MOTIVIC FUNCTORS

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1. Introduction

The purpose of this note is to give an overview about my contributions to the results developed by Bjørn Ian Dundas, Paul Arne Østvær and myself in [4]. These results show in particular that certain enriched functors constitute a well-behaved model for the stable motivic homotopy category. The latter category has been introduced by Vladimir Voevodsky in his proof of the Milnor conjecture [15]. Instead of giving proofs – which can be found in [4] – I will concentrate on the general ideas lying behind the setup of [4], with an emphasis on my influence. For the convenience of the reader, section 2 contains an exposition on motivic homotopy theory, including several of my results from [4]. These are concentrated in section 2.4. In section 4, I will indicate how this theory interacts with motivic cohomology. The application to motivic cohomology arose from an observation I made in the beginning of the whole process, and it is going to be treated in detail in a separate paper, which is joint with Bjørn Ian Dundas and Paul Arne Østvær. Thus I will concentrate on my observation – which motivated this work to some extent – in section 4.

Categories will occur frequently in this text, and if $\mathcal{C}$ is a category, the set of morphisms in $\mathcal{C}$ from the object A to the object B is denoted $\mathbf{Set}_{\mathcal{C}}(A, B)$. A category which has all limits and all colimits is called *bicomplete*.

2. Motivic Homotopy Theory

The fundamental gadget is the category $\mathbf{Sm}/S$ of smooth schemes of finite type over a fixed Noetherian base scheme S of finite dimension. The reader should think of S being the spectrum of some field k, for example the field $\mathbb{C}$ of complex numbers. The objects in $\mathbf{Sm}/S$ can be viewed as algebraic manifolds. To explain how these algebraic manifolds are going to be treated, I will mention some ideas from algebraic topology.

2.1. **Algebraic Topology.** From topology, one knows that in order to study (topological) manifolds, it is convenient to consider them as a part of something larger, namely the category **Top** of topological spaces. In **Top**, one can perform a lot of constructions (arbitrary limits and colimits) which lead out of the category of topological manifolds. Furthermore, the notions of *homotopy* and *weak homotopy equivalence* determine the homotopy category of topological spaces $\mathbf{Ho}(\mathbf{Top})$, which is more accessible, but nevertheless carries a lot of information. For example, any (generalized) homology theory of topological spaces just depends on the homotopy category. In order to simplify the situation further, one can exploit the fact that the effect of a *suspension* to a homology theory is just a dimension shift. Inverting the suspension functor on the homotopy category leads to the *stable homotopy category* of topological spaces. This category has a rich structure: the morphism sets are abelian groups, it has a smash product, it is triangulated, and every homology or cohomology theory on the category of topological spaces is represented by an object of the stable homotopy category.

To give a more precise description of the stable homotopy category, I want to use essentially two ingredients: the combinatorial model of CW-complexes given by simplicial sets which also leads to $\mathbf{Ho}(\mathbf{Top})$, and the theory of model categories. A reason for the first ingredient is that honest topological space may carry all kinds of pathologies, thereby leading to problems with function spaces. Furthermore, both ingredients are heavily used in the context of motivic homotopy theory.

2.2. **Simplicial Sets and Model Categories.** A *simplicial set* is a functor $K\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Here $\mathbf{Set}$ is the category of sets, and Δ has as objects the finite ordered sets $[n] := \{0 < 1 < \cdots < n\}$ with $n \geq 0$, and as morphisms the order-preserving maps. Any simplicial set K has as a *geometric realization* a topological space $|K|$ (in fact, a CW-complex). It can be described by the following two conditions.

- The geometric realization $|\Delta^n|$ of the simplicial set $\Delta^n = \mathbf{Set}_\Delta(-,[n])$ represented by $[n]$ is the standard n-simplex in $\mathbb{R}^{n+1}$.
- Geometric realization commutes with colimits.

The Yoneda lemma implies that every simplicial set is a colimit of representable ones, thus these two conditions determine a functor $|-|\colon \mathbf{sSet} \longrightarrow \mathbf{Top}$ from the category of simplicial sets to the category of topological spaces. Quillen proved in [12] that the category of simplicial sets has a homotopy category $\mathbf{Ho}(\mathbf{sSet})$ which is equivalent to the homotopy category $\mathbf{Ho}(\mathbf{Top})$ of topological spaces via geometric realization. For this purpose he used his theory of model categories. Hence if you are interested in the homotopy theory of topological spaces, you might as well try to work with simplicial sets – their simplicity is sometimes an advantage.

Model categories are very important in the construction of the stable motivic homotopy category mentioned in the beginning, thus I will spend a few words on them. Details can be found in [12] and the more modern book [5]. A *model category* is a bicomplete category $\mathcal{C}$ with extra structure, the *model structure*. Among this extra structure is a class $\mathcal{W}$ of morphisms called *weak equivalences*. The rest of this extra structure (which amounts to a class of *fibrations* and a class of *cofibrations*, and the validity of four axioms) allows one to construct the homotopy category $\mathbf{Ho}(\mathcal{C},\mathcal{W})$ of $\mathcal{C}$ with respect to $\mathcal{W}$. That is, one can describe a universal functor $\gamma\colon \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{C},\mathcal{W})$ such that the image of a weak equivalence under γ is always an isomorphism. This is a kind of localization procedure.

The derived category of a ring R is an example: it can be obtained as the homotopy category of a model category. The objects in this model category are chain complexes of R-modules, the weak equivalences are the chain maps inducing an isomorphism on homology groups. Further examples are topological spaces and simplicial sets. The weak equivalences in the first case are the weak homotopy equivalences I already mentioned, in the second case the weak equivalences are the maps which become homotopy equivalences after geometric realization. One should note that a category might be a model category in several distinct ways. An example is the category of topological spaces, which has a different interesting model structure in which the weak equivalences are chosen to be the honest homotopy equivalences. More examples will occur later.

2.3. **Spectra.** One of the simplest approaches to the stable homotopy category is given by Bousfield and Friedlander in [1]. They use simplicial sets which are – for technical reasons – equipped with a basepoint. To save me some writing, let me abbreviate this.

Definition 2.1. A *space* is a pointed simplicial set. In other words, a space is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}_*$, where $\mathbf{Set}_*$ is the category of pointed sets. Let $\mathcal{S}$ denote the category of spaces, where maps have to preserve the basepoint. A map in $\mathcal{S}$ is a *weak equivalence* if the map of underlying simplicial sets is a weak equivalence.

If K is a simplicial set, adding a disjoint basepoint yields a space K_+. A very important space is the circle $S^1 = \Delta^1/\partial\Delta^1$, which is defined as the simplicial interval Δ^1 with its two endpoints identified. Thus the geometric realization of S^1 is homeomorphic to the topological space of complex numbers of length one. The higher-dimensional spheres are built as suspensions of S^1, in the following sense. The *smash product* of two spaces K and L is the quotient $K \wedge L$ of the canonical embedding $K \vee L \hookrightarrow K \times L$. Here $K \vee L$ is the one-point-union of K and L. In particular, the functor $S^1 \wedge -$ given by smashing with the circle is a legitimate candidate for the suspension. Then the $n+1$-*sphere* is defined as $S^1 \wedge S^n$, the suspension of the n-sphere. Since geometric realization preserves the smash product, $|S^n|$ is homeomorphic to the topological space of unit vectors in $\mathbb{R}^{n+1}$. Now I recall the definition of spectra from [1, 2.1].

Definition 2.2. A *spectrum* E consists of a sequence $(E_0, E_1, \ldots, E_n, \ldots)$ of spaces and structure maps $\sigma_n^E\colon S^1 \wedge E_n \longrightarrow E_{n+1}$. A *map of spectra* is then a sequence of maps which are compatible with the structure maps. Let **Spectra** denote the category of spectra.

Any space K has a *suspension spectrum* $\Sigma^\infty K = (K, S^1 \wedge K, S^1 \wedge S^1 \wedge K, \ldots)$. All structure maps of $\Sigma^\infty K$ are identities. In particular, the suspension spectrum of the zero sphere $S^0 = \Delta^0_+$ is the sphere spectrum $(S^0, S^1, S^2, \ldots)$. The structure maps of a spectrum E allow the definition of stable homotopy groups. Since suspension is compatible with homotopy classes, one gets a natural map $\pi_i|K| \longrightarrow \pi_{i+1}|S^1 \wedge K|$, and the i-th stable homotopy group $\pi_i^{\mathrm{stable}}E$ of a spectrum E is the colimit of the sequence

$$\pi_{i+n}|E_n| \longrightarrow \pi_{i+n+1}|S^1 \wedge E_n| \xrightarrow{\pi_{i+n+1}|\sigma_n^E|} \pi_{i+n+1}|E_{n+1}| \longrightarrow \cdots$$

of abelian groups, where i is an arbitrary integer and n has to be big enough to make sense of the sequence.

Definition 2.3. A map of spectra is a *stable equivalence* if it induces an isomorphism on all stable homotopy groups.

Bousfield and Friedlander proved in [1] that the category of spectra is a model category in such a way that the weak equivalences are precisely the stable equivalences. Moreover, the associated homotopy category is then equivalent to the stable homotopy category mentioned before, which was originally constructed by Boardman according to [18].

2.4. **Motivic Spaces.** The great surprise now is that this program can be carried out in algebraic geometry, with the category $\mathbf{Sm}/S$ of smooth schemes of finite type over a base scheme S as input. It is not possible to do homotopy theory directly in $\mathbf{Sm}/S$, since a lot of constructions necessary for homotopy theory (that is, all limits and colimits) are not available. The simplest way to remove this is to consider *presheaves* on $\mathbf{Sm}/S$. A presheaf on $\mathbf{Sm}/S$ is a functor $\mathbf{Sm}/S^{\mathrm{op}} \longrightarrow \mathbf{Set}$, and a map of presheaves is a natural transformation. The category of presheaves on $\mathbf{Sm}/S$ is then bicomplete, because the category $\mathbf{Set}$ is so. Moreover, all limits and colimits are calculated *scheme-wise* in the category of sets. As a formula, this means

$$(\operatorname{colim} D)(U) = \operatorname{colim}(D(U))$$

for every smooth scheme U in $\mathbf{Sm}/S$ and every diagram D in the category of presheaves.

In some sense, the category of presheaves is obtained by adding formally all colimits to $\mathbf{Sm}/S$. To see this, one has to recall that the Yoneda lemma implies that $\mathbf{Sm}/S$ is a full subcategory of the category of presheaves (namely the representables $\mathbf{Set}_{\mathbf{Sm}/S}(-,U)$), and that every presheaf is a colimit of representables. Note further that the limits which already exist in $\mathbf{Sm}/S$ are the same in the category of presheaves. This category of set-valued presheaves has no direct connection to the category $\mathcal{S}$ of spaces, but this can be easily achieved by changing the target category.

Definition 2.4. A *motivic space* is a functor $\mathbf{Sm}/S^{\mathrm{op}} \longrightarrow \mathcal{S}$. The category of motivic spaces is denoted $\mathcal{M}$.

As the category of set-valued presheaves on $\mathbf{Sm}/S$, the category of motivic spaces (or space-valued presheaves on $\mathbf{Sm}/S$) is bicomplete. The difference is that there is now an easy starting point to do homotopy theory.[1] I should remark that three steps are required to pass to motivic homotopy theory, and the first step goes as follows.

