A Characterization of vNM-Stable Sets for Linear Production Games

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Abstract

We discuss linear production games or market games with a continuum of players which are represented as minima of finitely many nonatomic measures.

Within this contex we consider vNM-Stable Sets according to von Neumann and Morgenstern. We classify or characterize all solutions of this type which are convex polyhedra, i.e., which are the convex hull of finitely many imputations. Specifically, in each convex polyhedral vNM-Stable Set (and not only in the symmetric ones), the different types of traders must organize themselves into cartels. The vNM-Stable Set is then the convex hull of the utility distributions of the cartels.

Using the results from the continuum, we obtain a similar characterization also for finite glove market games .

1 Introduction

Within this paper we want to characterize the von Neumann and Morgenstern (vNM) stable sets as applied to linear production games. Such games represent production in fixed proportions, where the long side of the supply of the production factors is strictly greater than the short side. Within this context, most economic and game theoretical solution concepts yield zero profit to the long side, because of excess of supply.

By contrast, revitalizing the concept of both internal and external stability of vNM solutions, we obtain that each side will form a cartel (see Hart[HART74]) and the stable set will be the convex hull of the utility distributions of both cartels.

A vNM-Stable Set is (in the words of von Neumann and Morgenstern [vNM44]) seen as *a standard of behavior.* In particular each such standard of economic behavior will treat both sides equally, and in particular the long side of the market will end up with some positive amount of utility.

The games we consider are both nonatomic and finite totally balanced ones (cf. Shapley-Shubik[SHSH69]) represented as minima of finitely many measures. In the finite context this class is equivalent to either the class of market games or the class of linear production games. Formally, these games are represented by means of finitely many measures $\lambda^1, \ldots, \lambda^\rho$ via

$$v = \bigwedge \left\{ \lambda^1, \cdots, \lambda^r \right\} . \tag{1}$$

which means

$$v(S) \ := \ \min \left\{ \lambda^\rho(S) \mid \rho = 1, \ldots, r \right\} , \tag{2}$$

for every coalition S. A subclass of this class (the exact games) where the $\lambda^1, \ldots, \lambda^r$ are nonatomic probabilities has been extensively studied by Einy et al. ([EHMS96]), hence within this paper we will concentrate on the general (nonnormalized, nonexact) case describing economic situations where the short side of the market is different from the long side. We restrict ourselves to the *orthogonal* case (all measures involved have mutually disjoint carriers).

In [EHMS96] it was shown that the core is the unique vNM-Stable Set provided the game is exact. In our case, where the game is in general not exact, this cannot occur, by contrast it turns out that we have to provide a complete description of all vNM-Stable Sets.

Purely finite glove markets as well as our non-atomic games with orthogonal measures describe a pure exchange economy with different types of traders, each type commanding a "corner" of the market consisting of a certain variety of gloves. The core (as well as the Walrasian equilibria and other solution concepts) for such situations as described by Billera and Raanan([BILRA81]) assign zero utility to the long side of the market.

We emphasize that our results do much better:

Markets organized according to convex polyhedral vNM-Stable Sets treat all types symmetrically, that is, first of all each type decides how to distribute the amount of total production among its members (with some bound over the density of the distribution), and secondly, a standard of behavior is introduced by implementing the convex hull of r such distributions.

The main result of this paper is hence the characterization of all convex (polyhedral) vNM-Stable Sets of nonexact totally balanced games or, equivalently, the complete description of a solution concept not discriminating the long side of a large exchange economy unduly.

The paper is organized as follows. In Section 2 the model is introduced. Section 3 is devoted to the description of standard solutions. In Section 4 we prove the Main Characterization Theorem (of convex polyhedral vNM-Stable Sets). In order to achieve this result, some preparatory lemmata and theorems are obtained: we prove The Density Lemma and the Inheritance Lemma as well as the Support Theorem and the Orthogonality Theorem. Section 5 presents the Embedding Theorem which bridges the gap between finite glove games and our nonatomic linear production games. Finally, in Section 6 we offer the complete characterization for the finite case.

We would like to express our thanks to Sergiu Hart. He discussed the topic frequently with us and his comments and suggestions were most influential in the process of preparing this paper.

2 The Model

We consider ***games*** in coalitional function form written $(I, \underline{\underline{\mathbf{F}}}, v)$, here I is some interval in real space (the ***players***), $\underline{\underline{\mathbf{F}}}$ the $\sigma-$field of (Borel) measurable sets and v a real valued function on $\underline{\underline{\mathbf{F}}}$ which is absolutely continuous w.r.t Lebesgue measure λ.

We frequently refer to v as to *the game* without mentioning the environment. We are particularly interested in ***totally balanced games*** or ***market games*** generated by finitely many nonatomic measures, say $\lambda^1, \ldots, \lambda^r$ via

$$v(S) \; := \; \min\left\{\lambda^\rho(S) \mid \rho = 1, \ldots, r\right\}, \tag{1}$$

which is written

$$v = \bigwedge \left\{\lambda^1, \cdots, \lambda^r\right\}. \tag{2}$$

It constitutes no loss of generality to assume that the λ^ρ are restrictions of Lebesgue measure to certain intervals C^ρ, $(\rho = 1, \ldots, r)$. Most of the time we will assume that the λ^ρ are mutually orthogonal, hence the sets C^ρ are mutually disjoint.

