The Canonical Extensive Form of a Game Form: Part I - Symmetries

by

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Abstract

Within this series of papers we plan to exhibit to any noncooperative game in strategic or normal form a 'canonical' representation in extensive form that preserves all symmetries of the game. The operation defined this way will respect the restriction of games to subgames and yield a minimal total rank of the tree involved. Moreover, by the above requirements the 'canonical extensive game form' will be uniquely defined.

Part I is dealing with isomorphisms of game forms and games. An automorphism of the game is called motion. A symmetry of a game is a permutation which can be augmented to a motion. Some results on the existence of symmetry groups are presented. The context to the notion of symmetry for coalitional games is exhibited.

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0 Introduction

This paper is the first part of an investigation which is devoted to the study of the relationship between a game in strategic form and its possible representations by extensive games. (A representation of a strategic game G is an extensive game Γ whose normal form is G.) We should emphasize that our starting point is a game in strategic form. The transition from the extensive form to the strategic form as defined by VON NEUMANN and MORGENSTERN (1944) has already been investigated extensively (see KOHLBERG and MERTENS (1986) for a recent treatment of this topic). The transition in the opposite direction is considered 'trivial' and conceptually straightforward. It is the purpose of our work to show that this is not true: The choice of a method of representation of strategic games by extensive games which respects symmetries of strategic games leads to difficult conceptual problems and deep mathematical results. Some of the problems will be illustrated by the well known example of the Battle of the Sexes.

Example 0.1 *Consider the following version of the Battle of Sexes*

$$G = \begin{matrix} & C & S \\ C & 2 & 0 \\ S & 0 & 1 \end{matrix} \qquad \begin{matrix} & C & S \\ C & 1 & 0 \\ S & 0 & 2 \end{matrix}$$

(see, e.g., MYERSON (1991, p.98) for a verbal description of the game). Here C is going to a concert and S is going to a soccer game. Player 1 is the wife and player 2 is the husband. There are two potential focal equilibria in this game: (C,C) and (S,S) (see the beautiful discussion in MYERSON (1991, Section 3.5)). Now, according to the present standard convention of game theory one can represent G by an extensive game in the following two different ways (see Figure 0.1).

Figure 0.1: Two representations

The foregoing convention which leads to multiple representations has the following two problematic aspects.

(1) The transition to the extensive form might influence focality. Consider the extensive game Γ_1. It is common knowledge in this game that player 1 moves first. Therefore she has the option to choose C before player two makes his choice. Thus, it seems to us that in Γ_1 the pair (C, C) of strategies is more likely to be played than (S, S). Our feeling is supported by the experimental work of RAPOPORT (1994). Clearly, in Γ_2 the pair (S, S) may be the dominant focal equilibrium rather than (C, C).

(2) The transition to the extensive form may destroy symmetry. The game G is 'symmetric' in the following sense: It has an automorphism which permutes players 1 and 2 (for a definition of automorphism, i.e. an isomorphism of G to itself, see HARSANYI and SELTEN (1988, Section 3.4)). This automorphism is given explicitly in our Example 3.6 (1). However, Γ_1 and Γ_2 are totally asymmetric; more precisely, if $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_2$, then there is no non-trivial automorphism of Γ that respects the temporal ordering of moves in Γ.

The discussion in the last paragraph leads naturally to the following basic question:

Let G be a game in strategic form. When is G 'symmetric'? (In particular, is the Battle of Sexes a symmetric game?)

Quite surprisingly there is no satisfactory answer available to this question. If we follow our mathematical intuition and define a strategic game G to be symmetric if all possible joint renamings of players and strategies are automorphisms of G (see HARSANYI and SELTEN (1988, P.71) for the precise definition of renaming), then the class of symmetric games reduces to the trivial class of all games whose payoff functions are constant and equal. Also, this definition is incompatible with the definition of symmetric bimatrix games (see van DAMME (1987, p.211)).

Our answer to the basic question is indirect. A symmetry of G, according to our definition, is a *permutation* π *of the players* for which there exists an automorphism $\alpha = (\pi, \varphi)$ of G (here φ is a renaming of strategies which is compatible with π). Thus, our definition of symmetries (of strategic games) is different from that of HARSANYI and SELTEN (1988, p.73). The game G is *symmetric* if every permutation of the players is a symmetry of G. Thus, in particular, the Battle of Sexes is symmetric according to our definition. Our definition has the following desirable properties.

(1) The class of symmetric games is a nontrivial interesting class.

(2) It is possible to use similar ideas to define symmetries of extensive games (see Definition 3.16).

(3) It is possible to compare the symmetry groups of a strategic game and its coalitional form (see Theorem 3.11).

As far as we could check, symmetries of games in extensive form which preserve the partial ordering on the nodes that is induced by the game tree were not considered previously. Thus, our treatment of symmetry groups of extensive games is entirely new.

We now present our solution to the problem of representing the Battle of Sexes by an extensive game.

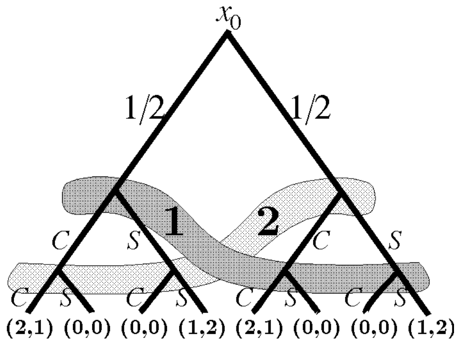

Figure 0.2: The 'canonical Battle of Sexes'

The above representation is *faithful* because, according to our definitions, it has the same symmetry group, namely $\sum(\{1,2\})$, as the Battle of Sexes. Also, it is quite obvious that it has 'minimum' graph in the class of all faithful representations (of the Battle of Sexes).

Our goal in this work is to generalize the foregoing construction to all finite strategic games. This task turned out to be very difficult. We just mention here two of the highest hurdles.

(1) Consider a 2×3 two-person game. Such a game has no symmetries. Therefore, there is no hope to find a 'canonical' representation just for this game. Thus, we have to add the requirement that our representations of 2×3 games are consistent with our representations of 2×2 games. Formally, the only way to get canonical representation is to consider mappings from game forms to extensive game forms, that are defined on rich enough domains.

