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Relaxing Perfectness: Which Graphs are "Almost" Perfect?[1]

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Relaxing Perfectness: Which Graphs are "Almost" Perfect?

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Abstract

For all perfect graphs, the stable set polytope STAB(G) coincides with the fractional stable set polytope QSTAB(G), whereas STAB(G) $\subset$ QSTAB(G) holds iff G is imperfect. Padberg asked in the early seventies for "almost" perfect graphs. He characterized those graphs for which the difference between STAB(G) and QSTAB(G) is smallest possible. We develop this idea further and define three polytopes between STAB(G) and QSTAB(G) by allowing certain sets of cutting planes only to cut off all the fractional vertices of QSTAB(G). The difference between QSTAB(G) and the largest of the three polytopes coinciding with STAB(G) gives some information on the stage of imperfectness of the graph G. We obtain a nested collection of three superclasses of perfect graphs and survey which graphs are known to belong to one of those superclasses. This answers the question: which graphs are "almost" perfect?

1 Introduction

Berge [1] proposed to call a graph $G = (V, E)$ **perfect** if, for each (node-induced) subgraph $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$. That is, for all $G' \subseteq G$, we need as many stable sets to cover all nodes of G' as the maximum clique of G' has nodes (a set $V' \subseteq V$ is a clique (stable set) if the nodes in V' are mutually (non-)adjacent; maximum cliques (stable sets) contain a maximal number of nodes).

Berge [1] conjectured two characterizations of perfect graphs. His first conjecture was that a graph G is perfect iff the clique covering number $\overline{\chi}(G')$ equals the stability number $\alpha(G')$ $\forall G' \subseteq G$ (i.e., that we need as many cliques

to cover all nodes of G' as a maximum stable set of G' has nodes). Since complementation transforms stable sets into cliques, we have $\alpha(G) = \omega(\overline{G})$ and $\chi(G) = \overline{\chi}(\overline{G})$ where $\overline{G}$ denotes the complement of G. Hence, BERGE [1] conjectured and LOVÁSZ [17] proved that *a graph G is perfect if and only if its complement $\overline{G}$ is (Perfect Graph Theorem).* The second BERGE conjecture concerns a characterization via forbidden subgraphs. It is a simple observation that chordless odd cycles C_{2k+1} with $k \geq 2$, termed **odd holes**, and their complements $\overline{C}_{2k+1}$, called **odd antiholes**, are imperfect. Clearly, each graph containing an odd hole or an odd antihole as subgraph is imperfect as well. BERGE conjectured in [1]: *a graph is perfect iff it contains neither odd holes nor odd antiholes as subgraphs (Strong Perfect Graph Conjecture).* This conjecture is still open and one of the most famous conjectures in graph theory.

PADBERG [21, 22] asked which graphs are "almost" perfect, i.e., which graphs are imperfect with the property that all of their proper induced subgraphs are perfect. Such graphs are nowadays called **minimally imperfect**. Using this term, the Strong Perfect Graph Conjecture reads: *odd holes and odd antiholes are the only minimally imperfect graphs.* In order to give a characterization of minimally imperfect graphs (and thereby to verify or falsify the Strong Perfect Graph Conjecture), many fascinating structures of such graphs have been discovered. First, the Perfect Graph Theorem implies that a graph is minimally imperfect iff its complement is. Further properties reflecting an extraordinary amount on symmetry of their maximum cliques and stable sets are given by the next two theorems.

Theorem 1.1 (LOVÁSZ [17]) *Every minimally imperfect graph G has exactly $\alpha\omega + 1$ nodes and, for every node x of G, the graph $G - x$ can be partitioned into α cliques of size ω and into ω stable sets of size α, where $\alpha = \alpha(G)$ and $\omega = \omega(G)$ holds.*

Theorem 1.2 (PADBERG [21]) *Every minimally imperfect graph G on n nodes has precisely n maximum stable sets and precisely n maximum cliques. Each node of G is contained in precisely $\alpha(G)$ maximum stable sets and in precisely $\omega(G)$ maximum cliques. For every maximum clique Q (maximum stable set S) there is a unique maximum stable set S (maximum clique Q) with $Q \cap S = \emptyset$.*

Unfortunately, minimally imperfect graphs are not characterized by those properties but share them with other graphs. BLAND, HUANG, and TROTTER suggested in [2] to call a graph **partitionable** if it satisfies the conditions of Theorem 1.1 for some integers α, ω and verified Theorem 1.2 for

all partitionable graphs (see Figure 1 for two partitionable graphs which are not minimally imperfect). Thus, the class of partitionable graphs contains all potential counterexamples to the Strong Perfect Graph Conjecture. One main interest is, therefore, to find so-called *genuine properties* satisfied by all minimally imperfect graphs but violated by at least one partitionable graph (see [25] for more information on minimally imperfect and partitionable graphs).

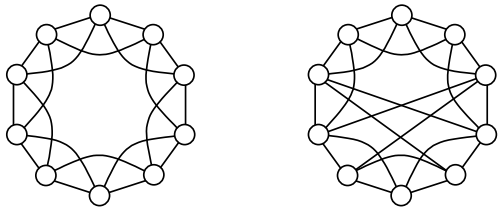

Figure 1: Examples of partitionable graphs.

Padberg [21, 22] investigated general set packing problems and studied the case when the polyhedron $P(A) = \{x \in \mathbb{R}^n_+ : Ax \leq \mathbb{1}\}$ associated with an $m \times n$ 0/1-matrix A has integral vertices only (where $\mathbb{1} = (1, \ldots, 1)$). Padberg proved in [21] that $P(A)$ coincides with $P_I(A)$, the convex hull of integer vertices of $P(A)$, if and only if A is a perfect 0/1-matrix. Translating this result in graph theoretic terms [21], consider the graph G associated with A where the nodes of G correspond to the n columns of A and two nodes of G are linked by an edge if the corresponding columns of A have a 1-entry in common. Consequently, A is the clique-node incidence matrix of G and $P(A)$ is the **fractional stable set polytope** QSTAB(G) given by the **nonnegativity constraints**

$$x_i \geq 0 \tag{0}$$

for all nodes i of G and by the **clique constraints**

$$\sum_{i \in Q} x_i \leq 1 \tag{1}$$

for all cliques $Q \subseteq G$. Furthermore, $P_I(A)$ corresponds to the **stable set polytope** STAB(G) which is defined as the convex hull of the incidence vectors of all stable sets of the graph G. Then the result on perfect 0/1-matrices says:

Theorem 1.3 (PADBERG [21]) *STAB(G) = QSTAB(G) if and only if G is perfect.*

Remark. In [4] CHVÁTAL noted that Theorem 1.1 implies that polyhedral characterization of perfect graphs, further references are FULKERSON [9, 10] and SACHS [27].

If G is an imperfect graph, $\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$ holds and the difference between $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ can be used as a tool in order to decide how far a graph is away from being perfect. In this sense, PADBERG [22, 23] introduced the notion of almost integral polyhedra defined with respect to $m \times n$ 0/1-matrices A: $P(A)$ is called **almost integral** if $P(A)$ possesses at least one fractional vertex, but the polyhedra obtained from $P(A)$ by projecting $P(A)$ into (strictly) lower-dimensional subspaces have integer vertices only. PADBERG proved several properties of almost integral polyhedra (see, e.g., Theorem 1.4 below) and showed recently an equivalent version of the Strong Perfect Graph Conjecture in terms of almost integral polyhedra [23]. (PADBERG introduced in [23] two kinds of orthogonal projections and proved that the Strong Perfect Graph Conjecture is correct if and only if the studied projections of almost integral polyhedra yield again almost integral polyhedra.)

