ON K-THEORY OF CLASSICAL-LIKE GROUPS

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`Why do we care about singers? Wherein lies the power of songs? Maybe it derives from the sheer strangeness of there being singing in the world. The note, the scale, the chord; melodies, harmonies, arrangements; symphonies, ragas, Chinese operas, jazz, the blues: that such things should exist, that we should have discovered the magical intervals and distances that yield the poor cluster of notes, all within the span of a human hand, from which we can build our cathedrals of sound, is as alchemical a mystery as mathematics, or wine, or love. Maybe the birds taught us. Maybe not. Maybe we are just creatures in search of exaltation. We don't have much of it. Our lives are not what we deserve; they are, let us agree, in many painful ways deficient. Song turns them into something else. Song shows us a world that is worthy of our yearning, it shows us our selves as they might be, if we were worthy of the world...`

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CONTENT.

INTRODUCTION.

`Our life shows us who we are...`

This thesis consists of two Parts. Part 1 is devoted to the study of nonstable K_1 of quadratic groups using Bak's dimension theory. This part is to appear in *K-Theory*. Part 2 is devoted to the study of the reduced K-theory of central simple algebras and is based on three papers which I published this year in the Journal of Algebra [6], Proc. Amer. Math. Soc. [7] and Algebra Colloq [8].

The main theme in Part 1 is the K-theory of classical groups. This theory has a rich history, beginning already in the 19th century when Jordan and then Dickson investigated certain matrix groups with coefficients in finite fields. Later Dieudonné considered families of "classical groups", namely general linear groups, symplectic groups, unitary groups and orthogonal groups over arbitrary fields and division rings and studied their subgroups and quotient groups. Progress over more general rings was very slow until the emergence of algebraic K-theory in the late 60'ies and the work of H. Bass. After that, activity exploded in the work of Bak, Suslin, Van der Kallen, Dennis, Stein, Vaserstein and many others. Their work extended that of Dieudonné and his predecessors, to classical and classical-like groups defined over general rings instead of just fields and division rings. In particular in the case of a module finite algebra R, Suslin showed that for $n \geq 3$, non-stable K_1 of the general linear group $GL_n(R)$ is a group, that is, the elementary subgroup $E_n(R)$ is a normal subgroup of the general linear group $GL_n(R)$. In general, the analysis of the nonstable K-group $K_{1,n}(R) = GL_n(R)/E_n(R)$ is a very difficult problem which involves the machinery of K-theory. In 1991 using his localisation-completion method, Bak obtained a functorial filtration of general linear groups, which shows that non-stable $K_{1,n}(R)(n \geq 3)$ is a nilpotent by abelian group when the Bass-Serre dimension of R is finite. This led Bak to conjecture that the same holds for any classical-like group, e.g. a general quadratic group, a general Hermitian group, a Chevalley group (the classical cases are covered already by the groups above), or a net version of any of the above (it being assumed that all of the groups have rank greater than 2).

The main aim of Part 1 is to show that non-stable K_1 of a general quadratic group is an abelian by nilpotent group when the Bass-Serre dimension of the ground ring is finite. To be precise, we investigate the non-stable quadratic K-group $K_{1,2n}(A,\Lambda) = G_{2n}(A,\Lambda)/E_{2n}(A,\Lambda)$, $n \geq 3$, where $G_{2n}(A,\Lambda)$ denotes the general quadratic group of rank n over a form ring (A,Λ) and $E_{2n}(A,\Lambda)$ its elementary subgroup. Considering form rings as a category with dimension in the sense of Bak (see 1.3), we obtain a dimension filtration $G_{2n}(A,\Lambda) \supseteq G^0_{2n}(A,\Lambda) \supseteq G^1_{2n}(A,\Lambda) \supseteq \cdots \supseteq E_{2n}(A,\Lambda)$ of the general quadratic group $G_{2n}(A,\Lambda)$ such that $G_{2n}(A,\Lambda)/G^0_{2n}(A,\Lambda)$ is abelian, $G^0_{2n}(A,\Lambda) \supseteq G^1_{2n}(A,\Lambda) \supseteq \cdots$ is a descending central series, and $G^{d(A)}_{2n}(A,\Lambda) = E_{2n}(A,\Lambda)$ whenever $d(A) =$ (Bass-Serre dimension of A) is finite. In particular $K_{1,2n}(A,\Lambda)$ is solvable when $d(A) < \infty$.

We also refer the reader to a new joint paper with Nikolai Vavilov, where we settle positively the conjecture for the case of Chevalley groups.

- R. Hazrat, N. Vavilov, K_1 of Chevalley groups are nilpotent, J. Pure Appl. Alg, To appear.

Part 2 concerns the theory of division algebras. Here we assume that the reader is comfortable with the theory of central simple algebras, valuation theory for fields and division algebras. A division algebra is a very elementary object. It is just a vector space with an associative product structure where each non-zero element has an inverse. A central simple algebra is just a set of matrices over a division algebra. A classical theorem of Wedderburn shows that central simple algebras are basic building blocks of ring theory. Despite the simplicity of the definition of a central simple algebra, the study of these objects has come to involve a large collection of mathematical tools and view points. The major classical theorems for them include some of the most splendid results of the great heros of Algebra: Wedderburn, Artin, Noether, Brauer, Albert, Jacobson, Serre and many others. During the past two decades new insight into this theory began arising under the influence of algebraic geometry and algebraic K-theory. It can be seen in the works of Suslin, Voevodsky, Friedlander, Merkurjev and others. Note especially that over the last few years, this work has resulted in the solution of the Milnor conjecture.

In Part 2, we focus on K_1 of central simple algebras. Here there is a classical reduced norm map which connects K_1 of a division algebra to that of a field. The kernel of the reduced norm $SK_1(D) = SL(1,D)/D'$ got extensive attention after the exciting work of Platonov, which answered Bass-Tannaka-Artin question. Platonov showed that SK_1 is not trivial in general, and developed a very complicated machinery to compute this group in certain cases.

In Part 2.1, using Wedderburn's factorization theorem, we provide a very simple and short way of obtaining some of the key points in Platonov's theory, including his congruence theorem. This is then used to study the descending central series of the multiplicative group of a division algebra. In section 2.2.1, we show that certain properties, which a term in the descending central series may have, can be lifted to the full multiplicative group. In 2.2.1, using valuation theory for division algebras, we determine quotients of consecutive terms in the descending central series for a tame and Henselian unramified or totally ramified division algebra.

In Section 2.3, we introduce the functor $G(A) = coker(K_1(F) \to K_1(A))$ from the category of central simple algebras to abelian groups. It is shown that this functor has the following three properties: it admits the transfer maps, is trivial on the center of the algebra and is n-torsion in the split case (2.3.1). Note that the functor $SK_1(D)$ shares these properties. (Merkurjev-Suslin $\bar{K}_i$, Draxl's CH^0 and Tignol's $G(D)$ also have these properties). We then show that G has the most important functorial properties of the reduced Whitehead group. We establish a fundamental connection between this group, its residue version, and relative valued

group when D is a Henselian division ring. The structure of $G(D)$ turns out to carry significant information concerning the arithmetic of D. As an application, we obtain theorems of reduced K-theory which previously required heavy machinery, as simple examples of our methods.