{K}^b(A\text{-proj})$) We take as input a full subspectroid of $\mathcal{K}^b(A\text{-proj})$ given by its objects $T_1, \dots, T_n$. The quiver of the spectroid is returned.

We label the points of Q by $1, \dots, n$ corresponding to the objects $T_1, \dots, T_n$ of T.

Following algorithm 5.3.2 we compute a k-basis of the homomorphism space $\mathcal{K}^b(A\text{-proj})(T_i, T_j)$ for any pair of summands T_i, T_j of T. These give us the morphism spaces $T(T_i, T_j)$ of the spectroid T. As a by-product we are also supplied with a basis for each space of null homotopic chain maps $T_i \to T_j$.

To determine the number of arrows $i \to j$ of Q, we have to compute a k-basis of the space $\operatorname{Rad} T/\operatorname{Rad}^2 T(T_i, T_j)$. If $T_i \neq T_j$, then $\operatorname{Rad} T(T_i, T_j)$ is equal to $\mathcal{K}^b(A\text{-proj})(T_i, T_j)$. Otherwise $\mathcal{K}^b(A\text{-proj})(T_i, T_i) = \operatorname{End}(T_i)$ is a local algebra because T_i is indecomposable. In section 6 we have seen how to compute a k-basis of $\operatorname{Rad}(\operatorname{End}(T_i))$ over arbitrary base fields k.

```
INPUT : a quiver Q and
        a map f sending a path to the corresponding morphism
OUTPUT: G

Z:= {}
P:= {}
P[1]:= the set of all arrows in Q
i:= 2
WHILE P[i-1] <> {} DO
  P[i]:= {}
  FOR p IN P[i-1] DO
    FOR all arrows a such that (ap) is path in Q DO
      IF f(ap) = 0 THEN Z:= Z union {ap}
      ELSE P[i]:= P[i] union {ap}
      END
    END
  END
  P:= P union P[i]
  i:= i + 1
END
R:= {}
FOR i, j points of Q DO
  P[i,j]:= set of all paths in P which start in i and end in j
  B:= a basis of the space of solutions to
         x[1]*f(p[1]) + ... x[r]*f(p[r]) = 0
      where {p[1],..., p[r]} = P[i,j]
  R:= R union {b[1]*p[1] + ... + b[r]*p[r] | b in B}
END
```

Figure 4: Finding the ideal of a spectroid

At this point we can decide if T is a factor of a path spectroid: For all $i \in \{1, \ldots, n\}$ let $d_i := \dim_k \operatorname{End}(T_i)$; if $\dim_k \operatorname{Rad}(\operatorname{End}(T_i)) < d_i - 1$ for some i then we cannot determine a quiver Q and ideal I such that $T \cong kQ/I$ and hence we stop the computations. If, however, we have $\dim_k \operatorname{Rad}(\operatorname{End}(T_i)) = d_i - 1$ for all $i \in \{1, \ldots, n\}$, we know that T is a factor of a path spectroid and we can proceed with our calculations.

For all pairs $T_i, T_j \in T_0$ we fix a k-basis B_{ij} of the space $\operatorname{Rad} T(T_i, T_j)$. Then $\operatorname{Rad}^2 T(T_i, T_j)$ is the subspace generated by all compositions gf where f is an element of B_{is} and g an element of B_{sj} for $s = 1, \ldots, n$. We compute a basis R_{ij} of a k-vector space complement of $\operatorname{Rad}^2 T(T_i, T_j)$ in $\operatorname{Rad} T(T_i, T_j)$. The number of arrows $i \to j$ in Q is the number of elements in R_{ij}.

So, by definition of Q, the elements of R_{ij} are in correspondence with the arrows $i \to j$ of Q, thus we can define a bijection $\Phi : Q_1 \to \bigcup_{i,j \in Q_0} R_{ij}$. We extend Φ to arbitrary paths of Q by mapping a path $\alpha_r \cdots \alpha_1$ to the composition of maps $\Phi(\alpha_r) \cdots \Phi(\alpha_1)$.

Finally, we have to determine I. As described in algorithm 7.1.1, we start with searching for zero relations, by induction on the length of paths in Q.

Since T is a full subcategory of $\mathcal{K}^b(A\text{-proj})$ a morphism in T is zero if and only if it is null homotopic as a chain map. For a path p in Q we can easily determine if $\Phi(p)$ is zero as a chain map, so we test for this first. Only if this is not the case we must do further (and more time-consuming) checks to determine if $\Phi(p)$ is null homotopic.

Although the basis sets R_{ij} $(i, j \in Q_0)$ do not contain null homotopic morphisms it may happen that the composition of two basis elements is null homotopic without being zero. This is due to the fact that we had to choose a complement of each subspace $\operatorname{Rad}^2 T(T_i, T_j)$ in $\operatorname{Rad} T(T_i, T_j)$, and the product fg of two elements $f \in R_{sj}$ and $g \in R_{is}$ may not be an element of the vector space spanned by R_{ij}.

But we have already computed a k-basis of the space of null homotopic chain maps $T_i \to T_j$ when we determined $\mathcal{K}^b(A\text{-proj})(T_i, T_j)$. Hence we can decide if $\Phi(p)$ is an element of this subspace. In fact, this amounts to solving a system of linear equations.

As described in 7.1.1 we determine a set Z of zero relations in I as well as the finite set P of non-zero paths in kQ/I.

It remains to decide which linear combinations of elements of P are zero in S and therefore elements of I. Following algorithm 7.1.1 we partition P into subsets P_{ij} containing the paths starting in i and ending in j. For each pair $i, j \in Q_0$ we must solve the linear equation $\sum_{p \in P_{ij}} \lambda_p \Phi(p) = 0$ in T meaning that we have to determine the null homotopic chain maps in the k-vector space spanned by $\{\Phi(p) \,|\, p \in P_{ij}\}$.

Let B_{ij} be a basis of the intersection of $\operatorname{span}\{\Phi(p) \,|\, p \in P_{ij}\}$ and the space of null homotopic morphisms $T_i \to T_j$ which we computed by solving a system of linear equations. Then we define

$$R := \{ \sum_{p \in P_{ij}} \lambda_p p \mid \sum_{p \in P_{ij}} \lambda_p \Phi(p) \in B_{ij} \}$$

and obtain $G := Z \cup R$ as a set of generators for I.

We remark that G may not be minimal if Q has cycles, so it might be advisable to apply BUCHBERGER's algorithm to G to get rid of redundant generators.

An implementation of the algorithms described in this chapter is provided by the function `pathalg::EndAlgebraComplex` which computes the quiver with relations of the endomorphism algebra of a complex of projectives in the homotopy category. Such a complex must be given by its indecomposable direct summands.

8 Extensions of the Base Field

In the previous chapters we have usually assumed k to be a finite field because we would like to carry out computations using a computer algebra system. In this chapter we show that the quiver with relations of a k-algebra "remains the same" over any field extension of k. Moreover, this is also true for the quiver of the endomorphism algebra of a chain complex considered over k and over a field extension of k, respectively. Therefore it is possible to derive results over an algebraically closed base field from computations over a suitable subfield.

Throughout this chapter, L will be a field extension of k. Let A be a k-algebra. In this section, by an A-module we mean an arbitrary left A-module which may have infinite k-dimension. The category of all left A-modules is denoted by A-Mod.

8.1 The quiver of an algebra

The tensor product yields an endofunctor $L \otimes_k -$ of k-Mod, the category of k-vector spaces. Moreover, $L \otimes_k -$ is exact since L is free and therefore flat as a k-module.

Let M be an A-module. If k is central in A (this is obviously true if A is a factor of a path algebra) then $L \otimes_k M$ carries a left A-module structure given by $a \cdot (l \otimes m) := l \otimes (a \cdot m)$ for $a \in A$, $l \in L$ and $m \in M$. Therefore $L \otimes_k -$ can also be considered as a functor $A\text{-Mod} \longrightarrow A\text{-Mod}$. We note the following fact:

8.1.1 Lemma. *If k is central in A then $L \otimes_k -$ is an exact endofunctor of A-Mod.*

Let $A = kQ/I$. We are concerned with the question if the quiver of A stays the same if we change the base field of A to an extension field L of k, i.e., we want to know the quiver with relations of the L-algebra $L \otimes_k A$. Note that A is a subalgebra of $L \otimes_k A$ via the inclusion $A \cong k \otimes_k A \hookrightarrow L \otimes_k A$.