Theorem 1.4 (PADBERG [22]) *If $P(A)$ is almost integral, then the following conditions are simultaneously satisfied. Every fractional vertex has exactly n adjacent integer vertices. $P(A)$ has exactly one fractional vertex. $P_I(A) = P(A) \cap \{x \in \mathbb{R}^n_+ : \sum_{1 \le i \le n} x_i \le \overline{\alpha}\}$ with $\overline{\alpha} = \max\{\sum_{1 \le i \le n} x_i : x \in P(A)\}$.*

PADBERG called a 0/1-matrix A almost perfect if $P(A)$ is almost integral and showed that A is almost perfect if and only if it is the clique-node incidence matrix of an almost perfect (i.e., minimally imperfect) graph. In graph theoretic terms, the above theorem implies, therefore, the following characterization of minimally imperfect graphs.

Theorem 1.5 (PADBERG [21, 22]) *G is minimally imperfect if and only if QSTAB(G) has exactly one fractional vertex (adjacent to the $|G|$ integer vertices coming from the maximum stable sets of G) and $STAB(G) = QSTAB(G) \cap \{x \in \mathbb{R}^{|G|}_+ : \sum_{i \in G} x_i \le \alpha(G)\}$.*

(Note that $\overline{\alpha} = \max\{\sum_{i \in G} x_i : x \in \mathrm{QSTAB}(G)\}$ is sometimes called the fractional stability number of G.) That means, G is minimally imperfect if

and only if QSTAB(G) has *precisely one* fractional vertex which can be cut off by *exactly one* cutting plane, namely, the so-called **full rank constraint**

$$\sum_{i \in G} x_i \leq \alpha(G) \tag{2}$$

associated with G. The above theorem implies, therefore, a most beautiful nontrivial genuine property that holds *exactly* for all minimally imperfect graphs and for *none* of the other partitionable graphs:

Theorem 1.6 (Padberg [21, 22]) *A partitionable graph G is minimally imperfect if and only if QSTAB(G) has exactly one fractional vertex.*

In the case of minimally imperfect graphs G, the polytope QSTAB(G) is the smallest possible relaxation of STAB(G) and, hence, minimally imperfect graphs are indeed "almost perfect". The next possible relaxation of STAB(G) is the case when QSTAB(G) may have more than one fractional vertex but, again, the full rank constraint is required as only cutting plane to cut off all those fractional vertices. This lead Shepherd [30], inspired by Padberg's results, to the definition of near-perfect matrices: an $m \times n$ 0/1-matrix A is called near-perfect if $P_I(A)$ coincides with $P(A) \cap \{x \in \mathbb{R}^n_+ : \sum_{1 \leq i \leq n} x_i \leq \alpha(G)\}$ where G is again the graph with clique-node incidence matrix A. Let denote FSTAB(G) the polytope given by all nonnegativity constraints (0), all clique constraints (1), and the full rank constraint (2). Shepherd [30] called a graph G **near-perfect** if STAB(G) = FSTAB(G). Minimally imperfect graphs are obviously near-perfect. Since there is no requirement that QSTAB(G) has at least one fractional vertex but only that all fractional vertices are cut off by the full rank constraint, perfect graphs are all near-perfect, too. Figure 2 shows near-perfect graphs which are neither perfect nor minimally imperfect. More examples and considerations on near-perfect graphs can be found in Section 3.

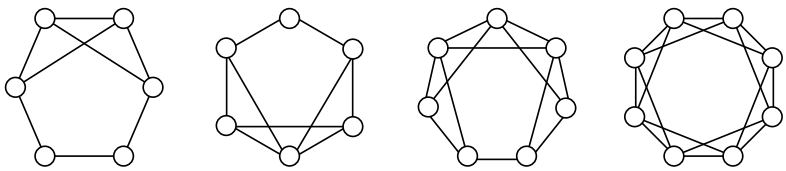

Figure 2: Examples of near-perfect graphs.

Following a suggestion of GRÖTSCHEL, LOVÁSZ, and SCHRIJVER [15] one may relax the notion of perfectness further by generalizing clique constraints to other classes of inequalities valid for the stable set polytope and then by investigating all graphs such that their stable set polytope is entirely described by nonnegativity constraints and the inequalities in question. A natural way to generalize both clique constraints and the full rank constraint is to consider *all* 0/1-inequalities, i.e., to investigate the **rank constraints**

$$\sum_{i \in G'} x_i \;\leq\; \alpha(G') \tag{3}$$

associated with arbitrary induced subgraphs $G' \subseteq G$ (note $\alpha(G') = 1$ holds iff G' is a clique). Every rank constraint is obviously valid for the stable set polytope and defines in some cases also a facet (see next section for examples of graphs G where the full rank constraint is facet-defining). Hence, the polytope RSTAB(G) given by all nonnegativity constraints (0) and all rank constraints (3) is a further relaxation of STAB(G) but contained in FSTAB(G). We define all graphs G with STAB(G) = RSTAB(G) to be **rank-perfect** (i.e., if we need only 0/1-inequalities of the form (3) to cut off all fractional vertices of QSTAB(G)). Every perfect, every minimally imperfect, and every near-perfect graph is obviously also rank-perfect. Further classes of rank-perfect graphs are discussed in Section 4.

If a rank constraint is associated with a *proper* subgraph $G' \subset G$, then it does not yield a facet of STAB(G) in general, even if $\sum_{i \in G'} x_i \leq \alpha(G')$ is facet-defining for STAB(G'). In the latter case, we can determine a facet

$$\sum_{i \in G'} x_i \;+ \sum_{i \in G - G'} a_i \, x_i \;\leq\; \alpha(G') \tag{4}$$

of the stable set polytope of the whole graph G by computing appropriate coefficients a_i for all nodes i in $G - G'$ via sequential lifting [20] (see Section 2). We call facets of the form (4) **weak rank constraints** if the base rank constraint associated with G' is facet-defining for STAB(G'). (That means, a lifted rank constraint $\sum_{i \in G'} x_i + \sum_{i \in G - G'} a_i x_i \leq \alpha(G')$ is a weak rank constraint if an orthogonal projection is the full rank facet of STAB(G').) Clearly, facet-defining rank constraints are weak rank constraints with $a_i = 0$ for $i \in G - G'$. Let WSTAB(G) be the polytope given by all nonnegativity constraints (0) and all weak rank constraints (4). WSTAB(G) is a further relaxation of STAB(G) but contained in RSTAB(G) (since we allow more general cutting planes than rank constraints only). We define all graphs G with STAB(G) = WSTAB(G) to be **weakly rank-perfect** (see Section 5

for classes of weakly rank-perfect graphs). Moreover, the stable set polytope itself is entirely described by all "trivial" facets (0) and all "nontrivial" facets of the general form

$$\sum_{i \in G} a_i \, x_i \;\leq\; \alpha(G, a) \tag{5}$$

where we interpret the vector $a = (a_1, \ldots, a_n)$ to be a node weighting of G associating the weight a_i to $i \in G$ and denote the weighted graph by (G, a). Furthermore, $\alpha(G, a) = \max\{\sum_{i \in S} a_i : S \subseteq G \text{ stable set}\}$ stands for the weighted stability number. Thus, there is no further relaxation of STAB(G) possible that way beyond WSTAB(G). By the chain of relaxations of STAB(G)

$$\mathrm{STAB}(G) \subseteq \mathrm{WSTAB}(G) \subseteq \mathrm{RSTAB}(G) \subseteq \mathrm{FSTAB}(G) \subseteq \mathrm{QSTAB}(G)$$

we have finally obtained a nested collection of superclasses of perfect graphs: near-perfect, rank-perfect, and weakly rank-perfect graphs. The difference between QSTAB(G) and the largest polytope coinciding with STAB(G) increases, hence each superclass contains graphs which are successively "less" perfect. This gives us some information on the stage of imperfectness or answers in a wider sense the question: which graphs are "almost" perfect? Our considerations will have a special stress on near-perfect graphs (which are closest to perfect graphs) while we only list known classes of rank-perfect and weakly rank-perfect graphs in Section 4 and Section 5. We close with some final remarks and open problems in Section 6.