(2) Given a 'square' n-person strategic game form, that is a game form with the property that all the players have the same number of strategies, it is not clear how to find for it a minimal and faithful representation. (Observe that a square game form allows for complete symmetry between the players.) When the number of players is greater than two, then there is no obvious solution to problem of representing square game forms. Indeed, we started with the simplest ('atomic') representations of square game forms and built 'symmetrizations' of such 'atoms' in order to obtain faithful (i.e., symmetry-preserving) representations.

We now review the contents of the paper. Game trees and their isomorphisms are presented in Section 1. Isomorphisms respect all partitions as well as the partial ordering of the tree. Section 2 introduces strategic and extensive preforms. A strategic preform specifies only the set of players and the strategy sets. Extensive preforms, similarly, specify only the set of players and the game tree. The definitions of strategic and extensive games and game forms are also reviewed in Section 2.

Isomorphisms of strategic and extensive preforms, game forms, and games are introduced in Section 3. Our definition of isomorphisms of extensive games seems to be new. An automorphism of a game G is called *motion.* A *symmetry* of G is a permutation of the players that is applied by some motion. After defining formally the symmetry group of a game we present some results on existence of symmetry groups.

The existence of canonical representations of game forms will be investigated in Part II.

1 Prerequisits

The structure of strategic games and game forms will be discussed in Section 2. As for extensive games and forms some prerequisites are necessary which we will deal with presently. Most readers familiar with the topic could just browse or entirely skip this section.

We start out with a pair $(E, \prec)$ where E is a finite set (the ***nodes***) and $\prec$ is a binary relation on E such that $(E, \prec)$ is a ***tree.*** The ***root*** of the tree is denoted by x_0, the generic element is ξ and the set of ***endpoints*** is ∂E. A ***play*** is a sequence

$$x = (x_0, x_1, ..., x_T)$$

satisfying $x_i \prec x_{i+1}$ $(i = 0, ..., T-1)$, such that x_0 is the root and $x_T \in \partial E$ holds true. A ***path*** is a sequence of consecutive nodes.

The distance of nodes is measured by the ***rank function*** defined via $r(x_0) = 0$ and $r(\xi') = r(\xi) + 1$ if $\xi \prec \xi'$. Then the nodes

$$\{\xi \mid r(\xi) = t\} \quad =: \; \mathcal{L}(E, \prec, t)$$

constitute the ***level*** t; thus $\mathcal{L}$ is the ***level function.*** In addition we shall employ the notion of ***maximal rank***

$$r_{max}(E, \prec) := max\ \{r(\xi) \mid \xi \in E\}$$

as well as the one of ***total rank***

$$R(E, \prec) = \sum_{\xi \in \partial E} r(\xi).$$

An extensive game (form) is also formulated with the aid of partitions; we introduce three partitions as follows:

P is the ***player partition*** (a partition of $E - \partial E$). The names of the players will be assigned later (Section 2); however, we assume that there is a distinguished element $P_0 \in \mathbf{P}$ (which may be the empty set) representing the ***chance moves.*** All other player sets are assumed to be nonvoid. Let $p = (p^\xi)_{\xi \in P_0}$ be a family of probability distributions (of chance moves), i.e., let p^ξ be a probability on the successors of ξ for every $\xi \in P_0$. We assume that $p^\xi(\xi')$ is positive for every successor ξ' of ξ.

Q represents the ***information partition.*** **Q** is a refinement of **P**; thus an element $Q \in \mathbf{Q}$, $Q \subseteq P$, is an information set of the player who commands the elements of P. In particular it is required that **Q** refines P_0 to singletons, that is,

$$Q \in \mathbf{Q} \ , \ Q \subseteq P_0 \quad \Longrightarrow \quad Q = \{\xi\} \ \ for \ \ some \ \ \xi \in E.$$

Also, any two elements $\xi, \xi' \in Q \in \mathbf{Q}$ belonging to the same element Q of **Q** should have the same number of successors. Moreover, every information set contains at most one node of any play.

C is a family of partitions representing ***choices***. This item refers to the above defined structures $(E, \prec), \mathbf{P}, \mathbf{Q}$. Denote by

$$C(\xi) \ := \ \{\xi' \mid \xi \prec \xi'\} \tag{1.1}$$

the successors of $\xi \in E$ and write for $Q \in \mathbf{Q}$

$$C(Q) \ := \ \bigcup_{\xi \in Q} C(\xi). \tag{1.2}$$

Now we assume that for any $Q \in \mathbf{Q}$ we are given a partition $\mathbf{C}(Q)$ of $C(Q)$ such that

$$S \ \in \ \mathbf{C}(Q) \ \Longrightarrow \ |S \cap C(\xi)| = 1 \quad (\xi \in Q).$$

(We use $(\xi \in Q)$ to indicate the quantification *'for all* $\xi \in Q$'.)

Now $\mathbf{C} \ := \ (\mathbf{C}(Q))_{Q \in \mathbf{Q}}$ denotes the system of choices.

We will refer to the structure $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ as to a ***game tree.*** In addition we shall omit **Q** or p respectively, if $\mathbf{P} = \mathbf{Q}$ or $P_0 = \emptyset$ respectively. That is, we simplify the notation accordingly, if players have but one decision or there are no chance moves.

Now we continue by defining various operations to be performed on game trees. To this end let $(E, \prec)$ and $(E', \prec')$ be trees and consider a bijective mapping $\phi \ : \ E \ \longrightarrow \ E'$.

We shall say that a bijective mapping ϕ ***respects*** $(\prec, \prec')$ if

$$\xi \prec \eta \Longrightarrow \phi(\xi) \prec' \phi(\eta) \tag{1.3}$$

holds true.

Similarly, if **P** and **P'** are partitions of subsets B of E and B' of E' respectively, then we shall say that a bijective mapping ϕ ***respects*** $(\mathbf{P}, \mathbf{P}')$ if, for all $P' \in \mathbf{P}'$ it follows that $\phi^{-1}(P') \in \mathbf{P}$ and also $\phi^{-1}(B') = B$ hold true.

Thus, if we want to consider game trees, it is clear in which way a mapping respects the player partition and the information structure. To respect the decision structure is a property defined in a straightforward way.