8.1.2 Proposition. *Let $A = kQ/I$. Then $L \otimes_k A \cong LQ/(L \otimes_k I)$ as L-algebras.*

Proof. We have the following short exact sequence:

$$0 \longrightarrow I \longrightarrow kQ \longrightarrow kQ/I \longrightarrow 0$$

Since $L \otimes_k -$ is an exact functor by lemma 8.1.1, the sequence

$$0 \longrightarrow L \otimes I \longrightarrow L \otimes kQ \longrightarrow L \otimes kQ/I \longrightarrow 0$$

is still exact. Therefore $L \otimes (kQ/I)$ is isomorphic to $(L \otimes kQ)/(L \otimes I)$ as an A-module. Moreover, this isomorphism respects the L-algebra structure.

It remains to show that $L \otimes kQ \cong LQ$. We define a map $L \otimes kQ \to LQ$ by $\sum_i (l_i \otimes \sum_j \mu_j p_j) \mapsto \sum_j \sum_i (l_i \mu_j) p_j$. One easily verifies that this is an isomorphism of L-algebras. $\square$

This lemma shows that the quiver with relations of an algebra (if it exists) does not change when we extend the base field. We would like to apply this result to the endomorphism algebra of a tilting complex T. Before we can do this, we have to prove that $L \otimes_k \operatorname{End}_A(T) \cong \operatorname{End}_{L \otimes A}(L \otimes T)$. We will start by showing that this is true if T is a module.

8.2 The endomorphism algebra of a chain complex

8.2.1 Lemma. *Let A be a k-algebra and let M, N be (not necessarily finite dimensional) A-modules. Then*

$$\begin{array}{rcl} L \otimes_k \operatorname{Hom}_A(M,N) & \longrightarrow & \operatorname{Hom}_{L\otimes_k A}(L\otimes_k M, L\otimes_k N) \\ l\otimes f & \longmapsto & l\cdot(L\otimes f) \end{array}$$

is an isomorphism of L-vector spaces if M or L has finite k-dimension.

Proof. We have the following isomorphisms of L-vector spaces:

$$\begin{array}{rcl} \operatorname{Hom}_{L\otimes_k A}(L\otimes_k M, L\otimes_k N) & \cong & \operatorname{Hom}_{L\otimes_k A}(L\otimes_k A\otimes_A M, L\otimes_k N) \\ & \cong & \operatorname{Hom}_A(M, L\otimes_k N). \end{array}$$

The last isomorphism follows from $(L\otimes_A A)\otimes_k -$ being left adjoint to the forgetful functor $(L\otimes_k A)$-Mod $\to (k\otimes_k A)$-Mod $\cong A$-Mod. So it suffices to show that $L\otimes_k \operatorname{Hom}_A(M,N)$ is isomorphic to $\operatorname{Hom}_A(M, L\otimes_k N)$.

Let $(l_j)_{j\in J}$ be a k-basis of L. Then

$$L\otimes N \cong (\bigoplus_{j\in J} l_j\cdot k)\otimes N \cong \bigoplus_{j\in J}(l_j\cdot k\otimes N) \cong \bigoplus_{j\in J} N$$

as A-modules. Now we define a morphism

$$\begin{array}{rcl} \Phi : L\otimes \operatorname{Hom}_A(M,N) & \longrightarrow & \operatorname{Hom}_A(M, L\otimes N) \cong \operatorname{Hom}_A(M, \bigoplus_{j\in J} N) \\ l\otimes f & \longmapsto & (m\mapsto l\otimes f(m)). \end{array}$$

Then an inverse of Φ is given by the map which sends $g : M \longrightarrow \bigoplus_{j\in J} N$ to $\sum_{j\in J} l_j\otimes g_j$ (where g_j is the composition of g with the j-th projection $\bigoplus_{j\in J} N \xrightarrow{\pi_j} N$). We remark that the sum $\sum_{j\in J} l_j\otimes g_j$ is finite if $\dim_k L<\infty$ (so J is finite) or if M has finite k-dimension. In the latter case the image $g(M)$ in $\bigoplus_{j\in J} N$ is finite dimensional over k. Therefore almost all maps g_j must be zero. □

Let m and n be integers. We define a new quiver $Q^{[m,n]}$ by gluing together $n-m$ copies $Q^{(i)}$ of Q, indexed by $i=m,\ldots,n$.

We have two types of arrows in $Q^{[m,n]}$: All arrows $\alpha^{(i)}$ of $Q^{(i)}$ for $i=m,\ldots,n$, and arrows $\beta_x^{(i)} : x^{(i)} \to x^{(i+1)}$ for all $x\in Q_0$ and $i=m,\ldots,n-1$ which connect two copies $Q^{(i)}$ and $Q^{(i+1)}$.

The relations are given by all relations in $Q^{(i)}$ for $i=m,\ldots,n$ together with all relations $\alpha^2=0$ and all relations $\beta\alpha=\alpha\beta$. We denote the ideal of relations by $I^{[m,n]}$.

Let C, D be two complexes in $\mathcal{C}^b(A\text{-mod})$ and let (m_C,n_C) and (m_D,n_D) be the bounds of C and D respectively. We define $m:=\min(m_C,m_D)$ and $n:=\max(n_C,n_D)$. Then we can view C and D as finite dimensional modules over $\bar{A}=kQ^{[m,n]}/I^{[m,n]}$. Hence, $\mathcal{C}^b(A\text{-mod})(C,D)$ and $\operatorname{Hom}_{\bar{A}}(C,D)$ are isomorphic as k-vector spaces.

8.2.2 Corollary. *Let $A = kQ/I$ and let C, D be chain complexes over A-mod. Then*

$$\begin{aligned} \Phi : L \otimes_k \mathcal{C}^b(A\text{-mod})(C,D) &\longrightarrow \mathcal{C}^b((L \otimes_k A)\text{-mod})(L \otimes_k C, L \otimes_k D) \\ \sum \lambda_j \otimes f_j &\longmapsto \sum \lambda_j \cdot (L \otimes f_j) \end{aligned}$$

is an isomorphism of L-vector spaces.

Proof. Let $\bar{A} := kQ^{[m,n]}/I^{[m,n]}$ as defined above. Now we can apply proposition 8.2.1 to the finite dimensional $\bar{A}$-modules C and D: It follows that $L \otimes \mathrm{Hom}_{\bar{A}}(C,D) \cong \mathrm{Hom}_{L\otimes\bar{A}}(L \otimes C, L \otimes D)$.

By lemma 8.1.2 we have $L\otimes\bar{A} \cong L\bar{Q}/(L\otimes\bar{I})$ as well as $L\otimes A \cong LQ/(L\otimes I)$, thus $L\otimes C$ and $L\otimes D$ can be considered as chain complexes in $\mathcal{C}^b((L\otimes A)\text{-mod})$. This yields the required isomorphism of L-vector spaces. □

8.3 The endomorphism algebra of a complex in $\mathcal{K}^b(A\text{-proj})$

We remark that $L \otimes_k -$ maps projective A-modules to projective $(L \otimes A)$-modules: If we decompose $A = \bigoplus P_i$ as an A-module then

$$L \otimes_k A = L \otimes_k (\bigoplus P_i) \cong \bigoplus (L \otimes_k P_i).$$

Therefore $L \otimes_k -$ is also a functor $\mathcal{C}^b(A\text{-proj}) \to \mathcal{C}^b((L \otimes_k A)\text{-proj})$.

8.3.1 Lemma. *Let $A = kQ/I$ and let C and D be complexes in $\mathcal{C}^b(A\text{-proj})$. Let*

$$\Phi : L \otimes_k \mathcal{C}^b(A\text{-proj})(C,D) \longrightarrow \mathcal{C}^b((L \otimes_k A)\text{-proj})(L \otimes_k C, L \otimes_k D)$$

be the isomorphism given by corollary 8.2.2. If $h \in L \otimes_k \mathcal{C}^b(A\text{-proj})(C,D)$ such that $\Phi(h)$ is null homotopic, then there exist null homotopic chain maps $f_j \in \mathcal{C}^b(A\text{-proj})(C,D)$ and elements $\lambda_j \in L$ $(j \in J)$ with $h = \sum_{j\in J} \lambda_j \otimes f_j$.