[1]An amazing property of motivic homotopy theory is that one can nevertheless work with set-valued presheaves as motivic spaces, which is essentially due to the cosimplicial structure of the collection of affine spaces $\mathbb{A}^n$ in $\mathbf{Sm}/S$ and the "contractibility" of the affine line.

Definition 2.5. A map $f\colon A \longrightarrow B$ of motivic spaces is a *scheme-wise weak equivalence* if the map $f(U)\colon A(U) \longrightarrow B(U)$ is a weak equivalence for every smooth scheme U in $\mathbf{Sm}/S$.

It is easy to work out that scheme-wise weak equivalences are part of a model structure on motivic spaces. However, there are two canonical choices to complete the model structure, that is, to define a class of fibrations and a class of cofibrations. It is important for the techniques used in [4] to pick the right choice. The standard approach – as pursued by Jardine in [8], and thus also in the fundamental paper [11] of Morel and Voevodsky in the disguise of sheaves – is to say that a map of motivic spaces is a cofibration if it is a monomorphism (or, equivalently, a *scheme-wise cofibration*). This leads to a complicated notion of fibrations. For example, the cardinal number of the set of morphisms in $\mathbf{Sm}/S$ then plays a role. However, the cofibrations are simple, and every motivic space A is cofibrant in this sense, meaning that the unique map $* \longrightarrow A$ is always a monomorphism.

The approach used in [4] is to define a *scheme-wise fibration* as a map of motivic spaces $f\colon A \longrightarrow B$ such that $f(U)\colon A(U) \longrightarrow B(U)$ is a Kan fibration of underlying simplicial sets for every scheme U in $\mathbf{Sm}/S$. A consequence of this simple definition is that scheme-wise fibrations are closed under filtered colimits, since this is true for Kan fibrations. The corresponding notion of cofibrations is then more complicated, but the situation is good enough: every cofibration can be built in a specific way from maps of the form

$$(\mathbf{Set}_{\mathbf{Sm}/S}(-,U) \times (\partial\Delta^n \hookrightarrow \Delta^n))_+,$$

where U varies through $\mathbf{Sm}/S$ and $\partial\Delta^n \hookrightarrow \Delta^n$ is the inclusion of the boundary of Δ^n. I will refer to these cofibrations as *projective cofibrations.* In particular, the representable motivic spaces are *projectively cofibrant.* Note further that the axioms of a model category imply that every motivic space A has a projectively cofibrant replacement $K(A)$ which maps via a natural scheme-wise acyclic fibration $K(A) \xrightarrow{\sim} A$ to A. Details can be found in [4, 9.16].

The next step is to take into account the internal structure of the category $\mathbf{Sm}/S$ by means of a *Grothendieck topology.* In fact, a lot of people are a lot more interested in sheaves than in presheaves. Fortunately, the difference between sheaves and presheaves can be erased by homotopy-theoretical means. In more detail, for every Grothendieck topology there is a notion of *local weak equivalence*, such that the canonical map from a (space-valued) presheaf to its associated (space-valued) sheaf is always a local weak equivalence. Several Grothendieck topologies exist on $\mathbf{Sm}/S$, reflecting various algebraic properties of the schemes in question. The "correct" choice – as explained in the beginning of [11, section 3] – is the Nisnevich topology, which sits in between the étale topology and the Zariski topology. It can be characterized as follows.

Definition 2.6. An *elementary distinguished square* is a pullback square of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d, "\phi"] \\ U \arrow[r, "\psi"] & X \end{tikzcd}$$

where ϕ is an étale map, ψ is an open embedding and $\phi^{-1}(X-U) \longrightarrow (X-U)$ is an isomorphism of schemes (with the reduced structure). A (set-valued) presheaf A is a *Nisnevich sheaf* if $A(\emptyset)$ is a point, where $\emptyset$ is the empty scheme, and if the image

$$\begin{tikzcd} A(P) & A(Y) \arrow[l] \\ A(U) \arrow[u] & A(X) \arrow[l, "A(\psi)"] \arrow[u, "A(\phi)"'] \end{tikzcd}$$

of every elementary distinguished square is a pullback square in the category of sets.

Similarly, one can speak of space-valued and group-valued Nisnevich sheaves. The latter is relevant for the definition of local weak equivalences, which uses Nisnevich sheaves of homotopy groups of spaces. A precise definition can be found in [4, 9.22], which is probably too technical for the purpose of this note. The idea is as follows.

Definition 2.7. A map of motivic spaces is a *Nisnevich local weak equivalence* if the map induced on Nisnevich sheaves associated to the presheaves of homotopy groups is an isomorphism. This has to be true for every possible basepoint and every index of homotopy groups.

Instead of describing the *Nisnevich fibrations*, I will focus on the Nisnevich fibrant motivic spaces (that is, those motivic spaces A such that the map $A \longrightarrow *$ is a Nisnevich fibration). The reason is that they are simply a homotopy-theoretical variant of the Nisnevich sheaves.

Definition 2.8. A motivic space A is *Nisnevich fibrant* if

- it is scheme-wise fibrant, meaning that $A(U)$ is a Kan simplicial set for every scheme U in $\mathbf{Sm}/S$,
- $A(\emptyset)$ is contractible, and
- the image

$$\begin{tikzcd} A(P) & A(Y) \arrow[l] \\ A(U) \arrow[u] & A(X) \arrow[l, "A(\psi)"] \arrow[u, "A(\phi)"'] \end{tikzcd}$$

of every elementary distinguished square is a homotopy pullback square of spaces.

Using this, one can deduce the general definition of Nisnevich fibration. A nice aspect of the theory then is that a map which is both a Nisnevich fibration and a Nisnevich local weak equivalence is even a scheme-wise weak equivalence and a scheme-wise fibration. In a model category, the class of cofibrations is always determined by the class of *acyclic fibrations*, that is, the intersection of the class of weak equivalences and the class of fibrations. Thus the cofibrations completing this model structure are the projective cofibrations. Another nice aspect is that, as the scheme-wise fibrations, the Nisnevich fibrations are very well-behaved. For example, they are closed under filtered colimits, as one can see from [4, 9.30 and 3.11].

The final step now is to use the affine line $\mathbb{A}^1$ as a variant of the real interval $[0,1]$, to allow something like algebraic homotopies. If the affine line has to serve as an interval, it should better be contractible. But it is not the case that the map from the affine line to the point is a Nisnevich local weak equivalence. Fortunately, there is a way around this using the theory of model categories. To describe this way, I have to mention the smash product of motivic spaces first.

Definition 2.9. The smash product of motivic spaces is defined scheme-wise. That is, if A and B are motivic spaces, their smash product satisfies

$$(A \wedge B)(U) = A(U) \wedge B(U)$$

for every smooth scheme U in $\mathbf{Sm}/S$.

As explained in [4, 9.3 and 9.6], this smash product is in fact a *closed symmetric monoidal product* in the sense of enriched category theory. The "closed" here means that the functor $- \wedge A\colon \mathcal{M} \longrightarrow \mathcal{M}$ obtained by smashing with a fixed motivic space A has a right adjoint $\mathcal{M}(A,-)\colon \mathcal{M} \longrightarrow \mathcal{M}$. One should think of the motivic space $\mathcal{M}(A,B)$ (the value of $\mathcal{M}(A,-)$ at B) as the *motivic space of maps from A to B*. This is not only justified by results from enriched category theory, but by the fact that the value of the presheaf $\mathcal{M}(A,B)\colon \mathbf{Sm}/S^{\mathrm{op}} \longrightarrow \mathcal{S}$ at the base scheme S has as its zero-simplices the (pointed) set of maps in $\mathcal{M}$ from A to B. As a formula:

$$\mathcal{M}(A,B)(S)_0 = \mathbf{Set}_{\mathcal{M}}(A,B).$$

Consequently, I will speak of $\mathcal{M}(A,B)(S) =: \mathcal{S}_{\mathcal{M}}(A,B)$ as the *space of maps in $\mathcal{M}$ from A to B*.

Definition 2.10. A Nisnevich fibrant motivic space C is called *motivic Nisnevich fibrant* if the map of spaces $C(U) \longrightarrow C(U \times \mathbb{A}^1)$ induced by the canonical projection $U \times \mathbb{A}^1 \longrightarrow U$ is a weak equivalence for every smooth scheme U in $\mathbf{Sm}/S$. A map $f\colon A \longrightarrow B$ of motivic spaces is called *motivic weak equivalence* if the map of spaces

$$\mathcal{S}(K(f),C)\colon \mathcal{S}_{\mathcal{M}}(K(B),C) \longrightarrow \mathcal{S}_{\mathcal{M}}(K(A),C)$$

is a weak equivalence for every motivic Nisnevich fibrant C. Here K is the projectively cofibrant replacement in the scheme-wise model structure on $\mathcal{M}$.

Motivic weak equivalences are called $\mathbb{A}$-*equivalences* in [4, 9.35]. The main result of [4, section 9] is then the following.

Theorem 2.11. *Motivic weak equivalences and projective cofibrations are part of a model structure on the category of motivic spaces. The associated homotopy category is equivalent to the (pointed) motivic homotopy category as constructed by Morel and Voevodsky in* [11].

Certain properties of this motivic model structure imply that motivic weak equivalences are closed under filtered colimits. Moreover, the *motivic Nisnevich fibrations* in this model structure are well-behaved, although possibly not as well-behaved as the Nisnevich fibrations. A reference for these facts is [4, 3.11].

Lemma 2.12. *Motivic Nisnevich fibrations having a motivic Nisnevich fibrant codomain are closed under filtered colimits.*

This lemma is of great importance in the construction of the stable model structure on motivic functors, and I want to emphasize that the standard model structure yielding the motivic homotopy category does not satisfy this property. In fact, the way how Jardine gets around some problems in his discussion [9] of *motivic symmetric spectra* shows that although he uses the standard approach to motivic homotopy theory, he makes implicitly use of the motivic Nisnevich fibrations, and in particular of 2.12.

Finally, the smash product and the model structure interact very well.

Theorem 2.13. *The motivic model structure described in 2.11 turns $\mathcal{M}$ into a monoidal model category satisfying the monoid axiom. Moreover, smashing with any motivic space preserves motivic weak equivalences.*

The notion of *monoidal model categories* and the *monoid axiom* are explained in [4, 3.13 and 3.14]. To justify them, I will just mention two theorems. A closed symmetric monoidal product on a model category descends to a closed symmetric monoidal product on the homotopy category if the model category is monoidal.[2] If a monoidal model category satisfies the monoid axiom, the model structure lifts in a canonical way to give model structures on the category of modules over a fixed monoid, and on the category of all monoids.[3] These phenomena will be discussed in more detail later.

[2]To be honest, the unit of the closed symmetric monoidal product has to satisfy a certain technical property. But this property is automatically satisfied if the unit is cofibrant, and this is the case in the motivic model structure.

[3]Again I am sweeping a detail under the rug. This detail concerning *smallness* is satisfied in our case, because of the combinatorial approach through simplicial sets.