The concept of a ***vNM-Stable Set*** dates back to von Neumann-Morgenstern([vNM44]), intuitively this is a set $\mathcal{S}$ of imputations such that no internal domination occurs while any feasible payoff measure outside of $\mathcal{S}$ can be dominated from inside. The concept of ***domination*** in the continuous context is described as follows:

Definition 2.1. *Let $(I, \underline{\underline{\mathbf{F}}}, v)$ be a game. An* ***imputation*** *is a measure ξ such that $\xi(I) = v(I)$ holds true. An imputation ξ* ***dominates*** *an imputation η w.r.t a coalition $S \in \underline{\underline{\mathbf{F}}}$ if ξ is* ***effective*** *for S, i.e.,*

$$\lambda(S) > 0 \quad and \quad \xi(S) \leq v(S) \tag{3}$$

and if

$$\xi(T) > \eta(T) \quad (T \in \underline{\underline{\mathbf{F}}},\; T \subseteq S, \lambda(S) > 0) \tag{4}$$

holds true, that is, every subcoalition of S (almost every player in S) strictly improves its payoff at ξ versus η. We write $\xi \;\mathrm{dom}_S\; \eta$ to indicate domination.

We allow domination also to take place between 'subimputations', i.e., measures with total mass less than $v(I)$.

Definition 2.2. *Let v be a game. A set $\mathcal{S}$ of imputations is called a **vNM-Stable Set** if*

- *there is no pair $\xi, \mu \in \mathcal{S}$ such that ξ dom_S μ takes place w.r.t. some coalition $S \in \underline{\underline{\mathbf{F}}}$,*
- *for every imputation $\eta \notin \mathcal{S}$ there exists $\xi \in \mathcal{S}$ such that , for some $S \in \underline{\underline{\mathbf{F}}}$ the relation ξ dom_S η is satisfied.*

Since domination requires in particular effectiveness, we will restrict ourselves to vNM-Stable Sets containing measures only which are absolutely continuous w.r.t. to a 'reference measure'. This reference measure is supplied in a natural way by the form of the coalitional function, which is given by

$$v = \bigwedge \left\{ \lambda^1, \cdots, \lambda^r \right\}. \tag{5}$$

In this context, we may use e.g.

$$\lambda^0 := \sum_{\rho=1}^{r} \lambda^\rho \tag{6}$$

to serve as a reference measure. Actually, it is no severe restriction to assume that all the λ^ρ are restrictions of Lebesgue measure and hence the reference measure is Lebesgue measure.

The assumption of an underlying reference measure and existing densities for the members of a vNM-Stable Set may be justified by various considerations.

First of all, if for two measures η and ϑ domination takes place, say, $\vartheta \operatorname{dom}_S \eta$, then we may as well assume that $\lambda^1(S) = \ldots = \lambda^r(S) = v(S) \geq \vartheta(S)$ holds true. Now, as $T \subseteq S$ domination requires $\vartheta(T) > \eta(T)$. *Intuitively* one is tempted to expect that $\vartheta(T) \leq v(T)$. If so, it makes sense to postulate absolute continuity of ϑ with a density bounded by 1. And this would be true for all elements of a vNM-Stable Set.

In the exact case, i.e., if

$$\lambda^1(I) = \ldots = \lambda^r(I)$$

it follows from a result of Billera-Raanan ([BILRA81]), that the core equals the convex hull of the λ^ρ. As has been shown in Einy et. al.

[EHMS96], the core in this case is the unique vNM-Stable Set, hence every element of the solution is *ex ante* absolutely continuous w.r.t the reference measure provided by (6).

In our present context it will eventually turn out to be a *result* of this paper that a vNM-Stable Set for the game given by (2) contains measures μ of bounded densities only, more precisely, we find

$$\frac{d\mu}{d\lambda^\rho} \leq 1 \tag{7}$$

only.

Let us mention some notational conventions. We use $\bigwedge$ in order to denote the min operation in a lattice (e.g. the measures on I); equation (2) provides the standard usage. The ***carrier*** of a measure μ is denoted by $C(\mu)$, in the context of a game given by (2) we use the abbreviation $C^\rho := C(\lambda^\rho)$.

The Radon Nikodym derivative of some measure μ with respect to Lebesgue measure (reference measure) is denoted by $\overset{\bullet}{\mu}$ and we shall frequently neglect the fact that it is defined only almost everywhere as this does not influence our arguments. Thus the reader may miss the repeated iteration of the abbreviation 'a.e.'.

For sets S, T (say $\in \underline{\underline{\mathbf{F}}}$), we use the notation $S+T$ instead of $S \cup T$ *if and only if* S and T are *disjoint.* This way additivity of a measure μ conveniently writes e.g. $\sum_{\rho=1}^{r} \mu(S^\rho) = \mu(\sum_{\rho=1}^{r} S^\rho)$.

3 The Class of Standard Stable Sets

In what follows we consider a totally balanced game $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ which is not necessarily exact, hence the total mass of the measures λ^ρ may differ. It constitutes no loss of generality to assume that λ^1 attaines the minimal total mass and that λ^1 (and hence v) is normalized, i.e., we assume

$$(1) \qquad 1 = v(I) = \lambda^1(I) \leq \lambda^\rho(I) \quad (\rho = 1, \ldots, r).$$

Our first observation is true for a general situation of this type, however, later on we shall assume that the measures λ^ρ are in addition orthogonal. Presently we introduce a certain type of vNM-stable set which, as it turns out fully describes the relevant class of **convex** vNM-Stable Sets.

As a prerequisit we first turn to external stability of the solution concept we have in mind.