Proof. We write $h = \sum_{s\in S} \mu_s \otimes h_s$ for some $\mu_s \in L$ and $h_s \in \mathcal{C}^b(A\text{-proj})(C,D)$. Let $(l_j)_{j\in J}$ be a k-basis of L. Then $\mu_s = \sum_{j\in J} \mu'_{sj} l_j$ for some $\mu'_{sj} \in k$ and an easy calculation shows that we can write h as $\sum_{j\in J} l_j \otimes f_j$ where $f_j = \sum_{s\in S} \mu'_{sj} h_s$.

By assumption $\Phi(h)$ is null homotopic, so it induces the zero map on the homology modules. For any $n \in \mathbb{Z}$ we infer

$$\begin{aligned} 0 &= H_n(\Phi(h)) = H_n(\sum_{j\in J} l_j(L \otimes f_j)) \\ &= \sum_{j\in J} l_j H_n(L \otimes f_j) \\ &= \sum_{j\in J} l_j(L \otimes H_n(f_j)). \end{aligned} \tag{1}$$

The last equality follows by exactness of $L \otimes_k -$. Now $\sum_{j \in J} l_j(L \otimes H_n(f_j))$ is an element of $\mathrm{Hom}_{L\otimes A}(L \otimes H_n(C), L \otimes H_n(D))$ which is, by lemma 8.2.1, isomorphic to $L \otimes \mathrm{Hom}_A(H_n(C), H_n(D))$. Thus (1) is equivalent to

$$0 = \sum_{j \in J} l_j \otimes H_n(f_j).$$

Let $(b_t)_{t\in T}$ be a k-basis of $\mathrm{Hom}_A(H_n(C), H_n(D))$. We write each $H_n(f_j)$ as a linear combination $\sum_{t\in T} \nu_{tj} b_t$ and obtain

$$\begin{aligned} 0 &= \sum_{j\in J} l_j \otimes H_n(f_j) \\ &= \sum_{j \in J} l_j \otimes \sum_{t\in T} \nu_{tj} b_t \\ &= \sum_{j\in J}\sum_{t \in T} \nu_{tj}(l_j \otimes b_t) \end{aligned}$$

But $(l_j \otimes b_t)_{j\in J, t\in T}$ is a k-basis for $L \otimes \mathrm{Hom}_A(H_n(C), H_n(D))$ and thus $\nu_{tj} = 0$ for all $t \in T$ and all $j \in J$. Since $H_n(f_j) = \sum_{t\in T} \nu_{tj} b_j$ it follows that $H_n(f_j) = 0$ for all $j \in J$.

Finally, we conclude from $H_n(f_j) = 0$ that f_j is null homotopic: This is true since $\mathcal{K}^b(A\text{-proj})(C, D) \cong \mathcal{D}^b(A)(C, D)$ for the bounded complexes of projectives C and D. Then $H_n(f_j) = 0$ means that f_j is mapped to zero under this isomorphism. Therefore f_j must be zero in $\mathcal{K}^b(A\text{-proj})(C, D)$, so f_j is null homotopic as a chain map. □

8.3.2 Lemma. *Let $A = kQ/I$ and let C and D be complexes in $\mathcal{C}^b(A\text{-proj})$. Then $L \otimes_k \mathcal{K}^b(A\text{-mod})(C, D)$ and $\mathcal{K}^b((L \otimes_k A)\text{-mod})(L \otimes_k C, L \otimes_k D)$ are isomorphic as L-vector spaces.*

Proof. By corollary 8.2.2 we know that $L \otimes \mathcal{C}^b(A\text{-proj})(C, D)$ is isomorphic to $\mathcal{C}^b((L \otimes A)\text{-proj})(L \otimes C, L \otimes D)$ via the isomorphism Φ which induces the map

$$\begin{aligned} \Phi' : L \otimes \mathcal{K}^b(A\text{-proj})(C, D) &\longrightarrow \mathcal{K}^b((L \otimes_k A)\text{-proj})(L \otimes C, L \otimes D) \\ \sum \lambda_j \otimes [f_j] &\longmapsto \sum \lambda_j [L \otimes f_j]. \end{aligned}$$

We check that Φ' is well defined: Let $f \in L \otimes_k \mathcal{C}^b(A\text{-mod})(C, D)$ with $f = \sum \lambda_j \otimes f_j$ such that all f_j are null homotopic. Then $\Phi'(\sum \lambda_j \otimes [f_j]) = \sum \lambda_j [L \otimes f_j]$. By exactness of $L \otimes_k -$ we have

$$H_n(\sum \lambda_j (L \otimes f_j)) = \sum \lambda_j (L \otimes H_n(f_j)) = 0$$

and a similar argument as in the proof of 8.3.1 shows that $\sum \lambda_j(L \otimes f_j)$ is null homotopic.

Lemma 8.3.1 implies that Φ' is injective. So it remains to show that Φ' is surjective. This follows from the commutative diagram

$$\begin{tikzcd} L \otimes \mathcal{C}^b(A\text{-proj})(C,D) \arrow[r, "\Phi"] \arrow[d] & \mathcal{C}^b((L \otimes_k A)\text{-proj})(L \otimes C, L \otimes D) \arrow[d, "\pi"] \\ L \otimes \mathcal{K}^b(A\text{-proj})(C,D) \arrow[r, "\Phi'"] & \mathcal{K}^b((L \otimes_k A)\text{-proj})(L \otimes C, L \otimes D) \end{tikzcd}$$

since the composition of maps $\pi\Phi$ is surjective. □

8.3.3 Theorem. *Let $A = kQ/I$ and let T be a chain complex in $\mathcal{K}^b(A\text{-proj})$. If $\mathcal{K}^b(A\text{-proj})(T,T) \cong kQ'/I'$ as k-algebras then $\mathcal{K}^b((L \otimes_k A)\text{-proj})(L \otimes_k T, L \otimes_k T)$ is isomorphic to $LQ'/(L \otimes_k I')$ as an L-algebra.*

In other words, if the k-algebra $\mathrm{End}(T)$ has the quiver with relations (Q', I') then the L-algebra $\mathrm{End}(L \otimes_k T)$ is given by the same quiver with the same relations.

Proof. Application of lemma 8.3.2 shows that there exists an isomorphism

$$\Phi' : \mathcal{K}^b((L \otimes_k A)\text{-proj})(L \otimes T, L \otimes T) \longrightarrow L \otimes_k \mathcal{K}^b(A\text{-mod})(T,T)$$

of L-vector spaces. It is easy to check that Φ' respects the L-algebra structure and therefore is an isomorphism of L-algebras.

Now it follows from $\mathcal{K}^b(A\text{-proj})(T,T) \cong kQ'/I'$ and proposition 8.1.2 that $L \otimes \mathcal{K}^b(A\text{-proj})(T,T)$ is isomorphic to $LQ'/(L \otimes_k I')$ as an L-algebra. Together this yields

$$\mathcal{K}^b((L \otimes_k A)\text{-proj})(L \otimes T, L \otimes T) \cong LQ'/(L \otimes_k I').$$

□

The advantage is that we can choose k as a finite field or the field of rational numbers which can be handled with exact results by a computer algebra system. As we will see in the following examples, it is often sufficient to consider the corresponding prime field.

9 Some Examples

We now turn to the practical aspects of the algorithms described in this thesis. An implementation is available as the library package pathalg for the computer algebra system MuPAD 2.0, written in MuPAD's own programming language. This has the advantage that we can use the well-tried and optimized linear algebra methods provided by MuPAD, e.g. for solving systems of linear equations, matrix equations, or finding the basis of a vector space given by a set of generators. Moreover, all computations can be carried out over arbitrary finite fields or the field of rational, real or complex numbers, respectively.

The following example sessions demonstrate the use of the pathalg library in the study of derived equivalences by computing the endomorphism algebra of a tilting complex.