Definition 1.1 *A game tree* $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ *is* ***isomorphic*** *to a game tree* $(E', \prec', \mathbf{P}', \mathbf{Q}', \mathbf{C}', p')$ *if there is a bijective mapping* $\phi : E \longrightarrow E'$ *(an* ***isomorphism between*** *the game trees) which satisfies the following properties.*

(1) The mapping ϕ *respects* $(\prec, \prec')$, $(\mathbf{P}, \mathbf{P}')$, *and* $(\mathbf{Q}, \mathbf{Q}')$.

(2) $\phi(P_0) = P_0'$ *and* $p'^{\phi(\xi)}(\phi(\xi')) = p^{\xi}(\xi')$ *holds true for* $\xi \in P_0$ *and* $\xi' \in C(\xi)$.

(3) For $Q \in \mathbf{Q}$ *the mapping* ϕ *respects* $(\mathbf{C}(Q), \mathbf{C}'(Q'))$, *where* $Q' \in \mathbf{Q}'$ *is the unique information set which satisfies* $\phi(Q) = Q'$.

Note that *(2)* makes sense in view of *(1)*, the bijectivity of ϕ, and the underlying tree structure.

2 Forms and Games

There is a definite hierarchy within which we want to approach both 'versions' of a game, the strategic one and the extensive one. This hierarchy is characterized by the concepts of *preform, game form,* and *game.*

The *preform* is the pure underlying structure listing the environment of the players' actions.

Definition 2.1 *A* ***strategic preform*** *is a pair* $e = (N, S)$, *where* N *is a finite set (the set of players,* $\mid N \mid \geq 2$*) and* $S = \prod_{i \in N} S_i$ *is the product of finite sets* $S_i \neq \emptyset \quad (i \in N)$ *(the strategy sets).*

Thus, the strategic preform determines just the players and the domain of the payoff functions to be completed later.

Analogously, the extensive preform is described by an environment as follows.

Definition 2.2 *An* ***extensive preform*** *is a tuple*

$$\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota) \tag{2.1}$$

where N *is as in Definition 2.1 and the next six data have been described in Section 1. Moreover,* $\iota : \mathbf{P} - \{P_0\} \to N$ *is a bijective mapping;* ι *assings the nodes of* $P \in \mathbf{P} - \{P_0\}$ *to* $\iota(P) \in N$, *naturally we use* $\iota^{-1}(i) = P_i$ *to refer to these nodes.*

In order to discuss the concept of a game form, we assume that an abstract set of outcomes A is defined, together with a mapping h which specifies the outcome corresponding to the choice of every strategy profile by the players.

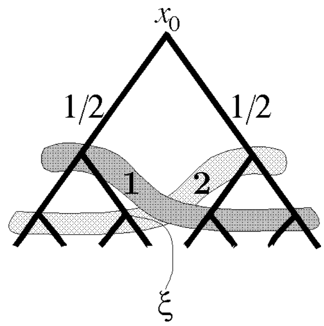

Figure 2.1: An extensive preform

Definition 2.3 *(1) A **strategic game form** is a quadruple $g = (e; A, h) = (N, S; A, h)$. Here e is a preform, A a finite set (the **outcomes**) and $h : S \to A$ is a surjective mapping called **outcome function.** g is called **general** if h is a bijection.*

*(2) An **extensive game form** is a tuple*

$$\gamma = (\epsilon; A, \eta) = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota; A, \eta). \tag{2.2}$$

Here again, ϵ is an extensive preform and A is a finite set while the surjective $\eta : \partial E \to A$ assigns outcomes to endpoints of the graph $(E, \prec)$.

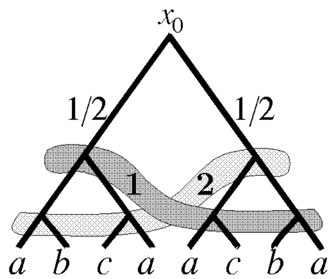

Figure 2.2: An extensive game form

Thus, we have augmented preforms with 'outcomes', 'results' of strategic activities, or, 'alternatives'. All of these do not result in 'utilities' for players but constitute a way to describe the consequences of strategic behavior.

Turning now to games, we want to convey the notion that payoffs are assigned in 'utils' - and that expected payoffs may occur as the result of strategic behavior.

Definition 2.4 *1. A **strategic game** is a tuple*

$$G = (e; u) = (N, S; u) \tag{2.3}$$

such that

$$u = (u_i)_{i \in N} : S \to \mathbb{R}^N \tag{2.4}$$

u_i is player i's utility function or payoff function.

*2. Analogously, an **extensive game** is a tuple*

$$\Gamma = (\epsilon; v) = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota; v) \tag{2.5}$$

such that

$$v = (v_i)_{i \in N} : \partial E \to I\!R^N. \tag{2.6}$$

v_i again is player i's payoff depending on endpoints of the graph $(E, \prec)$.

Remark 2.5 *Of course there is a close relation, between games and game forms. Formally, if g and γ are (strategic and extensive) game forms and*

$$U : A \to I\!R^N$$

is a ('utility') function defined on outcomes, then

$$u := U \circ h, \quad v := U \circ \eta \tag{2.7}$$

induce games G and Γ which we will conveniently denote by $U * g$ and $U * \gamma$, thus we have

$$G = U * g = U * (e; A, h) = (e; U \circ h) \tag{2.8}$$

as well as

$$\Gamma = U * \gamma = U * (\epsilon; A, \eta) = (\epsilon; U \circ \eta) \tag{2.9}$$

There are marked differences between games and game forms with respect to symmetries - even observed at this early stage.

To this end, consider Figure 2.3. Here the first sketch represents a game form with $A = \{a, b, c\}$. If we put $U : A \to I\!R^{\{1,2\}}$, $U(a) = (1, 1) \quad (a \in A)$, then the resulting game $\Gamma = U * \gamma$ is represented in the second sketch. It would appear that, whatever our definition of symmetries will be, this game is completely symmetric.

Now turn to the third sketch. This game in extensive form apparently has lost some of the symmetries. One can imagine (we haven't defined the normalized strategic form mapping as yet) that the strategic version looks exactly like the one of the second sketch - this fact being due to the possibility of forming expectations with respect to the probabilities involved. We may perceive that the game of the third sketch stems from the game form of the fourth sketch. Note however that a possible result of strategic behavior could be a pair (a, b) resulting from a 'lottery' choosing a and b with probability $(\frac{1}{2}, \frac{1}{2})$ - this procedure would greatly blur the picture of symmetries of a game.