Theorem 3.1. *Let $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ be a (normalized) totally balanced game and let $\mu^1, \ldots \mu^r$ be probabilities such μ^ρ is absolutely continuous w.r.t. λ^ρ, and satisfies $\frac{d\mu^\rho}{d\lambda^\rho} \leq 1 \quad (\rho = 1, \ldots, r)$. Then $\mathcal{S} = \mathrm{ConvH}\{\mu^1, \ldots \mu^\rho\}$ is an externally vNM-Stable Set for v.*

Proof: Consider the game

$$(2) \qquad w := \bigwedge\{\mu^1, \ldots \mu^r\}.$$

It follows from $\frac{d\mu^\rho}{d\lambda^\rho} \leq 1$ and from $\lambda^1(I) = 1$ that

$$(3) \qquad w \leq v, \quad w(I) = v(I).$$

The imputations for both games are the same and it follows from (3) that, for two imputations ξ and η, the relation $\xi \operatorname{dom}^w \eta$ always implies the relation $\xi \operatorname{dom}^v \eta$.

Now, according to the Main Theorem of Einy et. al ([EHMS96]), $\mathcal{S} = \mathrm{ConvH}\{\mu^1, \ldots, \mu^r\}$ is the unique vNM-Stable Set of w, hence dominates all imputations outside of $\mathcal{S}$ with respect to w. In view of the above observation it follows that $\mathcal{S}$ all the more dominates everything outside of $\mathcal{S}$ with respect to v. More precisely, if, for some imputation $\eta \notin \mathcal{S}$ we find $\xi \in \mathcal{S}$ satisfying

ξ *dom*$_S^w$ η, then it is not hard to see that $\xi \ \mathrm{dom}^v_{(S\cap D)} \ \eta$ where D is the union of the carriers of the $\mu^\rho(\rho = 1,\ldots,r)$.

qed.

We are now going to impose the additional requirement of orthogonality on the representation of a game v. Thus, we consider the class

$$(4) \qquad \mathbb{D} := \left\{ v \ = \ \bigwedge\{\lambda^1,\cdots,\lambda^r\} \mid \lambda^\rho \perp \lambda^\sigma, \rho,\sigma = 1,\ldots,r; \rho \neq \sigma \right\},$$

or rather the normalized subclass

$$(5) \qquad \mathbb{D}^1 := \left\{ v \ \in \ \mathbb{D} \mid v(I) \ = \ 1 \ = \ \lambda^1(I). \right\},$$

For this class of games we are now going to show that the type of solution introduced by the previous theorem is indeed a vNM-Stable Set. More precisely, we obtain the following theorem:

Theorem 3.2. *Let* $v \ = \ \bigwedge\{\lambda^1,\cdots,\lambda^r\} \in \mathbb{D}^1$ *and let* $\mu^1,\ldots,\mu^r$ *be probabilities such that* $\mu^\rho \ll \lambda^\rho, \ \ \rho = 1,\ldots,r$ *holds true. Then* $\mathcal{S} \ = \ \mathrm{ConvH}\{\mu^1,\ldots,\mu^r\}$ *is internally stable. Hence, if* $\frac{d\mu^\rho}{d\lambda^\rho} \leq 1 \ \ (\rho = 1,\ldots,r)$ *holds true, then* $\mathcal{S}$ *is a vNM-Stable Set.*

Proof: Assume *per absurdum* that, for two imputations $\mu = \sum_{\rho=1}^r c_\rho\mu^\rho \in \mathcal{S}$ and $\vartheta = \sum_{\rho=1}^r d_\rho\mu^\rho \in \mathcal{S}$ we have $\mu \ \mathrm{dom}_S \, \vartheta$ with suitable measurable S. Then we have necessarily $v(S) \geq w(S) > 0$ and hence $\mu^\rho(S) > 0 \ \ (\rho = 1,\ldots,r)$. Because of so much orthogonality we have assumed, there is $0 < \delta < 1$ and $S_0 \subseteq S$ such that $\mu^\rho(S_0) = \delta\mu^\rho(I) \ \ (\rho = 1,\ldots,r)$ holds true. Now consider the two imputations at hand; we find

$$(6) \qquad \begin{aligned} \mu(S_0) &= \textstyle\sum_{\rho=1}^r c_\rho\mu^\rho(S_0) &&= \textstyle\sum_{\rho=1}^r c_\rho\delta\mu^\rho(I) \\ &= \delta\textstyle\sum_{\rho=1}^r c_\rho\mu^\rho(I) &&= \delta\mu(I) \\ &= \delta v(I), \end{aligned}$$

as well as

$$(7) \qquad \begin{aligned} \vartheta(S_0) &= \textstyle\sum_{\rho=1}^r d_\rho\mu^\rho(S_0) &&= \textstyle\sum_{\rho=1}^r d_\rho\delta\mu^\rho(I) \\ &= \delta\textstyle\sum_{\rho=1}^r d_\rho\mu^\rho(I) &&= \delta\vartheta(I) \\ &= \delta v(I). \end{aligned}$$

As both, μ and ϑ are imputations, this result clearly contradicts the fact that $\mu(S_0) > \vartheta(S_0)$ is required in view of μ *dom*$_S$ ϑ, hence there can be no internal domination in $\mathcal{S}$, **qed.**

Definition 3.3. *Let* $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\} \in \mathbb{D}^1$ *and let* $\mu^1, \ldots, \mu^r$ *be probabilities such that* $\mu^\rho \ll \lambda^\rho$, $\frac{d\mu^\rho}{d\lambda^\rho} \leq 1$, $\rho = 1, \ldots, r$ *holds true. Then the vNM-Stable Set* $\mathcal{S} = \mathrm{ConvH}\{\mu^1, \ldots, \mu^r\}$ *is called a* ***standard solution***.