2.5. **Motivic Spectra.** Once the correct model category is in place, the construction of the stable motivic homotopy category can be performed as Bousfield and Friedlander did. For an explanation of this point, I like to recommend Voevodsky's ICM article [16]. There is just a slight peculiarity: motivic homotopy theory has two "circles". One is the ordinary simplicial circle S^1, considered as a constant (space-valued) presheaf on $\mathbf{Sm}/S$. The other one is called the *Tate circle*. It is the (discrete space-valued) presheaf represented by $\mathbb{A}^1 - \{0\}$, the scheme in $\mathbf{Sm}/S$ obtained from the affine line by removing the zero point. This presheaf is pointed by the point $\{1\}$. For example, if the base scheme S is the complex numbers, the non-zero complex numbers pointed by 1 form the Tate circle. It is worth noticing that the underlying topological space in this case is in fact homotopy equivalent to the ordinary circle of unit vectors in the complex plane.

Instead of treating these two circles separately, one can smash them together to yield a motivic space T. Another surprise – which is explained in detail in [11, section 3.2.2] – is that T is isomorphic in the motivic homotopy category to the motivic space $(\mathbb{P}^1, \infty)$, the projective line pointed at infinity. Thus also smashing with the projective line is a legitimate candidate for suspension. Nevertheless, I will choose $T \wedge -$ as a suspension combining the ordinary suspension and the Tate circle suspension.[4] Motivic spectra are then defined in complete analogy to classical spectra, cf. 2.2.

Definition 2.14. A *motivic spectrum*, or *T-spectrum*, consists of a sequence $(E_0, E_1, \ldots, E_n, \ldots)$ of motivic spaces and structure maps $\sigma_n^E \colon T \wedge E_n \longrightarrow E_{n+1}$. A *map of motivic spectra* is then a sequence of maps which are compatible with the structure maps.

Voevodsky also introduced a notion of *stable equivalence of motivic spectra* which is part of a model structure on motivic spectra. The associated homotopy category is then the *stable motivic homotopy category*, and as in the classical case, this is a triangulated category. Instead of giving the definition of stable equivalences, I just want to remark that they can be defined using a process similar to the one leading to the model category 2.11 of motivic spaces, rather than pursuing an analog of stable homotopy groups. Details can be found in [4, 6.14], or in Voevodsky's ICM article [16].

A great advantage of motivic spectra is that now cohomology theories of motivic spaces (thus cohomology theories of algebraic manifolds) can be described by a motivic spectrum, respectively the functor it represents in the stable motivic homotopy category. Voevodsky gives three examples in [16, section 6]: motivic

[4]For technical reasons, it is necessary to work with a model of T which is projectively cofibrant. If this T isn't so, this can easily achieved by a simplicial mapping cylinder construction I explained below the proof of [4, 9.43].

cohomology, algebraic K-theory and algebraic cobordism. The latter is an algebraic analog of complex cobordism and plays a role in Voevodsky's proof of the Milnor conjecture. I will talk about motivic cohomology later.

3. Enriched Functors and Stable Homotopy Theory

So far, I have advertised the simplicity of the spectra in the sense of Bousfield and Friedlander, but I have not mentioned a drawback of that model. I will do this now. The stable homotopy category, as constructed by Boardman over thirty years ago in [18], has a smash product which is both closed symmetric monoidal and compatible with the smash product of spaces. Since the category of spectra is a model for the stable homotopy category, it is natural to ask whether there is also a closed symmetric monoidal smash product of spectra which is a model for the smash product in the stable homotopy category.[5] Such a smash product seems not to exist on the model given by spectra, but there are several models for the stable homotopy category. However, even experts in this field once believed that none of these models for the stable homotopy category can be equipped with a well-behaved smash product. More is true: even experts who later constructed models with a well-behaved smash product shared this belief, although such constructions had been around implicitly.

In the last ten years, several models for the stable homotopy category with the correct smash product appeared. As a matter of life, these models are more complicated than spectra, but I will mention two which are quite simple. The category of *symmetric spectra* introduced in [7] is due to Jeff Smith, and Manos Lydakis developed the stable homotopy theory of *simplicial functors* in [10]. The latter inspired the work which is now contained in [4], thus it deserves an extra section.

3.1. **Simplicial Functors.** To give the definition of simplicial functors, I have to recall the smash product of spaces, which is in fact a closed symmetric monoidal product. The unit of the smash product is the zero sphere S^0. The right adjoint of $K \wedge -$ is $\mathcal{S}(K,-)$, the *space of maps* from K to some other space. The (pointed) set of maps from K to L sits inside this space as the set of zero-simplices. In particular, the identity map of K is a point in $\mathcal{S}(K,K)$. By enriched category theory or some direct calculation, one verifies the existence of a *composition map of spaces* $\mathcal{S}(L,M) \wedge \mathcal{S}(K,L) \longrightarrow \mathcal{S}(K,M)$ which is associative and compatible with identities and the usual composition. Simplicial functors are required to preserve this composition, but before I explain this in more detail, I will describe their domain.

[5]This would also be desirable to study the *homotopical algebra of ring spectra*, which will be explained later.

Definition 3.1. A space K is called *finite* if it has only finitely many non-degenerate simplices. Equivalently, K is finite if and only if the functor $\mathcal{S}(K,-)$ commutes with filtered colimits.

The full subcategory of $\mathcal{S}$ determined by the finite spaces is denoted $\mathbf{f}\mathcal{S}$. It is equivalent to a small category, thus I may view $\mathbf{f}\mathcal{S}$ as a domain in a functor category without causing set-theoretical difficulties. A functor $X\colon \mathbf{f}\mathcal{S} \longrightarrow \mathcal{S}$ assigns to every finite space K some space XK, and to every map of finite spaces $f\colon K \longrightarrow L$ a map $Xf\colon XK \longrightarrow XL$ such that identities and compositions are preserved. The assignment on maps can be subsumed as a map of sets $\mathbf{Set}_{\mathcal{S}}(K,L) \xrightarrow{\hom^X} \mathbf{Set}_{\mathcal{S}}(XK,XL)$ which is compatible with identities and composition.

Definition 3.2. A *simplicial functor* $X\colon \mathbf{f}\mathcal{S} \longrightarrow \mathcal{S}$ assigns to every finite space K a space XK, and is equipped with maps of *spaces*

$$\mathcal{S}(K,L) \xrightarrow{\hom^X} \mathcal{S}(XK,XL)$$

which are compatible with identities and composition maps of spaces. Here K and L are finite spaces. In short words, X is *enriched over* $\mathcal{S}$ or an $\mathcal{S}$-functor.

One should think of simplicial functors as being "continuous" in some sense. The good news from enriched category theory is that enriched functors are very similar to ordinary functors, when it comes to conceptual properties. For example, one gets at once a bicomplete category **SF** of simplicial functors, in which the morphisms are an appropriate simplicial analog of natural transformations. Moreover, there is an enriched version of the Yoneda lemma which specializes here to the fact that every simplicial functor is a colimit of representable simplicial functors. The latter ones are by definition of the form $\mathcal{S}(K,-)$, for K a finite space.

The connection between spectra and simplicial functors is very close. A spectrum is a simplicial functor which is defined only on spheres. To be more precise, if X is a simplicial functor, the sequence $\mathrm{ev}(X) = (XS^0, XS^1, XS^2, \dots)$ obtained from *evaluating at spheres* is a spectrum. The n-th structure map of $\mathrm{ev}(X)$ is obtained as the adjoint of the composition

$$S^1 \xrightarrow{\text{unit}} \mathcal{S}(S^n, S^1 \wedge S^n) = \mathcal{S}(S^n, S^{n+1}) \xrightarrow{\hom^X} \mathcal{S}(XS^n, XS^{n+1}).$$

For example, the evaluation of the inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{S} \hookrightarrow \mathcal{S}$ is the sphere spectrum. More generally, the evaluation of the simplicial functor $-\wedge K\colon \mathbf{f}\mathcal{S} \longrightarrow \mathcal{S}$ yields the suspension spectrum of K.

Theorem 3.3 (Lydakis)**.** *There exists a model structure on the category of simplicial functors. Furthermore, it is Quillen equivalent via the evaluation functor* ev *to the stable model structure on the category of spectra.*

For the precise definition of a Quillen equivalence, consider [5, 1.3.12]. The important thing about a Quillen equivalence – which is an adjoint pair of certain functors of model categories – is that it induces an equivalence on homotopy categories. Thus the homotopy category of simplicial functors is equivalent to the stable homotopy category.

From a conceptual point of view, simplicial functors are not more complicated than spectra. The advantage of requiring values for every finite space instead of just the spheres is that simplicial functors have a smash product, while spectra don't. As Lydakis also points out, the smash product of simplicial functors is a special case of a very general theorem by Brian Day in [3] which has been around for over thirty years by now. Due to its generality, this smash product is very simple to define. It is determined by the following two conditions.

- For a fixed simplicial functor X, both $X \wedge -$ and $- \wedge X$ commute with colimits.
- The smash product of representables is representable:

$$\mathcal{S}(K,-) \wedge \mathcal{S}(L,-) = \mathcal{S}(K \wedge L,-).$$

The latter condition is applicable, because the smash product of spaces (which is an $\mathcal{S}$-functor $\mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$) restricts to an $\mathcal{S}$-functor $\mathbf{f}\mathcal{S} \times \mathbf{f}\mathcal{S} \longrightarrow \mathcal{S}$ of finite spaces.[6] The unit of the smash product of simplicial functors is then the inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{S} \hookrightarrow \mathcal{S}$. In [4, section 2.3], the smash product of enriched functors X and Y is defined using enriched left Kan extension. In the special case of spaces, $X \wedge Y$ thus makes the diagram

$$\begin{tikzcd}
\mathbf{f}\mathcal{S} \times \mathbf{f}\mathcal{S} \arrow[r, "X \times Y"] \arrow[d, "\wedge"'] & \mathcal{S} \times \mathcal{S} \arrow[d, "\wedge"] \\
\mathbf{f}\mathcal{S} \arrow[r, dashed, "X \wedge Y"] & \mathcal{S}
\end{tikzcd}$$

commutative in a universal way. These two approaches to the smash product are easily seen to agree. The following theorem of Lydakis on the compatibility of his model structure and the smash product can also be obtained as a special case of [4, 6.25, 6.26 and 6.29]. The proof of the last statement relies on the fact that every space is a filtered colimit of finite spaces.

Theorem 3.4 (Lydakis)**.** *The model structure from 3.3 turns the category* **SF** *into a monoidal model category, and the monoid axiom holds.*

---

[6]By generalizing Lydakis' view of spectra given in [10, 4.3], I deduced statement [4, 2.12]. In the special case of spaces it shows that spectra are simplicial functors on a specific simplicial sub-$\mathcal{S}$-category Sph of the category of finite spaces, which has only spheres as objects *and* as spaces of maps. Moreover, I indicate below [4, 2.13] why the smash product of spaces does not restrict as an $\mathcal{S}$-functor to the category Sph.