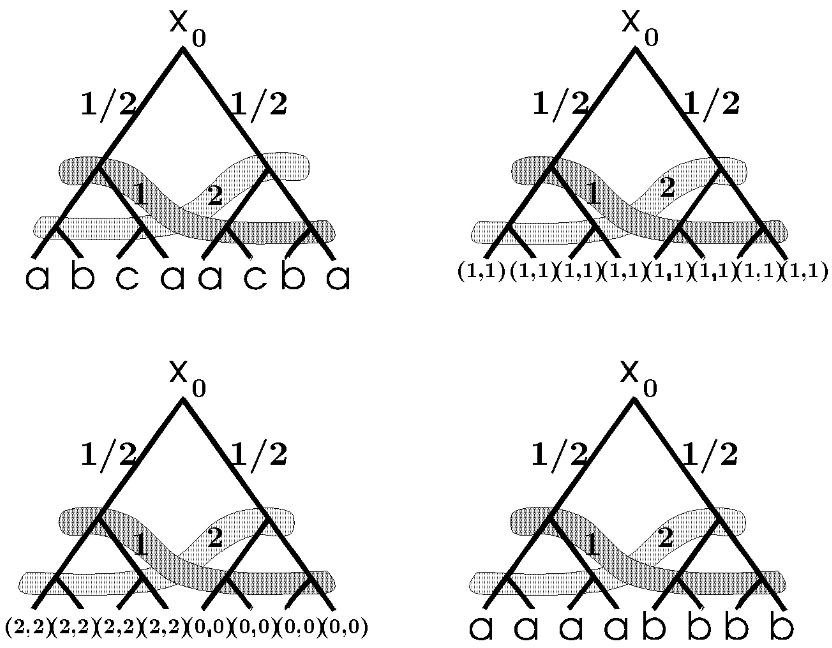

Figure 2.3: Extensive game forms and games

3 Isomorphisms, Motions and Symmetries

Isomorphisms are first of all defined for strategic preforms: we want to rename the players and simultaneously reshuffle the ordering according to which the strategies are listed.

Thus, given strategic preforms $e = (N, S)$ and $e' = (N, S')$ we consider bijective mappings $\pi : N \to N$ and $\varphi_i : S_i \to S_{\pi(i)}$ $(i \in N)$. π renames the players and φ_i maps strategies of player i into strategies of his double.

The pair (π, φ) of course induces a simultaneous reshuffling of strategies, i.e., a mapping

$$\varphi^\pi : S \to S', \; (\varphi^\pi(s))_{\pi(i)} \;:=\; \varphi_i(s_i) \quad (i \in N). \tag{3.1}$$

Definition 3.1 *An* ***isomorphism*** *between strategic preforms* e *and* e' *is a family* (π, φ) *of bijective mappings*

$$\pi : N \to N, \varphi_i : S_i \to S'_{\pi(i)} (i \in N)$$

such that

$$(\pi, \varphi)e = (\pi, \varphi)(N, S) := (\pi N, \varphi^\pi(S)) = (N, S') \tag{3.2}$$

holds true.

Now turn to extensive preforms. Here, we want to rename not only the players but also the nodes of the graph. Clearly, the decision structure as well as the information structure should be preserved. But in addition, as we have made it clear in Section 1, the 'order of play' should not be disturbed. Thus we come up with the following

Definition 3.2 *Let* $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota)$ *and* $\epsilon' = (N, E', \prec', \mathbf{P}', \mathbf{Q}', \mathbf{C}', p'; \iota')$ *be extensive preforms. A* ***isomorphism*** *between* ϵ *and* ϵ' *is a pair of mappings* (π, ϕ) *such that* π *is a permutation of* $N, \phi : E \to E'$ *with the following properties.*

(1) ϕ *is an isomorphism between the underlying game trees (cf. Definition 1.1).*

(2) $\pi(\iota(P)) = \iota'(\phi(P))$ $(P \in \mathbf{P})$

Thus, we write

$$\begin{aligned}(\pi, \phi)\epsilon &= (\pi, \phi)(N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota) \\ &:= (\pi(N), \phi E, \prec^{\phi}, \phi\mathbf{P}, \phi\mathbf{Q}, \phi\mathbf{C}, p_{\phi^{-1}}; \pi \circ \iota \circ \phi^{-1}) \\ &= (N, E', \prec', \mathbf{P}', \mathbf{Q}', \mathbf{C}', p'; \iota')\end{aligned} \tag{3.3}$$

Now, let us turn to isomorphisms of forms. Next to the structure of the preform we also want to preserve the nature of the pair (A, h) or (A, η) respectively. As we consider outcomes as basically different, we will regard a bijective mapping $\rho : A \to A$ not as particularly relevant. Thus for example we would like to consider game forms represented by

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & c \\ b & a \end{pmatrix}$$

as to be essentially equal, i.e., they are isomorphic; the appropriate mapping will involve a bijection $\rho : A \to A$, $a \to a$, $b \to c \to b$. This consideration motivates the following definition.

Definition 3.3 *Let* $g = (e; A, h)$ *and* $g' = (e'; A', h')$ *be strategic game forms. An* ***isomorphism*** *of* g *and* g' *is a triple* $(\pi, \varphi; \rho)$ *such that* (π, φ) *is an isomorphism between* e *and* e' *while* $\rho : A \to A'$ *is a bijection satisfying*

$$h' = \rho \circ h \circ (\varphi^{\pi})^{-1}.$$

Here, φ^{π} *is the mapping defined in (3.1). That is, we have*

$$(\pi, \varphi; \rho)(e; A, h) \;:=\; ((\pi, \varphi)e; \rho \circ h \circ (\varphi^{\pi})^{-1}) \;=\; (e'; A', h')$$

An isomorphism is ***outcome preserving*** *if* $A = A'$ *and* ρ *is the identity.*

Another way of looking at it is suggested by the observation that the following diagram is commutative.

$$\begin{array}{ccc} S & \xrightarrow{\varphi^\pi} & S' \\ h \downarrow & & \downarrow h' \\ A & \xrightarrow{\rho} & A' \end{array}$$

Analogously a reshuffling of outcomes should not disturb the nature of an extensive form. Since an isomorphism between extensive preforms carries endpoints into endpoints, the outcome function η' is defined on elements $\phi(\xi)$ of the image graph, the result should yield outcomes as previously up to a bijective ρ, i.e., we should have $\rho(\eta(\xi)) = \eta'(\phi(\xi))$ for $\xi \in \partial E$.

