The minimal quota for a complete and transitive majority relation [1]

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Abstract

In this paper, we show that $\frac{2}{3}$ is the minimal quota that guarantees the transitivity of a complete majority relation. We argue that this quota is important for the process of negotiation that may take place when a group has to take a clear-cut decision under a specific quota-rule.

Key words: quota-rules, completeness, communication

1 Introduction

The pairwise majority rule may lead to intransitivities (cycles) which makes it impossible to find a straightforward solution to pick up a winner from the set of alternatives. A way out of this problem is to require the support of more than a half of the population to consider that an alternative is better than another. We enter the domain of *quota-rules*. It is well known (Ferejohn & Grether 1974, Peleg 1978) that when n alternatives are in an electoral competition, it is sufficient and necessary to require the support of more than $(n-1)/n$ of the population to be sure that no cycle may appear in the *qualified* majority relation. Unfortunately, this fraction, that we shall call *quota* and denote λ in the remainder, tends to 1 as the number of alternatives increases, i.e. to be considered socially better than an alternative y, the

[1] This paper was prepared under the financial support of the European Community through TMR network "*Cooperation and Information*" contract N° FMRX CT96 0055, while the author was visiting the IMW. Professors Dominique Lepelley, Vincent Merlin and Fabrice Valognes are acknowledged for their comments and advice. *JEL Classification*: C7, D7

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alternative x must be preferred to y by "*almost*" all the individuals. The absence of intransitivities does not help the decision since it leaves the place to a likely empty binary relation. Indeed, when the quota is not obtained then the two involved alternatives should be considered "*incomparable*". [3] This conclusion is a pain when a clear-cut decision has to be taken. Let us give two examples.

For historical reasons, a pope is elected if he casts more than two-thirds of the votes and it is necessary that a pope is elected. This means that the cardinals have to deal and negotiate in order to attain the required quota of two-thirds. [4]

Another example may be found in Sidney Lumet's movie "*Twelve Angry Men*" (1957) of which action takes place on the stage of the jury room. The jurors have to decide whether a young Spanish-American is guilty or innocent of murdering his father. For a decision to be taken, the thirteen jurors have to agree (possibly at the expense of negotiation). After a first vote, the tally is 12:1 in favor of the condemnation to death. After some (hours of) negotiation, the minority position —acquittal— turns into a unanimity and the young man released.

In both examples, the negotiation goes on as long as the decision is not taken. [5] This article finds its motivation in this particular framework where the members have to negotiate until the group is able to perform a clear-cut decision on the basis of a λ-majority relation. Intuitively, it is clear that if the number of candidates and/or the quota increase then a decision might be more difficult to achieve. In other words, it is most likely that negotiation will be necessary to perform a decision.

As far as the quota is concerned, we face the following dilemma. When the quota is low (one-half) the pairwise majority relation is complete but not necessarily transitive. When the quota is high (more than $(n-1)/n$ where n is the number of candidates in competition), the pairwise *qualified-majority* relation is acyclic but not necessarily complete. Of course, when λ is intermediate, the relation may be incomplete and contain some cycles.

We show in this article that when the quota λ is greater than $\frac{2}{3}$, then a complete λ-majority relation is necessarily transitive. Remarkably, this quota is independent of the number alternatives in competition. This sharply contrasts with the result of

[3] Although each member of the group performs this comparison.

[4] This example was reported in Saari (1996).

[5] Of course some device may be set up to encourage a quick decision. This is actually done in a stringent way for the pope election (Fanning 1911)

> *"When the cardinals found themselves face to face with* [the situation where the $\frac{2}{3}$ has not been obtained] *on the death of Clement IV in 1268, they commissioned six cardinals as plenipotentiaries to decide on a candidate. The vacancy of the Holy See had lasted for two years and nine months. To prevent a recurrence of this evil, the Second Council of Lyons under Gregory X (1274) decreed that ten days after the pope's decease, the cardinals should assemble in the palace in the city in which the pope died, and there hold their electoral meetings, entirely shut out from all outside influences. If they did not come to an agreement on a candidate in three days, their victuals were to lessened, and after a further delay of five days, the food supply was to be still further restricted. This is the origin of conclaves."*

Ferejohn and Grether (1974).