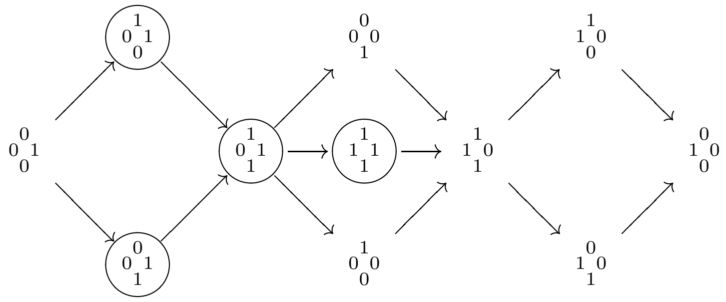
9.1 The commutative square

As a first example we compute the endomorphism algebra of a tilting module T over the algebra A with quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd}$$

and relation $\gamma\alpha = \delta\beta$.

The Auslander-Reiten quiver of A is



We define the tilting module T as the direct sum of the modules indicated in the Auslander-Reiten quiver. These are the indecomposable projective modules P_2 and P_3, the projective-injective P_1 and the indecomposable module M with dimension vector $\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}$.

We start our MuPAD session with loading the pathalg library from the appropriate path and exporting the functions of the library.

```
>> package("pathalg")

>> export(pathalg)
```

Now we choose a base field for A, here the field of rational numbers $\mathbb{Q}$, and define the algebra A by its quiver and the field. The quiver is a list of two elements: first the number of points, second a list of arrows. An arrow is defined by its source, its target and a name which is a positive integer. For example the arrow $1 \xrightarrow{1} 2$ is represented by the list [1,2,1]. Note that arrows and points are both numbered by positive integers.

```
>> Q:= Dom::Rational

                                Dom::Rational
>> A:= PathAlgebra(Q, [4, [[1,2,1], [1,3,2], [2,4,3], [3,4,4]]])

                         table(
                           "arrowsfrom4" = [],
                           "arrowsfrom3" = [4],
                           "arrowsfrom2" = [3],
                           "arrowsfrom1" = [1, 2],
                           "arrowlist" = [1, 2, 3, 4],
                           "arrow4" = [3, 4],
                           "arrow3" = [2, 4],
                           "arrow2" = [1, 3],
                           "arrow1" = [1, 2],
                           "points" = 4,
                           "field" = Dom::Rational
                         )
```

Now we have to compute a Gröbner basis of the ideal I which is generated by the set $S := \{\gamma\alpha - \delta\beta\}$. An algebra element is given as a list of paths with coefficients which represent a linear combination of paths. Arrows are composed from left to right, so the path $\gamma\alpha$ is written as [1,3]. The Gröbner basis is computed with respect to the length-lexicographic order.

```
>> S:= [ [[1, [1,3]], [-1, [2,4]]] ]

                         [[[-1, [1, 3]], [1, [2, 4]]]]
>> G:= GroebnerBasis( A, S )

                         [[[-1, [1, 3]], [1, [2, 4]]]]
```

It turns out that the generating set itself is a Gröbner basis. Given a Gröbner basis we can compute the set of nontips of I.

```
>> N:= Nontips( A, G )

                          [[1], [1, 3], [2], [3], [4]]
```

The paths in $\mathrm{NonTip}(I)$ form a $\mathbb{Q}$-basis of the algebra A.

Next we construct the direct summands of our tilting module: First, the indecomposable projective A-module P_1 corresponding to object 1.

```
>> P1:= Projective(A, G, N, 1)
```

```
--                     table(        --
|                            +- -+    |
|                        4 = | 1 |,   |
|                            +- -+    |
|                            +- -+    |
|  +-           -+      3 = | 1 |,   |
|  | 1, 1, 1, 1 |,          +- -+    |
|  +-           -+          +- -+    |
|                        2 = | 1 |,   |
|                            +- -+    |
|                            +- -+    |
|                        1 = | 1 |    |
|                            +- -+    |
--                     )             --
```

The function call `Projective` returns a module in the format of the `pathalg` library. The first element of the module is the dimension vector of the module. The second element represents the linear maps; entry i of the table is the map corresponding to arrow i.

The other indecomposable projective modules P_2 and P_3 are constructed by analogous function calls.

```
>> P2:= Projective(A, G, N, 2)
```

```
--                     table(        --
|                            +- -+    |
|                        4 = | 0 |,   |
|                            +- -+    |
|                            +- -+    |
|  +-           -+      3 = | 1 |,   |
|  | 0, 1, 0, 1 |,          +- -+    |
|  +-           -+          +- -+    |
|                        2 = | 0 |,   |
|                            +- -+    |
|                            +- -+    |
|                        1 = | 0 |    |
|                            +- -+    |
--                     )             --
```

```
>> P3:= Projective(A, G, N, 3)
```

```
--                     table(        --
|                            +- -+    |
|                        4 = | 1 |,   |
|                            +- -+    |
|                            +- -+    |
|  +-           -+      3 = | 0 |,   |
|  | 0, 0, 1, 1 |,          +- -+    |
|  +-           -+          +- -+    |
|                        2 = | 0 |,   |
|                            +- -+    |
|                            +- -+    |
|                        1 = | 0 |    |
|                            +- -+    |
--                     )             --
```

Finally, we need one non-projective module, the module M with the dimension vector ${}_{0}{\overset{1}{\underset{1}{}}}{}_{1}$. For the construction of arbitrary modules the function `Module` is provided.

```
>> M:= Module(A, [0,1,1,1], table(3=1, 4=1))

                --               table(       --
                |                    +- -+    |
                |                4 = | 1 |,   |
                |                    +- -+    |
                |                    +- -+    |
                |                3 = | 1 |,   |
                |  [0, 1, 1, 1],     +- -+    |
                |                    +- -+    |
                |                2 = | 0 |,   |
                |                    +- -+    |
                |                    +- -+    |
                |                1 = | 0 |    |
                |                    +- -+    |
                --               )            --
```

Here, the second argument is the dimension vector; the third argument is the table of maps where an entry α corresponds to the map $M(\alpha)$ for an arrow α. Arrow which do not occur in the table are assumed to correspond to zero.

Since the pathalg library computes the endomorphism algebra of a complex in $\mathcal{K}^b(A\text{-proj})$, we need to turn the modules into complexes of projectives. We find such a complex by computing a projective resolution.

```
>> C1:= projectiveResolution(A, G, N, P2, ProjComplex):

              Projective cover P[0] of input module: Done.

               Kernel K[0] of the projective cover: Done.

                 End of projective resolution reached.

                      Projective Dimension is 0.

                   --         table(              --
                   |  [0, 0],   0 = [2]  , table()  |
                   --         )                   --
```

The fifth argument determines the format of the return value of the function call. In this case, we receive a complex of projectives, in the internal format. The complex is represented by a list with its bounds, a table where entry i represents the projective module in degree i, and a table where entry i is the differential in degree i. The projective modules are given by a list of objects in A, representing the direct summands.

In the same way, we compute the other complexes of projectives.

```
>> C2:= projectiveResolution(A, G, N, P3, ProjComplex):
```

```
               Projective cover P[0] of input module: Done.

                Kernel K[0] of the projective cover: Done.

                  End of projective resolution reached.

                        Projective Dimension is 0.
>> C3:= projectiveResolution(A, G, N, M, ProjComplex):

               Projective cover P[0] of input module: Done.

                Kernel K[0] of the projective cover: Done.

                              P[1]: Done.

                              K[1]: Done.

                  End of projective resolution reached.

                        Projective Dimension is 1.
```

This shows that M is not a projective module, but its projective dimension is 1.

```
>> C4:= projectiveResolution(A, G, N, P1, ProjComplex):

               Projective cover P[0] of input module: Done.

                Kernel K[0] of the projective cover: Done.

                  End of projective resolution reached.

                        Projective Dimension is 0.
```

Now we have finished the preparations and can start the computation of the quiver with relations of $\mathrm{End}(T)$. The tilting complex T is given by a list of its direct summands.

```
>> B:= EndAlgebraComplex(A, G, N, [C1, C2, C3, C4])

                             Elapsed real time: 3 sec

         table(
           "field" = Dom::Rational,
           "arrowsfrom4" = [],
           "arrowsfrom3" = [3],
           "arrowsfrom2" = [2],
           "arrowsfrom1" = [1],    ,
           "arrowlist" = [1, 2, 3],
           "arrow3" = [3, 4],
           "arrow2" = [2, 3],
```

```
        "arrow1" = [1, 3],
        "points" = 4
      )
[4, [[1, 3], [2, 3], [3, 4]]], []
```

The first return value of the function call is the path algebra End(T) in the internal format, the second is the quiver given by the number of points, 4, and a list of arrows which assemble to

$$\begin{tikzcd} 1 \arrow[dr] & & \\ & 3 \arrow[r] & 4 \\ 2 \arrow[ur] & & \end{tikzcd}$$

The last return value is a Gröbner basis of the ideal. This list is empty, thus End(T) is the hereditary algebra with quiver of DYNKIN type $\vec{\mathbb{D}}_4$ in the above orientation. It follows that A, the algebra with quiver the commutative square, is derived equivalent to $k\vec{\mathbb{D}}_4^{op}$.