Thus we have

Definition 3.4 *An* ***isomorphism*** *of two extensive game forms* $\gamma = (\epsilon; A, \eta)$ *and* $\gamma' = (\epsilon'; A', \eta')$ *is a triple* $(\pi, \phi; \rho)$ *such that* (π, ϕ) *is an isomorphism between* ϵ *and* ϵ' *and* $\rho : A \to A'$ *is a bijective mapping such that* $\eta' \circ \phi = \rho \circ \eta$ *holds true. That is, we have*

$$(\pi, \phi, \rho)(\epsilon; A, \eta) = ((\pi, \phi)\epsilon; \rho(A), \rho \circ \eta \circ \phi^{-1}) = (\epsilon', A', \eta').$$

An isomorphism is ***outcome preserving*** *if* $A = A'$ *and* ρ *is the identity.*

The reader may benefit from the following diagram.

$$\begin{array}{ccc} \partial E & \xrightarrow{\phi} & \partial E' \\ \eta \downarrow & & \downarrow \eta' \\ A & \xrightarrow{\rho} & A' \end{array}$$

Finally let us turn to games. Of course the procedure is now familiar, all we have to do is to explain the kind of action a permutation (renaming) of the players formally performs on n-tuples of utility functions, i.e. on u or v respectively.

Clearly, if (π, φ) is an isomorphism between the strategic preforms e and e' and u and u' are tuples of utilities defined on S and S' respectively, then the utility of i's image $\pi(i) \in N$ should be given by

$$u'_{\pi(i)}(\varphi^\pi(s)) = u_i(s) \quad (i \in N), \tag{3.4}$$

thus indicating that we rename players and strategies simultaneously.

This defines the action of the pair (π, φ) on tuples of utility functions via

$$((\pi,\varphi)u)_{\pi(i)}(\varphi^{\pi}(s)) \;:=\; u_i(s). \tag{3.5}$$

We have thus explained

$$(\pi,\varphi)u : S' \to I\!R^N. \tag{3.6}$$

Analogously, within the extensive set-up, if we have an isomorphism (π,ϕ) of preforms ϵ and ϵ' , cf. Definition 3.2, and if $\upsilon : \partial E \to I\!R^N$ is a utility N-tuple defined on the endpoints of $(E,\prec)$, the action of (π,ϕ), i.e.

$$(\pi,\phi)\upsilon \;:\; \partial E' \;\to I\!R^N \tag{3.7}$$

is given by

$$((\pi,\phi)\upsilon)_{\pi(i)}(\phi(\xi)) := \upsilon_i(\xi) \;\; (\xi \in \partial E,\;\; i \in N). \tag{3.8}$$

Thus we have

Definition 3.5 *(1) Let $G = (e;u)$ and $G' = (e';u')$ be strategic games. An* ***isomorphism*** *between G and G' is a pair (π,φ) such that (π,φ) is an isomorphism between e and e' (see Definition 3.1 and (3.2)) and $u'_{\pi(i)}(\varphi^{\pi}(s)) = u_i(s)$ $(i \in N, s \in S)$. That is, we have*

$$(\pi,\varphi)G = (\pi,\varphi)(e;u) = ((\pi,\varphi)e;(\pi,\varphi)u) = (e',u') \tag{3.9}$$

(see (3.5) and (3.6)).

(2) Let $\Gamma = (\epsilon,\upsilon)$ and $\Gamma' = (\epsilon',\upsilon')$ be extensive games. An ***isomorphism*** *between Γ and Γ' is a pair (π,ϕ) such that (π,ϕ) is an isomorphism between ϵ and ϵ' (see Definition 3.2) and $\upsilon'_{\pi(i)}(\phi(\xi)) = \upsilon_i(\xi)$ $(\xi \in \partial E, i \in N)$. That is, we write*

$$(\pi,\phi)\Gamma = (\pi,\phi)(\epsilon;\upsilon) = ((\pi,\phi)\epsilon,(\pi,\phi)\upsilon) = (\epsilon',\upsilon') \tag{3.10}$$

(see (3.7) and (3.8)).

So far we have finished the presentation describing isomorphisms between preforms, game forms, and games.

We are now in the position to discuss symmetric games. At first, this notion might seem to be obvious: one is tempted to argue that a symmetry of a game G is just an automorphism of G , i.e., an isomorphism $(\pi,\varphi), \pi :\; N \to N, \varphi_i : S_i \to S_{\pi(i)}$, such that

$$(\pi,\varphi)G = G$$

holds true. However, this approach will not work appropriately. Keep in mind that we want to speak about symmetries with respect to *players,* for good reasons a rearrangement of the order of the of strategies is irrelevant. The reader is referred to the introduction for a discussion of this viewpoint.

Example 3.6 *(1) The 'Battle of Sexes' represented by the following pair of matrices*

$$G = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{3.11}$$

allows for two 'symmetries'. Consider the automorphism (π, φ) given by $\pi = id$, $\varphi_i = id$ $(i = 1, 2)$ as well as the one given by $\pi : 1 \to 2 \to 1$ and $\varphi_i(s_i) = 3 - s_i$ $(s_i \in S_i = \{1, 2\})$.

(2) If G is indicated by

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} \tag{3.12}$$

then we feel that there are absolutely no symmetries with respect to the players. Nevertheless, there are nontrivial automorphisms of G, e.g. $\pi = id$ combined with $\varphi_i : S_i \to S_i$, i.e. $\varphi_i : \{1, 2, 3\} \to \{1, 2, 3\}, \varphi_1 : 1 \to 2 \to 3 \to 1$ and $\varphi_2 = id$. I.e., if player 1 renames his strategies it does not change the game but this exhibits no 'symmetry' of the game.

Thus automorphisms cannot be 'symmetries' and 'symmetries' should disregard the pure rearrangement of strategies which does not involve interchanging players. Or more to the point, if we rename players **and** strategies, the 'symmetry' involved should only reflect the 'essential' similarities of players.

