ENRICHED FUNCTORS AND MOTIVIC STABLE HOMOTOPY THEORY

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1. Introduction

In this paper we introduce a functor model for Voevodsky's stable motivic homotopy category [18]. Its objects are called *motivic functors* and two such functors have a smash product reminiscent to the tensor product of modules. The stable motivic homotopy category has a symmetric monoidal smash product, which is important in many respects and makes it possible to consider "ring spectra" and modules over them. However, it is impossible to construct such a structure on the level of the underlying category of motivic spectra, if defined in its most naïve way. Voevodsky suggests that a way around this problem is to use symmetric spectra as in [10], and this is achieved by Jardine [12] and Hovey [9]. Let us summarize our solution to these questions. Motivic functors and their natural transformations form a closed symmetric monoidal model category whose homotopy category is equivalent to the stable motivic homotopy category. This model category has important additional properties, and we refer to 10.29 for the precise statement.

Theorem. *There is a monoidal model category $(\mathcal{F}, \wedge, \mathbb{I})$ satisfying the monoid axiom, and a right Quillen equivalence from $\mathcal{F}$ to the stable model category of T-spectra.*

The underlying category $\mathcal{M}$ of "motivic" spaces consists of pointed simplicial presheaves on the category of smooth schemes over a base scheme S. There is a preferred "circle" in $\mathcal{M}$ called the Tate object T. A T-spectrum is a sequence of motivic spaces $E_0, E_1, \ldots$ together with structure maps $T \wedge E_n \longrightarrow E_{n+1}$. A motivic functor X is a functor from a category of "finite" motivic spaces to all motivic spaces, which is "continuous" or "enriched" in the sense that it induces a map of motivic spaces of maps. In particular there is a map

$$v \wedge X(w) \longrightarrow X(v \wedge w)$$

which is natural in the finite motivic spaces v and w. The importance of this is that any motivic functor X gives rise to a T-spectrum $\mathrm{ev}(X)$ by "evaluating on spheres", that is

$$\mathrm{ev}(X)_n := X(T^{\wedge n}).$$

The motivic functors form the category $\mathcal{F}$ mentioned in the main theorem, and the evaluation on spheres is the Quillen equivalence.

Voevodsky uses the stable motivic homotopy category to introduce important cohomology theories for algebraic varieties [18]. The motivic Eilenberg-MacLane spectrum is of particular interest. It represents motivic cohomology, and the question arise as how to best describe the category of modules over this object. Motivic functors allow to give homotopical meaning to algebraic constructions such as algebras and modules over monoids in this context. The motivic Eilenberg-MacLane spectrum has a concrete representative $M\mathbb{Z}$ in $\mathcal{F}$ which is a commutative monoid. Our results interpret this object as the ground ring in a category where motivic algebra takes place [5]. This paper provides the necessary background for such constructions. In particular, the theorem above implies immediately that we can use all the machinery of homotopical algebra on the category of $M\mathbb{Z}$-modules.

Corollary. *The category of $M\mathbb{Z}$-modules in $\mathcal{F}$ is a monoidal model category and the monoid axiom holds.*

The smash product in $\mathcal{F}$ comes from an old result of Day [4] which gives monoidal structures to a wide range of enriched functor categories. Basically, the idea is that coherence problems which arise when one tries to make a smash product must be encoded in the category of spaces itself. This is different from the popular means of attack through symmetric spectra [10], where the controlling categories are much more restricted. Our approach is in analogy with Lydakis' development in [13] of simplicial functors as a model for the classical stable homotopy category, but the theory is complicated by the fact that we must do without many desirable features which are quite particular to simplicial sets. Contrary to the simplicial case, the choice of the category of finite spaces is not obvious, and it turns out to be a crucial point that we are free to pick several choices, and extract the best properties from each of them. However, the differences are at a technical level: the guiding principles may be elevated to a more general level. We hope the reader will find general enough results herein which he or she can prove to have further applications.

Here is an outline of the paper. Section 2 is a short review of the categorical background. There is a general isomorphism of categories between (both classical and symmetric) spectra in the sense of [9] and a category of enriched functors build from spheres. Section 3 introduces *weakly finitely generated* model categories. Such a model structure is in particular cofibrantly generated. For later use, we consider enriched and some not so enriched fibrant replacement functors. These constructions are of technical interest, and might be omitted on a first cursory reading.

In the next three sections we consider homotopy theory for general enriched functor categories $\mathcal{F}$. First we construct the *pointwise model structure*. This is a weakly finitely generated model structure provided some reasonable assumptions are satisfied. Moreover, the important *monoid axiom* holds in many cases of interest. Theorem 4.12 says that smashing with a cofibrant object preserves weak equivalences in the pointwise model structure. Algebraists may think of this property as follows. Replace *cofibrant* with *projective*, and recall that tensoring with a projective module preserves short exact sequences.

A homotopy functor is a functor preserving weak equivalences. In favorable situations, we associate to any enriched functor a canonically given homotopy functor. In this case, a map is a weak equivalence in the *homotopy functor model structure* if it induces a pointwise equivalence of associated homotopy functors. We need slightly more flexibility than this, and Section 5 explains the general setup, see 5.10 for the precise statement. Using the homotopy functor model structure, one can hope to discuss a motivic version of Goodwillie's calculus of functors. It should be noted that this section contains the only points in our general setup which have to be modified for the application to the motivic category.

In Section 6 the *stable model structure* is constructed by means of a general stabilization process. Stable fibrations are characterized by certain homotopy pullback squares, and stable acyclic fibrations are precisely the pointwise acyclic fibrations by 6.13. Theorem 6.25 gives conditions for the stable model structure to exist. Moreover, it is always a weakly finitely generated model structure. To prove this result we compare with spectra [9]. Our stabilization does not coincide with the one for spectra, and it requires some cruel details to compare them. These can be found in Appendix A. We note that the monoid axiom holds under an additional assumption on the source of the functor category. This assumption is not relevant for the existence of the stable model structure. There is an evaluation functor which to an enriched functor associates a spectrum by considering its values on spheres. For a particular choice of the source category, which is explained in Section 7, it follows that the evaluation functor is the right adjoint in a Quillen equivalence. Hence the highly structured category of enriched functors describes the same homotopy theory as spectra in many cases of interest. Section 8 gives a short summary of the important algebraic consequences of the previous sections.

Applications follow in the last part of the paper. The first example is motivic homotopy theory, and the second is equivariant homotopy theory. To explain how motivic homotopy theory fits into the general picture, we construct a specific model of it in Section 9. This part is quite detailed and contains more general results than we need, which might be useful when it comes to comparing various Grothendieck topologies on the category of schemes over a base scheme. Section 10 contains the main results on motivic stable homotopy presented above. There are several points to this section which are worth while mentioning. The first is that it does not properly follow as a special case from the general machinery. In particular, we employ a modified fibrant replacement functor. This part may also streamline the presentation for other examples. To prove the monoid axiom we consider two choices of finite motivic spaces, and combine the corresponding stable model structures to extract a third model structure. The latter puts us in the best of two worlds, and the previous results lead effectively to a proof of the theorem above.

Finally, in Section 11 we treat G-equivariant homotopy theory for G a finite group. The machinery is the same as for simplicial or motivic functors, and we obtain the following.

Theorem. *Let G be a finite group. Then there is a monoidal model category $(G\mathcal{F}, \wedge, \mathbf{S}^G)$ satisfying the monoid axiom and a right Quillen equivalence from $G\mathcal{F}$ to the category of G-spectra.*

2. Enriched Categories

This section starts with a brief introduction to enriched category theory and Day's work on enriched functor categories [4]. A short discussion of simplicial homotopies in categories enriched over simplicial sets follows. Finally, the categories of spectra and symmetric spectra [9] build from a closed symmetric monoidal category are expressed as categories of enriched functors. For a small category $\mathcal{C}$ and a category $\mathcal{D}$, denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$.

2.1. **Introduction.** Entry points to the literature on enriched category theory include [1] and [14]. A *monoidal* category consists of a category $\mathcal{V}$, and:

- a functor $\otimes\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and a natural associativity isomorphism
$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$
subject to the coherence law [1, 6.1],
- an object e of $\mathcal{V}$ called the *unit*, and natural unit isomorphisms
$$l_A\colon e \otimes A \longrightarrow A, \text{ and } r_A\colon A \otimes e \longrightarrow A$$
satisfying the coherence law [1, 6.2].

One refers to $\otimes$ as the *tensor product* or the *monoidal product*. A monoidal category is called *symmetric monoidal* if there exists a natural isomorphism $\sigma_{A,B}\colon A \otimes B \longrightarrow B \otimes A$ subject to the three coherence laws [1, 6.3, 6.4, 6.5]. These laws require the commutativity isomorphism to be compatible with the associativity and unit isomorphisms. A *closed* symmetric monoidal category is a symmetric monoidal category $(\mathcal{V}, \otimes, e)$ equipped with a right adjoint
$$\mathrm{Hom}_{\mathcal{V}}(A, -)\colon \mathcal{V} \longrightarrow \mathcal{V}$$
to the endofunctor $- \otimes A$ for every object A of $\mathcal{V}$.

A category with products is a symmetric monoidal category, albeit not closed. Consider the category of topological spaces. Then $- \times X$ does not preserve quotient projections in general, and hence it can not always have a right adjoint. The category **Set** of sets, with the Cartesian product, is a closed symmetric monoidal category. Likewise for the category of pointed sets $\mathbf{Set}_*$. Its monoidal product is the smash product of pointed sets, and the right adjoint of $- \wedge N$ is the set of pointed maps with domain N, pointed by the trivial map. Let Δ denote the simplicial category. Its objects are the finite ordered sets $[n] = \{0 < 1 < \ldots < n\}$ for $n \geq 0$, and morphisms are order-preserving maps. Consider the category of simplicial sets $\mathbf{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$. The monoidal product is the categorical product, which is formed degree-wise in **Set**. If $K, L \in \mathbf{sSet}$, the simplicial set of maps $\mathrm{Hom}_{\mathbf{sSet}}(K, L)$ has as n-simplices the set of maps in **sSet** from $K \times \Delta^n$ to L. Here Δ^n is the simplicial set represented by $[n]$. The unit of the product is the terminal object Δ^0.

Fix a closed symmetric monoidal category $(\mathcal{V}, \otimes, e)$. A *$\mathcal{V}$-category* $\mathcal{C}$ consists of a class $\mathrm{Ob}\,\mathcal{C}$ of objects, and:

- for any pair (a, b) of objects in $\mathcal{C}$, an object $\mathcal{V}_{\mathcal{C}}(a, b)$ of $\mathcal{V}$ (the $\mathcal{V}$-object of maps in $\mathcal{C}$ from a to b),
- a composition $\mathcal{V}_{\mathcal{C}}(b, c) \otimes \mathcal{V}_{\mathcal{C}}(a, b) \longrightarrow \mathcal{V}_{\mathcal{C}}(a, c)$ and an identity or unit map $e \longrightarrow \mathcal{V}_{\mathcal{C}}(a, a)$ which satisfy the associativity and unit coherence conditions [1, 6.9 and 6.10].

A $\mathcal{V}$-category is also called a category *enriched over* $\mathcal{V}$. The **Set**-categories are precisely categories in the usual sense. For a **Set**-category $\mathcal{C}$, denote the set of maps in $\mathcal{C}$ from a to b by $\mathbf{Set}_{\mathcal{C}}(a, b)$. Every closed symmetric monoidal category $\mathcal{V}$ is a $\mathcal{V}$-category due to its internal Hom objects, see [1, 6.2.6]. Write $\mathcal{V}(A, B)$ for the $\mathcal{V}$-object $\mathrm{Hom}_{\mathcal{V}}(A, B)$ of maps in $\mathcal{V}$. Note that a $\mathcal{V}$-category $\mathcal{C}$ defines a **Set**-category $\mathcal{U}\mathcal{C}$ as follows. Its class of objects is $\mathrm{Ob}\,\mathcal{C}$, and the morphism sets are $\mathbf{Set}_{\mathcal{U}\mathcal{C}}(a, b) = \mathbf{Set}_{\mathcal{V}}(e, \mathcal{V}_{\mathcal{C}}(a, b))$. For example, the **Set**-category obtained from a **sSet**-category $\mathcal{C}$ has morphism sets $\mathbf{Set}_{\mathbf{sSet}}(\Delta^0, \mathbf{sSet}_{\mathcal{C}}(a, b)) = \mathbf{sSet}_{\mathcal{C}}(a, b)_0$ the zero-simplices of the simplicial sets of maps.

A *$\mathcal{V}$-functor* F from $\mathcal{C}$ to $\mathcal{D}$ is an assignment $a \longmapsto F(a)$ from $\mathrm{Ob}\,\mathcal{C}$ to $\mathrm{Ob}\,\mathcal{D}$ together with morphisms $\mathrm{hom}^F_{a,b}\colon \mathcal{V}_{\mathcal{C}}(a, b) \longrightarrow \mathcal{V}_{\mathcal{D}}(F(a), F(b))$ in $\mathcal{V}$ which preserve composition and identities. A small **sSet**-category can also be described as a simplicial object in the category **Cat** of small categories which is constant on the objects. With this description, a **sSet**-functor is simply a natural transformation of functors from Δ^{op} to **Cat**, see [13, 3.2].

If F and G are $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{D}$, a *$\mathcal{V}$-natural transformation* $t\colon F \longrightarrow G$ consists of the following data. There is a morphism $t(a)\colon F(a) \longrightarrow G(a)$ in $\mathcal{U}\mathcal{D}$ for every $a \in \mathrm{Ob}\,\mathcal{C}$, and all the diagrams

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{C}}(a,b) \arrow[r, "\mathrm{hom}^F_{a,b}"] \arrow[d, "\mathrm{hom}^G_{a,b}"'] & \mathcal{V}_{\mathcal{D}}(F(a),F(b)) \arrow[d, "\mathcal{V}_{\mathcal{D}}(F(a),t(b))"] \\
\mathcal{V}_{\mathcal{D}}(G(a),G(b)) \arrow[r, "\mathcal{V}_{\mathcal{D}}(t(a),G(b))"] & \mathcal{V}_{\mathcal{D}}(F(a),G(b))
\end{tikzcd}$$

commute. One defines $\mathcal{V}$-natural isomorphisms and $\mathcal{V}$-adjoint pairs of $\mathcal{V}$-functors as in the classical case $\mathcal{V} = \mathbf{Set}$. The adjoint endofunctors $(- \otimes A, \mathcal{V}(A, -))$ on $\mathcal{V}$ explicate a $\mathcal{V}$-adjoint pair of $\mathcal{V}$-functors. Denote the unit of the adjunction by $\eta_A\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow \mathcal{V}(A, - \otimes A)$, and the counit by $\epsilon_A\colon \mathcal{V}(A, -) \otimes A \longrightarrow \mathrm{Id}_{\mathcal{V}}$. Details concerning the maps $\mathrm{hom}^{-\otimes A}$ and $\mathrm{hom}^{\mathcal{V}(A,-)}$ can be found in Appendix A.

Note that any $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ gives a functor $\mathcal{U}F\colon \mathcal{U}\mathcal{C} \longrightarrow \mathcal{U}\mathcal{D}$ having the same effect on objects as F, and similarly for $\mathcal{V}$-natural transformations. That is, one can consider $\mathcal{U}$ as a 2-functor from the 2-category of small $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations to the 2-category of small categories, functors and natural transformations. A $\mathcal{V}$-category $\mathcal{C}$ is small by definition if $\mathcal{U}\mathcal{C}$ is small. In fact, $\mathcal{U}$ is the base change along the lax symmetric monoidal functor

$\mathbf{Set}_{\mathcal{V}}(e,-)\colon \mathcal{V} \longrightarrow \mathbf{Set}$, see [1, 6.4]. If no confusion can arise, we will omit $\mathcal{U}$ from the notation.

The *monoidal product* $\mathcal{C} \otimes \mathcal{D}$ of two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathcal{V}$-category where $\mathrm{Ob}(\mathcal{C} \otimes \mathcal{D}) := \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$ and $\mathcal{V}_{\mathcal{C}\otimes\mathcal{D}}((a,x),(b,y)) := \mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{D}}(x,y)$. The monoidal product in $\mathcal{V}$ is then a $\mathcal{V}$-functor $\mathrm{mon}\colon \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{V}$. Now $\mathcal{V}$ is not only a monoid in the "category" of categories, but also in the "category" of $\mathcal{V}$-categories. Hence one can consider modules over it. See [7, 4.1.6] for the classical case of the following. A $\mathcal{V}$-category $\mathcal{C}$ is called a *right $\mathcal{V}$-module* if it is equipped with a $\mathcal{V}$-functor $\mathrm{act}\colon \mathcal{C} \otimes \mathcal{V} \longrightarrow \mathcal{C}$, often written as $(c, A) \longmapsto c \oslash A$, and a $\mathcal{V}$-natural unit isomorphism $r_c\colon \mathrm{act}(c,e) \longrightarrow c$ subject to the following conditions.

- There exists a natural coherent associativity isomorphism
$$\mathrm{act}(c, A \otimes B) \xrightarrow{\cong} \mathrm{act}(\mathrm{act}(c,A),B).$$
- The two canonical isomorphisms $\mathrm{act}(c, e \otimes A) \rightrightarrows \mathrm{act}(c,A)$ coincide.

A right $\mathcal{V}$-module $(\mathcal{C}, \mathrm{act}, r)$ is called *closed* if there is a $\mathcal{V}$-functor
$$\mathrm{coact}\colon \mathcal{V}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathcal{C}$$
such that for all $A \in \mathrm{Ob}\,\mathcal{V}$ and $c \in \mathrm{Ob}\,\mathcal{C}$, the $\mathcal{V}$-functor $\mathrm{act}(-,A)\colon \mathcal{C} \longrightarrow \mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathrm{coact}(A,-)$ and $\mathrm{act}(c,-)\colon \mathcal{V} \longrightarrow \mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathcal{V}_{\mathcal{C}}(c,-)$.

A *monoidal* $\mathcal{V}$-category is a $\mathcal{V}$-category $\mathcal{C}$ which is equipped with a $\mathcal{V}$-functor $\diamond\colon \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$, a unit $u \in \mathrm{Ob}\,\mathcal{C}$, a $\mathcal{V}$-natural associativity isomorphism and two $\mathcal{V}$-natural unit isomorphisms satisfying the conditions mentioned for $\mathcal{V} = \mathbf{Set}$. Similarly, one can define symmetric monoidal and closed symmetric monoidal $\mathcal{V}$-categories.

2.2. **Categories of enriched functors.** If $\mathcal{C}$ is a small $\mathcal{V}$-category, $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ and their $\mathcal{V}$-natural transformations form the category $[\mathcal{C}, \mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. If $\mathcal{V}$ is complete, then $[\mathcal{C}, \mathcal{V}]$ is also a $\mathcal{V}$-category. Denote this $\mathcal{V}$-category by $\mathcal{F}(\mathcal{C})$, or $\mathcal{F}$ if no confusion can arise. The morphism $\mathcal{V}$-object $\mathcal{V}_{\mathcal{F}}(X,Y)$ is the end
$$\int_{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(X(c), Y(c)).$$
See [1, 6.3.1] for details. Note that $\mathcal{U}\mathcal{F}$ is $[\mathcal{C}, \mathcal{V}]$. One can compare $\mathcal{F}$ with $\mathcal{C}$ and $\mathcal{V}$ as follows. Given $c \in \mathrm{Ob}\,\mathcal{C}$, one defines the $\mathcal{V}$-functor $\mathrm{Ev}_c\colon \mathcal{F} \longrightarrow \mathcal{V}$ called "evaluation at c" by $X \longmapsto X(c)$. The assignment $c \longmapsto \mathcal{V}_{\mathcal{C}}(c,-)$ from $\mathcal{C}$ to $\mathcal{F}$ is again a $\mathcal{V}$-functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{F}$, called the *$\mathcal{V}$-Yoneda embedding* [1, 6.3.6]. The $\mathcal{V}$-functor $\mathcal{V}_{\mathcal{C}}(c,-)$ is a representable functor, represented by c.

Lemma 2.1 (Enriched Yoneda lemma)**.** *Let $\mathcal{V}$ be a complete closed symmetric monoidal category and $\mathcal{C}$ a small $\mathcal{V}$-category. For a $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ and $c \in \mathrm{Ob}\,\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism $X(c) \cong \mathcal{V}_{\mathcal{F}}(\mathcal{V}_{\mathcal{C}}(c,-), X)$.*

The isomorphism in 2.1 is called the *Yoneda* isomorphism. It is described in [1, 6.3.5]. As a consequence of 2.1, every $\mathcal{V}$-functor can be expressed as a colimit of representable functors. See [1, 6.6.13 and 6.6.17] for the following result.

Lemma 2.2. *If $\mathcal{V}$ is a bicomplete closed symmetric monoidal category and $\mathcal{C}$ is a small $\mathcal{V}$-category, then $[\mathcal{C}, \mathcal{V}]$ is bicomplete: (co)limits are formed pointwise.*

Corollary 2.3. *Assume $\mathcal{V}$ is bicomplete, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then any $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ is $\mathcal{V}$-naturally isomorphic to the coend*

$$\int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}_{\mathcal{C}}(c, -) \otimes X(c).$$

A proof of 2.3 is given in [1, 6.6.18].

Proposition 2.4. *Let $\mathcal{V}$ be a closed symmetric monoidal category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then $\mathcal{F}$ is a closed $\mathcal{V}$-module.*

Proof. There is an obvious "pointwise" closed $\mathcal{V}$-module structure. The $\mathcal{V}$-functor $\mathcal{F} \otimes \mathcal{V} \longrightarrow \mathcal{F}$ defined by $(X, A) \longmapsto (- \otimes A) \circ X$ gives the action of $\mathcal{V}$ on $\mathcal{F}$. Next, the assignment $(A, X) \longmapsto \mathcal{V}(A, -) \circ X$ defines the coaction. There are $\mathcal{V}$-natural isomorphisms $\mathcal{V}_{\mathcal{F}}((-\otimes A)\circ X, Y) \cong \mathcal{V}_{\mathcal{F}}(X, \mathcal{V}(A, -)\circ Y) \cong \mathcal{V}(A, \mathcal{V}_{\mathcal{F}}(X, Y))$ induced from the natural closed $\mathcal{V}$-module structure on $\mathcal{V}$. From this, a routine check finishes the proof. □

Recall the notion of *left Kan extensions*.

Proposition 2.5. *Fix a bicomplete closed symmetric monoidal category $\mathcal{V}$, and a $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ of small $\mathcal{V}$-categories. For any $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$, there exists a $\mathcal{V}$-functor $F_*(X)\colon \mathcal{D} \longrightarrow \mathcal{V}$ and a $\mathcal{V}$-natural isomorphism*

$$\mathcal{V}_{\mathcal{F}(\mathcal{D})}(F_*X, Y) \cong \mathcal{V}_{\mathcal{F}(\mathcal{C})}(X, Y \circ F)$$

In other words, there exists a $\mathcal{V}$-adjoint pair of $\mathcal{V}$-functors

$$F_*\colon \mathcal{F}(\mathcal{C}) \rightleftarrows \mathcal{F}(\mathcal{D})\colon F^*$$

where F^* denotes pre-composition with F.

See [1, 6.7.7] for a proof of 2.5. The $\mathcal{V}$-functor F_*X is the left Kan extension of X along F. An explicit expression is given by the coend

$$F_*X = \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}_{\mathcal{D}}(F(c), -) \otimes X(c).$$

2.3. **Smash product of enriched functors.** Assume $(\mathcal{C}, \diamond, u)$ is a small symmetric monoidal $\mathcal{V}$-category where $\mathcal{V}$ is bicomplete. B. Day has constructed a closed symmetric monoidal product $\wedge$ on the category $[\mathcal{C}, \mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ in [4]. Because Day uses this as a technical condition, let us emphasize that $\mathcal{V}$ is normalized via the functor $\mathbf{Set}_{\mathcal{V}}(e, -)\colon \mathcal{V} \longrightarrow \mathbf{Set}$. Since the monoidal product in $\mathcal{V}$ is a $\mathcal{V}$-functor, one obtains for $X, Y \in \operatorname{Ob}[\mathcal{C}, \mathcal{V}]$ the $\mathcal{V}$-functor

$$X \overline{\wedge} Y \colon \mathcal{C} \otimes \mathcal{C} \xrightarrow{X \otimes Y} \mathcal{V} \otimes \mathcal{V} \xrightarrow{\text{mon}} \mathcal{V}$$

The smash product $X \wedge Y \in \operatorname{Ob}[\mathcal{C}, \mathcal{V}]$ is the left Kan extension

$$\diamond_*(X \overline{\wedge} Y) = \int^{\operatorname{Ob}(\mathcal{C} \otimes \mathcal{C})} \mathcal{V}_{\mathcal{C}}(c \diamond d, -) \otimes (X(c) \otimes Y(d))\colon \mathcal{C} \longrightarrow \mathcal{V}.$$

The next result is a special case of [4, 3.3], according to [4, 3.6 and 4.1].

Theorem 2.6 (Day). *Let $(\mathcal{V}, \otimes, e)$ be a bicomplete closed symmetric monoidal category and $(\mathcal{C}, \diamond, u)$ a small symmetric monoidal $\mathcal{V}$-category. Then the tuple $([\mathcal{C}, \mathcal{V}], \wedge, \mathcal{V}_{\mathcal{C}}(u, -))$ is a closed symmetric monoidal category.*

The $\mathcal{V}$-category $\mathcal{F}$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ is also a closed symmetric monoidal $\mathcal{V}$-category. The internal hom functor, right adjoint to $- \wedge X$, is given by

$$\mathcal{F}(X, Y)(c) = \mathcal{V}_{\mathcal{F}}(X, Y(c \diamond -)) = \int_{d \in \operatorname{Ob}\mathcal{C}} \mathcal{V}(X(d), Y(c \diamond d)).$$

Concerning smash products of representable functors, one has the following result.

Lemma 2.7. *The smash product of representable functors is again representable. There is a natural isomorphism $\mathcal{V}_{\mathcal{C}}(c, -) \wedge \mathcal{V}_{\mathcal{C}}(d, -) \cong \mathcal{V}_{\mathcal{C}}(c \diamond d, -)$.*

As a consequence of 2.7, one can define assembly maps in the special case where $\mathcal{C}$ is a full sub-$\mathcal{V}$-category of $\mathcal{V}$ closed under the monoidal product and containing the unit. In this case, the inclusion $\mathbb{I}\colon \mathcal{C} \hookrightarrow \mathcal{V}$ can be chosen as the unit of $[\mathcal{C}, \mathcal{V}]$. The left Kan extension of $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ along the inclusion $\mathbb{I}\colon \mathcal{C} \hookrightarrow \mathcal{V}$ is

$$\mathbb{I}_* X = \int^{\operatorname{Ob}\mathcal{C}} \mathcal{V}(c, -) \otimes X(c),$$

and the *composition* $X \circ Y$ of two $\mathcal{V}$-functors $X, Y\colon \mathcal{C} \longrightarrow \mathcal{V}$ is given by $\mathbb{I}_* X \circ Y$. Up to coherent natural isomorphisms, composition is associative and has the inclusion $\mathbb{I}\colon \mathcal{C} \hookrightarrow \mathcal{V}$ as a unit.

Corollary 2.8. *Given functors X and Y in $[\mathcal{C}, \mathcal{V}]$, there exists a natural assembly map $X \wedge Y \longrightarrow X \circ Y = \mathbb{I}_* X \circ Y$ which is an isomorphism if Y is representable.*

Proof. Suppose $X = \mathcal{V}(c, -)$ and $Y = \mathcal{V}(d, -)$, for $c, d \in \operatorname{Ob}\mathcal{C}$. Then $X \wedge Y$ is naturally isomorphic, by 2.7, to $\mathcal{V}(c \otimes d, -)$, which is naturally isomorphic to $\mathcal{V}(c, -) \circ \mathcal{V}(d, -)$. If X is arbitrary, it follows by 2.3 that $X \wedge \mathcal{V}(d, -)$ is naturally isomorphic to $\mathbb{I}_*X \circ \mathcal{V}(d,-)$. If also Y is arbitrary, use a coend of representables as in 2.3 and consider the natural map

$$\int^{\mathrm{Ob}\,\mathcal{C}} (\mathbb{I}_*X \circ \mathcal{V}(c,-)) \otimes Y(c) \longrightarrow \mathbb{I}_*X \circ \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c,-) \otimes Y(c).$$

□

In the language introduced at the beginning of the following section, the identity $\mathrm{Id}_{[\mathcal{C},\mathcal{V}]}$ is thus a lax monoidal functor from the monoidal category $([\mathcal{C},\mathcal{V}],\circ,\mathbb{I})$ to the (closed symmetric) monoidal category $([\mathcal{C},\mathcal{V}],\wedge,\mathbb{I})$ – just as in the case of simplicial functors, which is described in [13, 5.14].

2.4. **Categories enriched over simplicial sets.** A functor $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ of monoidal categories $(\mathcal{V},\otimes,e)$ and $(\mathcal{W},\otimes',e')$ is called *lax monoidal* if there is a natural transformation $t_{A,B}\colon F(A)\otimes' F(B) \longrightarrow F(A\otimes B)$ and a morphism $e' \longrightarrow F(e)$, such that the diagrams [1, 6.27 and 6.28] commute. The word "lax" is replaced by "*strict*" if $t_{A,B}$ is a natural isomorphism and $e' \longrightarrow F(e)$ is an isomorphism. The functor F is called *lax symmetric monoidal* if the equality $t_{B,A}\circ\sigma_{FA,FB} = F(\sigma_{A,B})\circ t_{A,B}$ holds. If $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ is a lax symmetric monoidal functor, then every $\mathcal{W}$-category is a $\mathcal{V}$-category in a natural way by [1, 6.4.3]. We used this in the construction of the forgetful 2-functor $\mathcal{U}$ induced by $\mathbf{Set}_{\mathcal{V}}(e,-)$.

Assume that there exists a lax symmetric monoidal functor $F\colon \mathbf{sSet} \longrightarrow \mathcal{V}$. For $\mathbf{sSet}$-categories there is the notion of *simplicial homotopy equivalence*, which extends to $\mathcal{V}$-categories using F.

Definition 2.9. Let $\mathcal{C}$ be a $\mathbf{sSet}$-category and $f,f'\colon \Delta^0 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c,d)$ maps in $\mathcal{C}$. Then $H\colon \Delta^1 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c,d)$ is a *simplicial homotopy* from f to f' if the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "i_0"] \arrow[dr, "f"'] & \Delta^1 \arrow[d, "H"] & \Delta^0 \arrow[l, "i_1"'] \arrow[dl, "f'"] \\ & \mathbf{sSet}_{\mathcal{C}}(c,d) & \end{tikzcd}$$

commutes. Here, i_0 and i_1 are the canonical inclusions. The map f is a *simplicial homotopy equivalence* provided there exists a map $g\colon \Delta^0 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(d,c)$, and simplicial homotopies from $g\circ f$ to id_c and from $f\circ g$ to id_d.

Simplicial homotopy equivalences are decorated as $f\colon c \xrightarrow{\simeq} d$. The notion of simplicial homotopy equivalence is in general not an equivalence relation. If $\mathcal{C}$ is a closed $\mathbf{sSet}$-module with action $(c,K) \longmapsto c\oslash K$ and coaction $(c,K) \longmapsto c^K$, then a simplicial homotopy can also be described as a map $c\oslash\Delta^1 \longrightarrow d$ or as a map $c \longrightarrow d^{\Delta^1}$. If $\mathcal{C}$ has additionally pushouts, maps in $\mathcal{C}$ can be factored using the *simplicial mapping cylinder*. Given a map $f\colon c \longrightarrow d$ in $\mathcal{C}$, let C_f be the pushout of the diagram

$$c \oslash \Delta^1 \xleftarrow{c \oslash i_1} c \oslash \Delta^0 \xleftarrow{\cong} c \xrightarrow{f} d.$$

The maps $c \oslash \Delta^1 \xrightarrow{c \oslash s} c \oslash \Delta^0 \xrightarrow{\cong} c \xrightarrow{f} d$ and id_d induce a map $p_f\colon C_f \xrightarrow{\simeq} d$ which is a simplicial homotopy equivalence, with homotopy inverse the canonical map $d \xrightarrow{\simeq} C_f$. Denote the composition $c \xrightarrow{\cong} c \oslash \Delta^0 \xrightarrow{c \oslash i_0} c \oslash \Delta^1 \longrightarrow C_f$, where the last map is the canonical map, by $i_f\colon c \longrightarrow C_f$. Note that $p_f \circ i_f = f$. The relevance of the factorization will become clear in the context of simplicial model categories: if c is cofibrant, then i_f is a cofibration.

Lemma 2.10. *A* **sSet**-*functor preserves simplicial homotopies, and therefore simplicial homotopy equivalences.*

Proof. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a **sSet**-functor, and let $H\colon \Delta^1 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c,d)$ be a simplicial homotopy from f to g. Then the composition

$$\Delta^1 \xrightarrow{H} \mathbf{sSet}_{\mathcal{C}}(c,d) \xrightarrow{\hom^F_{c,d}} \mathbf{sSet}_{\mathcal{D}}(F(c),F(d))$$

is a simplicial homotopy from $F(f)$ to $F(g)$. The second statement follows since functors preserve identities and compositions. ☐

Corollary 2.11. *Assume $F\colon$* **sSet** *$\longrightarrow \mathcal{V}$ is a lax monoidal functor. Then any $\mathcal{V}$-functor preserves simplicial homotopies and simplicial homotopy equivalences.*

2.5. **Spectra as enriched functors.** Let $(\mathcal{V}, \otimes, e)$ denote a bicomplete closed symmetric monoidal category with initial object $\emptyset$. For $T \in \operatorname{Ob}\mathcal{V}$, one can consider T-spectra in $\mathcal{V}$, see [9, 1.1]. A *T-spectrum* E is a sequence $(E_0, E_1, \dots)$ of objects in $\mathcal{V}$, together with structure maps $e_n\colon E_n \otimes T \longrightarrow E_{n+1}$ for all n. If E and F are T-spectra, a *map of T-spectra* $g\colon E \longrightarrow F$ is a collection of maps $g_n\colon E_n \longrightarrow F_n$ such that the diagram

$$\begin{array}{ccc} E_n \otimes T & \xrightarrow{e_n} & E_{n+1} \\ {\scriptstyle g_n \otimes T}\downarrow & & \downarrow{\scriptstyle g_{n+1}} \\ F_n \otimes T & \xrightarrow{f_n} & F_{n+1} \end{array}$$

commutes for all n. Thus T-spectra in $\mathcal{V}$ form a category $\mathrm{Sp}(\mathcal{V}, T)$, see [9, 1.3].

We claim $\mathrm{Sp}(\mathcal{V}, T)$ can be viewed as an enriched functor category, cf. [13, 4.3]. Its domain category is the $\mathcal{V}$-category $T\mathrm{Sph}$. The objects in $T\mathrm{Sph}$ are the objects T^n for $n \geq 0$, where $T^0 = e$ and $T^n := T \otimes T^{n-1}$ for $n > 0$. The $\mathcal{V}$-objects of maps in $T\mathrm{Sph}$ are

$$\mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) := \begin{cases} T^{n-m} & n \geq m \geq 0 \\ \emptyset & n < m. \end{cases}$$

Note that there are canonical unit maps $\mathrm{id}_{T^0}\colon T^0 \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^n, T^n)$ for all $n \geq 0$. It remains to describe the composition. For $k, l, m \geq 0$, there is a map

$$\mathcal{V}_{T\mathrm{Sph}}(T^{l+m}, T^{k+l+m}) \otimes \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{l+m}) \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{k+l+m})$$

determined by the associativity isomorphism $\alpha_{k,l}\colon T^k \otimes T^l \longrightarrow T^{k+l}$. In all other cases, the composition is uniquely determined. Using the associativity and unit coherence laws in $\mathcal{V}$, it follows that $T\mathrm{Sph}$ is a $\mathcal{V}$-category.

To illuminate this definition, let us describe a $\mathcal{V}$-functor $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$. Define π to be the identity on objects. Concerning morphisms, it suffices to define $\hom^{\pi}_{T^m, T^{k+m}}\colon \mathcal{V}_{T\mathrm{Sph}}(T^k, T^{k+m}) \longrightarrow \mathcal{V}(T^k, T^{k+m})$ for $k, m \geq 0$. The composition

$$i_{k,m}\colon T^k \xrightarrow{\eta_{T^m} T^k} \mathcal{V}(T^m, T^k \otimes T^m) \xrightarrow{\mathcal{V}(T^m, \alpha_{k,m})} \mathcal{V}(T^m, T^{k+m})$$

defines $\hom^{\pi}_{T^m, T^{k+m}}$. It is clear that π preserves identities. Associativity coherence and a calculation with adjoints imply that

$$T^k \otimes T^l \xrightarrow{i_{k,l+m} \otimes i_{l,m}} \mathcal{V}(T^{l+m}, T^{k+l+m}) \otimes \mathcal{V}(T^m, T^{l+m}) \xrightarrow{\mathrm{comp}} \mathcal{V}(T^m, T^{k+l+m})$$

factors as the composition $T^k \otimes T^l \xrightarrow{\alpha_{k,l}} T^{k+l} \xrightarrow{i_{k+l,m}} \mathcal{V}(T^m, T^{k+l+m})$. In the cases we are interested in, the maps $\eta_{T^m} T^k$ are always monomorphisms, hence $T\mathrm{Sph}$ can be regarded as a sub-$\mathcal{V}$-category of $\mathcal{V}$.

Proposition 2.12. *The categories* $\mathrm{Sp}(\mathcal{V}, T)$ *and* $[T\mathrm{Sph}, \mathcal{V}]$ *are isomorphic.*

Proof. For a $\mathcal{V}$-functor $X\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$, let $\Psi(X)$ be the spectrum with objects $\Psi(X)_n := X(T^n)$ and structure maps $\Psi(X)_n \otimes T \longrightarrow \Psi(X)_{n+1}$ being adjoint to $T = \mathcal{V}_{T\mathrm{Sph}}(T^n, T^{n+1}) \longrightarrow \mathcal{V}(X(T^n), X(T^{n+1}))$. If $f\colon X \longrightarrow Y$ is a $\mathcal{V}$-natural transformation, put $\Psi(f)_n := f(T^n)\colon X(T^n) \longrightarrow Y(T^n)$. The diagram

$$\begin{tikzcd} X(T^n) \otimes T \arrow[r] \arrow[d, "f(T^n)\otimes T"'] & X(T^{n+1}) \arrow[d, "f(T^{n+1})"] \\ Y(T^n) \otimes T \arrow[r] & Y(T^{n+1}) \end{tikzcd}$$

commutes by $\mathcal{V}$-naturality, and Ψ respects identities and composition.

Define $\Phi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ by $\Phi(E)(T^n) := E_n$. If $n = m + k$ with $m \geq 0$ and $k \geq 1$, the map $\hom^{\Phi(E)}_{T^m, T^n}\colon T^k = \mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) \longrightarrow \mathcal{V}(E_m, E_n)$ is the adjoint of the composition of the maps

$$E_m \otimes T^k \xrightarrow{\alpha^{-1}_{E_m, T, T^{k-1}}} (E_m \otimes T) \otimes T^{k-1} \xrightarrow{e_m \otimes T^{k-1}} E_{m+1} \otimes T^{k-1} \longrightarrow \cdots \longrightarrow E_n.$$

The maps $\hom^{\Phi(E)}_{T^n,T^n}$ are determined by the property that $\Phi(E)$ has to preserve identities. To prove that $\Phi(E)$ is a $\mathcal{V}$-functor, it remains to note that the diagram

$$\begin{tikzcd}
T^k \otimes T^l \arrow[r, "\alpha_{k,l}"] \arrow[d, "\hom^{\Phi(E)}_{T^{l+m},T^{k+l+m}} \otimes \hom^{\Phi(E)}_{T^m,T^{l+m}}"'] & T^{k+l} \arrow[d, "\hom^{\Phi(E)}_{T^m,T^{k+l+m}}"] \\
\mathcal{V}(E_{l+m},E_{k+l+m}) \otimes \mathcal{V}(E_m,E_{l+m}) \arrow[r, "\mathrm{comp}"] & \mathcal{V}(E_m,E_{k+l+m})
\end{tikzcd}$$

commutes for $m \geq 0$ and $k,l \geq 1$. This follows from the triangular identity $\epsilon_{E_m}(-\otimes E_m) \circ \eta_{E_m} \otimes E_m = \mathrm{id}_{-\otimes E_m}$, associativity coherence, and associativity of composition in $\mathcal{V}$. If g is a map of spectra, one defines $\Phi(g)(T^n)$ as g_n for $n \geq 0$. Then $\Phi(g)$ is $\mathcal{V}$-natural, essentially because g is a map of spectra. The functoriality of Φ is then obvious.

Note that $\Phi(\Psi(X)) = X$ holds on objects, i.e. for all T^n. The structure maps of X and $\Phi(\Psi(X))$ coincide, since the adjointness isomorphism used to construct the structure maps of $\Phi(\Psi(X))$ is inverse to the adjointness isomorphism used to construct the structure maps of $\Psi(X)$. The equality $\Phi(\Psi(f)) = f$ is obvious, hence $\Phi \circ \Psi$ is the identity functor. Likewise one finds $\Psi \circ \Phi = \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$. ☐

Remark 2.13. In the following, we will identify $\mathrm{Sp}(\mathcal{V},T)$ with $[T\mathrm{Sph},\mathcal{V}]$ via 2.12 implicitly. In particular, $\mathrm{Sp}(\mathcal{V},T)$ is a closed $\mathcal{V}$-module by 2.4. A consequence of A.1 is that the functor "suspension with T" obtained from the $\mathcal{V}$-module structure on $\mathrm{Sp}(\mathcal{V},T)$ can also be defined as the prolongation [9, 1.5] of the functor $-\otimes T\colon \mathcal{V} \longrightarrow \mathcal{V}$ using $t\colon (-\otimes T)\circ(-\otimes T) \longrightarrow (-\otimes T)\circ(-\otimes T)$, the natural transformation which twists the factors. It is defined as the coherence isomorphism

$$(A\otimes T)\otimes T \xrightarrow{\alpha_{A,T,T}} A\otimes(T\otimes T) \xrightarrow{A\otimes\sigma_{T,T}} A\otimes(T\otimes T) \xrightarrow{\alpha^{-1}_{A,T,T}} (A\otimes T)\otimes T.$$

Another functor "suspension with T" – which we denote by Σ_T – is obtained as the prolongation of $-\otimes T$ using the identity natural transformation $\mathrm{id}_{(-\otimes T)\circ(-\otimes T)}$. If $X\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor, then the nth structure map in the associated spectrum of $X\otimes T$ is the left hand side composition in the following diagram.

$$\begin{tikzcd}
 & (X_n\otimes T)\otimes T \arrow[dl, "\sigma_{X_n\otimes T,T}"'] \arrow[dr, "\sigma_{X_n,T}\otimes T"] & \\
T\otimes(X_n\otimes T) \arrow[rr, "\alpha^{-1}_{T,X_n,T}"] \arrow[d, "\hom^X_{n,n+1}\otimes(X_n\otimes T)"'] & & (T\otimes X_n)\otimes T \arrow[d, "(\hom^X_{n,n+1}\otimes X_n)\otimes T"] \\
\mathcal{V}(X_n,X_{n+1})\otimes(X_n\otimes T) \arrow[rr, "\alpha^{-1}_{\mathcal{V}(X_n,X_{n+1}),X_n,T}"] \arrow[d, "\hom^{-\otimes T}_{X_n,X_{n+1}}"'] & & (\mathcal{V}(X_n,X_{n+1})\otimes X_n)\otimes T \arrow[d, "(\epsilon_{X_n}X_{n+1})\otimes T"] \\
\mathcal{V}(X_n\otimes T,X_{n+1}\otimes T)\otimes(X_n\otimes T) \arrow[rr, "\epsilon_{X_n\otimes T}(X_{n+1}\otimes T)"] & & X_{n+1}\otimes T
\end{tikzcd}$$

The right hand side composition is the structure map of the spectrum $\Sigma_T X$. The lower square commutes by A.1, the middle square commutes by naturality, but the triangle does not commute in general. This will cause some complications in our comparison of stable model categories, cf. Section 6.

The monoidal product $\otimes$ does not restrict via $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ to a monoidal product of $T\mathrm{Sph}$ in interesting cases. To explain this, we may ignore the associativity isomorphisms for the rest of this section. The monoidal product $\otimes$ defines a $\mathcal{V}$-functor $\mathrm{mon}\colon \mathcal{V}\otimes\mathcal{V} \longrightarrow \mathcal{V}$ using as

$$\hom^{\mathrm{mon}}_{(A_1,A_2)(B_1,B_2)}\colon \mathcal{V}(A_1,B_1)\otimes\mathcal{V}(A_2,B_2) \longrightarrow \mathcal{V}(A_1\otimes A_2, B_1\otimes B_2)$$

the adjoint of the composition

$$\begin{array}{c} \mathcal{V}(A_1,B_1)\otimes\mathcal{V}(A_2,B_2)\otimes A_1\otimes A_2 \\ \Big\downarrow {\scriptstyle \mathcal{V}(A_1,B_1)\otimes\sigma_{\mathcal{V}(A_2,B_2),A_1}\otimes A_2} \\ \mathcal{V}(A_1,B_1)\otimes A_1\otimes\mathcal{V}(A_2,B_2)\otimes A_2 \xrightarrow{\epsilon_{A_1}B_1\otimes\epsilon_{A_2}B_2} B_1\otimes B_2. \end{array}$$

Suppose the monoidal product induces a $\mathcal{V}$-functor $\mathrm{mon}\colon T\mathrm{Sph}\otimes T\mathrm{Sph} \longrightarrow T\mathrm{Sph}$ (which on the nose is just possible by ignoring associativity isomorphisms). On objects we thus have $\mathrm{mon}(T^k,T^l) = T^k\otimes T^l = T^{k+l}$. On $\mathcal{V}$-objects of morphisms, we get a map

$$\mathcal{V}_{T\mathrm{Sph}}(T^m, T^k\otimes T^m)\otimes\mathcal{V}_{T\mathrm{Sph}}(T^n,T^l\otimes T^n) \xrightarrow{f} \mathcal{V}_{T\mathrm{Sph}}(T^m\otimes T^n, T^k\otimes T^m\otimes T^l\otimes T^n)$$

making the diagram

$$\begin{tikzcd} T^k\otimes T^l \arrow[r, "f"] \arrow[d, "\eta_{T^m}T^k\otimes\eta_{T^n}T^l"'] & T^k\otimes T^l \arrow[d, "\eta_{T^m\otimes T^n}(T^k\otimes T^l)"] \\ \mathcal{V}(T^m, T^k\otimes T^m)\otimes\mathcal{V}(T^n,T^l\otimes T^n) \arrow[r, "\hom^{\mathrm{mon}}"] & \mathcal{V}(T^m\otimes T^n, T^k\otimes T^m\otimes T^l\otimes T^n) \end{tikzcd}$$

commute. Switching over to the adjoints, a straightforward calculation shows that the maps

$$T^{k+l+m+n} = T^k\otimes T^l\otimes T^m\otimes T^n \underset{T^k\otimes\sigma_{T^l,T^m}\otimes T^n}{\overset{f\otimes T^m\otimes T^n}{\rightrightarrows}} T^k\otimes T^m\otimes T^l\otimes T^n = T^{k+l+n+m}$$

have to coincide, which is only possible if $\sigma_{T,T} = \mathrm{id}_{T^2}$ and $f = \mathrm{id}$. Following Lydakis' description [13, 5.15] of symmetric spectra, we enlarge $T\mathrm{Sph}$ as little as possible to remedy this defect.

2.6. **Symmetric spectra as enriched functors.** Still T is some object of $\mathcal{V}$. To simplify the notation, we ignore associativity and unit isomorphisms throughout this section. The set $\{1,\ldots,n\}$ is denoted $\{n\}$ for any $n\geq 0$, with $\{0\}$ being the empty set. Let Inj be the category with the sets $\{n\}$ as objects and injective maps as morphisms. We define

$$\mathrm{Inj}(m,n) := \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\{m\},\{n\})} T^0$$

for $m \leq n$, with T^0 being the unit in $\mathcal{V}$. Thus $\mathrm{Inj}(n,n)$ is a group object in $\mathcal{V}$, namely the symmetric group on n letters, and – regarding $\mathrm{Inj}(n,n)$ as a $\mathcal{V}$-category with a single object – a left $\mathrm{Inj}(n,n)$-action on $A \in \mathrm{Ob}\,\mathcal{V}$ is simply a $\mathcal{V}$-functor $\mathrm{Inj}(n,n) \longrightarrow \mathcal{V}$ with value A on the single object. One gets a left $\mathrm{Inj}(n,n)$-action on $T^n = T_n \otimes \cdots \otimes T_2 \otimes T_1$ using iterations of the commutativity isomorphism $\sigma_{T,T}$.