2 Rank Constraints and Sequential Lifting

Determining the system of facet-defining inequalities of STAB(G), i.e., to find all cutting planes required to cut off the fractional vertices of QSTAB(G), is very difficult in general. Thus one often tries to find classes of valid inequalities for STAB(G) and to investigate when those valid inequalities yield facets of STAB(G). One natural class of valid nontrivial inequalities are rank constraints (3) associated with induced subgraphs $G' \subseteq G$. For convenience, we often write (3) as $x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$, $x(G', \mathbb{1}) \leq \alpha(G')$, or just $x(G') \leq \alpha(G')$. The goal is to find out, for which subgraphs $G' \subseteq G$, the associated rank constraint $x(G') \leq \alpha(G')$ yields a facet of STAB(G). A first step towards this goal is to identify those graphs G for which their stable set polytope has the full rank constraint (2)

$$x(G, \mathbb{1}) \;\leq\; \alpha(G, \mathbb{1})$$

as a facet (we say that such graphs produce the full rank facet). PADBERG showed this if G is a clique [20] or minimally imperfect [21]. BLAND, HUANG, and TROTTER [2] generalized PADBERG's result [21] by showing that (2) is a facet of STAB(G) for all partitionable graphs G. Webs form a graph class with circular symmetry of their maximum cliques and stable sets which contain many partitionable graphs. A **web** W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (i.e., if $|i-j| \leq k \bmod n$); we assume $n \geq 2(k+1)$ in the following since W_n^k is a clique otherwise. Note that W_n^1 is a hole, W_{2k+1}^{k-1} an odd antihole for $k \geq 2$, and that $W_{\alpha\omega+1}^{\omega-1}$ is partitionable with $\alpha = \alpha(W_n^{k-1}) = \lfloor \frac{n}{k} \rfloor$ and $\omega = \omega(W_n^{k-1}) = k$. The partitionable web W_{10}^2 is shown in Figure 1(a), the near-perfect graph in Figure 2(d) is the web W_8^2.

Remark. Webs are also called circulant graphs C_n^k defined in [5]. Furthermore, graphs $W(n,k)$ with $n \geq 2$, $1 \leq k \leq \frac{1}{2}n$ and $W(n,k) = \overline{W}_n^{k-1}$ were introduced in [32].

TROTTER [32] studied when the complement of a web, called **antiweb**, produces the full rank facet: he showed that this is the case if and only if the antiweb $\overline{W}_n^{k-1}$ is **prime**, i.e., if k and n are relatively prime. In order to show which webs produce the full rank facet, we need the following result [4]. An edge e of a graph $G = (V, E)$ is **α-critical** if $\alpha(G) < \alpha(G - e)$. We call G **α-connected** if the graph on the same node set V having all α-critical edges of G is connected. CHVÁTAL [4] showed that *every α-connected graph produces the full rank facet* (see [29] for a survey and [18, 24] for further results).

Theorem 2.1 *W_n^{k-1} produces the full rank facet if and only if k is not a divisor of n.*

Proof. *If:* Consider the maximum stable set $S_i = \{i, i+k, \ldots, i+(\lfloor \frac{n}{k} \rfloor - 1)k\}$ of W_n^{k-1} (where all indices are taken modulo n). Since k is not a divisor of n, we have $\lfloor \frac{n}{k} \rfloor k < n$. Subtracting $k - i$ from both sides of this relation yields $(i + (\lfloor \frac{n}{k} \rfloor - 1)k < i + n - k$ where $i + n - k$ is the last neighbor of i in W_n^{k-1}. Consequently, $(S_i - \{i\}) \cup \{i-1\}$ is also a maximum stable set of W_n^{k-1} and the edge $i-1, i$ is, therefore, α-critical for $1 \leq i \leq n$ (where all indices are again taken modulo n). Thus, if k is not a divisor of n, then W_n^{k-1} is α-connected and produces the full rank facet due to CHVÁTAL [4]. $\diamond$
Only if: In the case that k is a divisor of n, there are only k maximum stable sets in W_n^{k-1} (of size $\frac{n}{k}$). Thus, W_n^{k-1} cannot contain n maximum stable sets the incidence vectors of which are affinely independent. $\square$

Remark. The *If*-part is along the proof in [32] that W_n^{k-1} produces the full rank facet if k and n are relatively prime. The weaker condition that k is not a divisor of n suffices for the argumentation. (E.g., W_{10}^3 produces the full rank facet but 4 and 10 are not relatively prime.)

Moreover, EDMONDS and PULLEYBLANK [7] established via matching theory that line graphs of 2-connected hypomatchable graphs have the full rank facet: H is called **hypomatchable** if, for all nodes v of H, the subgraph $H - v$ admits a matching (i.e., a set of disjoint edges) meeting all nodes. A graph is 2-connected if it is still connected after removing an arbitrary node. The **line graph** $L(F)$ of a graph F is obtained by taking the edges of F as nodes of $L(F)$ and connecting two nodes in $L(F)$ iff the corresponding edges of F are incident. (Note: matchings of H correspond to stable sets of its line graph $L(H)$ since the line operator transforms non-incident edges of H to non-adjacent nodes of $L(H)$).

For some cases, a sufficient condition is known when a rank constraint $x(G') \leq \alpha(G')$ associated with a *proper* subgraph $G' \subset G$ yields a facet of the stable set polytope of the whole graph G. PADBERG [20] showed that clique constraints $x(Q, \mathbb{1}) \leq 1$ are facet-inducing for STAB(G) iff Q is an (inclusion-wise) maximal clique of G. The result in [7] implies that a rank constraint

$$x(L(H), \mathbb{1}) \leq \frac{|H| - 1}{2} \tag{3a}$$

associated with the line graph of a 2-connected hypomatchable graph $H \subseteq F$ is a facet of STAB$(L(F))$ iff H is an induced subgraph of F. In general, a rank constraint associated with a proper subgraph $G' \subset G$ does not need to provide a facet of STAB(G), even if STAB(G') admits the full rank facet. This is the case for, e.g., **odd hole constraints**

$$x(C_{2k+1}, \mathbb{1}) \leq k \tag{3b}$$

with $C_{2k+1} \subset G$ and for **odd antihole constraints**

$$x(\overline{C}_{2k+1}, \mathbb{1}) \leq 2 \tag{3c}$$

with $\overline{C}_{2k+1} \subset G$. Figure 3(a) shows a graph with an induced C_5 (note a C_5 is both an odd hole and an odd antihole) but the rank constraint associated with this C_5 does not induce a facet of the stable set polytope of the whole graph.
However, rank constraints $x(G') \leq \alpha(G')$ with $G' \subset G$ may be strengthened to a facet of STAB(G) using **sequential lifting** introduced by PADBERG [20],