A *monoid* in **SF** is a simplicial functor X, together with a multiplication $\mu\colon X \wedge X \longrightarrow X$ and a unit $\nu\colon \mathbb{I} \longrightarrow X$, such that the diagrams

$$\begin{tikzcd} X \wedge X \wedge X \arrow[r, "X\wedge\mu"] \arrow[d, "\mu\wedge X"'] & X \wedge X \arrow[d, "\mu"] \\ X \wedge X \arrow[r, "\mu"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X = X \wedge \mathbb{I} \arrow[r, "X\wedge\nu"] \arrow[dr, "id"'] & X \wedge X \arrow[d, "\mu"] & \mathbb{I} \wedge X = X \arrow[l, "\nu\wedge X"'] \arrow[dl, "id"] \\ & X & \end{tikzcd}$$

commute. It should then be clear what a *commutative monoid* is. An example of a commutative monoid is given by

$$H_{\mathbb{Z}}(K) := \mathbb{Z}[K]/\mathbb{Z}[*]$$

which maps a finite space to the simplicial free abelian group it generates modulo the copy of the integers coming from the basepoint. Since $\mathbb{Z}$ is a commutative ring, $H_{\mathbb{Z}}$ is a commutative monoid in **SF**. The homotopy groups of $H_{\mathbb{Z}}(K)$ are the reduced singular homology groups of K with integer coefficients. The monoid axiom for the stable model structure on simplicial functors give a model structure on the modules over $H_{\mathbb{Z}}$. A general result of Schwede and Shipley in [13] shows that the homotopy category of these modules is the derived category of $\mathbb{Z}$. In this sense, homotopical algebra over ring spectra (meaning here the homotopy theory of monoids in **SF**) is a generalization of homological algebra.

3.2. **The Category of Motivic Functors.**

Jardine constructed a motivic analog of Smith's symmetric spectra in [9]. In this section I will describe how a motivic analog of Lydakis' simplicial functors looks like. It is quite easy to guess the right category, provided that one has a good notion of finiteness.

Definition 3.5. A motivic space A is *finite* if the functor $\mathcal{M}(A,-)\colon \mathcal{M} \longrightarrow \mathcal{M}$ commutes with filtered colimits. The full subcategory of finite motivic spaces is denoted $\mathbf{f}\mathcal{M}$.

From [4, 9.9 and 9.10] I can deduce that this notion of finiteness is indeed good: finite motivic spaces are closed under finite colimits and smash products. Furthermore, there are enough finite motivic spaces in the following two senses: every representable motivic space is finite, and any motivic space is a filtered colimit of finite motivic spaces. Finally, there are not too many finite motivic spaces in the sense that the category $\mathbf{f}\mathcal{M}$ is equivalent to a small category.

As I mentioned before, the category of motivic spaces is a closed symmetric monoidal category under the smash product. Thus a composition map of motivic spaces

$$\mathcal{M}(B,C) \wedge \mathcal{M}(A,B) \longrightarrow \mathcal{M}(A,C)$$

exists as in the case of spaces. And the identity id_A of some motivic space A gives rise to an identity map $S^0 \longrightarrow \mathcal{M}(A,A)$ from the constant presheaf with value S^0 (which is clearly the unit of the smash product of motivic spaces). This enables me to give the following definition.

Definition 3.6. A *motivic functor* is a functor $X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ which is enriched over motivic spaces. That is, for every finite motivic space A there is a motivic space XA. Furthermore, there is a map

$$\hom^X\colon \mathcal{M}(A,B) \longrightarrow \mathcal{M}(XA,XB)$$

for any pair of finite motivic spaces A and B, and this collection of maps is compatible with composition and identities.

Motivic functors form a category $\mathbf{MF}$ which has nice properties for formal reasons. It is bicomplete by [4, 2.2] – with colimits being constructed pointwise – and every motivic functor is a colimit of representable ones by the enriched Yoneda lemma [4, 2.1]. In an enriched setting, these colimits are usually written as coends:

$$X(-) = \int^{A\in \mathrm{Ob}\,\mathbf{f}\mathcal{M}} \mathcal{M}(A,-)\wedge XA.$$

Maybe the best thing that one gets for free is the smash product of motivic functors. It can be obtained as for simplicial functors because of Day's general theorem, which is cited in [4, 2.6]. Again the smash product of two motivic functors X and Y is the enriched left Kan extension of the composition

$$\mathbf{f}\mathcal{M}\times\mathbf{f}\mathcal{M} \xrightarrow{X\times Y} \mathcal{M}\times\mathcal{M} \xrightarrow{-\wedge-} \mathcal{M}$$

along the restriction $\mathbf{f}\mathcal{M}\times\mathbf{f}\mathcal{M} \xrightarrow{-\wedge-} \mathbf{f}\mathcal{M}$ of the smash product to finite motivic spaces. A coend formula is

$$(X\wedge Y)(-) = \int^{(A,B)\in \mathrm{Ob}\,\mathbf{f}\mathcal{M}\times\mathbf{f}\mathcal{M}} \mathcal{M}(A\wedge B,-)\wedge XA\wedge YB.$$

The unit of this smash product is again the inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$. Smashing with a fixed motivic functor X has as right adjoint the *motivic functor of maps from X to something*. If the something is a motivic functor Y, a pointwise formula is given by the end

$$\mathbf{MF}(X,Y)(A) = \int_{B\in \mathrm{Ob}\,\mathbf{f}\mathcal{M}} \mathcal{M}(XB, Y(A\wedge B)).$$

As in any symmetric monoidal category, one has notions of monoids and commutative monoids in the category of motivic functors. The diagrams which explained these notions in the category of simplicial functors transfer directly. If M is a monoid with multiplication μ and unit ν, a *left module over M* is a motivic functor X, together with a map $a\colon M\wedge X \longrightarrow X$ such that the diagrams

$$\begin{tikzcd} M\wedge M\wedge X \arrow[r, "M\wedge a"] \arrow[d, "\mu\wedge X"'] & M\wedge X \arrow[d, "a"] \\ M\wedge X \arrow[r, "a"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X=\mathbb{I}\wedge X \arrow[r, "\nu\wedge X"] \arrow[dr, "\mathrm{id}_X"'] & M\wedge X \arrow[d, "a"] \\ & X \end{tikzcd}$$

commute. In section 4 I will introduce a commutative monoid in the category of motivic functors which should have an interesting category of modules.

Now recall the definition of the motivic space T, which is a smash product of a simplicial circle S^1 and the Tate circle $\mathbb{A}^1 - \{0\}$. The above mentioned properties of finite motivic spaces show that T is one of them, as are the higher-dimensional spheres $T^{n+1} := T \wedge T^n$. With the convention that $T^0 = S^0$ coincides with the unit of the smash product, any motivic functor $X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ has as an evaluation a motivic spectrum $\mathrm{ev}(X) = (XT^0, XT^1, XT^2, \dots)$. The n-th structure map of $\mathrm{ev}(X)$ is the adjoint of the composition

$$T \xrightarrow{\alpha} \mathcal{M}(T^n, T \wedge T^n) = \mathcal{M}(T^n, T^{n+1}) \xrightarrow{\hom^X} \mathcal{M}(XT^n, XT^{n+1}).$$

Here α is the value at T of the unit of the adjunction $(- \wedge T^n, \mathcal{M}(T^n, -))$, or – equivalently – the adjoint of the identity $\mathrm{id}\colon T \wedge T^n \longrightarrow T^{n+1}$.

Before I switch over to the homotopy theory of motivic functors – which is considerably more difficult than these easy categorical constructions – I would like to remark that the evaluation functor ev from motivic functors to motivic spectra factors over the category of motivic symmetric spectra constructed in [9]. Another discussion of the categorical properties of motivic symmetric spectra can be obtained as special cases of the general results in [4, section 2.6]. The essence of this section is that the category of symmetric spectra is the smallest enlargement of the category of spectra with a smash product in the spirit of Day's theorem. With this philosophy in mind, the category of enriched functors is in some sense the biggest enlargement of spectra with a smash product á la Day.

3.3. **Homotopy Theory of Motivic Functors.** The stable homotopy theory of motivic functors is built in three steps. The first step, the pointwise model structure, is quite obvious and came to my mind at an early stage of the project. I also had ideas concerning the other two steps which turned out to be correct, but some of the relevant proofs relied heavily on the collaboration with Bjørn Ian Dundas and Paul Arne Østvær.

Definition 3.7. A map $f\colon X \longrightarrow Y$ of motivic functors is a

- *pointwise weak equivalence* if $fA\colon XA \longrightarrow YA$ is a motivic weak equivalence for all finite motivic spaces A,
- *pointwise fibration* if $fA\colon XA \longrightarrow YA$ is a motivic Nisnevich fibration for all finite motivic spaces A,
- *cofibration* if it has the left lifting property with respect to all *pointwise acyclic fibrations*, that is, maps which are both pointwise fibrations and pointwise weak equivalences.

It is quite straightforward to deduce that these classes define a model structure on the category of motivic functors. Essentially, one uses two ingredients. The first is the fact that the motivic model structure appearing in 2.11 is *cofibrantly generated* as defined in [4, 3.6]. Any generating cofibration $C \rightarrowtail D$ in $\mathcal{M}$

induces a generating cofibration $\mathcal{M}(A,-)\wedge C \longrightarrow \mathcal{M}(A,-)\wedge D$, where A varies through the finite motivic spaces, and similarly for generating acyclic cofibrations. The second ingredient is the monoid axiom for the motivic model structure. It enters the proof to deduce that the maps of motivic functors obtained from the generating acyclic cofibrations are in fact pointwise weak equivalence. The problem is that $\mathcal{M}(A,B)$ is not necessarily projectively cofibrant, which also justifies the following remark.

Remark 3.8. *A cofibration $f\colon X \rightarrowtail Y$ is not necessarily pointwise a cofibration in the sense that $fA\colon XA \longrightarrow YA$ is a projective cofibration for every finite motivic space A. However, every cofibration is pointwise a monomorphism (which is enough to ensure left properness for the pointwise model structure).*

The pointwise model structure has way too many homotopy types. One reason is the following: if $f\colon A \xrightarrow{\sim} B$ is a weak equivalence of finite motivic spaces, the induced map of representable motivic functors $\mathcal{M}(f,-)\colon \mathcal{M}(B,-) \longrightarrow \mathcal{M}(A,-)$ is not necessarily a pointwise weak equivalence. That is, although A and B are the same in the motivic homotopy category, their associated representable motivic functors are so far different. To get around this problem – but also for other reasons – Lydakis introduced a *homotopy functor model structure* in the case of simplicial functors. It is a consequence of several properties of the category of motivic spaces that an analogous model structure exists for motivic functors.

Suppose for the moment that $f\colon A \xrightarrow{\sim} B$ is a motivic weak equivalence of projectively cofibrant finite motivic spaces. Then $\mathcal{M}(f,C)\colon \mathcal{M}(B,C) \longrightarrow \mathcal{M}(A,C)$ is a motivic weak equivalence, provided that C is motivic Nisnevich fibrant. The reason is that $\mathcal{M}$ is a *monoidal* model category. Thus only the fibrant motivic spaces should count for a weak equivalence of motivic functors. Unfortunately, fibrant motivic spaces are usually not finite, hence this would lead out of the category of finite motivic spaces. But this is no problem, as long as it is kept track of the way out of $\mathbf{f}\mathcal{M}$ as long as possible.

Definition 3.9. A map of motivic spaces is called an *acyclic monomorphism* if it is both a monomorphism and a motivic weak equivalence. If A is a finite motivic space, let $\mathrm{ac}(A)$ be the category having as objects acyclic monomorphisms of finite motivic spaces $\beta\colon A \xhookrightarrow{\sim} B$ with domain A. A morphism in $\mathrm{ac}(A)$ from $\beta\colon A \xhookrightarrow{\sim} B$ to $\gamma\colon A \xhookrightarrow{\sim} C$ is an acyclic monomorphism $\delta\colon B \xhookrightarrow{\sim} C$ with $\delta\circ\beta = \gamma$. A map $f\colon X \longrightarrow Y$ of motivic functors is an *hf-equivalence* if the map

$$\operatorname*{colim}_{A \xhookrightarrow{\sim} B} f(B)$$

is a motivic weak equivalence for every finite motivic space A, where the colimit is taken over $\mathrm{ac}(A)$.