Remark 3.4. *Note that for* $\mu^\rho := \frac{\lambda^\rho}{\lambda^\rho(C^\rho)}$, $(\rho = 1, \ldots, r)$, *the vNM-stable set* $\mathcal{S} = \mathrm{ConvH}\{\mu^1, \ldots, \mu^r\}$ *supplies a symmetric standard solution, (cf. HART ([HART74]) and EINY ET. AL. ([EHMS96]).*

One of our main goals within this paper is to show that the class of standard solutions is large, actually it describes all convex (compact) vNM-Stable Sets with finitely many extreme points. The topic will be dealt with within the next section.

4 Characterizing Convex Stable Sets

Within this section we start to develop the converse direction or the 'characterization' of vNM-Stable Sets. That is we want to show that all convex (compact) polyhedral solutions of a game

$$v = \bigwedge \{\lambda^1, \cdots, \lambda^r\}$$

are elements of the standard class as defined in Definition 3.3 of Section 3. To this end, some prerequisits are necessary, most of which, however, may be considered to be of interest of their own.

The first lemma corroborates the idea that solutions to a game described by (1) are contain only measures that are absolutely continous w.r.t to the reference measure. More than that, we have the following result.

Lemma 4.1 (The Density Lemma). *Let* $v = \bigwedge \{\lambda^1, \cdots, \lambda^r\} \in \mathbb{D}^1$ *and let* $\mathcal{S}$ *be a vNM-stable set. Then, for all* $\mu \in \mathcal{S}$ *and all* $\rho = 1, \ldots, r$ *it follows that*

(1)
$$\frac{d\mu}{d\lambda^\rho} \leq 1$$

holds true on $C(\lambda^\rho) = C^\rho$.

Proof: Assume *per absurdum* that, for some $\mu \in \mathcal{S}$ and $\sigma \in \{\rho = 1, \ldots, r\}$, the set

(2)
$$R_\sigma := \left\{ t \in C^\sigma \mid \frac{d\mu}{d\lambda^\sigma}(t) > 1 \right\} \subseteq C^\sigma$$

has positive measure $\lambda^\sigma(R_\sigma) > 0$. Define a measure

(3)
$$\vartheta := \mu \,|_{I - R_\sigma} + \lambda^\sigma \,|_{R_\sigma},$$

then

(4)
$$\begin{aligned} \vartheta(I) &= \mu(I - R_\sigma) + \lambda^\sigma(R_\sigma) \\ &< \mu(I - R_\sigma) + \mu(R_\sigma) \\ &= \mu(I) = v(I). \end{aligned}$$

Thus, $\vartheta \notin \mathcal{S}$ and hence we can find some $\eta \in \mathcal{S}$ such that

$$\eta \ dom_A \ \vartheta \tag{5}$$

for a suitable measurable set A. This implies immediately

$$\eta(A) \leq v(A). \tag{6}$$

as well as

$$\eta(A \cap R_\sigma) > \vartheta(A \cap R_\sigma) = \lambda^\sigma(A \cap R_\sigma). \tag{7}$$

(If it so happens that $\lambda^\sigma(A \cap R_\sigma) = 0$ is the case, then we are done, as this would imply $\eta \ dom_A \ \mu$ contradicting $\eta, \ \mu \in \mathcal{S}$.)

Moreover, since we assume the measures λ^ρ to be mutually orthogonal, we know that

$$\lambda^\sigma(A \cap R_\sigma) = \max\{\lambda^\rho(A \cap R_\sigma) \mid \rho = 1, \ldots, r\}$$

and consequently

$$\begin{aligned} v(A - R_\sigma) &= \min\{\lambda^\rho(A - R_\sigma) \mid \rho = 1, \ldots, r\} \\ &= \min\{\lambda^\rho(A) - \lambda^\rho(A \cap R_\sigma) \mid \rho = 1, \ldots, r\} \\ &\geq \min\{\lambda^\rho(A)\} - \max\{\lambda^\rho(A \cap R_\sigma)\} \\ &= v(A) - \lambda^\sigma(A \cap R_\sigma). \end{aligned} \tag{8}$$

From this we proceed by the following chain of equations and inequalities, using (6) as well as (7) and (8) :

$$\begin{aligned} \eta(A - R_\sigma) &= \eta(A) - \eta(A \cap R_\sigma) \\ &\leq v(A) - \eta(A \cap R_\sigma) \\ &< v(A) - \lambda^\sigma(A \cap R_\sigma) \\ &\leq v(A - R_\sigma). \end{aligned} \tag{9}$$

Now we have found that $A - R_\sigma$ is effective for η. But on $A - R_\sigma$ it is true that $\eta > \vartheta = \mu$, which implies that we have

$$\eta \ dom_{A - R_\sigma} \ \mu,$$

a contradiction to the fact that $\mu, \ \eta \in \mathcal{S}$. Thus $A - R_\sigma$ is a set of Lebesgue measure 0 which is impossibly compatible with (9), **qed.**

We may draw some interesting conclusions from this lemma.