A *symmetric T-spectrum X in $\mathcal{V}$* as defined in [9, 7.2] consists of a sequence $(X_0, X_1, \ldots, X_n, \ldots)$, where X_n is an object in $\mathcal{V}$ with a left $\mathrm{Inj}(n,n)$-action, and structure maps $X_n \otimes T \longrightarrow X_{n+1}$ such that the composition

$$X_n \otimes T^m \longrightarrow X_{n+1} \otimes T^{m-1} \longrightarrow \cdots \longrightarrow X_{n+m}$$

is as equivariant as it can be, namely $\mathrm{Inj}(n,n) \otimes \mathrm{Inj}(m,m)$-equivariant. A map $f\colon X \longrightarrow Y$ of symmetric T-spectra is a collection of maps $f_n\colon X_n \longrightarrow Y_n$ both compatible with the $\mathrm{Inj}(n,n)$-action and the structure maps. The result is a category $\mathrm{Sp}^{\Sigma}(\mathcal{V},T)$. We will describe this category as a category of $\mathcal{V}$-functors with codomain $\mathcal{V}$ and domain denoted by $T\mathrm{Sph}^{\Sigma}$. The object set of $T\mathrm{Sph}^{\Sigma}$ coincides with the object set of $T\mathrm{Sph}$, but the morphism objects are different. If $n = k + m$ with $m \geq 0, k \geq 0$, the $\mathcal{V}$-object of maps in $T\mathrm{Sph}^{\Sigma}$ from T^m to T^n is

$$\mathcal{V}_{T\mathrm{Sph}^{\Sigma}}(T^m, T^n) := \mathrm{Inj}(m,n) \otimes T^k.$$

If $n < m$, then $\mathcal{V}_{T\mathrm{Sph}^{\Sigma}}(T^m, T^n)$ is defined as the initial object. The unit map

$$T^0 \longrightarrow \mathrm{Inj}(n,n) = \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\{n\},\{n\})} T^0 = \mathcal{V}_{T\mathrm{Sph}^{\Sigma}}(T^n, T^n)$$

is the canonical map of the summand indexed by $\mathrm{id}_{\{n\}}$. To describe the composition, it is useful to think of the summand of $\mathrm{Inj}(m, k+m) \otimes T^k$ indexed by $\beta\colon \{m\} \longrightarrow \{k+m\}$ as $T_{i_1^\beta} \otimes T_{i_2^\beta} \otimes \cdots T_{i_k^\beta} = T^k$, where $\{i_1^\beta, i_2^\beta, \ldots, i_k^\beta\}$ is the set $\{k+m\} - \beta(\{m\})$ reordered such that $i_1^\beta > i_2^\beta > \cdots > i_k^\beta$. If $n = k + l + m$ with $k, l, m \geq 0$, the composition

$$\mathrm{Inj}(l+m, n) \otimes T^k \otimes \mathrm{Inj}(m, l+m) \otimes T^l \longrightarrow \mathrm{Inj}(m,n) \otimes T^{k+l}$$

is given by first using the identification

$$\mathrm{Inj}(l+m,n) \otimes T^k \otimes \mathrm{Inj}(m,l+m) \otimes T^l \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\{l+m\},\{n\}) \times \mathbf{Set}_{\mathrm{Inj}}(\{m\},\{l+m\})} T^{k+l}$$

and then mapping the summand

$$T_{i_1^\gamma} \otimes T_{i_2^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes T_{i_1^\beta} \otimes T_{i_2^\beta} \otimes \cdots \otimes T_{i_l^\beta} = T^{k+l}$$

indexed by (γ, β) to the summand

$$T_{i_1^{\gamma\circ\beta}} \otimes T_{i_2^{\gamma\circ\beta}} \otimes \cdots \otimes T_{i_{k+l}^{\gamma\circ\beta}} = T^{k+l}$$

indexed by $\gamma \circ \beta$ using a specific isomorphism. Namely, one uses the unique isomorphism $T^{k+l} \longrightarrow T^{k+1}$ induced by the permutation which reorders the set

$$\{k + l + m\} - (\gamma \circ \beta)(\{m\}) = \{i_1^\gamma, i_2^\gamma, \ldots , i_k^\gamma, \gamma(i_1^\beta), \gamma(i_2^\beta), \ldots , \gamma(i_l^\beta)\}$$

as $\{i_1^{\gamma\circ\beta}, i_2^{\gamma\circ\beta}, \ldots , i_{k+l}^{\gamma\circ\beta}\}$. Maybe an example is useful to explain this composition. Let $k = 2, l = 3, m = 2$, so $n = 7$. If $\beta\colon \{1, 2\} \longrightarrow \{1, 2, 3, 4, 5\}$ maps 1 to 3 and 2 to 1 (indexing $T_5 \otimes T_4 \otimes T_2$) and $\gamma\colon \{1, 2, 3, 4, 5\} \longrightarrow \{1, 2, 3, 4, 5, 6, 7\}$ has the image $\{5, 4, 7, 3, 1\}$ (indexing the summand $T_6 \otimes T_2$), then the summand $T_6 \otimes T_2 \otimes T_5 \otimes T_4 \otimes T_2$ indexed by (γ, β) of the domain of the composition is written as $T_6 \otimes T_2 \otimes T_{\gamma(5)=1} \otimes T_{\gamma(4)=3} \otimes T_{\gamma(2)=4}$. This summand is mapped to the summand $T_6 \otimes T_4 \otimes T_3 \otimes T_2 \otimes T_1$ via the map $(T_6 \otimes \sigma_{T_3,T_4} \otimes T_2 \otimes T_1) \circ (T_6 \otimes \sigma_{T_2\otimes T_1,T_3\otimes T_4})$.

Lemma 2.14. *$T\mathrm{Sph}^\Sigma$ is a symmetric monoidal $\mathcal{V}$-category.*

Proof. First one should observe that the composition in $T\mathrm{Sph}^\Sigma$ is unital. Associativity might not be so obvious, but it is essentially due to the uniqueness of the permutation reordering $\{i_1^\gamma, i_2^\gamma, \ldots , i_k^\gamma, \gamma(i_1^\beta), \gamma(i_2^\beta), \ldots , \gamma(i_l^\beta)\}$. On objects, the monoidal product in $T\mathrm{Sph}^\Sigma$ is defined just as in $\mathcal{V}$ by setting $(T^l, T^k) \longmapsto T^{k+l}$. To describe the behavior on morphisms, take injections $\beta\colon \{n\} \longrightarrow \{l + n\}$ and $\gamma\colon \{m\} \longrightarrow \{k + m\}$, and define $\beta \triangleright \gamma\colon \{m + n\} \longrightarrow \{k + l + m + n\}$ via concatenation. That is,

$$\beta \triangleright \gamma(i) := \begin{cases} \gamma(i) & i \leq m \\ \beta(i - m) + k + m & i > m. \end{cases}$$

The ordered set $i_1^{\beta\triangleright\gamma} > \cdots > i_{k+l}^{\beta\triangleright\gamma}$ coincides with the ordered set

$$i_1^\beta + k + m > \cdots > i_l^\beta + k + m > i_1^\gamma > \cdots > i_k^\gamma.$$

On $\mathcal{V}$-objects of morphisms, the map

$$\mathrm{Inj}(n, l + n) \otimes T^l \otimes \mathrm{Inj}(m, k + m) \otimes T^k \longrightarrow \mathrm{Inj}(m + n, k + l + m + n) \otimes T^{k+l}$$

maps the summand

$$T^l \otimes T^k = T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$$

indexed by $(\{n\} \xrightarrow{\beta} \{l + n\}, \{m\} \xrightarrow{\gamma} \{k + m\})$ via the identity onto the summand T^{k+l} indexed by $\beta \triangleright \gamma$. One just rewrites the indices according to the above equality of ordered sets, that is, changing

$$T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$$

into

$$T_{i_1^\beta+k+m} \otimes \cdots \otimes T_{i_l^\beta+k+m} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}.$$

Let us check that this defines a $\mathcal{V}$-functor $\mathrm{mon}^\Sigma\colon T\mathrm{Sph}^\Sigma \otimes T\mathrm{Sph}^\Sigma \longrightarrow T\mathrm{Sph}^\Sigma$. The reason is essentially that $\rhd\colon \mathrm{Inj}\times\mathrm{Inj} \longrightarrow \mathrm{Inj}$ is a functor, but we will give some details. Since $\mathrm{id}_{\{n\}} \rhd \mathrm{id}_{\{m\}} = \mathrm{id}_{\{m+n\}}$, mon^Σ preserves identities. To check compatibility with composition, fix four injective maps $\alpha\colon \{m\} \longrightarrow \{l+m\}$, $\beta\colon \{l+m\} \longrightarrow \{k+l+m =: n\}$, $\gamma\colon \{r\} \longrightarrow \{q+r\}$ and $\delta\colon \{q+r\} \longrightarrow \{p+q+r\}$. The equality $(\delta\circ\gamma)\rhd(\beta\circ\alpha) = (\delta\rhd\beta)\circ(\gamma\rhd\alpha)$ holds, which implies that it remains to check compatibility on one summand. Composing reorders and rewrites

$$M = \{i_1^\delta, \ldots, i_p^\delta, i_1^\beta, \ldots, i_k^\beta, i_1^\gamma, \ldots, i_q^\gamma, i_1^\alpha, \ldots, i_l^\alpha\}$$

first as $\{i_1^\delta, \ldots, i_p^\delta, \delta(i_1^\gamma), \ldots, \delta(i_q^\gamma), i_1^\beta, \ldots, i_k^\beta, \beta(i_1^\alpha), \ldots, \beta(i_l^\alpha)\}$, then reorders it as $\{i_1^{\delta\circ\gamma}, \ldots, i_{p+q}^{\delta\circ\gamma}, i_1^{\beta\circ\alpha}, \ldots, i_{k+l}^{\beta\circ\alpha}\}$. Monoidal product rewrites this set as

$$\{i_1^{\delta\circ\gamma}+n, \ldots, i_{p+q}^{\delta\circ\gamma}+n, i_1^{\beta\circ\alpha}, \ldots, i_{k+l}^{\beta\circ\alpha}\} = M'$$

which determines a certain bijection $M \xrightarrow{\cong} M'$. Monoidal product rewrites M as $\{i_1^\delta+n, \ldots, i_p^\delta+n, i_1^\beta, \ldots, i_k^\beta, i_1^\gamma+l+m, \ldots, i_q^\gamma+l+m, i_1^\alpha, \ldots, i_l^\alpha\}$, which is then rewritten by composition as

$$\{i_1^\delta{+}n, \ldots, i_p^\delta{+}n, i_1^\beta, \ldots, i_k^\beta, \delta{\rhd}\beta(i_1^\gamma{+}l{+}m), \ldots, \delta{\rhd}\beta(i_q^\gamma{+}l{+}m), \delta{\rhd}\beta(i_1^\alpha), \ldots, \delta{\rhd}\beta(i_l^\alpha)\}.$$

By definition of $\rhd$, this coincides with the set

$$\{i_1^\delta+n, \ldots, i_p^\delta+n, i_1^\beta, \ldots, i_k^\beta, \delta(i_1^\gamma)+n, \ldots, \delta(i_q^\gamma)+n, \beta(i_1^\alpha), \ldots, \beta(i_l^\alpha)\}$$

which is then reordered as

$$\{i_1^{(\delta\rhd\beta)\circ(\gamma\rhd\alpha)}, \ldots, i_{k+l+p+q}^{(\delta\rhd\beta)\circ(\gamma\rhd\alpha)}\} = M'.$$

The resulting bijection $M \xrightarrow{\cong} M'$ is thus the same as the bijection above. Hence mon^Σ is a $\mathcal{V}$-functor.

By definition, we have equalities $\mathrm{mon}^\Sigma(T^0, T^m) = T^m = \mathrm{mon}^\Sigma(T^m, T^0)$ showing that T^0 is a strict unit in $T\mathrm{Sph}^\Sigma$. Similarly, strict associativity holds. The commutativity isomorphism $T^0 \longrightarrow \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^{m+n}, T^{m+n})$ is the canonical map of the summand indexed by the permutation $\{m+n\} \longrightarrow \{n+m\}$ which interchanges m and n. The relevant coherence conditions [1, 6.3, 6.4 and 6.5] can be checked by a straightforward calculation with permutations of $\{l+m+n\}$, thereby finishing the proof. □

To explain the composition in $T\mathrm{Sph}^\Sigma$ further, we will describe two $\mathcal{V}$-functors $\nu\colon T\mathrm{Sph} \longrightarrow T\mathrm{Sph}^\Sigma$ and $\sigma\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ such that $\sigma\circ\nu = \pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ as defined before. Both ν and σ are the identity on objects. The map

$$\mathrm{hom}^\nu_{T^m, T^{l+m}}\colon T^l \longrightarrow \mathrm{Inj}(m, l+m)\otimes T^l \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\{m\},\{l+m\})} T^l$$

is the canonical map of the summand indexed by the inclusion $\{m\} \hookrightarrow \{l+m\}$, which makes it at once clear that ν preserves identities. For the composition, consider the diagram

$$\begin{tikzcd}
T^k \otimes T^l \arrow[rr, "\mathrm{id}"] \arrow[d, "\hom^{\nu}_{T^{l+m},T^{k+l+m}}"', "\otimes \hom^{\nu}_{T^m,T^{l+m}}"] & & T^{k+l} \arrow[d, "\hom^{\nu}_{T^m,T^{k+l+m}}"] \\
\mathrm{Inj}(l+m,k+l+m)\otimes T^k \otimes \mathrm{Inj}(m,l+m)\otimes T^l \arrow[rr, "\mathrm{comp}"] & & \mathrm{Inj}(m,k+l+m)\otimes T^{k+l}
\end{tikzcd}$$

and observe that the left vertical map hits the summand

$$(T_{k+l+m} \otimes T_{k+l+m-1} \otimes \cdots \otimes T_{l+m+1}) \otimes (T_{l+m} \otimes T_{l+m-1} \otimes \cdots \otimes T_{m+1})$$

indexed by the inclusions $(\{l+m\} \hookrightarrow \{k+l+m\}, \{m\} \hookrightarrow \{l+m\})$. Composition in $T\mathrm{Sph}^\Sigma$ maps this summand by the identity to the summand $T_{k+l+m}\otimes\cdots\otimes T_{m+1}$ indexed by the composition $\{m\} \hookrightarrow \{l+m\} \hookrightarrow \{k+l+m\}$ (which is the inclusion), because the indices are already ordered in the correct fashion. Hence the diagram commutes and ν is a $\mathcal{V}$-functor, explaining thereby why we chose to order $\{k+m\} - \beta(\{k\})$ as $i_1^\beta > i_2^\beta > \cdots > i_k^\beta$. Furthermore, we have another explanation for the missing monoidal product in $T\mathrm{Sph}$: the concatenation of two inclusions is not an inclusion, just an injective map.

The map $\hom^\sigma_{T^m,T^{l+m}} \colon \mathrm{Inj}(m,l+m)\otimes T^l \longrightarrow \mathcal{V}(T^m, T^{l+m})$ has as adjoint

$$\mathrm{Inj}(m,l+m)\otimes T^l \otimes T^m \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\{m\},\{l+m\})} T^l \otimes T^m \longrightarrow T^{l+m}$$

which is defined on the summand $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_m \otimes \cdots \otimes T_1$ indexed by β as follows. First rewrite the indices as $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{\beta(m)} \otimes \cdots \otimes T_{\beta(1)}$, then map to $T_{l+m} \otimes \cdots \otimes T_2 \otimes T_1$ using the unique permutation which reorders the set $\{i_1^\beta, \dots, i_l^\beta, \beta(m), \dots, \beta(1)\}$. It is plausible that σ preserves identities. To conclude that σ is a $\mathcal{V}$-functor, one has to check that reordering the set $\{i_1^\gamma, \dots, i_k^\gamma, \gamma(i_1^\beta), \dots, \gamma(i_l^\beta), \gamma(\beta(m)), \dots, \gamma(\beta(1))\}$ is the same as first reordering $\{i_1^\gamma, \dots, i_l^\gamma, i_1^\beta, \dots, i_k^\beta, \beta(m), \dots, \beta(1)\}$ as $\{i_1^\gamma, \dots, i_l^\gamma, l+m, \dots, 2, 1\}$, and then reordering $\{i_1^\gamma, \dots, i_l^\gamma, \gamma(l+m), \dots, \gamma(2), \gamma(1)\}$. The monoidal product mon^Σ in $T\mathrm{Sph}^\Sigma$ is such that the diagram

$$\begin{tikzcd}
T\mathrm{Sph}^\Sigma \otimes T\mathrm{Sph}^\Sigma \arrow[r, "\sigma\otimes\sigma"] \arrow[d, "\mathrm{mon}^\Sigma"'] & \mathcal{V}\otimes\mathcal{V} \arrow[d, "\mathrm{mon}"] \\
T\mathrm{Sph}^\Sigma \arrow[r, "\sigma"] & \mathcal{V}
\end{tikzcd}$$

commutes, so σ is in fact a strict symmetric monoidal $\mathcal{V}$-functor.

Proposition 2.15. *The category $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is isomorphic to $\mathrm{Sp}^\Sigma(\mathcal{V}, T)$, the category of symmetric T-spectra in $\mathcal{V}$. The functor from symmetric T-spectra to T-spectra induced by ν is the forgetful functor. Furthermore, the smash product on $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ corresponds to the smash product of symmetric T-spectra.*

Proof. This is similar to 2.12, and some details will be left out in the proof. The functor $\Phi\colon [T\mathrm{Sph}^\Sigma, \mathcal{V}] \longrightarrow \mathrm{Sp}^\Sigma(\mathcal{V}, T)$ maps $X\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ to the sequence $(XT^0, XT^1, \dots)$. It is clear that XT^n has a left $\mathrm{Inj}(n,n)$-action. The structure map $XT^n \otimes T \longrightarrow XT^{n+1}$ has as adjoint the map

$$T \xrightarrow{\hom^{\nu}_{T^n,T^{n+1}}} \mathrm{Inj}(n, n+1) \otimes T \xrightarrow{\hom^{X}_{T^n,T^{n+1}}} \mathcal{V}(XT^n, XT^{n+1}).$$

More generally, the composition of structure maps $XT^n \otimes T^k \longrightarrow XT^{n+k}$ is the adjoint of

$$T^k \xrightarrow{\hom^{\nu}_{T^n,T^{k+n}}} \mathrm{Inj}(n, k+n) \otimes T^k \xrightarrow{\hom^{X}_{T^n,T^{k+n}}} \mathcal{V}(XT^n, XT^{k+n})$$

which proves the required equivariance. The definition of Φ on $\mathcal{V}$-natural transformations should be as clear as its functoriality.

The inverse $\Psi\colon \mathrm{Sp}^\Sigma(\mathcal{V}, T) \longrightarrow [T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is harder to define. If $(X_0, X_1, \dots)$ is a symmetric T-spectrum with structure maps $x_n^1\colon X_n \otimes T \longrightarrow X_{n+1}$, the $\mathcal{V}$-functor $\Psi(X)\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ has on the object T^n the value X_n. Since X_n has a left $\mathrm{Inj}(n,n)$-action, we have a map

$$\hom^{\Psi(X)}_{T^n,T^n}\colon \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, T^n) = \mathrm{Inj}(n,n) \longrightarrow \mathcal{V}(X_n, X_n).$$

To construct the map

$$\hom^{\Psi(X)}_{T^n,T^{k+n}}\colon \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, T^{k+n}) = \mathrm{Inj}(n, k+n) \longrightarrow \mathcal{V}(X_n, X_{k+n})$$

let us pick an injection $\beta\colon \{n\} \longrightarrow \{k+n\}$ and define the map on the summand $T_{i_1^\beta} \otimes \cdots \otimes T_{i_k^\beta}$ as the adjoint of the composition

$$T_{i_1^\beta} \otimes \cdots \otimes T_{i_k^\beta} \otimes X_n \xrightarrow{\sigma_{T^k,X_n}} X_n \otimes T_{i_1^\beta} \otimes \cdots \otimes T_{i_k^\beta} \xrightarrow{x_n^k} X_{k+n} \xrightarrow{p} X_{k+n}$$

where x_n^k is a composition of structure maps of the symmetric T-spectrum, and p is the isomorphism associated to the permutation $\{k+n\} \longrightarrow \{k+n\}$ which writes the set $\{k+n, \dots, 2, 1\}$ as $\{i_1^\beta, \dots, i_k^\beta, \beta(n), \dots, \beta(1)\}$ and reorders this as $\{k+n, \dots, 2, 1\}$. Here we use the left $\mathrm{Inj}(k+n, k+n)$-action on X_{k+n}.

The $\mathrm{Inj}(n,n)$-action on X_n is unital, hence $\Psi(X)$ preserves identities. To prove that $\Psi(X)\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor, pick injections $\beta\colon \{m\} \longrightarrow \{l+m\}$

and $\gamma\colon \{l+m\} \longrightarrow \{k+l+m\}$ and consider the diagram

$$\begin{tikzcd}
T_{i_1^\gamma}\otimes\cdots\otimes T_{i_k^\gamma}\otimes T_{i_1^\beta}\otimes\cdots\otimes T_{i_l^\beta}\otimes X_m \arrow[rr, "T^k\otimes(x^l_m\circ\sigma_{T^l,X_m})"] \arrow[d, "f\otimes X_m"'] \arrow[dr, "x_m^{k+l}\circ\sigma_{T^{k+l},X_m}"] & & T_{i_1^\gamma}\otimes\cdots\otimes T_{i_k^\gamma}\otimes X_{l+m} \arrow[dl, "x^k_{l+m}\circ\sigma_{T^k,X_{l+m}}"'] \arrow[d, "T^k\otimes b"] \\
T_{i_1^{\gamma\circ\beta}}\otimes\cdots\otimes T_{i_{k+l}^{\gamma\circ\beta}}\otimes X_m \arrow[d, "x_m^{k+l}\circ\sigma_{T^{k+l},X_m}"'] & X_{k+l+m} \arrow[d, "f'"] & T_{i_1^\gamma}\otimes\cdots\otimes T_{i_k^\gamma}\otimes X_{l+m} \arrow[d, "x^k_{l+m}\circ\sigma_{T^k,X_{l+m}}"] \\
X_{k+l+m} \arrow[dr, "cb"'] & X_{k+l+m} \arrow[d, "cb"] & X_{k+l+m} \arrow[dl, "c"] \\
& X_{k+l+m}. &
\end{tikzcd}$$

Here $b\colon X_{l+m} \longrightarrow X_{l+m}$ is the isomorphism obtained from reordering the set $\{i_1^\beta,\dots,i_k^\beta,\beta(m),\dots,\beta(1)\}$ as $\{l+m,\dots,2,1\}$, and similarly for c and cb. The isomorphism $f\colon T^{k+l} \longrightarrow T^{k+l}$ is induced by the permutation

$$\{i_1^\gamma,\dots,i_k^\gamma,\gamma(i_1^\beta),\dots,\gamma(i_l^\beta)\} \xrightarrow{\cong} \{i_1^{\gamma\circ\beta},\dots,i_{k+l}^{\gamma\circ\beta}\}.$$

Similarly, $f'\colon X_{k+l+m} \longrightarrow X_{k+l+m}$ is induced by the permutation

$$\{i_1^\gamma,\dots,i_k^\gamma,\gamma(i_1^\beta),\dots,\gamma(i_l^\beta),m,\dots,1\} \xrightarrow{\cong} \{i_1^{\gamma\circ\beta},\dots,i_{k+l}^{\gamma\circ\beta},m,\dots,1\}.$$

The triangle commutes by definition and commutativity coherence, the left parallelogram commutes, because x_m^{k+l} is $\mathrm{Inj}(k+l,k+l)\otimes\mathrm{Inj}(m,m)$-equivariant. The right parallelogram commutes, because x^k_{l+m} is $\mathrm{Inj}(k,k)\otimes\mathrm{Inj}(l+m,l+m)$-equivariant, the $\mathrm{Inj}(k+l+m,k+l+m)$-action on X_{k+l+m} is associative and the permutation obtained from reordering

$$\{i_1^\gamma,\dots,i_k^\gamma,\gamma(i_1^\beta),\dots,\gamma(i_l^\beta),\gamma(\beta(m)),\dots,\gamma(\beta(1))\}$$

is the same as the permutation obtained from first reordering

$$\{i_1^\gamma,\dots,i_k^\gamma,i_1^\beta,\dots,i_l^\beta,\beta(m),\dots,\beta(1)\}$$

as $\{i_1^\gamma,\dots,i_k^\gamma,\gamma(l+m),\dots,\gamma(1)\}$ and then as $\{k+l+m,\dots,2,1\}$.

The definition of Ψ on maps and its functoriality is straightforward to write down, hence we omit it. The equation $\Phi\circ\Psi = \mathrm{Id}_{\mathrm{Sp}^\Sigma(\mathcal{V},T)}$ follows in the same way as in the proof of 2.12, the equation $\Psi\circ\Phi = \mathrm{Id}_{[T\mathrm{Sph}^\Sigma,\mathcal{V}]}$ needs an extra argument. In fact, the only thing which is not completely obvious is whether the map $\hom^X_{T^m,T^{k+m}}\colon \mathrm{Inj}(m,k+m)\otimes T^k \longrightarrow \mathcal{V}(XT^m,XT^{m+k})$ coincides with the map $\hom^{\Psi(\Phi(X))}_{T^m,T^{k+m}}\colon \mathrm{Inj}(m,k+m)\otimes T^k \longrightarrow \mathcal{V}(XT^m,XT^{m+k})$. Fix an injection $\beta\colon\{m\} \longrightarrow \{k+m\}$ and let $\iota\colon \{m\} \hookrightarrow \{k+m\}$ denote the inclusion. The permutation $\gamma\colon \{k+m\} \longrightarrow \{k+m\}$ obtained from rewriting $\{k+m,\dots,1+m,m,\dots,1\}$ as $\{i_1^\beta,\dots,i_k^\beta,\beta(m),\dots,\beta(1)\}$ and reordering this as $\{k+m,\dots,2,1\}$ has the property that $\gamma\circ\iota=\beta$. Since X is a $\mathcal{V}$-functor, the map $\mathrm{hom}^X_{T^m,T^{k+m}}$ can be recovered from its restriction to the summand indexed by the inclusion $\{m\} \hookrightarrow \{k+m\}$ in precisely the way that guarantees the equality $\mathrm{hom}^X_{T^m,T^{k+m}} = \mathrm{hom}^{\Psi(\Phi(X))}_{T^m,T^{k+m}}$.

The statement about $[\nu, \mathcal{V}]\colon [T\mathrm{Sph}^\Sigma, \mathcal{V}] \longrightarrow [T\mathrm{Sph}, \mathcal{V}]$ is now obvious. It remains to prove the compatibility of the smash products. The monoidal product in $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is determined by its behavior on representatives described in 2.7, which simply says that

$$\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m, -) \wedge \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, -) \cong \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^{m+n}, -).$$

It is plausible that the isomorphism Φ maps $\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, -)$ to the symmetric T-spectrum F_nT^0 in the notation of [9, 7.3]. The smash product $\wedge'$ of symmetric T-spectra satisfies $F_mT^0 \wedge' F_nT^0 \cong F_{m+n}T^0$ by the remark below [9, 7.3], which completes the proof. $\square$

Corollary 2.16. *Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category of $\mathcal{V}$ closed under the monoidal product, containing the unit and the object T. Then $\sigma\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ factors over $\mathcal{C} \hookrightarrow \mathcal{V}$, and the resulting $\mathcal{V}$-functor $T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{C}$ induces a lax symmetric monoidal functor $[\mathcal{C}, \mathcal{V}] \longrightarrow \mathrm{Sp}^\Sigma(\mathcal{V}, T)$ which has a strict symmetric monoidal left adjoint.*

Proof. It is clear that σ factors as described. The closed symmetric monoidal product on $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ supplied by Day's theorem 2.6 coincides – under the above isomorphism – with the closed symmetric monoidal product on symmetric T-spectra. For formal reasons, the $\mathcal{V}$-functor induced by the first factor of σ is lax symmetric monoidal. By checking on representatives, it follows that its left $\mathcal{V}$-adjoint obtained by enriched Kan extension 2.5 is strict symmetric monoidal. $\square$

3. Model Categories

The word *model category* is to be understood in the sense of [7, 1.1.4], cf. [8]. Weak equivalences are denoted $\xrightarrow{\sim}$, fibrations $\twoheadrightarrow$ and cofibrations $\rightarrowtail$.

3.1. Types of model categories.

Some words found in [7, 2.1] will have slightly different meanings in this text. Let $\mathcal{C}$ be a category with all colimits.

Definition 3.1. Let λ be an ordinal. A *λ-sequence* is a colimit preserving functor $F\colon \lambda \longrightarrow \mathcal{C}$. The canonical map $F(\emptyset) \longrightarrow \operatorname{colim} F$ is the *sequential composition of F*. If $\mathcal{D}$ is a subclass of the morphisms in $\mathcal{C}$, a λ-sequence F is a *λ-sequence in $\mathcal{D}$* if $F(\beta) \longrightarrow F(\beta+1)$ is in $\mathcal{D}$ for every $\beta + 1 < \lambda$.

The most prominent example will be the well-ordered set of natural numbers.

Definition 3.2. The *cardinality* of a set M is the smallest ordinal $|M|$ admitting a bijection $|M| \xrightarrow{\cong} M$. Let κ be some cardinal, i.e. an ordinal having itself as its cardinality. An ordinal λ is *κ-filtered* if it is a limit ordinal, and if $M \subseteq \lambda$ with $|M| \leq \kappa$, then $\sup M < \lambda$.

Definition 3.3. Let $\mathcal{D}$ be a subclass of the morphisms in $\mathcal{C}$, $A \in \operatorname{Ob}\mathcal{C}$ and κ a cardinal. Then A is *κ-small relative to $\mathcal{D}$* if, for every κ-filtered ordinal λ, the functor $\mathbf{Set}_{\mathcal{V}}(A, -)$ commutes with colimits of λ-sequences in $\mathcal{D}$. Moreover, A is *small relative to $\mathcal{D}$* if there exists a cardinal κ such that A is κ-small relative to $\mathcal{D}$. Finally, A is *small* if it is small relative to the whole class of morphisms in $\mathcal{C}$.

The word "finite" is reserved for the following, see also [7, 2.1.4].

Definition 3.4. An object $A \in \operatorname{Ob}\mathcal{C}$ is *finite* if the set-valued Hom-functor $\mathbf{Set}_{\mathcal{C}}(A, -)$ commutes with all filtered colimits. If $\mathcal{C}$ is a $\mathcal{V}$-category, $A \in \operatorname{Ob}\mathcal{C}$ is *$\mathcal{V}$-finite* if the $\mathcal{V}$-valued Hom-functor $\mathcal{V}_{\mathcal{C}}(A, -)$ commutes with all filtered colimits.

A set is finite in the category of sets if and only if it is a finite set, whence the terminology. If the unit e in $\mathcal{V}$ is finite, any $\mathcal{V}$-finite object is finite.

Definition 3.5. If I is a set of maps in $\mathcal{C}$, a *relative I-cell complex* is a sequential composition of cobase changes of elements of I. The class I-cell is the class of relative I-cell complexes. Let I-inj be the class of maps having the right lifting property with respect to every map $i\colon si \longrightarrow ti$ in I.

Definition 3.6. A model category $\mathcal{C}$ is called *cofibrantly generated* if there are sets of maps I and J such that the following holds.

- The domains of the maps in I are small relative to I-cell.
- The domains of the maps in J are small relative to J-cell.
- The class of fibrations is J-inj.
- The class of acyclic fibrations is I-inj.

Maps in I are called *generating cofibrations*, and maps in J are called *generating acyclic cofibrations*. The (co)domains of I and J may have important additional properties. See [9, 4.1] for 3.7, 3.8 and 3.9.

Definition 3.7. An object A in $\mathcal{C}$ is called *finitely presented* if it is κ-small for a finite cardinal κ.

Definition 3.8. A cofibrantly generated model category $\mathcal{C}$ is *finitely generated* if I and J can be chosen such that the (co)domains of the maps in I and J are finitely presented.

Finitely generated model categories are not necessarily closed under Bousfield localization, as Hovey explains in [9, section 4]. This changes after weakening the definition slightly, which was suggested by Voevodsky.

Definition 3.9. The cofibrantly generated model category $\mathcal{C}$ is *almost finitely generated* if I and J can be chosen such that the domains and codomains of the maps in I are finitely presented, and there exists a subset J' of J for which:

- the domains and codomains of the maps in J' are finitely presented, and
- a map $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that B is fibrant is a fibration if and only if it is contained in J'-inj.

Left Bousfield localization with respect to a set of maps with **sSet**-small domains and codomains preserves the structure of almost finitely generated cellular left proper simplicial model categories [6, Chapters 3,4]. In an almost finitely generated model category, the classes of weak equivalences and fibrant objects are closed under sequential colimits. We require these classes to be closed under filtered colimits. This condition holds for model categories of the following type.

Definition 3.10. A cofibrantly generated model category $\mathcal{V}$ is *weakly finitely generated* if I and J can be chosen such that the following conditions hold.

- The domains and the codomains of the maps in I are finite.
- The domains of the maps in J are small.
- There exists a subset J' of J of maps with finite domains and codomains, such that a map $f\colon A \longrightarrow B$ in $\mathcal{V}$ with fibrant codomain B is a fibration if and only if it is contained in J'-inj.

The choices of I, J and J' will often be left implicit. A weakly finitely generated model category is always almost finitely generated. Examples of weakly finitely generated model categories are simplicial sets with the usual model structure, simplicial sets with action of a finite group as explained in 11.5, and the category of pointed simplicial presheaves on the Nisnevich site, equipped with the projective $\mathbb{A}^1$-model structure described in 9.36.

Lemma 3.11. *Assume $\mathcal{V}$ is a weakly finitely generated model category. Then the following classes are closed under filtered colimits: weak equivalences, acyclic fibrations, fibrations with fibrant codomain, and fibrant objects.*

Proof. Since $\mathcal{V}$ is cofibrantly generated, there is a cofibrantly generated model structure on the category $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ of diagrams of shape $\mathcal{I}$ in $\mathcal{V}$ for any small category $\mathcal{I}$, as one can deduce from [7, 2.1.19]. In this model structure, the weak equivalences and fibrations are defined pointwise. The colimit functor is therefore a left Quillen functor. Any weak equivalence $f\colon A \xrightarrow{\sim} B$ in $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ factors as an acyclic cofibration $g\colon A \rightarrowtail{\sim} T$, followed by a fibration $p\colon T \twoheadrightarrow B$. By "two out of three", p is an acyclic fibration. Consider the induced factorization $\operatorname{colim} f = \operatorname{colim} p \circ \operatorname{colim} g$. Note that $\operatorname{colim} g$ is an acyclic cofibration, since colim is a left Quillen functor. Therefore, the second claim will imply the first claim.

Assume $\mathcal{I}$ is filtered. We will prove that $\operatorname{colim} p$ is an acyclic fibration. Let

$$\begin{tikzcd} si \arrow[r, "\alpha"] \arrow[d, tail, "i"'] & \operatorname{colim} T \arrow[d, "\operatorname{colim} p"] \\ ti \arrow[r, "\beta"] & \operatorname{colim} B \end{tikzcd}$$

be a lifting problem, where $i \in I$. The existence of a lift in this diagram for all choices of α and β is equivalent to the surjectivity of the canonical map

$$\phi\colon \mathbf{Set}_{\mathcal{V}}(ti, \operatorname{colim} T) \longrightarrow \mathbf{Set}_{\mathcal{V}}(si, \operatorname{colim} T) \times_{\mathbf{Set}_{\mathcal{V}}(si, \operatorname{colim} B)} \mathbf{Set}_{\mathcal{V}}(ti, \operatorname{colim} B)$$

induced by composition with $\operatorname{colim} p$ and pre-composition with i. Since si and ti are finite by assumption and filtered colimits commute with pullbacks in **Set**, the canonical map ϕ is the filtered colimit of the canonical maps

$$\phi_d\colon \mathbf{Set}_{\mathcal{V}}(ti, T(d)) \longrightarrow \mathbf{Set}_{\mathcal{V}}(si, T(d)) \times_{\mathbf{Set}_{\mathcal{V}}(si, B(d))} \mathbf{Set}_{\mathcal{V}}(ti, B(d))$$

induced by composition with $p(d)$ and pre-composition with i. Note that ϕ_d is surjective, since i is a cofibration and p is a pointwise acyclic fibration. It follows that ϕ is surjective and $\operatorname{colim} p$ is an acyclic fibration. The proof of the third claim – including the last claim as a special case – is analogous. □

Definition 3.12. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category, and $\mathcal{D}$ a right $\mathcal{V}$-module with action $(C, A) \longmapsto C \oslash A$. Consider, for $f\colon C \longrightarrow D$ a map in $\mathcal{D}$ and $g\colon A \longrightarrow B$ a map in $\mathcal{V}$, the diagram

$$\begin{tikzcd} C \oslash A \arrow[r, "C \oslash g"] \arrow[d, "f \oslash A"'] & C \oslash B \arrow[d, "f \oslash B"] \\ D \oslash A \arrow[r, "D \oslash f"] & D \oslash B. \end{tikzcd}$$

If $\mathcal{D}$ has pushouts, the induced map from the pushout of the diagram to the terminal corner is denoted $f \square g\colon D \oslash A \cup_{C \oslash A} C \oslash B \longrightarrow D \oslash B$. It is the *pushout product map* of f and g.

Note that 3.12 applies to $(\mathcal{V}, \otimes, e)$ considered as a right $\mathcal{V}$-module with the obvious action. The next axiom guarantees that a model structure and a monoidal structure interact well.

Definition 3.13. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category and a model category. It is a *monoidal model category* if the pushout product $f \Box g$ of two cofibrations f and g is a cofibration, which is acyclic if either f or g is acyclic.

This is the pushout product axiom [16, 3.1]. Monoidal model categories with a cofibrant unit are also symmetric monoidal model categories in the sense of [7, 4.2.6]. Given a monoidal model category $\mathcal{V}$, one can consider (1) the category of monoids in $\mathcal{V}$ and (2) the category of modules for some fixed monoid. Reference [16] gives conditions for these categories to inherit model structures from $\mathcal{V}$. See [16, 3.3] for the following.

Definition 3.14. Let $\mathcal{V}$ be a monoidal model category. For a class $\mathcal{K}$ of maps in $\mathcal{V}$, define $\mathcal{K} \otimes \mathcal{V}$ as the class of maps $f \otimes A$, where f is a map in $\mathcal{K}$ and $A \in \operatorname{Ob} \mathcal{V}$. Let $\operatorname{aCof}(\mathcal{V})$ be the class of acyclic cofibrations in $\mathcal{V}$. The *monoid axiom* holds if every map in $\operatorname{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell is a weak equivalence.

In the proof of 4.2, we will use the monoid axiom to construct the pointwise model structure on $[\mathcal{C}, \mathcal{V}]$, where $\mathcal{V}$ is a monoidal model category and $\mathcal{C}$ is a small $\mathcal{V}$-category. This section ends with the definition of two other types of model categories.

Definition 3.15. Let $\mathcal{V}$ be a monoidal model category, and $\mathcal{C}$ a closed $\mathcal{V}$-module which is also a model category. The action of $\mathcal{V}$ on $\mathcal{C}$ allows to consider the pushout product of a map in $\mathcal{C}$ and a map in $\mathcal{V}$. Then $\mathcal{C}$ is called a *$\mathcal{V}$-model category* if the pushout product of a cofibration f in $\mathcal{C}$ and a cofibration g in $\mathcal{V}$ is a cofibration in $\mathcal{C}$, which is acyclic if either f or g is acyclic.

A simplicial model category is just a **sSet**-model category. All the model categories we will consider are **sSet**-model categories. Recall the definition of simplicial homotopy equivalences from 2.9.

Lemma 3.16. *A simplicial homotopy equivalence in a* **sSet***-model category is a weak equivalence.*

Proof. This follows from [6, 10.4.6], because simplicial homotopy equivalences in our sense are also simplicial homotopy equivalences as defined in [6, 10.4.3]. ☐

Definition 3.17. Let $\mathcal{V}$ and $\mathcal{C}$ be monoidal model categories, and $F \colon \mathcal{V} \longrightarrow \mathcal{C}$ a strict symmetric monoidal functor. If F is a left Quillen functor, then $\mathcal{C}$ is called a *monoidal $\mathcal{V}$-model category*. This language is consistent in the sense that every monoidal $\mathcal{V}$-model category is a $\mathcal{V}$-model category in a canonical way.

3.2. **Homotopy pullback squares.** Homotopy pullback squares will be used to characterize fibrations. Definition 3.18 is equivalent to [7, 7.1.12].

Definition 3.18. Let $\mathcal{C}$ be a model category. A commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

in $\mathcal{C}$ is a *homotopy pullback square* if, for any commutative diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, tail, "\sim"'] & D \arrow[d, tail, "\sim"'] & B \arrow[l, "f"'] \arrow[d, tail, "\sim"] \\ C' \arrow[r] & D' & B' \arrow[l, two heads] \end{tikzcd}$$

with C' and D' being fibrant, the canonical map $A \longrightarrow C' \times_{D'} B'$ is a weak equivalence.

Although this definition seems to by asymmetric, one of the squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "i"] \\ B \arrow[r, "h"] & D \end{tikzcd}$$

is a homotopy pullback square if and only if the other is so. We list further elementary properties of homotopy pullback squares.

Lemma 3.19. *All diagrams below are commutative diagrams in $\mathcal{C}$.*

1. *The diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\sim"] & D \end{tikzcd}$$

   is a homotopy pullback square if and only if f is a weak equivalence.
2. *Consider a natural transformation $f\colon A \longrightarrow B$ of diagrams*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & A_3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_3 \end{tikzcd}$$

   which is a pointwise weak equivalence. That is, $f_i\colon A_i \longrightarrow B_i$ is a weak equivalence for all $i \in \{0,1,2,3\}$. Then A is a homotopy pullback square if and only if B is a homotopy pullback square.

3. *Let*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & A_2 \arrow[d] \\ B_0 \arrow[r] & B_1 \arrow[r] & B_2 \end{tikzcd}$$

be a diagram where (2) is a homotopy pullback square. Then the composed square (12) is a homotopy pullback square if and only if the square (1) is a homotopy pullback square.

If the model category in question is right proper (which means that the base change of a weak equivalence along a fibration is always a weak equivalence), it is easier to test whether a diagram is a homotopy pullback square.

Lemma 3.20. *Assume $\mathcal{C}$ is right proper. Then*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pullback square if and only if for some factorization of f as a weak equivalence $B \xrightarrow{\sim} E$ followed by a fibration $E \twoheadrightarrow D$, the induced map $A \longrightarrow C \times_D E$ is a weak equivalence.

Proof. Since $\mathcal{C}$ is right proper, the dual of the gluing lemma holds. The statement follows easily from this. ☐

3.3. **Fibrant replacement functors.** In every model category, any object may be replaced by a fibrant object in a functorial way up to an acyclic cofibration. It is often desirable to explicate fibrant replacement functors. Quillen's small object argument is the classical method we recall in 3.3.1. We place emphasis on enriched fibrant replacement functors in 3.3.2. Another fibrant replacement functor is constructed in 3.3.3 as a certain filtered colimit.

3.3.1. *Classical.* Let us start fairly general, with a cocomplete category $\mathcal{V}$ and a set $K = \{sk \longrightarrow tk\}_{k \in K}$ of maps in $\mathcal{V}$ with finite domains and finite codomains. The set K gives rise to a natural transformation of endofunctors on $\mathcal{V}$, namely

$$\coprod_{k \in K} \coprod_{f \in \mathbf{Set}_{\mathcal{V}}(sk,-)} sk \xrightarrow{\coprod_{k\in K}\coprod_{f\in\mathbf{Set}_{\mathcal{V}}(sk,-)} k} \coprod_{k \in K} \coprod_{f \in \mathbf{Set}_{\mathcal{V}}(sk,-)} tk.$$

Consider also

$$\coprod_{k \in K} \coprod_{f \in \mathbf{Set}_{\mathcal{V}}(sk,-)} sk \xrightarrow{\coprod_{k\in K}\coprod_{f\in\mathbf{Set}_{\mathcal{V}}(sk,-)} f} \coprod_{k \in K} \coprod_{f \in \mathbf{Set}_{\mathcal{V}}(sk,-)} \mathrm{Id}_{\mathcal{V}} \xrightarrow{\text{codiagonal}} \mathrm{Id}_{\mathcal{V}}$$

and take the pushout of the two natural transformations. Let $\iota_1^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F_1^K$ denote the natural transformation which is obtained as the canonical map to the colimit. This construction can be iterated. Suppose a natural transformation $\iota_n^K\colon F_{n-1}^K \longrightarrow F_n^K$ of endofunctors on $\mathcal{V}$ has been defined. Then F_{n+1}^K is defined as the pushout of

$$\coprod_{k\in K}\ \coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} tk \xleftarrow{\coprod_{k\in K}\coprod_{f\in\mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} k} \coprod_{k\in K}\ \coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} sk \longrightarrow F_n.$$

Let $F^K\colon \mathcal{V} \longrightarrow \mathcal{V}$ denote the colimit of the sequence

$$\mathrm{Id}_{\mathcal{V}} = F_0 \xrightarrow{\iota_1^K} F_1^K \xrightarrow{\iota_2^K} \cdots \xrightarrow{\iota_n^K} F_n^K \xrightarrow{\iota_{n+1}^K} \cdots$$

and let $\iota^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F^K$ be the canonical natural transformation. The following statement is a special case of [7, 2.1.14].

Lemma 3.21. *For every object A in $\mathcal{V}$, the map $\iota^K(A)\colon A \longrightarrow F^K(A)$ is in K-cell, and the map $F^K(A) \longrightarrow *$ is in K-inj.*

Corollary 3.22. *Suppose that $\mathcal{V}$ is a weakly finitely generated model category. Then $\iota^{J'}\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F^{J'}$ is a fibrant replacement functor in the sense that $F^{J'}(A)$ is fibrant, and the natural map $\iota^{J'}(A)\colon A \longrightarrow F^{J'}(A)$ is an acyclic cofibration for all $A \in \mathrm{Ob}\,\mathcal{V}$.*

This fibrant replacement functor yields big objects. We will make this more precise after recalling some definitions from [6, 12.6].

Definition 3.23. A relative K-cell complex $f\colon A \longrightarrow B$ is called *finite* if f is obtained by attaching finitely many cells from K. That is, f is a composition

$$A = B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} B_n = B$$

where each f_m is a cobase change of a map in K.

It is possible to include choices.

Definition 3.24. A relative K-cell complex $f\colon A \longrightarrow B$ is called *presented* if it is equipped with an explicit choice of the data involved [6, 12.6.3]. In detail, one chooses a limit ordinal λ, a λ-sequence

$$A = B_0 \xrightarrow{f_0} B_1 \longrightarrow \cdots \xrightarrow{f_\beta} B_{\beta+1} \longrightarrow \cdots$$

whose sequential composition is f, and pushout diagrams

$$\begin{tikzcd}
\coprod_{m\in M_\beta} si_m \arrow[r] \arrow[d, "\coprod i_m"'] & B_\beta \arrow[d, "f_\beta"] \\
\coprod_{m\in M_\beta} ti_m \arrow[r] & B_{\beta+1}
\end{tikzcd}$$

for every $\beta < \lambda$. We omit the choice from the notation. Let $f\colon A \longrightarrow B$ be a presented relative K-cell complex. A *subcomplex* of f is a presented K-cell complex $g\colon A \longrightarrow C$ relative to A such that the explicit choice relevant for g is a subset of the explicit choice for f, see [6, 12.6.7]. In particular, there exists a map $h\colon C \longrightarrow B$ in K-cell such that $h \circ g = f$. A subcomplex is called *finite* if it is a finite K-cell complex relative to A, using the explicit choice.

Any relative K-cell complex can be turned into a presented one, but there may be many different choices. If we consider $\iota^K(A)\colon A \longrightarrow F^K(A)$ as a presented relative K-cell complex, we will use the explicit choices appearing in its construction.

Lemma 3.25. *Let $f\colon A \longrightarrow B$ be a finite relative K-cell complex. Then f has the structure of a finite subcomplex of $\iota^K(K)\colon A \longrightarrow F^K(A)$.*

Proof. We will prove this using induction on the number of cells attached. By convention, a map obtained by attaching zero cells is an identity map. Suppose the following is true. If $f\colon A \longrightarrow B$ is obtained by attaching n cells from K, with $n \geq 0$, then f has the structure of a finite subcomplex of the presented relative K-cell complex

$$A \xrightarrow{\iota_1^K(A)} \cdots \xrightarrow{\iota_n^K(A)} F_n^K(A)$$

Now let $f\colon A \longrightarrow B$ be obtained by attaching $n+1$ cells from K. By definition, there is a factorization $A \xrightarrow[g]{} C \xrightarrow[h]{} B$ where g is obtained by attaching n cells from K and h is the cobase change of some $k \in K$ along a map $\alpha\colon sk \longrightarrow C$. From the induction hypothesis, α induces a map $sk \xrightarrow{\alpha} C \longrightarrow F_nA$, hence an element in $\mathbf{Set}_{\mathcal{V}}(sk, F_nA)$. It follows that there is a map $\beta\colon tk \longrightarrow F_{n+1}A$ rendering the diagram

$$\begin{tikzcd}
sk \arrow[r, "\alpha"] \arrow[d, "k"', "\sim"] & C \arrow[r] & F_n^K(A) \arrow[d, "\iota_{n+1}^K(A)"] \\
tk \arrow[rr, "\beta"] & & F_{n+1}^K(A)
\end{tikzcd}$$

commutative. Since h is the cobase change of k along α, there is a unique induced map $B \longrightarrow F_{n+1}^K(A)$ which endows f with the structure of a subcomplex of

$$A \xrightarrow{\iota_1^K(A)} \cdots \xrightarrow{\iota_{n+1}^K(A)} F_{n+1}^K(A).$$

□

3.3.2. *Enriched.* Now suppose that $\mathcal{V}$ is a cocomplete closed symmetric monoidal category, and that K is a set of maps with finite domains and codomains. The fibrant replacement functor defined in 3.3.1 is a priori not a $\mathcal{V}$-functor, but one can remedy this as follows. Each k in K induces a $\mathcal{V}$-natural transformation

$\mathcal{V}(sk,-)\otimes k\colon \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathcal{V}(sk,-)\otimes tk$ of endo-$\mathcal{V}$-functors. On the other hand, the counit $\epsilon_{sk}\colon \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathrm{Id}_{\mathcal{V}}$ is also a $\mathcal{V}$-natural transformation. By taking the coproduct over all $k\in K$ one gets the diagram of $\mathcal{V}$-functors

$$\coprod_{k\in K}\mathcal{V}(sk,-)\otimes tk \xleftarrow{\coprod \mathcal{V}(sk,-)\otimes k} \coprod_{k\in K}\mathcal{V}(sk,-)\otimes sk \longrightarrow \mathrm{Id}_{\mathcal{V}}.$$

Denote the pushout by R_1^K. It is clear that one can iterate this construction. Suppose a $\mathcal{V}$-natural transformation $\rho_n^K\colon R_{n-1}^K \longrightarrow R_n^K$ of endo-$\mathcal{V}$-functors on $\mathcal{V}$ has been constructed. Then R_{n+1}^K is defined as the pushout of

$$\coprod_{k\in K}\mathcal{V}(sk,R_n^K(-))\otimes tk \xleftarrow{\coprod_{k\in K}\mathcal{V}(sk,R_n^K(-))\otimes k} \coprod_{k\in K}\mathcal{V}(sk,R_n^K(-))\otimes sk \longrightarrow R_n^K.$$

Let $R^K\colon \mathcal{V}\longrightarrow\mathcal{V}$ denote the colimit of the diagram

$$\mathrm{Id}_{\mathcal{V}} = R_0 \xrightarrow{\rho_1^K} R_1^K \xrightarrow{\rho_2^K} \cdots \xrightarrow{\rho_n^K} R_n^K \xrightarrow{\rho_{n+1}^K} \cdots$$

and let $\rho^K\colon \mathrm{Id}_{\mathcal{V}}\longrightarrow R^K$ be the canonical $\mathcal{V}$-natural transformation.