This construction is essentially due to Dundas. It works because of the following properties, which can be deduced from [4, 3.29, 3.33 and 10.6].

- For any finite motivic space A, the colimit
$$\Phi(A) = \operatorname*{colim}_{A \overset{\sim}{\hookrightarrow} B} B$$
is motivic Nisnevich fibrant.
- A slight modification of $\mathrm{ac}(A)$ yields that $\Phi(A)$ is a filtered colimit over motivic weak equivalences of finite motivic spaces.
- Any motivic functor commutes with filtered colimits, because the representables $\mathcal{M}(A, -)$ do so, according to the finiteness of the motivic spaces in $\mathbf{f}\mathcal{M}$.
- If $\phi\colon A \xrightarrow{\sim} B$ is a motivic weak equivalence in $\mathbf{f}\mathcal{M}$, the induced map of motivic functors $\mathcal{M}(\phi, -)\colon \mathcal{M}(B, -) \longrightarrow \mathcal{M}(A, -)$ is an hf-equivalence.

The last item shows that a restriction to projectively cofibrant finite motivic spaces is not necessary. Thus the disadvantage of the motivic model structure in 2.11 concerning the cofibrations is really not too bad. Moreover, pointwise weak equivalences are hf-equivalences, because of the second item and the fact proven in [4, 9.36 and 3.11] that motivic weak equivalences are closed under filtered colimits.

Theorem 3.10. *The class of hf-equivalences and the class of cofibrations are part of a model structure on the category of motivic functors. The resulting model category is cofibrantly generated, monoidal and satisfies the monoid axiom.*

This is [4, 10.9]. Fortunately, the according *hf-fibrations* have a very easy definition.

Definition 3.11. A pointwise fibration $f\colon X \twoheadrightarrow Y$ of motivic functors is an *hf-fibration* if the square

$$\begin{array}{ccc} XA & \xrightarrow{X\phi} & XB \\ {\scriptstyle fA}\downarrow & & \downarrow{\scriptstyle fB} \\ YA & \xrightarrow{Y\phi} & YB \end{array}$$

is a homotopy pullback square of motivic spaces for every motivic weak equivalence $\phi\colon A \xrightarrow{\sim} B$.[7]

In particular, the hf-fibrant motivic functors are the ones which are pointwise fibrant and homotopy functors, where a homotopy functor is a functor preserving weak equivalences. Thus the letters "hf" stand for "homotopy functor", and the situation is very similar to the one in Lydakis' paper [10]. In particular, I will

[7] I should remark that is is possible to pass from acyclic monomorphisms to all motivic weak equivalences of finite motivic spaces via a simplicial mapping cylinder argument.

refer to the model structure in 3.10 as the *homotopy functor model structure.* The main input in the proof of 3.10 is that an hf-fibration which is also an hf-equivalence is even a pointwise acyclic fibration, which is [4, 10.8]. This in turn requires in an essential way 2.12.

With the homotopy functor model structure in place, the stable model structure needs essentially one further ingredient which has not appeared before. This ingredient is the *cyclic permutation condition* for the motivic space T which is used to suspend. The cyclic permutation condition roughly says that the cyclic permutation on the three-fold smash power $T \wedge T \wedge T \longrightarrow T \wedge T \wedge T$ becomes the identity in the motivic homotopy category. I will explain this a bit. Since the motivic sphere spectrum $(T^0, T^1, T^2, \dots)$ is the unit of the closed symmetric monoidal product in the motivic homotopy category, it is in particular a commutative monoid. The commutativity comes precisely from the cyclic permutation condition for T which has been verified by Voevodsky. Motivic functors shall model this situation, thus the cyclic permutation condition has to enter somewhere. Fortunately, this condition is not only necessary, but also sufficient to build the stable model structure on motivic functors, just as for motivic symmetric spectra, cf. [9, 3.13].

There are two ways to describe the stabilization with respect to T. The first description goes via the map

$$\tau_A \colon \mathcal{M}(T \wedge A, -) \wedge T \xrightarrow{\cong} \mathcal{M}(T, \mathcal{M}(A, -)) \wedge T \longrightarrow \mathcal{M}(A, -)$$

where A is some finite motivic space, and the second map is the counit of the adjunction $(- \wedge T, \mathcal{M}(T, -))$. Via the Yoneda lemma, τ_A induces a natural map of motivic functors

$$t(X)_A \colon XA \longrightarrow \mathcal{M}(T, X(T \wedge A)) = (\mathcal{M}(T, -) \circ X \circ (T \wedge -))(A)$$

The latter motivic functor is denoted $\mathbb{T}(X)$, and $\mathbb{T}$ can be viewed as an endofunctor $\mathbb{T} \colon \mathbf{MF} \longrightarrow \mathbf{MF}$ which comes with a natural transformation $t \colon \mathrm{Id}_{\mathbf{MF}} \longrightarrow \mathbb{T}$. This natural transformation is similar to the natural transformation of spectra $E \longrightarrow \Omega(\mathrm{shift}(E))$ obtained as the collection of the adjoints of the structure maps $S^1 \wedge E_n \longrightarrow E_{n+1}$. It is similar, but not directly compatible, for reasons explained in [4, section 6.3]. Let $\mathbb{T}^\infty(X)$ be the colimit of the sequence

$$X \xrightarrow{t(X)} \mathbb{T}(X) \xrightarrow{\mathbb{T}(t(X))} \mathbb{T}(\mathbb{T}(X)) \longrightarrow \cdots$$

and let $t^\infty \colon \mathrm{Id}_{\mathbf{MF}} \longrightarrow \mathbb{T}^\infty$ denote the resulting natural transformation. Exploiting the similarity mentioned above, $\mathbb{T}^\infty(X)$ is something like the Ω-spectrum associated to X.

Definition 3.12. Let $X \longmapsto X^h$ denote a fibrant replacement in the homotopy functor model structure. Thus X^h is a pointwise fibrant homotopy functor for every motivic functor X, and there is a natural hf-equivalence $X \xrightarrow{\sim \mathrm{hf}} X^h$. A map $f\colon X \longrightarrow Y$ of motivic functors is a *stable equivalence* if and only if the map $T^\infty(f^h)\colon T^\infty(X^h) \longrightarrow T^\infty(Y^h)$ is a pointwise weak equivalence.

Since T is projectively cofibrant and $\mathbb{T}^\infty$ is defined as a sequential colimit, hf-equivalences are stable equivalences. Stable fibrations of motivic functors are then defined in complete analogy to stable fibrations of simplicial functors (respectively spectra, modulo the missing compatibility mentioned above).

Definition 3.13. An hf-fibration $f\colon X \xrightarrow{\mathrm{hf}} Y$ is a *stable fibration* if the square

$$\begin{tikzcd} X \arrow[r, "t(X)"] \arrow[d, "f"'] & \mathbb{T}(X) \arrow[d, "\mathbb{T}(f)"] \\ Y \arrow[r, "t(Y)"] & \mathbb{T}(Y) \end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure on motivic functors. That is, it is a homotopy pullback square of motivic spaces at each finite motivic space A.

There are now two basic properties which have to be fulfilled, before stable equivalences, stable fibrations and cofibrations can form a model category. The first – having to do with generating stable acyclic cofibrations – is checked in [4, 10.14], which essentially relies on the existence of the stable model category of motivic spectra and the cyclic permutation condition. The second is [4, 10.13], saying that a stable fibration which is also a stable equivalence is in fact a pointwise acyclic fibration. This result uses similar arguments as the corresponding fact for the homotopy functor model structure, but one first has to replace the abstract fibrant replacement $X \longmapsto X^h$ by something concrete one can work with. The latter in turn requires a different description of the stabilization process $t\colon \mathrm{Id}_{\mathbf{MF}} \longrightarrow \mathbb{T}$. This description uses the adjoint of the map $T \wedge XA \longrightarrow X(T \wedge A)$ which is adjoint to the composition

$$T \xrightarrow{\alpha} \mathcal{M}(A, T \wedge A) \xrightarrow{\hom^X} \mathcal{M}(XA, X(T \wedge A)).$$

The map α is the value at T of the unit of the adjunction $(- \wedge A, \mathcal{M}(A, -))$. I proved that these two different descriptions of the stabilization map t coincide with a long and boring computation in an arbitrary closed symmetric monoidal category. It can be found in [4, Appendix A].

Theorem 3.14. *Stable equivalences, stable fibrations and cofibrations form a cofibrantly generated monoidal model structure on the category of motivic functors. The monoid axiom holds.*

The question is now whether this model category is Quillen equivalent to the stable model category of motivic spectra via the evaluation functor ev. I believe that this is not the case. To justify my belief, I will first sketch my point of view of Lydakis' impressive proof that the evaluation functor is a Quillen equivalence in the case of spaces. Let i_* denote the left adjoint of the evaluation functor ev. To check that the Quillen pair (i_*, ev) is a Quillen equivalence, it suffices – by general arguments which can be found in [5, 1.3.16] – to prove two things:

- The unit $E \longrightarrow \mathrm{ev}((i_*E)^s)$ of the adjunction is a stable equivalence of spectra, provided that E is a cofibrant spectrum and the simplicial functor i_*E is made stably fibrant, as indicated.
- A map $f\colon X \longrightarrow Y$ of stably fibrant simplicial functors is a stable equivalence if its evaluation $\mathrm{ev}(f)$ is a stable equivalence of spectra.

Lydakis proved the first condition with the Blakers-Massey theorem, which is not available in motivic homotopy theory. But as it turns out, the first condition holds also for motivic functors, essentially because of the cyclic permutation condition (see [4, section 7.2] for details). Lydakis had to work a lot to prove the second condition, thus I will spend a few more sentences on that. General arguments show that a stable equivalence of stably fibrant spectra or stably fibrant simplicial functors is indeed a pointwise weak equivalence. Thus the situation is as follows: $f\colon X \longrightarrow Y$ is a map of stably fibrant simplicial functors such that $f(S^n)$ is a weak equivalence for every $n \geq 0$, but $f(K)$ has to be a weak equivalence for every finite space K. Fortunately, every finite space is built from finitely many spheres in the sense that the quotient of the n-skeleton modulo the $(n-1)$-skeleton is a finite wedge $\bigvee S^n$ of the n-dimensional sphere. Since stably fibrant simplicial functors map cofibration sequences to homotopy fiber sequences, one can proceed by induction on the skeleton to conclude the proof.

Here comes the reason why the evaluation functor should not be a Quillen equivalence: Finite motivic spaces are not built from the spheres $T^0, T^1, T^2, \ldots$ in the above sense. Instead, one has to use all schemes in $\mathbf{Sm}/S$ in general. Thus the argument I just sketched for simplicial functors cannot work. However, one can modify the stable model structure from 3.14 to get around this problem. Just replace in every definition of an equivalence or a fibration of motivic functors (pointwise, hf-, and stable) the words "for every finite motivic space A" by the words "for every finite motivic space A *weakly equivalent to some* T^n*, where* $n \geq 0$". I will then speak of *sphere-wise weak equivalences*, *sphere-wise hf-fibrations* and so on. The theorems 3.10 and 3.14 then remain true, with an obvious change from cofibrations to *sphere-wise cofibrations*, and the next theorem essentially follows by the previous work and the definition. I want to emphasize that this result is due to the wonderful cooperation, and details can be found in [4, 10.6].