Definition 3.7 *A* ***motion*** *of a strategic game G is an automorphism (π, φ) of G. A motion (π, φ) is* ***impersonal*** *if π is the identity (and $\varphi_i : S_i \to S_i (i \in N)$).*

Remark 3.8 *It is not hard to see that motions enjoy the structure of a* ***group***. *Indeed, if (π, φ) and (σ, ψ) are motions of a strategic game G, then define $\psi \otimes \varphi$ via*

$$(\psi \otimes \varphi)_i := \psi_{\pi(i)} \circ \varphi_i \tag{3.13}$$

such that

$$(\varphi \otimes \psi)_i : S_i \to S_{\sigma(\pi(i))} = S_{\sigma \circ \pi(i)} (i \in I) \tag{3.14}$$

The product of (π, φ) and (σ, ψ) is then

$$(\sigma, \psi)(\pi, \varphi) = (\sigma\pi, \psi \otimes \varphi) \tag{3.15}$$

Clearly, the **unit motion** *is given by*

$$(id, id) = (id_N, (id_{S_i})_{i \in N})$$

This way the motions of G constitute a group. Clearly, the impersonal motions constitute a ***subgroup.***

Remark 3.9 *The subgroup of impersonal motions of the group of motions of a game is normal.*

The proof of this statement will be left to the reader. However, it now becomes conceivable that 'disregarding the impersonal part of a motion' as suggested by Example 3.6 mathematically amounts to introducing the **quotient group** of motions with respect to impersonal motions.

Definition 3.10 *Let G be a strategic game. $\mathcal{M} = \mathcal{M}(G)$ denotes the group of* ***motions****. $\mathcal{I} = \mathcal{I}(G) \subseteq \mathcal{M}$ denotes the subgroup of G constituted by the* ***impersonal*** *motions. The group of* ***symmetries*** *of G is the quotient group*

$$\mathcal{S} = \mathcal{S}(G) = \mathcal{M}/\mathcal{I} \; = \; \frac{\mathcal{M}(G)}{\mathcal{I}(G)} \tag{3.16}$$

A game G is ***symmetric*** *if its symmetry group is isomorphic to the full group of permutations of N called $\Sigma(N)$, i.e., if*

$$\mathcal{S}(G) \cong \Sigma(N) \tag{3.17}$$

holds true.

We would like to further motivate the definition of symmetries by pointing out that there is a natural relation between the concept of a symmetry we have developped so far for noncooperative games and the 'natural' concept of symmetry used in the framework of cooperative games represented by the 'coalitional' or 'characteristic' function.

To this end, let $G = (N, S; u)$ be a strategic game, and let (π, φ) be a motion of G. We consider the coalitional game (N, v) which is derived from G in the standard way (see ,e.g., McKinsey [4] , Chapter 17).

Theorem 3.11 *Let (π, φ) be a motion of a strategic game G. Then π is a symmetry of (N, v), that is, $v(\pi(T)) = v(T)$ for all $T \subseteq N$.*

Proof: Let $T \subseteq N, T \neq \emptyset, N$. For $Q \subseteq N, Q \neq \emptyset, N$, denote $S_Q = \prod_{i \in Q} S_i$. Then $\Delta(S_T)(\Delta(S_{N \setminus T}))$ is the set of all correlated strategies of $T(N \setminus T)$. By definition

$$v(T) = \max_{\sigma \in \Delta(S_T)} \min_{\tau \in \Delta(S_{N \setminus T})} \sum_{i \in T} u_i(\sigma, \tau).$$

Now let $i \in T, \sigma \in \Delta(S_T)$ and $\tau \in \Delta(S_{N\setminus T})$. Then

$$u_i(\sigma,\tau) = \sum_{s\in S_T} \sum_{t\in S_{N\setminus T}} \sigma(s)\tau(t)u_i(s,t).$$

Furthermore let us tentatively use the notation $(\varphi_i)_{i\in T} =: \varphi_T^\pi$ and $(\varphi_i)_{i\in N\setminus T} =: \varphi_{N\setminus T}^\pi$. Then $\varphi_T^\pi : S_T \to S_{\pi(T)}$ and $\varphi_{N\setminus T}^\pi : S_{N\setminus T} \to S_{\pi(N\setminus T)}$ are bijections and for each $i \in T, s \in S_T$, and $t \in S_{N\setminus T}$

$$u_i(s,t) = u_{\pi(i)}(\varphi_T^\pi(s), \varphi_{N\setminus T}^\pi(t)).$$

holds true.

Hence, for $\sigma \in \Delta(S_T)$ and $\tau \in \Delta(S_{N\setminus T})$ we obtain

$$\sum_{i\in T}\sum_{s\in S_T}\sum_{t\in S_{N\setminus T}} \sigma(s)\tau(t)u_i(s,t) =$$

$$\sum_{i\in T}\sum_{s\in S_T}\sum_{t\in S_{N\setminus T}} \sigma(s)\tau(t)u_{\pi(i)}(\varphi_T^\pi(s), \varphi_{N\setminus T}^\pi(t))$$

$$\sum_{i\in\pi(T)}\sum_{s\in S_{\pi(T)}}\sum_{t\in S_{\pi(N\setminus T)}} \sigma((\varphi_T^\pi)^{-1}(s))\tau((\varphi_{N\setminus T}^\pi)^{-1}(t))u_i(s,t).$$

The mapping $\hat{\varphi}_T^\pi : \Delta(S_T) \to \Delta(S_{\pi(T)})$ given by $\hat{\varphi}_T^\pi(\sigma)(s) = \sigma((\varphi_T^\pi)^{-1}(s))$ is an affine isomorphism between $\Delta(S_T)$ and $\Delta(S_{\pi(T)})$. Similarly we define the affine isomorphism $\hat{\varphi}_{N\setminus T}^\pi : \Delta(S_{N\setminus T}) \to \Delta(S_{\pi(N\setminus T)})$. Using these notations we obtain

$$\sum_{i\in T}\sum_{s\in S_T}\sum_{t\in S_{N\setminus T}} \sigma(s)\tau(t)u_i(s,t) = \sum_{i\in T} u_i(\sigma,\tau)$$

$$\sum_{i\in\pi(T)}\sum_{s\in S_{\pi(T)}}\sum_{t\in S_{\pi(N\setminus T)}} \hat{\varphi}_T^\pi(\sigma)(s)\hat{\varphi}_{N\setminus T}^\pi(\tau)(t)u_i(s,t) = \sum_{i\in\pi(T)} u_i(\hat{\varphi}_T^\pi(\sigma), \hat{\varphi}_{N\setminus T}^\pi(\tau)).$$