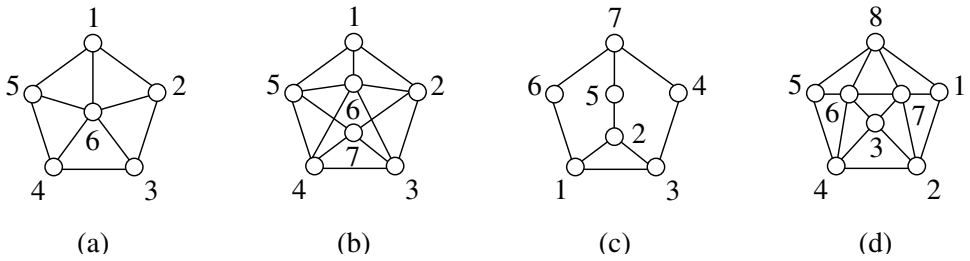

Figure 3

i.e., by determining appropriate lifting coefficients a_i for all nodes i in $G - G'$ such that the right hand side $\alpha(G')$ of the inequality is still satisfied and that there are $|G|$ many stable sets of weight $\alpha(G')$ the incidence vectors of which are linearly independent. Every inequality

$$x(G', \mathbb{1}) + x(G - G', a) \leq \alpha(G', \mathbb{1}) \tag{4}$$

is a weak rank constraint if it is obtained by lifting a base rank constraint $x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$ which is facet-defining for STAB(G'), i.e., if G' produces the full rank facet. The graph G depicted in Figure 3(a) yields a weak rank constraint basing on an odd hole constraint by using a lifting coefficient $\neq 0, 1$ (thus G is not rank-perfect in particular). G consists of an odd hole (nodes 1, ..., 5) and a central node adjacent to all nodes of the odd hole (such graphs are termed **odd wheels**). The C_5 yields 5 stable sets of weight 2 the incidence vectors of which are linearly independent. In order to construct the remaining stable set of weight 2 containing the central node 6, we have to choose lifting coefficient $a_6 = 2$. The resulting facet $x(C_5, \mathbb{1}) + 2x_6 \leq 2$ of STAB(G) is a special weak rank constraint, called **odd wheel constraint**

$$x(C_{2k+1}, \mathbb{1}) + kx_c \leq k \tag{4a}$$

where c is the central node adjacent to all nodes of the odd hole C_{2k+1} and $k \geq 2$. (See Padberg [20] for a general description how to lift odd hole constraints associated with proper subgraphs to weak rank facets of the whole graph.) Shepherd [31] studied a more general weak rank constraint

$$\sum_{i \leq k} \frac{1}{\alpha(\overline{W}_i)} x(\overline{W}_i) + x(Q, \mathbb{1}) \leq 1 \tag{4b}$$

associated with the complete join of prime antiwebs $\overline{W}_1, \ldots, \overline{W}_k$ and a clique Q. (The **complete join** of two disjoint graphs G_1 and G_2 is obtained by joining every node of G_1 and every node of G_2 by an edge. E.g., every odd

wheel is the complete join of an odd hole and a single node. Note that the support graph of such facets arise by the complete join of graphs which all produce their full rank facet, i.e., we put together disjoint facet blocks. The obtained constraints can be scaled in such a way that they have the form (4) with a base rank constraint $x(G') \leq \alpha(G')$ and noninteger coefficients a_i for $i \in G - G'$. In this sense, (4b) can be seen as a lifted clique constraint.) SHEPHERD [31] showed that odd antiholes are the only prime antiwebs that occur in complements of line graphs. Thus their stable set polytopes admit weak rank constraints

$$\sum_{i \leq k} x(A_i) + 2x(Q) \leq 2 \tag{4c}$$

associated with the complete join of odd antiholes $A_1, \ldots, A_k$ and a clique Q. COOK studied (in an unpublished manuscript, see [30]) the stable set polytopes of graphs G with $\alpha(G) = 2$. He showed that the inequality

$$x(\tilde{N}(Q)) + 2x(Q) \leq 2 \tag{4d}$$

is valid for STAB(G) for every clique Q where $\tilde{N}(Q)$ is the set of all nodes v of G with $Q \subseteq N(v)$ (note $\tilde{N}(Q) = V(G)$ if $Q = \emptyset$ and $N(v)$ denotes the set of all neighbors of v). COOK showed that (4d) is a facet of STAB(G) iff no component of $\tilde{N}(Q)$ in the complementary graph $\overline{G}$ is bipartite (see [30]). Since $\alpha(G) = 2$ implies $\omega(\overline{G}) = 2$, a component of $\overline{G}$ is not bipartite iff it contains an odd hole. Hence, (4d) is a facet iff $\tilde{N}(Q)$ is the complete join of subgraphs all containing an odd antihole and (4d) is, therefore, as lifting of (4c) a special weak rank constraint.

GILES and TROTTER [14] studied further weak rank constraints: Consider the webs W_n^{k+1} and W_n^k with $n = 2k(k+2)+1$ where $V(W_n^{k+1}) = \{1, \ldots, n\}$ and $V(W_n^k) = \{1', \ldots, n'\}$. Construct the graph G^k by taking W_n^{k+1} and W_n^k as induced subgraphs and adding the edges $\{i, i'\}$, $\{i, (i+1)'\}$, $\ldots$, $\{i, (i+2k+1)'\}$ for $1 \leq i \leq n$ where all indices are taken modulo n. (The graph G^1 is shown in Figure 4, here the odd antihole W_7^2 and the odd hole W_7^1 are emphasized with bold lines). Then

$$(k+1)\ x(W_n^{k+1}) + k\ x(W_n^k) \leq 2k(k+1) \tag{4e}$$

is a facet of STAB(G^k) by [14]. W_n^{k+1} has stability number $2k$ and produces the full rank facet by Theorem 2.1 (since $k+2$ is not a divisor of $n = 2k(k+2)+1$). Hence (4e) is a class of weak rank constraints.
Note that the weak rank facet obtained by lifting may depend on the order in which the nodes are lifted [20]. Hence, lifting a base rank constraint may result in *several* weak rank constraints. The graph G in Figure 3(b), e.g.,

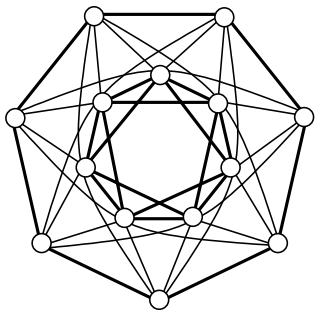

Figure 4

contains the 5-wheel from Figure 3(a) as induced subgraph and the associated odd wheel constraint $x(C_5, \mathbb{1}) + 2x_6 \leq 2$ is also a facet of STAB(G). Furthermore, there is another way to lift the rank constraint associated with the C_5 to a facet of STAB(G), namely, by choosing $a_6 = 1$ and $a_7 = 1$ (i.e., STAB(G) also admits the full rank facet).