Lemma 3.26. *Given any object A in $\mathcal{V}$, the map $\rho^K(A)\colon A\longrightarrow R^K(A)$ is in $K\otimes\mathcal{V}$-cell, and the map $R^K(A)\longrightarrow *$ is in K-inj.*

Proof. The first statement is obvious. To prove that $R^K(A)\longrightarrow *$ is in K-inj, it suffices to find a lift in the diagram

$$\begin{tikzcd} sk \arrow[r, "f"] \arrow[d, "k"', "\sim"] & R^K(A) \arrow[d] \\ tk \arrow[r] & * \end{tikzcd}$$

for $k\in K$, $f\colon sk\longrightarrow R^K(A)$. Note that f factors as $sk\xrightarrow{g} R_n^K(A)\longrightarrow R^K(A)$ for some n, since sk is finite and R is a sequential colimit. The adjoint of g, tensored with sk, gives the map $h\colon sk\longrightarrow \mathcal{V}(sk,R_n^K(A))\otimes sk$ rendering the diagram

$$\begin{tikzcd} & \mathcal{V}(sk,R_n^K(A))\otimes sk \arrow[dr, "\epsilon_{sk}(R_n^K(A))"] & \\ sk \arrow[ur, "h"] \arrow[rr, "g"] & & R_n^K(A) \end{tikzcd}$$

commutative. Let h' denote the canonical map to the coproduct given by h. Then the diagram

$$\begin{tikzcd} sk \arrow[r, "h'"] \arrow[d, "k"', "\sim"] & \coprod_{k\in K}\mathcal{V}(sk,R_n^K(A))\otimes sk \arrow[r] \arrow[d] & R_n^K(A) \arrow[d] \\ tk \arrow[r] & \coprod_{k\in K}\mathcal{V}(sk,R_n^K(A))\otimes tk \arrow[r] & R_{n+1}^K(A) \end{tikzcd}$$

shows that a lift in the diagram we are interested in is given by the composition $tk \longrightarrow \coprod_{k\in K} \mathcal{V}(sk, R_n^K(A)) \longrightarrow R_{n+1}^K(A) \longrightarrow R^K(A)$. Hence $R^K(A) \longrightarrow *$ is in K-inj. □

Corollary 3.27. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category satisfying the monoid axiom. Then $\rho^{J'}\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R^{J'}$ is a fibrant replacement $\mathcal{V}$-functor, i.e. $R^{J'}(A)$ is fibrant and $\rho^{J'}(A)$ is a weak equivalence for all $A \in \mathrm{Ob}\,\mathcal{V}$.*

Proof. It remains to prove that the natural map $\rho_A^{J'}\colon A \longrightarrow R^{J'}(A)$ is a weak equivalence for every $A \in \mathrm{Ob}\,\mathcal{V}$. Note that $\rho_A^{J'}$ is contained in $\mathrm{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell. Thus the monoid axiom for $\mathcal{V}$ implies that $\rho_A^{J'}$ is a weak equivalence. □

Remark 3.28. The map $\rho^{J'}(A)$ is in general not a cofibration, and even if A is cofibrant, $R^{J'}(A)$ need not be cofibrant. However, if all objects in $\mathcal{V}$ are cofibrant, then $\rho^{J'}(A)$ is a cofibration.

3.3.3. *Filtered.* Still $\mathcal{V}$ denotes a cocomplete closed symmetric monoidal category, and K is some set of maps with finite domains and codomains. We define for each object $A \in \mathrm{Ob}\,\mathcal{V}$ three categories $\mathrm{ac}^K(A), \mathrm{ac}^K(A,R)$ and $\mathrm{ac}^K(A,F)$, and functors $U_R\colon \mathrm{ac}^K(A,R) \longrightarrow \mathrm{ac}^K(A)$ and $U_F\colon \mathrm{ac}^K(A,F) \longrightarrow \mathrm{ac}^K(A)$ comparing these.

Objects in $\mathrm{ac}^K(A)$ are finite K-cell complexes $\beta\colon A \longrightarrow B$ relative to A. The morphisms in $\mathrm{ac}^K(A)$ from $\beta\colon A \longrightarrow B$ to $\gamma\colon A \longrightarrow C$ are finite K-cell complexes $\tau\colon B \longrightarrow C$ satisfying $\tau \circ \beta = \gamma$. The identity id_A is the initial object in $\mathrm{ac}^K(A)$. Consider the functor $\Psi_A\colon \mathrm{ac}^K(A) \longrightarrow \mathcal{V}$ which sends $\beta\colon A \longrightarrow B$ to B and $\tau\colon B \longrightarrow C$ to τ. The colimit of Ψ_A will define the desired fibrant replacement of A (up to isomorphism).

Objects in $\mathrm{ac}^K(A,R)$ are pairs $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow R^K(A))$ such that β is a finite K-cell complex relative to A, and that $\beta' \circ \beta = \rho^K(A)$ holds. A map from $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow R^K(A))$ to $(\gamma\colon A \rightarrowtail{\sim} C, \gamma'\colon C \xrightarrow{\sim} R^K(A))$ is a finite K-cell complex $\tau\colon B \rightarrowtail C$ such that

$$\begin{tikzcd} & B \arrow[dr, "\beta'"] \arrow[dd, "\tau"] & \\ A \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & R^K(A) \\ & C \arrow[ur, "\gamma'"'] & \end{tikzcd}$$

commutes. The initial object in $\mathrm{ac}^K(A,R)$ is (id_A, ρ_A). Denote the forgetful functor $\mathrm{ac}^K(A,R) \longrightarrow \mathrm{ac}^K(A)$ which maps the pair (β,β') to β by U_R.

The category $\mathrm{ac}^K(A,F)$ has as objects all finite subcomplexes of the relative K-cell complex $\iota^K(A)\colon A \longrightarrow F^K(A)$. Such a subcomplex $\beta\colon A \longrightarrow B$ comes with a map $\beta'\colon B \longrightarrow F^K(A)$ in K-cell satisfying $\beta' \circ \beta = \iota^K(A)$, whence the objects in $\mathrm{ac}^K(A,F)$ are also denoted (β,β'). Maps are defined as for $\mathrm{ac}^K(A,R)$. Here $(\mathrm{id}_A, \iota(A))$ is the initial object. Let $U_F\colon \mathrm{ac}^K(A,F) \longrightarrow \mathrm{ac}^K(A)$ be the forgetful functor. Recall from [14, IX.3] the notion of a final functor.

Lemma 3.29. *Assume that maps in K-cell are monomorphisms. Then U_R and U_F are final functors. Furthermore, $\mathrm{ac}^K(A,F)$ is a small filtered preorder for every object A in $\mathcal{V}$.*

Proof. To prove that U_R is final, let $\beta\colon A \longrightarrow B$ be an object of $\mathrm{ac}^K(A)$. Since $R^K(A) \longrightarrow *$ is in K-inj, a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "\rho^K(A)"] \arrow[d, "\beta"'] & R^K(A) \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

exists, which shows that the comma category $\beta \downarrow U_R$ is nonempty. Consider objects (γ,γ') and (δ,δ') in $\mathrm{ac}^K(A,R)$, and maps $\sigma\colon \beta \longrightarrow \gamma$, $\tau\colon \beta \longrightarrow \delta$ in $\mathrm{ac}^K(A)$. The pushout

$$\begin{tikzcd} B \arrow[r, "\tau"] \arrow[d, "\sigma"'] & D \arrow[d] \\ C \arrow[r] & C \cup_B D \end{tikzcd}$$

yields a finite K-cell complex $\alpha\colon A \longrightarrow C\cup_B D$. The maps γ' and δ' induce a map $\alpha'\colon C\cup_B D \longrightarrow R^K(A)$ turning (α,α') into an object in $\mathrm{ac}^K(A,R)$. Moreover, the cobase changes of σ and τ are maps $(\gamma,\gamma') \longrightarrow (\alpha,\alpha')$ and $(\delta,\delta') \longrightarrow (\alpha,\alpha')$ in $\mathrm{ac}^K(A,R)$ which prove that $\beta\downarrow U_R$ is connected. Hence U_R is final. Next we consider U_F. Let $\beta\colon A \longrightarrow B$ be an object of $\mathrm{ac}^K(A)$. By 3.25, $\beta \downarrow U_F$ is nonempty. Since maps in K-cell are monomorphisms, the union of two finite subcomplexes is again a finite subcomplex by [6, 12.6.11], which proves connectedness.

A category is a preorder if there is at most one map between any two objects. Suppose that there exist two different maps $\sigma\colon B \longrightarrow C$ and $\tau\colon B \longrightarrow C$ from $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow F^K(A))$ to $(\gamma\colon A \longrightarrow C, \gamma'\colon C \longrightarrow F^K(A))$. By definition, we have $\gamma'\circ\sigma = \beta' = \gamma'\circ\tau$. Since γ' is a monomorphism by assumption, it follows that $\sigma=\tau$. Note that $\mathrm{ac}^K(A,F)$ is nonempty, since it contains the initial object (id_A, ι_A). Hence to prove that $\mathrm{ac}^K(A,F)$ is filtered, it remains to observe that any two objects have a common upper bound, which is given by the union of subcomplexes (see again [6, 12.6.11]). □

The colimits of Ψ_A, $\Psi_A\circ U_R$ and $\Psi_A \circ U_F$ are isomorphic via the canonical maps $\operatorname{colim}\Psi_A\circ U_R \longrightarrow \operatorname{colim}\Psi_A \longleftarrow \operatorname{colim}\Psi_A\circ U_F$ by Theorem 1 of [14, IX.3]. To have a natural comparison with the $\mathcal{V}$-functor R^K, we let $\Phi^K(A)$ be the colimit of the functor $\Psi_A\circ U_R\colon \mathrm{ac}^K(A,R) \longrightarrow \mathcal{V}$. There is a canonical map $\varphi_A\colon A \longrightarrow \Phi^K A$ induced by the object $(\mathrm{id}_A, \rho^K(A))$ in $\mathrm{ac}^K(A,R)$, and likewise there is a canonical map $\omega_A\colon \Phi^K(A) \longrightarrow R^K(A)$ induced by the maps β'.

Proposition 3.30. *Suppose that relative K-cell complexes are monomorphisms in $\mathcal{V}$. The map $\Phi^K(A) \longrightarrow *$ is in K-inj, and the map $\varphi_A\colon A \longrightarrow \Phi^K(A)$ is a filtered colimit of finite relative K-cell complexes. Moreover, $A \longmapsto \Phi^K(A)$ defines a functor $\Phi^K\colon \mathcal{V} \longrightarrow \mathcal{V}$, and the maps $A \xrightarrow{\varphi_A} \Phi^K(A)$ and $\Phi^K(A) \xrightarrow{\omega_A} R^K(A)$ are natural transformations of functors $\mathrm{Id}_{\mathcal{V}} \xrightarrow{\varphi} \Phi^K \xrightarrow{\omega} R^K$ with $\omega \circ \varphi = \rho^K$.*

Proof. Consider the first claim. Since ΦA is isomorphic to $\tilde{A} := \operatorname{colim} \Psi_A \circ U_F$, it suffices to prove that $\tilde{A} \longrightarrow *$ is in K-inj. For $k \in K$, let

$$\begin{tikzcd} sk \arrow[r, "f"] \arrow[d, "k"', "\sim"] & \tilde{A} \arrow[d] \\ tk \arrow[r] & * \end{tikzcd}$$

be a lifting problem. The object sk is finite and $\tilde{A}$ is a filtered colimit by 3.29, hence the map f factors as $sk \xrightarrow{g} a \longrightarrow \tilde{A}$ for some finite subcomplex $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow F^K(A))$ in $\mathrm{ac}^K(A, F)$. Take the pushout of the diagram

$$\begin{tikzcd} sk \arrow[r, "g"] \arrow[d, "k"'] & B \arrow[d, "\tau"] \\ tk \arrow[r, "h"] & C \end{tikzcd}$$

and define $\gamma := \tau \circ \beta\colon A \longrightarrow C$. Since τ is the cobase change of $k \in K$, 3.25 implies that $\gamma\colon A \longrightarrow C$ is a finite subcomplex of ι_A such that $\tau\colon \beta \longrightarrow \gamma$ is a map in $\mathrm{ac}^K(A, F)$. The canonical map $C \longrightarrow \tilde{A}$ belonging to γ then defines a lift $tk \xrightarrow{h} C \longrightarrow \tilde{A}$ in the square in question. Hence $\tilde{A} \longrightarrow *$ is in K-inj.

To prove the second assertion, it suffices to note that the constant diagram $c_A\colon \mathrm{ac}^K(A, F) \longrightarrow \mathcal{V}$ having value A factors over the canonical projection $\mathrm{ac}^K(A, F) \longrightarrow \pi_0 \mathrm{ac}^K(A, F)$ onto the set of connected components of $\mathrm{ac}^K(A, F)$. This projection is a final functor. Since $\mathrm{ac}^K(A, F)$ is connected, $\operatorname{colim} c_A \cong A$ follows. Note the natural transformation of diagrams $c_A \longrightarrow \Psi_A \circ U_F$ with value β at (β, β'). The induced map $\operatorname{colim} c_A \longrightarrow \Phi^K(A)$ coincides with the canonical map $\varphi_A\colon A \longrightarrow \Phi^K(A)$ up to the isomorphism mentioned above. The proof of the final statement is left to the reader. □

Corollary 3.31. *Let $\mathcal{V}$ be a monoidal model category satisfying the monoid axiom, and suppose that $\mathcal{V}$ is weakly finitely generated. If relative J'-cell complexes are monomorphisms, the maps*

$$A \xrightarrow{\varphi_A} \Phi^{J'}(A) \xrightarrow{\omega_A} R^{J'}(A)$$

are weak equivalences with fibrant codomains.

If $\mathcal{V}$ is pointed by $*$, any endo-$\mathcal{V}$-functor on $\mathcal{V}$ maps the point to the point. On the other hand, ι_* is not an isomorphism and there may be non-trivial finite subcomplexes of ι_*. So Φ^K is not a $\mathcal{V}$-functor in general. To define the stable model structure we need Φ to be "enriched with respect to spheres". This requires a natural map $T \otimes \Phi(A) \longrightarrow \Phi(T \otimes A)$ for some finite object T, such that

$$\begin{tikzcd} T \otimes \Phi^K(A) \arrow[r] \arrow[d, "T\otimes\omega_A"'] & \Phi^K(T \otimes A) \arrow[d, "\omega_{T\otimes A}"] \\ T \otimes R^K(A) \arrow[r, "\mathrm{move}_T^R(A)"] & R^K(T \otimes A) \end{tikzcd}$$

is commutative. Here the lower horizontal map is the adjoint of

$$T \xrightarrow{\eta_A(T)} \mathcal{V}(A, T \otimes A) \xrightarrow{\hom^{R^K}_{A,T\otimes A}} \mathcal{V}(R^K(A), R^K(T \otimes A)).$$

More details on the map move can be found in Appendix A. Suppose $T \otimes -$ maps finite K-cell complexes relative to A to finite K-cell complexes relative to $T \otimes A$, for every $A \in \mathrm{Ob}\,\mathcal{V}$. If T is $\mathcal{V}$-finite, finite and cofibrant, then the set K can be modified to achieve this, for example by defining K_T as the union

$$K \cup T \otimes K \cup T \otimes (T \otimes K)) \cup \dots$$

and using K_T instead of K. Tensoring with T then defines a functor

$$\Theta_A \colon \mathrm{ac}^K(A, R) \longrightarrow \mathrm{ac}(T \otimes A, R)$$

by sending $(\beta \colon A \longrightarrow B, \beta' \colon B \longrightarrow R^K(A))$ to

$$(T \otimes A \xrightarrow{T\otimes\beta} T \otimes B, T \otimes B \xrightarrow{T\otimes\beta'} T \otimes R^K(A) \xrightarrow{r(T,A)} R^K(T \otimes A)).$$

This functor induces a map

$$\begin{array}{rcl} T \otimes \Phi^K(A) & = & T \otimes \operatorname{colim}_{\mathrm{ac}^K(A,R)} \Psi_A \circ U_R \\ & \cong & \operatorname{colim}_{\mathrm{ac}^K(A,R)} (T \otimes -) \circ \Psi_A \circ U_R \\ & = & \operatorname{colim}_{\mathrm{ac}^K(A,R)} \Psi_{T\otimes A} \circ U_R \circ \Theta_A \\ & \longrightarrow & \operatorname{colim}_{\mathrm{ac}(T\otimes A,R)} \Psi_{T\otimes A} \circ U_R \\ & = & \Phi^K(T \otimes A) \end{array}$$

via the identity natural transformation $(T \otimes -) \circ \Psi_A \circ U_R \longrightarrow \Psi_{T\otimes A} \circ U_R \circ \Theta_A$.

Lemma 3.32. *Let $\mathcal{V}$ be a monoidal model category satisfying the monoid axiom. Suppose that $\mathcal{V}$ is weakly finitely generated, and that relative J'-cell complexes are monomorphisms. If A is an object of $\mathcal{V}$, the map $\theta_A \colon T \otimes \Phi^{J'}(A) \longrightarrow \Phi^{J'}(T \otimes A)$*

is a weak equivalence making the diagram

$$\begin{tikzcd} T \otimes \Phi^{J'}(A) \arrow[r, "\theta_A"] \arrow[d, "T\otimes\omega_A"'] & \Phi^{J'}(T\otimes A) \arrow[d, "\omega_{T\otimes A}"] \\ T\otimes R^{J'}(A) \arrow[r, "r(T,A)"] & R^{J'}(T\otimes A) \end{tikzcd}$$

commutative. Furthermore, θ_A is natural in A.

Proof. Commutativity and naturality follow by the construction of θ_A. Note that $\rho_{T\otimes A}$ and $T \otimes \rho_A^{J'} \in \mathrm{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell are weak equivalences. It follows that $r(T, A)$ is a weak equivalence, since $r(T, A) \circ T \otimes \rho_A^{J'} = \rho_{T\otimes A}^{J'}$. The map $\varphi_{T\otimes A}$ is a weak equivalence, and $T \otimes \varphi_A$ is a filtered colimit of weak equivalences. Hence the vertical maps in the diagram are weak equivalences. □

Finally, we relate $\mathcal{V}$-functors and Φ^K in the case where $\mathbb{I}\colon \mathcal{C} \hookrightarrow \mathcal{V}$ is a full sub-$\mathcal{V}$-category and all objects in $\mathcal{C}$ are $\mathcal{V}$-finite.

Lemma 3.33. *Suppose that relative K-cell complexes are monomorphisms. Let $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$-functor and let $A \in \mathrm{Ob}\,\mathcal{C}$. There is an isomorphism*

$$\mathbb{I}_*X(\Phi^K(A)) \cong \mathrm{colim}(X \circ \Psi_A \circ U_R\colon \mathrm{ac}^K(A, R) \longrightarrow \mathcal{V})$$

which is natural in X.

Proof. This follows by 2.3 (which expresses any $\mathcal{V}$-functor as a coend of representables), 3.29 and the $\mathcal{V}$-finiteness of the objects in $\mathcal{C}$. Then

$$\mathbb{I}_*X(\Phi^K(A)) = \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, \Phi^K(A)) \otimes X(c) \cong \mathrm{colim} \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, \Psi_A \circ U_R) \otimes X(c),$$

and the claim follows. □

4. The pointwise model structure

Let $\mathcal{V}$ be a monoidal model category. We introduce the pointwise model structure on the category $[\mathcal{C},\mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$, where $\mathcal{C}$ is some small $\mathcal{V}$-category. For this purpose, assume $\mathcal{V}$ is weakly finitely generated and that the monoid axiom holds. To prove left properness of the pointwise model structure, we have to assume something slightly stronger than left properness of $\mathcal{V}$. Of particular interest is the case where $\mathcal{C}$ is a full sub-$\mathcal{V}$-category of $\mathcal{V}$, and $\mathcal{C}$ satisfies the following properties.

f0 Every object in $\mathcal{V}$ is a filtered colimit of objects in $\mathcal{C}$.
f1 Every object in $\mathcal{C}$ is $\mathcal{V}$-finite.
f2 The unit e is in $\mathcal{C}$, and $\mathcal{C}$ is closed under the monoidal product in $\mathcal{V}$.

4.1. **The general case.** Our pointwise notions of weak equivalences, fibrations and cofibrations are as follows.

Definition 4.1. A morphism $f\colon X \longrightarrow Y$ in $[\mathcal{C},\mathcal{V}]$ is a

- *pointwise weak equivalence* if $f(c)$ is a weak equivalence in $\mathcal{V}$ for all $c \in \operatorname{Ob}\mathcal{C}$.
- *pointwise fibration* if $f(c)$ is a fibration in $\mathcal{V}$ for all $c \in \operatorname{Ob}\mathcal{C}$.
- *cofibration* if it has the left lifting property with respect to all pointwise acyclic fibrations.

Theorem 4.2. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Suppose the monoid axiom holds in $\mathcal{V}$. Then $[\mathcal{C},\mathcal{V}]$, with the classes of maps in 4.1, is a weakly finitely generated model category.*

Proof. We will use [7, 2.1.19]. The category $[\mathcal{C},\mathcal{V}]$ is bicomplete by 2.2. The class of pointwise weak equivalences is closed under retracts and satisfies the "two out of three" axiom. Let I be the generating cofibrations in $\mathcal{V}$, and J the generating acyclic cofibrations in $\mathcal{V}$. Let $\mathcal{P}_I$ be the set of maps in $[\mathcal{C},\mathcal{V}]$

$$\{\mathcal{V}_{\mathcal{C}}(c,-)\otimes si \xrightarrow{\mathcal{V}_{\mathcal{C}}(c,-)\otimes i} \mathcal{V}_{\mathcal{C}}(c,-)\otimes ti \mid i\in I, c\in \operatorname{Ob}\mathcal{C}\}$$

and likewise we let $\mathcal{P}_J$ denote

$$\{\mathcal{V}_{\mathcal{C}}(c,-)\otimes sj \xrightarrow{\mathcal{V}_{\mathcal{C}}(c,-)\otimes j} \mathcal{V}_{\mathcal{C}}(c,-)\otimes tj \mid j\in J, c\in \operatorname{Ob}\mathcal{C}\}.$$

Since $\mathcal{V}$ is assumed to be cofibrantly generated, it follows from adjointness that $\mathcal{P}_J$-inj coincides with the class of pointwise fibrations, and that $\mathcal{P}_I$-inj coincides with the class of pointwise fibrations which are also pointwise weak equivalences. If A is finite (small) in $\mathcal{V}$, then $\mathcal{V}_{\mathcal{C}}(c,-)\otimes A$ is finite (small) in $[\mathcal{C},\mathcal{V}]$ for any $c\in \operatorname{Ob}\mathcal{C}$, since colimits are computed pointwise according to 2.2. The smallness conditions listed in 3.10 are therefore satisfied. It remains to check that maps in $\mathcal{P}_J$-cell are pointwise weak equivalences. Every map in $\mathcal{P}_J$-cell is pointwise a map in $J\otimes\mathcal{V}$-cell, and the latter class consists of weak equivalences by the monoid axiom. □

We refer to the model structure in 4.2 as the *pointwise* model structure.

Lemma 4.3. *Suppose the pointwise model structure exists. Then the functor from $\mathcal{V}$ to $[\mathcal{C}, \mathcal{V}]$ which maps A to $\mathcal{V}_{\mathcal{C}}(c,-) \otimes A$ is a left Quillen functor for every $c \in \mathrm{Ob}\,\mathcal{C}$. The functor Ev_c which evaluates at c is its right adjoint.*

If the unit in $\mathcal{V}$ is cofibrant, then the representable functors $\mathcal{V}(c,-)$ are all cofibrant in the pointwise model structure. Furthermore, the pointwise model structure is compatible with the enrichment.

Theorem 4.4. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.2. Then $[\mathcal{C}, \mathcal{V}]$, with its pointwise model structure, is a $\mathcal{V}$-model category. If $\mathcal{C}$ is a symmetric monoidal $\mathcal{V}$-category, then $[\mathcal{C}, \mathcal{V}]$ is a monoidal $\mathcal{V}$-model category.*

Proof. Recall from 2.4 that $[\mathcal{C}, \mathcal{V}]$ is a closed $\mathcal{V}$-module. By [7, 4.2.5] it suffices to check the following conditions.

- Let $\mathcal{V}_{\mathcal{C}}(c,-) \otimes i\colon \mathcal{V}_{\mathcal{C}}(c,-) \otimes si \rightarrowtail \mathcal{V}_{\mathcal{C}}(c,-) \otimes ti$ be a map in $\mathcal{P}_I$, and let $j\colon sj \rightarrowtail tj$ be a map in I. Then the pushout product map $(\mathcal{V}_{\mathcal{C}}(c,-) \otimes i)\Box j$ is a cofibration.
- If either i or j in the above sentence are generating acyclic cofibrations, then the pushout product map $(\mathcal{V}_{\mathcal{C}}(c,-) \otimes i)\Box j$ is a pointwise acyclic cofibration.

Since $\mathcal{V}_{\mathcal{C}}(c,-) \otimes -$ is a left adjoint, the pushout product map in question is of the form $\mathcal{V}_{\mathcal{C}}(c,-) \otimes (i \Box j)$. Hence the conditions hold, because $\mathcal{V}$ is a monoidal model category and 4.3 holds.
The second statement is proven similarly using [7, 4.2.5]. Because a smash product of representables is representable by 2.7, the compatibility of $\wedge$ and $\otimes$ implies that the pushout product map of $\mathcal{V}_{\mathcal{C}}(c,-) \otimes i$ and $\mathcal{V}_{\mathcal{C}}(d,-) \otimes j$ is isomorphic to $\mathcal{V}_{\mathcal{C}}(c \diamond d,-) \otimes (i \Box j)$. Here $\diamond$ denotes the monoidal product in $\mathcal{C}$. If u is the unit in $\mathcal{C}$, the functor $\mathcal{V}_{\mathcal{C}}(u,-) \otimes -$ is a left Quillen functor by 4.3, and also strict symmetric monoidal. □

Remark 4.5. Via 2.12, the pointwise model structure on T-spectra corresponds to the pointwise model structure on $[T\mathrm{Sph}, \mathcal{V}]$.

For a discussion of properness of the pointwise model structure, we introduce the following definition.

Definition 4.6. A monoidal model category $\mathcal{V}$ is *strongly left proper* if the cobase change of a weak equivalence along any map in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell is again a weak equivalence, where $\mathrm{Cof}(\mathcal{V})$ is the class of cofibrations in $\mathcal{V}$.

Strongly left proper monoidal model categories are left proper. Conversely, if every object in a model category is cofibrant, then the model structure is left proper and strongly left properness follows. The relevance of 4.6 is explained by the following lemma.

Lemma 4.7. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.2. If f is a cofibration in $[\mathcal{C},\mathcal{V}]$, then $f(c)$ is a retract of a map in $\mathrm{Cof}(\mathcal{V})\otimes\mathcal{V}$-cell for every $c\in\mathrm{Ob}\,\mathcal{C}$.*

Proof. Any cofibration in $[\mathcal{C},\mathcal{V}]$ is a retract of a relative $\mathcal{P}_I$-cell complex, and $\mathcal{V}_{\mathcal{C}}(d,c)\otimes i$ is a map in $I\otimes\mathcal{V}$-cell for every $c\in\mathrm{Ob}\,\mathcal{C}$. Hence the claim follows. □

Corollary 4.8. *The pointwise model structure on $[\mathcal{C},\mathcal{V}]$ is right proper if $\mathcal{V}$ is right proper, and left proper if $\mathcal{V}$ is strongly left proper.*

4.2. **The subcategory case.** Recall that if $\mathcal{C}$ is a full sub-$\mathcal{V}$-category of $\mathcal{V}$, the left Kan extension along the inclusion functor $\mathbb{I}\colon\mathcal{C}\hookrightarrow\mathcal{V}$ is

$$\mathbb{I}_*X=\int^{\mathrm{Ob}\,\mathcal{C}}\mathcal{V}(c,-)\otimes X(c).$$

First we derive the monoid axiom for the pointwise model structure.

Lemma 4.9. *Assume $\mathcal{V}$ is a weakly finitely generated monoidal model category, and let $\mathcal{C}$ be a small full sub-$\mathcal{V}$-category of $\mathcal{V}$ satisfying* **f2**. *If the monoid axiom holds in $\mathcal{V}$, then it holds also in the pointwise model structure on $[\mathcal{C},\mathcal{V}]$.*

Proof. Abbreviate $[\mathcal{C},\mathcal{V}]$ by $\mathcal{F}$. Since $\mathcal{F}$ is cofibrantly generated, it suffices to check that every map in the class $\mathcal{P}_J\wedge\mathcal{F}$-cell is a pointwise weak equivalence. Let $c\in\mathrm{Ob}\,\mathcal{C}$, $j\in J$ and $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ a $\mathcal{V}$-functor. Then $(\mathcal{V}(c,-)\otimes j)\wedge X$ coincides up to isomorphism with the map $\mathbb{I}_*X(\mathcal{V}(c,-))\otimes j$ by 2.8. In particular, $\mathbb{I}_*X(\mathcal{V}(c,d))\otimes j$ is contained in $J\otimes\mathcal{V}$ for every $d\in\mathrm{Ob}\,\mathcal{C}$ and every $X\in\mathrm{Ob}\,\mathcal{F}$. For a map f in $\mathcal{P}_J\otimes\mathcal{F}$-cell, $f(d)$ belongs to $J\otimes\mathcal{V}$-cell because colimits are formed pointwise. Since the monoid axiom holds in $\mathcal{V}$, f is a pointwise weak equivalence. □

In particular, smashing with any $\mathcal{V}$-functor maps acyclic cofibrations to weak equivalences in the pointwise structure. Under stronger assumptions on $\mathcal{C}$ and $\mathcal{V}$, smashing with any cofibrant $\mathcal{V}$-functor preserves pointwise weak equivalences. This fact will be used to prove the monoid axiom in the stable model structure.

Lemma 4.10. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category, and assume $\mathcal{C}$ is a small and full sub-$\mathcal{V}$-category of $\mathcal{V}$ satisfying* **f0** *and* **f1**. *If f is a pointwise weak equivalence in $[\mathcal{C},\mathcal{V}]$, then so is $\mathbb{I}_*f$.*

Proof. We have to check that, for any $A\in\mathrm{Ob}\,\mathcal{V}$, $\mathbb{I}_*f(A)$ is a pointwise weak equivalence. Write A as the colimit of $C\colon\mathcal{I}\longrightarrow\mathcal{C}$ for $\mathcal{I}$ filtering by **f0**. Since coends commute with colimits and **f1** holds, it follows that

$$\begin{aligned}
\mathbb{I}_* f(A) &= \int^B \mathcal{V}(B, A) \otimes f(B) \\
&\cong \int^B \mathcal{V}(B, \operatorname{colim}_{i\in\mathcal{I}} C) \otimes f(B) \\
&\cong \operatorname{colim}_{i\in\mathcal{I}} \int^B \mathcal{V}(B, C(i)) \otimes f(B) \\
&\cong \operatorname{colim}_{i\in\mathcal{I}}\ f \circ C.
\end{aligned}$$

Note that because f is a pointwise weak equivalence, $\mathbb{I}_* f(A)$ is a filtered colimit of weak equivalences, and hence a weak equivalence by 3.11. □

Corollary 4.11. *Let $\mathcal{V}$ and $\mathcal{C}$ be as in 4.10. If $\mathcal{C}$ satisfies* **f2** *and f is a pointwise weak equivalence in $[\mathcal{C}, \mathcal{V}]$, then $f \wedge \mathcal{V}(c, -)$ is a pointwise weak equivalence for every $c \in \operatorname{Ob}\mathcal{C}$.*

Proof. Axiom **f2** implies that the smash product of two enriched functors exists. By 2.8, $f \wedge \mathcal{V}(c, -)$ is isomorphic to $\mathbb{I}_* f \circ \mathcal{V}(c, -)$. Since f is a pointwise weak equivalence by 4.10, $\mathbb{I}_* f(\mathcal{V}(c, d))$ is a weak equivalence for any $d \in \operatorname{Ob}\mathcal{C}$. □

Theorem 4.12. *Let $\mathcal{V}$ and $\mathcal{C}$ be as in 4.11. Assume $\mathcal{V}$ satisfies the monoid axiom and is strongly left proper, and tensoring with the domains and the codomains of the generating cofibrations in $\mathcal{V}$ preserves weak equivalences. Then smashing with a cofibrant object preserves weak equivalences in the pointwise model structure.*

Proof. Let $f : X \longrightarrow Y$ be a pointwise weak equivalence, and let Z be a cofibrant $\mathcal{V}$-functor. Since Z is cofibrant, it is a retract of some $\mathcal{V}$-functor Z', such that $* \longrightarrow Z'$ is the sequential composition of a sequence

$$* = Z'_0 \xrightarrow{g_0} Z'_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} Z'_n \xrightarrow{g_n} \cdots$$

where g_n is the cobase change of a coproduct of maps in $\mathcal{P}_I$. It suffices to consider sequences indexed by the natural numbers, since the domains of the maps in $\mathcal{P}_I$ are finite. The map $f \wedge Z$ is a retract of $f \wedge Z'$, hence it remains to prove that the latter is a pointwise weak equivalence. We will prove this by induction on n. Consider the following diagram.

$$\begin{tikzcd}
X \wedge \coprod_{m\in M} \mathcal{V}(c_m, -) \otimes ti_m \arrow[d, "f\wedge\coprod\mathcal{V}(c_m,-)\otimes ti_m"] & X \wedge \coprod_{m\in M} \mathcal{V}(c_m, -) \otimes si_m \arrow[l, "X\wedge\coprod\mathcal{V}(c_m,-)\otimes i_m"'] \arrow[r] \arrow[d, "f\wedge\coprod\mathcal{V}(c_m,-)\otimes si_m"] & X \wedge Z_n \arrow[d, "f\wedge Z_n"] \\
Y \wedge \coprod_{m\in M} \mathcal{V}(c_m, -) \otimes ti_m & Y \wedge \coprod_{m\in M} \mathcal{V}(c_m, -) \otimes si_m \arrow[l, "X\wedge\coprod\mathcal{V}(c_m,-)\otimes i_m"'] \arrow[r] & Y \wedge Z_n.
\end{tikzcd}$$

The map induced on the pushouts of the upper and lower row is $f \wedge Z_{n+1}$. Suppose $f \wedge Z'_n$ is a pointwise weak equivalence. By 4.11 and the hypothesis on I, it follows that both of the other vertical maps are pointwise weak equivalences. The horizontal maps on the left hand side are not necessarily cofibrations. However, evaluation at some $d \in \mathrm{Ob}\,\mathcal{C}$, gives maps in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell. Since $\mathcal{V}$ is strongly left proper, this implies that the map induced on the pushouts – which is computed pointwise – is a pointwise weak equivalence. □

Remark 4.13. Suppose that $\mathcal{V}$ is a monoidal model category which is weakly finitely generated. If the generating cofibrations I can be chosen such that $- \otimes si$ and $- \otimes ti$ preserve weak equivalences for every $i \in I$, and if $\mathcal{V}$ is strongly left proper, then $- \otimes A$ preserves weak equivalences for any cofibrant object A in $\mathcal{V}$. The argument is the same as in the proof of 4.12. We define a *strongly monoidal* model category to be a monoidal model category which is strongly left proper, and in which $- \otimes A$ preserves weak equivalences either for any cofibrant object A, or for A running through the domains and codomains of the generating cofibrations (if they exist). A monoidal model category in which every object is cofibrant is strongly monoidal and satisfies the monoid axiom.

5. The homotopy functor model structure

Suppose $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor of categories with chosen subclasses of weak equivalences. If F maps weak equivalences to weak equivalences, then F is called a *homotopy functor*. As a first step towards the stable model structure on enriched functors, we define a model structure in which every enriched functor is weakly equivalent to a homotopy functor.

Let $\mathcal{V}$ be a weakly finitely generated monoidal **sSet**-model category with cofibrant unit. Assume $\mathcal{V}$ satisfies the monoid axiom and is strongly monoidal as defined in 4.13. Moreover, we suppose that filtered colimits commute with pullbacks in $\mathcal{V}$, and that cofibrations are monomorphisms in $\mathcal{V}$. The simplicial structure is needed for a simplicial mapping cylinder construction, and we assume also that Δ^1 is finite in $\mathcal{V}$. Furthermore, $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ denotes a small full sub-$\mathcal{V}$-category satisfying the following axioms.

f1 Every object in $\mathbf{f}\mathcal{V}$ is $\mathcal{V}$-finite.
f2 The unit e is in $\mathbf{f}\mathcal{V}$, and $\mathbf{f}\mathcal{V}$ is closed under the monoidal product.
f3 If $tj \xleftarrow{\sim} sj \longrightarrow v$ is a diagram in $\mathcal{V}$ where $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$ and $j \in J'$, then the pushout $tj \cup_{sj} v$ is in $\mathbf{f}\mathcal{V}$.

Some additional axioms will occasionally be used.

f0 Every object in $\mathcal{V}$ is a filtered colimit of objects in $\mathbf{f}\mathcal{V}$.
f4 All objects in $\mathbf{f}\mathcal{V}$ are cofibrant.
f5 The simplicial mapping cylinder is available in $\mathbf{f}\mathcal{V}$.

Objects in $\mathbf{f}\mathcal{V}$ will usually be denoted by small letters. Let $\mathcal{F}$ be short for $[\mathbf{f}\mathcal{V}, \mathcal{V}]$ and recall the left Kan extension $\mathbb{I}_*X$ of $X \in \operatorname{Ob}\mathcal{F}$ along $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$.

5.1. **Equivalences of homotopy functors.** The fibrant replacement functor $\Phi^{J'}$ from 3.3.3 defines a functor $\Phi\colon \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$, ignoring the set J' for some time. It is in general not a $\mathcal{V}$-functor, hence the composition $\mathbb{I}_*X \circ \Phi$ is just a functor which we denote by $\hbar(X)\colon \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$. The functor $\hbar(X)$ comes with a natural transformation $X \longrightarrow \hbar(X)$ induced by the canonical maps $\varphi_v\colon v \longrightarrow \Phi(v)$, where v varies through the set of objects in $\mathbf{f}\mathcal{V}$.

Lemma 5.1. *The assignment $X \longmapsto \hbar(X)$ described above defines a functor $\hbar\colon \mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ which commutes with colimits and the action of $\mathcal{V}$. The natural transformations $X \longrightarrow \hbar(X)$ define a natural transformation from the forgetful functor $\mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ to $\hbar$.*

Proof. The claim follows, because the left Kan extension along $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ is constructed as a coend. $\square$

Definition 5.2. A map $f\colon X \longrightarrow Y$ is an *hf-equivalence* if the map $\hbar(f)(v)$ is a weak equivalence in $\mathcal{V}$ for every $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$.

Lemma 5.3. *A pointwise weak equivalences is in particular an hf-equivalence. The class of hf-equivalences satisfies the "two out of three" axiom.*

Proof. The first statement follows as in 4.10, since $\hbar(f)(v) = \mathbb{I}_*(f)(\Phi v)$ is a filtered colimit of weak equivalences provided f is a pointwise weak equivalence. The second statement follows from 5.1 and the corresponding statement in $\mathcal{V}$. □

Lemma 5.4. *Let $f\colon X \rightarrowtail Y$ be a cofibration in $\mathcal{F}$. Then $\hbar(f)(v)$ is a retract of a map in* $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$*-cell for every $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. If $\mathcal{V}(v,-) \otimes i$ is a generating cofibration, then $\hbar(\mathcal{V}(v,w) \otimes i)$ coincides with $\mathcal{V}(v,\Phi(w)) \otimes i$, which is in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell. The general case follows, since $\hbar$ commutes with colimits. □

5.2. **Fibrations of homotopy functors.** If $\phi\colon v \longrightarrow w$ is an acyclic cofibration in $\mathbf{f}\mathcal{V}$, one can factor the induced map $\mathcal{V}(\phi,-)\colon \mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ using the simplicial mapping cylinder as a cofibration $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ followed by a simplicial homotopy equivalence. This uses that $\mathcal{V}(w,-)$ is a cofibrant functor, since the unit in $\mathcal{V}$ is cofibrant, and $\mathcal{V}$ (hence $\mathcal{F}$ by 4.4) is a **sSet**-model category. Take any generating cofibration $i\colon si \longrightarrow ti$ in $\mathcal{V}$ and form the pushout product

$$c_\phi \square i\colon \mathcal{V}(w,-) \otimes ti \cup_{\mathcal{V}(w,-)\otimes si} C_\phi \otimes si \longrightarrow C_\phi \otimes ti.$$

Let $\mathcal{H}$ denote the set $\{c_\phi \square i\}$, where ϕ runs through the set of acyclic cofibrations in $\mathbf{f}\mathcal{V}$ and i runs through the set I of generating cofibrations in $\mathcal{V}$.

Definition 5.5. A map $f\colon X \longrightarrow Y$ is an *hf-fibration* if it is a pointwise fibration having the right lifting property with respect to the set $\mathcal{H}$.

Lemma 5.6. *Let $f\colon X \longrightarrow Y$ be a pointwise fibration. Then f is an hf-fibration if and only if the following square is a homotopy pullback square in $\mathcal{V}$ for every acyclic cofibration $\phi\colon v \overset{\sim}{\rightarrowtail} w$ in $\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} X(v) \arrow[r, "X(\phi)"] \arrow[d, "f(v)"'] & X(w) \arrow[d, "f(w)"] \\ Y(v) \arrow[r, "Y(\phi)"] & Y(w) \end{tikzcd}$$

Proof. Let $\mathcal{V}_{\mathcal{F}}(X,Y)$ denote the $\mathcal{V}$-object of maps in $\mathcal{F}$ from X to Y. For a map of $\mathcal{V}$-functors $f\colon X \longrightarrow Y$, the square in the statement of the lemma is naturally isomorphic, by the Yoneda lemma 2.1, to the square

$$\begin{tikzcd} \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),X) \arrow[r, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(\phi,-),X)"] \arrow[d, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),f)"'] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),X) \arrow[d, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),f)"] \\ \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),Y) \arrow[r, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(\phi,-),Y)"] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y). \end{tikzcd}$$

The factorization of $\mathcal{V}(\phi,-)$ as a cofibration $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ followed by a simplicial homotopy equivalence $C_\phi \xrightarrow{\simeq} \mathcal{V}(v,-)$ induces a factorization of the square above into two squares. Since $\mathcal{V}_{\mathcal{F}}(-,X)$ preserves simplicial homotopy equivalences by 2.11, which are pointwise weak equivalences by 3.16, the square above is a homotopy pullback square if and only if

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{F}}(C_\phi,X) \arrow[r, "\mathcal{V}_{\mathcal{F}}(c_\phi,X)"] \arrow[d, "\mathcal{V}_{\mathcal{F}}(C_\phi,f)"'] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),X) \arrow[d, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),f)"] \\
\mathcal{V}_{\mathcal{F}}(C_\phi,Y) \arrow[r, "\mathcal{V}_{\mathcal{F}}(c_\phi,Y)"] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y)
\end{tikzcd}$$

is a homotopy pullback square. If $f\colon X \longrightarrow Y$ is a pointwise fibration, then the induced map $g\colon \mathcal{V}_{\mathcal{F}}(C_\phi,X) \longrightarrow \mathcal{V}_{\mathcal{F}}(C_\phi,Y)\times_{\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y)}\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),X)$ is a fibration in $\mathcal{V}$. Here we use **f2**, so $\mathcal{F}$ is a monoidal model category when equipped with the pointwise model structure, see 4.4. The square in 5.6 is therefore a homotopy pullback square if and only if g has the right lifting property with respect to the generating cofibrations in $\mathcal{V}$. By adjointness, this holds if and only if f has the right lifting property with respect to $\mathcal{H}$. □

Lemma 5.7. *Let $f\colon X \longrightarrow Y$ be an hf-fibration. Then*

$$\begin{tikzcd}
X(v) \arrow[r] \arrow[d, "f(v)"'] & \hbar(X)(v) \arrow[d, "\hbar(f)(v)"] \\
Y(v) \arrow[r] & \hbar(Y)(v)
\end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for every object v in $\mathbf{f}\mathcal{V}$.

Proof. Let $f\colon X \twoheadrightarrow Y$ be a pointwise fibration of pointwise fibrant functors. From 3.33, $\hbar(f)(v)$ is a filtered colimit of fibrations of fibrant objects in $\mathcal{V}$, and therefore a fibration of fibrant objects by 3.11. This uses properties **f1** and **f3**. The square in the lemma is therefore a homotopy pullback square if and only if $X(v) \longrightarrow Y(v)\times_{\operatorname{colim} Y(a)} \operatorname{colim} X(a)$ is a weak equivalence in $\mathcal{V}$. Up to isomorphism, the colimit is taken over the category of finite subcomplexes $(\alpha\colon v \overset{\sim}{\rightarrowtail} a, \alpha'\colon a \overset{\sim}{\rightarrowtail} Fv)$ in $\operatorname{Ob}\operatorname{ac}(v,F)$. Note that this colimit is filtered by 3.29. Filtered colimits commute with pullbacks in $\mathcal{V}$ by assumption, so the map in question is a filtered colimit of maps $X(v) \longrightarrow Y(v)\times_{Y(a)} X(a)$ for acyclic cofibrations $\alpha\colon v \overset{\sim}{\rightarrowtail} a$ in $\mathbf{f}\mathcal{V}$. If f is an hf-fibration, then by 5.6 the map in question is a filtered colimit of weak equivalences in $\mathcal{V}$, and hence a weak equivalence.

If $f\colon X \twoheadrightarrow Y$ is any pointwise fibration, use the factorizations in the pointwise model structure to construct a commutative square

$$\begin{tikzcd} X \arrow[r, tail, "\sim", "g"'] \arrow[d, two heads, "f"'] & X' \arrow[d, two heads, "f'"] \\ Y \arrow[r, tail, "\sim", "h"'] & Y' \end{tikzcd}$$

for f' a pointwise fibration of pointwise fibrant functors, and g and f pointwise acyclic cofibrations. Note that f is an hf-fibration if and only f' is, see 3.19 and 5.6. The maps g and h are pointwise weak equivalences, hence $\hbar(g)(v)$ and $\hbar(h)(v)$ are weak equivalences in $\mathcal{V}$ for every v. The square in question is therefore a homotopy pullback square by the previous case. □

Corollary 5.8. *A map is an hf-fibration and an hf-equivalence if and only if it is a pointwise acyclic fibration.*

Proof. If f is an hf-fibration, it is a pointwise fibration by definition. If f is also an hf-equivalence, it is a pointwise weak equivalence by 3.19 and 5.7. A pointwise acyclic fibration is a fibration of homotopy functors by 3.19 and 5.6. □

5.3. **The homotopy functor theorem.** Before we prove that hf-equivalences and hf-fibrations are part of a model structure, let us first consider the relative $\mathcal{H}$-cell complexes. In order to apply [7, 2.1.19], these have to be hf-equivalences.

Lemma 5.9. *The maps in $\mathcal{H}$-cell are hf-equivalences.*

Proof. Let $\phi\colon v \xrightarrow{\sim} w$ be an acyclic cofibration in $\mathbf{f}\mathcal{V}$. Then the induced map $\mathcal{V}(\phi,-)\colon \mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ is an hf-equivalence, because $\hbar(\mathcal{V}(\phi,-))(v)$ is naturally isomorphic to $\mathcal{V}(\phi,\Phi v)$, ϕ is an acyclic cofibration and Φv is fibrant. Pointwise weak equivalences are hf-equivalences by 5.3, so $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ is an hf-equivalence.
Let $i\colon si \rightarrowtail ti$ be a generating cofibration in $\mathcal{V}$. Consider the diagram

$$\begin{tikzcd} \mathcal{V}(w,-)\otimes si \arrow[r, "c_\phi\otimes si"] \arrow[d, "\mathcal{V}(w,-)\otimes i"'] & C_\phi \otimes si \arrow[d, "C_\phi\otimes i"] \\ \mathcal{V}(w,-)\otimes ti \arrow[r, "c_\phi\otimes ti"] & C_\phi \otimes ti \end{tikzcd}$$

and the pushout product map $c_\phi\Box i$. The functor $\hbar$ commutes with pushouts and the action of $\mathcal{V}$ by 5.1, thus $\hbar(c_\phi\Box i)(u)$ is the pushout product map obtained from $\hbar(c_\phi)(u)$ and i. Since $\mathcal{V}$ is strongly monoidal, it follows that $\hbar(c_\phi)(u)\Box i$ is a weak equivalence. Hence the maps in $\mathcal{H}$ are hf-equivalences. The general case of a map in $\mathcal{H}$-cell follows using similar arguments and 5.4. □

Theorem 5.10. *Let $\mathcal{V}$ be a monoidal* **sSet***-model category which is weakly finitely generated, and let* $\mathbf{f}\mathcal{V}$ *be a full sub-$\mathcal{V}$-category of $\mathcal{V}$ satisfying* **f1**, **f2** *and* **f3**. *Suppose the monoid axiom holds in $\mathcal{V}$, pullbacks commute with filtered colimits in $\mathcal{V}$, and Δ^1 is finite in $\mathcal{V}$. Suppose also that $\mathcal{V}$ is strongly monoidal, and that relative J'-cell complexes are monomorphisms. Then $\mathcal{F}$ is a weakly finitely generated model category, with hf-equivalences as weak equivalences, hf-fibrations as fibrations, and cofibrations as cofibrations.*

Proof. Again we use [7, 2.1.19]. The set of generating cofibrations is $\mathcal{P}_I$, and the set of generating acyclic cofibrations is the union $\mathcal{P}_J \cup \mathcal{H}$. It is clear that the class of equivalences of homotopy functors has the "two out of three" property and is closed under retracts. It is also clear that the domains of the maps in $\mathcal{H}$ are finite, because finite objects are closed under pushouts and tensoring with finite objects. Here we use the assumption that Δ^1 is finite in $\mathcal{V}$. The other properties which have to be checked are either obvious or follow from 5.8, 5.9 and the corresponding theorem for the pointwise model structure 4.2. □

The model structure in 5.10 is the *homotopy functor* model structure. To emphasize the model structure, we use the notation $\mathcal{F}_{\mathrm{hf}}$. An hf-equivalence is denoted by $\xrightarrow{\sim\mathrm{hf}}$ and an hf-fibration by $\overset{\mathrm{hf}}{\twoheadrightarrow}$. Likewise, we use the notations $\mathcal{F}_{\mathrm{pt}}$, $\xrightarrow{\sim\mathrm{pt}}$ and $\overset{\mathrm{pt}}{\twoheadrightarrow}$ for the pointwise model structure. The identity induces a left Quillen functor $\mathrm{Id}_{\mathcal{F}}\colon \mathcal{F}_{\mathrm{pt}} \longrightarrow \mathcal{F}_{\mathrm{hf}}$.