Theorem 3.15. *The category of motivic functors becomes a monoidal model category with the classes of sphere-wise stable equivalences, sphere-wise stable fibrations and sphere-wise cofibrations. The monoid axioms holds, and the evaluation functor to the stable model category of motivic spectra is a Quillen equivalence.*

3.4. **Equivariant Simplicial Functors.** When I started thinking about the generalization of Lydakis' results on simplicial functors to motivic homotopy theory, I hoped for a quite general machinery to stabilize a lot of monoidal model categories, similar to the one Hovey developed in [6]. Then several limitations occurred. Nevertheless, I realized that equivariant homotopy theory is a valid candidate. The details and proofs I will leave out in the following summary can be found in [4, section 11].

In the following, let G be a finite group with multiplication $\mu\colon G \times G \longrightarrow G$ and unit $\nu\colon \Delta^0 \longrightarrow G$. The unit map should make it clear that I think of G as a (finite, thus discrete) simplicial group. A lot of the following is also true for an arbitrary simplicial group, but it will become apparent later why G should better be finite. Still $\mathcal{S}$ is the category of spaces.

Definition 3.16. A *G-space* K consists of an underlying space uK and an action map $a_K\colon G_+ \wedge uK \longrightarrow uK$, such that the diagrams

$$\begin{tikzcd} G_+ \wedge G_+ \wedge uK \arrow[r, "G_+\wedge a_K"] \arrow[d, "\mu_+\wedge uK"'] & G_+ \wedge uK \arrow[d, "a_K"] \\ G_+ \wedge uK \arrow[r, "a_K"] & uK \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} uK = S^0 \wedge uK \arrow[r, "\nu_+\wedge uK"] \arrow[dr, "id_{uK}"'] & G_+ \wedge uK \arrow[d, "a_K"] \\ & uK \end{tikzcd}$$

commute. Note that $G_+ \wedge G_+ = (G \times G)_+$. A *map of G-spaces* $f\colon K \longrightarrow L$ is a map of the underlying spaces $f\colon uK \longrightarrow uL$ making the diagram

$$\begin{tikzcd} G_+ \wedge uK \arrow[r, "a_K"] \arrow[d, "G_+\wedge f"'] & uK \arrow[d, "f"] \\ G_+ \wedge uL \arrow[r, "a_L"] & uL \end{tikzcd}$$

commutative. Let $G\mathcal{S}$ denote the category of G-spaces.

Any space K determines the *constant* G-space $\mathrm{ct}(K)$ with $u\mathrm{ct}(K) = K$ and $a_{\mathrm{ct}(K)}\colon G_+ \wedge K \xrightarrow{c_+\wedge K} S^0 \wedge K = K$, where $c\colon G \longrightarrow \Delta^0$ is the constant map. Sometimes I will not distinguish between $\mathrm{ct}(K)$ and K. The *smash product* $K \wedge L$ of two G-spaces K and L is defined on underlying spaces. In detail, $u(K \wedge L) = uK \wedge uL$ and $a_{K\wedge L}\colon G_+ \wedge uK \wedge uL \longrightarrow uK \wedge uL$ is the composition

$$G_+\wedge uK\wedge uL \xrightarrow{d\wedge uK\wedge uL} G_+\wedge G_+\wedge uK\wedge uL \xrightarrow{\sigma} G_+\wedge uK\wedge G_+\wedge uL \xrightarrow{a_K\wedge a_L} uK\wedge uL$$

where $d\colon G_+ \longrightarrow (G \times G)_+$ is the diagonal and σ is the natural isomorphism which interchanges G_+ and uK, and fixes the two outer terms. This smash product is as nice as the ones before.

Proposition 3.17. *The smash product of G-spaces is in fact a closed symmetric monoidal product on $G\mathcal{S}$.*

It is quite obvious that the smash product is associative and commutative. The unit is the space S^0 with a constant action. The right adjoint of $- \wedge K$ is denoted $G\mathcal{S}(K, -)$. If L is a G-space, $uG\mathcal{S}(K, L) = \mathcal{S}(uK, uL)$. Thus the maps which build $G\mathcal{S}(K,L)$ are not necessarily equivariant. The action of G on $G\mathcal{S}(K,L)$ is the map

$$g \wedge (f\colon uK \wedge \Delta^n_+ \longrightarrow uL) \longmapsto (g \cdot f\colon uK \wedge \Delta^n_+ \longrightarrow uL)$$

where $g \cdot f(x \wedge \alpha) := a_L(g \wedge f(a_K(g^{-1} \wedge x) \wedge \alpha))$. Here g is an element of G, f is a map of spaces, x is in uK and $\alpha\colon [m] \longrightarrow [n]$ is in Δ^n. Note that if G is a commutative group, Day's theorem [4, 2.6] supplies a different closed symmetric monoidal product of G-spaces.[8]

Definition 3.18. A G-space K is called *finite* if uK is a finite space. Let $\mathbf{f}G\mathcal{S}$ be the full subcategory of finite G-spaces.

The result [4, 11.2] shows that K is a finite G-space if and only if $G\mathcal{S}(K,-)$ commutes with filtered colimits. Furthermore, every G-space is a filtered colimit of finite ones by [4, 11.3], hence this notion of finiteness fits perfectly well in the general scheme.

Definition 3.19. A *G-simplicial functor* is a functor $X\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$ which is enriched over $G\mathcal{S}$. Let $\mathbf{GF}$ be the category of G-simplicial functors.

As in the case of motivic and simplicial functors, G-simplicial functors have a smash product by Day's theorem [4, 2.6] with the same formal properties.

Corollary 3.20. *The category* $\mathbf{GF}$ *of G-simplicial functors is bicomplete and has a closed symmetric monoidal product.*

To set up a connection of G-simplicial functors with spectra in equivariant homotopy theory, the spheres come into play. From one point of view, there might be a lot of them. From another, there is just one. I have to switch to topological spaces to explain this. Let V be some *G-representation*, that is, a finite-dimensional real vector space V with an $\mathbb{R}$-linear action of G. The one-point-compactification $S(V)$ of V then yields a pointed topological space – homeomorphic to a $\dim V$-sphere – with an action of G. This is an equivariant sphere. Fortunately, one can gather all the possibly different ones. Let $\mathbb{R}^G$ denote the *regular representation*, the real vector space of maps $G \longrightarrow \mathbb{R}$ with the canonical action. A classical result from representation theory says that any G-representation is a direct summand of finite copies of the regular representation. Thus the one-point-compactification of $\mathbb{R}^G$ accounts for all the equivariant spheres.

Let S^G be the G-space which has as underlying space the G-fold smash product of the simplicial circle S^1. The action of G on S^G is by permuting the factors. It can also be defined by the condition that the geometric realization of S^G shall be homeomorphic as a topological G-space to the one-point-compactification of the regular representation. I will call S^G the *G-sphere*. Since G is finite, S^G is a finite G-space.

[8] Well, if G is the trivial group, it is not different, but otherwise it is.

Definition 3.21. A *G-spectrum* E consists of a sequence $(E_0, E_1, \dots, E_n, \dots)$ of G-spaces, together with structure maps $S^G \wedge E_n \longrightarrow E_{n+1}$. A *map* $f\colon E \longrightarrow F$ of G-spectra is a sequence of maps $f_n\colon E_n \longrightarrow F_n$ which are compatible with the structure maps.

G-spectra appear in topological disguise as *G-prespectra* in Carlsson's survey article [2]. Similar to the non-equivariant case, G-spectra have a stable model structure whose homotopy category is the *equivariant stable homotopy category*. By evaluating a G-simplicial functor X on the smash powers of the G-sphere, one gets a G-spectrum $\mathrm{ev}(X)$ by the usual argument.

3.5. **Homotopy Theory of Equivariant Simplicial Functors.** Before I describe the homotopy theory of G-simplicial functors, I will review some unstable equivariant homotopy theory. If $\phi\colon H \hookrightarrow G$ is a subgroup of G, any G-space K becomes an H-space ϕ^*K with underlying space $u\phi^*K = uK$ and action $H_+ \wedge uK \xrightarrow{\phi_+ \wedge uK} G_+ \wedge uK \xrightarrow{a_K} uK$.

Definition 3.22. If K is a G-space and $H \hookrightarrow G$ is a subgroup of G, the *H-fixed point space* $\mathrm{Fix}(H, K)$ is the subspace of uK which is fixed under the induced action of H.

For example, uK is the fixed point space of K for the subgroup of G given by the unit. The G-fixed point space of the G-sphere is the space S^1 which sits inside S^G as the diagonal. The G-fixed point space of the G-space $G\mathcal{S}(K, L)$ of maps from uK to uL is the *space of maps in $G\mathcal{S}$ from K to L*. As a formula:

$$\mathrm{Fix}(G, G\mathcal{S}(K, L)) = \mathcal{S}_{G\mathcal{S}}(K, L).$$

Definition 3.23. A map $f\colon K \longrightarrow L$ of G-spaces is a *G-weak equivalence* if the map $\mathrm{Fix}(H, f)$ is a weak equivalence of spaces for every subgroup $H \hookrightarrow G$. It is a *G-fibration* if the map $\mathrm{Fix}(H, f)$ is a Kan fibration for every subgroup $H \hookrightarrow G$.

To conclude that these classes are part of a model structure on $G\mathcal{S}$, it is useful to note two facts. Firstly, the functor $\mathrm{Fix}(H, -)\colon G\mathcal{S} \longrightarrow \mathcal{S}$ has a left adjoint $G/H_+ \wedge -\colon \mathcal{S} \longrightarrow \mathrm{Fix}(H, -)$ for every subgroup $H \hookrightarrow G$. Here G/H is the set of left cosets, with action of G induced by the left multiplication action on G. Secondly, the model category of spaces has generating cofibrations

$$I = (\partial\Delta^n \hookrightarrow \Delta^n)_+$$

and generating acyclic cofibrations

$$J = (\Lambda^n_i \hookrightarrow \Delta^n)_+$$

where Λ^n_i is the sub-simplicial set of Δ^n containing all but the i-th face.

Theorem 3.24. *G-weak equivalences and G-fibrations are part of a model structure on the category of G-spaces. The cofibrations are the monomorphisms. The resulting model category is monoidal and satisfies the monoid axiom.*

This familiar theorem is proven in [4, 11.5] using the sets

$$I_G := \{G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n\geq 0,\, H \text{ subgroup of } G}$$

and

$$J_G := \{G/H_+ \wedge (\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{n\geq 1,\, 0\leq i\leq n,\, H \text{ subgroup of } G}.$$

Because all the G-spaces appearing in these two sets are finite (which relies on G being finite), G-weak equivalences and G-fibrations are closed under filtered colimits. Together with the fact that all G-spaces are cofibrant, this simplifies several of the arguments needed to construct the stable homotopy theory of enriched functors considerably. An example is the following *enriched fibrant replacement* construction, which will lead to a nice fibrant replacement in the homotopy functor model structure on G-simplicial functors.