Finally, STAB(G) may admit nontrivial facets which are not weak rank constraints. The stable set polytope of the graph G in Figure 3(c), e.g., has the facet $\sum_{i \leq 6} x_i + 2x_7 \leq 3$ which is not a weak rank constraint: among the nodes of G with coefficient 1, there is no subgraph G' such that $x(G') \leq 3$ is a facet of STAB(G'). (That means, there is no facet-inducing structure of a proper subgraph $G' \subset G$ which we could lift to a facet of STAB(G).) In particular, the graph G in Figure 3(c) is an example of a graph which is not weakly rank-perfect. (Checking the stable set polytopes of small imperfect graphs yields that G and $\overline{G}$ are the only two not weakly rank-perfect graphs on up to seven nodes.) The graph G in Figure 3(d) is a so-called wedge introduced in [14]. Wedges are further example of graphs which are not weakly rank-perfect. The stable set polytope of G has, e.g., the facet $\sum_{i \leq 5} x_i + 2 \sum_{6 \leq i \leq 8} x_i \leq 3$ which is not a weak rank constraint, too.

Oriolo [19] introduced a new class of inequalities valid for the stable set polytope of every graph. Let $G = (V, E)$ be a graph and $\mathcal{Q}$ be a family of (at least three) maximal cliques of G. Let $k \leq |\mathcal{Q}|$ be an integer, $\lambda(|\mathcal{Q}|, k) = \frac{k-l-1}{k-l}$ with $l = |\mathcal{Q}| - k\lfloor \frac{|\mathcal{Q}|}{k} \rfloor$, and define the following two sets: $I(\mathcal{Q}, k) = \{v \in V : |\{Q \in \mathcal{Q} : v \in Q\}| \geq k\}$ and $O(\mathcal{Q}, k) = \{v \in V : |\{Q \in \mathcal{Q} : v \in Q\}| = k - 1\}$. It is known from [19] that

$$x(I(\mathcal{Q}, k)) + \lambda(|\mathcal{Q}|, k)\ x(O(\mathcal{Q}, k)) \leq \lfloor \frac{|\mathcal{Q}|}{k} \rfloor \tag{5a}$$

is valid for the stable set polytope of every graph G and that (5a) is a common

generalization of the rank constraints (3a) associated with line graphs of 2-connected hypomatchable graphs, the full rank constraints associated with webs W_n^{k-1} where k is not a divisor of n, and of the weak rank constraints (4e) asociated with graphs G^k introduced in [14]. However, it is not known so far whether a *facet* of the form (5a) is a weak rank constraint in general.

3 Near-Perfect Graphs

The subject of this section is a class of graphs which is, in a polyhedral sense, the smallest superclass of perfect graphs: the class of near-perfect graphs G where only one cutting plane, namely the full rank constraint, is required to cut off all fractional vertices of QSTAB(G) [30]. That means, for near-perfect graphs G we only have to add the full rank constraint (2) to the nonnegativity (0) and clique constraints (1) in order to arrive at STAB(G). Since there is no requirement that QSTAB(G) has at least one fractional vertex, all perfect graphs are near-perfect in particular (here the full rank constraint is not a facet except in the case of a clique). Hence near-perfect graphs are indeed the closest superclass of perfect graphs.

Minimally imperfect graphs are further examples of near-perfect graphs by Padberg [21, 22], see Theorem 1.5. While the characterization of minimally imperfect graphs via the Strong Perfect Graph Conjecture is still open, there is, besides Theorem 1.5, a further polyhedral characterization of minimally imperfect graphs in terms of near-perfection.

Theorem 3.1 (Shepherd [30]) *An imperfect graph G is minimally imperfect if and only if both G and $\overline{G}$ are near-perfect.*

That means, the part of the class of near-perfect graphs which is closed under complementation consists exactly in all perfect and all minimally imperfect graphs. For every partitionable graph G we know that G and $\overline{G}$ produce the full rank facet by Bland, Huang, and Trotter [2], but at most one of G and $\overline{G}$ is near-perfect. We have even more:

Theorem 3.2 *A partitionable graph G is minimally imperfect if and only if G is near-perfect.*

Proof. Every minimally imperfect graph is near-perfect by Padberg [21, 22]. We show that a partitionable graph G which is not minimally imperfect cannot be near-perfect either. G properly contains a minimally imperfect subgraph $G' \subset G$ with $\alpha(G') < \alpha(G)$ by [30]. The rank constraint associated

with G' yields a nontrivial facet of STAB(G) which is different from a clique facet and the full rank facet of G. $\Box$

Hence, we have, in addition to Theorem 1.6, a further nontrivial genuine property that holds *exactly* for all minimally imperfect graphs and for *none* of the other partitionable graphs. That means: if G is partitionable but not minimally imperfect, then QSTAB(G) has at least two fractional vertices by Theorem 1.6 and at least two cutting planes are required to arrive at STAB(G) (recall that every partitionable graph G produces the full rank facet by [2], but the full rank facet does not suffice to cut off all fractional vertices of QSTAB(G) by the above Theorem 3.2).

In order to be near-perfect, an imperfect graph G has obviously to satisfy the condition that every minimally imperfect subgraph of G has the same stability number as G. A further property was conjectured to characterize near-perfect graphs in [30].

Conjecture 3.3 (Shepherd [30]) *A graph G is near-perfect if and only iff each lifting of a rank constraint associated with a minimally imperfect subgraph of G yields the full rank facet $x(G) \leq \alpha(G)$.*

Besides perfect and minimally imperfect graphs, no other class is known so far to belong (completely) to the class of near-perfect graphs. In addition to Theorem 3.2, we give characterizations of all the near-perfect graphs in three graph classes. We start with a result from [30] on graphs G with stability number $\alpha(G) = 2$.

Theorem 3.4 (Shepherd [30]) *A graph G with $\alpha(G) = 2$ is near-perfect if and only if the neighborhood of every node of G induces a perfect graph.*

Next we study two classes which contain all odd holes, all odd antiholes, and many partitionable graphs: webs and antiwebs. Recall from Section 2 that a web W_n^{k-1} produces the full rank facet iff k is not a divisor of n (Theorem 2.1) while the same is true for antiwebs $\overline{W}_n^{k-1}$ iff k and n are relatively prime (Trotter [32]). We now determine for which webs and antiwebs the full rank facet is the only facet of the stable set polytope besides facets of type (0) and (1).

Theorem 3.5 *A web is near-perfect if and only if it is perfect, an odd hole, W_{11}^2, or if it has stability number two.*

Proof. *If:* The assertion is trivial if W_n^{k-1} is perfect and follows for odd holes from PADBERG [21]. In the case $\alpha(W_n^{k-1}) = 2$, we apply Theorem 3.4 due to SHEPHERD [30]. The neighborhood $N(i)$ of every node i of W_n^{k-1} consists of two disjoint cliques, namely, $\{i-(k-1), \ldots, i-1\}$ and $\{i+1, \ldots, i+(k-1)\}$ where all indices are taken modulo n. Thus $N(i)$ induces the complement of a bipartite graph and is, therefore, perfect for all nodes i. Hence, W_n^{k-1} is near-perfect by Theorem 3.4 if $\alpha(W_n^{k-1}) = \lfloor \frac{n}{k} \rfloor = 2$ holds. Checking the stable set polytope of W_{11}^2 explicitely shows that W_{11}^2 is near-perfect, too. (Note: W_{11}^2 has C_7 as only minimally imperfect subgraphs and $\alpha(C_7) = 3 = \alpha(W_{11}^2)$ holds.) $\diamond$

Only if: W_n^{k-1} is a stable set if $k = 1$ and a hole if $k = 2$, hence either perfect or minimally imperfect and, in the latter case, near-perfect by PADBERG [21]. W_{2k}^{k-1} is the complement of the graph consisting of k disjoint edges (recall that we assume $n \geq 2k$ since W_n^{k-1} is a clique whenever $n < 2k$). W_{2k+1}^{k-1} is an odd antihole if $k \geq 2$, hence near-perfect by PADBERG [21]. We have to show that, for $k \geq 3$ and $n \geq 2k+2$, the web W_{11}^2 is the only near-perfect web W_n^{k-1} with stability number $\lfloor \frac{n}{k} \rfloor > 2$.