Lemma 5.11. *Let $\mathcal{V}$ and* $\mathbf{f}\mathcal{V}$ *be as in 5.10, and assume* $\mathbf{f}\mathcal{V}$ *satisfies* **f4**. *Then $\mathcal{F}_{\mathrm{hf}}$ is a monoidal $\mathcal{F}_{\mathrm{pt}}$-model category.*

Proof. Condition **f2** is used to construct the smash product on $\mathcal{F}$, and 4.4 holds. To prove that the homotopy functor model structure is monoidal, it suffices to show that the pushout product map of a map $c_\phi \Box i$ in $\mathcal{H}$ (where $\phi\colon v \overset{\sim}{\rightarrowtail} w$) and a generating cofibration $\mathcal{V}(u,-)\otimes j$ is an hf-equivalence. It is straightforward to check that this pushout product map coincides with the pushout product map $c_{\phi\otimes u}\Box f$, where f is the pushout product map in $\mathcal{V}$ of i and j. Since $\mathcal{V}$ is a monoidal model category and $\mathbf{f}\mathcal{V}$ satisfies **f2** and **f4**, $\phi\otimes u$ is an acyclic cofibration in $\mathbf{f}\mathcal{V}$. Hence the map in question is an hf-equivalence. □

Lemma 5.12. *The homotopy functor model structure is left proper. If $\mathcal{V}$ is right proper, then the homotopy functor model structure is right proper.*

Proof. For left properness, let $i\colon Y \longrightarrow Z \cup_X Y$ be the cobase change of an hf-equivalence $g\colon X \xrightarrow{\sim\mathrm{hf}} Z$ along a cofibration $f\colon X \rightarrowtail Y$. Factor g as a cofibration $h\colon X \rightarrowtail T$, followed by a pointwise acyclic fibration $p\colon T \overset{\sim}{\twoheadrightarrow} Z$. Then g is an hf-equivalence, so in fact an acyclic cofibration in the homotopy functor model structure. Since these maps are closed under cobase change, i factors as an acyclic cofibration (in the homotopy functor model structure), followed by the cobase change of p. Now the cobase change of p is a pointwise weak equivalence, because the cobase change of f along g is a cofibration and the pointwise model structure is left proper by 4.8 if $\mathcal{V}$ is strongly left proper.

A slightly stronger property than right properness holds. Consider the maps $f\colon X \xrightarrow{\mathrm{pt}} Z$ and $g\colon Y \xrightarrow{\sim \mathrm{hf}} Z$. We claim the base change i of g along f is an hf-equivalence. Let us shorten the notation for a moment by denoting the fibrant replacement $\mathcal{V}$-functor $R^{J'}$ from 3.27 simply as R. To prove that i is an hf-equivalence, factor $Rf\colon RX \longrightarrow RZ$ as $h\colon RX \xrightarrow{\sim \mathrm{pt}} T$ followed by $p\colon T \xrightarrow{\mathrm{pt}} RZ$. Then $\hbar(p)(v)$ is a fibration of fibrant objects for any v by 3.11. Moreover, $\hbar(h)(v)$ and $\hbar(\rho_{Z(v)})\colon \hbar(Z)(v) \longrightarrow \hbar(RZ)(v)$ are weak equivalences for any v. Hence the base change of the weak equivalence $\hbar(\rho_{Z(v)}) \circ \hbar(g)(v)$ along $\hbar(b)(v)$ is a weak equivalence, using that $\mathcal{V}$ is right proper. Note that the base change map factors as $\hbar(h)(v)$ composed with the base change of $\hbar(g)(v)$ along $\hbar(f)(v)$, i.e. $\hbar(i)(v)$ since pullbacks commute with filtered colimits. It follows that i is an hf-equivalence. □

5.4. **Homotopy functors.** We end this section with a discussion of homotopy functors. If **f4** and **f5** hold, then a fibrant $\mathcal{V}$-functor X in $\mathcal{F}_{\mathrm{hf}}$ is a homotopy functor for the following reason. By 5.6, X maps acyclic cofibrations to weak equivalences. Using **f4** and **f5**, every weak equivalence in $\mathbf{f}\mathcal{V}$ can be factored as an acyclic cofibration in $\mathbf{f}\mathcal{V}$, followed by a simplicial homotopy equivalence in $\mathbf{f}\mathcal{V}$. Because X preserves simplicial homotopy equivalences by 2.11, X thus preserves arbitrary weak equivalences in $\mathbf{f}\mathcal{V}$. Conversely, any $\mathcal{V}$-functor which is pointwise fibrant and a homotopy functor is fibrant in $\mathcal{F}_{\mathrm{hf}}$. Therefore we regard the expressions “pointwise fibrant homotopy functor” and “fibrant in $\mathcal{F}_{\mathrm{hf}}$” as synonymous.

Next we define a fibrant replacement functor in $\mathcal{F}_{\mathrm{hf}}$ which allows to replace our definition of hf-equivalence by a better one. Unfortunately, we can only prove that this construction works under additional assumptions.

Definition 5.13. For $X \in \mathrm{Ob}\,\mathcal{F}$, define X^h as the composition $\mathbb{I}_*X \circ R$, where $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ is the inclusion and $R := R^{J'}$ is the fibrant replacement $\mathcal{V}$-functor constructed in 3.3.2.

For pointed simplicial sets, Lydakis uses in [13, 8.6] the composition “singular complex” applied to “geometric realization” as an enriched fibrant replacement functor. This functor is very well-behaved, since it preserves fibrations, weak equivalences and finite limits.

Lemma 5.14. *The map $X \longrightarrow X^h$ is a $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors, which is part of a natural transformation* $\mathrm{Id} \longrightarrow (-)^h$ *of endofunctors on $\mathcal{F}$. The functor $(-)^h$ commutes with colimits and the action of $\mathcal{V}$.*

Proof. The first two statements follow from 3.31 and properties of enriched Kan extension. For the last statement, use that coends commute with colimits and the action of $\mathcal{V}$, which are pointwise constructions. □

Lemma 5.15. *Assume that* **f4** *holds, and let $X \in \mathrm{Ob}\,\mathcal{F}$ be cofibrant. Then the weak equivalence $\omega_v\colon \Phi v \xrightarrow{\sim} Rv$ induces a weak equivalence $\hbar(X)(v) \longrightarrow X^h(v)$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Since $\mathcal{V}$ is monoidal and **f4** holds, the map $\mathcal{V}(w, \Phi v) \longrightarrow \mathcal{V}(w, Rv)$ induced by the canonical weak equivalence $\omega_v\colon \Phi v \xrightarrow{\sim} Rv$ is a weak equivalence for every $w \in \mathbf{f}\mathcal{V}$. Now express X as a retract of a $\mathcal{P}_I$-cell complex. The lemma follows then by induction, because $\mathcal{V}$ is strongly monoidal. □

Corollary 5.16. *Assume* **f4** *is satisfied. If X is cofibrant, then $R \circ X^h$ is a pointwise fibrant homotopy functor and the natural map $X \longrightarrow R \circ X^h$ is an hf-equivalence. A map f of cofibrant functors is an hf-equivalence if and only if f^h is a pointwise weak equivalence.*

Proof. The second claim follows directly from 5.15, while the first claim requires just a slight variation of the proof of 5.15. □

Lemma 5.17. *Suppose* **f0** *and* **f4** *are satisfied. For any $X \in \mathrm{Ob}\,\mathcal{V}$, the canonical map $\hbar(X)(v) \longrightarrow X^h(v)$ induced by the weak equivalence $\omega_v\colon \Phi v \xrightarrow{\sim} Rv$ is then a weak equivalence for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Note that 4.10 holds. Hence, using a cofibrant replacement in $\mathcal{F}_{\mathrm{pt}}$, it suffices to consider cofibrant functors. This case follows from 5.15. □

Corollary 5.18. *Assume* **f0** *and* **f4** *hold. A map f is then an hf-equivalence if and only if f^h is a pointwise weak equivalence. Furthermore, for any $X \in \mathrm{Ob}\,\mathcal{F}$, the functor $R \circ X^h$ is a pointwise fibrant homotopy functor and the natural map $X \longrightarrow R \circ X^h$ is an hf-equivalence.*

Proof. As in 5.16, using 4.10. □

Remark 5.19. If $\rho\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R$ has the property that its restriction to $\mathbf{f}\mathcal{V}$ takes values in cofibrant objects, then X^h is always a homotopy functor. The reason is that in a **sSet**-model category, any weak equivalence of objects which are both fibrant and cofibrant is automatically a simplicial homotopy equivalence, and $\mathcal{V}$-functors preserve such.

6. The stable model structure

We will construct the stable model structure on the category $\mathcal{F} = [\mathbf{f}\mathcal{V}, \mathcal{V}]$ with respect to some cofibrant object T of $\mathbf{f}\mathcal{V}$. For this, assume $\mathcal{V}$ and $\mathbf{f}\mathcal{V}$ are as in 5.10. But this is not enough. In addition, $\mathcal{V}$ has to be right proper and cellular, although it is quite probable that the result 6.15 is true without assuming cellularity, see the remark below [9, 4.12]. Furthermore, $\mathbf{f}\mathcal{V}$ has to satisfy **f4**, in order to apply a nice fibrant replacement in $\mathcal{F}_{\mathrm{hf}}$. Finally, we assume the adjoint pair $(-\otimes T, \mathcal{V}(T,-))$ is a Quillen equivalence on the stable model structure on spectra described in 6.15. This will be explained in detail below. Since T is contained in $\mathbf{f}\mathcal{V}$ and **f2** holds, the canonical functor $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ factors over the inclusion as $i\colon T\mathrm{Sph} \longrightarrow \mathbf{f}\mathcal{V}$. The resulting adjoint pair of functors is denoted (i_*, ev). One should think of ev as "evaluation on spheres".

6.1. **Stable equivalences.** We start by describing the stabilization process. For each $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$, the counit $\epsilon_T\mathcal{V}(v,-)\colon \mathcal{V}(T,\mathcal{V}(v,-))\otimes T \longrightarrow \mathcal{V}(v,-)$ and the natural isomorphism $\mathcal{V}(T,\mathcal{V}(v,-)) \cong \mathcal{V}(T\otimes v,-)$ allow us to define a map $\tau_v\colon \mathcal{V}(T\otimes v,-)\otimes T \longrightarrow \mathcal{V}(v,-)$ which is natural in v. If X is a $\mathcal{V}$-functor, then the induced map $\mathcal{V}_{\mathcal{F}}(\tau_v, X)\colon \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),X) \longrightarrow \mathcal{V}_{\mathcal{F}}(\mathcal{V}(T\otimes v,-)\otimes T, X)$ is natural in v and X. Using the enriched Yoneda lemma 2.1, one obtains a map $t_X(v)\colon X(v) \longrightarrow \mathcal{V}(T, X(T\otimes v))$. Let $\mathrm{Sh}\colon \mathcal{F} \longrightarrow \mathcal{F}$ denote the functor obtained by pre-composing with the $\mathcal{V}$-functor $T\otimes -\colon \mathbf{f}\mathcal{V} \longrightarrow \mathbf{f}\mathcal{V}$. Define $\mathbb{T}\colon \mathcal{F} \longrightarrow \mathcal{F}$ to be the composition $\mathcal{V}(T,-)\circ \mathrm{Sh}$, so that $\mathbb{T}(X)(v) = \mathcal{V}(T, X(T\otimes v))$. The maps $t_X(v)$ assemble to a $\mathcal{V}$-natural transformation $t\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \mathbb{T}$. Let $\mathbb{T}^\infty(X)$ denote the colimit of the sequence

$$X \xrightarrow{t(X)} \mathbb{T}(X) \xrightarrow{\mathbb{T}(t(X))} \mathbb{T}(\mathbb{T}(X)) \longrightarrow \cdots$$

Denote the canonical map $X \longrightarrow \mathbb{T}^\infty(X)$ by t_X^∞, and note that these maps assemble to a natural transformation $t^\infty\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \mathbb{T}^\infty$.

At first sight, the following seems to be a quite natural definition of stable weak equivalences: say f is a stable equivalence if $\mathbb{T}^\infty(f)$ is a pointwise weak equivalence. This works, provided f is a map of pointwise fibrant homotopy functors, but not in general. To define stable weak equivalences, we will use the filtered fibrant replacement $\Phi^{J'}$ described in 3.3.3. To simplify the notation, we set $\Phi := \Phi^{J'}$, and similarly for R and F, unless otherwise specified. Recall that $\hbar(X)$ is not necessarily a $\mathcal{V}$-functor for $X \in \mathrm{Ob}\,\mathcal{F}$. Nevertheless, we can stabilize $\hbar(X)$, since there are natural maps

$$\theta_v\colon T\otimes\Phi(v) \xrightarrow{\sim} \Phi(T\otimes v)$$

which are weak equivalences due to 3.32. Define $T'\colon \mathrm{Fun}(\mathbf{f}\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{Fun}(\mathbf{f}\mathcal{V},\mathcal{V})$ to be the functor that maps $X\colon \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$ to the composition

$$\mathbf{f}\mathcal{V} \xrightarrow{T\otimes -} \mathbf{f}\mathcal{V} \xrightarrow{X} \mathcal{V} \xrightarrow{\mathcal{V}(T,-)} \mathcal{V}$$

Consider the map $t'_{\hbar(X)}\colon \hbar X \longrightarrow \mathbb{T}'(\hbar(X))$ obtained pointwise as the adjoint of

$$\mathbb{I}_*X(\Phi(v)) \otimes T \longrightarrow \mathbb{I}_*X(T \otimes \Phi(v)) \xrightarrow{\mathbb{I}_*X(\theta_v)} \mathbb{I}_*X(\Phi(T \otimes v)).$$

Here the first map is the adjoint of

$$T \xrightarrow{\eta_{\Phi(v)}(T)} \mathcal{V}(\Phi(v), T \otimes \Phi(v)) \xrightarrow{\mathrm{hom}^{\mathbb{I}_*X}_{\Phi(v), T\otimes\Phi(v)}} \mathcal{V}(\mathbb{I}_*X(\Phi(v)), \mathbb{I}_*X(T \otimes \Phi(v))).$$

Lemma 6.1. *There is a natural transformation $t'_{\hbar}\colon \hbar \longrightarrow T' \circ \hbar$. The natural transformation $u\colon U \longrightarrow \hbar$, where $U\colon \mathcal{F} \longrightarrow \mathrm{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ is the forgetful functor, makes the following diagram commutative.*

$$\begin{tikzcd}
U \circ \mathrm{Id}_{\mathcal{F}} \arrow[r, "U\circ t"] \arrow[d, "u"'] & U \circ T = T' \circ U \arrow[d, "T'\circ u"] \\
\hbar \arrow[r, "t'_{\hbar}"] & T' \circ \hbar
\end{tikzcd}$$

Proof. The claim follows, once it is clear that the map $t_X(v)\colon X(v) \longrightarrow \mathbb{T}(X)(v)$ can also be defined as the adjoint (under tensoring with T) of the adjoint (under tensoring with $X(v)$) of the map

$$T \xrightarrow{\eta_v(T)} \mathcal{V}(v, T \otimes v) \xrightarrow{\mathrm{hom}^X_{v,T\otimes v}} \mathcal{V}(X(v), X(T \otimes v)).$$

This is proven in A.8. $\square$

Let $\mathbb{T}^\infty(\hbar(X))$ denote the colimit of the diagram

$$\hbar(X) \xrightarrow{t'_{\hbar(X)}} \mathbb{T}'(\hbar(X)) \xrightarrow{\mathbb{T}'(t'_{\hbar(X)})} \mathbb{T}'(\mathbb{T}'(\hbar(X))) \longrightarrow \cdots$$

and $t^\infty_{\hbar(X)}$ the canonical map $\hbar(X) \longrightarrow \mathbb{T}^\infty(\hbar(X))$.

Definition 6.2. A map f in $\mathcal{F}$ is a *stable equivalence* if $\mathbb{T}^\infty(\hbar(Rf))(v)$ is a weak equivalence in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Lemma 6.3. *Each hf-equivalence is a stable equivalence. The class of stable equivalences satisfies the "two out of three" axiom.*

There are canonical maps $X(v) \longrightarrow RX(v) \longrightarrow \hbar(RX)(v)$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. Consider the induced map $\mathbb{T}^\infty(X) \longrightarrow \mathbb{T}^\infty(\hbar(RX))$. The latter is a pointwise weak equivalence in some favorable cases.

Lemma 6.4. *Assume $X \in \mathrm{Ob}\,\mathcal{F}$ is a pointwise fibrant homotopy functor. Then $\mathbb{T}^\infty(X)(v) \longrightarrow \mathbb{T}^\infty(\hbar(RX))(v)$ is a weak equivalence in $\mathcal{V}$ for every $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Note that $X(v) \longrightarrow \hbar(RX)(v)$ is a weak equivalence of fibrant objects by the assumption on X. The functor $\mathcal{V}(T, -)$ preserves weak equivalences of fibrant objects since T is cofibrant and $\mathcal{V}$ is a monoidal model category. Consequently, $\mathbb{T}^\infty(X(v) \longrightarrow \hbar(RX)(v))$ is a sequential colimit of weak equivalences, and the result follows by 3.11. $\square$

Corollary 6.5. *A map f between pointwise fibrant homotopy functors is a stable equivalence if and only if $\mathbb{T}^\infty(f)$ is a pointwise weak equivalence.*

6.2. **Stable fibrations.** The set of generating acyclic cofibrations for the class of stable equivalences is defined as follows. Let $\tau_v\colon \mathcal{V}(T\otimes v,-)\otimes T \longrightarrow \mathcal{V}(v,-)$ be the canonical map described in 6.1. By using the simplicial mapping cylinder, one can factor τ_v as a cofibration $d_v\colon \mathcal{V}(T\otimes v,-)\otimes T \rightarrowtail D_v$ followed by a simplicial homotopy equivalence. Take any generating cofibration $i\colon si \rightarrowtail ti$ in $\mathcal{V}$ and form the pushout product map $d_v\Box i$. Let $\mathcal{D}$ denote the set $\{d_v\Box i\}$, where v runs through the set of objects in $\mathbf{f}\mathcal{V}$ and i runs through the set I of generating cofibrations in $\mathcal{V}$.

Definition 6.6. A map is called a *stable fibration* if it is an hf-fibration having the right lifting property with respect to the set $\mathcal{D}$.

Lemma 6.7. *If $f\colon X \longrightarrow Y$ is an hf-fibration, it is a stable fibration if and only if*

$$\begin{tikzcd} X(v) \arrow[r, "t_X(v)"] \arrow[d, "f(v)"'] & \mathbb{T}(X)(v) \arrow[d, "\mathbb{T}(f)(v)"] \\ Y(v) \arrow[r, "t_Y(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for any object v in $\mathbf{f}\mathcal{V}$.

Proof. The proof is formally the same as the proof of 5.6. $\square$

The rest of this section is devoted to prove that a stable fibration which is also a stable equivalence is a pointwise weak equivalence.

Lemma 6.8. *Assume $\mathcal{V}$ is right proper, and that filtered colimits commute with pullbacks in $\mathcal{V}$. Let $f\colon X \longrightarrow Y$ be a stable fibration. Then*

$$\begin{tikzcd} \hbar(RX)(v) \arrow[r, "t'_{\hbar(RX)}(v)"] \arrow[d, "\hbar(Rf)(v)"'] & \mathbb{T}'(\hbar(RX))(v) \arrow[d, "\mathbb{T}'(\hbar(Rf))(v)"] \\ \hbar(RY)(v) \arrow[r, "t'_{\hbar(RY)}(v)"] & \mathbb{T}'(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for any $v\in \operatorname{Ob}\mathbf{f}\mathcal{V}$.

The proof of 6.8 uses 6.9, 6.10 and 6.11.

Lemma 6.9. *Let $f\colon X \longrightarrow Y$ be a pointwise fibration, and assume $\mathcal{V}$ is right proper. Then the following is a homotopy pullback square in $\mathcal{V}$ for any $v\in \operatorname{Ob}\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} \mathcal{V}(T,X(v)) \arrow[r, "\mathcal{V}(T,\rho_{X(v)})"] \arrow[d, "\mathcal{V}(T,f(v))"'] & \mathcal{V}(T,RX(v)) \arrow[d, "\mathcal{V}(T,Rf(v))"] \\ \mathcal{V}(T,Y(v)) \arrow[r, "\mathcal{V}(T,\rho_{Y(v)})"] & \mathcal{V}(T,RY(v)) \end{tikzcd}$$

Proof. We can factor the composition $X(v) \xrightarrow{f(v)} Y(v) \underset{\rho_{Y(v)}}{\overset{\sim}{\rightarrowtail}} RY(v)$ as an acyclic cofibration $g\colon X(v) \overset{\sim}{\rightarrowtail} A$ followed by a fibration $q\colon A \twoheadrightarrow RY(v)$. Then there exists a lift in the diagram

$$\begin{tikzcd} X(v) \arrow[r, tail, "\sim", "g"'] \arrow[d, tail, "\rho_{X(v)}"', "\sim"] & A \arrow[d, two heads, "q"] \\ RX(v) \arrow[r, "Rf(v)"] & RY(v). \end{tikzcd}$$

That is, A allows a factorization $RX(v) \xrightarrow{\sim} A \overset{q}{\twoheadrightarrow} RY(v)$ of $Rf(v)$ as a weak equivalence followed by a fibration. Let B denote the pullback of the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, two heads, "q'"'] & A \arrow[d, two heads, "q"] \\ Y(v) \arrow[r, tail, "\sim", "\rho(Y(v))"'] & RY(v) \end{tikzcd}$$

Then g and $f(v)$ induce a map $h\colon X(v) \longrightarrow B$. The model category $\mathcal{V}$ is right proper, so $B \longrightarrow A$ is a weak equivalence. Therefore h is a weak equivalence. To conclude, it suffices by 3.20 to prove that $\mathcal{V}(T, h)$ is a weak equivalence. Because T is cofibrant and $\mathcal{V}$ is monoidal, $\mathcal{V}(T, -)$ is a right Quillen functor. The first step is to consider the pullback

$$\begin{tikzcd} C \arrow[r, two heads, "f(v)'"] \arrow[d, two heads, "q''"'] & B \arrow[d, two heads, "q'"] \\ X(v) \arrow[r, two heads, "f(v)"] & Y(v) \end{tikzcd}$$

Note that $\mathrm{id}_{X(v)}$ and h induce a map $X(v) \longrightarrow C$, which can be factored as an acyclic cofibration $s\colon X(v) \overset{\sim}{\rightarrowtail} D$ followed by a fibration $p\colon D \twoheadrightarrow C$. Since $q\colon A \twoheadrightarrow RY(v)$ is a fibration, also its base change $C \twoheadrightarrow X(v)$ is a fibration. Hence the composition $r\colon D \overset{p}{\twoheadrightarrow} C \overset{q''}{\twoheadrightarrow} X(v)$ is a fibration having as section the acyclic cofibration $s\colon X(v) \overset{\sim}{\rightarrowtail} D$. It follows that r is an acyclic fibration, so $\mathcal{V}(T, r)$ is an acyclic fibration with section $\mathcal{V}(T, s)$. Hence $\mathcal{V}(T, s)$ is a weak equivalence. Note that $\mathcal{V}(T, h)$ factors over $\mathcal{V}(T, s)$, so it remains to observe that the image under $\mathcal{V}(T, -)$ of the composition $c\colon D \overset{p}{\twoheadrightarrow} C \overset{f(v)'}{\twoheadrightarrow} B$ is a weak equivalence. That composition is a fibration since f is a pointwise fibration, and a weak equivalence because h and s are weak equivalences. Hence $\mathcal{V}(T, c)$ is an acyclic fibration, completing the proof. $\square$

Corollary 6.10. *Suppose $\mathcal{V}$ is right proper and $f\colon X \longrightarrow Y$ is a stable fibration. Then the following is a homotopy pullback square for every $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd}
RX(v) \arrow[r, "t_{RX}(v)"] \arrow[d, "Rf(v)"'] & \mathbb{T}(RX)(v) \arrow[d, "\mathbb{T}(Rf)(v)"] \\
RY(v) \arrow[r, "t_{RY}(v)"] & \mathbb{T}(RY)(v)
\end{tikzcd}$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
X(v) \arrow[rr, "t_X(v)"] \arrow[dr, "\rho_{X(v)}"'] \arrow[dd, "f(v)"'] & & \mathbb{T}(X)(v) \arrow[dr, "\mathbb{T}(\rho_{X(v)})"] \arrow[dd, "\mathbb{T}(f)(v)"' near start] & \\
& RX(v) \arrow[rr, "t_{RX}(v)" near start] \arrow[dd, "Rf(v)"' near start] & & \mathbb{T}(RX)(v) \arrow[dd, "\mathbb{T}(Rf)(v)"] \\
Y(v) \arrow[rr, "t_Y(v)" near end] \arrow[dr, "\rho_{Y(v)}"'] & & \mathbb{T}(Y)(v) \arrow[dr, "\mathbb{T}(\rho_{Y(v)})"] & \\
& RY(v) \arrow[rr, "t_{RY}(v)"] & & \mathbb{T}(RY)(v)
\end{tikzcd}$$

The right hand square of the cube is a homotopy pullback square by 6.9. Likewise for the square in the back using the assumption on f. In the left hand square the horizontal maps are weak equivalences. Hence the square in question is a homotopy pullback square by 3.19. □

Lemma 6.11. *Assume $f\colon X \longrightarrow Y$ is a stable fibration of pointwise fibrant functors, and filtered colimits commute with pullbacks in $\mathcal{V}$. Then*

$$\begin{tikzcd}
\hbar(X)(v) \arrow[r, "t'_{\hbar(X)}(v)"] \arrow[d, "\hbar(f)(v)"'] & \mathbb{T}'(\hbar(X))(v) \arrow[d, "\mathbb{T}'(\hbar(f))(v)"] \\
\hbar(Y)(v) \arrow[r, "t'_{\hbar(Y)}(v)"] & \mathbb{T}'(\hbar(Y))(v)
\end{tikzcd}$$

is a homotopy pullback square for every $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.

Proof. The square above decomposes up to isomorphism into two squares.

$$\begin{tikzcd}
\operatorname{colim}_{\mathrm{ac}(v,R)} X(a) \arrow[r, "\operatorname{colim} t_X(a)"] \arrow[d, "\operatorname{colim} f(a)"'] & \operatorname{colim}_{\mathrm{ac}(v,R)} \mathbb{T}(X)(a) \arrow[r, "c_X"] \arrow[d, "\operatorname{colim} \mathbb{T}(f)(a)"'] & \operatorname{colim}_{\mathrm{ac}(T\otimes v,R)} \mathcal{V}(T, X(b)) \arrow[d, "\operatorname{colim} \mathcal{V}(T,f(b))"] \\
\operatorname{colim}_{\mathrm{ac}(v,R)} Y(a) \arrow[r, "\operatorname{colim} t_Y(a)"] & \operatorname{colim}_{\mathrm{ac}(v,R)} \mathbb{T}(Y)(a) \arrow[r, "c_Y"] & \operatorname{colim}_{\mathrm{ac}(T\otimes v,R)} \mathcal{V}(T, Y(b))
\end{tikzcd}$$

Recall from 3.3.3 that $\mathrm{ac}(v, R)$ consists of certain factorizations $v \overset{\sim}{\rightarrowtail} a \longrightarrow Rv$. Similarly, objects in $\mathrm{ac}(T \otimes v, R)$ are of the form $T \otimes v \overset{\sim}{\rightarrowtail} b \longrightarrow R(T \otimes v)$.

The maps c_X and c_Y are obtained from $\Theta_v\colon \mathrm{ac}(v,R) \longrightarrow \mathrm{ac}(T\otimes v,R)$, which maps $v \overset{\sim}{\rightarrowtail} a \longrightarrow Rv$ to $T\otimes v \overset{\sim}{\rightarrowtail} T\otimes a \longrightarrow T\otimes Rv \longrightarrow R(T\otimes v)$, and the natural transformation $\mathcal{V}(T, X(T\otimes\Phi_v)) \longrightarrow \mathcal{V}(T, X(\Phi_{T\otimes v}\circ\Theta_v))$ which consists of identity maps. From 3.29, one can replace the indexing categories $\mathrm{ac}(v,R)$ and $\mathrm{ac}(T\otimes v,R)$ by filtered ones, namely $\mathrm{ac}(v,F)$ and $\mathrm{ac}(T\otimes v,F)$. Then all the vertical maps are fibrations of fibrant objects, because f is a pointwise fibration of pointwise fibrant functors and 3.11 holds. Thus the left hand square is a homotopy pullback square if and only if the canonical map g from $\mathrm{colim}_{\mathrm{ac}(v,F)} X(a)$ to the pullback is a weak equivalence. Filtered colimits commute with pullbacks, so g is the filtered colimit of the canonical maps induced by the squares

$$\begin{tikzcd}
X(a) \arrow[r, "t_X(a)"] \arrow[d, "f(a)"'] & \mathbb{T}(X)(a) \arrow[d, "\mathbb{T}(f)(a)"] \\
Y(a) \arrow[r, "t_Y(a)"] & \mathbb{T}(Y)(a)
\end{tikzcd}$$

for finite sub-complexes $v \overset{\sim}{\rightarrowtail} a$ of $v \overset{\iota_v}{\rightarrowtail} Fv$. These squares are all homotopy pullback squares: f is a stable fibration, and the vertical maps are fibrations of fibrant objects. It follows that g is a filtered colimit of weak equivalences, so the left hand square is a homotopy pullback square.

That the right hand square is a homotopy pullback depends on whether f is an hf-fibration. We claim that

$$\begin{tikzcd}
X(T\otimes a) \arrow[r] \arrow[d, "f(T\otimes a)"'] & \mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} X(b) \arrow[d, "\mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} f(b)"] \\
Y(T\otimes a) \arrow[r] & \mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} Y(b)
\end{tikzcd}$$

is a homotopy pullback square for all $v \underset{\alpha}{\overset{\sim}{\rightarrowtail}} a \longrightarrow Rv$ in $\mathrm{ac}(v,R)$. Denote the full subcategory of $\mathrm{ac}(T\otimes v,R)$ consisting of $T\otimes v \underset{\beta}{\overset{\sim}{\rightarrowtail}} b \longrightarrow R(T\otimes v)$ where β factors as $T\otimes v \overset{T\otimes\alpha}{\rightarrowtail} T\otimes a \rightarrowtail b$ by $\mathrm{ac}(T\otimes v,R)_a$. Using pushouts, this category is a final subcategory of $\mathrm{ac}(T\otimes v,R)$. Hence we may assume the colimit in the square above is indexed by $\mathrm{ac}(T\otimes v,R)_a$. As in the proof of 5.7, it follows that the square above is a homotopy pullback square for all $v \underset{\alpha}{\overset{\sim}{\rightarrowtail}} a \longrightarrow Rv$ in $\mathrm{ac}(v,R)$. Here we use that f is an hf-fibration. The right hand square in the main diagram is then a homotopy pullback square, using the by now standard argument for filtered colimits of homotopy pullback squares. $\square$

A proof of 6.8 follows.

Proof. By 6.10, we may assume f is a stable fibration of pointwise fibrant functors. The result is therefore a consequence of 6.11. $\square$

Lemma 6.12. *Suppose that $\mathcal{V}$ is right proper and filtered colimits commute with pullbacks in $\mathcal{V}$. Let $f\colon X \longrightarrow Y$ be a stable fibration. Then*

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \mathbb{T}^\infty(\hbar(RX))(v) \arrow[d, "\mathbb{T}^\infty(\hbar(Rf))(v)"] \\ Y(v) \arrow[r] & \mathbb{T}^\infty(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$.

Proof. Let $f\colon X \longrightarrow Y$ be a stable fibration. Factor Rf as $RX \xrightarrow{\sim \mathrm{pt}} Z$ followed by $g\colon Z \xrightarrow{\mathrm{pt}} RY$. Here g is a stable fibration of pointwise fibrant functors by 5.6 and 6.10. The square in 6.12 is a homotopy pullback square if and only if

$$\begin{tikzcd} Z(v) \arrow[r] \arrow[d, "g(v)"'] & \mathbb{T}^\infty(\hbar(Z))(v) \arrow[d, "\mathbb{T}^\infty(\hbar(g))(v)"] \\ RY(v) \arrow[r] & \mathbb{T}^\infty(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square, where the vertical maps are fibrations of fibrant objects. This square decomposes as

$$\begin{tikzcd} Z(v) \arrow[r] \arrow[d, "g(v)"'] & \hbar(Z)(v) \arrow[r] \arrow[d, "\hbar(g)(v)"'] & \mathbb{T}^\infty(\hbar(Z))(v) \arrow[d, "\mathbb{T}^\infty(\hbar(g))(v)"] \\ RY(v) \arrow[r] & \hbar(RY)(v) \arrow[r] & \mathbb{T}^\infty(\hbar(RY))(v) \end{tikzcd}$$

The left hand square is a homotopy pullback square by 5.7. The functor $\mathcal{V}(T,-)$ preserves homotopy pullback squares for which the vertical maps are fibrations of fibrant objects. It follows, using 6.8, that the right hand square is a homotopy pullback square. □

Corollary 6.13. *A map is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. Let $f\colon X \longrightarrow Y$ be a stable fibration and a stable equivalence. Stable fibrations are in particular pointwise fibrations, so it remains to prove that f is a pointwise weak equivalence. By 6.12, the diagram

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \mathbb{T}^\infty(\hbar(RX))(v) \arrow[d, "\mathbb{T}^\infty(\hbar(Rf))(v)"] \\ Y(v) \arrow[r] & \mathbb{T}^\infty(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$. Since f is a stable equivalence, the vertical map on the right hand side is a weak equivalence. Hence f is a pointwise weak equivalence.

Conversely, let $f\colon X \longrightarrow Y$ be a pointwise acyclic fibration. Then f is a stable equivalence according to 6.3. By 5.8, it follows that f is an hf-fibration. Using 6.7, it remains to prove that

$$\begin{tikzcd} X(v) \arrow[r, "t(X)(v)"] \arrow[d, "f(v)"'] & \mathbb{T}(X)(v) \arrow[d, "\mathbb{T}(f)(v)"] \\ Y(v) \arrow[r, "t(Y)(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathbf{f}\mathcal{V}$. The maps $f(v)$ and $f(T \otimes v)$ are acyclic fibrations and $\mathcal{V}(T, -)$ preserves acyclic fibrations. This implies that $\mathbb{T}(f)(v)$ is an acyclic fibration. $\square$

6.3. **Comparison with T-spectra.** To proceed with the stable model structure, we compare the stabilization of enriched functors in 6.1 with the stabilization of spectra as described in [9, section 4]. Notations from [9] will be used throughout. Recall the two functors "suspension with T" on the category $\mathrm{Sp}(\mathcal{V}, T)$, the functor $- \otimes T$ coming from the $\mathcal{V}$-module structure on $\mathrm{Sp}(\mathcal{V}, T)$ and Σ_T defined as the prolongation of $- \otimes T\colon \mathcal{V} \longrightarrow \mathcal{V}$ using the identity natural transformation $\mathrm{id}\colon (- \otimes T) \circ (- \otimes T) \longrightarrow (- \otimes T) \circ (- \otimes T)$. In fact, $- \otimes T\colon \mathrm{Sp}(\mathcal{V}, T) \longrightarrow \mathrm{Sp}(\mathcal{V}, T)$ is the prolongation of $- \otimes T\colon \mathcal{V} \longrightarrow \mathcal{V}$ using the natural transformation $t\colon (- \otimes T) \circ (- \otimes T) \longrightarrow (- \otimes T) \circ (- \otimes T)$ defined by the coherence isomorphism

$$(A \otimes T) \otimes T \xrightarrow{\alpha_{A,T,T}} A \otimes (T \otimes T) \xrightarrow{A \otimes \sigma_{T,T}} A \otimes (T \otimes T) \xrightarrow{\alpha^{-1}_{A,T,T}} (A \otimes T) \otimes T.$$

See 2.13 for details. Let $\mathrm{Ev}_n := \mathrm{Ev}_{T^n}$ denote the functor evaluating a spectrum on T^n. Then the diagram

$$\begin{tikzcd} \mathrm{Sp}(\mathcal{V}, T) \arrow[r, "- \otimes T"] \arrow[d, "\Sigma_T"'] & \mathrm{Sp}(\mathcal{V}, T) \arrow[d, "\mathrm{Ev}_n"] \\ \mathrm{Sp}(\mathcal{V}, T) \arrow[r, "\mathrm{Ev}_n"] & \mathcal{V} \end{tikzcd}$$

commutes for any $n \geq 0$. Whence for any spectrum E, the spectra $E \otimes T$ and $\Sigma_T E$ differ only in their structure maps. This statement carries over to the adjoints $\mathcal{V}(T, -)\colon \mathrm{Sp}(\mathcal{V}, T) \longrightarrow \mathrm{Sp}(\mathcal{V}, T)$ and $\Omega_T\colon \mathrm{Sp}(\mathcal{V}, T) \longrightarrow \mathrm{Sp}(\mathcal{V}, T)$.

A technical problem is that the composition $T\mathrm{Sph} \hookrightarrow \mathcal{V} \xrightarrow{T \otimes -} \mathcal{V}$ does not factor over the inclusion $T\mathrm{Sph} \hookrightarrow \mathcal{V}$. Hence the shift functor $\mathrm{Sh}\colon \mathcal{F} \longrightarrow \mathcal{F}$ does not have a compatible analog in the category of spectra. Instead, one has the shift functor $\mathrm{sh}\colon \mathrm{Sp}(\mathcal{V}, T) \longrightarrow \mathrm{Sp}(\mathcal{V}, T)$ mapping the spectrum E to the spectrum $\mathrm{sh}(E)$ given by $(\mathrm{sh}(E))_n = E_{n+1}$. The nth structure map of $\mathrm{sh}(E)$ is simply e_{n+1}. Nevertheless the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{Sh}"] \arrow[d, "\mathrm{ev}"'] & \mathcal{F} \arrow[r, "\mathrm{ev}"] & \mathrm{Sp}(\mathcal{V},T) \arrow[d, "\mathrm{Ev}_n"] \\ \mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{sh}"] & \mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{Ev}_n"] & \mathcal{V} \end{tikzcd}$$

commutes for all $n \geq 0$.

The stabilization functor for spectra uses the fact that, for any spectrum E, the structure maps e define a map $s(E)\colon E \longrightarrow \mathrm{sh}(\Omega_T E)$ which is natural in E. In general, there is no map $E \longrightarrow \mathrm{sh}(\mathcal{V}(T,E))$, which is a motivation to introduce the functors Σ_T and Ω_T. Let us abbreviate the functor $\mathrm{sh} \circ \Omega_T$ by $S\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$. Then $s(E)_n\colon E_n \longrightarrow S(E)_n = \Omega_T E_{n+1}$ is adjoint to e_n. The stabilization $S^\infty(E)$ of a spectrum E is defined as the colimit of the diagram

$$E \xrightarrow{s(E)} S(E) \xrightarrow{S(s(E))} S(S(E)) \longrightarrow \cdots$$

and $s^\infty\colon E \longrightarrow S^\infty(E)$ is the canonical map. In [9, section 4], the notation $\iota\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow \Theta$ is used instead of $s\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow S$, and $j\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow \Theta^\infty$ instead of $s^\infty\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow S^\infty$. We let $l\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow L'$ denote a fibrant replacement functor for the pointwise model structure on $\mathrm{Sp}(\mathcal{V},T) = [T\mathrm{Sph},\mathcal{V}]$.

Definition 6.14. A map $f\colon E \longrightarrow F$ of spectra is a *stable equivalence* if $S^\infty(L'f)$ is a pointwise weak equivalence, and a *stable fibration* if it is a pointwise fibration and

$$\begin{tikzcd} E \arrow[r, "s^\infty(L'E)\circ l(E)"] \arrow[d, "f"'] & S^\infty(L'E) \arrow[d, "S^\infty(L'f)"] \\ F \arrow[r, "s^\infty(L'F)\circ l(F)"] & S^\infty(L'F) \end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure.

We summarize the results [9, 4.12, 4.14 and 6.5].

Theorem 6.15 (Hovey)**.** *Let $\mathcal{V}$ be an almost finitely generated, pointed, proper, and cellular monoidal model category. Let T be a cofibrant and $\mathcal{V}$-finite object in $\mathcal{V}$. Assume sequential colimits commute with pullbacks in $\mathcal{V}$. Then* $\mathrm{Sp}(\mathcal{V},T)$ *is an almost finitely generated proper $\mathcal{V}$-model category with stable equivalences as weak equivalences, stable fibrations as fibrations and cofibrations as cofibrations.*

The main technical input in the proof of 6.15 is the following lemma.

Lemma 6.16. *If E is pointwise fibrant, then $s^\infty(E)$ is a stable equivalence with a stably fibrant codomain.*

The natural maps $\mathrm{ev}(t^\infty(X))$ and $s^\infty(\mathrm{ev}(X))$ do not coincide. Compatibility of the two stabilization processes is therefore an issue, see 6.17 and 6.18 below.

Lemma 6.17. *Let $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$, and denote the functor $\mathcal{F} \longrightarrow \mathcal{F}$ mapping X to $X \circ (v \otimes -)$ by v^*. For $X \in \operatorname{Ob} \mathcal{F}$, the diagram*

$$\begin{tikzcd} X(v \otimes T^n) \arrow[r, "t(X)(v\otimes T^n)"] \arrow[dr, "s(\mathrm{ev}(v^*(X)))_n"'] & \mathcal{V}(T, X(T \otimes (v \otimes T^n))) = \mathbb{T}(X)(v \otimes T^n) \arrow[d, "\mathcal{V}(T,X(\gamma_{v,n}))"] \\ & \mathcal{V}(T, X(v \otimes T^{n+1})) \end{tikzcd}$$

commutes naturally in X, where $\gamma_{v,n}$ is the coherence isomorphism

$$T \otimes (v \otimes T^n) \xrightarrow{\alpha_{T,V,T^n}} (T \otimes v) \otimes T^n \xrightarrow{\sigma_{T,v}\otimes T^n} (v \otimes T) \otimes T^n \xrightarrow{\alpha^{-1}_{v,T,T^n}} v \otimes T^{n+1}.$$

Proof. The map $t(X)(v \otimes T^n)$ is by definition the adjoint of the composition

$$T \xrightarrow{\eta_{v\otimes T^n}(T)} \mathcal{V}(v\otimes T^n, T\otimes(v\otimes T^n)) \xrightarrow{\hom^X_{v\otimes T^n, T\otimes(v\otimes T^n)}} \mathcal{V}(X(v\otimes T^n), X(T\otimes(v\otimes T^n)))$$

and the map $s(\mathrm{ev}(v^*(X)))_n$ is by definition the adjoint of the composition

$$T \xrightarrow{\eta_{T^n}(T)} \mathcal{V}(T^n, T^{n+1}) \xrightarrow{\hom^{v^*(X)}_{T^n,T^{n+1}}} \mathcal{V}(X(v \otimes T^n), X(v \otimes T^{n+1}))$$

where $\hom^{v^*(X)}_{T^n,T^{n+1}} = \hom^{X\circ(v\otimes -)}_{T^n,T^{n+1}} = \hom^X_{v\otimes T^n, v\otimes T^{n+1}} \circ \hom^{v\otimes -}_{T^n,T^{n+1}}$. The claim follows from A.2. □

Corollary 6.18. *For all $X \in \operatorname{Ob} \mathcal{F}$ and $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$, there exists an isomorphism $\gamma \colon \mathbb{T}^\infty(X)(v \otimes T^n) \longrightarrow S^\infty(\mathrm{ev}(v^*(X)))_n$ rendering the diagram*

$$\begin{tikzcd} X(v \otimes T^n) \arrow[r, "t^\infty(X)(v\otimes T^n)"] \arrow[dr, "s^\infty(\mathrm{ev}(v^*(X)))_n"'] & \mathbb{T}^\infty(X)(v \otimes T^n) \arrow[d, "\gamma"] \\ & S^\infty(\mathrm{ev}(v^*(X)))_n \end{tikzcd}$$

commutative. Furthermore, the diagram is natural in X.

Proof. The two non-vertical maps in the diagram above are the vertical sequential compositions in the commutative diagram

$$\begin{tikzcd} X(v \otimes T^n) \arrow[r, "\mathrm{id}_{X(v\otimes T^n)}"] \arrow[d, "t(X)(v\otimes T^n)"'] & X(v \otimes T^n) \arrow[d, "s(\mathrm{ev}(v^*(X)))_n"] \\ \mathbb{T}(X)(v \otimes T^n) \arrow[r, "\mathcal{V}(T,X(\gamma^1))"] \arrow[d, "\mathbb{T}(t(X))(v\otimes T^n)"'] & \mathcal{V}(T, X(v \otimes T^{n+1})) \arrow[d, "\mathcal{V}(T,s(\mathrm{ev}(v^*(X)))_{n+1})"] \\ \mathbb{T}(\mathbb{T}(X))(v \otimes T^n) \arrow[r, "\mathcal{V}(T,\mathcal{V}(T,X(\gamma^2)))"] \arrow[d] & \mathcal{V}(T, \mathcal{V}(T, X(v \otimes T^{n+2}))) \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

Here γ^1 is the map $\gamma_{v,n}$ in 6.17. Note that the second square above is $\mathcal{V}(T,-)$ applied to the square

$$\begin{array}{ccc} X(T\otimes(v\otimes T^n)) & \xrightarrow{t(X)(t\otimes(v\otimes T^n))} & \mathbb{T}(X)(T\otimes(v\otimes T^n)) \\ \downarrow{\scriptstyle \mathcal{V}(T,X(\gamma_1))} & & \downarrow{\scriptstyle \mathcal{V}(T,X(\gamma_2))} \\ X(v\otimes T^{n+1}) & \xrightarrow{s(\mathrm{ev}(v^*(X)))_{n+1}} & \mathcal{V}(T,X(v\otimes T^{n+1})). \end{array}$$

Define γ^2 as the composition

$$T\otimes(T\otimes(v\otimes T^n)) \xrightarrow{T\otimes\gamma_{v,n}} T\otimes(v\otimes T^{n+1}) \xrightarrow{\gamma_{v,n+1}} v\otimes T^{n+1}.$$

Then the lower square commutes because of naturality and 6.17. Likewise one constructs γ^n, and γ is defined to be $\mathrm{colim}_n\,\gamma^n$. The result follows. $\square$

To use 6.18, we make the following definition. If v is an object of $\mathbf{f}\mathcal{V}$, the composition of functors $\mathbb{I}_*X\circ\Phi\circ(v\otimes-)=\hbar(X)\circ(v\otimes-)$ determines a T-spectrum $\overline{v^*X}$ for every $\mathcal{V}$-functor X. The nth term of $\overline{v^*X}$ is $\mathbb{I}_*X(\Phi(v\otimes T^n))$, the structure map $\overline{v^*X}_n\otimes T\longrightarrow \overline{v^*X}_{n+1}$ is the composition

$$\mathbb{I}_*X(\Phi(v\otimes T^n))\otimes T \xrightarrow{\mathrm{move}_T^{\mathbb{I}_*X}} \mathbb{I}_*X(\Phi(v\otimes T^n)\otimes T) \xrightarrow{\mathbb{I}_*X(\theta_{v\otimes T^n})} \mathbb{I}_*X(\Phi(v\otimes T^{n+1}))$$

up to an associativity isomorphism. This construction is functorial and commutes with colimits and the closed $\mathcal{V}$-module structures.

Lemma 6.19. *A map $f\colon X\longrightarrow Y$ in $\mathcal{F}$ is a stable equivalence if and only if the map $\overline{v^*f}\colon\overline{v^*X}\longrightarrow\overline{v^*Y}$ is a stable equivalence of T-spectra for every $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Let $f\colon X\longrightarrow Y$ be a stable equivalence, and pick $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$. By definition, the map $\mathbb{T}^\infty(\hbar(R\circ f))(v\otimes T^n)$ is a weak equivalence in $\mathcal{V}$. The isomorphism mentioned in 6.18 implies that this map is isomorphic to the map $S^\infty(\overline{v^*(R\circ f)'})_n$. Thus $\overline{v^*(R\circ f)}$ is a stable equivalence of T-spectra. Since $\overline{v^*(R\circ f)}$ is a pointwise fibrant replacement of $\overline{v^*f}$ in $\mathrm{Sp}(\mathcal{V},T)$, this implies that $\overline{v^*f}$ is a stable equivalence. The converse holds by running the argument backwards. $\square$

If additional conditions are satisfied, the characterization of stable equivalences can be improved in the sense that $\hbar$ becomes redundant. Note that the last characterization uses the axiom **f4**.

Corollary 6.20. *A map $f\colon X\longrightarrow Y$ of pointwise fibrant homotopy functors is a stable equivalence if and only if $\mathrm{ev}(v^*(f))$ is a stable equivalence of spectra for every object v of $\mathbf{f}\mathcal{V}$.*

Proof. By 6.4, f is a stable equivalence if and only if $\mathbb{T}^\infty(f)(w)\colon X(w)\longrightarrow Y(w))$ is a weak equivalence in $\mathcal{V}$ for every w. The proof now proceeds as for 6.19. $\square$

Corollary 6.21. *A map f of cofibrant functors is a stable equivalence if and only if $\mathrm{ev}(v^*(f^h))$ is a stable equivalence of spectra for all $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\sim\mathrm{hf}"] \arrow[d, "f"'] & RX^h \arrow[d, "Rf^h"] \\ Y \arrow[r, "\sim\mathrm{hf}"] & RY^h \end{tikzcd}$$

From 5.16 – which uses **f4** – and 6.3 we have that f is a stable equivalence if and only if Rf^h is a stable equivalence. Note that the latter map is a map of pointwise fibrant homotopy functors. It follows by 6.20 that f is a stable equivalence if and only if $\mathrm{ev}(v^*(Rf^h))$ is a stable equivalence in $\mathrm{Sp}(\mathcal{V},T)$ for every $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. From the equality $\mathrm{ev}(v^*(Rf^h)) = R\,\mathrm{ev}(v^*(f^h))$, and since pointwise weak equivalences of spectra are stable equivalences, one deduces that f is a stable equivalence if and only if $\mathrm{ev}(v^*(f^h))$ is a stable equivalence of spectra. □

6.4. **The generating stable equivalences.** Recall the stable model structure on $\mathrm{Sp}(\mathcal{V},T)$ from 6.15. In this structure, the adjoint pair of functors (Σ_T,Ω_T) is a Quillen equivalence by [9, 3.9]. In general, it is not clear whether the pair $(-\otimes T,\mathcal{V}(T,-))$ – which is more natural to consider when viewing spectra as $\mathcal{V}$-functors – is a Quillen equivalence. However, if the object T is *symmetric*, which roughly means that the cyclic permutation on $T\otimes T\otimes T$ is homotopic to the identity, then also the adjoint pair $(-\otimes T,\mathcal{V}(T,-))$ is a Quillen equivalence on $\mathrm{Sp}(\mathcal{V},T)$ by [9, 10.3]. We formulate the last property as a working hypothesis.