Therefore it follows that

$$v(T) = \max_{\sigma\in\Delta(S_T)} \min_{\tau\in\Delta(S_{N\setminus T})} \sum_{i\in T} u_i(\sigma,\tau) =$$

$$\max_{\sigma\in\Delta(S_T)} \min_{\tau\in\Delta(S_{N\setminus T})} \sum_{i\in\pi(T)} u_i(\hat{\varphi}_T^\pi(\sigma), \hat{\varphi}_{N\setminus T}^\pi(\tau)) =$$

$$\max_{\hat{\sigma}\in\Delta(S_{\pi(T)})} \min_{\hat{\tau}\in\Delta(S_{\pi(N\setminus T)})} \sum_{i\in\pi(T)} u_i(\hat{\sigma},\hat{\tau}) = v(\pi(T))$$

holds true in view of the fact that $\hat{\varphi}_T^\pi$ and $\hat{\varphi}_{N\setminus T}^\pi$ are bijections. **q.e.d.**

Corollary 3.12 *The symmetry group of G is contained in the symmetry group of (N,v).*

Example 3.13 *Let G be given by $(0,0)(0,0)$. Then the symmetry group of G is trivial ($|S_1| \neq |S_2|$), and (N,v) is symmetric. Thus, the Corollary is sharp.*

Remark 3.14 *One obtains similar results by considering the α or β NTU coalitional games which are derived from G.*

Example 3.15 *Next, let G be described as follows. There are 3 players, each of them having 2 strategies. Player 1 chooses strategies from $S_1 = \{small, BIG\}$. His payoff is always -1, and does not depend on the choice of strategy triples. The payoffs of players 2 and 3 in the two situations player 1 may choose are represented as follows; player 2 chooses t(op) or b(ottom) and player 3 chooses l(eft) or r(ight).*

$$\begin{array}{ccc} & & \begin{array}{cc} l & r \end{array} \quad \begin{array}{cc} l & r \end{array} \\ small & \begin{array}{c} t \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \\ & & \begin{array}{cc} l & r \end{array} \quad \begin{array}{cc} l & r \end{array} \\ BIG & \begin{array}{c} t \\ b \end{array} & \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \tag{3.18}$$

The symmetries of the Battle of Sexes are to some extend revived within this example: player 2 may play his role in the 'small' version of a modified Battle of Sexes as well as player 1 in the 'BIG' version.

More precisely, there is a (personal) motion (π, φ) given by $\pi : 1 \to 1 \ , \ 2 \to 3 \to 2 \ , \ \varphi_1 :$ $BIG \to small \to BIG$; $\varphi_2 : t \to r \ , \ b \to l \ ; \ \varphi_3 : r \to t \ , \ l \to b$, which leaves G untouched. This motion involves player 1's exchange of strategies. It can be seen that the only further motion is (id, id) (one cannot exchange player 1 with any of the others and exchanging the others, while player 1 remains fixed requires the above φ).

Thus, the quotient group is $\{id, \pi\}$ or, loosely speaking, 'players 2 and 3 are symmetric'. Thus, motions have to be incorporated in the definition of symmetries but, as the next example shows, this is not a sufficiently comprehensive definition depending on the context.

On the other hand consider the following example where a strategic game G for 2 players is indicated by

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}.$$

G admits of exactly four motions (π, φ). Indeed, if we denote the nontrivial permutation of two indices by $\tau : 1 \longrightarrow 2 \longrightarrow 1$ and the trivial permutation by id, then we can list these motions to be $(id, (id, id))$, $(id, (\tau, \tau))$, $(\tau, (id, \tau))$, and $(\tau, (\tau, id))$.

Thus, we have four motions, but G enjoys only two symmetries (the first and second pair of motions coincide on the player set). It should be pointed out that, contrary to the 'Battle of Sexes' example (cf. Example 3.6), both the nontrivial motions $(\tau, (id, \tau))$ and $(\tau, (\tau, id))$ are ***noncyclic,*** i.e. $\tau^k = id$ for some $k \in I\!N$ does *not* imply that $(\varphi^\tau)^k(s) = s$ holds true for all $s \in S$.

Within the framework of extensive structures, the definition of symmetries is now rather analogous to those given in the context of strategic structures.

Definition 3.16 *Let* $\Gamma = (\epsilon, v)$ *be an extensive game. A* ***motion*** (π, ϕ) *of* Γ *is an automorphism of* Γ *and* $\mathcal{M} = \mathcal{M}(\Gamma)$ *denotes the group of motions.* $\mathcal{I} = \mathcal{I}(\Gamma)$ *is the normal subgroup of* ***impersonal motions****, i.e., automorphisms of shape* (id, ϕ)*. The* ***group of symmetries*** *is the quotient group*

$$\mathcal{S} = \mathcal{S}(\Gamma) = \mathcal{M}/\mathcal{I} = \frac{\mathcal{M}(\Gamma)}{\mathcal{I}(\Gamma)} \tag{3.19}$$

and Γ *is symmetric if* $\mathcal{S}(\Gamma) \cong \Sigma(N)$ *holds true.*

In the remainder of this section we will shortly indicate a characterization of all possible groups of motions. Let $G = (N, S; u)$ be a strategic game and let $\mathcal{M}$ be the set of motions of G. We define an equivalence relation $\sim_{\mathcal{M}}$ on $B = N \times S$ in the following way:

$$(i, s) \sim_{\mathcal{M}} (j, t) \text{ if there exists } (\pi, \varphi) \in \mathcal{M} \text{ such that } (\pi(i), \varphi^\pi(s)) = (j, t) \tag{3.20}$$

$\sim_{\mathcal{M}}$ is an equivalence relation because $\mathcal{M}$ is a group.

Theorem 3.17 $\mathcal{M}$ *is covering, that is, if* (π, φ) *is an* $(n+1)$*-tuple of bijections,* $\pi : N \to N, \varphi_i : S_i \to S_{\pi(i)} (i \in N)$*, such that*

$$(\pi(i), \varphi^\pi(s)) \sim_{\mathcal{M}} (i, s) \text{ for all } (i, s) \in B \tag{3.21}$$

then $(\pi, \varphi) \in \mathcal{M}$*. (Here and in the sequel we use* $n = \mid N \mid$ *.)*

Proof: By 3.20 and 3.21 for each $(i, s) \in B$ there exists $(\tau, \rho) \in \mathcal{M}$ such that $(\tau(i), \rho^\tau(s)) = (\pi(i), \varphi^\pi(s))$.