Rewrite the set J_G from above as $\{j\colon sj \xrightarrow{\sim} tj\}_{j\in J_G}$. Each $j \in J_G$ gives rise to a diagram

$$G\mathcal{S}(sj,K)\wedge tj \xleftarrow[\sim]{G\mathcal{S}(sj,K)\wedge j} G\mathcal{S}(sj,K)\wedge sj \xrightarrow{\text{counit}} K.$$

Taking the coproduct over all $j \in J_G$ determines a diagram

$$\coprod_{j\in J_G} G\mathcal{S}(sj,K)\wedge tj \xleftarrow[\sim]{\coprod_{j\in J_G} G\mathcal{S}(sj,K)\wedge j} \coprod_{j\in J_G} G\mathcal{S}(sj,K)\wedge sj \longrightarrow K$$

of which the pushout is denoted $R^1(K)$. Now replace K in the above diagram with $R^1(K)$ to get the pushout $R^2(K)$, and so on. Let RK be the colimit of the sequence

$$K \xrightarrow{\sim} R^1(K) \xrightarrow{\sim} R^2(K) \xrightarrow{\sim} \cdots$$

then the canonical map $\rho(K)\colon K \longrightarrow RK$ is in fact an enriched natural transformation $\rho\colon \mathrm{Id}_{G\mathcal{S}} \longrightarrow R$, that is, a map of G-simplicial functors. To the reader who is familiar with Quillen's small object argument, the following consequence of [4, 3.26] should be no surprise.

Lemma 3.25. *For every G-space K, the G-space RK is G-fibrant and the map $\rho(K)\colon K \longrightarrow RK$ is a G-weak equivalence and a monomorphism.*

I will not distinguish between the functor $R\colon G\mathcal{S} \longrightarrow G\mathcal{S}$ and its restriction $R\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$ to finite G-spaces. Before this functor comes into play, I have to mention that every G-simplicial functor $X\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$ extends to a G-simplicial functor $X'\colon G\mathcal{S} \longrightarrow G\mathcal{S}$, which is defined as a left Kan extension via the coend

$$X' = \int^{K\in \mathrm{Ob}\,\mathbf{f}G\mathcal{S}} G\mathcal{S}(K,-)\wedge XK.$$

When restricted to the category of finite G-spaces, this G-simplicial functor X' coincides with X. Now the composition $X'\circ R$ is possible.

Definition 3.26. A map $f\colon X \longrightarrow Y$ in $\mathbf{GF}$ is a

- *pointwise weak equivalence* if $f(K)$ is a G-weak equivalence for all finite G-spaces K,
- *pointwise fibration* if $f(K)$ is a G-fibration for all finite G-spaces K,
- *cofibration* if it has the left lifting property with respect to all pointwise acyclic fibrations,
- *hf-equivalence* if $f' \circ R\colon X' \circ R \longrightarrow Y' \circ R$ is a pointwise weak equivalence,
- *hf-fibration* if it is a pointwise fibration such that

$$\begin{tikzcd} XK \arrow[r] \arrow[d, "f(K)"'] & XL \arrow[d, "f(L)"] \\ YK \arrow[r] & YL \end{tikzcd}$$

  is a homotopy pullback square for every G-weak equivalence $K \xrightarrow{\sim} L$ of finite G-spaces.

The pointwise model structure then exists by [4, 11.7], the homotopy functor model structure by [4, 11.10]. In the proof of the latter theorem, one has to use that hf-equivalences of G-simplicial functors can also be defined through a filtered colimit analogous to the one in 3.9. This implies that hf-equivalences which are also hf-fibrations are even pointwise acyclic fibrations in $\mathbf{GF}$.

Since all G-spaces are cofibrant, it is quite easy to see that $X' \circ R$ is a homotopy functor for any $X\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$. Here is the argument. If $f\colon K \xrightarrow{\sim} L$ is a G-weak equivalence of finite G-spaces, $Rf\colon RK \longrightarrow RL$ is a G-weak equivalence of G-spaces which are both G-fibrant and cofibrant. By general arguments from model category theory, Rf is thus even a *simplicial homotopy equivalence* as defined in [4, 2.9]. That is, there exists a homotopy inverse $g\colon RL \longrightarrow RK$ and simplicial homotopies

$$\begin{tikzcd} RK \arrow[r, hook, "i_0"] \arrow[dr, "g\circ Rf"'] & RK \wedge \Delta^1_+ \arrow[d, "H"] & RK \arrow[l, hook', "i_1"'] \arrow[dl, "\mathrm{id}_{RK}"] \\ & RK & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} RL \arrow[r, hook, "i_0"] \arrow[dr, "Rf\circ g"'] & RL \wedge \Delta^1_+ \arrow[d, "H'"] & RL \arrow[l, hook', "i_1"'] \arrow[dl, "\mathrm{id}_{RL}"] \\ & RL & \end{tikzcd}$$

where i_0 and i_1 denote the two possible natural inclusions. Either directly, or by the general [4, 2.11], it is easy to see that any G-simplicial functor preserves simplicial homotopy equivalences. Thus $X'(Rf)$ is a simplicial homotopy equivalence, hence in particular a G-weak equivalence.[9]

Since the canonical map $Y \longrightarrow R \circ Y$ induced by $\rho\colon \mathrm{Id}_{GS} \longrightarrow R$ is a pointwise weak equivalence with a pointwise fibrant target, it remains to observe that the canonical map $X \longrightarrow X' \circ R$ is an hf-equivalence to conclude the next lemma, which is a special case of [4, 5.18]. Lydakis uses in [10] as R the composition of "singular complex" and "geometric realization".

[9]Also motivic functors preserve simplicial homotopy equivalences. The problem is that the motivic variant of R does not necessarily take its values in projectively cofibrant motivic spaces.

Lemma 3.27. *The functor $X \longmapsto R \circ X' \circ R$ is a fibrant replacement in the homotopy functor model structure on the category* $\mathbf{GF}$.

Stabilization proceeds as already described in the case of motivic functors, with T replaced by the G-sphere S^G. In detail, $\mathbf{S}\colon \mathbf{GF} \longrightarrow \mathbf{GF}$ is the endofunctor mapping X to the composition $G\mathcal{S}(S^G,-)\circ X\circ(S^G\wedge -)$, and $s\colon \mathrm{Id}_{\mathbf{GF}} \longrightarrow \mathbf{S}$ is the natural transformation obtained pointwise as the adjoint of the map of G-spaces $XK \wedge S^G \longrightarrow X(S^G \wedge K)$, which is in turn the adjoint of the composition

$$S^G \xrightarrow{\alpha} G\mathcal{S}(K, S^G \wedge K) \xrightarrow{\mathrm{hom}^X} G\mathcal{S}(XK, X(S^G \wedge K)).$$

Here α is the unit of an adjunction. Let $\mathbf{S}^\infty(X)$ denote the colimit of

$$X \xrightarrow{s(X)} \mathbf{S}(X) \xrightarrow{\mathbf{S}(s(X))} \mathbf{S}(\mathbf{S}(X)) \longrightarrow \cdots$$

Again one should insert pointwise fibrant homotopy functors into this stabilization process. By 3.27 a concrete choice is possible.

Definition 3.28. A map $f\colon X \longrightarrow Y$ of G-simplicial functors is a

- *stable equivalence* if $\mathbf{S}^\infty(R \circ f' \circ R)$ is a pointwise weak equivalence, and a
- *stable fibration* if it is an hf-fibration such that

$$\begin{tikzcd} X \arrow[r, "s(X)"] \arrow[d, "f"'] & \mathbf{S}(X) \arrow[d, "\mathbf{S}(f)"] \\ X \arrow[r, "s(X)"] & \mathbf{S}(X) \end{tikzcd}$$

  is a homotopy pullback square in the pointwise model structure on $\mathbf{GF}$.

To conclude the existence of the stable model structure on $\mathbf{GF}$, it remains to observe that also the G-sphere S^G satisfies the cyclic permutation condition. This is verified in the proof of [4, 11.15]. Together with the fact that stable fibrations which are stable equivalences are even pointwise weak equivalences (again this involves a filtered colimit of certain homotopy pullback squares that is well-behaved due to properties of the model structure on G-spaces), this implies theorem 11.16 of [4].

Theorem 3.29. *The category $\mathbf{GF}$, equipped with the classes of stable equivalences, stable fibrations and cofibrations, is a monoidal model category satisfying the monoid axiom. Both stable fibrations and stable equivalences are closed under filtered colimits, and smashing with a cofibrant G-simplicial functor preserves stable equivalences.*

If G is the trivial group, this is a new proof of Lydakis' result on the existence of the stable model structure on simplicial functors. For an arbitrary finite group G, I do not know so far whether the evaluation functor to the stable model category of G-spectra is a Quillen equivalence, but I am quite convinced of this hypothesis. The reason is that there is an equivariant version of the Blakers-Massey-Theorem that entered Lydakis' proof on the Quillen equivalence of model categories at several points. Nevertheless, one can perform the same trick I already described in the context of motivic functors to get the following theorem.

Theorem 3.30. *There is a monoidal model structure on the category* **GF** *of G-simplicial functors, which is Quillen equivalent via the evaluation functor to the stable model category of G-spectra. The monoid axiom holds.*

The identity functor $\mathrm{Id}_{\mathbf{GF}}$ maps the stable fibrations and stable equivalences of 3.29 to the fibrations and weak equivalences in the model structure of 3.30. I hope to prove in another paper that $\mathrm{Id}_{\mathbf{GF}}$ is in fact a Quillen equivalence, which then would yield a complete generalization of Lydakis' results on simplicial functors to the equivariant case.[10]

4. Motivic Cohomology

The basic observation is that the motivic spectrum Voevodsky describes in section 6.1 of his ICM-article [16] to represent motivic cohomology has a very natural extension to a motivic functor $M_{\mathbb{Z}}\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$. Moreover, this motivic functor is in fact a commutative monoid in **MF**. To indicate why this is true, I will assume throughout that the base scheme S is regular, although the result holds more generally. The rough idea behind the following already appeared in the description of singular homology as a simplicial functor $H_{\mathbb{Z}}$. Homology is some kind of abelianization process. This alone, however, is not sufficient for smooth schemes over S. One reason is the following. From topology, one knows that covering projections induce *transfer maps* in homology theory which "point in the other way". In the algebraic setting, such transfer maps have to be supplied by means of correspondences.

Definition 4.1. Given two smooth schemes U and V in $\mathbf{Sm}/S$, a *correspondence* C from U to V is a closed irreducible subscheme C of $U \times V$ which is finite and surjective over a connected component of U.

For example, if $f\colon U \longrightarrow V$ is a map in $\mathbf{Sm}/S$, its graph $\Gamma(f) \subseteq U \times V$ is a correspondence. The graph of the identity map id_U is the diagonal in $U \times U$.

Definition 4.2. For two smooth schemes U and V in $\mathbf{Sm}/S$, the free abelian group on the set of correspondences from U to V is denoted $\mathbf{Cor}/S(U,V)$.

The passage from the set of correspondences to the free abelian group of correspondences allows both a composition and a product of correspondences, the latter being introduced as the *external product of relative cycles* in [14].

[10] To a lot of people, a "complete generalization" has to include the case of compact Lie groups. So far, this is not possible with the methods from [4], which are somehow tied to a simplicial environment.