Hypothesis: The adjoint functor pair $(-\otimes T,\mathcal{V}(T,-))$ is a Quillen equivalence on the model category $\mathrm{Sp}(\mathcal{V},T)$ described in 6.15.

Lemma 6.22. *The maps in $\mathcal{D}$ are stable equivalences.*

Proof. A map $d_v\Box i$ in $\mathcal{D}$ is a cofibration of cofibrant functors, and by 6.21 a stable equivalence if and only if $\mathrm{ev}(w^*(d_v\Box i)^h) = \mathrm{ev}(w^*(d_v)^h)\Box i$ is a stable equivalence of spectra for all $w \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. We claim the latter holds if $f := \mathrm{ev}(w^*(d_v)^h)$ is a stable equivalence. Factor f as a cofibration g followed by a pointwise acyclic fibration p. The stable model structure on spectra is a $\mathcal{V}$-model structure by 6.15, hence $f\Box i$ factors as a cobase change of the stable acyclic cofibration $g\Box i$, followed by $p\Box i$. Since $\mathcal{V}$ is assumed to be strongly monoidal, $p\Box i$ is a pointwise weak equivalence.

It remains to prove that d_v is a stable equivalence. This condition is equivalent to τ_v being a stable equivalence. The latter factors by definition as

$$\mathcal{V}(T\otimes v,-)\otimes T \xrightarrow{\cong} \mathcal{V}(T,\mathcal{V}(v,-))\otimes T \xrightarrow{\epsilon_T V(v,-)} \mathcal{V}(v,-)$$

where ϵ_T is the counit $(-\otimes T)\circ\mathcal{V}(T,-) \longrightarrow \mathrm{Id}_{\mathcal{F}}$. We are reduced to prove that $\epsilon_T V(v,-)$ is a stable equivalence. By 6.21, this map is a stable equivalence if and only if $\mathrm{ev}(w^*(\epsilon_T\mathcal{V}(v,-))^h)$ is a stable equivalence of spectra for every w. Since ev commutes with the action and coaction of $\mathcal{V}$, $\mathrm{ev}(w^*(\epsilon_T\mathcal{V}(v,-))^h)$ coincides with $\epsilon_T\mathrm{ev}(w^*(\mathcal{V}(v,-))^h)$.

Let $q\colon Q \longrightarrow \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$ be a cofibrant replacement functor in the category of spectra, so $q(E)_n$ is an acyclic fibration in $\mathcal{V}$ for every spectrum E and $n \geq 0$. Consider the following diagram, where the notation is simplified.

$$\begin{tikzcd}
(Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))) \otimes T \arrow[r, "q\otimes T"] \arrow[d, "Q\epsilon_T"'] & \mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))) \otimes T \arrow[d, "\epsilon_T"] \\
Q\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))) \arrow[r, "q"] & \mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))
\end{tikzcd}$$

The composition

$$s^\infty \circ q \circ Q\epsilon_T\colon (Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))) \otimes T \longrightarrow S^\infty(\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))$$

is a stable equivalence of spectra by the hypothesis. The target of $s^\infty \circ q \circ Q\epsilon_T$ is stably fibrant, and its domain is cofibrant. Recall that $\mathcal{V}(T,-)$ commutes with filtered colimits. Up to an isomorphism, the stable weak equivalence

$$s^\infty \circ q\colon Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))) \longrightarrow S^\infty(\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))$$

is an adjoint of $s^\infty \circ q \circ Q\epsilon_T$. Thus $q \circ Q\epsilon_T$ is a stable equivalence of spectra. The map $q\otimes T$ is a pointwise weak equivalence, since $\mathcal{V}$ is strongly monoidal. Hence ϵ_T is a stable equivalence of spectra, which ends the proof. $\square$

Before considering maps in $\mathcal{D}$-cell, we need to record the following property of the stable model structure of spectra.

Lemma 6.23. *Let $f\colon E \longrightarrow F$ be a stable equivalence of spectra such that f_n is a retract of a map in* $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$*-cell for every $n \geq 0$. Then any cobase change of f is a stable equivalence.*

Proof. Let $g\colon E \longrightarrow G$ be a map of spectra. Factor g as $i\colon E \rightarrowtail T$ followed by $p\colon T \xrightarrow{\sim\ \mathrm{pt}} G$, and consider the diagram

$$\begin{tikzcd}
E \arrow[r, tail, "i"] \arrow[d, "f"'] & T \arrow[r, two heads, "p", "\sim\ \mathrm{pt}"'] \arrow[d, "f'"] & G \arrow[d, "f''"] \\
F \arrow[r, tail] & F\cup_E T \arrow[r, "p'"] & F\cup_E G.
\end{tikzcd}$$

The stable model structure on spectra is left proper by 6.15, hence f' is a stable equivalence. Pushouts are formed pointwise, so f'_n is a retract of a map belonging to $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell for every $n \geq 0$. By the assumption that $\mathcal{V}$ is strongly left proper, the cobase change p' of the pointwise weak equivalence p along f' is again a pointwise weak equivalence. Hence the map f'' is a stable equivalence. $\square$

Lemma 6.24. *The maps in $\mathcal{D}$-cell are stable equivalences.*

Proof. Let $f\colon X \longrightarrow Y$ be a map in $\mathcal{D}$-cell. First suppose that X is cofibrant. Then Y is automatically cofibrant. By 6.21, it suffices to prove that $\mathrm{ev}(v^*f^h)$ is a stable equivalence of spectra for every v. The functors ev, v^* and $(-)^h$ preserve colimits, hence $\mathrm{ev}(v^*f^h)$ is in $\mathrm{ev}(v^*(\mathcal{D}^h))$-cell. Every map in $\mathrm{ev}(v^*(\mathcal{D}^h))$ is of the type considered in 6.23, so cobase changes of these are stable equivalences of spectra. The stable model structure on spectra is almost finitely generated, which implies that stable equivalences of spectra are closed under sequential compositions. This proves the lemma for maps in $\mathcal{D}$-cell with cofibrant domain.

For f arbitrary, we will construct a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\sim"', "\mathrm{pt}"] & Y' \arrow[d, "\mathrm{pt}"', "\sim"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where X' is cofibrant and f' is a map in $\mathcal{P}_J \cup \mathcal{D}$-cell. It allows to finish the proof using the special case treated above. Without loss of generality, f is the sequential composition of $X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \longrightarrow X_n \longrightarrow \cdots$ where each f_n is a cobase change of a coproduct of maps in $\mathcal{D}$. We construct f' as a sequential composition. Consider a cofibrant replacement $g_0 \colon X_0' = X' \xrightarrow{\sim \mathrm{pt}} X$ of X. Let $z_0 \colon sZ_0 \longrightarrow tZ_0$ be the coproduct of maps in $\mathcal{D}$ of which f_0 is the cobase change, and let $a_0 \colon sZ_0 \longrightarrow X$ be the attaching map. The functor sZ_0 is cofibrant, hence a_0 lifts to a map $a_0' \colon sZ_0 \longrightarrow X_0'$. Constructing pushouts in the commutative diagram

$$\begin{tikzcd} tZ_0 \arrow[d, "\mathrm{id}"'] & sZ_0 \arrow[l, "z_0"'] \arrow[r, "a_0'"] \arrow[d, "\mathrm{Id}"'] & X_0' \arrow[d, "g_0"] \\ tZ_0 & sZ_0 \arrow[l, "z_0"'] \arrow[r, "a_0"] & X_0 \end{tikzcd}$$

gives a pointwise weak equivalence $tZ_0 \cup_{sZ_0} X_0' \xrightarrow{\sim \mathrm{pt}} X_1$. One can factor it as a map in $\mathcal{P}_J$-cell, followed by a pointwise fibration $g_1 \colon X_1' \xrightarrow{\sim \mathrm{pt}} X_1$, which is then acyclic. By iterating the above, one finds a pointwise acyclic fibration $g_n \colon X_n' \xrightarrow{\sim \mathrm{pt}} X_n$ for all n. Forming the colimit gives $f' \colon X' \longrightarrow Y := \mathrm{colim}_n X_n'$. $\square$

6.5. **The main theorem.** Before stating our main theorem, we summarize the list of assumptions. First of all, $\mathcal{V}$ is a weakly finitely generated monoidal **sSet**-model category for which the monoid axiom holds. Furthermore, $\mathcal{V}$ has to be strongly monoidal as defined in 4.13, right proper and cellular. Suppose that filtered colimits commute with pullbacks in $\mathcal{V}$, and that relative J'-cell complexes are monomorphisms. Assume $\mathbf{f}\mathcal{V}$ satisfies **f1** – **f4**, and let T be a cofibrant object of $\mathbf{f}\mathcal{V}$ such that $(- \otimes T, \mathcal{V}(T, -))$ is a Quillen equivalence on the stable model structure on $\mathrm{Sp}(\mathcal{V}, T)$.

Theorem 6.25. *Under the assumptions above, the classes of stable equivalences, stable fibrations and cofibrations give $\mathcal{F} = [\mathbf{f}\mathcal{V}, \mathcal{V}]$ the structure of a weakly finitely generated model category.*

Proof. The proof is analogous to the proof of 5.10, using 6.13 and 6.24. □

We refer to the model structure in 6.25 as the *stable* model structure. If $\mathcal{F}$ is equipped with the stable model structure, we indicate this by the subscript "st".

Lemma 6.26. *The model category $\mathcal{F}_{\mathrm{st}}$ is a monoidal $\mathcal{F}_{\mathrm{hf}}$-model category.*

Proof. Property **f4** guarantees that the homotopy functor structure is monoidal. Let $d_v \Box i$ be a map in $\mathcal{D}$, and let $\mathcal{V}(w, -) \otimes j$ be a map in $\mathcal{P}_I$. Then the pushout product map in $\mathcal{F}$ is isomorphic to the map $d_{v \otimes w} \Box (i \Box j)$. The latter is a stable equivalence. □

Lemma 6.27. *The stable model structure is proper.*

Proof. Left properness is straightforward to prove, using that the pointwise model structure is left proper. To prove right properness, note that it remains to check, using 5.12, that the base change of a stable equivalence of pointwise fibrant homotopy functors along an hf-fibration of pointwise fibrant homotopy functors is a stable equivalence. The statement follows then by 6.5, since $\mathbb{T}^\infty$ preserves pullbacks and pointwise fibrations of pointwise fibrant functors, and $\mathcal{F}_{\mathrm{pt}}$ is right proper. □

In the stable structure we have the following important result, analogous to and in fact an easy a consequence of Theorem 4.12.

Theorem 6.28. *If* $\mathbf{f}\mathcal{V}$ *satisfies* **f0**, *then smashing with a cofibrant $\mathcal{V}$-functor in $\mathcal{F}$ preserves stable equivalences.*

Proof. Factor a stable equivalence f as a stable acyclic cofibration followed by a stable acyclic fibration, i.e. a pointwise acyclic fibration. By 4.12, we may assume f is a stable acyclic cofibration. The claim follows since $\mathcal{F}_{\mathrm{st}}$ is monoidal. □

Lemma 6.29. *Suppose* $\mathbf{f}\mathcal{V}$ *satisfies* **f0**. *Then the monoid axiom holds in $\mathcal{F}_{\mathrm{st}}$.*

Proof. The domains of the generating acyclic cofibrations $\mathcal{D}' := \mathcal{P}_J \cup \mathcal{H} \cup \mathcal{D}$ for the stable model structure on $\mathcal{F}$ are cofibrant. Because **f0** holds, 4.12 implies that every map in $\mathcal{D}' \wedge \mathcal{F}$ is a stable equivalence. The case of a map in $\mathcal{D}' \wedge \mathcal{F}$-cell follows similarly as in the proof of 6.24. □

7. A Quillen equivalence

In this section, we will discuss two natural choices for the domain category $\mathbf{f}\mathcal{V}$. One of the choices gives a Quillen equivalence between the stable model structure $\mathrm{Sp}(\mathcal{V},T)_{\mathrm{st}}$ on spectra and the stable model structure $\mathcal{F}_{\mathrm{st}}$ on enriched functors.

7.1. **The choices.** Let $\mathbf{f}\mathcal{V}$ be the category of all cofibrant $\mathcal{V}$-finite objects, and $\mathbf{e}\mathcal{V}$ the full subcategory of $\mathbf{f}\mathcal{V}$ given by the objects v for which there exists an acyclic cofibration $T^n \overset{\sim}{\rightarrowtail} v$ for some $n \geq 0$. In the applications, the category of $\mathcal{V}$-finite objects is equivalent to a small category, hence its subcategories are valid domain categories. Axioms **f1**, **f2** and **f4** hold in both cases. If the domains and codomains of the maps in J' are $\mathcal{V}$-finite, then **f3** holds. The minimal choice satisfies a property which does not hold for the maximal choice in general.

Lemma 7.1. *A map f of pointwise fibrant homotopy functors in $\mathcal{F} = [\mathbf{e}\mathcal{V}, \mathcal{V}]$ is a stable equivalence if and only if $\mathrm{ev}(f)$ is a stable equivalence of spectra.*

Proof. This follows by definition of $\mathbf{e}\mathcal{V}$. □

Nevertheless, the evaluation functor is a right Quillen functor for both choices.

Lemma 7.2. *Evaluation* $\mathrm{ev}\colon \mathcal{F}_{\mathrm{st}} \longrightarrow \mathrm{Sp}(\mathcal{V},T)_{\mathrm{st}}$ *is a right Quillen functor.*

Proof. Pointwise fibrations and pointwise acyclic fibrations are preserved by ev. The characterizations of stable fibrations using homotopy pullback squares can be compared using 6.18, which implies that ev preserves stable fibrations. □

To deduce that ev is the right adjoint of a Quillen equivalence for the minimal choice, we prove a property of the stable model structure of spectra which is independent of the choice of the domain category.

7.2. **The unit of the adjunction.** The following lemma is a crucial observation which depends on the hypothesis on $(-\otimes T, \mathcal{V}(T,-))$, the $\mathcal{V}$-finiteness of T and the stabilization functor of spectra 6.16. Recall that i_* is the left adjoint of ev.

Lemma 7.3. *The canonical map*

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow \mathcal{V}(T^n,-)^h = \mathrm{ev}((i_*\mathcal{V}_{T\mathrm{Sph}}(T^n,-))^h)$$

is a stable equivalence of T-spectra for all $n \geq 0$.

Proof. It suffices, by 6.16, to prove that $\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow S^\infty((\mathcal{V}(T^n,-))^h)$ is a stable equivalence. This map can be described in terms of other maps. There is a canonical map $f\colon \mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes T^n \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^0,-)$ consisting of isomorphisms in degree n and above. In particular, it is a stable equivalence. The canonical map of T-spectra $\mathcal{V}_{T\mathrm{Sph}}(T^0,-) \longrightarrow \mathcal{V}(T^0,-)$ is just the identity, and the evaluation of the canonical map $\mathcal{V}(T^0,-) \longrightarrow (\mathcal{V}(T^0,-))^h$ is pointwise an acyclic cofibration, hence also a stable equivalence. By 6.16 the map $\mathcal{V}(T^0,-)^h \longrightarrow S^\infty((\mathcal{V}(T^0,-))^h)$ is a stable equivalence to a stably fibrant T-spectrum. This uses that $(\mathcal{V}(T^0,-))^h$ is pointwise fibrant, which follows since T^0 is cofibrant and the input is fibrant. Hence the composition

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes T^n \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^0,-) \longrightarrow \mathcal{V}(T^0,-)^h \longrightarrow S^\infty(\mathcal{V}(T^0,-)^h)$$

is a stable equivalence from the n-fold T-suspension of a cofibrant T-spectrum to a stably fibrant T-spectrum. Note here that $\mathcal{V}_{T\mathrm{Sph}}(T^0,-)=\mathcal{V}(T^0,-)$. Since $-\otimes T$ is a Quillen equivalence for T-spectra by the hypothesis, its adjoint

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow \mathcal{V}(T^n, S^\infty((\mathcal{V}(T^0,-))^h))$$

is a stable equivalence. The object T is $\mathcal{V}$-finite, thus the latter T-spectrum is isomorphic to the T-spectrum $S^\infty((\mathcal{V}(T^n,-))^h)$. It is straightforward to check that the map we have constructed is really the map we are interested in. □

Corollary 7.4. *The canonical map $c_E\colon E \longrightarrow \mathrm{ev}((i_*E)^h)$ is a stable equivalence for every cofibrant T-spectrum E.*

Proof. For E a T-spectrum and $A \in \mathrm{Ob}\,\mathcal{V}$, the map $c_{E\otimes A}$ is isomorphic to the map $(c_E)\otimes A$. Since tensoring with the domains and codomains of the generating cofibrations preserves weak equivalences in $\mathcal{V}$, also tensoring with the domains and codomains of the generating cofibrations preserves stable equivalences of spectra. The cofibrant T-spectra are precisely retracts of $\mathrm{Sph}(I)$-cell complexes, where $\mathrm{Sph}(I)$ denotes the set of generating cofibrations $\{\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes i\}_{n\geq 0, i\in I}$ from [9, 1.8]. So it suffices to consider $\mathrm{Sph}(I)$-cell complexes. Recall that ev, i_* and $(-)^h$ preserve colimits, and stable equivalences of T-spectra are closed under sequential compositions. This allows to use transfinite induction. The induction step follows from the diagram

$$\begin{array}{ccccc}
\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes ti & \longleftarrow & \mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes si & \longrightarrow & E \\
\sim\downarrow & & \sim\downarrow & & \downarrow\sim \\
\mathcal{V}(T^n,-)^h\otimes ti & \xleftarrow{f} & \mathcal{V}(T^n,-)^h\otimes si & \longrightarrow & \mathrm{ev}((i_*E)^h)
\end{array}$$

together with the following observations. The right hand vertical map is a stable equivalence by the induction hypothesis, and likewise for the other vertical maps by 7.3 and the argument given above. Note that f is not necessarily a cofibration of spectra, but it is pointwise in $\mathrm{Cof}(\mathcal{V})\otimes\mathcal{V}$. Finally, one can prove that strong left properness of $\mathcal{V}$ implies that the map induced on the pushouts of the rows in the diagram is again a stable equivalence. □

Corollary 7.5. *The functor* $\mathrm{ev}\colon [\mathbf{e}\mathcal{V},\mathcal{V}] \longrightarrow \mathrm{Sp}(\mathcal{V},T)$ *is the right adjoint in a Quillen equivalence.*

Proof. Use 7.1 and 7.4. □

8. Algebraic structure

Let us discuss the algebraic structures which the monoidal model category $\mathcal{F}_{\mathrm{st}}$ supports in the presence of the monoid axiom. Recall that $\mathcal{F}_{\mathrm{st}}$ satisfies the monoid axiom provided $\mathbf{f}\mathcal{V}$ satisfies **f0**, cf. 6.29.

Fix a $\mathcal{V}$ which satisfies all the assumptions mentioned at the beginning of Section 5, and a small full sub-$\mathcal{V}$-category $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ which satisfies **f1**–**f4**.

8.1. **Recollections.** In a symmetric monoidal category like $(\mathcal{F}, \wedge, \mathbb{I})$ there are notions of algebras and modules over the algebras. Recall that an $\mathbb{I}$-algebra is a monoid in $(\mathcal{F}, \wedge, \mathbb{I})$ or alternatively an $\mathcal{F}$-category with only one object.

If A is an $\mathbb{I}$-algebra, a (left) A-module M is an object in $\mathcal{F}$ with an appropriate action of A. It can alternatively be described as an $\mathcal{F}$-functor from A to $\mathcal{F}$. The category of A-modules mod_A is then an $\mathcal{F}$-category. There is a smash product $\wedge_A\colon \mathrm{mod}_{A^{\mathrm{op}}} \wedge \mathrm{mod}_A \longrightarrow \mathcal{F}$ by the usual formula. For $M \in \mathrm{mod}_{A^{\mathrm{op}}}$ and $N \in \mathrm{mod}_A$, $M \wedge_A N$ is given as the coequalizer of

$$M \wedge A \wedge N \rightrightarrows M \wedge N.$$

If M and N are A-modules, there is the function object $\mathrm{mod}_A(M, N)$ in $\mathcal{F}$ given as the equalizer of

$$\mathcal{F}(M, N) \rightrightarrows \mathcal{F}(A \wedge M, N).$$

If k is a *commutative* $\mathbb{I}$-algebra, recall that $\mathrm{mod}_k \cong \mathrm{mod}_{k^{\mathrm{op}}}$ and mod_k is a closed symmetric monoidal category under $\wedge_k$ with internal morphism object $\mathrm{mod}_k(M, N)$. A k-algebra is a monoid in $(\mathrm{mod}_k, \wedge_k, k)$ or a mod_k-category with one object. With this notation, notice that another name for $\mathcal{F}$ is $\mathrm{mod}_{\mathbb{I}}$.

8.2. **The model structures.**

Definition 8.1. If A is an $\mathbb{I}$-algebra and k is a commutative $\mathbb{I}$-algebra, then a map in mod_A or alg_k is called a *weak equivalence* (resp. *fibration*) if it is so when considered in $\mathcal{F}_{\mathrm{st}}$. Cofibrations are defined by the left lifting property.

Remark 8.2. Note that we chose the stable model structure as our basis. This is fixed in the following (at least on the top level), so the missing prefix "stable" from fibrations and weak equivalences should not be a source of confusion.

The next result is due to Schwede and Shipley [16, 4.1].

Theorem 8.3. *Suppose that $\mathcal{F}_{\mathrm{st}}$ satisfies the monoid axiom. With the structures described above, the following is true.*

- *Let $A \in \mathcal{F}$ be an $\mathbb{I}$-algebra. Then the category mod_A of (left) A-modules is a cofibrantly generated model category.*
- *Let $k \in \mathcal{F}$ be a commutative $\mathbb{I}$-algebra. Then the category of k-modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*

- *Let $k \in \mathcal{F}$ be a commutative $\mathbb{I}$-algebra. Then the category alg_k of k-algebras is a cofibrantly generated model category.*

Note that we did not state the hypothesis that all objects in $\mathcal{F}$ are small. Since $\mathcal{V}$ is weakly finitely generated, $\mathcal{F}_{\mathrm{st}}$ is so, too. The smallness of the domains and codomains of the generating cofibrations and generating acyclic cofibrations in $\mathcal{F}_{\mathrm{st}}$ carries over to the relevant smallness conditions needed to prove the theorem. See also [16, 2.4].

Lemma 8.4. *Suppose that* $\mathbf{f}\mathcal{V}$ *satisfies* $\mathbf{f0}$. *Let A be an $\mathbb{I}$-algebra. Then for any cofibrant A-module N, the functor $- \wedge_A N$ takes weak equivalences in $mod_{A^{\mathrm{op}}}$ to weak equivalences in $\mathcal{F}_{\mathrm{st}}$.*

Proof. Given a weak equivalence in mod_A, factor it as an acyclic cofibration followed by an acyclic fibration. The only trouble is with the acyclic fibration, but this is a pointwise acyclic fibration, and the argument can be phrased in the analogous theory for A-modules built on the pointwise structure $\mathcal{F}_{\mathrm{pt}}$. In this case, the generating cofibrations in mod_A are of the form $A \wedge S \xrightarrow{A \wedge i} A \wedge T$ where $S \xrightarrow{i} T$ is a generating cofibration in $\mathcal{F}_{\mathrm{pt}}$. The argument of 4.12 goes through verbatim: smashing commutes with colimits and $A \wedge_A S \cong S$. $\square$

Lemma 8.4 and [16, 4.3] imply:

Corollary 8.5. *Suppose that* $\mathbf{f}\mathcal{V}$ *satisfies* $\mathbf{f0}$. *Let $f\colon A \xrightarrow{\sim} B$ be a weak equivalence of $\mathbb{I}$-algebras. Then extension and restriction of scalars define a Quillen equivalence*

$$mod_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} mod_B.$$

If A and B are commutative, there is the Quillen equivalence

$$alg_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} alg_B.$$

9. Motivic Spaces

9.1. **The Category.** Let $\mathcal{I}$ be a small category with pullbacks, a terminal object S and an initial object $\emptyset$. For $U, V \in \operatorname{Ob}\mathcal{I}$, denote the set of maps from V to U in $\mathcal{I}$ by $\mathbf{Set}_{\mathcal{I}}(V,U)$. Let $\mathbf{sSet}$ be the category of simplicial sets, and let $\mathcal{S}$ be the category of pointed simplicial sets. Write $\mathcal{M} := \operatorname{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{S})$ for the functor category of *pointed simplicial presheaves on* $\mathcal{I}$. We will often refer to its objects simply as *presheaves.* It is isomorphic to $\operatorname{Fun}(\mathcal{I}^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{Set}_*)$, $\operatorname{Fun}(\Delta^{\mathrm{op}}, \operatorname{Fun}(\mathcal{I}^{\mathrm{op}}, \mathbf{Set}_*))$, and $* \downarrow \operatorname{Fun}(\mathcal{I}^{\mathrm{op}}, \mathbf{sSet})$, where $\operatorname{Fun}(\mathcal{I}^{\mathrm{op}}, \mathbf{sSet})$ is the category of *simplicial presheaves on* $\mathcal{I}$ with the indicated terminal simplicial presheaf $*$. The evaluation functor $\operatorname{Ev}_U \colon \mathcal{M} \longrightarrow \mathcal{S}$ for $U \in \operatorname{Ob}\mathcal{I}$ sends the presheaf $A \in \operatorname{Ob}\mathcal{M}$ to $A(U)$.

Proposition 9.1. *The category $\mathcal{M}$ is bicomplete. Finite limits commute with filtered colimits, and Ev_U preserves limits and colimits for every $U \in \operatorname{Ob}\mathcal{I}$.*

Recall that $\mathcal{S}$ is a closed symmetric monoidal category using the smash product $K \wedge L := K \times L / K \vee L$, with $K \vee L \hookrightarrow K \times L$ the canonical injection. The unit is $S^0 := \Delta^0_+$, where K_+ denotes the simplicial set K equipped with a disjoint basepoint, and Δ^n is the simplicial set represented by $[n] \in \operatorname{Ob}\Delta$. Associativity and commutativity isomorphisms are inherited from the categorical product. The right adjoint of $- \wedge K$ is given as follows. Consider the pointed simplicial set $\mathcal{S}(K,L)$ which has as its n-simplices the pointed set of maps in $\mathcal{S}$ from $K \wedge \Delta^n_+$ to L. This set is pointed by the trivial map. Its simplicial structure is induced from the cosimplicial object $\Delta^\bullet_+ : \Delta \longrightarrow \mathcal{S}$ where $(\Delta^\bullet_+)_n = \Delta^n_+$.

Define an action of $\mathcal{S}$ on $\mathcal{M}$ via $A \oslash K := (- \wedge K) \circ A$. The pointed simplicial set of maps in $\mathcal{M}$ from A to B is $\mathcal{S}_{\mathcal{M}}(A,B)_n := \mathbf{Set}_{*\mathcal{M}}(A \oslash \Delta^n_+, B)$, and the basepoint and the simplicial structure are obtained likewise as for $\mathcal{S}(K,L)$. Next, the coaction is given by $A^K := \mathcal{S}(K,-)\circ A$. For $U \in \operatorname{Ob}\mathcal{I}$, denote by $\mathbf{Set}_{\mathcal{I}}(-,U)_+$ the discrete pointed simplicial presheaf mapping V to $\mathbf{Set}_{\mathcal{I}}(V,U)$ with a disjoint basepoint added. The left adjoint of Ev_U is the functor $\operatorname{Fr}_U \colon \mathcal{S} \longrightarrow \mathcal{M}$ which maps K to $\mathbf{Set}_{\mathcal{I}}(-,U)_+ \oslash K$. Note that Fr_S is the constant diagram functor, where S is the terminal object in $\mathcal{I}$.

Proposition 9.2. *The category $\mathcal{M}$ is a closed $\mathcal{S}$-module, and $(\operatorname{Fr}_U, \operatorname{Ev}_U)$ is a $\mathcal{S}$-adjoint pair for every $U \in \operatorname{Ob}\mathcal{I}$.*

Proof. The first claim is straightforward to prove. Since Ev_U coincides with the functor $\operatorname{Fun}(U \hookrightarrow \mathcal{I}, \mathcal{S}) \colon \mathcal{M} \longrightarrow \mathcal{S}$, where $U \hookrightarrow \mathcal{I}$ is regarded as a subcategory having only the morphism id_U, Ev_U is a $\mathcal{S}$-functor. The functor Fr_U is then obtained as a left Kan extension along $U \hookrightarrow \mathcal{I}$. By examining 2.5, one can see that the ordinary left Kan extension coincides with the version enriched over $\mathcal{S}$. This all works, because $U \hookrightarrow \mathcal{I}$ can be viewed as a $\mathcal{S}$-functor of discrete $\mathcal{S}$-categories. ∎

Recall the symmetric monoidal structure on $\mathcal{M}$. It is defined pointwise as $(A \wedge B)(U) := A(U) \wedge B(U)$, thus $A \wedge B = A \times B / A \vee B$ where $A \vee B \hookrightarrow A \times B$ is the canonical injection. The unit is the constant presheaf $\mathrm{Fr}_S S^0$. Associativity and commutativity isomorphisms are inherited from the categorical product.

Proposition 9.3. *The category $(\mathcal{M}, \wedge, \mathrm{Fr}_S S^0)$ is symmetric monoidal. There is a natural isomorphism $\mathrm{Fr}_U(K) \wedge \mathrm{Fr}_V(L) \longrightarrow \mathrm{Fr}_{U \times_S V}(K \wedge L)$. The functors Fr_S and Ev_U are strict symmetric monoidal, while Fr_U is lax symmetric monoidal.*

We want to turn $(\mathcal{M}, \wedge, \mathrm{Fr}_S S^0)$ into a closed symmetric monoidal category. For $U \in \mathrm{Ob}\,\mathcal{I}$, let $\mathcal{I} \downarrow U$ be the category of objects in $\mathcal{I}$ over U. Note that $\mathcal{I} \downarrow U$ has pullbacks, its terminal object is id_U, and its initial object is $\emptyset \longrightarrow U$. Hence everything above holds for $\mathcal{M}|_U := \mathrm{Fun}((\mathcal{I} \downarrow U)^{\mathrm{op}}, \mathcal{S})$. Consider the functor $\mathrm{pr}_U \colon \mathcal{I} \downarrow U \longrightarrow \mathcal{I}$ mapping $\phi \colon V \longrightarrow U$ to V. Denote the induced "restriction" functor by $\mathrm{res}_U := \mathrm{Fun}(\mathrm{pr}_U^{\mathrm{op}}, \mathcal{S}) \colon \mathcal{M} \longrightarrow \mathcal{M}|_U$. The value $\mathcal{M}(A, B)(U)$ of the pointed simplicial presheaf of maps from A to B at $U \in \mathrm{Ob}\,\mathcal{I}$ is $\mathcal{S}_{\mathcal{M}|_U}(\mathrm{res}_U A, \mathrm{res}_U B)$. Fix $\phi \colon V \longrightarrow U$ and let $\mathcal{I} \downarrow \phi \colon \mathcal{I} \downarrow V \longrightarrow \mathcal{I} \downarrow U$ send $\psi \in \mathbf{Set}_{\mathcal{I}}(W, V)$ to $\phi \circ \psi$. It induces $\mathrm{res}_\phi := \mathrm{Fun}((\mathcal{I} \downarrow \phi)^{\mathrm{op}}, \mathcal{S}) \colon \mathcal{M}|_U \longrightarrow \mathcal{M}|_V$.

Lemma 9.4. *The functors res_ϕ and res_U are $\mathcal{S}$-functors.*

Lemma 9.4 gives a presheaf structure on $\mathcal{M}(A, B)$. For $\phi \in \mathbf{Set}_{\mathcal{I}}(V, U)$, we consider $\mathcal{M}(A, B)(\phi) := \mathcal{S}_{\mathcal{M}|_U}(\mathrm{res}_U A, \mathrm{res}_U B) \longrightarrow \mathcal{S}_{\mathcal{M}|_V}(\mathrm{res}_V A, \mathrm{res}_V B)$ induced by the $\mathcal{S}$-functor res_ϕ. The identities $\mathrm{pr}_U \circ \mathcal{I} \downarrow \phi = \mathrm{pr}_V$, $\mathcal{I} \downarrow \mathrm{id}_U = \mathrm{Id}_{\mathcal{I} \downarrow U}$ and $\mathcal{I} \downarrow (\phi \circ \psi) = (\mathcal{I} \downarrow \phi) \circ (\mathcal{I} \downarrow \psi)$ imply that $\mathcal{M}(A, B)$ is a functor defined on $\mathcal{I}^{\mathrm{op}}$, hence a pointed simplicial presheaf on $\mathcal{I}$.

Proposition 9.5. *The functor $\mathcal{M}(A, -)$ is right adjoint to $- \wedge A$.*

Proof. First we define the unit $\eta_A(B) \colon B \longrightarrow \mathcal{M}(A, B \wedge A)$ of the adjunction. Its U-component $B(U) \longrightarrow \mathcal{S}_{\mathcal{M}|_U}(\mathrm{res}_U A, \mathrm{res}_U(B \wedge A))$ maps $b \in B(U)_n$ to $f(b)$. Here $f(b)(V \xrightarrow{\phi} U) \colon A(V) \wedge \Delta^n_+ \longrightarrow B(V) \wedge A(V)$ is defined for representatives $a \in A(V)_m$ and $\alpha \in \Delta^n_m$ by $[a, \alpha] \mapsto [\alpha^*(B(\phi)(b)), a]$. The U-component of the counit $\epsilon_A(B) \colon \mathcal{M}(A, B) \wedge A \longrightarrow B$ is a map of pointed simplicial sets $\mathcal{S}_{\mathcal{M}|_U}(\mathrm{res}_U A, \mathrm{res}_U B) \wedge A(U) \longrightarrow B(U)$ which assigns $f(\mathrm{id}_U)([a, \mathrm{id}_{[n]}])$ to $[f, a]$, where $a \in A(U)_n$ and $f \colon \mathrm{res}_U A \oslash \Delta^n_+ \longrightarrow \mathrm{res}_U B$. A proof follows easily. $\square$

Since pullbacks exist in $\mathcal{I}$, there is a functor $V \times_S - \colon \mathcal{I} \longrightarrow \mathcal{I}$ for every $V \in \mathrm{Ob}\,\mathcal{I}$ where $U \mapsto V \times_S U$. Hence the induced functor $V^* \colon \mathcal{M} \longrightarrow \mathcal{M}$ maps the presheaf A to the presheaf $V^*(A)$ given by $U \longmapsto A(V \times_S U)$. We will use this presheaf to express $\mathcal{M}(\mathrm{Fr}_V K, A)$ in another way.

Lemma 9.6. *Let U, V be objects of $\mathcal{I}$, and let $\phi \colon U \times_S V \longrightarrow U$ be the canonical map. Then the $\mathcal{S}$-functors $\mathrm{res}_U \circ \mathrm{Fr}_V$ and Fr_ϕ are naturally isomorphic.*

Proof. There is a bijection $f_\psi\colon \mathbf{Set}_{\mathcal{I}}(W,V) \longrightarrow \mathbf{Set}_{\mathcal{I}\downarrow U}(\psi,\phi)$ for $\psi\colon W \longrightarrow U$. It takes $\sigma\colon W \longrightarrow V$ to $W \longrightarrow V \times_S U$ determined by σ and ψ, and is natural for maps in $\mathcal{I} \downarrow U$. The claim follows by considering the induced $\mathcal{S}$-natural isomorphism $f(K)\colon \mathrm{res}_U \circ \mathrm{Fr}_V K \longrightarrow \mathrm{Fr}_\phi K$ with ψ-component $f_\psi \oslash K$. □

Corollary 9.7. *There is a natural isomorphism* $\mathcal{M}(\mathrm{Fr}_V K, A) \cong V^*(A^K)$.

Proof. By adjointness and 9.6, we have natural isomorphisms of pointed simplicial presheaves $\mathcal{M}(\mathrm{Fr}_V K, A)(U) \cong \mathcal{S}_{\mathcal{M}\downarrow U}(\mathrm{Fr}_\phi K, \mathrm{res}_U A) \cong \mathcal{S}(K, A(V \times_S U))$. Naturality in A and V follows readily. □

Let us turn to smallness referring to 3.1.

Lemma 9.8. *A pointed simplicial set is finite if and only if it has only finitely many non-degenerate simplices. A pointed simplicial set is $\mathcal{S}$-finite if and only if it is finite. The class of finite pointed simplicial sets is closed under finite limits and colimits, hence under smashing with finite pointed simplicial sets.*

Lemma 9.9. *Every pointed simplicial presheaf is a filtered colimit of finite colimits of presheaves of the form* $\mathrm{Fr}_U \Delta^n_+$.

Proof. By the Yoneda lemma, $A\colon \mathcal{I}^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}_*$ can be expressed as a colimit of representables. The representables are the presheaves $\mathrm{Fr}_U \Delta^n_+$, and the indexing category for the colimit is $\mathcal{Y}_A := (\mathcal{I}^{\mathrm{op}} \times \Delta^{\mathrm{op}}) \downarrow A$. An object of $\mathcal{Y}_A$ is a triple $(U \in \mathrm{Ob}\,\mathcal{I}, n \geq 0, a \in A(U)_n)$. A map from (U,n,a) to (V,m,b) is a pair $(\phi\colon U \longrightarrow V, \beta\colon [n] \longrightarrow [m])$ where $a = A(\phi,\beta)(b)$. The functor Y of which A is the colimit maps (U,n,a) to $\mathrm{Fr}_U \Delta^n_+$, and this colimit is the coequalizer of

$$\coprod_{(\phi,\beta)\in\mathcal{Y}_A} \mathrm{Fr}_U \Delta^n_+ \rightrightarrows \coprod_{(W,k,a)\in\mathcal{Y}_A} \mathrm{Fr}_W \Delta^k_+$$

where on a fixed summand $\mathrm{Fr}_U \Delta^n_+$ indexed by (ϕ,β), the upper map is the identity and the lower map is induced by ϕ and β.

Let $\mathcal{J}$ be the set of finite subsets of the morphism set of $\mathcal{Y}_A$, which is partially ordered by inclusion, and regard it as a category. Note that $\mathcal{J}$ is filtered. If M is a finite subset in $\mathcal{J}$, let M' denote the set given by the domains and codomains of the maps in M. Denote by Y_M the coequalizer of

$$\coprod_{(\phi,\beta)\in M} \mathrm{Fr}_U \Delta^n_+ \rightrightarrows \coprod_{(W,k,a)\in M'} \mathrm{Fr}_W \Delta^k_+$$

with maps as above. It induces a functor $Y'\colon \mathcal{J} \longrightarrow \mathcal{M}$, namely $Y'(M) := Y_M$. It remains to prove that the colimit of Y is isomorphic to the colimit of Y'. Since every set is the filtered colimit of its finite subsets, there is an isomorphism

$$\mathrm{colim}_{M\in\mathcal{J}} \coprod_{(\phi,\beta)\in M} \mathrm{Fr}_U \Delta^n_+ \cong \coprod_{(\phi,\beta)\in\mathcal{Y}_A} \mathrm{Fr}_U \Delta^n_+.$$

Hence the result follows from the analogous isomorphism

$$\operatorname{colim}_{M\in\mathcal{J}} \coprod_{(W,k,a)\in M'} \mathrm{Fr}_W\Delta^k_+ \cong \coprod_{(W,k,a)\in\mathcal{Y}_A} \mathrm{Fr}_W\Delta^k_+$$

which uses the final functor from $\mathcal{J}$ to the category of finite subsets of the object set of $\mathcal{Y}_A$ mapping M to M'. □

Lemma 9.10. *If K is finite, then $\mathrm{Fr}_U(K)$ is $\mathcal{M}$-finite for all $U \in \mathrm{Ob}\,\mathcal{I}$. The class of $\mathcal{M}$-finite pointed simplicial presheaves is closed under retracts, finite colimits and smashing with $\mathcal{M}$-finite pointed simplicial presheaves. A pointed simplicial presheaf is $\mathcal{M}$-finite if and only if it is finite.*

Proof. The first claim follows from 9.7, since for $D\colon \mathcal{J} \longrightarrow \mathcal{M}$ a filtered diagram:

$$\begin{aligned}\mathcal{M}(\mathrm{Fr}_U(K), \operatorname{colim} D)(V) &\cong \mathcal{S}(K, \operatorname{colim} \mathrm{Ev}_{V\times_S U} \circ D)\\ &\cong \operatorname{colim} \mathcal{S}(K, \mathrm{Ev}_{V\times_S U} \circ D)\\ &\cong \operatorname{colim} \mathcal{M}(\mathrm{Fr}_U(K), D)(V).\end{aligned}$$

The remaining claims are straightforward to prove. A $\mathcal{M}$-finite presheaf is finite, because $\mathrm{Fr}_S S^0$ is finite. Conversely, by 9.9, a finite presheaf is isomorphic to a filtered colimit of $\mathcal{M}$-finite presheaves. By finiteness, this isomorphism factors over some $\mathcal{M}$-finite presheaf appearing in the colimit. It follows that any finite presheaf is a retract of a $\mathcal{M}$-finite presheaf, hence $\mathcal{M}$-finite. □

The full subcategory of finite pointed simplicial presheaves in $\mathcal{M}$ is equivalent to a small category by [1, 5.3.8], cf. [1, 5.3.3] and the pointed version of [1, 5.2.2b]. Since this category appears as a domain category of functors later on, one should better pick a certain small choice once and for all. However, we will neglect this point. The following lemma is a special case of [1, 5.2.10].

Lemma 9.11. *Every pointed simplicial presheaf is small.*

We end the categorical considerations by noting that the $\mathcal{M}$-category $A\mathrm{Sph}$ of A-spheres is a sub-$\mathcal{M}$-category of $\mathcal{M}$ for any presheaf A. For $A \in \mathrm{Ob}\,\mathcal{M}$, define $A^0 := \mathrm{Fr}_S S^0$ with the exception $*^0 := *$, $A^1 := A$ and $A^n := A \wedge A^{n-1}$ for $n \geq 1$.

Lemma 9.12. *For $A \in \mathrm{Ob}\,\mathcal{M}$, the map $\eta_{A^l}(A^k)\colon A^k \longrightarrow \mathcal{M}(A^l, A^k \wedge A^l)$ is a monomorphism for $k, l \geq 0$.*

9.2. **The Model Structures.** The starting point for constructing model structures on pointed simplicial presheaves is the following.

Definition 9.13. A map of pointed simplicial sets is a *weak equivalence* if its geometric realization is a homotopy equivalence of topological spaces. It is a *cofibration* if it is injective, and a *fibration* if it has the right lifting property with respect to the set of inclusions $\{(\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{n\geq 1, 0\leq i\leq n}$ where Λ^n_i denotes the sub-simplicial set of Δ^n generated by the faces containing the i-th vertex.

Theorem 9.14 (Quillen)**.** *The above classes define a monoidal model structure on $\mathcal{S}$ which is proper and cellular. The inclusions $\{(\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n\geq 0}$ are a set of generating cofibrations, and all objects are cofibrant. The domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite.*

Definition 9.15. A map $f\colon A \longrightarrow B$ in $\mathcal{M}$ is an *objectwise weak equivalence* if, for all $U \in \mathrm{Ob}\,\mathcal{I}$, $f(U)$ is a weak equivalence in $\mathcal{S}$. Likewise for *objectwise fibrations* and *objectwise cofibrations.*

Note that the objectwise cofibrations are simply the monomorphisms.

Theorem 9.16. *The objectwise weak equivalences and objectwise fibrations give $\mathcal{M}$ the structure of a monoidal $\mathcal{S}$-model category which is proper and cellular. A set of generating cofibrations is $\{\mathrm{Fr}_U(\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{U\in\mathrm{Ob}\,\mathcal{I},n\geq 0}$ and a set of generating acyclic cofibrations is $\{\mathrm{Fr}_U(\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{U\in\mathrm{Ob}\,\mathcal{I},n\geq 1,\,0\leq i\leq n}$. Their domains and codomains are finite.*

Proof. The existence of the model structure follows from [7, 2.1.19], using the generating cofibrations and generating acyclic cofibrations described above. The properties which have to be checked are either straightforward or follow from 9.10 and 9.14. Properness follows from properness in $\mathcal{S}$, where we use that a cofibration is in particular an objectwise cofibration. Note that for f, g in $\mathcal{S}$ there exists an isomorphism $\mathrm{Fr}_U(f)\Box\mathrm{Fr}_V(g) \cong \mathrm{Fr}_{U\times_S V}(f\Box g)$ by 9.3. Since Fr_U is a left Quillen functor for any $U \in \mathrm{Ob}\,\mathcal{I}$, it follows that $\mathcal{M}$ is a monoidal $\mathcal{S}$-model category via the functor Fr_S. Because $\mathcal{S}$ is a cellular model category, the model structure is cellular by [6, 14.1.5]. □

We call the model structure in 9.16 the *objectwise projective* model structure, and refer to *projective* cofibrations for its cofibrations. If $\mathcal{M}$ is equipped with this model structure, we use the subscript $\mathcal{M}_{\mathrm{ob}}$, and arrows $\xrightarrow{\sim\mathrm{ob}}$ and $\overset{\mathrm{ob}}{\twoheadrightarrow}$. A projective cofibration is denoted by $\rightarrowtail$. The adjective "projective" is here to avoid confusion with another model structure on $\mathcal{M}$, which has objectwise weak equivalences and objectwise cofibrations. These two model structures are Quillen equivalent via the identity functor: the weak equivalences coincide, and every projective cofibration is an objectwise cofibration.

Lemma 9.17. *The functor* $\mathrm{res}_U\colon \mathcal{M} \longrightarrow \mathcal{M}|_U$ *preserves objectwise weak equivalences, objectwise fibrations and projective cofibrations.*

Proof. The very last statement follows from 9.6, the rest is clear. □

Lemma 9.18. *Smashing with a pointed simplicial presheaf preserves objectwise weak equivalences.*

Proof. This follows, because all definitions involved are objectwise and smashing with any pointed simplicial set preserves weak equivalences. □

Lemma 9.19. *The model category $\mathcal{M}_{\mathrm{ob}}$ is strongly left proper, and the monoid axiom holds.*

Proof. If $f\colon A \overset{\sim}{\rightarrowtail} B$ is an objectwise projective acyclic cofibration and C is a presheaf, then $f \wedge C$ is an objectwise acyclic cofibration. The class of objectwise acyclic cofibrations is closed under cobase changes and sequential compositions by 9.14, hence the monoid axiom holds.

To prove that $\mathcal{M}_{\mathrm{ob}}$ is strongly left proper, it suffices to note three facts which follow from 9.14. If $f\colon A \rightarrowtail B$ is a projective cofibration and C is some presheaf, then $f \wedge C$ is an objectwise cofibration. Objectwise cofibrations are closed under cobase changes and sequential compositions. The cobase change of an objectwise weak equivalence along an objectwise cofibration is an objectwise weak equivalence. ☐

In the following, we give $\mathcal{I}$ a Grothendieck topology, see for example [2, 3.2.4].

Definition 9.20. A *Grothendieck topology* $\mathbf{T}$ on $\mathcal{I}$ is a family $L(U)$ of subfunctors of the representable functor $\mathbf{Set}_{\mathcal{I}}(-,U)$ for every $U \in \mathrm{Ob}\,\mathcal{I}$, such that:

- $L(U)$ contains $\mathbf{Set}_{\mathcal{I}}(-,U)$ for every $U \in \mathrm{Ob}\,\mathcal{I}$.
- If $\phi\colon V \longrightarrow U$ and $F \hookrightarrow \mathbf{Set}_{\mathcal{I}}(-,U) \in L(U)$, then $F_\phi \hookrightarrow \mathbf{Set}_{\mathcal{I}}(-,V)$ defined by the following pullback diagram belongs to $L(V)$.

$$\begin{tikzcd} F_\phi \arrow[r] \arrow[d, hook] & F \arrow[d, hook] \\ \mathbf{Set}_{\mathcal{I}}(-,V) \arrow[r, "\mathbf{Set}_{\mathcal{I}}(-,\phi)"] & \mathbf{Set}_{\mathcal{I}}(-,U) \end{tikzcd}$$

- Consider $G \hookrightarrow \mathbf{Set}_{\mathcal{I}}(-,U)$ and $F \in L(U)$. If the induced subfunctor $G_\phi \hookrightarrow \mathbf{Set}_{\mathcal{I}}(-,V)$ belongs to $L(V)$ for all $\phi\colon V \longrightarrow U$ in $F(V)$, then $G \in L(U)$.

If $\mathbf{T}$ is a Grothendieck topology on $\mathcal{I}$, the pair $(\mathcal{I},\mathbf{T})$ is called a *site*, and an element of the family $L(U)$ in $\mathbf{T}$ is called a *covering sieve.* Note that any covering sieve is a (discrete) pointed simplicial presheaf in a natural way, by adjoining an extra basepoint. A pointed simplicial presheaf A on $\mathcal{I}$ is a pointed simplicial *sheaf* on $(\mathcal{I},\mathbf{T})$ if, for every $U \in \mathrm{Ob}\,\mathcal{I}$ and $F \in L(U)$, any map $f\colon F_+ \oslash \Delta^n_+ \longrightarrow A$ extends uniquely to a map $\mathrm{Fr}_U \Delta^n_+ \longrightarrow A$. We are interested in Grothendieck topologies defined by cd-structures, as introduced by Voevodsky in [19].