We believe that this result may offer an interesting framework to negotiation processes.

After having introduced the necessary definitions and notation (section 2), we give our main result (section 3) and conclude by discussing of open problems related to negotiation (section 4).

2 Definitions and notation

We shall always consider X as a finite set of n alternatives and V a finite set of v voters.

A *binary relation R over X* is a collection of couples (x, y) such that both x and y belong to X. When the couple (x, y) belongs to the binary relation R, then we shall write xRy. A *path* is a sequence of alternatives $x_0, \ldots, x_k \in X$ such that $x_j R x_{j+1}$ for every $j \in \{0, \ldots, k-1\}$. The *length* of such a path is k. If there exists a path encompassing the whole set X and of length $n-1$, then the relation is *hamiltonian*. A *cycle* is a sequence of alternatives $x_0, \ldots, x_k \in X$ describing a path and such that $x_k R x_0$. A binary relation R is *quasi-complete* if for every distinct $x, y \in X, (\neg xRy) \Longrightarrow yRx$. It is *complete* if it is quasi-complete and reflexive. It is *anti-symmetric* if for every $x, y \in X$, we have xRy and yRx if and only if $x = y$. It is *transitive* if for every $x, y, z \in X$, xRy and yRz implies xRz. It is *acyclic* if it contains no cycle, i.e. if for every distinct $x_0, x_1, \ldots, x_k$ such that $x_j R x_{j+1}$ for every $j \in \{0, \ldots, k-1\}$, we have $\neg x_k R x_0$. It is *connected* if for every $x, y \in X$, there exists a sequence $x = x_0, x_1, \ldots, x_k = y \in X$ such that $x_j R x_{j+1}$ or $x_{j+1} R x_j$ for every $j \in \{0, \ldots, k-1\}$.

We denote $Bin(X)$ the set of binary relations defined over X, $Acy(X)$ the set of acyclic binary relations, $Ord(X)$ the set of complete, anti-symmetric and transitive binary relations (linear ordering), $Tour(X)$ is the set of complete and anti-symmetric binary relations (tournaments).

An alternative $x \in X$ is a *maximal element* of the relation R if there is no alternative $y \in X$ $(y \neq x)$ such that yRx.

We assume that each individual $i \in V$ is endowed with a *preference* $P_i \in Ord(X)$ and we define *a profile* $\pi = (P_1, \ldots, P_v)$ as the list of all the individual preferences.

For any $\lambda \in [\frac{1}{2}, 1[$, given a profile $\pi \in Ord(X)^V$, we define the λ-majority relation as follows: $\forall x, y \in X : xM_\lambda(\pi)y \iff \#\{i \in V : xP_iy\} > \lambda.v$ where $\#Y$ is the cardinality of the set Y.

We define the *range of the λ-majority* as the set of binary relations than can be obtained through the λ-majority rule. Formally, $Ran(\lambda, X) = \{R \in Bin(X) : \exists V$ and $\pi \in Ord(X)^V$ such that for every $x, y \in X$, $xRy \iff xM_\lambda(\pi)y\}$. Notice that we don't restrict the number of voters.

3 A transitive $\frac{2}{3}$-majority relation

McGarvey (1953) has shown that for any finite set X, every tournament could be obtained from a pairwise (simple) majority voting, i.e. $Ran(\frac{1}{2}, X) = Tour(X)$ and allowed the study of tournaments from a voting theoretical point of view.[6] Mala (1998) proved that there exists some tournaments that can not be obtained through pairwise λ-majority relation, as soon as λ is strictly greater than a $\frac{1}{2}$, i.e. $Ran(\lambda, X) \subsetneqq Tour(X)$. This latter result excludes a systematic study of tournaments under the arguments of quota-rules.