9.2 Derived equivalent tame blocks

In this section we present a verification of some results of T. HOLM (cf [Ho]) studying the algebraic structure of blocks of group algebras and their module categories. Given a defect group one would like to classify all blocks with that defect group up to a suitable notion of equivalence of algebras. In a series of papers K. ERDMANN classified all blocks of tame representation type up to Morita equivalence. Based on this work Holm obtains a classification of tame blocks up to derived equivalence.

The article [Ho] is split into several parts presenting classes of derived equivalent block algebras of dihedral type, semidihedral type and quaternion type. We will consider two examples here.

Example 1: Dihedral type — two simple modules

For any integer $s \geq 1$ and $c \in \{0, 1\}$ we define the algebra $D(2\mathcal{A})^{s,c}$ by the quiver

$$\begin{tikzcd} 1\bullet \arrow[loop left, "\alpha"] \arrow[r, shift right, "\gamma"'] & \bullet 2 \arrow[l, shift right, "\beta"'] \end{tikzcd}$$

and the relations $\gamma\beta = 0$, $\alpha^2 = c(\alpha\beta\gamma)^s$, $(\alpha\beta\gamma)^s = (\beta\gamma\alpha)^s$.

For any $s \geq 1$ and $c \in \{0, 1\}$ the algebra $D(2\mathcal{B})^{s,c}$ is given by the quiver

$$\begin{tikzcd} 1\bullet \arrow[loop left, "\alpha"] \arrow[r, shift right, "\gamma"'] & \bullet 2 \arrow[l, shift right, "\beta"'] \arrow[loop right, "\eta"] \end{tikzcd}$$

with relations $\beta\eta = 0$, $\eta\gamma = 0$, $\gamma\beta = 0$, $\alpha^2 = c(\alpha\beta\gamma)$, $\alpha\beta\gamma = \beta\gamma\alpha$, $\gamma\alpha\beta = \eta^s$.

9.2.1 Proposition. *[Ho, 2.1] Let k be an algebraically closed field of characteristic 2. For any integer $s \geq 1$ and $c \in \{0,1\}$ the k-algebras $D(2\mathcal{A})^{s,c}$ and $D(2\mathcal{B})^{s,c}$ are derived equivalent.*

This is proved by constructing a tilting complex for $D(2\mathcal{A})^{s,c}$ whose endomorphism algebra is isomorphic to $(D(2\mathcal{B})^{s,c})^{op}$, for fixed s and c. The tilting complex is the direct sum of the complex

$$Q_0 = \ 0 \longrightarrow P_2 \oplus P_2 \xrightarrow{(\gamma,\gamma\alpha)} P_1 \longrightarrow 0$$

(in degrees 0 and 1) and the complex

$$Q_1 = 0 \longrightarrow P_2 \longrightarrow 0$$

(in degree 0).

In [Ho] the ground field k is required to be algebraically closed of characteristic 2. Using the pathalg library we can give a computational proof of a variant of 9.2.1 over a given finite field of characteristic 2 for $c = 1$ and $s = 2$. Then theorem 8.3.3 states that this proof implies the result over the algebraic closure. We start our MuPAD session with loading and exporting the pathalg library.

```
>> package("pathalg")

>> export(pathalg)
```

Let $A := D(2\mathcal{A})^{2,1}$ over the field $F := \mathbb{F}_2$ and let I be the ideal generated by the set $S := \{\gamma\beta,\ \alpha^2 - (\alpha\beta\gamma)^2,\ (\alpha\beta\gamma)^2 - (\beta\gamma\alpha)^2\}$. The path algebra A is initialized for the pathalg library by passing its ground field and its quiver to the function PathAlgebra. The quiver is a list of two elements: first the number of points, second a list of arrows.

```
>> F:= Dom::IntegerMod(2)

                                 Dom::IntegerMod(2)

>> A:= PathAlgebra(F, [2, [[1,1,1],[2,1,2],[1,2,3]]])

                           table(
                             "arrowsfrom2" = [2],
                             "arrowsfrom1" = [1, 3],
                             "arrowlist" = [1, 2, 3],
                             "arrow3" = [1, 2],
                             "arrow2" = [2, 1],
                             "arrow1" = [1, 1],
                             "points" = 2,
                             "field" = Dom::IntegerMod(2)
                           )
```

Now we can compute a Gröbner basis of the ideal I with respect to the length-lexicographic order.

```
>> G:= GroebnerBasis(A, [ [[1, [2,3]]],
[[1, [1,1]], [-1,[3,2,1,3,2,1]]],
[[1, [3,2,1,3,2,1]],[-1, [1,3,2,1,3,2]]] ], 30)

[[[1 mod 2, [2, 3]]], [[1 mod 2, [1, 1]],

   [1 mod 2, [3, 2, 1, 3, 2, 1]]],

   [[1 mod 2, [1, 3, 2, 1, 3, 2]], [1 mod 2, [1, 1]]],

   [[1 mod 2, [2, 1, 1]]], [[1 mod 2, [1, 1, 3]]],

   [[1 mod 2, [1, 1, 1]]]]
```

Then we construct the set of nontips of I.

```
>> N:= Nontips(A, G)

[[1], [1, 1], [1, 3], [1, 3, 2], [1, 3, 2, 1], [1, 3, 2, 1, 3],

   [2], [2, 1], [2, 1, 3], [2, 1, 3, 2], [2, 1, 3, 2, 1],

   [2, 1, 3, 2, 1, 3], [3], [3, 2], [3, 2, 1], [3, 2, 1, 3],

   [3, 2, 1, 3, 2]]
```

The function returns a list of paths in A.

Now we can define the complex Q_0: First we convert $(\gamma, \gamma\alpha)$ into a morphism of projective modules $P_2 \oplus P_2 \longrightarrow P_1$, i.e., a natural transformation.

For this we define a (1,2)-MuPAD array with entries the elements γ and $\gamma\alpha$ of A.

```
>> a:= array(1..1, 1..2, [[ [[F(1),[3]]], [[F(1),[1,3]]] ]])

              +-                                   -+
              | [[1 mod 2, [3]]], [[1 mod 2, [1, 3]]] |
              +-                                   -+
```

Then we can convert `a` into a module homomorphism:

```
>> f:= homOfProjectives(A, G, N, [2,2], [1], a):
```

Here, a list with two elements is returned where element i represents the F-linear map $P_2 \oplus P_2(i) \longrightarrow P_1(i)$.

Now construct Q_0:

```
>> Q0:= [ [0,1], table( 0=[2,2], 1=[1] ), table(0=f) ]:
```

A complex is a list with three entries: The first one `[0,1]` sets the bounds of the complex. In this case, we must provide modules in degrees from 0 up to 1. The second element of the complex is a table where entry i represents the module in degree i. Each table entry is a list of objects of A. These objects correspond to indecomposable projective modules, for example `[2,2]` represents the projective module which is a direct sum of P_2 and P_2. The third element of the complex is a table where entry i is the differential in degree i.

In the same way we define Q_1:

```
>> Q1:= [ [0,0], table( 0=[2] ), table() ]:
```

If a complex is defined in the bounds $[m, n]$ then the differentials must be defined for $m, \ldots, n-1$.