In the case $k \geq 3$ and $n \geq 2k+2$, W_n^{k-1} *properly* contains an odd hole or an odd antihole by TROTTER [32]. If one of those odd holes or odd antiholes has a stability number $< \alpha(W_n^{k-1})$ then STAB(W_n^{k-1}) has a nontrivial facet which is neither associated with a clique nor with W_n^{k-1} itself. Hence W_n^{k-1} is near-perfect only if it has stability number two or if it contains only odd holes $W_{n'}^1$ with stability number $\lfloor \frac{n'}{2} \rfloor = \lfloor \frac{n}{k} \rfloor > 2$ but no odd antihole. We show that W_n^{k-1} with $k \geq 3$ and $n \geq 3k$ has odd holes with stability number $< \alpha(W_n^{k-1})$ except the case $k = 3$ and $n = 11$.

Claim 1. *W_n^{k-1} contains odd holes of different length if $k = 3, 4$ and $n \geq 24$, if $k = 5$ and $n \geq 27$, if $k \geq 6$ and $n \geq 5k$.*

Proof of Claim 1. Due to TROTTER [32], we have $W_{n'}^1 \subseteq W_n^{k-1}$ iff $2n \geq n'k$ and $n \leq n'(k-1)$, i.e., iff

$$\frac{n}{k-1} \leq n' \leq 2\frac{n}{k}. \tag{i}$$

If $\frac{n}{k-1} + 4 \leq 2\frac{n}{k}$, there exist at least two odd n' which satisfy (i). Thus, determine n s.t.

$$\frac{4k(k-1)}{k-2} \leq n$$

holds. We obtain $n \geq 24$ if $k \in \{3, 4\}$, $n \geq 27$ if $k = 5$, and $n \geq 5k$ if $k \geq 6$.

Moreover, $\frac{n}{k-1} > 5$ holds in all those cases, hence W_n^{k-1} contains odd holes of different length.

Claim 2. *W_n^{k-1} contains only odd holes with stability number $\lfloor \frac{n}{k} \rfloor \geq 3$ only if $k = 3$ and $n = 11$.*

Proof of Claim 2. We consider W_n^{k-1} containing odd holes of length n' only, i.e., we get

$$n' - 2 < \frac{n}{k-1} \leq n' \leq 2\frac{n}{k} < n' + 2$$

by (i) . Replacing n' by $2\lfloor \frac{n}{k} \rfloor + 1$ (since $\lfloor \frac{n'}{2} \rfloor = \frac{n'-1}{2} = \lfloor \frac{n}{k} \rfloor$ is required), we obtain

$$2\lfloor \frac{n}{k} \rfloor - 1 < \frac{n}{k-1} \leq 2\lfloor \frac{n}{k} \rfloor + 1 \leq 2\frac{n}{k} < 2\lfloor \frac{n}{k} \rfloor + 3 \qquad \text{(ii)}$$

in order to guarantee $\alpha(W_{n'}^1) = \alpha(W_n^{k-1}) \geq 3$. We first observe that $2\frac{n}{k} < 2\lfloor \frac{n}{k} \rfloor + 3$ is true for all k and n (since $\frac{n}{k} < \lfloor \frac{n}{k} \rfloor + 1$). Further, $2\lfloor \frac{n}{k} \rfloor + 1 \leq 2\frac{n}{k}$ means $\lfloor \frac{n}{k} \rfloor + \frac{1}{2} \leq \frac{n}{k}$ and is fulfilled whenever $ik + \frac{k}{2} \leq n < (i+1)k$ for some i. If $i = 3$, we consider $2\lfloor \frac{n}{k} \rfloor - 1 = 5 < \frac{n}{k-1} = \frac{3k+l}{k-1}$ with $\frac{k}{2} \leq l < k$ and obtain $2k < 5 + l$ which is true only if $k < 4$. If $i = 4$, then $2\lfloor \frac{n}{k} \rfloor - 1 = 7 < \frac{n}{k-1} = \frac{4k+l}{k-1}$ with $\frac{k}{2} \leq l < k$ yields $3k < 7 + l$ which is true only if $k < 3$. If $i = 5$, we only have to check $k = 3, 4$ by Claim 1 (note $5k + \frac{k}{2} > 27$ if $k = 5$) but W_{17}^2, W_{22}^3, and W_{23}^3 all contain a C_9 and a C_{11} (which is implied by (i)). If $i = 6, 7$ we only have to check $k = 3$ by Claim 1, but we obtain $C_{11}, C_{13} \subseteq W_{20}^2$ and $C_{13}, C_{15} \subseteq W_{23}^2$ by (i). The case $i \geq 8$ has not to be checked for any $k \geq 3$ by Claim 1, thus we have only left $i = 3$ and $k = 3$.

The observation that W_n^{k-1} with $n < 3k$ cannot contain an odd hole different from a C_5 (since $\alpha(W_n^{k-1}) = 2$) finishes the proof. □

Theorem 3.6 *An antiweb is near-perfect if and only if it is perfect, an odd hole, or an odd antihole.*

Proof. In the case that $\overline{W}_n^{k-1}$ is perfect or minimally imperfect, then $\overline{W}_n^{k-1}$ is clearly near-perfect. We show that there are no other near-perfect antiwebs. $\overline{W}_n^{k-1}$ is a clique if $k = 1$ and an antihole if $k = 2$. $\overline{W}_n^{k-1}$ consists of k disjoint edges (and is perfect) if $n = 2k$. Trotter [32] has shown that $\overline{W}_n^{k-1}$ contains an odd hole or odd antihole as induced subgraph if $k \geq 3$ and $n > 2k$. If $n = 2k + 1$, then $\overline{W}_n^{k-1}$ is isomorphic to an odd hole. If $n > 2k + 1$, then $\overline{W}_n^{k-1}$ *properly* contains an odd hole $\overline{W}_{2k'+1}^{k'-1}$ or an odd

antihole $\overline{W}^1_{n'}$ since $\overline{W}^{l-1}_n \subseteq \overline{W}^{k-1}_n$ implies $l = k$ by [32] and $\overline{W}^1_{n'} \neq \overline{W}^{k-1}_n$ follows by $k \geq 3$. Then STAB($\overline{W}^{k-1}_n$) has the corresponding (zero-lifted) odd hole or odd antihole facet by TROTTER [32]. This facet is different from the full rank constraint associated with $\overline{W}^{k-1}_n$ since the stability number of the odd hole or odd antihole in $\overline{W}^{k-1}_n$ is *strictly* less than $k = \alpha(\overline{W}^{k-1}_n)$ (note that $\overline{W}^{k-1}_{n'} \subseteq \overline{W}^{k-1}_n$ implies $n = n'$ by [32] again). Hence, $\overline{W}^{k-1}_n$ is not near-perfect if $k \geq 3$ and $n > 2k + 1$. □

4 Rank-Perfect Graphs

We now turn to the next superclass of perfect graphs: the class of rank-perfect graphs G where 0/1-inequalities of the form (3)

$$x(G') \leq \alpha(G')$$

with $G' \subseteq G$ are needed as only nontrivial facets to describe STAB(G). Since clique constraints are special rank constraints (namely those with $\alpha(G') = 1$), all perfect graphs are rank-perfect in particular. Furthermore, all near-perfect graphs are obviously rank-perfect, too. There are further classes of rank-perfect graphs known.