Corollary 4.2. *Let* $v = \bigwedge \{\lambda^1, \cdots, \lambda^r\}$ *and let* $\mathcal{S}$ *be a convex vNM-Stable Set for* v. *If there are probabilities* $\mu^1, \ldots \mu^r \in \mathcal{S}$, *such that* $\mu^\rho \perp \mu^\sigma$ $(\rho, \sigma = 1, \ldots, r;\ \rho \neq \sigma)$ *and* $\mu^\rho \ll \lambda^\rho$ $(\rho = 1, \ldots r)$, *then*

$$\mathcal{S} \ = \ \text{ConvH}\left\{\mu^1, \ldots \mu^r\right\}. \tag{10}$$

Proof: By virtue of Lemma 4.1 it follows that $\frac{d\mu^\rho}{d\mu^\rho} \leq 1$ holds true and from Theorem 3.1 we conclude that

$$\mathcal{S}^0 \ := \ \text{ConvH}\left\{\mu^1 \ldots \mu^r\right\}$$

is a vNM-Stable Set for v. As $\mathcal{S}$ is assumed to be convex, we may infer that $\mathcal{S}^0 \subseteq \mathcal{S}$ holds true, hence we have necessarily $\mathcal{S}^0 = \mathcal{S}$ **qed.**

Corollary 4.3. *Let* $v = \bigwedge \{\lambda^1, \cdots, \lambda^r\} \in \mathbb{D}^1$ *and let* $\mathcal{S}$ *be a vNM-stable set. Then*

$$\{\lambda^\rho \mid \lambda^\rho(I) = 1\} \subseteq \mathcal{S}, \tag{11}$$

i.e., $\mathcal{S}$ *contains all normalized probabilities among the* $\lambda^\rho, \rho = 1, \ldots, r$. *If* $\mathcal{S}$ *is convex, then all these normalized measures are extreme points in* $\mathcal{S}$. *(Note that they constitute the extreme points of the core.)*

Proof: Assume $\lambda^1 \notin \mathcal{S}$, then there is some $\mu \in \mathcal{S}$ dominating λ^1, say

$$\mu \ \text{dom}_S \ \lambda^1, \tag{12}$$

where S is a suitable measurable set. It would then follow that

$$\lambda^1(S) \geq v(S) \geq \mu(S) > \lambda^1(S), \tag{13}$$

which is impossible. If $\mathcal{S}$ is convex, then λ^1 has to be extreme. For any convex combination of λ^1 by means of members of $\mathcal{S}$, say $\lambda^1 = \frac{1}{2}(\mu^1 + \mu^2)$ involves probabilities μ^1 and μ^2 which, according to Lemma 4.1 are bounded in density by λ^1 from which it follows at once that they equal λ^1, **qed.**

Remark 4.4. *Let* $v = \bigwedge \left\{\lambda^1, \cdots, \lambda^r\right\} \in \mathbb{D}^1$ *and let* $\mathcal{S} = \text{ConvH}\{\lambda^1, \mu\}$ *be a vNM-stable set with two extreme points. It is then easily seen, that the carriers of* λ^1 *and* μ *are disjoint. For, as* μ *differs from* λ^1 *and* $\dot{\mu} \leq 1$ *on* $C^1 = C(\lambda^1)$ *(Lemma 4.1), it is necessarily true that* μ *has positive mass on the complement of* C^1. *Hence,*

$$\frac{\mu \mid_{C^{1c}}}{\mu(C^{1c})}$$

is a probability. This probability strictly exceeds μ *on* $C^{1c} \cap C(\mu)$ *if the carriers are not disjoint, hence it cannot be dominated by any element of* $\mathcal{S}$. *Extending this argument, it is clear that* $C(\mu)$ *has to be in the carrier of* λ^2 *and that* r *necessarily has to be two. Thus we see, that in this case* $\mathcal{S}$ *is a* standard solution.

For the subsequent discussion we may, therefore, always assume that a convex solution has at least three extremepoints, one of which is λ^1.

In order to proceed with the next Theorem, some auxiliary lemma is necessary. The lemma shows that domination between two imputations with respect to some measurable set S is always inherited by an arbitrarily small subset of S.

Lemma 4.5 (The Inheritance Lemma). *Let* $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ *and let* ϑ *and* η *be probabilities such that for some* $T = \sum_{\rho=1}^{r} T^\rho,\ T^\rho \subseteq C^\rho,\quad (\rho = 1, \ldots, r)$ *we have* $\vartheta \ \mathrm{dom}_T\ \eta$. *Then, for every* $\varepsilon > 0$, *There exist coalitions* $S^1, \ldots, S^r$ *such that the following holds true:*

$$S^\rho \subseteq T^\rho \quad (\rho = 1, \ldots, r), \tag{14}$$

$$\lambda^\rho(S^\rho) < \varepsilon \quad (\rho = 1, \ldots, r) \tag{15}$$

$$\vartheta \ \mathrm{dom}_{\sum_{\rho=1}^{r} S^\sigma}\ \eta. \tag{16}$$

Proof: W.l.g. assume that $\lambda^1(T^1) = \ldots = \lambda^r(T^r)$ holds true and that

$$\vartheta(T^1 + \ldots + T^r) \leq \lambda^1(T^1) = \ldots = \lambda^r(T^r) = v(T) \tag{17}$$

is the case.