Finally we can start the computation of the endomorphism algebra. The tilting complex is given by a list of its indecomposable summands.

```
>> B:= EndAlgebraComplex(A, G, N, [Q0, Q1])

                              Elapsed real time: 305 sec

table(
  "field" = Dom::IntegerMod(2),
  "arrowsfrom2" = [3, 4],
  "arrowsfrom1" = [1, 2],
  "arrowlist" = [1, 2, 3, 4],
  "arrow4" = [2, 2],           ,
  "arrow3" = [2, 1],
  "arrow2" = [1, 2],
  "arrow1" = [1, 1],
  "points" = 2
)

   [2, [[1, 1], [1, 2], [2, 1], [2, 2]]],

   [[[1 mod 2, [1, 1, 1]]], [[1 mod 2, [1, 1, 2]]],

   [[1 mod 2, [2, 4]]], [[1 mod 2, [3, 1, 1]]],

   [[1 mod 2, [3, 2]]], [[1 mod 2, [4, 3]]],

   [[1 mod 2, [4, 4, 4]]],

   [[1 mod 2, [1, 1]], [1 mod 2, [1, 2, 3]]],

   [[1 mod 2, [1, 1]], [1 mod 2, [2, 3, 1]]],

   [[1 mod 2, [3, 1, 2]], [1 mod 2, [4, 4]]]]
```

Here, the first return value is $\mathrm{End}(T)$ in the pathalg format. The second return value is the quiver of the computed endomorphism algebra. So in this case, we have the quiver

$$\begin{tikzcd} 1\bullet \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \bullet 2 \arrow[l, shift left, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd}$$

and the third return value gives us a Gröbner basis of the ideal which is the set $\{\alpha^3, \beta\alpha^2, \beta\eta, \alpha^2\gamma, \beta\gamma, \gamma\eta, \eta^3, \alpha^2 + \gamma\beta\alpha, \alpha^2 + \alpha\gamma\beta, \beta\alpha\gamma + \eta^2\}$.

Obviously the quiver we received is the opposite quiver of $D(2\mathcal{B})^{2,1}$. To check if the ideal is as required, we have to compute a Gröbner basis of the generating set given in the definition of $D(2\mathcal{B})^{2,1}$.

```
>> s:= [ [[1, [2,4]]], [[1, [4,3]]], [[1, [3,2]]],
[[1,[1,1]], [1,[1,2,3]]], [[1, [1,2,3]], [1, [2,3,1]]],
[[1, [3,1,2]], [1, [4,4]]] ]:

>> GroebnerBasis(B[1], s)

[[[1 mod 2, [2, 4]]], [[1 mod 2, [4, 3]]], [[1 mod 2, [3, 2]]],

   [[1 mod 2, [1, 1]], [1 mod 2, [1, 2, 3]]],

   [[1 mod 2, [1, 2, 3]], [1 mod 2, [2, 3, 1]]],

   [[1 mod 2, [3, 1, 2]], [1 mod 2, [4, 4]]],

   [[1 mod 2, [4, 4, 4]]], [[1 mod 2, [3, 1, 1]]],

   [[1 mod 2, [1, 1, 2]]], [[1 mod 2, [1, 1, 1]]]]
```

Comparing the two Gröbner bases we find that they generate the same ideal. Thus we have proved proposition 9.2.1 in case $s = 2$ and $c = 1$.

Example 2: Semidihedral type — three simple modules

For any integer $s \geq 1$ let $SD(3\mathcal{A})_1^s$ be the algebra given by the quiver

$$1 \underset{\gamma}{\overset{\beta}{\leftrightarrows}} 2 \underset{\eta}{\overset{\delta}{\leftrightarrows}} 3$$

and the relations $\beta\gamma = 0$, $\delta\eta\delta = (\gamma\beta\delta\eta)^{s-1}\gamma\beta\delta$, $\eta\delta\eta = (\eta\gamma\beta\delta)^{s-1}\eta\gamma\beta$.

For any integer $s \geq 2$ let $SD(3\mathcal{B})_1^s$ be the algebra defined by the quiver

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\leftrightarrows}} 2 \underset{\eta}{\overset{\delta}{\leftrightarrows}} 3$$

with relations $\alpha\beta = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$, $\alpha^s = \beta\delta\eta\gamma$, $\eta\delta\eta = \eta\gamma\beta$, $\delta\eta\delta = \gamma\beta\delta$.

9.2.2 Proposition. *[Ho, 3.2.1] Let k be an algebraically closed field of characteristic 2. For any integer $s \geq 2$ the k-algebras $SD(3\mathcal{A})_1^s$ and $SD(3\mathcal{B})_1^s$ are derived equivalent.*

This is proved using the tilting complex $Q = Q_1 \oplus Q_2 \oplus Q_3$ over $SD(3\mathcal{A})_1^s$ where

$$Q_1 = 0 \longrightarrow P_1 \longrightarrow 0$$

(in degree 0),

$$Q_2 = 0 \longrightarrow P_1 \oplus P_1 \overset{(\beta,\beta\delta\eta)}{\longrightarrow} P_2 \longrightarrow 0$$

(in degrees 0 and 1) and

$$Q_3 = 0 \longrightarrow P_1 \xrightarrow{\beta\delta} P_3 \longrightarrow 0.$$

We will verify proposition 9.2.2 for the parameter $s = 2$. Again, we start our MuPAD session by loading and exporting the pathalg library.

```
>> package("pathalg")

>> export(pathalg)
```

We define the base field $\mathbb{F}_2$ and the path algebra A with the quiver of $SD(3\mathcal{A})_1^2$.

```
>> F:= Dom::IntegerMod(2)

                              Dom::IntegerMod(2)
>> A:= PathAlgebra(F, [3, [[2,1,1], [3,2,2], [2,3,3], [1,2,4]]])

                        table(
                          "arrowsfrom3" = [2],
                          "arrowsfrom2" = [1, 3],
                          "arrowsfrom1" = [4],
                          "arrowlist" = [1, 2, 3, 4],
                          "arrow4" = [1, 2],
                          "arrow3" = [2, 3],
                          "arrow2" = [3, 2],
                          "arrow1" = [2, 1],
                          "points" = 3,
                          "field" = Dom::IntegerMod(2)
                        )
```

We define the generating set S of the ideal I as above and compute a Gröbner basis of I with respect to the length-lexicographic order.

```
>> S:= [ [[1, [4,1]]], [[1, [2,3,2]], [-1, [2,1,4,3,2,1,4]]],
[[1,[3,2,3]], [-1, [1,4,3,2,1,4,3]]] ]:
>> G:= GroebnerBasis(A, S, 30)

[[[1 mod 2, [4, 1]]], [[1 mod 2, [2, 3, 2]],

   [1 mod 2, [2, 1, 4, 3, 2, 1, 4]]],

   [[1 mod 2, [3, 2, 3]], [1 mod 2, [1, 4, 3, 2, 1, 4, 3]]],

   [[1 mod 2, [4, 3, 2, 3]]], [[1 mod 2, [2, 3, 2, 1]]],

   [[1 mod 2, [3, 2, 3, 2, 3]]], [[1 mod 2, [2, 3, 2, 3, 2]]]]
```

With the Gröbner basis G we can compute the list of nontips of I.

```
>> N:= Nontips(A, G)
```

```
[[1], [1, 4], [1, 4, 3], [1, 4, 3, 2], [1, 4, 3, 2, 1],

   [1, 4, 3, 2, 1, 4], [2], [2, 1], [2, 1, 4], [2, 1, 4, 3],

   [2, 1, 4, 3, 2], [2, 1, 4, 3, 2, 1], [2, 3], [2, 3, 2],

   [2, 3, 2, 3], [3], [3, 2], [3, 2, 1], [3, 2, 1, 4],

   [3, 2, 1, 4, 3], [3, 2, 1, 4, 3, 2], [3, 2, 1, 4, 3, 2, 1],

   [3, 2, 3], [3, 2, 3, 2], [4], [4, 3], [4, 3, 2], [4, 3, 2, 1],

   [4, 3, 2, 1, 4], [4, 3, 2, 1, 4, 3], [4, 3, 2, 1, 4, 3, 2],

   [4, 3, 2, 1, 4, 3, 2, 1]]
```

Now we have to define the direct summands of the tilting complex. The first summand Q_1 is only defined in degree zero.

```
>> Q1:= [ [0,0], table(0=[1]), table() ]

                      --         table(               --
                      |  [0, 0],   0 = [1]  , table()  |
                      --         )                    --
```