CHVÁTAL [4] defined graphs G to be **t-perfect** if STAB(G) has rank constraints associated with edges and odd holes as only nontrivial facets. (Note that "t" stands for "trou", the French word for hole, and that every C_{2k+1} with $k \geq 1$ is here considered to be a hole.) Bipartite graphs without isolated nodes are obviously t-perfect. CHVÁTAL conjectured in [4] and BOULALA and UHRY proved in [3] that **series-parallel graphs** are t-perfect (that are graphs obtained from disjoint cycle-free subgraphs by repeated application of the following two operations: adding a new edge parallel to an existing edge and subdividing edges, i.e., replacing edges by a path). Further examples of t-perfect graphs are **almost bipartite graphs** (having a node the deletion of which leaves the graph bipartite) due to FONLUPT and UHRY [8] and **strongly t-perfect graphs** (having no subgraph obtained from subdividing edges of a K_4 such that all four cycles corresponding to the triangles of the K_4 are odd) due to GERARDS and SCHRIJVER [12]. Further investigations of t-perfect graphs without certain subdivisions of K_4 can be found in GERARDS and SHEPHERD [13].
By definition [15], a natural generalization of t-perfect graphs is the class of **h-perfect graphs** (from hole-perfect) where, besides nonnegativity constraints (0), all clique constraints (1) and odd hole constraints (3b) suffice

to describe the associated stable set polytopes. At present, there are no interesting classes of h-perfect graphs known that are neither perfect, nor t-perfect, nor combinations of these. (For combinations, see FONLUPT and UHRY [8] and SBIHI and UHRY [28].)

Line graphs are a further class of rank-perfect graphs due to a result of EDMONDS and PULLEYBLANK [7]. Their result implies that the stable set polytopes of line graphs are given by nonnegativity constraints (0), clique constraints (1), and rank constraints (3a) associated with the line graphs of 2-connected hypomatchable graphs. Note that line graphs are a "natural" graph class which is proven to contain rank-perfect graphs only (while near-perfect, t-perfect, and h-perfect graphs are rank-perfect by definition).
It is worth noting that line graphs seem to be a maximal class of rank-perfect graphs. The closest superclass of line graphs consists of all **quasi-line graphs** where the neighborhood of each node partitions into two cliques. (Quasi-line graphs were first investigated by BEN REBEA in his PhD thesis. Tragically, he died shortly after completing his thesis and all the efforts to reorganize and publish his results have been unsuccessful so far.) It is easy to check that, besides all line graphs, each web is a quasi-line graph. We know which webs are near-perfect due to Theorem 3.5. DAHL [6] showed that webs W^2_n for all $n \geq 4$ are rank-perfect. But there are webs with clique number > 4 (e.g. W^5_{25}) the stable set polytopes of which have non-rank facets (see KIND [16]). The graphs G^k introduced in [14] are further quasi-line graphs which produce non-rank facets (4e). Thus, quasi-line graphs are not rank-perfect. Furthermore, we studied in [33] critical edges with respect to perfectness (that are edges of perfect graphs the deletion of which yields an imperfect graph). We investigated the case of deleting critical edges e from perfect line graphs G. Besides 0/1-liftings of rank constraints (3a) associated with line graphs of 2-connected hypomatchable graphs, there also appear odd wheel constraints (4a) associated with 5-wheels as facets of STAB$(G-e)$, see [33]. Thus deleting edges from line graphs destroys the property of being rank-perfect, too. However, the 5-wheel constraints does not appear if we restrict our consideration to line graphs of *bipartite* graphs. Thus, $G-e$ might be rank-perfect if G is the line graph of a bipartite graph [33].

5 Weakly Rank-Perfect Graphs

This section deals with weakly rank-perfect graphs G where, besides nonnegativity constraints (0), only weak rank constraints (4) of the form

$$x(G', \mathbb{1}) + x(G - G', a) \leq \alpha(G', \mathbb{1})$$

are required to describe STAB(G). (Recall that the above inequality is obtained by lifting the base rank constraint associated with $G' \subseteq G$ and that $x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$ produces the full rank facet of STAB(G') by the definition of a weak rank constraint.) Since every facet-defining rank constraint $x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$ is a weak rank constraint with $a_i = 0$ for $i \in G - G'$, the class of weakly rank-perfect graphs contains all rank-perfect graphs (and, therefore, all near-perfect and all perfect graphs). One general way to arrive at classes of weakly rank-perfect graphs goes as follows: Consider a class of rank-perfect graphs where only nonnegativity constraints and special rank constraints are needed to describe the stable set polytope. Then define the "corresponding" class of weakly rank-perfect graphs by allowing weak rank constraints based on those special rank constraints as the only nontrivial facets of the stable set polytope. E.g., the class of **weakly h-perfect graphs** can be defined that way to contain all graphs whose stable set polytope is given by nonnegativity constraints (0), clique constraints (1), and lifted odd hole constraints. (See PADBERG [20] for a general description how to lift odd hole constraints to weak rank facets.) The 5-wheel in Figure 3(a) and the graph in Figure 3(b) are examples of weakly h-perfect graphs which are not h-perfect. (Note that the classes of weakly t-perfect and weakly h-perfect graphs coincide since clique constraints are liftings of edge constraints.)

Two natural graph classes are known to consist of weakly rank-perfect graphs only due to SHEPHERD [31]: so-called near-bipartite graphs and complements of line graphs. A graph G is **near-bipartite** if removing all neighbors of an arbitrary node leaves the graph bipartite. (That means, $G - N(v)$ can be partitioned into two stable sets for all nodes v of G and near-bipartite graphs are, therefore, the complements of quasi-line graphs.) The stable set polytope of near-bipartite graphs has facets of type (4b)

$$\sum_{i \leq k} \frac{1}{\alpha(\overline{W}_i)} x(\overline{W}_i) + x(Q, \mathbb{1}) \leq 1$$

associated with the complete join of prime antiwebs $\overline{W}_1, \ldots, \overline{W}_k$ and a clique Q as its only nontrivial facets [31]. The class of near-bipartite graphs contains all complements of line graphs (the non-neighbors of a node v in $\overline{L(F)}$ correspond to the edges incident to the edge v in F, hence to two cliques in $L(F)$ and to two stable sets in $\overline{L(F)}$). SHEPHERD [31] showed that odd antiholes are the only prime antiwebs that occur in complements of line graphs. Thus the only nontrivial facets of their stable set polytope are weak rank constraints (4c) associated with the complete join of odd antiholes and a clique.