Definition 9.21. A *cd-structure* on $\mathcal{I}$ is a collection P of commutative squares

$$Q \quad = \quad \begin{tikzcd} Q_1 \arrow[r] \arrow[d] & Q_2 \arrow[d, "\phi_Q"] \\ Q_3 \arrow[r, "\psi_Q"] & Q_4 \end{tikzcd}$$

such that any commutative square in $\mathcal{I}$ which is isomorphic to some square in P is also in P. The squares in P are called *distinguished.*

Every cd-structure P determines a Grothendieck topology $\mathbf{T}(P)$, namely the smallest one such that the following two conditions hold.

1. The empty functor is a covering sieve of the initial object.
2. For each distinguished square Q, the functor which maps U to the set of maps $U \longrightarrow Q_4$ which factor either over ϕ_Q or over ψ_Q is a covering sieve of Q_4.

The first condition implies that any sheaf A on $(\mathcal{I}, \mathbf{T}(P))$ has a one-point set as value at $\emptyset$.

The following three properties of cd-structures are discussed in detail in [19]. *Bounded* cd-structures are the ones in which every object has finite dimension. In a *regular* cd-structure, sheaves map distinguished squares to pullback squares. A presheaf A which has this property and satisfies $A(\emptyset) = *$ is a sheaf if the cd-structure is *complete.* An example of a Grothendieck topology generated by a cd-structure is the Nisnevich topology on the site Sm/S of smooth schemes of finite type over a Noetherian base scheme S of finite dimension. The distinguished squares are pullback squares of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d, "\phi"] \\ U \arrow[r, "\psi"] & X \end{tikzcd}$$

where ϕ is étale, ψ is an open embedding and $\phi^{-1}(X - U) \longrightarrow (X - U)$ is an isomorphism of schemes (with the reduced structure). In [20] Voevodsky proves that this cd-structure is bounded, complete and regular.

If $\mathbf{T}$ is a Grothendieck topology on $\mathcal{I}$ and U an object in $\mathcal{I}$, then $\mathbf{T}$ determines a Grothendieck topology $\mathbf{T}|_U$ on $\mathcal{I} \downarrow U$. The functor res_U maps pointed simplicial sheaves on $\mathcal{I}$ to pointed simplicial sheaves on $\mathcal{I} \downarrow U$. If additionally $\mathbf{T}$ is generated by a cd-structure P, the topology $\mathbf{T}|_U$ is generated by the cd-structure $P|_U$, which is given by the commutative squares Q in $\mathcal{I} \downarrow U$ such that $\mathrm{pr}_U(Q)$ is in P.

To any presheaf A one can assign a sheaf $a(A)$ in a functorial way such that a is left adjoint to the inclusion of the full subcategory of sheaves, cf. [2, 3.3]. We will apply this to the presheaf of homotopy groups of a pointed simplicial presheaf. If A is a pointed simplicial presheaf, regard it as a simplicial presheaf by forgetting the basepoint. Now let $\pi_0 A$ be the presheaf (with values in sets) which maps $U \in \mathrm{Ob}\,\mathcal{I}$ to the set of connected components of $A(U)$. If $U \in \mathrm{Ob}\,\mathcal{I}$ and $x \in A(U)_0$, consider the pair $(\mathrm{res}_U A, x)$ as a pointed simplicial presheaf on the site $\mathcal{I} \downarrow U$. For any $n > 0$, the presheaf of groups $\pi_n(\mathrm{res}_U A, x)$ (on the same site) maps $\phi\colon V \longrightarrow U$ to the n-th homotopy group of the geometric realization of the simplicial set $A(V)$ pointed by $A(\phi)(x)$. A map $f\colon A \longrightarrow B$ induces

$\pi_0 f\colon \pi_0 A \longrightarrow \pi_0 B$ and $\pi_n(\mathrm{res}_U f, x)\colon \pi_n(\mathrm{res}_U A, x) \longrightarrow \pi_n(\mathrm{res}_U B, f(U)_0(x))$ in a functorial way. To develop a homotopy theory of presheaves which takes into account the Grothendieck topology $\mathbf{T}$, one uses local weak equivalences.

Definition 9.22. A map $f\colon A \longrightarrow B$ in $\mathcal{M}$ is a *local* weak equivalence if the map $a(\pi_0 f)$ is an isomorphism and, for every $U \in \mathrm{Ob}\,\mathcal{I}$, $x \in A(U)_0$ and $n > 0$, the map $a(\pi_n(\mathrm{res}_U f, x))$ is an isomorphism. If we want to emphasize the rôle of the Grothendieck topology $\mathbf{T}$, we speak of $\mathbf{T}$-local weak equivalences, or P-local weak equivalences, if $\mathbf{T}$ is generated by a cd-structure P.

The class of local weak equivalences is closed under retracts and satisfies the "two out of three" axiom.

Lemma 9.23. *Objectwise weak equivalences are local weak equivalences.*

Proof. If $f\colon A \longrightarrow B$ is an objectwise weak equivalence, the maps $\pi_0 f$ and $\pi_n(\mathrm{res}_U f, x)$ are isomorphisms of presheaves for every $U \in \mathrm{Ob}\,\mathcal{I}$, $x \in A(U)_0$ and $n > 0$. The result follows, because a is a functor. □

An important set of local weak equivalences can be constructed as follows, provided the Grothendieck topology is given by a regular cd-structure P. Under this assumption, ψ_Q is a monomorphism for each distinguished square Q in P. Hence the associated square of pointed simplicial presheaves is a pullback square, with the base change $\mathrm{Fr}_{Q_1} S^0 \longrightarrow \mathrm{Fr}_{Q_2} S^0$ of $\mathrm{Fr}_{\psi_Q} S^0$ being a monomorphism. The simplicial mapping cylinder factors this map as a projective cofibration $\mathrm{Fr}_{Q_1} S^0 \rightarrowtail C$, followed by a simplicial homotopy equivalence. Consider the pushout diagram

$$\begin{tikzcd} \mathrm{Fr}_{Q_1} S^0 \arrow[r, tail] \arrow[d] & C \arrow[d] \\ \mathrm{Fr}_{Q_3} S^0 \arrow[r, tail] & sq \end{tikzcd}$$

and the induced map $sq \longrightarrow \mathrm{Fr}_{Q_4} S^0$. Factor this map as a projective cofibration $q\colon sq \rightarrowtail tq$, followed by a simplicial homotopy equivalence. Note that q is a projective cofibration of projectively cofibrant finite presheaves.

Lemma 9.24. *For every distinguished square Q in a regular cd-structure P, the map q is a P-local weak equivalence, as is the map $* \rightarrowtail \mathrm{Fr}_\emptyset S^0$.*

Proof. The map $sq \overset{q}{\rightarrowtail} tq \xrightarrow{\simeq} \mathrm{Fr}_{Q_4} S^0$ is the pointed version of a map which is a P-local weak equivalence by [19, 3.7]. Hence the lemma follows from 9.23 and the "two out of three" property for local weak equivalences. □

Let J_P be the set of maps $q\Box(\partial\Delta^n \hookrightarrow \Delta^n)_+$ for every distinguished square $Q \in P$ and $n \geq 0$, together with the map $\mathrm{Fr}_\emptyset(\partial\Delta^n \hookrightarrow \Delta^n)_+$ for every $n \geq 0$. It will serve as a set of generating acyclic cofibrations.

Lemma 9.25. *The maps in J_P-cell are P-local weak equivalences.*

Proof. By [11, 2.3], local weak equivalences of simplicial presheaves which are objectwise cofibrations are closed under tensoring with simplicial sets, cobase changes and sequential compositions. This holds also in the pointed case. □

Definition 9.26. A *P-fibration* is an objectwise fibration having the right lifting property with respect to J_P.

By adjointness, an objectwise fibration $f\colon A \xrightarrow{\text{ob}} B$ is a P-fibration if and only if $f(\emptyset)$ is a weak equivalence and

$$\begin{tikzcd} \mathcal{S}_{\mathcal{M}}(tq, A) \arrow[r] \arrow[d] & \mathcal{S}_{\mathcal{M}}(tq, B) \arrow[d] \\ \mathcal{S}_{\mathcal{M}}(sq, A) \arrow[r] & \mathcal{S}_{\mathcal{M}}(sq, B) \end{tikzcd}$$

is a homotopy pullback square for every distinguished square Q in P. Using the construction of the map q and the $\mathcal{S}$-Yoneda lemma, this diagram is a homotopy pullback square if and only if

$$\begin{tikzcd} A(Q_4) \arrow[r] \arrow[d] & B(Q_4) \arrow[d] \\ \mathcal{S}_{\mathcal{M}}(C, A) \times_{A(Q_1)} A(Q_3) \arrow[r] & \mathcal{S}_{\mathcal{M}}(C, B) \times_{B(Q_1)} B(Q_3) \end{tikzcd}$$

is a homotopy pullback square, where C is the simplicial mapping cylinder of the map $\mathrm{Fr}_{Q_1} S^0 \longrightarrow \mathrm{Fr}_{Q_2} S^0$. In particular, an objectwise fibrant presheaf A is P-fibrant if and only if $A(\emptyset)$ is contractible and

$$\begin{tikzcd} A(Q_4) \arrow[r] \arrow[d] & A(Q_3) \arrow[d] \\ A(Q_2) \arrow[r] & A(Q_1) \end{tikzcd}$$

is a homotopy pullback square for every distinguished square Q in P. Such presheaves are often called *flasque*. If the cd-structure is bounded and complete, these presheaves have the following property, cf. [19, 3.5] for the unpointed case.

Lemma 9.27. *If the cd-structure P is bounded and complete, P-local weak equivalences of P-fibrant pointed simplicial presheaves are objectwise weak equivalences.*

Before we can apply 9.27, we need a property of the restriction functor.

Lemma 9.28. *The functor $\mathrm{res}_U\colon \mathcal{M} \longrightarrow \mathcal{M}|_U$ preserves local weak equivalences and maps P-fibrations to $P|_U$-fibrations for every $U \in \mathrm{Ob}\,\mathcal{I}$.*

Proof. Let $f\colon A \longrightarrow B$ be a local weak equivalence. Then $\pi_0(\mathrm{res}_U f) = \mathrm{res}_U(\pi_0 f)$. Since res_U preserves sheaves, $a(\pi_0(\mathrm{res}_U f))$ is isomorphic to $\mathrm{res}_U(a(\pi_0 f))$. If x is a zero-simplex in $(\mathrm{res}_U A)(V \xrightarrow{\phi} U) = A(V)$, the map of pointed simplicial presheaves $(\mathrm{res}_\phi(\mathrm{res}_U f), x)$ coincides with $(\mathrm{res}_V f, x)$. Hence the first statement follows.

The second statement follows from the characterization of P-fibrations given below 9.26 and the fact that res_U preserves objectwise fibrations. Essentially, one has to check that if Q is a distinguished square in $P|_U$, then

$$\begin{tikzcd}
\mathrm{res}_U A(Q_4) \arrow[r] \arrow[d] & \mathcal{S}_{\mathcal{M}|_U}(C, \mathrm{res}_U A) \times_{\mathrm{res}_U A(Q_1)} \mathrm{res}_U A(Q_3) \arrow[d] \\
\mathrm{res}_U B(Q_4) \arrow[r] & \mathcal{S}_{\mathcal{M}|_U}(C, \mathrm{res}_U B) \times_{\mathrm{res}_U B(Q_1)} \mathrm{res}_U B(Q_3)
\end{tikzcd}$$

is a homotopy pullback square, where C is the simplicial mapping cylinder of the map $\mathrm{Fr}_{Q_1} S^0 \longrightarrow \mathrm{Fr}_{Q_2} S^0$. By definition, $\mathrm{pr}_U(Q)$ is a distinguished square in P, and the above square translates into

$$\begin{tikzcd}
A(\mathrm{pr}_U Q_4) \arrow[r] \arrow[d] & \mathcal{S}_{\mathcal{M}|_U}(C, \mathrm{res}_U A) \times_{A(\mathrm{pr}_U Q_1)} A(\mathrm{pr}_U Q_3) \arrow[d] \\
B(\mathrm{pr}_U Q_4) \arrow[r] & \mathcal{S}_{\mathcal{M}|_U}(C, \mathrm{res}_U B) \times_{B(\mathrm{pr}_U Q_1)} B(\mathrm{pr}_U Q_3).
\end{tikzcd}$$

For any presheaf D in $\mathcal{M}$, $\mathcal{S}_{\mathcal{M}|_U}(C, \mathrm{res}_U D)$ coincides with $\mathcal{S}_{\mathcal{M}}(C', D)$, where C' is the simplicial mapping cylinder of $\mathrm{Fr}_{\mathrm{pr}_U Q_1} S^0 \longrightarrow \mathrm{Fr}_{\mathrm{pr}_U Q_2} S^0$. The square above is then the same as the homotopy pullback square obtained from f and the distinguished square $\mathrm{pr}_U(Q)$, which completes the proof. $\square$

Corollary 9.29. *Let P be a bounded, complete and regular cd-structure. Then a map is a P-fibration and a P-local weak equivalence if and only if it is an objectwise acyclic fibration.*

Proof. The proof we give is similar to [19, 4.4]. Objectwise acyclic fibrations have the right lifting property with respect to all projective cofibrations, in particular to those in J_P. Hence one implication follows from 9.23. Let $f\colon A \longrightarrow B$ be a P-local weak equivalence and a P-fibration. It remains to prove that f is an objectwise weak equivalence, that is, $f(U)\colon A(U) \longrightarrow B(U)$ is a weak equivalence for every $U \in \mathrm{Ob}\,\mathcal{I}$. Since $f(U)$ is a fibration of pointed simplicial sets, it suffices to prove that the fiber over every zero-simplex $x\colon \Delta^0 \longrightarrow B(U)$ is contractible. The map $f(U)$ coincides with $\mathrm{Ev}_{\mathrm{id}_U}(\mathrm{res}_U f)$, where id_U is the terminal object in $\mathcal{I} \downarrow U$. Hence the above situation translates into the following pullback square

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d] & \mathrm{Fr}_{\mathrm{id}_U}\Delta^0 = * \arrow[d, "g"] \\ \mathrm{res}_U A \arrow[r, "\mathrm{res}_U f"] & \mathrm{res}_U B \end{tikzcd}$$

with g being the composition of a counit and $\mathrm{Fr}_{\mathrm{id}_U} x$. Note that $F(\mathrm{id}_U)$ coincides with the fiber of $f(U)$ over x. By 9.28, the map $\mathrm{res}_U f$ is a $P|_U$-fibration. Hence F is $P|_U$-fibrant. To prove that f' is a $P|_U$-local weak equivalence, note that because $\mathcal{M}_{\mathrm{ob}}$ is right proper, one can assume that $\mathrm{res}_U f$ is a map of objectwise fibrant presheaves. Since objectwise fibrations are in particular local fibrations in the sense of [11], one can apply [11, 1.12] to conclude that f' is a $P|_U$-local weak equivalence. From 9.27, f' is an objectwise weak equivalence, using that if P is bounded and complete, then so is $P|_U$. This completes the proof. □

Theorem 9.30. *Let P be a bounded, complete and regular cd-structure. Then $\mathcal{M}$ has a cofibrantly generated model structure with P-local weak equivalences as weak equivalences, P-fibrations as fibrations and projective cofibrations as cofibrations. This model structure is additionally proper and also cellular. One can choose generating cofibrations and generating acyclic cofibrations with finite domains and codomains.*

Proof. The claim of a cofibrantly generated model structure with finite domains and codomains follows from [7, 2.1.19], using 9.25 and 9.29. Since the objectwise projective model structure – which has the same cofibrations – is left proper, the same is true here. Right properness can be deduced from [11, 1.13]. To be more precise, since $\mathcal{M}_{\mathrm{ob}}$ is right proper, it suffices to consider the base change of a P-local weak equivalence of objectwise fibrant objects along a P-fibration. This base change is then a P-local weak equivalence, because objectwise fibrations are in particular local fibrations, and local fibrations which are local weak equivalences are closed under base change. This model structure is the Bousfield localization of the cellular model category $\mathcal{M}_{\mathrm{ob}}$ with respect to the set of projective cofibrations $\{q\}_{Q\in P} \cup \{* \rightarrowtail \mathrm{Fr}_\emptyset S^0\}$, hence cellular. □

This model structure is called the *P-local projective* model structure. Quillen equivalent model structures for simplicial presheaves and simplicial sheaves on a general site are described in [11] using local weak equivalences and objectwise cofibrations, and these model structures are the basis for $\mathbb{A}^1$-homotopy theory developed in [15]. The P-local projective model structure is emphasized by the subscript $\mathcal{M}_P$ and arrows $\xrightarrow{\sim P}$ and $\overset{P}{\twoheadrightarrow}$. To prove that the P-local projective model structure is monoidal, we have to assume that the cd-structure P is closed under products. That is, if Q is a distinguished square and U some object in $\mathcal{I}$, then also $Q \times_S U$ is a distinguished square. This weak assumption is satisfied in many cases of interest, see [20].

Lemma 9.31. *If P is bounded, complete, regular and closed under products, then $\mathcal{M}_P$ is a monoidal $\mathcal{M}_{\mathrm{ob}}$-model category.*

Proof. We have to check that the pushout product of a generating cofibration $\mathrm{Fr}_U(\partial\Delta^n \hookrightarrow \Delta^n)_+$ and a map in J_P is a P-local weak equivalence. If the map in J_P is of the form $\mathrm{Fr}_\emptyset(\partial\Delta^m \hookrightarrow \Delta^m)_+$, the pushout product is $\mathrm{Fr}_\emptyset i_+^{n,m}$ where $i^{n,m}$ is the pushout product $(\partial\Delta^n \hookrightarrow \Delta^n)\Box(\partial\Delta^m \hookrightarrow \Delta^m)$. Hence this map is a P-local weak equivalence. If we take a map in J_P of the form $q\Box(\partial\Delta^m \hookrightarrow \Delta^m)_+$, there is an isomorphism

$$\mathrm{Fr}_U(\partial\Delta^n \hookrightarrow \Delta^n)_+\Box(q\Box(\partial\Delta^m \hookrightarrow \Delta^m)_+) \cong (q\times_S U)\Box i_+^{m,n}$$

where $q\times_S U$ is the map obtained from the distinguished square $Q\times_S U$. The statement follows. □

Lemma 9.32. *The functor $-\wedge A$ preserves P-local weak equivalences for every pointed simplicial presheaf A.*

Proof. Factor $f\colon B \xrightarrow{\sim P} C$ as a projective cofibration $i\colon B \rightarrowtail T$ followed by an objectwise acyclic fibration $p\colon T \xrightarrow{\sim \mathrm{ob}} C$. Furthermore, replace A by a projectively cofibrant object $\kappa_A\colon KA \xrightarrow{\sim\mathrm{ob}} A$. The result is a diagram

$$\begin{tikzcd}
B\wedge KA \arrow[r, tail, "i\wedge KA"] \arrow[d, "B\wedge\kappa_A"'] & T\wedge KA \arrow[r, "p\wedge KA"] \arrow[d, "T\wedge\kappa_A"'] & C\wedge KA \arrow[d, "C\wedge\kappa_A"] \\
B\wedge A \arrow[r, "i\wedge A"] & T\wedge A \arrow[r, "p\wedge A"] & C\wedge A
\end{tikzcd}$$

where the vertical maps are objectwise weak equivalences by 9.18, and likewise for the two horizontal maps on the right hand side. The map $i\wedge KA$ is a P-local weak equivalence according to 9.31. Hence $f\wedge A = p\wedge A\circ i\wedge A$ is a P-local weak equivalence by 9.23. □

Lemma 9.33. *The P-local projective model structure is strongly left proper and satisfies the monoid axiom.*

Proof. The first claim follows from 9.19, by factoring a P-local weak equivalence as a projective cofibration followed by an objectwise acyclic fibration. For the second claim, let $f\colon A \overset{\sim P}{\rightarrowtail} B$ be an acyclic cofibration in $\mathcal{M}_P$ and C any presheaf. Then $f\wedge C$ is an objectwise cofibration, and a P-local equivalence by 9.32. The class of these maps is closed under cobase changes and sequential composition, cf. [11, 2.3]. Hence the monoid axiom holds. □

One further model structure will appear. Suppose additionally that $\mathcal{I}$ has a *represented interval*. This notion is a special case of a site with interval [15, 2.3].

Definition 9.34. A *represented interval* is an object $\mathbb{A}$ in $\mathcal{I}$, together with maps $\mu\colon \mathbb{A}\times_S\mathbb{A} \longrightarrow \mathbb{A}$ and $i_0, i_1\colon S \longrightarrow \mathbb{A}$, such that the following conditions hold.

1. The following diagrams commute.

$$\begin{tikzcd} S \times_S \mathbb{A} \arrow[r, "i_0 \times_S \mathbb{A}"] \arrow[d] & \mathbb{A} \times_S \mathbb{A} \arrow[d, "\mu"'] & \mathbb{A} \times_S S \arrow[l, "\mathbb{A} \times_S i_0"'] \arrow[d] \\ S \arrow[r, "i_0"] & \mathbb{A} & S \arrow[l, "i_0"'] \end{tikzcd} \qquad \begin{tikzcd} S \times_S \mathbb{A} \arrow[r, "i_1 \times_S \mathbb{A}"] \arrow[dr, "\mathrm{id}_{\mathbb{A}}"'] & \mathbb{A} \times_S \mathbb{A} \arrow[d, "\mu"'] & \mathbb{A} \times_S S \arrow[l, "\mathbb{A} \times_S i_1"'] \arrow[dl, "\mathrm{id}_{\mathbb{A}}"] \\ & \mathbb{A} & \end{tikzcd}$$

2. The map $\mathrm{Fr}_S S^0 \coprod \mathrm{Fr}_S S^0 \longrightarrow \mathrm{Fr}_{\mathbb{A}} S^0$ induced by i_0 and i_1 is an objectwise cofibration.

The represented interval should be homotopically trivial, meaning that the map $\mathrm{Fr}_{i_0} S^0 \colon \mathrm{Fr}_S S^0 \longrightarrow \mathrm{Fr}_{\mathbb{A}} S^0$ should be a weak equivalence. This can be achieved by a localization procedure.

Definition 9.35. A pointed simplicial presheaf Z which is fibrant in the P-local projective model structure is called *projectively* $\mathbb{A}$*-fibrant* if the map

$$\mathcal{M}(\mathrm{Fr}_{i_0} S^0, Z) \colon \mathcal{M}(\mathrm{Fr}_{\mathbb{A}} S^0, Z) \longrightarrow \mathcal{M}(\mathrm{Fr}_S S^0, Z)$$

is an objectwise weak equivalence. A map $f \colon A \longrightarrow B$ is an $\mathbb{A}$*-equivalence* if

$$\mathcal{S}_{\mathcal{M}}(Kf, Z) \colon \mathcal{S}_{\mathcal{M}}(KB, Z) \longrightarrow \mathcal{S}_{\mathcal{M}}(KA, Z)$$

is a weak equivalence for each projectively $\mathbb{A}$-fibrant Z. As above, $\kappa \colon K \longrightarrow \mathrm{Id}_{\mathcal{M}}$ is a cofibrant replacement functor for $\mathcal{M}_{\mathrm{ob}}$.

By 9.7, a presheaf Z which is fibrant in $\mathcal{M}_P$ is projectively $\mathbb{A}$-fibrant if and only if $Z(i_0 \times_S U) \colon Z(\mathbb{A} \times_S U) \longrightarrow Z(U)$ is a weak equivalence for every $U \in \mathrm{Ob}\,\mathcal{I}$. A map f of projectively cofibrant objects is an $\mathbb{A}$-equivalence if and only if $\mathcal{S}_{\mathcal{M}}(f, Z)$ is an objectwise weak equivalence for every projectively $\mathbb{A}$-fibrant Z. Local weak equivalences are $\mathbb{A}$-equivalences, since $\mathcal{M}_P$ is a $\mathcal{S}$-model category.

Theorem 9.36. *Let P be a bounded, complete and regular cd-structure on $\mathcal{I}$. There exists a weakly finitely generated model structure on the category $\mathcal{M}$ with $\mathbb{A}$-equivalences as weak equivalences and projective cofibrations as cofibrations. The fibrant objects are precisely the projectively $\mathbb{A}$-fibrant objects. The model structure is left proper and cellular.*

Proof. The model structure exists by [6, 4.1.1]. To be more precise, since $\mathcal{M}_P$ is a left proper cellular model category by 9.30, it suffices to identify the class of projectively $\mathbb{A}$-fibrant presheaves as the class of S-local objects in the sense of [6, 3.2.4] for some set of maps S in $\mathcal{M}$. Since $\mathcal{M}_P$ is a $\mathcal{S}$-model category, the homotopy function complexes can be chosen to agree with the pointed simplicial sets $\mathcal{S}_{\mathcal{M}}(A, B)$, provided A is projectively cofibrant and B is P-fibrant. Using the characterization of projectively $\mathbb{A}$-fibrant presheaves given below 9.35 and the $\mathcal{S}$-Yoneda lemma, we can choose S to be the set of maps $\mathrm{Fr}_U S^0 \longrightarrow \mathrm{Fr}_{\mathbb{A} \times_S U} S^0$ induced by $i_0 \times_S U$, where U varies through the set of objects of $\mathcal{I}$.

The model structure is left proper and cellular by [6, 4.1.1], hence cofibrantly generated. From [6, 3.4.16], $f\colon A \longrightarrow B$, with B being projectively $\mathbb{A}$-fibrant, is a fibration in this model structure if and only if f is a P-fibration and A is projectively $\mathbb{A}$-fibrant. We will use this to construct a set J' with finite domains and codomains which detects fibrations with projectively $\mathbb{A}$-fibrant codomain.

Factor the map $\mathrm{Fr}_{i_0}S^0\colon \mathrm{Fr}_S S^0 \longrightarrow \mathrm{Fr}_{\mathbb{A}} S^0$ using the simplicial mapping cylinder as a projective cofibration $c_{\mathbb{A}}\colon \mathrm{Fr}_S S^0 \rightarrowtail C_{\mathbb{A}}$, followed by a simplicial homotopy equivalence. Consequently, $\mathcal{M}(\mathrm{Fr}_{i_0}, Z)$ is a weak equivalence if and only if $\mathcal{M}(c_{\mathbb{A}}, Z)$ is a weak equivalence. Since $\mathcal{M}_{\mathrm{ob}}$ is monoidal, $\mathcal{M}(c_{\mathbb{A}}, Z)$ is an objectwise fibration. Hence a P-fibrant object Z is projectively $\mathbb{A}$-fibrant if and only if it the map $Z \xrightarrow{P} *$ has the right lifting property with respect to the set $c_{\mathbb{A}}\square \mathrm{Fr}_U(\partial\Delta^n \hookrightarrow \Delta^n)_+$ of pushout products. Let J' be the union of this set of pushout products and the set of generating acyclic cofibrations for the P-local projective model structure. Since the domain and codomain of $c_{\mathbb{A}}$ are finite, the maps in J' have finite domains and codomains. If f has the right lifting property with respect to J', then f is a P-fibration and the composition $A \xrightarrow{f} B \longrightarrow *$ has the right lifting property with respect to J'. In particular, also A is projectively $\mathbb{A}$-fibrant, and it follows that the model structure is weakly finitely generated. ◻

This model structure is called the *projective $\mathbb{A}$-model structure*, and its use is indicated by $\mathcal{M}_{\mathbb{A}}$, $\xrightarrow{\sim\mathbb{A}}$ and $\xrightarrow{\mathbb{A}}$, leaving the cd-structure understood. The fibrations are called *projective $\mathbb{A}$-fibrations*. Let us state in a formal way that this model structure is Quillen equivalent to the pointed version of the model structure [12, 1.1].

Theorem 9.37. *Let P be a bounded, complete and regular cd-structure which is additionally closed under products. By choosing monomorphisms as cofibrations and $\mathbb{A}$-equivalences as weak equivalences, we get the model structure from* [12, 1.1] *on $\mathcal{M}$. Let $\mathcal{M}_m$ denote the resulting model category, then the identity $\mathrm{Id}_{\mathcal{M}}\colon \mathcal{M}_{\mathbb{A}} \longrightarrow \mathcal{M}_m$ is the left adjoint of a Quillen equivalence.*

Proof. The statement about the Quillen equivalence is clear, once we set up the model structure $\mathcal{M}_m$. Following Jardine, we call the fibrations in the pointed version of the model structure constructed in [11] *global fibrations*. The weak equivalences in this model structure are the local weak equivalences, the cofibrations are the objectwise ones. Say that a globally fibrant presheaf Z is *i_0-fibrant* if the map $\mathcal{M}(\mathrm{Fr}_{i_0}S^0, Z)\colon \mathcal{M}(\mathrm{Fr}_S S^0, Z) \longrightarrow \mathcal{M}(\mathrm{Fr}_{\mathbb{A}} S^0, Z)$ is a global acyclic fibration. This is equivalent to the pointed version of the definition of $\mathrm{Fr}_{i_0}S^0$-local simplicial presheaves given in [11, section 1.2], using that $\mathrm{Fr}_{i_0}S^0$ is a monomorphism. A map $f\colon A \longrightarrow B$ is called an *i_0-equivalence* if the map $\mathcal{S}_{\mathcal{M}}(f, Z)\colon \mathcal{S}_{\mathcal{M}}(B, Z) \longrightarrow \mathcal{S}_{\mathcal{M}}(A, Z)$ is a weak equivalence of pointed simplicial sets for every i_0-fibrant presheaf Z. The i_0-equivalences are the weak equivalences in $\mathcal{M}_m$.

First we prove that any $\mathbb{A}$-equivalence is an i_0-equivalence. Let $f\colon A \xrightarrow{\sim\mathbb{A}} B$ be an $\mathbb{A}$-equivalence and Z be i_0-fibrant. Since globally fibrant presheaves are in particular P-fibrant, it is clear that Z is projectively $\mathbb{A}$-fibrant. Hence $\mathcal{S}_{\mathcal{M}}(Kf, Z)$ is a weak equivalence, which proves that Kf is an i_0-equivalence. Since Kf is connected to f via objectwise weak equivalences, f is an i_0-equivalence. This proves that $\mathbb{A}$-equivalences are i_0-equivalences.

Now let $f\colon A \longrightarrow B$ be an i_0-equivalence and Z be projectively $\mathbb{A}$-fibrant. Choose an objectwise weak equivalence $Z \xrightarrow{\sim\mathrm{ob}} Z'$, with Z' being globally fibrant. Such a map exists as a local weak equivalence, because of the model structure in [11], and it is an objectwise weak equivalence by 9.27, because both the domain and codomain are P-fibrant. The domain and codomain of $\mathrm{Fr}_{i_0}S^0$ are projectively cofibrant, thus we can deduce from 9.16 that Z' is i_0-fibrant. Since $\mathcal{M}_{\mathrm{ob}}$ is a $\mathcal{S}$-model category, the diagram

$$\begin{tikzcd} \mathcal{S}_{\mathcal{M}}(KB, Z) \arrow[r, "\mathcal{S}_{\mathcal{M}}(Kf,Z)"] \arrow[d, "\sim"', "\mathrm{ob}"] & \mathcal{S}_{\mathcal{M}}(KA, Z) \arrow[d, "\mathrm{ob}"', "\sim"] \\ \mathcal{S}_{\mathcal{M}}(KB, Z') \arrow[r, "\mathcal{S}_{\mathcal{M}}(Kf,Z')"] & \mathcal{S}_{\mathcal{M}}(KA, Z') \end{tikzcd}$$

has objectwise weak equivalences as its vertical maps. Hence it remains to prove that $\mathcal{S}_{\mathcal{M}}(Kf, Z')$ is a weak equivalence. Since f is an i_0-equivalence, Kf is so, too, which shows that $\mathcal{S}_{\mathcal{M}}(Kf, Z')$ is a weak equivalence. $\square$

Hence $\mathcal{M}_A$ is also Quillen equivalent, by [12, 1.2], to the Morel-Voevodsky model category for motivic homotopy theory, if the site is Sm/S equipped with the Nisnevich topology, and the represented interval is the affine line $\mathbb{A}^1$. This justifies the title of the section. From now on, we will sometimes refer to "motivic spaces" instead of "pointed simplicial presheaves on Sm/S".

Lemma 9.38. *Assume P is a bounded, complete and regular cd-structure which is closed under products. Then $- \wedge C$ preserves projective $\mathbb{A}$-equivalences for every projectively cofibrant presheaf C.*

Proof. Let Z be projectively $\mathbb{A}$-fibrant and C be projectively cofibrant. We will prove first that $\mathcal{M}(C, Z)$ is projectively $\mathbb{A}$-fibrant. Since $\mathcal{M}_P$ is monoidal, $\mathcal{M}(C, Z)$ is P-fibrant. The map $\mathcal{M}(\mathrm{Fr}_{i_0}S^0, \mathcal{M}(C, Z))$ is isomorphic to the map $\mathcal{M}(C, \mathcal{M}(\mathrm{Fr}_{i_0}S^0, Z))$, and $\mathcal{M}(\mathrm{Fr}_{i_0}S^0, Z)$ is a weak equivalence of objectwise fibrant objects. This uses that both $\mathrm{Fr}_S S^0$ and $\mathrm{Fr}_{\mathbb{A}}S^0$ are projectively cofibrant. The functor $\mathcal{M}(C, -)$ preserves such maps for C cofibrant, since $\mathcal{M}_{\mathrm{ob}}$ is monoidal. Hence $\mathcal{M}(C, Z)$ is projectively $\mathbb{A}$-fibrant.

Let $f\colon A \xrightarrow{\sim\mathbb{A}} B$ be an $\mathbb{A}$-equivalence. We have to prove that the map $\mathcal{S}_{\mathcal{M}}(K(f \wedge C), Z)$ is a weak equivalence for every $\mathbb{A}$-fibrant Z. By the argument above and the hypothesis on f, the map $\mathcal{S}_{\mathcal{M}}(Kf \wedge C, Z) \cong \mathcal{S}_{\mathcal{M}}(Kf, \mathcal{M}(C, Z))$ is a weak equivalence, thus $Kf \wedge C$ is an $\mathbb{A}$-equivalence. Using 9.18, the commutative diagram

$$\begin{tikzcd} K(A \wedge C) \arrow[r, two heads, "\sim \mathrm{ob}"] \arrow[d, "K(f\wedge C)"'] & A \wedge C \arrow[d, "f\wedge C"'] & KA \wedge C \arrow[l, "\sim \mathrm{ob}"'] \arrow[d, "Kf\wedge C"] \\ K(B \wedge C) \arrow[r, two heads, "\sim \mathrm{ob}"] & B \wedge C & KB \wedge C \arrow[l, "\sim \mathrm{ob}"'] \end{tikzcd}$$

shows that $K(f \wedge C)$ is an $\mathbb{A}$-equivalence if and only $Kf \wedge C$ is so. □

Corollary 9.39. *If P is a bounded, complete and regular cd-structure which is closed under products, then $\mathcal{M}_{\mathbb{A}}$ is a monoidal $\mathcal{M}_P$-model category.*

Proof. We have to check that the pushout product of $\mathrm{Fr}_U(\partial\Delta^n \hookrightarrow \Delta^n)_+$ and a generating acyclic cofibration in $\mathcal{M}_{\mathbb{A}}$ is an $\mathbb{A}$-equivalence for every $U \in \mathrm{Ob}\,\mathcal{I}$ and $n \geq 0$. Since Fr_U takes values in projectively cofibrant presheaves, the result follows from 9.38 and left properness of $\mathcal{M}_{\mathbb{A}}$. □

We can now extend 9.38 to all pointed simplicial presheaves.

Lemma 9.40. *The functor $- \wedge A$ preserves $\mathbb{A}$-equivalences for all $A \in \mathrm{Ob}\,\mathcal{M}$.*

Proof. Similar to the proof of 9.32. □

Lemma 9.41. *The model category $\mathcal{M}_{\mathbb{A}}$ is strongly left proper, and the monoid axiom holds.*

Proof. The proof of the first claim is similar to the proof of the corresponding claim in 9.33. The second claim requires extra arguments, but the proof of the monoid axiom for the P-local projective model structure shows that $f \wedge C$ is an objectwise cofibration which is also a weak equivalence in $\mathcal{M}_{\mathbb{A}}$ (an *objectwise $\mathbb{A}$-cofibration* for the rest of this proof). Therefore it suffices to prove that the class of objectwise $\mathbb{A}$-cofibrations is closed under cobase changes and sequential compositions. The latter is clear by 3.11, since $\mathcal{M}_{\mathbb{A}}$ is weakly finitely generated.

Let $f\colon A \xrightarrow{\sim \mathbb{A}} B$ be an objectwise $\mathbb{A}$-cofibration and $g\colon A \longrightarrow C$ any map. Factor g as a projective cofibration $i\colon A \rightarrowtail T$, followed by an objectwise acyclic fibration $p\colon T \xrightarrow{\sim \mathrm{ob}} C$. The cobase change of f along i is clearly an objectwise cofibration, but also an $\mathbb{A}$-equivalence, because $\mathcal{M}_{\mathbb{A}}$ is left proper. The cobase change of p along the cobase change of f along i is an objectwise weak equivalence by 9.14. Hence the cobase change of f along g is an $\mathbb{A}$-equivalence by the "two out of three" axiom. This completes the proof. □

Lemma 9.42. *If P is a bounded, complete and regular cd-structure which is closed under products, then $\mathcal{M}_{\mathbb{A}}$ is right proper.*

Proof. The proof is quite long and similar to the proof of the corresponding fact for Jardine's model structure on simplicial presheaves [12]. We prove the claim for the category $\mathcal{N}$ of simplicial presheaves on $\mathcal{I}$, which implies the statement for pointed simplicial presheaves. All results we have derived for $(\mathcal{M}, \wedge, \mathrm{Fr}_S S^0)$ have obvious analogs for $(\mathcal{N}, \times, \mathrm{Fr}_S \Delta^0)$, where $\times$ is the categorical product and $\mathrm{Fr}_S \colon \mathbf{sSet} \longrightarrow \mathcal{N}$ is the left adjoint of the functor which evaluates at S. The model structures we constructed on $\mathcal{M}$ can then also be obtained via [7, 1.1.8].

Let $\mathcal{K}$ be the class of $\mathbb{A}$-acyclic projective cofibrations f in $\mathcal{N}$ such that the base change of f along any projective $\mathbb{A}$-fibration is an $\mathbb{A}$-equivalence. Since the P-local projective model structure is right proper, $\mathcal{K}$ contains the class of local acyclic projective cofibrations. Sequential colimits commute with pullbacks, and sequential compositions of $\mathbb{A}$-equivalences are again $\mathbb{A}$-equivalences. Thus $\mathcal{K}$ is closed under sequential compositions. As for the class of $\mathbb{A}$-equivalences, $\mathcal{K}$ satisfies the "two out of three" axiom.

A more involved argument is needed to prove that $\mathcal{K}$ is closed under cobase changes. Assume $f \colon A \overset{\sim\mathbb{A}}{\rightarrowtail} B$ is in $\mathcal{K}$, and let $g \colon C \overset{\sim\mathbb{A}}{\rightarrowtail} C \cup_A B =: D$ be the cobase change of f along $h \colon A \longrightarrow C$. Let $p \colon E \overset{\mathbb{A}}{\twoheadrightarrow} D$ be a projective $\mathbb{A}$-fibration. We have to prove that the base change g' of g along p is a $\mathbb{A}$-equivalence. By factoring h as a projective cofibration, followed by an objectwise acyclic fibration, it suffices to prove the result when h is a projective cofibration. Taking base changes, one obtains the cube

$$\begin{tikzcd}
A \times_D E \arrow[rr, "f'"] \arrow[dr] \arrow[dd, two heads, "\mathbb{A}"'] & & B \times_D E \arrow[dr] \arrow[dd, two heads, "\mathbb{A}" near end] & \\
& C \times_D E \arrow[rr, "g'" near start] \arrow[dd, two heads, "\mathbb{A}"' near end] & & E \arrow[dd, two heads, "p"', "\mathbb{A}"] \\
A \arrow[rr, tail, "\sim\mathbb{A}", "f"' near end] \arrow[dr, tail, "h"'] & & B \arrow[dr] & \\
& C \arrow[rr, tail, "\sim\mathbb{A}", "g"'] & & D
\end{tikzcd}$$

where all faces are pullback squares. This is true by construction for all faces except the upper and the lower. Since the lower square is a pointwise pushout square of injective maps of sets, it is also a pullback square. The upper face is then pointwise a pullback square of injective maps of sets. Furthermore, pointwise the terminal corner of the upper face is the union of the two sets which inject in directly. Hence the upper face is also a pushout square, which implies that g' is a cobase change of f'. Now f' is an $\mathbb{A}$-equivalence and an objectwise cofibration, and such maps are closed under cobase changes, as we deduced in the proof of 9.41. This proves that g' is in $\mathcal{K}$.

The next argument uses that $\mathcal{M}$ and $\mathcal{N}$ are different. Recall that a P-fibrant simplicial presheaf A is projectively $\mathbb{A}$-fibrant if and only if $A \xrightarrow{P} *$ has the right lifting property with respect to the set $J_{\mathbb{A}}$ of maps $c_{\mathbb{A}} \Box \mathrm{Fr}_U(\partial\Delta^n \hookrightarrow \Delta^n)$, where $\mathrm{Fr}_S\Delta^0 \xrightarrow{c_{\mathbb{A}}} C_{\mathbb{A}} \xrightarrow{\simeq} \mathrm{Fr}_{\mathbb{A}}\Delta^0$ is the factorization of $\mathrm{Fr}_{i_0}\Delta^0$ obtained from the simplicial mapping cylinder. We will prove that this set of maps is contained in $\mathcal{K}$. Note that also the class $\mathcal{K}'$ of $\mathbb{A}$-equivalences having the property that their base change along any projective $\mathbb{A}$-fibration is again an $\mathbb{A}$-equivalence satisfies the "two out of three" axiom. Since $i_0\colon S \longrightarrow \mathbb{A}$ is a section of the canonical map $\mathbb{A} \longrightarrow S$, it suffices to prove that the projection to the second factor $\mathrm{Fr}_{\mathbb{A}}\Delta^0 \times \mathrm{Fr}_U L \longrightarrow \mathrm{Fr}_U L$ is in $\mathcal{K}'$ for every simplicial set L and $U \in \mathrm{Ob}\,\mathcal{I}$. For a projective $\mathbb{A}$-fibration $p\colon E \xrightarrow{\mathbb{A}} \mathrm{Fr}_U L$, the corresponding pullback diagram is

$$\begin{tikzcd}
\mathrm{Fr}_{\mathbb{A}}\Delta^0 \times E \arrow[r, "q"] \arrow[d, "\mathrm{Fr}_{\mathbb{A}}\times p"', "\mathbb{A}"] & E \arrow[d, "\mathbb{A}"', "p"] \\
\mathrm{Fr}_{\mathbb{A}}\Delta^0 \times \mathrm{Fr}_U L \arrow[r] & \mathrm{Fr}_U L
\end{tikzcd}$$

with q being the projection to the second factor. By the unpointed version of 9.40, the map q is an $\mathbb{A}$-equivalence, and hence $J_{\mathbb{A}}$ is contained in $\mathcal{K}$.

This enables us to prove right properness. Let $f\colon A \xrightarrow{\sim\mathbb{A}} B$ be an $\mathbb{A}$-equivalence and $p\colon E \xrightarrow{\mathbb{A}} B$ be a projective $\mathbb{A}$-fibration. We have to prove that the base change of f along p is an $\mathbb{A}$-equivalence. Using the small object argument with respect to the union J of the set of generating acyclic cofibrations for the P-local projective model structure and the set $J_{\mathbb{A}}$ which detects projectively $\mathbb{A}$-fibrant simplicial presheaves, we replace B by a projectively $\mathbb{A}$-fibrant simplicial presheaf $j_B\colon B \xrightarrow{\sim\mathbb{A}} B'$ such that j_B is in $\mathcal{K}$. Furthermore, we replace $j_B \circ f$ by the composition $f' \circ j_A$, where $j_A\colon A \xrightarrow{\sim\mathbb{A}} A'$ is in $\mathcal{K}$ and f' has the right lifting property with respect to J. In particular, A' is a projectively $\mathbb{A}$-fibrant simplicial presheaf, and f' is an $\mathbb{A}$-equivalence of such. This implies that f' is an objectwise weak equivalence. Finally, we factor $j_B \circ p$ as an $\mathbb{A}$-acyclic projective cofibration $i\colon E \xrightarrow{\sim\mathbb{A}} D$ followed by a projective $\mathbb{A}$-fibration $q\colon D \xrightarrow{\mathbb{A}} B'$.

The base change of f' along q is an objectwise weak equivalence, because $\mathcal{M}_{\mathrm{ob}}$ is right proper. The base change of j_A along the base change of q along f' is an $\mathbb{A}$-equivalence, because j_A is in $\mathcal{K}$. Since j_B is also in $\mathcal{K}$, the base change $B \times_{B'} D \longrightarrow D$ of j_B along q is also an $\mathbb{A}$-equivalence. Hence the induced map $g\colon E \longrightarrow B \times_{B'} D$ is an $\mathbb{A}$-equivalence of projective $\mathbb{A}$-fibrations with codomain B'. One can regard g as a map of fibrant objects in the over category $\mathcal{N} \downarrow B$, with model structure inherited from the projective $\mathbb{A}$-model structure. This implies that g is even an objectwise weak equivalence. By right properness of the objectwise projective model structure, we can finally conclude that the base change of f along p is an $\mathbb{A}$-equivalence. $\square$

For a bounded, complete and regular cd-structure closed under products, the projective $\mathbb{A}$-model structure has all the properties needed to apply the results in [9, 4]. That is, the stable model structure on $\mathrm{Sp}(\mathcal{M}_{\mathbb{A}}, A)$ exists as described in 6.15, where A is any projectively cofibrant and finite pointed simplicial presheaf. The most interesting candidate is one giving rise to the motivic stable homotopy category, in which the "circle" is obtained from the represented interval $\mathbb{A}^1$, the affine line. One choice, as described in [15, 3.2.2], is the pushout of the diagram

$$* \longleftarrow \mathrm{Fr}_S S^1 \xrightarrow{\mathrm{Fr}_{i_1} S^1} \mathrm{Fr}_{\mathbb{A}^1-\{0\}} S^1$$

Since $\mathrm{Fr}_{i_1} S^1$ might not be a projective cofibration, we replace it by a projective cofibration using the simplicial mapping cylinder to get a projectively cofibrant and finite presheaf T. By [9, 5.7], the choice of T – up to $\mathbb{A}^1$-equivalence – does not matter, cf. 9.40. Moreover, because the identity $\mathrm{Id}_{\mathcal{M}}$ is a left Quillen equivalence from $\mathcal{M}_{\mathbb{A}^1}$ to the pointed version of Jardine's model structure on $\mathcal{M}$ by 9.37, it follows from [9, 5.7] that the stable model structure on $\mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T)$ is Quillen equivalent to Jardine's model of the motivic stable homotopy category. Using Voevodsky's result on the cyclic permutation map, we get the following.

Lemma 9.43. *The functor* $- \wedge T\colon \mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T) \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T)$ *is a Quillen equivalence for the stable model structure.*

Proof. The identity $\mathrm{Id}_{\mathcal{M}}$ induces a commutative diagram of left Quillen functors

$$\begin{tikzcd}
\mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T) \arrow[r] \arrow[d, "-\wedge T"'] & \mathrm{Sp}(\mathcal{M}_m, T) \arrow[d, "-\wedge T"] \\
\mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T) \arrow[r] & \mathrm{Sp}(\mathcal{M}_m, T)
\end{tikzcd}$$

in which the horizontal arrows are Quillen equivalences. Here $\mathcal{M}_m$ denotes the pointed version of Jardine's model structure on $\mathcal{M}$. In particular, the cofibrations are the objectwise cofibrations, and hence every presheaf is cofibrant in $\mathcal{M}_m$. By [9, 10.3], it suffices to prove that T is weakly equivalent to a symmetric presheaf, that is, to a presheaf A rendering a diagram

$$\begin{tikzcd}
A^3 \wedge \mathrm{Fr}_S S^0 \arrow[r, "A^3\wedge \mathrm{Fr}_{i_0} S^0"] \arrow[dr, "\mathrm{id}_{A^3\wedge \mathrm{Fr}_S S^0}"'] & A^3 \wedge \mathrm{Fr}_{\mathbb{A}^1} S^0 \arrow[d, "H"] & A^3 \wedge \mathrm{Fr}_S S^0 \arrow[l, "A^3\wedge \mathrm{Fr}_{i_1} S^0"'] \arrow[dl, "\mathrm{cyc}\wedge \mathrm{Fr}_S S^0"] \\
& A^3 \wedge \mathrm{Fr}_S S^0 &
\end{tikzcd}$$

commutative, where $\mathrm{cyc}\colon A^3 \longrightarrow A^3$ is the cyclic permutation map, cf. [9, 10.2]. From [12, 3.13], the presheaf $\mathbb{A}^1/\mathbb{A}^1 - \{0\}$, which is weakly equivalent to T, is symmetric. Hence $- \wedge T$ on the right hand side is a Quillen equivalence, which implies the same statement for the functor $- \wedge T$ on the left hand side. $\square$

10. Motivic functors

Recall the category of motivic spaces $\mathcal{M}$ discussed in the previous section. Let $\mathbf{f}\mathcal{M}$ be a category of finite motivic spaces which contains the projectively cofibrant version T of the Tate object introduced before 9.43, the unit $T^0 = \mathrm{Fr}_S S^0$ and is closed under the smash product $\wedge$. To emphasize the choice of $\mathbf{f}\mathcal{M}$, we let $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ denote the category of $\mathcal{M}$-functors from $\mathbf{f}\mathcal{M}$ to $\mathcal{M}$. These were called "motivic functors" in the introduction, see Section 2 for details on $\mathcal{M}$-functors.

We follow the general setup, by first defining the pointwise structure, then the homotopy functor structure, and finally the stable structure. Note that the pointwise structure is defined entirely in terms of the weakly finitely generated model structure $\mathcal{M}_{\mathbb{A}^1}$ which is a model for the pointed motivic homotopy category.

10.1. The pointwise structure.

Definition 10.1. A map $f\colon X \longrightarrow Y$ in $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ is a

- *pointwise weak equivalence* if $f(v)\colon X(v) \longrightarrow Y(v)$ is a weak equivalence in $\mathcal{M}_{\mathbb{A}^1}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$,
- *pointwise fibration* if $f(v)\colon X(v) \longrightarrow Y(v)$ is a fibration in $\mathcal{M}_{\mathbb{A}^1}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$,
- *cofibration* if it has the left lifting property with respect to all pointwise acyclic fibrations.