For Q_2 we first determine the map $f_2 = (\beta, \beta\delta\eta) : P_1 \oplus P_1 \longrightarrow P_2$.

```
>> a:= array(1..1, 1..2, [[ [[F(1), [1]]], [[F(1), [3,2,1]]] ]])

              +-                                      -+
              | [[1 mod 2, [1]]], [[1 mod 2, [3, 2, 1]]] |
              +-                                      -+
>> f2:= homOfProjectives(A, G, N, [1,1], [2], a):
```

Now we can define Q_2.

```
>> Q2:= [ [0,1], table(0=[1,1], 1=[2]), table(0=f2) ]:
```

Similarly we define Q_3. First, we determine the map $f_3 = \delta\beta : P_1 \longrightarrow P_3$.

```
>> b:= array(1..1, 1..1, [[ [[F(1), [2,1]]] ]])

                      +-                    -+
                      | [[1 mod 2, [2, 1]]] |
                      +-                    -+
>> f3:= homOfProjectives(A, G, N, [1], [3], b):
>> Q3:= [ [0,1],table(0=[1], 1=[3]), table(0=f3) ]:
```

Now that we have defined all summands of the complex Q we can check if Q satisfies the first condition on a tilting complex: $\mathrm{Hom}(Q, Q[i]) = 0$ for any $i \neq 0$.

```
>> isTiltingComplex(A, G, N, [Q1, Q2, Q3])

                                TRUE
```

Since this is the case, we start the computation of the quiver with relations of the endomorphism algebra of Q.

```
>> B:= EndAlgebraComplex( A, G, N, [Q1, Q2, Q3] )

                              Elapsed real time: 435 sec

table(
  "field" = Dom::IntegerMod(2),
  "arrowsfrom3" = [5],
  "arrowsfrom2" = [3, 4],
  "arrowsfrom1" = [1, 2],
  "arrowlist" = [1, 2, 3, 4, 5],
  "arrow5" = [3, 2],            ,
  "arrow4" = [2, 3],
  "arrow3" = [2, 1],
  "arrow2" = [1, 2],
  "arrow1" = [1, 1],
  "points" = 3
)

   [3, [[1, 1], [1, 2], [2, 1], [2, 3], [3, 2]]],

   [[[1 mod 2, [1, 1, 1]]], [[1 mod 2, [1, 2]]],

   [[1 mod 2, [2, 3]]], [[1 mod 2, [3, 1]]],

   [[1 mod 2, [5, 3, 2, 4, 5]]], [[1 mod 2, [1, 1]],

   [1 mod 2, [2, 4, 5, 3]]], [[1 mod 2, [3, 2, 4, 5]],

   [1 mod 2, [4, 5, 3, 2]]],

   [[1 mod 2, [3, 2, 4]], [1 mod 2, [4, 5, 4]]],

   [[1 mod 2, [5, 3, 2]], [1 mod 2, [5, 4, 5]]]]
```

The second return value gives us the quiver of $\mathrm{End}(Q)$, in this case

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \underset{\eta}{\overset{\delta}{\rightleftarrows}} 3$$

and the third return value is a Gröbner basis of the ideal of relations. Now we check if this is the required ideal of $(SD(3\mathcal{B})_1^2)^{op}$ by computing a Gröbner basis of the ideal generated by $\{\beta\alpha, \alpha\gamma, \gamma\beta, \alpha^2 + \gamma\eta\delta\beta, \eta\delta\eta + \beta\gamma\eta, \delta\eta\delta + \delta\beta\gamma\}$.

```
>> s:= [ [[1, [1,2]]], [[1, [3,1]]], [[1, [2,3]]],
[[1, [1,1]], [1, [2,4,5,3]]], [[1, [5,4,5]], [1, [5,3,2]]],
[[1, [4,5,4]], [1, [3,2,4]]] ]:
```

```
>> GroebnerBasis(B[1], s)

[[[1 mod 2, [1, 2]]], [[1 mod 2, [3, 1]]], [[1 mod 2, [2, 3]]],

   [[1 mod 2, [1, 1]], [1 mod 2, [2, 4, 5, 3]]],

   [[1 mod 2, [5, 4, 5]], [1 mod 2, [5, 3, 2]]],

   [[1 mod 2, [4, 5, 4]], [1 mod 2, [3, 2, 4]]],

   [[1 mod 2, [1, 1, 1]]], [[1 mod 2, [5, 3, 2, 4, 5]]],

   [[1 mod 2, [4, 5, 3, 2]], [1 mod 2, [3, 2, 4, 5]]]]
```

Comparing this result with the Gröbner basis returned by `EndAlgebraComplex` we find that they both generate the same ideal. This proves proposition 9.2.2 in case $s = 2$.

9.3 A covering technique for derived equivalences

In [As] H. Asashiba develops a "covering technique" for derived equivalences to give a new approach to the following problem: For a finite group G, is a block kG with abelian defect derived equivalent to its Brauer correspondent?

As an example he proved the following proposition using his covering technique.

9.3.1 Proposition. *[As, 5.2] Let k be an algebraically closed field of characteristic 2. Then the principal block Λ of $kSL(2,4)$ is derived equivalent to its Brauer correspondent Γ.*

The algebra Λ is given by the quiver

$$\begin{tikzcd} 2 \arrow[r, shift left, "\alpha_1"] & 1 \arrow[l, shift left, "\alpha_2"] \arrow[r, shift right, "\beta_2"'] & 3 \arrow[l, shift right, "\beta_1"'] \end{tikzcd}$$

and the relations $\beta_2\beta_1\alpha_2\alpha_1 = \alpha_2\alpha_1\beta_2\beta_1$, $\alpha_1\alpha_2 = 0$ and $\beta_1\beta_2 = 0$.

The algebra Γ is given by the quiver

$$\begin{tikzcd} & 1 \arrow[dl, shift left, "\alpha_2"] \arrow[dr, shift left, "\gamma_1"] & \\ 2 \arrow[ur, shift left, "\alpha_1"] \arrow[rr, shift left, "\beta_2"] & & 3 \arrow[ul, shift left, "\gamma_2"] \arrow[ll, shift left, "\beta_1"] \end{tikzcd}$$

and the relations $\alpha_2\alpha_1 = \gamma_1\gamma_2$, $\beta_2\beta_1 = \alpha_1\alpha_2$, $\gamma_2\gamma_1 = \beta_1\beta_2$, $\beta_1\alpha_1 = 0$, $\alpha_2\beta_2 = 0$, $\gamma_1\beta_1 = 0$, $\beta_2\gamma_2 = 0$, $\alpha_1\gamma_1 = 0$ and $\gamma_2\alpha_2 = 0$.

The proof that Λ and Γ are derived equivalent is carried out via $\mathbb{Z}$-coverings of both spectroids.

Let A be the algebra defined by the quiver

$$\begin{tikzcd}
2 & & 3 \\
& 1 \arrow[ul, "\alpha"'] \arrow[ur, "\beta"] & \\
5 \arrow[ur, "\gamma"] & & 6 \arrow[ul, "\delta"'] \\
& 4 \arrow[ul, "\epsilon"] \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

with relations $\alpha\gamma = 0$ and $\beta\delta = 0$. Then A is a $\mathbb{Z}$-covering of Λ. Define a tilting spectroid T over A as the direct sum of the chain complexes

$$T_1 = 0 \longrightarrow P_2 \oplus P_3 \xrightarrow{(\alpha,\beta)} P_1 \longrightarrow 0$$

in degrees 0 and 1,

$$T_2 = 0 \longrightarrow P_2 \longrightarrow 0,$$
$$T_3 = 0 \longrightarrow P_3 \longrightarrow 0,$$

both in degree 0,

$$T_4 = 0 \longrightarrow P_5 \oplus P_6 \xrightarrow{(\epsilon,\zeta)} P_4 \longrightarrow 0$$

in degrees 0 and 1,

$$T_5 = 0 \longrightarrow P_5 \longrightarrow 0,$$
$$T_6 = 0 \longrightarrow P_6 \longrightarrow 0$$

both in degree 0. Let B be the algebra given by the quiver

$$\begin{tikzcd}
& 1 & \\
2 \arrow[ur, "\alpha"] & 4 & 3 \arrow[ul, "\beta"'] \\
6 \arrow[u, "\gamma"] \arrow[ur, "\zeta"'] & & 5 \arrow[u, "\delta"'] \arrow[ul, "\epsilon"]
\end{tikzcd}$$

and relations $\alpha\gamma = 0$ and $\beta\delta = 0$. It can be shown that B is a $\mathbb{Z}$-covering of Γ.