We studied in [33] critical edges with respect to perfectness (recall: that are edges of perfect graphs the deletion of which yields an imperfect graph). We investigated the case of deleting critical edges e from complements G of perfect line graphs. We showed that odd antiholes are the only minimally imperfect subgraphs of $G - e$ and how to lift the corresponding odd antihole constraints to facets of STAB$(G - e)$. We were able to prove that those lifted odd antihole constraints are, besides clique constraints (1), the only nontrivial facets of STAB$(G - e)$ if G is the complement of the line graph of a *bipartite* graph. Thus: every graph obtained by deleting a critical edge from the complement of the line graph of a bipartite graph is weakly rank-perfect [33]. That means deleting edges from complements of line graphs of bipartite graphs leaves the resulting graphs in the same stage of imperfectness as general complements of line graphs, see [33] for more details.
Finally, a description of the facet-system of STAB(G) for all graphs G with $\alpha(G) = 2$ was found (but not published) by Cook, see [30]. He showed that the stable set polytope of graphs G with $\alpha(G) = 2$ is given by nonnegativity constraints (0) and weak rank constraints of the form (4d)

$$x(\tilde{N}(Q)) + 2x(Q) \leq 2$$

for every clique Q (recall that $\tilde{N}(Q)$ denotes the set of all nodes v of G with $Q \subseteq N(v)$). That means: graphs G with $\alpha(G) = 2$ are weakly rank-perfect, too. In order to figure out which graphs G with $\alpha(G) = 2$ are rank-perfect, we determine which rank facets may appear. The inequalities (4d) can be scaled to have no coefficients different from 0 and 1 only if Q is maximal (then $\tilde{N}(Q) = \emptyset$ follows) or Q is empty (then $\tilde{N}(Q) = V(G)$ follows). Thus, the only possible rank facets are maximal clique facets and the full rank facet. Hence, we have obtained: *a graph G with $\alpha(G) = 2$ is near-perfect if and only if G is rank-perfect.*

6 Concluding Remarks

For all perfect graphs, the stable set polytope coincides with the fractional stable set polytope, whereas STAB$(G) \subset$ QSTAB(G) holds iff G is imperfect. We used the difference between STAB(G) and QSTAB(G) to decide how far an imperfect graph is away from being perfect. For that, we introduced three polytopes that contain STAB(G) but are contained in QSTAB(G). The fractional stable set polytope QSTAB(G) is given by nonnegativity constraints (0) and clique constraints (1)

$$x(G', \mathbb{1}) \leq 1$$

for all cliques $G' \subseteq G$. We discussed which additional cutting planes are required to cut off all fractional vertices of QSTAB(G). We defined FSTAB(G) to be the polytope where the full rank constraint (2) is the only additional cutting plane. Next, we defined RSTAB(G) as the polytope given by non-negativity constraints (0) and all 0/1-inequalities (3)

$$x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1})$$

for arbitrary induced subgraphs $G' \subseteq G$. The last step was to allow in WSTAB(G) as nontrivial facets more general inequalities of the form (4)

$$x(G\,, a) \leq \alpha(G', \mathbb{1})$$

where $G' \subseteq G$ with $V(G') \subseteq \{v_i \in V(G) : a_i = 1\}$ and STAB(G') has the full rank facet. Since STAB(G) is given by (0) and all general inequalities (5)

$$x(G\,, a) \leq \alpha(G\,, a)$$

there is no further relaxation of STAB(G) possible that way than WSTAB(G):

$$\text{STAB}(G) \subseteq \text{WSTAB}(G) \subseteq \text{RSTAB}(G) \subseteq \text{FSTAB}(G) \subseteq \text{QSTAB}(G)$$

The difference between QSTAB(G) and the largest polytope coinciding with STAB(G) gives us some information on the stage of imperfectness of the graph G. This answers the question: which graphs are "almost" perfect? Closest to perfect graphs are, all near-perfect graphs G with STAB(G) = FSTAB(G). The next superclass contains all rank-perfect graphs G with STAB(G) = RSTAB(G). "Less perfect" are all weakly rank-perfect graphs G with STAB(G) = RSTAB(G). The discussion which graphs are known to belong to one of those superclasses of perfect graphs is summarized in Figure 5. For some interesting graph classes strongly related to minimally imperfect graphs, we do not know so far to which of the three superclasses they belong to: partitionable graphs and antiwebs. They are not all near-perfect, see Section 3, but there is some hope to prove that they are all rank-perfect.

Furthermore, perfect graphs are closed under complementation, but none of the superclasses of perfect graphs under consideration is: Theorem 3.1 by Shepherd [30] implies that for near-perfect graphs. The 5-wheel is not rank-perfect but its complement is; the wedge depicted in Figure 3(d) is not weakly rank-perfect but its complement is.

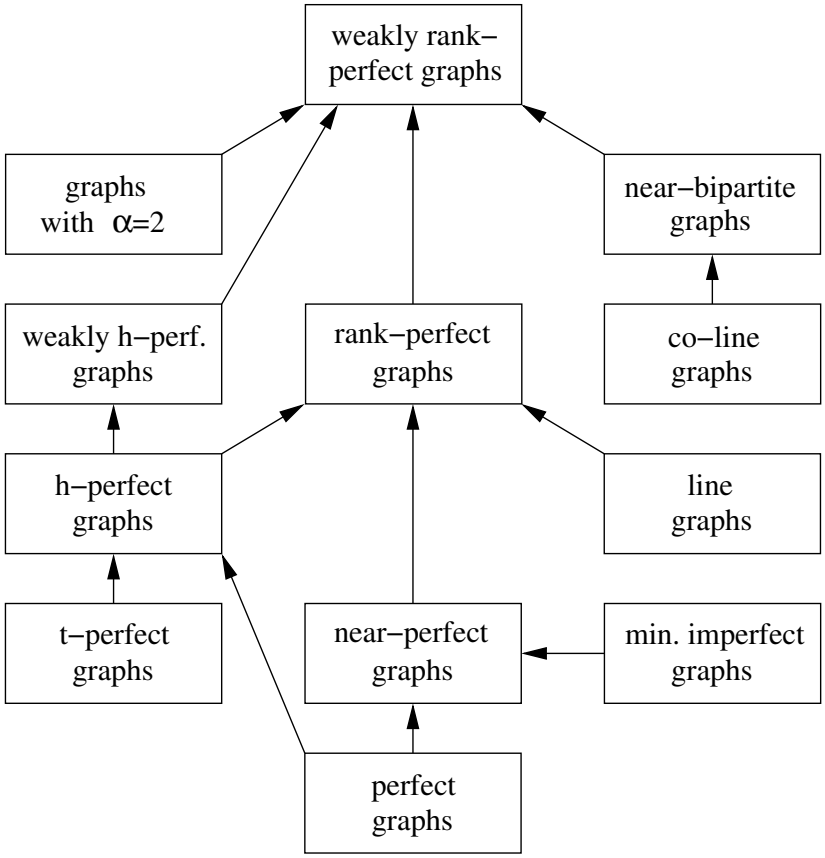

Figure 5

Finally, other than the perfect graphs, line graphs constitute the only natural class of graphs for which we have a polyhedral description for the stable set polytope for the class as well as for the complementary class. The question of polyhedral descriptions for quasi-line graphs and, more general, for claw-free graphs (having no node with a stable set of size three in its neighborhood) remains one of the interesting open problems in polyhedral combinatorics. We already know that quasi-line graphs are not rank-perfect, see the web W_{25}^5 and the graphs G^k introduced in [14]. Oriolo [19] conjectured that the only nontrivial facets of the stable set polytope of quasi-line graphs have the form (5a), but we even do not know whether these are weak rank constraints. We already know that claw-free graphs are not weakly rank-perfect, since all wedges are claw-free but produce facets which are not weak rank constraints by Giles and Trotter [14], see Section 2. Pulleyblank and Shepherd [26] showed that all wedges belong to a subclass of claw-free

graphs, so-called distance claw-free graphs (where the nodes at distance exactly two from a node do not contain a stable set of size three). Hence, distance claw-free graphs are not weakly rank-perfect, too. But there is a complete description of all rank facet producing claw-free graphs due to Galluccio and Sassano [11]. They showed that the rank facets of claw-free graphs essentially come from cliques, line graphs of 2-connected hypomatchable graphs, and partitionable webs.

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