The category $[\mathbf{f}\mathcal{M}, \mathcal{M}]$, together with these classes of morphisms, is denoted $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{pt}}$ and referred to as the *pointwise structure* on $[\mathbf{f}\mathcal{M}, \mathcal{M}]$.

Theorem 10.2. *The pointwise structure* $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{pt}}$ *is a weakly finitely generated proper monoidal model category satisfying the monoid axiom.*

Proof. The assumptions on $\mathbf{f}\mathcal{M}$ are exactly conditions **f1** and **f2** of Section 4, so the theorem follows from 4.2, 4.8 and 4.9. ☐

Remark 10.3. The symmetric monoidal product in $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ is written $\wedge$ (or $\wedge^{\mathbf{f}}$ if it is important to emphasize the rôle of $\mathbf{f}\mathcal{M}$), and the unit is the inclusion $\mathbb{I} = \mathbb{I}_{\mathbf{f}}\colon \mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$.

Theorem 10.4. *Let* $\mathbf{f}\mathcal{M}$ *be the category of all finite motivic spaces, then smashing with a cofibrant object in* $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{pt}}$ *preserves pointwise equivalences.*

Proof. From 9.9 and 9.10: every motivic space is a filtered colimit of finite motivic spaces, so **f0**, **f1** and **f2** are satisfied. The claim follows from 4.12. ☐

10.2. The homotopy functor structure.

Let $\mathbf{f}\mathcal{M}$ be a full sub-$\mathcal{M}$-category of $\mathcal{M}$ which satisfies the following conditions.

f1 Every object in $\mathbf{f}\mathcal{M}$ is finite.
f2 The motivic space T is in $\mathbf{f}\mathcal{M}$, and $\mathbf{f}\mathcal{M}$ is closed under the smash product.

f3 If $w \xleftarrow[j]{\sim} v \longrightarrow u$ is a diagram in $\mathcal{M}$, where $v \xrightarrow[j]{\sim} w$ is an acyclic monomorphism of finite motivic spaces and $u \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$, then the pushout $w \cup_v u$ is in $\mathbf{f}\mathcal{M}$.

f5 The simplicial mapping cylinder is available in $\mathbf{f}\mathcal{M}$.

Note that this corresponds to the old axioms **f1**–**f5**, but with respect to the model structure $\mathcal{M}_m$. Axiom **f4** is missing, since all motivic spaces are cofibrant in $\mathcal{M}_m$, and accordingly **f3** now refers to acyclic monomorphisms.

The following property is desirable, but it is not true for all our examples:

f0 Every object in $\mathcal{M}$ is a filtered colimit of objects in $\mathbf{f}\mathcal{M}$.

Let M be the set of acyclic monomorphisms of finite motivic spaces. The techniques developed in Section 3.3 yield an $\mathcal{M}$-natural transformation of $\mathcal{M}$-functors $\rho^M\colon \mathrm{Id}_{\mathcal{M}} \longrightarrow R^M$ and natural transformations $\mathrm{Id}_{\mathcal{M}} \xrightarrow{\varphi} \Phi^M \xrightarrow{\omega} R^M$ satisfying the following properties according to 3.26, 3.29 and 3.30.

Lemma 10.5. *For every finite motivic space v, the motivic spaces $\Phi^M(v)$ and $R^M(v)$ are fibrant in $\mathcal{M}_{\mathbb{A}^1}$, and the maps $\rho^M(v)$ and $\varphi(v)$ are acyclic monomorphisms.*

The use of the set M instead of the set J' has an important advantage.

Lemma 10.6. *Let A be a motivic space such that $A \longrightarrow *$ is in M-inj. If $\phi\colon v \longrightarrow w$ is an acyclic monomorphism of finite motivic spaces, the induced map $\mathcal{M}(\phi, A)\colon \mathcal{M}(w, A) \longrightarrow \mathcal{M}(v, A)$ is a weak equivalence in $\mathcal{M}_{\mathbb{A}^1}$.*

Proof. Let $\phi\colon v \longrightarrow w$ be an acyclic monomorphism of finite motivic spaces, and let A be a motivic space such that $A \longrightarrow *$ is in M-inj. We will prove that the map $\mathcal{M}(\phi, A)$ is even an acyclic fibration in $\mathcal{M}_{\mathbb{A}^1}$. Consider a lifting problem

$$\begin{tikzcd} si \arrow[r] \arrow[d, "i"'] & \mathcal{M}(w, A) \arrow[d, "\mathcal{M}(\phi, A)"] \\ ti \arrow[r] & \mathcal{M}(v, A). \end{tikzcd}$$

By adjointness, this lifting problem is equivalent to the lifting problem

$$\begin{tikzcd} si \wedge w \cup_{si \wedge v} ti \wedge v \arrow[r] \arrow[d, "i \Box \phi"'] & A \arrow[d] \\ ti \wedge w \arrow[r] & * \end{tikzcd}$$

where $i \Box \phi$ is the pushout product map of a generating cofibration $i \in I$ and ϕ. The map $i \Box \phi$ is an acyclic monomorphism of finite motivic spaces, and $A \longrightarrow *$ has the right lifting property with respect to such maps. We are done. $\square$

Given $X \in \mathrm{Ob}\,[\mathbf{f}\mathcal{M}, \mathcal{M}]$, we define $X^h := \mathbb{I}_* X \circ R^M \circ \mathbb{I}$. The map ρ^M induces a natural transformation $\mathrm{Id}_{[\mathbf{f}\mathcal{M},\mathcal{M}]} \longrightarrow (-)^h$. Similarly, we define the functor $\hbar_M(X)\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ via $\hbar_M(X)(v) := \mathbb{I}_* X(\Phi^M(v))$, and φ and ω induce natural transformations of functors $\mathrm{Id}_{[\mathbf{f}\mathcal{M}\mathcal{M}]} \longrightarrow \hbar_M \longrightarrow (-)^h$.

Definition 10.7. A map $f\colon X \longrightarrow Y$ in $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ is an

- hf-*weak equivalence* if the map $\hbar_M(X)(v)$ is a weak equivalence in $\mathcal{M}_{\mathbb{A}^1}$ for every $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$,
- hf-*fibration* if it is a pointwise fibration and for all acyclic monomorphisms $\phi\colon v \overset{\sim}{\hookrightarrow} w \in \mathbf{f}\mathcal{M}$ the diagram

$$\begin{array}{ccc} X(v) & \xrightarrow{X(\phi)} & X(w) \\ {\scriptstyle f(v)}\downarrow & & \downarrow{\scriptstyle f(w)} \\ Y(v) & \xrightarrow{Y(\phi)} & Y(w) \end{array}$$

  is a homotopy pullback square in $\mathcal{M}_{\mathbb{A}^1}$.

These two classes of morphisms, together with the class of cofibrations, is referred to as the *homotopy functor structure* on $[\mathbf{f}\mathcal{M}, \mathcal{M}]$. It is indicated by the subscript $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{hf}}$.

Lemma 10.8. *A map in* $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ *is both an* hf-*fibration and an* hf-*equivalence if and only if it is a pointwise acyclic fibration.*

Proof. The proofs of 5.7 and 5.8 work as before using Φ^M instead of $\Phi^{J'}$. □

Theorem 10.9. *The homotopy functor structure* $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{hf}}$ *is a weakly finitely generated and proper monoidal model category.*

Proof. First we establish the weakly finitely generated model structure. Making the obvious adjustments to allow all monomorphisms as cofibrations in the proof of 5.10 and its subsidiaries, we note that the only point which needs mentioning is the claim in 5.9: if $\phi\colon v \overset{\sim}{\hookrightarrow} w$ is an acyclic monomorphism in $\mathbf{f}\mathcal{M}$ and u is an object in $\mathbf{f}\mathcal{M}$, then the induced map

$$\mathcal{M}(w, \Phi^M(u)) \longrightarrow \mathcal{M}(v, \Phi^M(u))$$

is a weak equivalence in $\mathcal{M}$. But this is the content of 10.6.

Concerning the monoidal part, in 5.11 the requirement **f4** is superfluous since every motivic space is cofibrant in $\mathcal{M}_m$. The crucial observation is that if $\phi\colon v \overset{\sim}{\hookrightarrow} w$ is an acyclic monomorphism in $\mathbf{f}\mathcal{M}$ and $u \in \mathbf{f}\mathcal{M}$, then the map $\phi \wedge u\colon v \wedge u \overset{\sim}{\hookrightarrow} w \wedge u$ is still an acyclic monomorphism in $\mathbf{f}\mathcal{M}$. Properness follows exactly as in 5.12. □

Theorem 10.10. *Let* $\mathbf{f}\mathcal{M}$ *be the category of all finite objects in* $\mathcal{M}$*, then smashing with a cofibrant object in* $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{hf}}$ *preserves* hf*-equivalences and* $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{hf}}$ *satisfies the monoid axiom.*

Proof. Factorize the hf-equivalence into an hf-acyclic cofibration followed by an hf-acyclic fibration. Since hf-acyclic fibrations are pointwise acyclic fibrations, and smashing with a cofibrant object preserves pointwise weak equivalences by 10.4, we may assume the hf-equivalence is a cofibration. But smashing with a cofibrant object preserves hf-acyclic cofibrations since $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{hf}}$ is monoidal.

The monoid axiom is shown to hold as follows. Let $X \overset{\sim\mathrm{hf}}{\rightarrowtail} Y$ be a generating hf-acyclic cofibration, and Z any object in $[\mathbf{f}\mathcal{M},\mathcal{M}]$. Let $W \xrightarrow{\sim\mathrm{pt}} Z$ be a cofibrant replacement. Since both X and Y are cofibrant, there is the diagram

$$\begin{tikzcd} X\wedge W \arrow[r, "\sim\mathrm{hf}"] \arrow[d, "\sim"', "\mathrm{hf}"] & Y\wedge W \arrow[d, "\sim"', "\mathrm{hf}"] \\ X\wedge Z \arrow[r] & Y\wedge Z \end{tikzcd}$$

and hence $X\wedge Z \xrightarrow{\sim\mathrm{hf}} Y\wedge Z$. The full monoid axiom 3.14 follows by the method in 6.24. ☐

Remark 10.11. The validity of axiom **f5** implies that a pointwise fibration $f\colon X \xrightarrow{\mathrm{pt}} Y$ is an hf-fibration if and only if the square

$$\begin{tikzcd} X(v) \arrow[r, "X(\phi)"] \arrow[d, "f(v)"'] & X(w) \arrow[d, "f(w)"] \\ Y(v) \arrow[r, "Y(\phi)"] & Y(w) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathbb{A}^1}$ for any weak equivalence $\phi\colon v \xrightarrow{\sim} w$ in $\mathbf{f}\mathcal{M}$. In particular, the fibrant functors in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{hf}}$ are the pointwise fibrant homotopy functors.

10.3. **The stable structure.** Once the hf-model structure is in place, the stable model structure is defined more or less exactly as in the general case in Section 6. The stable equivalences are the maps which become pointwise weak equivalences after a stabilization process, and the stably fibrant objects are morally the "Ω-spectra". The stabilization process uses the $\mathcal{M}$-natural transformation

$$t_X\colon X \longrightarrow \mathbb{T}(X) = \mathcal{M}(T, X(T\wedge -))$$

defined in 6.1 and the functor

$$\mathbb{T}^\infty\colon [\mathbf{f}\mathcal{M},\mathcal{M}] \longrightarrow [\mathbf{f}\mathcal{M},\mathcal{M}]$$

used in 6.5 to characterize stable equivalences of pointwise fibrant homotopy functors. Note further that it is also possible to stabilize the composition of functors $\mathbb{I}_*X \circ \Phi^M \colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ – although Φ^M is not an $\mathcal{M}$-functor – because of the natural maps $\theta_v \colon \Phi^M(v) \wedge T \longrightarrow \Phi^M(v \wedge T)$ which are supplied by 3.32.

We keep the same blanket assumptions concerning some category $\mathbf{f}\mathcal{M}$ of finite motivic spaces as in Section 10.2.

Definition 10.12. A morphism $f\colon X \longrightarrow Y$ in $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ is a

- *stable equivalence* if the induced map $\mathbb{T}^\infty(\hbar_M(R \circ X)) \longrightarrow \mathbb{T}^\infty(\hbar(R \circ Y))$ is a pointwise weak equivalence,
- *stable fibration* if it is an hf-fibration, and for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{M}$ the diagram

$$\begin{tikzcd} X(v) \arrow[r, "t_X(v)"] \arrow[d] & \mathbb{T}(X)(v) \arrow[d] \\ Y(v) \arrow[r, "t_Y(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

  is a homotopy pullback square in $\mathcal{M}_{\mathbb{A}^1}$.

The class of cofibrations is not changed. The category $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ together with these classes of morphisms is denoted $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$ and referred to as the *stable structure* on $[\mathbf{f}\mathcal{M}, \mathcal{M}]$.

The category $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ together with these classes of morphisms is denoted $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$ and referred to as the *stable structure* on $[\mathbf{f}\mathcal{M}, \mathcal{M}]$. Some statements concerning the general stable structure will be modified in the following, but the first important ingredient does not require a new proof.

Lemma 10.13. *A map is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

For any finite motivic space v, let τ_v denote the composition

$$\mathcal{M}(T \wedge v, -) \wedge T \xrightarrow{\cong} \mathcal{M}(T, \mathcal{M}(v, -)) \wedge T \xrightarrow{\epsilon_T \mathcal{M}(v,-)} \mathcal{M}(v, -)$$

and factor it using the simplicial mapping cylinder as $d_v \colon \mathcal{M}(T \wedge v, -) \wedge T \rightarrowtail D_v$, followed by a simplicial homotopy equivalence. The set $\mathcal{D}$ is the set of pushout product maps $d_v \square i$, where $i\colon si \rightarrowtail ti$ is some generating cofibration in $\mathcal{M}_{\mathbb{A}^1}$ and v is some motivic space in $\mathbf{f}\mathcal{M}$. To deduce that the stable structure is indeed a model structure, the maps in $\mathcal{D}$-cell have to be stable equivalences. The corresponding general statement 6.24 relies on the characterization of stable equivalences of cofibrant functors 6.21, which in turn requires **f4** to hold. Therefore, we have to supply a new proof.

Lemma 10.14. *The maps in $\mathcal{D}$-cell are stable equivalences.*

Proof. First we will prove that the maps in $\mathcal{D}$ are stable equivalences, using 6.19 instead of 6.21. The same argument as in the beginning of the proof of 6.22 shows that it is sufficient to prove that the map $\epsilon_T \mathcal{M}(v, -)$ is a stable equivalence for every $v \in \operatorname{Ob}\mathbf{f}\mathcal{M}$. By 6.19, we therefore have to prove that the map of spectra $\overline{w^*(\epsilon_T\mathcal{M}(v,-))} \cong \epsilon_T(\overline{w^*\mathcal{M}(v,-)})$ is a stable equivalence of spectra for every $w \in \operatorname{Ob}\mathbf{f}\mathcal{M}$. As in the general part, the nth term of $\overline{w^*\mathcal{M}(v,-)}$ is $\mathcal{M}(v,\Phi^M(w \wedge T^n))$, and the structure maps of this spectrum involve the map $\Phi^M(u) \wedge T \longrightarrow \Phi^M(u \wedge T)$ which is natural in u. To simplify the notation, let us denote this spectrum as $\mathcal{M}(v,\Phi^M(w \wedge -))$.

Let $q\colon Q \longrightarrow \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$ be a cofibrant replacement functor in the category of spectra, so $q(E)_n$ is an acyclic fibration in $\mathcal{M}$ for every spectrum E and $n \geq 0$. Consider the diagram

$$\begin{tikzcd}
(Q\mathcal{M}(T,\mathcal{M}(v,\Phi^M(w\otimes -))))\otimes T \arrow[r, "q\otimes T"] \arrow[d, "Q\epsilon_T"'] & \mathcal{M}(T,\mathcal{M}(v,\Phi^M(w\otimes -)))\otimes T \arrow[d, "\epsilon_T"] \\
Q\mathcal{M}(v,\Phi^M(w\otimes -)) \arrow[r, "q"] & \mathcal{M}(v,\Phi^M(w\otimes -)).
\end{tikzcd}$$

The composition

$$s^\infty \circ q \circ Q\epsilon_T\colon (Q\mathcal{M}(T,\mathcal{M}(v,\Phi^M(w\otimes -))))\otimes T \longrightarrow S^\infty(\mathcal{M}(v,\Phi^M(w\otimes -)))$$

is a stable equivalence of spectra by the hypothesis, for the following reason. Since $\Phi^M(u) \longrightarrow *$ has the right lifting property with respect to set M of acyclic monomorphisms of finite motivic spaces, an adjointness argument shows that also $\mathcal{M}(v,\Phi^M(u))$ has this property. The reason is that M is closed under smashing with finite motivic spaces. In particular, the spectrum $\mathcal{M}(v,\Phi^M(w\wedge -))$ is pointwise fibrant, hence the codomain of $s^\infty \circ q \circ Q\epsilon_T$ is stably fibrant by 6.16. Its domain is cofibrant by construction. Recall that $\mathcal{M}(T,-)$ commutes with filtered colimits. Up to an isomorphism, the stable weak equivalence

$$s^\infty \circ q\colon Q\mathcal{M}(T,\mathcal{M}(v,\Phi^M(w\otimes -))) \longrightarrow S^\infty(\mathcal{M}(T,\mathcal{M}(v,\Phi^M(w\otimes -))))$$

is an adjoint of $s^\infty \circ q \circ Q\epsilon_T$. Thus $q \circ Q\epsilon_T$ is a stable equivalence. The map $q \otimes T$ is a pointwise weak equivalence, since $\mathcal{M}$ is strongly monoidal. Hence $\epsilon_T\mathcal{M}(v,\Phi^M(w\wedge -))$ is a stable equivalence of spectra, which implies that every map in $\mathcal{D}$ is a stable equivalence.

To extend this result to a map in $\mathcal{D}$-cell, once again 6.19 replaces 6.21. The proof then proceeds as the proof of 6.24, with the help of 6.23 and the simplification that it is not necessary to restrict to a map with cofibrant domain first. □

Theorem 10.15. *The stable structure* $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ *is a weakly finitely generated, proper and monoidal model category.*

Proof. The model structure follows from the hf-structure as in 6.25, using 10.13 and 10.14. The reference to property **f4** in 6.26 – apart from its use in 6.25 – was only to assure that the hf-structure was monoidal. Because smashing with any motivic space preserves weak equivalences (see 9.40) and monomorphisms, we get that $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ is a monoidal model category. Properness follows as before. □

By an argument exactly as in the hf-structure 10.10, we get

Theorem 10.16. *Let $\mathbf{f}\mathcal{M}$ be the category of all finite objects in $\mathcal{M}$. Then smashing with a cofibrant object in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ preserves stable equivalences, and $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ satisfies the monoid axiom.*

From this, we get the following theorem by using the techniques in Section 7. Note however that again certain statements need an extra argument, namely 7.3 and 7.4. The extra argument is that for the representable $\mathcal{M}$-functors $\mathcal{M}(T^n,-)$, the canonical map $\mathcal{M}(T^n,\Phi^M(v)) \longrightarrow \mathcal{M}(T^n,R^M(v))$ is a weak equivalence, because T^n is projectively cofibrant for every $n \geq 0$ and 10.5 holds. As a consequence, $\mathcal{M}(T^n,-) \xrightarrow{\sim \mathrm{hf}} \mathcal{M}(T^n,-)^h = \mathcal{M}(T^n,R^M(-))$ is a fibrant replacement in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{hf}}$ and the proofs of 7.3 and 7.4 go through as before.

Theorem 10.17. *Let $\mathbf{e}\mathcal{M} \hookrightarrow \mathcal{M}$ be the category of all finite motivic spaces $v \in \mathcal{M}$ such that there exists an acyclic monomorphism $T^n \overset{\sim}{\hookrightarrow} v$ for some $n \geq 0$. Then the evaluation functor*

$$[\mathbf{e}\mathcal{M},\mathcal{M}] \xrightarrow{\mathrm{ev}} \mathrm{Sp}(\mathcal{M},T), \qquad \mathrm{ev}(X)_n = X(T^n)$$

is the right adjoint in a Quillen equivalence between $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ and the stable model structure on T-spectra.

Remark 10.18. We need an explicit choice of generating (acyclic) cofibrations in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$. Let I and J be the generating cofibrations and acyclic cofibrations of the weakly finitely generated model category $\mathcal{M}_{\mathbb{A}^1}$ constructed in 9.36. Then the common generating cofibrations of $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{pt}}$, $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{hf}}$ and $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ are the maps

$$\mathcal{M}(v,-)\wedge i\colon \mathcal{M}(v,-)\wedge si \longrightarrow \mathcal{M}(v,-)\wedge ti$$

where v varies over $\mathbf{f}\mathcal{M}$ (it is sufficient to choose one for each element in the set of isomorphism classes) and $i\colon si \rightarrowtail ti \in I$ varies over the generating cofibrations in $\mathcal{M}_{\mathbb{A}^1}$.

The generating acyclic cofibrations are a bit more complicated. In the pointwise structure, we use the maps

$$\mathcal{M}(v,-)\wedge j\colon \mathcal{M}(v,-)\wedge sj \longrightarrow \mathcal{M}(v,-)\wedge tj$$

where v varies over $\mathbf{f}\mathcal{M}$ and $j\colon sj \rightarrowtail tj \in J$ varies over the generating acyclic cofibrations in $\mathcal{M}_{\mathbb{A}^1}$.

For the hf-structure, consider the acyclic monomorphisms $\phi\colon v \xrightarrow{\sim} w \in \mathbf{f}\mathcal{M}$, and let $\mathcal{M}(w,-) \overset{c_\phi}{\rightarrowtail} C_\phi \xrightarrow{\simeq} \mathcal{M}(v,-)$ be the factorization through the simplicial mapping cylinder. The hf-acyclic cofibrations are generated by the pointwise acyclic cofibrations, together with the pushout product maps

$$c_\phi \Box i\colon \mathcal{M}(w,-)\wedge ti \cup_{\mathcal{M}(w,-)\wedge si} C_\phi \wedge si \longrightarrow C_\phi \wedge ti$$

where ϕ varies over the acyclic monomorphisms in $\mathbf{f}\mathcal{M}$ and $i\colon si \rightarrowtail ti \in I$ varies over the generating cofibrations in $\mathcal{M}_{\mathbb{A}^1}$.

Let $v \in \mathbf{f}\mathcal{M}$ and consider the map $\tau_v\colon \mathcal{M}(T \wedge v, -) \wedge T \longrightarrow \mathcal{M}(v, -)$ of 6.1. Factor it as a cofibration $\mathcal{M}(T \wedge v, -) \wedge T \overset{d_v}{\rightarrowtail} D_v$ followed by a simplicial homotopy equivalence $D_v \xrightarrow{\simeq} \mathcal{M}(v, -)$ through the simplicial mapping cylinder. The stable acyclic cofibrations are generated by the hf-acyclic cofibrations together with the set of pushout product maps

$$d_v \Box i\colon (\mathcal{M}(T \wedge v, -) \wedge T) \wedge ti \cup_{(\mathcal{M}(T\wedge v,-)\wedge T)\wedge si} D_v \wedge si \longrightarrow D_v \wedge ti$$

where v varies over $\mathbf{f}\mathcal{M}$, and where $i\colon si \rightarrowtail ti \in I$ varies over the generating cofibrations in $\mathcal{M}_{\mathbb{A}^1}$. Note that all of these maps have cofibrant domains and codomains.

10.4. **Algebraic structure.** Recall the notation introduced in Section 8. Since the setting is slightly modified, we restate for the convenience of the reader the results in the motivic setting. All statements take place in $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$.

Definition 10.19. If A is an $\mathbb{I}$-algebra and k a commutative $\mathbb{I}$-algebra, then a map in mod_A or alg_k is called a weak equivalence (resp. fibration) if it is so when forgetting all the way down to $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$. Cofibrations are defined by the left lifting property.

Remark 10.20. We chose the stable model structure as our basis. This is fixed throughout (at least on the top level), and so the missing prefix "stable" from fibrations and weak equivalences in these categories should not be a source of confusion.

Theorem 10.21. *Suppose that $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$ satisfies the monoid axiom.*

- *If $A \in [\mathbf{f}\mathcal{M}, \mathcal{M}]$ is an $\mathbb{I}$-algebra, then the category mod_A of left A-modules with the stable model structure is a cofibrantly generated model category.*
- *If $k \in [\mathbf{f}\mathcal{M}, \mathcal{M}]$ is a commutative $\mathbb{I}$-algebra, then the category of k-modules with the stable model structure is a cofibrantly generated monoidal model category satisfying the monoid axiom.*
- *Let $k \in [\mathbf{f}\mathcal{M}, \mathcal{M}]$ be a commutative $\mathbb{I}$-algebra. Then the category alg_k of k-algebras with the stable model structure is a cofibrantly generated model category.*

Proof. This follows immediately from 8.3. $\square$

By the argument for 8.4, we have

Lemma 10.22. *Suppose that $\mathbf{f}\mathcal{M}$ satisfies* **f0**. *Let A be an $\mathbb{I}_{\mathbf{f}}$-algebra. Then for any cofibrant A-module N, the functor $- \wedge_A N$ takes weak equivalences in $\mathrm{mod}_{A^{\mathrm{op}}}$ to weak equivalences in $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$.*

Corollary 10.23. *Suppose all objects of $\mathcal{M}$ are small and that $\mathbf{f}\mathcal{M}$ satisfies* **f0**. *Let $f\colon A \xrightarrow{\sim} B$ be a weak equivalence of $\mathbb{I}$-algebras. Then extension and restriction of scalars define a Quillen equivalence*

$$mod_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} mod_B.$$

If A and B are commutative, there is the Quillen equivalence

$$alg_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} alg_B.$$

10.5. **The mixing of structures.** We will need to compare two choices of a category of finite motivic spaces. Consider the case with a refinement

$$\mathbf{e}\mathcal{M} \overset{i}{\hookrightarrow} \mathbf{f}\mathcal{M} \overset{\mathbb{I}_\mathbf{f}}{\hookrightarrow} \mathcal{M}$$

where both $\mathbf{e}\mathcal{M}$ and $\mathbf{f}\mathcal{M}$ satisfy the axioms **f1**, **f2**, **f3** and **f5**. Then there is a pair of adjoint functors

$$[\mathbf{e}\mathcal{M}, \mathcal{M}] \underset{i^*}{\overset{i_*}{\rightleftarrows}} [\mathbf{f}\mathcal{M}, \mathcal{M}]$$

where $i^*X(v) = X(i(v))$ and $i_*Y(w) = \int^{v\in\mathbf{e}\mathcal{M}} \mathcal{M}(v,w)_+ \wedge Y(v)$ for $X \in [\mathbf{f}\mathcal{M}, \mathcal{M}]$ and $v \in \mathbf{e}\mathcal{M}$. Recall that the right adjoint of $- \wedge_f X$ is denoted $[\mathbf{f}\mathcal{M}, \mathcal{M}](X, -)$. A direct calculation implies the next result.

Lemma 10.24. *The left adjoint i_* is strict symmetric monoidal: for all $\mathcal{M}$-functors $X, Y \in [\mathbf{e}\mathcal{M}, \mathcal{M}]$ there are natural isomorphisms*

$$(i_*X \wedge i_*Y) \cong i_*(X \wedge Y) \text{ and } i_*\mathbb{I}_\mathbf{e} \cong \mathbb{I}_\mathbf{f}$$

The right adjoint i^* is only lax symmetric monoidal, but satisfies $i^*\mathbb{I}_\mathbf{f} = \mathbb{I}_\mathbf{e}$. The composite i^*i_* is naturally isomorphic to the identity on $[\mathbf{e}\mathcal{M}, \mathcal{M}]$. Furthermore, the adjunction is monoidal in the sense that there is a natural isomorphism

$$i^*[\mathbf{f}\mathcal{M}, \mathcal{M}](i_*X, Y) \cong [\mathbf{e}\mathcal{M}, \mathcal{M}](X, i^*Y) \in [\mathbf{e}\mathcal{M}, \mathcal{M}]$$

for $X\colon \mathbf{e}\mathcal{M} \longrightarrow \mathcal{M}$ and $Y\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$.

The following is straightforward, and is true also in the pointwise and in the homotopy functor model structure.

Lemma 10.25. *The following is a Quillen pair.*

$$[\mathbf{e}\mathcal{M}, \mathcal{M}]_{\mathrm{st}} \underset{i^*}{\overset{i_*}{\rightleftarrows}} [\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$$

We use this pair to define a model structure on $\mathbf{f}\mathcal{M}$ which combines the best of the model structures $[\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$ and $[\mathbf{e}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$.

Definition 10.26. A morphism f in $[\mathbf{f}\mathcal{M}, \mathcal{M}]$ is an

- $\mathbf{e}$-*stable equivalence* if i^*f is a stable equivalence in $[\mathbf{e}\mathcal{M}, \mathcal{M}]_{\mathrm{st}}$,

- $\mathbf{e}$-*stable fibration* if i^*f is a stable fibration in $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$,
- $\mathbf{e}$-*cofibration* if it has the left lifting property with respect to maps that are both $\mathbf{e}$-stable equivalences and $\mathbf{e}$-stable fibrations.

The category $[\mathbf{f}\mathcal{M},\mathcal{M}]$ equipped with these classes of morphisms is denoted $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ and referred to as the $\mathbf{e}$-stable structure on $[\mathbf{f}\mathcal{M},\mathcal{M}]$.

Theorem 10.27. *The* $\mathbf{e}$*-stable structure* $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ *is a weakly finitely generated and proper monoidal model category. The adjoint pair*

$$[\mathbf{e}\mathcal{M},\mathcal{M}] \underset{i^*}{\overset{i_*}{\rightleftarrows}} [\mathbf{f}\mathcal{M},\mathcal{M}]$$

induces a Quillen equivalence between $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ *and* $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$.

Proof. First note that we may lift the generating cofibrations and generating acyclic cofibrations in $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ as described in 10.18 to $[\mathbf{f}\mathcal{M},\mathcal{M}]$ through the functor i_*. Since, by abuse of notation, there is a canonical isomorphism $i_*\mathcal{M}(v,-)\cong\mathcal{M}(v,-)$, the lifted (acyclic) cofibrations are exactly the maps in 10.18 for $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$, except that all reference to $\mathbf{f}\mathcal{M}$ has to be changed to $\mathbf{e}\mathcal{M}$. The composition i^*i_* is naturally isomorphic to the identity on $[\mathbf{e}\mathcal{M},\mathcal{M}]$, thus [6, 13.4.2] gives $[\mathbf{f}\mathcal{M},\mathcal{M}]$ a cofibrantly generated model structure, where f in $[\mathbf{f}\mathcal{M},\mathcal{M}]$ is a weak equivalence or fibration if and only if $i^*f\in[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ is.

Since $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ was weakly finitely generated, we automatically get that $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ is. Recall that i^*i_* is naturally isomorphic to the identity on $[\mathbf{e}\mathcal{M},\mathcal{M}]$, so i_* preserves fibrations and weak equivalences and i^* preserves cofibrations. In summary: both i^* and i_* preserve (co)fibrations and weak equivalences. Properness for $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ follows from properness for $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$: i^* preserves (co)limits, (co)fibrations and weak equivalences.

To prove monoidality, it suffices to check properties of the pushout product map on the generating (acyclic) cofibrations. Since i_* commutes with smash and pushouts, monoidality of $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ implies monoidality of $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$. To prove the adjunction induces a Quillen equivalence, let $B\colon\mathbf{e}\mathcal{M}\longrightarrow\mathcal{M}$ and $X\colon\mathbf{f}\mathcal{M}\longrightarrow\mathcal{M}$, and observe that, since $i^*i_*B\cong B$, the definition of weak equivalences in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ is such that a map $i_*B\longrightarrow X$ is a weak equivalence in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ if and only if the adjoint map $B\longrightarrow i^*X$ is a weak equivalence in $[\mathbf{e}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$. □

Notice that with these structures, there are the following relations:

- a fibration in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ is a fibration in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$,
- a weak equivalence in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ is a weak equivalence in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$, and
- an acyclic cofibration in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ is an acyclic cofibration in $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$.

Theorem 10.28. *If the model category* $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ *satisfies the monoid axiom, then so does* $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$.

Proof. Assume that $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}$ satisfies the monoid axiom. Then

$$\begin{aligned}&((\text{acyclic cofibrations in } [\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}) \wedge [\mathbf{f}\mathcal{M},\mathcal{M}]) - \text{cell}\\ \subseteq\ &((\text{acyclic cofibrations in } [\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}) \wedge [\mathbf{f}\mathcal{M},\mathcal{M}]) - \text{cell}\\ \subseteq\ &\text{weak equivalences in } [\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathrm{st}}\\ \subseteq\ &\text{weak equivalences in } [\mathbf{f}\mathcal{M},M]_{\mathbf{e}}\end{aligned}$$

and so $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ satisfies the monoid axiom too. □

10.6. **The main theorem of motivic functors.** In the following we let $\mathbf{f}\mathcal{M}$ be the category of all finite motivic spaces, and $\mathbf{e}\mathcal{M} \xhookrightarrow{i} \mathbf{f}\mathcal{M}$ the category of motivic spaces $v \in \mathbf{f}\mathcal{M}$ such that $T^n \xhookrightarrow{\sim} v$ for some $n \geq 0$.

Theorem 10.29. *With the choices of $\mathbf{f}\mathcal{M}$ and $\mathbf{e}\mathcal{M}$ above, the category $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$ is a weakly finitely generated and proper monoidal model category satisfying the monoid axiom. The evaluation functor*

$$\mathrm{ev}\colon [\mathbf{f}\mathcal{M},\mathcal{M}] \longrightarrow \mathrm{Sp}(\mathcal{M},T)$$

is the right adjoint in a Quillen equivalence between $[\mathbf{f}\mathcal{M},M]_{\mathbf{e}}$ and the stable model category structure on T-spectra.

Proof. This follows from 10.16, 10.17, 10.27 and 10.28. □

When working over the $\mathbf{e}$-stable model structure, we use the following definitions.

Definition 10.30. If A is an $\mathbb{I}$-algebra and k a commutative $\mathbb{I}$-algebra, then a map in mod_A or alg_k is called a weak equivalence (resp. fibration) if it is so when forgetting all the way down to $[\mathbf{f}\mathcal{M},\mathcal{M}]_{\mathbf{e}}$. Cofibrations are defined by the left lifting property. These structures will be referred to as the $\mathbf{e}$-*stable model structures.*

Remark 10.31. Note that there is a conflict of notation with the stable model structures from Subsection 10.4. If there is any chance for confusion, we will put a prefix $\mathbf{e}$ in front.

As before, we have

Theorem 10.32. *With the structures described above, the following is true.*

- *Let $A \in [\mathbf{f}\mathcal{M},\mathcal{M}]$ be an $\mathbb{I}$-algebra. Then the $\mathbf{e}$-stable model structure on mod_A is a cofibrantly generated model structure.*
- *Let $k \in [\mathbf{f}\mathcal{M},\mathcal{M}]$ be a commutative $\mathbb{I}$-algebra. Then the $\mathbf{e}$-stable model structure on mod_k is a cofibrantly generated monoidal model structure satisfying the monoid axiom.*
- *Let $k \in [\mathbf{f}\mathcal{M},\mathcal{M}]$ be a commutative $\mathbb{I}$-algebra. Then the $\mathbf{e}$-stable model structure on alg_k is a cofibrantly generated model category.*

The following corollary is somewhat weaker then could be desired, since the analog of 10.22 for the **e**-stable model structure is not known to hold. Hence we have to resort to stable equivalences – which are somewhat more special than **e**-equivalences – to obtain Quillen equivalences. It would be interesting to know whether this is necessary.

Corollary 10.33. *Let $f\colon A \xrightarrow{\sim} B$ be a stable equivalence of $\mathbb{I}$-algebras. Then extension and restriction of scalars define a Quillen equivalence*

$$mod_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} mod_B.$$

If A and B are commutative, there is the Quillen equivalence

$$alg_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} alg_B.$$

Proof. Follows by 10.22, cf. [16, 4.3 and 4.4]. □

10.7. **Comparison with motivic symmetric spectra.** Of course the discussion of motivic functors would be incomplete without describing the relation to the category of motivic symmetric spectra introduced in [12]. Fortunately, this is now quite simple, since all the work has been done elsewhere. Still $\mathbf{f}\mathcal{M}$ is the category of finite motivic spaces. By 2.15 and 2.16, the evaluation functor $\mathrm{ev}\colon [\mathbf{f}\mathcal{M}, \mathcal{M}] \longrightarrow \mathrm{Sp}(\mathcal{M}, T)$ factors as

$$[\mathbf{f}\mathcal{M}, \mathcal{M}] \xrightarrow{\mathrm{ev}'} \mathrm{Sp}^\Sigma(\mathcal{M}, T) \xrightarrow{U} \mathrm{Sp}(\mathcal{M}, T)$$

with U being the forgetful functor (induced by the inclusion of $\mathcal{M}$-categories $T\mathrm{Sph} \hookrightarrow T\mathrm{Sph}^\Sigma$). Furthermore, ev' is lax symmetric monoidal and has a strict symmetric monoidal left adjoint. The stable model structure on the category of symmetric spectra is described in [9, 8.7] and coincides with the stable model structure on motivic symmetric spectra constructed in [12], which uses the model category $\mathcal{M}_m$ we mentioned in 9.37 as input. The Quillen equivalence $\mathcal{M}_m \longrightarrow \mathcal{M}_{\mathbb{A}^1}$ given by the identity $\mathrm{Id}_\mathcal{M}$ induces a commutative square

$$\begin{array}{ccc} \mathrm{Sp}^\Sigma(\mathcal{M}_m, T) & \xrightarrow{U} & \mathrm{Sp}(\mathcal{M}_m, T) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^\Sigma(\mathcal{M}_{\mathbb{A}^1}, T) & \xrightarrow{U} & \mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T) \end{array}$$

in which the vertical functors are Quillen equivalences by [9, 5.7 and 9.3]. The upper forgetful functor is a Quillen equivalence by [12, 4.31], hence the lower forgetful functor is also a Quillen equivalence. Since $\mathrm{ev}\colon [\mathbf{f}\mathcal{M}, M]_{\mathbf{e}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathbb{A}^1}, T)$ is a Quillen equivalence by 10.29, it suffices to prove the following.

Lemma 10.34. *The functor $\mathrm{ev}'\colon [\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathbf{e}} \longrightarrow \mathrm{Sp}^\Sigma(\mathcal{M}_{\mathbb{A}^1}, T)$ is a right Quillen functor.*

Proof. Since the $\mathbf{e}$-stable acyclic fibrations are the maps f such that $f(v)$ is an acyclic fibration in $\mathcal{M}_{\mathbb{A}^1}$ for every v weakly equivalent to some T^n, ev' preserves stable acyclic fibrations. Similarly, any $\mathbf{e}$-stable fibration gets mapped to a stable fibration, because its evaluation on some T^n is a fibration and the square

$$\begin{tikzcd} XT^n \arrow[r] \arrow[d, "f(T^n)"'] & \mathcal{M}(T, XT^{n+1}) \arrow[d, "\mathcal{M}(T, f(T^{n+1}))"] \\ YT^n \arrow[r] & \mathcal{M}(T, YT^{n+1}) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathbb{A}^1}$ for every $n \geq 0$. The definition of stable fibrations of symmetric T-spectra given in Secion [12, 4.2] – which also applies if the model structure $\mathcal{M}_{\mathbb{A}^1}$ instead of $\mathcal{M}_m$ is used – then shows that ev' also preserves stable fibrations. □

Corollary 10.35. *The lax symmetric monoidal functor*

$$\mathrm{ev}' \colon [\mathbf{f}\mathcal{M}, \mathcal{M}]_{\mathbf{e}} \longrightarrow \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathbb{A}^1}, T)$$

is the right adjoint in a Quillen equivalence, and has a strict symmetric monoidal left adjoint. The induced pair on homotopy categories is a monoidal equivalence.

Proof. The statement about the Quillen equivalence is clear by now, the monoidality statements follow from 2.16 and [7, 4.3.3]. □

11. Equivariant Stable Homotopy Theory

11.1. **Equivariant spaces.** This part gathers the properties of equivariant homotopy theory which are relevant for the machinery developed in the main part. The basis is again the category $\mathcal{S}$ of pointed simplicial sets, or *spaces* for short, with its model structure described in 9.14. For technical reasons, we restrict ourselves to a finite group G.

Let $G\mathcal{S}$ denote the category of pointed simplicial sets with a basepoint preserving left G-action, or G-*spaces* for short. By considering G_+ as a $\mathcal{S}$-category with one object and composition $G_+ \wedge G_+ \longrightarrow G_+$ induced by the multiplication in G, one identifies $G\mathcal{S}$ as the category of $\mathcal{S}$-functors $[G_+, \mathcal{S}]$. That is, a G-space is a $\mathcal{S}$-functor $G_+ \xrightarrow{K} \mathcal{S}$. Nevertheless, a G-space $K\colon G_+ \longrightarrow \mathcal{S}$ is sometimes written as (uK, a_K), where $uK \in \operatorname{Ob}\mathcal{S}$ is the underlying space (the value of K on the single object) and $a_K\colon G_+ \wedge uK \longrightarrow uK$ is the left G-action (adjoint to the map $\hom^K\colon G_+ \longrightarrow \mathcal{S}(uK, uK)$). The image of $g \in G_+$ under $\hom^K$ is sometimes denoted as $g\colon uK \longrightarrow uK$.

By 2.4, $G\mathcal{S}$ is a closed $\mathcal{S}$-module. The underlying space functor $u\colon G\mathcal{S} \longrightarrow \mathcal{S}$ has a left $\mathcal{S}$-adjoint $G_+ \wedge -\colon \mathcal{S} \longrightarrow G\mathcal{S}$ obtained by Kan extension 2.5. The G-space $G_+ \wedge K\colon G_+ \longrightarrow \mathcal{S}$ has as underlying space $G_+ \wedge K$, the left G-action is the composition

$$G_+ \wedge (G_+ \wedge K) \xrightarrow{\cong} (G \times G)_+ \wedge K \xrightarrow{\mu_+ \wedge K} G_+ \wedge K$$

with $\mu\colon G \times G \longrightarrow G$ being the multiplication. For similar reasons, u has a right $\mathcal{S}$-adjoint which maps the space K to the G-fold product $\times_{g\in G} K_g = K$. The group G acts via permutation of factors. To be more precise, $h \in G$ maps K_g via the identity to K_{hg}. A similar functor, denoted $(-)^{\wedge G}\colon \mathcal{S} \longrightarrow G\mathcal{S}$, will be important later on. The value of this functor on the space K is the G-fold smash product $K^{\wedge G} = \wedge_{g\in G} K_g = K$, and G acts by permutation of factors as before. Another relevant functor is $\mathrm{ct}\colon \mathcal{S} \longrightarrow G\mathcal{S}$. The G-space $\mathrm{ct}K$ is constant, that is, the underlying space is K and the map $\hom^{\mathrm{ct}K}\colon G_+ \longrightarrow \mathcal{S}(K, K)$ hits the identity id_K and the trivial map.

The *smash product* $K \wedge L$ of two G-spaces $G_+ \xrightarrow{K} \mathcal{S}$ and $G_+ \xrightarrow{L} \mathcal{S}$ is defined as the composition

$$G_+ \xrightarrow{\mathrm{diag}_+} (G \times G)_+ \cong G_+ \wedge G_+ \xrightarrow{K \wedge L} \mathcal{S} \wedge \mathcal{S} \xrightarrow{\wedge} \mathcal{S}$$

where $\mathrm{diag}\colon G \longrightarrow G \times G$ is the diagonal, and the smash product $\mathcal{C} \wedge \mathcal{D}$ of two $\mathcal{S}$-categories $\mathcal{C}$ and $\mathcal{D}$ is formed according to the general rule [1, 6.2.9]. In other words, the smash product of G-spaces is formed on underlying spaces, and G acts via the diagonal. However, if G is commutative, a different closed symmetric monoidal product of G-spaces exists by Day's theorem 2.6.

Proposition 11.1. *The triple $(G\mathcal{S}, \wedge, \mathrm{ct}S^0)$ is a closed symmetric monoidal category. The functors u, ct and $(-)^{\wedge G}$ are strict symmetric monoidal, the functor $G_+ \wedge -$ is lax symmetric monoidal.*

Proof. It is clear that $\wedge$ is symmetric monoidal and has $\mathrm{ct}S^0$ as a unit. We will construct the right $\mathcal{S}$-adjoint $G\mathcal{S}(K, -)$ of smashing with $G_+ \xrightarrow{K} \mathcal{S}$. The value of $G\mathcal{S}(K, -)$ on $G_+ \xrightarrow{L} \mathcal{S}$ has as underlying space $\mathcal{S}(uK, uL)$. The left G-action is given by the map

$$\begin{aligned} G_+ \wedge \mathcal{S}(uK, uL) &\longrightarrow \mathcal{S}(uK, uL) \\ \overline{(g, f)} &\longmapsto g \circ f \circ (g^{-1} \wedge \Delta^n_+) \end{aligned}$$

where $\overline{(g, f)}$ denotes the equivalence class of $(g, f) \in (G_+ \times \mathcal{S}(uK, uL))_n$ in $(G_+ \wedge \mathcal{S}(uK, uL))_n$. The unit and counit of the adjunction are defined using the unit and counit of the adjunction

$$uK \wedge - \colon \mathcal{S} \rightleftarrows \mathcal{S} \colon \mathcal{S}(uK, -).$$

In detail, $u(\eta_K L) = \eta_{uK} uL \colon uL \longrightarrow \mathcal{S}(uK, \underline{u(L \wedge K)} = uL \wedge uK)$, and this map is equivariant because the value of $(\eta_{uK} uL)(g \cdot x)$ at $\overline{(y, \alpha)}$ is

$$\overline{(\alpha^*(g \cdot x), y)} = \overline{(g \cdot \alpha^* x, y)} = g \cdot \overline{(\alpha^* x, g^{-1} \cdot y)}.$$

This coincides with the value of $g \cdot (\eta_{uK} uL)(x)$ at $\overline{(y, \alpha)}$. Similarly, we have $u(\epsilon_K L) = \epsilon_{uK} uL \colon \mathcal{S}(uK, uL) \wedge uK \longrightarrow uL$, which is a G-map since

$$(\epsilon_{uK} uL)(g \cdot \overline{(f, y)}) = (g \cdot f)(\overline{(g \cdot y, \mathrm{id}_{[n]})}) = g \cdot f(g^{-1} \cdot \overline{(g \cdot y, \mathrm{id}_{[n]})}) = g \cdot f\overline{(y, \mathrm{id}_{[n]})}.$$

The triangular identities hold on underlying spaces, which proves the first statement. By definition, u and ct are (very) strict symmetric monoidal. The natural isomorphism $K^{\wedge G} \wedge L^{\wedge G} \xrightarrow{\cong} (K \wedge L)^{\wedge G}$ is the composition of commutativity and associativity isomorphisms which rearranges the factors in the correct way. By coherence, this isomorphism is equivariant. Similar considerations hold for the isomorphism of G-spaces $\mathrm{ct}S^0 \xrightarrow{\cong} (S^0)^{\wedge G}$. The natural map $(G_+ \wedge K) \wedge (G_+ \wedge L) \longrightarrow G_+ \wedge (K \wedge L)$ is induced by a commutativity isomorphism and the multiplication of G, the unit in G determines $\mathrm{ct}S^0 \longrightarrow G_+ \wedge S^0$. The relevant conditions hold by commutativity coherence. □

Lemma 11.2. *Let $K \colon G_+ \longrightarrow \mathcal{S}$ be a G-space. The following are equivalent.*

1. *K is $G\mathcal{S}$-finite.*
2. *K is finite.*
3. *uK is finite.*

Proof. As we will see again later, the $\mathcal{S}$-functor $\mathrm{ct} \colon \mathcal{S} \longrightarrow G\mathcal{S}$ has a right $\mathcal{S}$-adjoint, namely the $\mathcal{S}$-functor $\mathrm{Fix}(G, -)$ which maps a G-space (uK, a_K) to the subspace fixed under the action of G. That is,

$$\mathrm{Fix}(G, K) = \{x \in uK \mid a_K\overline{(g, x)} = x \text{ for all } g \in G\}.$$

Now $\mathrm{Fix}(G, K)$ can equally described as $\lim(K\colon G_+ \longrightarrow \mathcal{S})$, and since G_+ is a finite indexing category, $\mathrm{Fix}(G, -)$ commutes with filtered colimits. Hence ct preserves finiteness. In particular, the G-space $\mathrm{ct}S^0$ is finite, which proves the implication $1 \Rightarrow 2$. Also the right adjoint of the underlying space functor u commutes with filtered colimits, thus we have the implication $2 \Rightarrow 3$.

It remains to prove that 3 implies 1. Let $D\colon \mathcal{I} \longrightarrow G\mathcal{S}$ be a functor with $\mathcal{I}$ filtered, and consider the canonical map

$$f_K\colon \operatorname{colim}_{\mathcal{I}} G\mathcal{S}(K, D) \longrightarrow G\mathcal{S}(K, \operatorname{colim}_{\mathcal{I}} D).$$

Since colimits in $G\mathcal{S}$ are formed on underlying spaces, $u(f_K)$ is the canonical map

$$\operatorname{colim}_{\mathcal{I}} \mathcal{S}(uK, u \circ D) \longrightarrow \mathcal{S}(uK, \operatorname{colim}_{\mathcal{I}} u \circ D).$$

If uK is finite, $u(f_K)$ is an isomorphism, which implies that also f_K is an isomorphism, because the G-action on the domain coincides with the G-action on the codomain. This proves $3 \Rightarrow 1$. □

The full subcategory of finite G-spaces is equivalent to some small category. As in [13], one can pick a choice which is closed under the smash product.

Lemma 11.3. *Any G-space is the filtered colimit of its finite sub-G-spaces.*

11.2. **Unstable equivariant homotopy theory.** Theorems 4.2 and 4.4 apply in this situation, hence there is a *course* model structure on $G\mathcal{S}$ with weak equivalences and fibrations defined on underlying spaces. Cofibrations in this model structure can be characterized as those injective maps $f\colon K \longrightarrow L$ such that G acts free on the complement of $f(K)$ in L. Consequently, the cofibrant G-spaces are the ones in which G acts free away from the basepoint.