Decompose via Ljapounoff each set $T^\rho = T_1^\rho + T_2^\rho$ as to yield

$$\lambda^\rho(T_1^\rho) = \lambda^\rho(T_2^\rho) = \frac{1}{2}\lambda^\rho(T^\rho) \quad (\rho = 1, \ldots, r). \tag{18}$$

Assume without loss of generality that we have

$$\vartheta(T_1^\rho) \leq \vartheta(T_2^\rho) \quad (\rho = 1, \ldots, r) \tag{19}$$

holds true. Then we obtain necessarily

$$\vartheta(\sum_{\rho=1}^{r} T_1^\rho) \leq \lambda^1(T_1^1) = \ldots = \lambda^r(T_1^r). \tag{20}$$

For, otherwise we would have

$$\vartheta(\sum_{\rho=1}^{r} T_2^\rho) \geq \vartheta(\sum_{\rho=1}^{r} T_1^\rho) > \lambda^1(T_1^1) = \lambda^1(T_2^1) \tag{21}$$

and

$$\begin{array}{rcl} \vartheta(\sum_{\rho=1}^{r} T^\rho) & = & \vartheta(\sum_{\rho=1}^{r} T_1^\rho) + \vartheta(\sum_{\rho=1}^{r} T_2^\rho) \\ & > & \lambda^1(T_1^1) + \lambda^1(T_2^1) \\ & = & \lambda^1(T^1), \end{array} \tag{22}$$

contradicting (17). Clearly it follows that $\vartheta\ dom_{T_1}\ \eta$ for $T_1 = \sum_{\rho=1}^{r} T_1^\rho$. This way we may continue splitting $T, T_1, ...$ always decreasing the measure by $\frac{1}{2}$. **qed.**

Theorem 4.6 (The Support Theorem). *Let* $v = \bigwedge \{\lambda^1, \cdots, \lambda^r\} \in \mathbb{D}^1$ *and let* $\mathcal{S}$ *be a convex vNM-stable set for* v*. Let* $\xi, \eta \in \mathcal{S}$*. If, for some* $\sigma \in \{(\rho = 1, \ldots, r)\}$ *it is true that* $\xi(C^\sigma) > 0, \eta(C^\sigma) > 0$*, then it follows that* $C(\xi) \cap C^\sigma = C(\eta) \cap C^\sigma$ *is the case. That is, if* ξ *and* η *have positive mass on* $C^\sigma = C(\lambda^\sigma)$ *at all, then both have the same carrier inside* C^σ*.*

Proof:

1ˢᵗSTEP : We assume for simplicity that $\sigma = 1$.

Let $R \subseteq C^1$ be a set of positive $\xi-$measure such that $\eta(R) = 0$ and $\dot{\xi} > 0$ on R is obtained. This we shall exploit to lead to a contradiction.

Define η_0 to be the restriction of η to the complement of C^1, increased by a suitable constant as to render it in total measure smaller than 1, i.e.,

$$\eta_0 \ = \ (\eta + \alpha\lambda) \mid_{C^{1c}}, \tag{23}$$

where λ is the Lebesgue measure and $\alpha > 0$ is chosen such that $\eta_0(I) < 1$ holds true. Clearly we have

$$\eta_0 = \eta + \alpha \ \text{ on } \ C^{1c}. \tag{24}$$

2ⁿᵈSTEP : We now know that $\eta_0 \notin \mathcal{S}$ holds true. Hence there exists $\vartheta \in \mathcal{S}$ and $T = T^1 + \ldots + T^r$ such that we have

$$\vartheta \ \mathrm{dom}_T \ \eta_0 \tag{25}$$

for suitable $T^\rho \subseteq C^\rho (\rho = 1, \ldots , r)$. By means of the Inheritance Lemma 4.5, we may choose $S^\rho \subseteq T^\rho (\rho = 1, \ldots , r)$ such that for $S := S^1 + \ldots + S^r$ we obtain

$$\vartheta \ \mathrm{dom}_S \ \eta_0 \tag{26}$$

while, in addition, $\lambda^1(S^1) = \ldots = \lambda^r(S^r) < \lambda^1(R)$ holds true. Now we choose $R^1 \subseteq R$ satisfying

$$\lambda^1(S^1) = \ldots = \lambda^r(S^r) = \lambda^1(R^1). \tag{27}$$

Clearly it follows from (24) that we have

$$\begin{array}{rcl} \vartheta & > & \eta + \alpha \text{ on } S^2 + \ldots + S^r \\ \vartheta & \geq & \eta = 0 \text{ on } R^1 \end{array} \tag{28}$$

as η vanishes on R. On the other hand we observe that we have

$$\overset{\bullet}{\vartheta} > 0 \text{ on } S^1, \ \vartheta(S^1) > 0. \tag{29}$$

Moreover, we know that we may as well assume that

$$\vartheta(S^1 + \ldots + S^r) \leq \lambda^1(S^1) = \ldots = \lambda^r(S^r) = \lambda^1(R^1). \tag{30}$$

3rdSTEP :

For small $\varepsilon > 0$ consider now the probability

$$\mu^\varepsilon \ := \ \varepsilon\vartheta + (1 - \varepsilon)\eta, \tag{31}$$

from (28) it is immediately inferred that we have

$$\begin{array}{rcl} \mu^\varepsilon & > & \eta + \varepsilon\alpha \text{ on } S^2 + \ldots + S^r \\ \mu^\varepsilon & \geq & \eta = 0 \text{ on } R^1. \end{array} \tag{32}$$

For sufficiently small $\varepsilon > 0$ we find that

$$\mu^\varepsilon(R^1) = \varepsilon\vartheta(R^1) < \vartheta(S^1), \tag{33}$$

as the last quantity is positive by (29). Next, in view of (28) we see immediately that

$$\begin{array}{rcl} \mu^\varepsilon(R^1 + S^2 + \ldots + S^r) & < & \\ & < & \vartheta(S^1 + S^2 + \ldots + S^r) \\ & & \leq \lambda(R^1) = \lambda(S^2) = \ldots = \lambda(S^r) \\ & & = v(R^1 + S^2 + \ldots + S^r) \end{array} \tag{34}$$