Ferejohn and Grether (1974) proved that for any finite set X of $n \leq m$ elements, then for every $\pi \in Acy(X)^V$ and every $\lambda \in [\frac{m-1}{m}, 1[$, we have $M_\lambda(\pi) \in Acy(X)$. This means that the λ-majority relation contains no cycle as soon as the quota is greater than or equal to $\frac{n-1}{n}$. This result is the lower bound of λ that guarantees the existence of a maximal element in the λ-majority relation, but not its uniqueness. The next theorem states that if the quota λ is greater than or equal to $\frac{2}{3}$, for any number of alternatives, if the λ-majority relation is complete, then it is transitive, that is to say, it contains a unique maximal element.

Theorem 1 *For any finite set X of $n \geq 3$ alternatives.*

i. $\forall \lambda \in [\frac{1}{2}, \frac{2}{3}[, Ran(\lambda, X) \cap Comp(X) \supsetneqq Ord(X)$

ii. $\forall \lambda \in [\frac{2}{3}, 1[, Ran(\lambda, X) \cap Comp(X) = Ord(X)$

Proof. First, we need to show that for any finite set X and any quota $\lambda \in [\frac{1}{2}, 1[$, $Ord(X) \subseteq Ran(\lambda, X)$. In that purpose, consider any binary relation $P \in Ord(X)$ and the unanimous profile $\pi = (P, \ldots, P) \in Ord(X)^V$. For any $\lambda \in [\frac{1}{2}, 1[$ and every $x, y \in X : \#\{i \in V : xP_iy\} = v > \lambda v$. The λ-majority relation is identical to P so that $P \in Ran(\lambda, X) \supseteq Ord(X)$. Then, because $Ord(X) \subset Comp(X)$, it is clear that $\forall \lambda \in [\frac{1}{2}, 1[, Ord(X) \subseteq Ran(\lambda, X) \cap Comp(X)$.

To prove *i.*, we show that for any set X of $n \geq 3$ alternatives and any $\lambda \in [\frac{1}{2}, \frac{2}{3}[$ there exists a profile π such that the $\lambda-$majority relation is complete but not transitive.

[6] See Laslier (1997) for an extensive exposition on the topic.

Let $X = \{x_1, \ldots, x_n\}$ be the set of alternatives and consider a set $V = \{1, 2, 3\}$ of three individuals. Let $\pi = (P_1, P_2, P_3)$ be the following profile :

$Individual$	$Preference$
1	$x_1 P_1 x_2 P_1 x_3 P_1 x_4 P_1 \ldots P_1 x_n$
2	$x_2 P_2 x_3 P_2 x_1 P_2 x_4 P_2 \ldots P_2 x_n$
3	$x_3 P_3 x_1 P_3 x_2 P_3 x_4 P_3 \ldots P_3 x_n$

We observe that $\#\{i \in V : x_1 P_i x_2\} = \#\{i \in V : x_2 P_i x_3\} = \#\{i \in V : x_3 P_i x_1\} = 2$ and for every other pair, we have $\#\{i \in V : x P_i y\} = 3$ or $\#\{i \in V : y P_i x\} = 3$. This implies that for any $\lambda \in [\frac{1}{2}, \frac{2}{3}[$, we have $x_1 M_\lambda(\pi) x_2$, $x_2 M_\lambda(\pi) x_3$, $x_3 M_\lambda(\pi) x_1$ and for every other pair $x M_\lambda(\pi) y$ or $y M_\lambda(\pi) x$. The relation $M_\lambda(\pi) \in Comp(X) \backslash Ord(X)$. This proves *i.*

To prove *ii.*, we consider any non-transitive and complete λ-majority relation T and suppose that $\lambda > \frac{2}{3}$. Hence, there must exist a set of voters $V = \{1, \ldots, v\}$ and a profile $\pi = (P_1, \ldots, P_v) \in Ord(X)^V$ that lead to $M_\lambda(\pi) = T$ for a $\lambda > \frac{2}{3}$.