For the proof of proposition 9.3.1 Asashiba shows that $\mathrm{End}(T)^{op} \cong B$ and that the tilting complex T is compatible with the coverings A of Λ and B of Γ in a certain sense. It follows that Λ is derived equivalent to Γ.

We will prove the part $\mathrm{End}(T)^{op} \cong B$ over the field $\mathbb{F}_2$ using the pathalg library.

```
>> package("pathalg")

>> export(pathalg)

>> F:= Dom::IntegerMod(2):
```

First, we define the algebra A as described above.

```
>> A:= PathAlgebra(F, [6, [[1,2,1], [1,3,2], [5,1,3], [6,1,4],
[4,5,5], [4,6,6]]])

                        table(
                          "arrowsfrom6" = [4],
                          "arrowsfrom5" = [3],
                          "arrowsfrom4" = [5, 6],
                          "arrowsfrom3" = [],
                          "arrowsfrom2" = [],
                          "arrowsfrom1" = [1, 2],
                          "arrowlist" = [1, 2, 3, 4, 5, 6],
                          "arrow6" = [4, 6],
                          "arrow5" = [4, 5],
                          "arrow4" = [6, 1],
                          "arrow3" = [5, 1],
                          "arrow2" = [1, 3],
                          "arrow1" = [1, 2],
                          "points" = 6,
                          "field" = Dom::IntegerMod(2)
                        )
```

Next, we compute a Gröbner basis of the ideal I generated by the set $\{\alpha\gamma, \beta\delta\}$.

```
>> G:= GroebnerBasis(A, [ [[1, [3,1]]], [[1, [4,2]]] ])

               [[[1 mod 2, [3, 1]]], [[1 mod 2, [4, 2]]]]
```

The generating set itself turns out to be a Gröbner basis of I. Given G we can compute the set $\mathrm{NonTip}(I)$.

```
>> N:= Nontips(A, G)

[[1], [2], [3], [3, 2], [4], [4, 1], [5], [5, 3], [5, 3, 2],

   [6], [6, 4], [6, 4, 1]]
```

Before we can start defining the direct summands of our tilting complex T, we have to construct the map $f := (\alpha, \beta) : P_2 \oplus P_3 \longrightarrow P_1$ explicitly as a module homomorphism.

```
>> a:= array(1..1, 1..2, [[ [[F(1), [1]]], [[F(1), [2]]] ]])

                       +-                                    -+
                       | [[1 mod 2, [1]]], [[1 mod 2, [2]]] |
                       +-                                    -+
>> f:= homOfProjectives(A, G, N, [2,3], [1], a)

-- +-         -+  +-         -+  +-         -+  +-         -+
|  | 0 mod 2 |, | 1 mod 2 |, | 1 mod 2 |, | 0 mod 2 |,
-- +-         -+  +-         -+  +-         -+  +-         -+


   +-         -+  +-         -+ --
   | 0 mod 2 |, | 0 mod 2 |  |
   +-         -+  +-         -+ --
```

Such a module homomorphism f is a natural transformation and in the pathalg library it is represented by a list of matrices where the i-th matrix is the linear map $f_i : (P_2 \oplus P_3)(i) \longrightarrow P_1(i)$.

The same holds for the map $g := (\zeta, \epsilon) : P_5 \oplus P_6 \longrightarrow P_4$.

```
>> b:= array(1..1, 1..2, [[ [[F(1), [5]]], [[F(1), [6]]] ]])

                     +-                                  -+
                     | [[1 mod 2, [5]]], [[1 mod 2, [6]]] |
                     +-                                  -+
>> g:= homOfProjectives(A, G, N, [5,6], [4], b):
```

Now we are ready to define the complexes $T_1, \ldots, T_6$.

```
>> T1:= [ [0,1], table(0=[2,3], 1=[1]), table(0=f) ]:
>> T2:= [ [0,0], table(0=[2]), table() ]:
>> T3:= [ [0,0], table(0=[3]), table() ]:
>> T4:= [ [0,1], table(0=[5,6], 1=[4]), table(0=g) ]:
>> T5:= [ [0,0], table(0=[5]), table() ]:
>> T6:= [ [0,0], table(0=[6]), table() ]:
```

We check if their direct sum T satisfies the condition on a tilting complex $\mathcal{K}^b(A\text{-proj})(T, T[i]) = 0$ for all $i \neq 0$.

```
>> isTiltingComplex(A, G, N, [T1, T2, T3, T4, T5, T6])

                                TRUE
```

This being the case, we can trigger the computation of the endomorphism algebra of T.

```
>> EndAlgebraComplex(A, G, N, [T1, T2, T3, T4, T5, T6])

                           Elapsed real time: 9 sec

table(
  "field" = Dom::IntegerMod(2),
  "arrowsfrom6" = [],
  "arrowsfrom5" = [],
  "arrowsfrom4" = [5, 6],
  "arrowsfrom3" = [4],
  "arrowsfrom2" = [3],
  "arrowsfrom1" = [1, 2],
  "arrowlist" = [1, 2, 3, 4, 5, 6],,
  "arrow6" = [4, 6],
  "arrow5" = [4, 5],
  "arrow4" = [3, 5],
  "arrow3" = [2, 6],
  "arrow2" = [1, 3],
  "arrow1" = [1, 2],
  "points" = 6
)
```

```
[6, [[1, 2], [1, 3], [2, 6], [3, 5], [4, 5], [4, 6]]],

[[[1 mod 2, [1, 3]]], [[1 mod 2, [2, 4]]]]
```

The second return value represents the quiver of $\mathrm{End}(T)$ which looks like

$$\begin{tikzcd}
& 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 \arrow[d, "\gamma"'] & 4 \arrow[dl, "\zeta"] \arrow[dr, "\epsilon"'] & 3 \arrow[d, "\delta"] \\
6 & & 5
\end{tikzcd}$$

and is obviously the opposite of the quiver of B. The third return value gives the relations $\gamma\alpha = 0$ and $\delta\beta = 0$ as required. This shows that A is derived equivalent to B and thus Λ is derived equivalent to Γ by [As].

References

[As] H. Asashiba, *A covering technique for derived equivalences*, Preprint 95–119, Preprintreihe des SFB 343, Universität Bielefeld (1995)

[CIW] A. Cohen, G. Ivanyos, D. Wales, *Finding the radical of an algebra of linear transformations*, Journal of Pure and Applied Algebra 117 & 118 (1997) 177-193

[CLS] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties and Algorithms*, UTM Series, Springer-Verlag (1992)

[GR] P. Gabriel, A.V. Roiter, *Representations of finite-dimensional algebras*, Encyclopaedia of Math. Sci. 73, Algebra VIII, Berlin, Heidelberg, New York (1992)

[Gr] E. Green, *Noncommutative Gröbner Bases, and Projective Resolutions*, Progress in Mathematics, Vol.173 (1999), 29-60

[Ha] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, LMS Lecture notes series 119 (1988)

[Ho] T. Holm, *Derived equivalent tame blocks*, Preprint Nr. 12 (1996), Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik

[Kr] A. Krause, *pathalg — A library for MuPAD*, (2001), available at `http://www.mathematik.uni-bielefeld.de/~akrause/PATHALG`

[Mo] T. Mora, *An introduction to commutative and non-commutative Gröbner Bases*

[Mu1] The MuPAD Group, *MuPAD User's Manual*, Wiley-Teubner (1996)

[Mu2] J. Gerhard, W. Oevel, F. Postel, S. Wehmeier, *MuPAD Tutorial*, Springer-Verlag (2000)

[Rc] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) 39 (1989) 436-456

[Ri] C.M. Ringel, *Tame algebras and integral quadratic forms*, Springer Lecture Notes in Mathematics 1099 (1984)

[We] C. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics 38 (1994)