However, we are interested in another model structure on $G\mathcal{S}$, which will be defined soon. If $H \hookrightarrow G$ is a subgroup, let G/H_+ denote the pointed G-space with action $g \cdot g'H := (gg')H$. The $\mathcal{S}$-functor $G/H_+\wedge\colon \mathcal{S}_+ \longrightarrow G\mathcal{S}_+$, mapping K to $G/H_+ \wedge \mathrm{ct}K$, has as right $\mathcal{S}$-adjoint the functor $\mathrm{Fix}(H, -)\colon G\mathcal{S}_+ \longrightarrow \mathcal{S}_+$ which maps L to the space of fixed points under the action of H on L. Note that $\mathrm{Fix}(H, -)$ coincides with $\mathcal{S}_{G\mathcal{S}}(G/H_+, -)$. Furthermore, if H and H' are two subgroups of G, there is an isomorphism $\mathrm{Fix}(H', G/H_+ \wedge K) \cong \mathrm{Fix}(H', G/H_+) \wedge K$ which is natural in K.

Definition 11.4. A map $f\colon K \longrightarrow L$ in $G\mathcal{S}$ is a *G-weak equivalence* if the map $\mathrm{Fix}(H, f)$ is a weak equivalence in $\mathcal{S}$ for every subgroup $H \hookrightarrow G$. It is a *G-fibration* if the map $\mathrm{Fix}(H, f)$ is a fibration in $\mathcal{S}$ for every subgroup $H \hookrightarrow G$.

Theorem 11.5. *Let G be a finite group. There is a monoidal model structure on $G\mathcal{S}$ with G-weak equivalences as weak equivalences and G-fibrations as fibrations. The cofibrations are precisely the injective maps. This model structure is* *proper and satisfies the monoid axiom. One can choose generating cofibrations and generating acyclic cofibrations with finite domains and codomains.*

This theorem is well-known. Nevertheless, we give a proof.

Proof. To prove the existence of the model structure, we will apply [7, 2.1.19]. Let

$$I_G := \{G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n\geq 0,\, H \text{ subgroup of } G}$$

and

$$J_G := \{G/H_+ \wedge (\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{n\geq 1,\, 0\leq i\leq n,\, H \text{ subgroup of } G}.$$

It is clear by adjointness that a map is a G-fibration if and only if it is in J_G-inj, and a G-fibration and a G-weak equivalence if and only if it is in I_G-inj. Furthermore, the (co)domains of the maps in I_G and J_G are finite by 11.2. Because of the natural isomorphism $\mathrm{Fix}(H', G/H_+ \wedge K) \cong \mathrm{Fix}(H', G/H_+) \wedge K$ for subgroups H and H' of G, the maps in J_G consists of G-weak equivalences. The existence of the model structure follows, if every map in J_G-cell is a G-weak equivalence. Since $\mathrm{Fix}(H', -)$ commutes with sequential colimits, it therefore suffices to check that the cobase change of a map in J_G is again a G-weak equivalence. Thus let

$$\begin{tikzcd} A := (G/H \times \Lambda^n_i)_+ \arrow[r] \arrow[d, hook] & K \arrow[d, hook] \\ B := (G/H \times \Delta^n)_+ \arrow[r] & L \end{tikzcd}$$

be a pushout diagram, and fix a subgroup H'. The induced map of spaces $\mathrm{Fix}(H', B) \cup_{\mathrm{Fix}(H',A)} \mathrm{Fix}(H', K) \longrightarrow \mathrm{Fix}(H', L)$ is injective. To prove that it is also surjective, let $[x]$ be an element in $\mathrm{Fix}(H', L)$. Because $\mathrm{Fix}(H', -)$ preserves injective maps, we can deduce that if a representative x of $[x]$ lives in K, then this representative is even in $\mathrm{Fix}(H', K)$. A similar statement holds if a representative lives in $(G/H \times \Delta^n)_+$, which proves that the map above is an isomorphism. Thus any cobase change of a map in J_G is a G-weak equivalence, and the model structure exists.

It is clear that every map in I_G-cell is an injective map. Conversely, any injective map is contained in I_G-cell, as one can deduce by looking at one fixed point space at a time. The statement about the cofibrations follows, as does the left properness. Right properness follows by definition, because $\mathrm{Fix}(H, -)$ commutes with pullbacks and $\mathcal{S}$ is right proper.

The pushout product map of injective maps is again injective, so let us consider the pushout product map

$$G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+)\Box G/H'_+ \wedge (\Lambda^m_i \hookrightarrow \Delta^m)_+ \cong (G/H \times G/H')_+ \wedge i$$

with i being a weak equivalence of spaces. This pushout product map is a G-weak equivalence, because $\mathrm{Fix}(H'', (G/H \times G/H')_+ \wedge i) \cong \mathrm{Fix}(H'', (G/H \times G/H')_+) \wedge i$.

Hence the model structure is monoidal. The monoid axiom holds, since every G-space is cofibrant. □

This is the *fine* model structure. For the rest of this section, $G\mathcal{S}$ is equipped with the fine model structure. So far, just one item is missing to apply the machinery of the main part, namely the circle. Our choice is the regular representation $S^{\wedge G}$, the G-fold smash product of $S^1 = \Delta^1/\partial\Delta^1$, with G acting by permutation of the factors. For example, if $G = \mathbb{Z}/2\mathbb{Z}$, then $S^{\wedge G}$ is $S^1 \wedge S^1$, with $\bar{O} \in G$ acting as the identity, and $\bar{1} \in G$ acting via the commutativity isomorphism. The name "regular representation" for $S^{\wedge G}$ is justified, because its geometric realization is homeomorphic – as a G-space – to the one-point-compactification of the real vector space $\mathbb{R}^G$ of real-valued maps $G \longrightarrow \mathbb{R}$, with G acting by $g \cdot f(h) := f(gh)$. The G-space $S^{\wedge G}$ is finite, since G is a finite group.

11.3. **Stable equivariant homotopy theory.** The full sub-$G\mathcal{S}$-category of $G\mathcal{S}$ given by the finite G-spaces is denoted by $\mathbf{f}G\mathcal{S}$, and it is equivalent to a small $G\mathcal{S}$-category. Let $G\mathcal{F} = [\mathbf{f}G\mathcal{S}, G\mathcal{S}]$ be the category of $G\mathcal{S}$-functors from finite G-spaces to all G-spaces. Usually we will refer to these functors as *G-simplicial functors.* If G is the trivial group, we recover Lydakis' category of simplicial functors defined in [13, 4.4]. The smash product of G-simplicial functors derived from 2.6 is again denoted $\wedge$. The unit of this smash product is the inclusion $\mathbb{I}\colon \mathbf{f}G\mathcal{S} \hookrightarrow G\mathcal{S}$. Note that all G-spaces are cofibrant, which simplifies the following discussion according to the general setup considerably.

Definition 11.6. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is a

- *pointwise weak equivalence* if $f(K)$ is a G-weak equivalence for all finite G-spaces K,
- *pointwise fibration* if $f(K)$ is a G-fibration for all finite G-spaces K,
- *cofibration* if it has the left lifting property with respect to all pointwise acyclic fibrations.

Theorem 11.7. *The category $G\mathcal{F}$, equipped with these classes, is a monoidal proper model category satisfying the monoid axiom. Generating cofibrations and generating acyclic cofibrations can be chosen with finite domains and codomains. Finally, smashing with a cofibrant G-simplicial functor preserves pointwise weak equivalences.*

Proof. The generating cofibrations

$$\mathcal{P}_{I_G} = \{G\mathcal{S}(K,-) \wedge i\}_{i \in I_G,\, K \text{ finite}}$$

and generating acyclic cofibrations

$$\mathcal{P}_{J_G} = \{G\mathcal{S}(K,-) \wedge j\}_{j \in J_G,\, K \text{ finite}}$$

do the job, as 4.2 shows. By 4.4, the model structure is monoidal, properness follows from 4.8. The monoid axiom is provided by 4.9, and $- \wedge X$ preserves pointwise weak equivalences for cofibrant X, since 4.12 holds. The latter uses in particular 11.3. □

The usage of this *pointwise* model structure is indicated by the subscript $G\mathcal{F}_{\mathrm{pt}}$. To define the homotopy functor model structure on $G\mathcal{F}$, let $\rho\colon \mathrm{Id}_{G\mathcal{S}} \longrightarrow R$ denote the enriched fibrant replacement functor obtained by the methods from section 3.3.2 applied to the set J_G of generating acyclic cofibrations. Furthermore, $\mathbb{I}_*X$ is the enriched left Kan extension 2.5 of $X\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$ along $\mathbb{I}\colon \mathbf{f}G\mathcal{S} \hookrightarrow G\mathcal{S}$. Finally, let $X^h := \mathbb{I}_*X \circ R \circ \mathbb{I}$ and note that this defines an endofunctor of $G\mathcal{F}$ coming with a natural transformation $\mathrm{Id}_{G\mathcal{F}} \longrightarrow (-)^h$.

Definition 11.8. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is an

- *hf-equivalence* if f^h is a pointwise weak equivalence
- *hf-fibration* if it is a pointwise fibration such that

$$\begin{tikzcd} XK \arrow[r] \arrow[d, "f(K)"'] & XL \arrow[d, "f(L)"] \\ YK \arrow[r] & YL \end{tikzcd}$$

is a homotopy pullback square for every G-weak equivalence $K \xrightarrow{\sim} L$ of finite G-spaces.

Lemma 11.9. *A map of G-simplicial functors is a pointwise acyclic fibration if and only if it is an hf-equivalence and an hf-fibration.*

Proof. The general definition of hf-equivalences given in 5.2 uses the filtered fibrant replacement functor Φ^{J_G} from section 3.3.3 which enters the definition of $\hbar(X) = \mathbb{I}_*X \circ \Phi^{J_G}$. Since all G-spaces are cofibrant, the canonical map $\omega_K\colon \Phi^{J_G}K \longrightarrow RK$ is a G-weak equivalence of fibrant G-spaces, hence a simplicial homotopy equivalence. The induced map $\mathbb{I}_*X(\omega_K)\colon \hbar(X)(K) \longrightarrow X^hK$ is then a simplicial homotopy equivalence by 2.11, in particular a G-weak equivalence, for any finite G-space K. It follows that f^h is a pointwise weak equivalence if and only if $\hbar(f)(K)$ is a G-weak equivalence for every finite G-space K.

Using the argument given in section 5.4, the characterization 5.6 of hf-fibrations in general is equivalent to the definition of hf-fibrations given here. Again we use that any G-space is cofibrant, hence any G-weak equivalence of finite G-spaces can be factored as an acyclic cofibration of finite G-spaces and a simplicial homotopy equivalence. The lemma now follows from 5.8. □

Theorem 11.10. *The category $G\mathcal{F}$, equipped with the classes of hf-equivalences, hf-fibrations and cofibrations, is a proper monoidal model category satisfying the monoid axiom. Smashing with a cofibrant G-simplicial functor preserves hf-equivalences. One can choose generating cofibrations and generating acyclic cofibrations with finite domains and codomains.*

Proof. The model structure exists by 5.10, is monoidal by 5.11 and proper by 5.12. Since smashing with a cofibrant G-simplicial functor preserves pointwise weak equivalences, it preserves hf-equivalences, as one can see from factoring. The monoid axiom can be deduced quite easily from this statement. We describe the generating acyclic cofibrations which have to be added to the set $\mathcal{P}_{J_G}$.

A G-weak equivalence of finite G-spaces $w\colon K \xrightarrow{\sim} L$ induces an hf-equivalence $G\mathcal{S}(w,-)\colon G\mathcal{S}(L,-) \longrightarrow G\mathcal{S}(K,-)$. Using the simplicial mapping cylinder, we can factor $G\mathcal{S}(w,-)$ as a cofibration $c_w\colon G\mathcal{S}(L,-) \rightarrowtail C_w$ followed by a simplicial homotopy equivalence $C_w \xrightarrow{\simeq} G\mathcal{S}(K,-)$. The set of additional generating acyclic cofibrations $\mathcal{H}_G$ is defined as the set of pushout product maps $c_w \Box i$, where w is a G-weak equivalence of finite G-spaces and $i \in I_G$ is a generating cofibration. Restricting w to the set of acyclic cofibrations of finite G-spaces is also possible without changing the model structure. □

This is the *homotopy functor* model structure, and the homotopy functor model category is $G\mathcal{F}_{\mathrm{hf}}$. A fibrant replacement of X (that is, an hf-equivalence to some G-simplicial functor which is pointwise fibrant and preserves G-weak equivalences) is given by the natural transformation $X \longrightarrow R \circ X^h$. To define the stable model structure, recall the functor $\mathbf{S}\colon G\mathcal{F} \longrightarrow G\mathcal{F}$ mapping X to $G\mathcal{S}(S^{\wedge G}, X(S^{\wedge G} \wedge -))$ and the natural transformation $s\colon \mathrm{Id}_{G\mathcal{F}} \longrightarrow \mathbf{S}$ obtained pointwise as the adjoint of the map

$$XK \wedge S^{\wedge G} \xrightarrow{\mathrm{move}^X_{S^{\wedge G}}(K)} X(S^{\wedge G} \wedge K)$$

which is in turn the adjoint of

$$S^{\wedge G} \xrightarrow{\eta_K S^{\wedge G}} G\mathcal{S}(K, S^{\wedge G} \wedge K) \xrightarrow{\mathrm{hom}^X_{K, S^{\wedge G} \wedge K}} G\mathcal{S}(XK, X(S^{\wedge G} \wedge K)).$$

Let $\mathbf{S}^\infty(X)$ denote the colimit of

$$X \xrightarrow{s(X)} \mathbf{S}(X) \xrightarrow{\mathbf{S}(s(X))} \mathbf{S}(\mathbf{S}(X)) \longrightarrow \cdots$$

and call the canonical natural transformation $s\colon \mathrm{Id}_{G\mathcal{F}} \longrightarrow \mathbf{S}^\infty$.

Definition 11.11. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is a

- *stable equivalence* if $\mathbf{S}^\infty(R \circ f^h)$ is a pointwise weak equivalence,
- *stable fibration* if it is an hf-fibration such that

$$\begin{array}{ccc} X & \xrightarrow{s(X)} & \mathbf{S}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{S}(f)} \\ X & \xrightarrow{s(X)} & \mathbf{S}(X) \end{array}$$

is a homotopy pullback square in $G\mathcal{F}_{\mathrm{pt}}$.

Lemma 11.12. *A map of G-simplicial functors is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. By the observations done in the proof of 11.9, one sees that the definition of stable equivalences here agrees with the general one given in 6.2. The result follows from 6.13. □

The definition of stable fibrations leads to a set of generating stable acyclic cofibrations, namely the set of pushout product maps of some map in I_G with the factorization of

$$G\mathcal{S}(S^{\wedge G}\wedge K,-)\wedge S^{\wedge G} \xrightarrow{\cong} G\mathcal{S}(S^{\wedge G}, G\mathcal{S}(K,-))\wedge S^{\wedge G} \xrightarrow{\epsilon_{S^{\wedge G}} G\mathcal{S}(K,-)} G\mathcal{S}(K,-)$$

via the simplicial mapping cylinder for some finite G-space K. Recall that we used a certain property of the stable model category $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ of $S^{\wedge G}$-spectra in $G\mathcal{S}$ to derive that these maps are in fact stable equivalences. Before we prove it, let us recall the definition of $S^{\wedge G}$-spectra in $G\mathcal{S}$ and justify why they model G-equivariant stable homotopy theory.

Definition 11.13. A *G-spectrum* E consists of a sequence $(E_0, E_1, \dots)$ of G-spaces, together with structure maps $E_n \wedge S^{\wedge G} \longrightarrow E_{n+1}$. A *map* $f\colon E \longrightarrow F$ of G-spectra is a sequence of maps $f_n\colon E_n \longrightarrow F_n$ making the diagram

$$\begin{array}{ccc} E_n \wedge S^{\wedge G} & \longrightarrow & E_{n+1} \\ {\scriptstyle f_n \wedge S^{\wedge G}}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ F_n \wedge S^{\wedge G} & \longrightarrow & F_{n+1} \end{array}$$

commutative for every $n \geq 0$.

The resulting category $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ of G-spectra is by 2.12 isomorphic to the category $[S^{\wedge G}\mathrm{Sph}, G\mathcal{S}]$ of G-simplicial functors from the category of $S^{\wedge G}$-spheres $S^{\wedge G}\mathrm{Sph}$ to the category $G\mathcal{S}$. Thus 4.2 supplies $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ with the pointwise model structure. A fibrant replacement functor in this model structure is $[S^{\wedge G}\mathrm{Sph}, R]$, where R is the enriched fibrant replacement functor introduced earlier on.

Recall that the adjoints of the structure maps of a G-spectrum E can be viewed as a natural map $E \longrightarrow \Omega_G\mathrm{sh}E$, where the n-th structure map of the G-spectrum $\Omega_G\mathrm{sh}E$ is

$$G\mathcal{S}(S^{\wedge G}, E_{n+1} \longrightarrow G\mathcal{S}(S^{\wedge G}, E_{n+2})),$$

the $S^{\wedge G}$-loops of the $n+1$-th structure map of E. Denote this natural transformation as $\mathrm{st}\colon \mathrm{Id} \longrightarrow \mathrm{St} = \Omega_G \circ \mathrm{sh}$, and let $\mathrm{st}^\infty\colon \mathrm{Id} \longrightarrow \mathrm{St}^\infty$ be the colimit of the sequence

$$\mathrm{Id} \xrightarrow{\mathrm{st}} \mathrm{St} \xrightarrow{\mathrm{St(st)}} \mathrm{St}^2 \longrightarrow \cdots$$

The stable model structure on G-spectra – which has the same cofibrations as the pointwise model structure – is defined as follows.

Definition 11.14. A map $f\colon E \longrightarrow F$ of G-spectra is a *stable equivalence* if the map $\mathrm{St}^\infty(Rf)$ is a pointwise weak equivalence, and a *stable fibration* if it is a pointwise fibration such that the square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f"'] & \mathrm{St}(RE) \arrow[d, "\mathrm{St}(Rf)"] \\ F \arrow[r] & \mathrm{St}(RF) \end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure.

From now on we always equip the category of G-spectra with the stable model structure. Geometric realization induces a functor from $\mathrm{Sp}(S^{\wedge G}, G\mathcal{S})$ to the category of G-prespectra introduced for example in Carlsson's survey article [3, 3.2]. It is plausible that this functor induces an equivalence of homotopy categories. Hence the homotopy category of $\mathrm{Sp}(S^{\wedge G}, G\mathcal{S})$ is the G-equivariant stable homotopy category. Let us now return to the last ingredient for the proof of 11.16.

Proposition 11.15. *The functor* $-\wedge S^{\wedge G}\colon \mathrm{Sp}(G\mathcal{S}, S^{\wedge G}) \longrightarrow \mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ *is a Quillen equivalence.*

Proof. We want to apply [9, 10.3]. Hence we have to check whether $S^{\wedge G}$ is weakly equivalent to a symmetric G-space. Here a symmetric G-space K is one equipped with a commutative diagram

$$\begin{tikzcd} K^3 \wedge \mathrm{ct}S^0 \arrow[r, "K^3\wedge \mathrm{ct}i_0"] \arrow[dr, "\mathrm{id}_{K^3\wedge \mathrm{ct}S^0}"'] & K^3 \wedge \mathrm{ct}\Delta^1_+ \arrow[d, "H"] & K^3 \wedge \mathrm{ct}S^0 \arrow[l, "K^3\wedge \mathrm{ct}i_1"'] \arrow[dl, "\mathrm{cyc}\wedge \mathrm{st}S^0"] \\ & K^3 \wedge \mathrm{ct}S^0 & \end{tikzcd}$$

commutative, where $\mathrm{cyc}\colon K^3 \longrightarrow K^3$ is the cyclic permutation map mentioned in [9, 10.2]. Taking $G = 1$ to be the trivial group for a start, the space S^1 is weakly equivalent to a symmetric space. Indeed, the cyclic permutation map on the geometric realization $|S^1|^3$ is homotopic to the identity, hence we can deduce that its singular complex $\mathrm{sing}|S^1|$ is a symmetric space which allows a weak equivalence $S^1 \xrightarrow{\sim} \mathrm{sing}|S^1|$. Let us abbreviate the codomain of this weak equivalence by K, and assume G to be any finite group now. The induced map $S^{\wedge G} \longrightarrow K^{\wedge G}$ of G-fold smash products is then a G-weak equivalence. To see this, choose a subgroup H of G and write the underlying set of G as the union of the cosets gH. View an element

$$x = x_{g_1} \wedge x_{g_2} \wedge \ldots \wedge x_{g_n}$$

of $L^{\wedge G}$, L some space, where n is the order of G and $x_{g_k} \in L$ for every k. This x is invariant under the action of all $h \in H$ if and only if $x_{g_k} = x_{g_j}$ for all g_k, g_j lying in the same coset of H. Thus $\mathrm{Fix}(H, L^{\wedge G})$ is – up to a natural isomorphism – the G/H-fold smash product of the space L. In particular, we get an expression of the "diagonal" $d_L\colon \mathrm{ct}L \longrightarrow L^{\wedge G}$ as the adjoint of the isomorphism $L \longrightarrow \mathrm{Fix}(G, L^{\wedge G})$.

We will define a homotopy from the cyclic permutation map to the identity on $K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G}$. First recall from 11.1 that there is a natural isomorphism of G-spaces $f_{K,L}\colon K^{\wedge G} \wedge L^{\wedge G} \xrightarrow{\cong} (K \wedge L)^{\wedge G}$ which rearranges the factors. Using the map $d = d_{\Delta^1_+}\colon \mathrm{ct}\Delta^n_+ \longrightarrow (\Delta^n_+)^{\wedge G}$, we get a composition

$$\begin{tikzcd}
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G} \wedge \mathrm{ct}\Delta^1_+ \arrow[r, "\mathrm{id}\wedge d"] & K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G} \wedge (\Delta^1_+)^{\wedge G} \arrow[d, "\cong"'] \\
(K \wedge K \wedge K)^{\wedge G} & (K \wedge K \wedge K \wedge \Delta^1_+)^{\wedge G} \arrow[l, "F^{\wedge G}"']
\end{tikzcd}$$

where F is the homotopy from the cyclic permutation map to the identity. Up to a further rearrangement of factors, this composition gives the desired homotopy which shows that $K^{\wedge G}$ is a symmetric G-space. Hence $S^{\wedge G}$ is G-weakly equivalent to a symmetric G-space, which completes the proof. □

Theorem 11.16. *The category $G\mathcal{F}$, equipped with the classes of stable equivalences, stable fibrations and cofibrations, is a proper monoidal model category satisfying the monoid axiom. One can choose generating cofibrations and generating acyclic cofibrations with finite domains and codomains. Smashing with a cofibrant G-simplicial functor preserves stable equivalences.*

Proof. This is 6.25, 6.26, 6.27, 6.28 and 6.29. □

The use of this *stable* model structure is indicated by $G\mathcal{F}_{\mathrm{st}}$. We compare this model category with the stable model category of G-spectra. The category $S^{\wedge G}\mathrm{Sph}$ introduced in section 2.5 is a sub-$G\mathcal{S}$-category of $\mathbf{f}G\mathcal{S}$ via the inclusion $i\colon S^{\wedge G}\mathrm{Sph} \hookrightarrow \mathbf{f}G\mathcal{S}$, and i_* is the enriched left Kan extension along i, left adjoint to pre-composition with i, which we denote as ev.

Lemma 11.17. *The evaluation functor* $\mathrm{ev}\colon G\mathcal{F}_{\mathrm{st}} \longrightarrow \mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ *is a right Quillen functor.*

Proof. See 7.2. □

Furthermore, the unit of the adjunction has a desirable property by the general machinery developed in section 7.

Lemma 11.18. *The canonical map $E \longrightarrow \mathrm{ev}(i_*E)^h$ is a stable equivalence of G-spectra for every cofibrant G-spectrum E.*

Proof. This is 7.4 in the special case. □

Remark 11.19. If one modifies the stable model structure on G-simplicial functors as in the example of motivic functors 10.5, one gets a well-behaved model for the G-equivariant stable homotopy category. We leave the details to the reader. However, it would definitely be interesting to get the generalization of the Quillen equivalence [13, 11.3] between simplicial functors and spectra for the model structure given in 11.16. The statements we have already proven imply that it is enough to verify the equivariant analog of [13, 11.6], which uses essentially all of [13, section 11]. To achieve this, an equivariant version of the Blakers-Massey theorem seems to be useful. Since Manos Lydakis proposed to consider the equivariant analog of [13, 11.3], we do not pursue this question any further at the moment.

Appendix A. Calculations

In this appendix we discuss proofs for the remaining claims in the main part of this paper. It contains a lot of diagrams, thus it is a good place to mention Paul Taylor's diagram package [17]. The structure map $\hom_{A,B}^{-\otimes T}$ of the $\mathcal{V}$-functor $-\otimes T\colon \mathcal{V} \longrightarrow \mathcal{V}$ is defined as the adjoint of the composition

$$\mathcal{V}(A,B)\otimes(A\otimes T) \xrightarrow[\cong]{\alpha^{-1}_{\mathcal{V}(A,B),A,T}} (\mathcal{V}(A,B)\otimes A)\otimes T \xrightarrow{(\epsilon_A B)\otimes T} T\otimes A$$

where $\alpha^{-1}_{\mathcal{V}(B,T),B,A}$ is the associativity isomorphism. The next lemma shows that there are in general two different suspension functors for T-spectra.

Lemma A.1. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B)\otimes(A\otimes T) \arrow[r, "\hom_{A,B}^{-\otimes T}\otimes(A\otimes T)"] \arrow[d, "\alpha^{-1}_{\mathcal{V}(A,B),A,T}"'] & \mathcal{V}(A\otimes T, B\otimes T)\otimes(A\otimes T) \arrow[d, "\epsilon_{A\otimes T}(B\otimes T)"] \\
(\mathcal{V}(A,B)\otimes A)\otimes T \arrow[r, "(\epsilon_A B)\otimes T"] & B\otimes T
\end{tikzcd}$$

Proof. Use naturality and the triangular identity
$\epsilon_{A\otimes T}(\mathcal{V}(A,B)\otimes(A\otimes T))\circ(\eta_{A\otimes T}\mathcal{V}(A,B))\otimes(A\otimes T) = \mathrm{id}_{\mathcal{V}(A,B)\otimes(A\otimes T)}$. □

A similar statement is used to show that the stabilization of enriched functors and the stabilization of spectra can be compared.

Lemma A.2. *The following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta_B A"] \arrow[d, "\eta_{B\otimes T} A"'] & \mathcal{V}(B, A\otimes B) \arrow[d, "\hom_{B,A\otimes B}^{-\otimes T}"] \\
\mathcal{V}(B\otimes T, A\otimes(B\otimes T)) \arrow[r, "\mathcal{V}(B\otimes T, \alpha^{-1}_{A,B,T})"] & \mathcal{V}(B\otimes T, (A\otimes B)\otimes T)
\end{tikzcd}$$

Proof. Similar to A.1, using $\epsilon_B(A\otimes B)\circ(\eta_B A)\otimes B = \mathrm{id}_{A\otimes B}$. □

Next we will prove that the two natural stabilization maps $X \longrightarrow T(X)$ described in 6.1 coincide. It is lengthy and perhaps not very illuminating. The map $\hom_{A,B}^{\mathcal{V}(T,-)} : \mathcal{V}(A,B) \longrightarrow \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B))$ is given as the adjoint of $\mathrm{comp}\colon \mathcal{V}(A,B)\otimes\mathcal{V}(T,A) \longrightarrow \mathcal{V}(T,B)$ which, up to an associativity isomorphism, is the adjoint of $\mathcal{V}(A,B)\otimes(\mathcal{V}(T,A)\otimes T) \xrightarrow{\mathcal{V}(A,B)\otimes\epsilon_T A} \mathcal{V}(A,B)\otimes A \xrightarrow{\epsilon_A B} B$.

Lemma A.3. *The diagram*

$$\begin{tikzcd}
\mathcal{V}(A,B) \arrow[rr, "\hom_{A,B}^{\mathcal{V}(T,-)}"] \arrow[dr, "\mathcal{V}(\epsilon_T A, B)"'] & & \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B)) \\
& \mathcal{V}(\mathcal{V}(T,A)\otimes T, B) \arrow[ur, "\cong", "\int"'] &
\end{tikzcd}$$

commutes, where f is the adjointness isomorphism.

Proof. Here, f is $\mathcal{V}(\eta_T\mathcal{V}(T,A),\mathcal{V}(T,B))\circ \mathrm{hom}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}$. The diagram

$$\begin{tikzcd}
\mathcal{V}(A,B) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}_{A,B}"] \arrow[d, "\mathcal{V}(\epsilon_T A,B)"'] & \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B)) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\epsilon_T A),\mathcal{V}(T,B))"] \\
\mathcal{V}(\mathcal{V}(T,A)\otimes T,B) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}"] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(T,A)\otimes T),\mathcal{V}(T,B))
\end{tikzcd}$$

commutes, because $\mathcal{V}(T,-)$ is a $\mathcal{V}$-functor. Therefore, $f\circ\mathcal{V}(\epsilon_T(A),B)$ coincides with $\mathcal{V}(\eta_T\mathcal{V}(T,A),\mathcal{V}(T,B))\circ\mathcal{V}(\mathcal{V}(T,\epsilon_T(A)),\mathcal{V}(T,B))\circ\mathrm{hom}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}$ and the triangular identity $\mathcal{V}(T,\epsilon_T(A))\circ\eta_T\mathcal{V}(T,A)=\mathrm{id}_{\mathcal{V}(T,A)}$ completes the proof. $\square$

Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category of $\mathcal{V}$ satisfying **f2**. For $v\in\mathrm{Ob}\,\mathcal{C}$ and $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ is a $\mathcal{V}$-functor, one can consider the adjoint $X(c)\otimes v\longrightarrow X(v\otimes c)$ of the map $v\xrightarrow{\eta_c v}\mathcal{V}(c,v\otimes c)\xrightarrow{\mathrm{hom}^X_{c,v\otimes c}}\mathcal{V}(X(c),X(v\otimes c))$. It defines a $\mathcal{V}$-natural transformation $\mathrm{move}^X_v\colon X\otimes v\longrightarrow X\circ(v\otimes -)$. One of the maps $X\longrightarrow T(X)$ is defined using the map move^X_T. By pre-composing with a commutativity isomorphism we get a map $v\otimes X\longrightarrow X\circ(v\otimes -)$ which will also be denoted move^X_v. An interesting case is $\mathcal{V}(T,-)\colon\mathcal{V}\longrightarrow\mathcal{V}$. Then $\mathrm{move}^{\mathcal{V}(T,-)}_A(B)\colon A\otimes\mathcal{V}(T,B)\longrightarrow\mathcal{V}(T,A\otimes B)$ is the adjoint of $(A\otimes\mathcal{V}(T,B))\otimes T\xrightarrow{\alpha_{A,\mathcal{V}(T,B)}}A\otimes(\mathcal{V}(T,B)\otimes T)\xrightarrow{A\otimes\epsilon_T B}A\otimes B$.

Lemma A.4. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta_{\mathcal{V}(T,B)}A"] \arrow[d, "\eta_B A"'] & \mathcal{V}(\mathcal{V}(T,B),A\otimes\mathcal{V}(T,B)) \arrow[d, "\mathcal{V}(\mathcal{V}(T,B),\mathrm{move}^{\mathcal{V}(T,-)}_A(B))"] \\
\mathcal{V}(B,A\otimes B) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}_{B,A\otimes B}"] & \mathcal{V}(\mathcal{V}(T,B),V(T,A\otimes B))
\end{tikzcd}$$

Proof. Using the definition of $\mathrm{hom}^{\mathcal{V}(T,-)}$ and the description of $\mathrm{move}^{\mathcal{V}(T,-)}_A$ from above, one gets a large diagram which commutes by naturality and the triangular identity $\epsilon_B(A\otimes B)\circ(\eta_B A)\otimes B=\mathrm{id}_{A\otimes B}$. $\square$

Lemma A.5. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B)\otimes\mathcal{V}(T,A) \arrow[r, "\mathrm{move}^{\mathcal{V}(T,-)}_{\mathcal{V}(A,B)}(A)"] \arrow[d, "\mathrm{hom}^{\mathcal{V}(T,-)}_{A,B}\otimes\mathcal{V}(T,A)"'] & \mathcal{V}(T,\mathcal{V}(A,B)\otimes A) \arrow[d, "\mathcal{V}(T,\epsilon_A B)"] \\
\mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B))\otimes\mathcal{V}(T,A) \arrow[r, "\epsilon_{\mathcal{V}(T,A)}\mathcal{V}(T,B)"] & \mathcal{V}(T,B)
\end{tikzcd}$$

Proof. This is similar to the proof of A.4. The relevant triangular identity is $\epsilon_{\mathcal{V}(T,A)}(\mathcal{V}(A,B)\otimes\mathcal{V}(T,A))\circ(\eta_{\mathcal{V}(T,A)}\mathcal{V}(A,B))\otimes\mathcal{V}(T,A)=\mathrm{id}_{\mathcal{V}(A,B)\otimes\mathcal{V}(T,A)}$. $\square$

Lemma A.6. *Let $A, B, T \in \operatorname{Ob}\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
A \otimes \mathcal{V}(T,B) \arrow[r, "\eta_T A \otimes \mathcal{V}(T,B)"] \arrow[d, "\mathrm{move}_A^{\mathcal{V}(T,-)}(B)"'] & \mathcal{V}(T, A \otimes T) \otimes \mathcal{V}(T,B) \arrow[d, "\mathrm{move}_{\mathcal{V}(T,B)}^{\mathcal{V}(T,-)}(A\otimes T)"] \\
\mathcal{V}(T, A \otimes B) \arrow[d, "\mathcal{V}(T,\sigma_{A,B})"'] & \mathcal{V}(T, \mathcal{V}(T,B) \otimes (A \otimes T)) \arrow[d, "\mathcal{V}(T,c)"] \\
\mathcal{V}(T, B \otimes A) & \mathcal{V}(T, (\mathcal{V}(T,B) \otimes T) \otimes A) \arrow[l, "\mathcal{V}(T,(\epsilon_T B)\otimes A)"']
\end{tikzcd}$$

Here c is the composition of an associativity and a commutativity isomorphism.

Proof. Insert the description of $\mathrm{move}^{\mathcal{V}(T,-)}$ and use naturality, associativity and commutativity coherence, and $\epsilon_T(A \otimes T) \circ (\eta_T A) \otimes T = \mathrm{id}_{A\otimes T}$. $\square$

Lemma A.7. *Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category of $\mathcal{V}$ satisfying* **f2**, *$X \colon \mathcal{C} \longrightarrow \mathcal{V}$ a $\mathcal{V}$-functor, $v, w, T \in \operatorname{Ob}\mathcal{C}$ and $f \colon \mathcal{V}(T, \mathcal{V}(v,w)) \longrightarrow \mathcal{V}(T \otimes v, w)$ the adjointness isomorphism. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes (T \otimes Xv) \arrow[rr, "\alpha^{-1}"] \arrow[d, "f\otimes \mathrm{move}_T^X(v)"'] & & (\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T) \otimes Xv \arrow[d, "(\epsilon_T \mathcal{V}(v,w))\otimes Xv"] \\
\mathcal{V}(T\otimes v, w) \otimes X(T \otimes v) \arrow[d, "\mathrm{hom}^X_{T\otimes v,w} \otimes X(T\otimes v)"'] & & \mathcal{V}(v,w) \otimes Xv \arrow[d, "\mathrm{hom}^X_{v,w}\otimes Xv"] \\
\mathcal{V}(X(T\otimes v), Xw) \otimes X(T \otimes v) \arrow[dr, "\epsilon_{X(T\otimes v)}Xw"'] & & \mathcal{V}(Xv, Xw) \otimes Xv \arrow[dl, "\epsilon_{Xv}Xw"] \\
& Xw &
\end{tikzcd}$$

Proof. The proof is divided into two steps. First we note that

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes (T \otimes Xv) \arrow[rr, "\alpha^{-1}"] \arrow[d, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes(T\otimes Xv)"'] & & (\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T) \otimes Xv \arrow[d, "(\epsilon_T \mathcal{V}(v,w))\otimes Xv"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes (T \otimes Xv) \arrow[d, "g\otimes(T\otimes Xv)"'] & & \mathcal{V}(v,w) \otimes Xv \arrow[d, "\mathrm{hom}^X_{v,w}\otimes Xv"] \\
\mathcal{V}(T\otimes Xv, Xw) \otimes (T \otimes Xv) \arrow[dr, "\epsilon_{T\otimes Xv}Xw"'] & & \mathcal{V}(Xv, Xw) \otimes Xv \arrow[dl, "\epsilon_{Xv}Xw"] \\
& Xw &
\end{tikzcd}$$

commutes, where g is the adjointness isomorphism

$$\mathcal{V}(T \otimes Xv, \epsilon_{Xv}Xw) \circ \hom^{-\otimes Xv}_{T,\mathcal{V}(Xv,Xw)}.$$

Commutativity of the diagram follows from the definition of $\hom^{-\otimes Xv}$, naturality and the triangular identity $\epsilon_{T\otimes Xv}(- \otimes (T \otimes Xv)) \circ (\eta_{T\otimes Xv}) \otimes (T \otimes Xv) = \mathrm{id}$ applied to $\mathcal{V}(T, \mathcal{V}(Xv, Xw)) \otimes (T \otimes Xv)$. That X is a $\mathcal{V}$-functor is irrelevant for this result.

In the second step, we will prove that the diagram

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T \otimes Xv \arrow[r, "\mathcal{V}(T,\hom^X_{v,w})\otimes T\otimes Xv"] \arrow[d, "f\otimes \mathrm{move}^X_T"'] & \mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes T \otimes Xv \arrow[d, "g\otimes T\otimes Xv"] \\
\mathcal{V}(T\otimes v,w) \otimes X(T\otimes v) \arrow[d, "\hom^X_{T\otimes v,w}\otimes X(T\otimes v)"'] & \mathcal{V}(T\otimes Xv,Xw) \otimes T \otimes Xv \arrow[d, "\epsilon_{T\otimes Xv}Xw"] \\
\mathcal{V}(X(T\otimes v),Xw) \otimes X(T\otimes v) \arrow[r, "\epsilon_{X(T\otimes v)}Xw"] & Xw
\end{tikzcd}$$

commutes. The adjoint of the composition $\epsilon_{X(T\otimes v)}Xw \circ (\hom^X_{T\otimes v,w} \otimes \mathrm{move}^X_T)$ coincides with the composition $\mathrm{comp}\circ(\hom^X_{T\otimes v,w} \otimes \hom^X_{v,T\otimes v}) \circ \mathcal{V}(T \otimes v, w) \otimes \eta_v T$. Because X is a $\mathcal{V}$-functor, the latter composition coincides with the composition $\hom^X_{v,w} \circ \mathrm{comp} \circ \mathcal{V}(T \otimes v, w) \otimes \eta_v T$. Hence the diagram above commutes if and only if

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T \otimes Xv \arrow[rr, "\mathcal{V}(T,\hom^X_{v,w})\otimes T\otimes Xv"] \arrow[d, "f\otimes \eta_v T\otimes Xv"'] & & \mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes T \otimes Xv \arrow[d, "g\otimes T\otimes Xv"] \\
\mathcal{V}(T\otimes v,w) \otimes \mathcal{V}(v,T\otimes v) \otimes Xv \arrow[d, "\mathrm{comp}\otimes Xv"'] & & \mathcal{V}(T\otimes Xv,Xw) \otimes T \otimes Xv \arrow[d, "\epsilon_{T\otimes Xv}Xw"] \\
\mathcal{V}(v,w) \otimes Xv \arrow[dr, "\hom^X_{v,w}\otimes Xv"'] & & Xw \\
& \mathcal{V}(Xv,Xw) \otimes Xv \arrow[ur, "\epsilon_{Xv}Xw"'] &
\end{tikzcd}$$

commutes. The shortest composition is $\hom^X_{v,w} \otimes Xw \circ (\epsilon_T \mathcal{V}(v, w)) \otimes Xw$, as the triangular identity $\epsilon_{T\otimes Xv}(- \otimes (T \otimes Xv)) \circ (\eta_{T\otimes Xv}) \otimes (T \otimes Xv) = \mathrm{id}_{-\otimes(T\otimes Xv)}$ evaluated at $\mathcal{V}(T, \mathcal{V}(Xv, Xw))$ shows. Therefore it remains to prove that the map $\mathrm{comp} \circ (f \otimes \eta_v T)$ coincides with the map $\epsilon_T \mathcal{V}(v, w)$. It is equivalent to switch to the adjoints (under tensoring with v), and here naturality and the triangular identity $\epsilon_{T\otimes v}(-\otimes(T\otimes v))\circ(\eta_{T\otimes v})\otimes(T\otimes v) = \mathrm{id}_{-\otimes(T\otimes v)}$ evaluated at $\mathcal{V}(T, \mathcal{V}(v, w))$ give the desired identification. $\square$

Proposition A.8. *The two maps* $X \longrightarrow T(X)$ *coincide.*

Proof. Most for notational convenience, we will often leave out associativity and commutativity constraints. Recall that the two maps in question are determined by $\tau_v(w)\colon \mathcal{V}(T\otimes v,w)\otimes T \xrightarrow{f\otimes T} \mathcal{V}(T,\mathcal{V}(v,w))\otimes T \xrightarrow{\epsilon_T\mathcal{V}(v,w)} \mathcal{V}(v,w)$ up to Yoneda isomorphism, and $Xv \xrightarrow{\eta_T Xv} V(T, Xv\otimes T) \xrightarrow{V(T,\mathrm{move}^X_T(v))} \mathcal{V}(T, X(T\otimes v))$. The Yoneda isomorphism

$$Xv \xrightarrow{\cong} \int_{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(\mathcal{V}(v,w),Xw)$$

is induced the natural transformation $y^X_v(w)\colon Xv \longrightarrow \mathcal{V}(\mathcal{V}(v,w),Xw)$, that is, the composition

$$\begin{aligned} Xv &\xrightarrow{\eta_{\mathcal{V}(v,w)}Xv} \mathcal{V}(\mathcal{V}(v,w), Xv\otimes\mathcal{V}(v,w)) \\ &\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),Xv\otimes\mathrm{hom}^X_{v,w})} \mathcal{V}(\mathcal{V}(v,w), Xv\otimes\mathcal{V}(Xv,Xw)) \\ &\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),\sigma_{Xv,\mathcal{V}(Xv,Xw)})} \mathcal{V}(\mathcal{V}(v,w), \mathcal{V}(Xv,Xw)\otimes Xv) \\ &\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),\epsilon_{Xv}Xw)} \mathcal{V}(\mathcal{V}(v,w),Xw). \end{aligned}$$

Hence it suffices to prove that, for every $w\in\mathrm{Ob}\,\mathcal{C}$, the composition

$$a(w) := y^{\mathcal{V}(T,X)}_{T\otimes v}(w)\circ\mathcal{V}(T,\mathrm{move}^X_T(v))\circ\eta_T Xv$$

coincides with the composition $b(w) := h\circ\mathcal{V}(\tau_v,Xw)\circ y^X_v(w)$. Here h denotes the adjointness isomorphism $\mathcal{V}(\mathcal{V}(T\otimes v,w)\otimes T,Xw) \longrightarrow \mathcal{V}(\mathcal{V}(T\otimes v,w),\mathcal{V}(T,Xw))$. The isomorphism $f\colon \mathcal{V}(T,\mathcal{V}(v,w)) \longrightarrow \mathcal{V}(T\otimes v,w)$ will be used in the proof. The diagram

$$\begin{tikzcd} \mathcal{V}(\mathcal{V}(v,w),Xw) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}"] \arrow[d, "\mathcal{V}(\tau_v,Xw)"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),\mathcal{V}(T,Xw)) \\ \mathcal{V}(\mathcal{V}(T\otimes v,w))\otimes T,Xw) \arrow[r, "\mathcal{V}(f\otimes T,Xw)"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w))\otimes T,Xw) \arrow[u, "\cong"'] \end{tikzcd}$$

commutes by A.3, where the vertical map on the right hand side is the adjointness isomorphism. Then by naturality and A.4, $b(w)$ coincides with the composition $\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),\mathcal{V}(T,\epsilon_{Xv}Xw\circ\mathrm{hom}^X_{v,w})\circ\mathrm{move}^{\mathcal{V}(T,-)}_{Xv})\circ\eta_{\mathcal{V}(T,\mathcal{V}(v,w))}Xv$. The diagram

$$\begin{tikzcd} Xv \arrow[r, "\eta_{\mathcal{V}(T,\mathcal{V}(v,w))}Xv"] \arrow[d, "\eta_T Xv"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), Xv\otimes\mathcal{V}(T,\mathcal{V}(v,w))) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),(\eta_T Xv)\otimes\mathcal{V}(T,\mathcal{V}(v,w)))"] \\ \mathcal{V}(T,Xv\otimes T) \arrow[d, "\mathcal{V}(T,\mathrm{move}^X_T(v))"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),\mathcal{V}(T,Xv\otimes T)\otimes\mathcal{V}(T,\mathcal{V}(v,w))) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),\mathcal{V}(T,\mathrm{move}^X_T(v))\otimes\mathcal{V}(T,\mathcal{V}(v,w)))"] \\ \mathcal{V}(T,X(T\otimes v)) \arrow[r, "\eta_{\mathcal{V}(T,\mathcal{V}(v,w))}"] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),\mathcal{V}(T,X(T\otimes v))\otimes\mathcal{V}(T,\mathcal{V}(v,w))) \end{tikzcd}$$

commutes by naturality. Hence the maps $a(w)$ and $b(w)$ coincide if

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes Xv \arrow[r, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\eta_T Xv"] \arrow[d, "\mathrm{move}^{\mathcal{V}(T,-)}_{Xv}"'] & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,Xv\otimes T) \arrow[d, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,\mathrm{move}^X_T(v))"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(v,w)) \arrow[d, "\mathcal{V}(T,Xv\otimes\mathrm{hom}^X_{v,w})"'] & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,X(T\otimes v)) \arrow[d, "f\otimes\mathcal{V}(T,X(T\otimes v))"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(Xv,Xw)) \arrow[d, "\mathcal{V}(T,\sigma)"'] & \mathcal{V}(T\otimes v,w)\otimes\mathcal{V}(T,X(T\otimes v)) \arrow[d, "\mathrm{hom}^X\otimes\mathcal{V}(T,X(T\otimes v))"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw)\otimes Xv) \arrow[d, "\mathcal{V}(T,\epsilon_{Xv}Xw)"'] & \mathcal{V}(X(T\otimes v),Xw)\otimes\mathcal{V}(T,X(T\otimes v)) \arrow[d, "\mathrm{hom}^{\mathcal{V}(T,-)}\otimes\mathcal{V}(T,X(T\otimes v))"] \\
\mathcal{V}(T,Xw) & \mathcal{V}(\mathcal{V}(T,X(T\otimes v)),V(T,Xw))\otimes V(T,X(T\otimes v)) \arrow[l, "\epsilon_{\mathcal{V}(T,X(T\otimes v))}"']
\end{tikzcd}$$

commutes for all w. Now use A.5 (with $A = X(T\otimes v)$ and $B = Xw$), naturality of the map $\mathrm{move}^{\mathcal{V}(T,-)}$ and the isomorphism f to replace the composition from the upper corner on the right hand side to the lower corner on the left hand side. The result is the diagram

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes Xv \arrow[rr, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\eta_T Xv"] \arrow[d, "\mathrm{move}^{\mathcal{V}(T,-)}_{Xv}"'] & & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,Xv\otimes T) \arrow[d, "\mathrm{move}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,\mathcal{V}(v,w))}"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(v,w)) \arrow[d, "\mathcal{V}(T,Xv\otimes\mathrm{hom}^X_{v,w})"'] & & \mathcal{V}(T,\mathcal{V}(T,\mathcal{V}(v,w))\otimes X(T\otimes v)) \arrow[ll, "\mathcal{V}(T,Xv\otimes\epsilon_T\mathcal{V}(v,w))"'] \arrow[d, "\mathcal{V}(T,f\otimes X(T\otimes v))"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(Xv,Xw)) \arrow[d, "\mathcal{V}(T,\sigma)"'] & & \mathcal{V}(T,\mathcal{V}(T\otimes v,w)\otimes X(T\otimes v)) \arrow[d, "\mathcal{V}(T,\mathrm{hom}^X\otimes X(T\otimes v))"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw)\otimes Xv) \arrow[dr, "\mathcal{V}(T,\epsilon_{Xv}Xw)"'] & & \mathcal{V}(T,\mathcal{V}(X(T\otimes v),Xw)\otimes X(T\otimes v)) \arrow[dl, "\mathcal{V}(T,\epsilon_{X(T\otimes v)}Xw)"] \\
& \mathcal{V}(T,Xw) &
\end{tikzcd}$$

The upper part commutes by A.6 (with $A = Xv$ and $B = \mathcal{V}(v,w)$), the lower part is $\mathcal{V}(T,-)$ applied to a diagram which commutes by A.7. This completes the proof. □

References

[1] Francis Borceux. *Handbook of categorical algebra. 2.* Cambridge University Press, Cambridge, 1994. Categories and structures.

[2] Francis Borceux. *Handbook of categorical algebra. 3.* Cambridge University Press, Cambridge, 1994. Categories of sheaves.

[3] Gunnar Carlsson. A survey of equivariant stable homotopy theory. *Topology*, 31(1):1–27, 1992.

[4] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, pages 1–38. Springer, Berlin, 1970.

[5] Bjørn Ian Dundas, Oliver Röndigs, and Paul Arne Østvær. Motivic algebra. In preparation.

[6] Philip S. Hirschhorn. Localization of model categories. Book in preparation, October 2001.

[7] Mark Hovey. *Model categories.* American Mathematical Society, Providence, RI, 1999.

[8] Mark Hovey. Errata to Model Categories. Preprint, December 2001.

[9] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[10] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[11] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[12] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[13] Manos Lydakis. Simplicial functors and stable homotopy theory. Preprint 98-049, SFB 343, Bielefeld, June 1998.

[14] Saunders MacLane. *Categories for the working mathematician.* Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[15] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[16] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[17] Paul Taylor. Commutative diagrams in TEX. Macro package, September 2000.

[18] Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, pages 579–604 (electronic), 1998.

[19] Vladimir Voevodsky. Homotopy theory of simplicial sheaves in completely decomposable topologies. Available via http://www.math.uiuc.edu/K-theory/, August 2000.

[20] Vladimir Voevodsky. Unstable motivic homotopy categories in Nisnevich and cdh-topologies. Available via http://www.math.uiuc.edu/K-theory/, August 2000.