Harary and Moser (1966) have shown that such a relation T must contain a $3-$cycle, i.e. a cycle involving three alternatives. Without loss of generality, we suppose that this 3-cycle can be written $x_1 T x_2 T x_3 T x_1$.

Let us denote $C_1 = \{i \in V : x_1 P_i x_2\}$, $C_2 = \{i \in V : x_2 P_i x_3\}$ and $C_3 = \{i \in V : x_3 P_i x_1\}$.

By assumption, $\#C_1 > \frac{2}{3}v$, $\#C_2 > \frac{2}{3}v$ and $\#C_3 > \frac{2}{3}v$. By basic set theoretical properties, we find that $\#C_1 + \#C_2 - \#(C_1 \cap C_2) = \#(C_1 \cup C_2) \leq v$, which implies $\#(C_1 \cap C_2) > \frac{1}{3}v$. By the same arguments, we have $\#(C_1 \cap C_3) > \frac{1}{3}v$ and $\#(C_2 \cap C_3) > \frac{1}{3}v$.

Finally, $v \geq \#((C_1 \cap C_2) \cup (C_2 \cap C_3) \cup (C_1 \cap C_3)) = \#(C_1 \cap C_2) + \#(C_1 \cap C_3) + \#(C_2 \cap C_3) - \#(C_1 \cap C_2 \cap C_3)$. It must be that $\#(C_1 \cap C_2 \cap C_3) > 0$.

This latter inequality implies that there exists at least one individual that belongs simultaneously to C_1, C_2 et C_3, which is impossible since this individual would exhibit non-transitive preferences. The profile V supposed to have induced the relation T through the λ-majority rule does not exists. We conclude that any intransitivity in the λ-majority relation is impossible when $\lambda \in [\frac{2}{3}, 1[$ as soon as one assumes it is complete. $\square$

This result is of course in no contradiction with Ferejohn and Grether (1974).

4 The problems related to negotiation

Let us now consider the problem of a clear-cut decision to be taken by a committee. We consider a set V of v voters and a set X of n alternatives. Each voter is endowed with a complete linear ordering over X. The decision rule consists in choosing the maximal (non dominated) element of a λ-majority relation. If such an element does not exist or is not unique, then the voters have to negotiate in order to obtain a unique maximal element. Of course, we consider that the quota λ is given *a priori* and should not be changed during the negotiation.

The most important arising question is to know whether there exists a quota that guarantees the absence of negotiation. The answer is clearly negative except in the trivial and particular case where only two alternatives are in competition and an odd number of voters have to decide under majority rule. When more than three alternatives are in competition, it can be the case that no Condorcet's winner exists under the majority rule. A Condorcet's winner is an alternative that defeats any other alternative in pairwise λ-majority [7]. For any number of alternatives, the λ-majority relation may not be complete which implies that several maximal elements may coexist.

Negotiation is intrinsically a dynamic process and should then be treated as such. We believe that game theory may constitute a fruitful approach to this problem. Many authors, see for instance Ellison (1993), Kandori et al. (1993), Young (1993), Blume et al. (1993) or Blume (1998), have studied global or local interactions between the members of a group. The general context is that of a group of agents having to play repeatedly a 2×2 symmetric game against a random opponent. At each period, the strategy of a player is chosen according to its current relative success. In order to depart from the deterministic evolution, a random "*noise*" is introduced so that, on rare occasions, a player may not follow the deterministic rule. Under various assumptions, they study the way the system converges to a state where the game is played at some equilibrium.

The question is to know whether is is possible to define a similar setting that may converge towards a situation in which a clear-cut λ-majority decision can be taken. One may expect that different settings lead to different profiles of strategies and hence describe different negotiation processes. An important question, raised in Ellison (1993), deals with the rate at which the system converges. Does the system converge more rapidly towards a clear-cut decision when the quota is $\frac{2}{3}$ or $\frac{n-1}{n}$? In other word, is it true that a complete binary relation obtained through a quota of two-thirds is easier to obtain through a negotiation process than an acyclic hamiltonian binary relation obtained through a quota of $\frac{n-1}{n}$?