(compare (27)) holds true. Here the strict inequality provides the clue. For, we may now choose $\delta > 0$ sufficiently small such that

$$\mu^{\varepsilon,\delta} := \delta\xi + (1-\delta)\mu^{\varepsilon} \tag{35}$$

still satisfies

$$\mu^{\varepsilon,\delta}(R^1 + S^2 + \ldots + S^r) < v(R^1 + S^2 + \ldots + S^r) \tag{36}$$

But $\mu^{\varepsilon,\delta}$ also yields

$$\begin{array}{lcl} \mu^{\varepsilon,\delta} & > & \eta + (1-\delta)\varepsilon\alpha \text{ on } S^2 + \ldots + S^r \\ \mu^{\varepsilon,\delta} & > & \eta = 0 \text{ on } R^1, \end{array} \tag{37}$$

this follows from (32) and the fact that ξ is assumed to be positive on $R \supseteq R^1$.

The two equations (36) and (37) show that $\mu^{\varepsilon,\delta} dom_{R^1+S^2+\ldots+S^r} \eta$. This contradiction proves the theorem.

qed.

Remark 4.7. *In the situation of the Support Theorem, whenever $\lambda^\sigma(C^\sigma) = 1$, then $C(\eta) \supseteq C^\sigma$ holds true.*

We are now in the position to do a major step towards our main goal: we can show that the extreme points in $\mathcal{S}$ are essentially orthogonal. More precisely, we have

Theorem 4.8 (The Orthogonality Theorem). *Let $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\} \in \mathbb{D}_1$ and let $\mathcal{S} = \text{ConvH}\{\eta^1, \ldots \eta^s\}$ be a convex polyhedral vNM-Stable Set for v. Pick any two extremals of $\mathcal{S}$, say η^τ and η^σ for some $\tau, \sigma \in \{1, \ldots, s\}$. If, for some $\rho \in \{1, \ldots, r\}$, the sets $\{\eta^\tau > 0\} \cap C^\rho$ and $\{\eta^\sigma > 0\} \cap C^\rho$ have positive Lebesgue measure, then*

$$\eta^\tau \mid_{C^\rho} = \eta^\sigma \mid_{C^\rho} \tag{38}$$

holds true.

Proof: For simplicity we assume $\rho = 1$, thus the carrier involved happens to be C^1. Define the measurable function

$$\psi = \bigvee \left\{ \dot{\eta}^1, \ldots, \dot{\eta}^s \right\} \mid_{C^1}, \tag{39}$$

an index set

$$\mathbf{S} = \{\sigma \in \{1, \dots, s\} \mid \eta^\sigma(R) > 0 \text{ for some } R \subseteq C^1\}, \tag{40}$$

and another measurable function

$$\varphi = \bigwedge \left\{ \overset{\bullet}{\eta}^\sigma \mid \sigma \in \mathbf{S} \right\} |_{C^1} . \tag{41}$$

We claim that $\psi = \varphi$, obviously this implies our present theorem.

By the Support Theorem 4.6 we know that on C^1 the functions φ and ψ as well as the densities of all η^σ $(\sigma \in \mathbf{S})$ have the same carrier. Now, if our claim is wrong, then $\lambda(\{\varphi < \psi\}) > 0$ and as

$$\{\varphi < \psi\} = \bigcup_{\sigma \in \mathbf{S}} \bigcup_{\tau \in \mathbf{S}} \{\varphi = \eta^\sigma < \eta^\tau \;=\; \psi, \}$$

one of the sets on the right side has positive measure, w.l.g. we assume that this is

$$E := \{\varphi = \eta^1 < \eta^s = \psi\}. \tag{42}$$

Using this (and the Support Theorem 4.6), we know that

$$\lambda^1(E) > 0,\ \eta^1 \leq \eta^\sigma \ (\sigma = 1, \dots s),\ 0 < \eta^1 < \eta^s \text{ on E} \tag{43}$$

holds true. We may now choose small positive constants $\alpha > 0, \beta > 0$ as well as a measurable subset F of E such that the measure η_F defined by the density

$$\overset{\bullet}{\eta}_F := \overset{\bullet}{\eta}^1 - \alpha 1_F + \beta 1_{F^c} \tag{44}$$

is nonnegative and has total mass $\eta_F(I) < 1$. Note that η_F exceeds η^1 on the complement of F and is smaller than η^1 only on F. But on F it is true, that all other η^σ are at least as large as η^1. Now, as η_F has total mass smaller than 1, there is $\vartheta \in \mathcal{S}$ dominating it, we write

$$\vartheta \ \mathrm{dom}_{S^1 + \dots + S^r} \ \eta_F \tag{45}$$

with suitable $S \;=\; S^1 + \dots + S^r,\ S^\rho \subseteq C^\rho (\rho = 1, \dots , r)$.

Next, if $\lambda^1(S^1 \cap F) = 0$, then we can immediately see that $\vartheta \ \mathrm{dom}_{S^1 + \dots + S^r} \ \eta^1$ holds true, contradicting internal stability. The difficult case is the one in which this is not so.