Definition 4.3. Let $\mathbf{Cor}/S$ denote the category which has the same objects as the category $\mathbf{Sm}/S$ of smooth schemes of finite type over S. The set of morphisms from U to V shall be the group $\mathbf{Cor}/S(U,V)$.

Theorem 4.4 (Suslin, Voevodsky). *$\mathbf{Cor}/S$ is a symmetric monoidal additive category. The graph is then a functor $\Gamma\colon \mathbf{Sm}/S \longrightarrow \mathbf{Cor}/S$ which is the identity on objects.*

This can be derived from results in [14]. An additive category is a category enriched over the category $\mathbf{Ab}$ of abelian groups, which is a closed symmetric monoidal category via the tensor product $\otimes$. The symmetric monoidal product in $\mathbf{Cor}/S$ is defined on objects via the categorical product $(U,V) \longmapsto U \times V$, and the base scheme S is then obviously the unit. The hard work is to show that this product extends to the groups of correspondences.

Definition 4.5. A *presheaf with transfer* is a functor $F\colon \mathbf{Cor}/S^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ which is additive.

Any scheme U in $\mathbf{Sm}/S$ determines the representable presheaf with transfer $[U]\colon \mathbf{Cor}/S \longrightarrow \mathbf{Ab}$ which maps V to the abelian group $\mathbf{Cor}/S(V,U)$. It is worth noticing that the presheaves with transfers form a category $\mathbf{Tr}$ of functors enriched over $\mathbf{Ab}$. Thus the general machinery implies that $\mathbf{Tr}$ is bicomplete and any presheaf with transfer F can be written as the coend

$$F = \int^{U \in \mathrm{Ob}\,\mathbf{Cor}/S} \mathbf{Cor}/S(-,U) \otimes F(U) = \int^{U \in \mathrm{Ob}\,\mathbf{Cor}/S} [U] \otimes F(U).$$

Moreover, Day's theorem [4, 2.6] provides $\mathbf{Tr}$ with a closed symmetric monoidal product, because 4.4 holds. Every presheaf with transfer F gives rise to a (discrete) space-valued presheaf γF on $\mathbf{Sm}/S$ via the graph functor. That is, $\gamma F(U)$ is the underlying set of the abelian group $F(U)$, which is pointed by zero and considered as a discrete space. Let the resulting functor to motivic spaces be $\gamma\colon \mathbf{Tr} \longrightarrow \mathcal{M}$.

Now I am able to formulate Voevodsky's description of motivic cohomology as a motivic spectrum from [16, section 6.1].[11] In fact, he describes a functor L first and evaluates it on spheres, which is in the spirit of this note. If U is a scheme in $\mathbf{Sm}/S$, $L(U)$ is $\gamma[U]$, the underlying motivic space of the presheaf with transfer represented by U. Considering U as a motivic space, there is a map $U \longrightarrow \gamma[U]$ given by the graph $\mathbf{Set}_{\mathbf{Sm}/S}(V,U) \longrightarrow \mathbf{Cor}/S(V,U)$. This map is natural in U and will define the unit of the monoid $M_{\mathbb{Z}}\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ in $\mathbf{MF}$.

To define the motivic cohomology spectrum, L has to take values on the spheres $T^0, T^1, T^2, \ldots$ which are not schemes. Therefore Voevodsky considers motivic spaces which are the quotient of a scheme modulo a finite union of subschemes,

[11] Instead of using space-valued presheaves, he uses set-valued Nisnevich sheaves, but this does not make a difference up to Quillen equivalence.

He calls these motivic spaces *scheme-like* and observes that all spheres are scheme-like. The effect of L on such a scheme-like motivic space is then

$$A = U/(\cup_{1\leq i\leq n} V_i) \longmapsto L(A) = \gamma([U]/\oplus_{1\leq i\leq n}[V_i]).$$

In categorical terms, Voevodsky looks at motivic spaces which are certain colimits of representables, and uses these certain colimits to define L via the values of L on representables. It is obvious how to generalize this to arbitrary motivic spaces, and I will do it in a minute. I just have to remark that the result 4.4 on the symmetric monoidal structure on $\mathbf{Cor}/S$ implies the existence of a map

$$\mathbf{Cor}/S(W,U)\otimes\mathbf{Cor}/S(W,V) \longrightarrow \mathbf{Cor}/S(W\times W, U\times V) \longrightarrow \mathbf{Cor}/S(W, U\times V)$$

where the second arrow is induced from the diagonal $W \longrightarrow W\times W$. This map is natural in W and induces a map

$$L(U)\wedge L(V) = \gamma[U]\wedge\gamma[V] \longrightarrow \gamma[U\times V] = L(U\times V)$$

which will become the multiplication of $M_{\mathbb{Z}}\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ in $\mathbf{MF}$. The motivic cohomology spectrum is then $(L(T^0), L(T^1), L(T^2), \dots)$, with structure maps $T\wedge L(T^n) \longrightarrow L(T)\wedge L(T^n) \longrightarrow L(T\wedge T^n)$ induced by the unit and the multiplication.

Here comes the generalization to finite motivic spaces. Let A be a finite motivic space. It can be written in a natural way as the colimit

$$\operatorname*{colim}_{(U\times\Delta^n)_+ \longrightarrow A)} (U\times\Delta^n)_+$$

which is indexed over the category of maps in $\mathcal{M}$ with representable domain and codomain A.

Definition 4.6 (Motivic Cohomology)**.**

$$M_{\mathbb{Z}}(A) := \gamma(\operatorname*{colim}_{(U\times\Delta^n)_+ \longrightarrow A}[U]\otimes H_{\mathbb{Z}}(\Delta^n_+))$$

Colimit and tensor product appearing on the right hand side are scheme-wise taken in the category of simplicial abelian groups. That is, if one specializes to a scheme V in $\mathbf{Sm}/S$ and a simplicial degree $m\geq 0$, the formula

$$M_{\mathbb{Z}}(A)(V)_m = \operatorname*{colim}_{(U\times\Delta^n)_+ \longrightarrow A)} \mathbf{Cor}/S(V,U)\otimes\mathbb{Z}[\mathbf{Set}_{\Delta}([m],[n])]$$

holds. In particular, $M_{\mathbb{Z}}$ has the same values at schemes as L does, which also implies the compatibility for scheme-like motivic spaces.[12]

Lemma 4.7. *$M_{\mathbb{Z}}$ is a commutative monoid in* $\mathbf{MF}$.

[12]To be honest, one also has to take into account the difference between the set-valued Nisnevich sheaves on $\mathbf{Sm}/S$ Voevodsky uses, and the motivic spaces $\mathcal{M}$. As it turns out, the $\mathbb{A}^1$-realization of the sheafification of $M_{\mathbb{Z}}$ then coincides with the functor L. Both $\mathbb{A}^1$-realization and sheafification are Quillen equivalences by [9, B.5], hence this compatibility is good enough.

Proof. I have to describe a unit map $\nu\colon \mathbb{I} \longrightarrow M_{\mathbb{Z}}$. The definition of $M_{\mathbb{Z}}$ indicates that it is enough to construct the unit for representable motivic spaces $A = (U \times \Delta^n)_+$. For such, the unit map $(U \times \Delta^n)_+ \longrightarrow \gamma([U] \otimes H_{\mathbb{Z}}(\Delta^n_+))$ is defined via the graph map $U \longrightarrow \gamma[U]$ and the unit $\Delta^n_+ \longrightarrow H_{\mathbb{Z}}(\Delta^n_+)$ of singular cohomology.

The multiplication μ is a little bit harder to describe. As the smash product of motivic functors is defined as an enriched left Kan extension, a map of motivic functors $\mu\colon M_{\mathbb{Z}} \wedge M_{\mathbb{Z}} \longrightarrow M_{\mathbb{Z}}$ is the same as a certain collection of maps of motivic spaces $M_{\mathbb{Z}}(A) \wedge M_{\mathbb{Z}}(B) \longrightarrow M_{\mathbb{Z}}(A \wedge B)$ which is natural in the finite motivic spaces A and B. It suffices to describe such a map for $A = (U \times \Delta^n)_+$ and $B = (V \times \Delta^m)_+$. In this case, the multiplication

$$([U] \otimes H_{\mathbb{Z}}(\Delta^n_+)) \wedge ([V] \otimes H_{\mathbb{Z}}(\Delta^m_+)) \longrightarrow [U \times V] \otimes H_{\mathbb{Z}}(\Delta^n_+ \wedge \Delta^m_+)$$

is induced by the monoidal product of $\mathbf{Cor}/S$ from 4.4 and the multiplication of $H_{\mathbb{Z}}$. Both are commutative, thus $M_{\mathbb{Z}}$ is a commutative monoid.

I should say why $M_{\mathbb{Z}}$ is indeed a motivic functor $\mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$, instead of just a functor. The map

$$\hom^{M_{\mathbb{Z}}}\colon \mathcal{M}(A,B) \longrightarrow \mathcal{M}(M_{\mathbb{Z}}(A), M_{\mathbb{Z}}(B))$$

is given as follows. First observe that the unit and the multiplication of $M_{\mathbb{Z}}$ can at once be extended to all motivic spaces, for example to such of the form $\mathcal{M}(A,B)$. Then use as $\hom^{M_{\mathbb{Z}}}$ the adjoint of the composition

$$\mathcal{M}(A,B) \wedge M_{\mathbb{Z}}(A) \xrightarrow{\nu \wedge M_{\mathbb{Z}}(A)} M_{\mathbb{Z}}(\mathcal{M}(A,B)) \wedge M_{\mathbb{Z}}(A) \xrightarrow{\mu} M_{\mathbb{Z}}(\mathcal{M}(A,B) \wedge A \xrightarrow{\text{counit}} B)$$

and check that it does the job. □

The monoid axiom for the stable model structure on motivic functors then implies that the category $M_{\mathbb{Z}}$-**Mod** of $M_{\mathbb{Z}}$-modules has a model structure, with weak equivalences and fibrations defined on underlying motivic functors. Conjecturally, there is an interesting connection between its homotopy category and the derived category of motives constructed by Voevodsky in [17] for S the spectrum of a perfect field.

References

[1] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, pages 80–130. Springer, Berlin, 1978.

[2] Gunnar Carlsson. A survey of equivariant stable homotopy theory. *Topology*, 31(1):1–27, 1992.

[3] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, pages 1–38. Springer, Berlin, 1970.

[4] Bjørn Ian Dundas, Oliver Röndigs, and Paul Arne Østvær. Enriched functors and motivic stable homotopy theory. Anlage, April 2002.

[5] Mark Hovey. *Model categories.* American Mathematical Society, Providence, RI, 1999.

[6] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[7] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[8] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[9] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[10] Manos Lydakis. Simplicial functors and stable homotopy theory. Preprint 98-049, SFB 343, Bielefeld, June 1998.

[11] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[12] Daniel G. Quillen. *Homotopical algebra.* Springer-Verlag, Berlin, 1967.

[13] Stefan Schwede and Brooke Shipley. Classification of stable model categories. Accepted in Topology, December 2001.

[14] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.

[15] Vladimir Voevodsky. The Milnor conjecture. Preprint, December 1996.

[16] Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, pages 579–604 (electronic), 1998.

[17] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[18] Rainer Vogt. *Boardman's stable homotopy category.* Matematisk Institut, Aarhus Universitet, Aarhus, 1970.