Hence

$$u_{\pi(i)}(\varphi^\pi(s)) = u_{\tau(i)}(\rho^\tau(s)) = u_i(s)$$

that is, (π, φ) is a motion of G. **q.e.d.**

We now prove the converse result. Let $e = (N, S)$ be a strategic preform. A **transformation** of e is an $(n + 1)$-tuple (π, φ) of bijections, $\pi : N \to N, \varphi_i : S_i \to S_{\pi(i)} (i \in N)$, that is, it is an automorphism of e. Denote by $Aut(e)$ the group of all transformations.

For $H \subset Aut(e)$ we define $\sim_H$ by 3.20 (with $\mathcal{M}$ replaced by H). Similarly, H is covering if each $(\pi, \varphi) \in Aut(e)$ that satisfies

$$(\pi(i), \varphi^{\pi}(s)) \sim_H (i, s) \text{ for all } (i, s) \in B, \tag{3.22}$$

is a member of H.

Theorem 3.18 *If a subgroup $H \subset Aut(e)$ is covering, then there exists a vector u^H of payoff functions for e, $u^H : S \to R^N$, such that $\mathcal{M}(N, S; u^H) = H$.*

Proof: Let $I_1, ..., I_k$ be the equivalence classes of $\sim_H$. Choose real numbers $c_1 < ... < c_k$ and define $u^H : S \to R^N$ by $u_i(s) = c_l$ if $(i, s) \in I_l$.

Claim 1: If $(\pi, \varphi) \in H$ then $(\pi, \varphi) \in \mathcal{M}(N, S; u^H)$. Indeed, let $i \in N$ and $s \in S$. Then $(\pi(i), \varphi^{\pi}(s)) \sim_H (i, s)$. Hence $u_{\pi(i)}(\varphi^{\pi}(s)) = u_i(s)$ by the definition of u^H.

Claim 2: If $(\tau, \psi) \in \mathcal{M}(N, S; u^H)$ then $(\tau, \psi) \in H$. Indeed, for every $(i, s) \in B$ we have $u_{\tau(i)}(\psi^{\tau}(s)) = u_i(s)$. By the choice of u^H, $(\tau(i), \psi^{\tau}(s)) \sim_H (i, s)$. Because H is covering, $(\tau, \psi) \in H$. **q.e.d.**

Given a strategic preform $e = (N, S)$ some subgroups of $Aut(e)$ may not be the group of motions of $(N, S; u^H)$ for any $u^H : S \to R^N$.

Example 3.19 *Let $N = \{1, ..., n\}$, and $S_1 = ... = S_n = \{1, ..., l\}$ where $2 \le l \le n - 2$. Denote by A_n the group of even permutations of N. Then*

$$H = \{(\pi, (id, ..., id)) \mid \pi \in A_n\}$$

is a subgroup of Aut(e) which is not covering. Indeed, if π is an odd permutation, then $(\pi, (id, ..., id))$ is not in H. However, for each $(i, s) \in B$ there exists $\tau \in A_n$ such that $(\pi(i), id^{\pi}(s)) = (\tau(i), id^{\tau}(s))$, that is, $(\pi(i), id^{\pi}(s)) \sim_H (i, s)$.

The following result is a corollary of Theorems 3.17 and 3.18.

Corollary 3.20 *Let $e = (N, S)$ be a strateigc preform and let $H \subset \sum(N)$ be a group of permutations. Then there exists a vector of payoff functions $u^H : S \to R^N$ such that $H = \mathcal{S}(N, S; u^H)$ if and only if there is a covering group H^* of automorphisms of e such that $H = H^*/\mathcal{I}(H^*)$, where $\mathcal{I}(H^*)$ is the (normal) subgroup of impersonal automorphisms in H^*.*

Proof: Necessity. If $H = \mathcal{S}(N, S; u^H)$ then $H = \mathcal{M}(N, S; u^H)/\mathcal{I}(N, S; u^H)$ and $\mathcal{M}(N, S; u^H)$ is covering by Theorem 3.17.

Sufficiency. If $H = H^*/\mathcal{I}(H^*)$ and H^* is covering then, by Theorem 3.18, there exists a vector of payoff functions $u^H : S \to R^N$ such that $H^* = \mathcal{M}(N, S; u^H)$. Because $H = H^*/\mathcal{I}(H^*)$, $H = \mathcal{S}(N, S; u^H)$ by definition. **q.e.d.**

Example 3.21 *Let, again,* $N = \{1, ..., n\}$ *and* $S_1 = \cdots = S_n = \{1, ..., l\}$ *with* $l \geq n-1$. *We claim that for every subgroup* H *of* $\sum(N)$ *there exists a vector* $u^H : S \to R^N$ *such that* $H = \mathcal{S}(N, S; u^H)$. *Indeed, let* $H \subset \sum(N)$ *be a group. Consider the following group* H^* *of automorphisms of e*

$$H^* = \{(\pi, (id, ..., id)) \mid \pi \in H\}$$

We shall prove that H^* *is covering. Indeed, let* $(\pi, (\varphi_1, ..., \varphi_n))$ *satisfy* $(\pi, \varphi^\pi(s)) \sim_{H^*} (i, s)$ *for all* $(i, s) \in B$. *Consider a pair* (i, s_m) *where* $i \in N$ *and* $s_m = (m, ..., m)$. *By assumption*

$$(\pi(i), \varphi^\pi(s_m)) = (\pi(i), (\varphi_{\pi^{-1}(1)}(m), ..., \varphi_{\pi^{-1}(n)}(m)) \sim_{H^*} (i, s_m)$$

for all $i \in N$. *Therefore* $\varphi_{\pi^{-1}(i)}(m) = m$ *for all* $i \in N$ *and* $1 \leq m \leq l$. *Thus* $\varphi_1 = \cdots = \varphi_n = id$. *Now choose* $(i, s) = (n, (1, 2, ..., n-1, n-1))$. *By our assumption there is* $\hat{\pi} \in H$ *such that*

$$(\pi(n), id^\pi(s)) = (\hat{\pi}(n), id^{\hat{\pi}}(s))$$

Therefore, $\pi = \hat{\pi}$. *Thus* H^* *is the group of motions of some game* $(N, S; u^H)$ *whose group of symmetries is clearly* H.

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