[7] The original definition of a Condorcet's winner (Condorcet (1785)) was given for $\lambda = \frac{1}{2}$, but we think that it would have been splitting-hairs and misleading to define a λ-Condorcet's winner.

We believe that the problem of rate of convergence is related to the "*distance*" between the preference profile (before any negotiation) and the profile of strategies adopted (after the negotiation process is over) when the clear cut-decision can be taken. Let $\pi = (P_1, \ldots, P_v)$ be a profile of preferences and $D_\lambda \subset Ord(X)^N$ be the set of profiles such that a clear-cut decision can be taken through the λ-majority relation. For any $R, R' \in Ord(X)$, let $\delta(R, R') = \#\{(a, b) \in X \times X : aRb \iff bR'a\}$ be the Kemeny-Young (Kemeny 1959, Young 1988) distance between R and R'. We define the distance [8] $\Delta(\pi, S) = \sum_{i \in V} \delta(P_i, S_i)$ between the profile of preferences π and the profile of strategies adopted after the negotiation process $S \in D_\lambda$.

Several questions arise : does there exist some negotiation process converging rapidly toward a profile of strategies that is at a minimal distance form the preference profile? Does the choice obtained from a negotiation process depend on the quota? Is the criterion of minimal distance a good criterion from an axiomatic point of view? In the light of these considerations, is there a best quota?

References

Blume, A. (1998), 'Communication, risk, and efficiency in games', *Games and Economic Behavior* **22**(2), 171–202.

Blume, A., Kim, Y.-G. & Sobel, J. (1993), 'Evolutionary stability in games of communication', *Games and Economic Behavior* **5**(4), 547–575.

Condorcet, (Marquis de), M.-J.-A. (1785), *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix.*, Imprimerie Royale, Paris.

Ellison, G. (1993), 'Learning, local interaction, and coordination', *Econometrica* **61**(5), 1047–1071.

Fanning, W. H. (1911), Papal election, *in* C. G. Herbermann, E. A. Pace, C. B. Pallen, T. J. Shahan & J. J. Wynne, eds, 'The Catholic Encyclopedia', Vol. XI, Robert Appleton Company, New York, pp. 456–457.

Ferejohn, J. A. & Grether, D. M. (1974), 'On a class of rational social decision procedures', *Journal of Economic Theory* **8**, 471–482.

Harary, F. & Moser, L. (1966), 'The theory of round robin tournaments', *The American Mathematical Monthly* **73**, 231–246.

Kandori, M., Mailath, G. J. & Rob, R. (1993), 'Learning, mutation and long run equilibria in games', *Econometrica* **61**(1), 29–56.

Kemeny, J. (1959), 'Mathematics without Numbers', *Dædalus* **88**, 571–591.

[8] This distance is computed between two n-tuples. It is equal to the classical Kemeny distance in the only case where the profile S is unanimous.

Laslier, J.-F. (1997), *Tournament Solutions and Majority Voting*, Vol. 7 of *Studies in Economic Theory*, Springer, Berlin, Heidelberg, New York.

Mala, J. (1998), 'On $\lambda-$majority voting paradoxes', *Mathematical Social Sciences* **37**(1), 45–57.

McGarvey, D. (1953), 'A theorem on the construction of voting paradoxes', *Econometrica* **21**, 608–610.

Peleg, B. (1978), 'Consistent voting systems', *Econometrica* **46**, 153–161.

Saari, D. G. (1996), *Basics in Geometry of Voting*, Springer-Verlag.

Young, H. P. (1988), 'Condorcet's theory of voting', *American Political Science Review* **82**, 1231–1244.

Young, H. P. (1993), 'The evolution of conventions', *Econometrica* **61**(1), 57–84.