Consider, therefore, the case that we have $\lambda^1(S^1 \cap F) > 0$. Because of our construction we know that

$$\overset{\bullet}{\vartheta} \geq \overset{\bullet}{\eta}^1 \text{ on } S^1 \cap F \text{ and } \overset{\bullet}{\vartheta} \geq \overset{\bullet}{\eta}^1 + \beta \text{ on } (S^1 - F) + S^2 + \ldots + S^r. \tag{46}$$

By means of equation (45) , ϑ is effective for $S = S^1 + \ldots + S^r$, i.e., w.l.g

$$\vartheta(S^1 + \ldots + S^r) \leq \lambda^1(S^1) = \ldots = \lambda^r(S^r) = v(S^1 + \ldots + S^r) = v(S) \tag{47}$$

holds true. Take a convex combination, say

$$\mu := \frac{1}{2}(\vartheta + \eta^1), \tag{48}$$

then from (46) we obtain a similar version reading

$$\overset{\bullet}{\mu} \geq \overset{\bullet}{\eta}^1 \text{ on } S^1 \cap F \text{ and } \overset{\bullet}{\mu} \geq \overset{\bullet}{\eta}^1 + \frac{\beta}{2} \text{ on } (S^1 - F) + S^2 + \ldots + S^r, \tag{49}$$

while (47) becomes a strict inequality (ϑ exceeds η^1 on $(S^1 - F) + S^2 + \ldots + S^r$,) which reads

$$\mu(S^1 + \ldots + S^r) < \lambda^1(S^1) = \ldots = \lambda^r(S^r) = v(S^1 + \ldots + S^r) = v(S) \tag{50}$$

Now recall the role of η^s as specified in the definition (42) and in (43), and choose a small $\varepsilon > 0$ in order to define

$$\mu^\varepsilon := \varepsilon\eta^s + (1 - \varepsilon)\mu. \tag{51}$$

If ε is sufficiently small we can still guaranty effectiveness, for (50) will ensure

$$\mu^\varepsilon(S) < v(S). \tag{52}$$

Also, for ε decreased again if necessary, (49) ensures that

$$\overset{\bullet}{\mu}^\varepsilon > \overset{\bullet}{\eta}^1 \text{ on } (S^1 - F) + S^2 + \ldots + S^r, \tag{53}$$

while the fact that $\eta^s > \eta^1$ on $S^1 \cap F \subseteq E$ implies

$$\overset{\bullet}{\mu}^\varepsilon > \overset{\bullet}{\eta}^1 \text{ on } S^1 \cap F. \tag{54}$$

Indeed, equations (52), (53), and (54) show that we have constructed a situation in which $\mu^\varepsilon \ \mathrm{dom}_S \ \eta^1$, but both are elements of $\mathcal{S}$ which is contradictory. This finally proves the Theorem. **qed.**

We are now in the position to prove one of our main results:

Theorem 4.9 (The Main Theorem of Characterization). *Let $v \in \mathbb{D}_1$ be given by $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ and let $\mathcal{S}$ be a convex polyhedral vNM-Stable Set for v. Then there are probabilities $\mu^1, \ldots \mu^r$ such that μ^ρ is absolutely continuous w.r.t. λ^ρ, and $\frac{d\mu^\rho}{d\lambda^\rho} \leq 1$ $(\rho = 1, \ldots, r)$ holds true yielding $\mathcal{S} = \operatorname{ConvH}\{\mu^1, \ldots \mu^r\}$. That is,* ***every convex polyhedral solution is standard.***

Proof: This follows now easily from the Orthogonality Theorem 4.8. Indeed, any two of the extreme points of $\mathcal{S}$ on any carrier C^ρ will either coincide or one of them will vanish. Non of them will have total mass strictly between 0 and 1 on some C^ρ, for otherwise the normalization on this carrier could never be blocked. Therefore, we have at most r extreme points in $\mathcal{S}$ each of which has support completely contained in some C^ρ. But less than r such extreme points cannot occur, because this obviously would not satisfy external stability. **qed.**

Remark 4.10 (The General Density Property). *In passing we note that we have obtained a property which strengthens the Density Lemma 4.1. Obviously it follows from the last theorem that the following holds true:*

For any element μ of a convex polyhedral vNM-Stable Set $\mathcal{S}$ and for every

$$t_1 \in C^1, \ldots, t_r \in C^r$$

it follows that

$$\sum_{\rho=1}^{r} \frac{d\mu}{d\lambda^\rho}(t_\rho) \leq 1 \tag{55}$$

holds true.

5 Finite Solutions versus Large Solutions

Within this section we discuss the 'injection' or 'embedding'of a finite game into the continuum. This way we obtain a game with a continuum of players within which full intervals corresponding to the weight of finite players have the same power as those players in the finite game. Naturally the question arises as to whether vNM-Stable Sets are compatible with the embedding procedure and our result is affirmative.

At first sight, this result may draw limited enthusiasm. For, while it is nice that all vNM-Stable Sets of finite games (obeying a natural condition) induce vNM-Stable Sets of continuous games it also tells us clearly that there is little hope for 'classifying' all continuous vNM-Stable Sets as this seems to be out of the question in the finite case - for the time being.

On second thought there is also good news: it turns out that our description of all convex polyhedral vNM-Stable Sets surprisingly also induces a description of all convex polyhedral vNM-Stable Sets for finite games. Clearly, this has nice consequences for it means that we can offer a class of solution concepts for finite general glove games - apart from the fact that no classification of this type so far has been attempted in the finite context.

We will come back to these nice conclusions in a subsequent section. At present we start out to present the 'embedding procedure'.

Let $N = \{1, , ..., n\}$ be the set of players in the finite context and let

$\mathring{\lambda} = (\mathring{\lambda}_1, ..., \mathring{\lambda}_n)$ be an integer vector. For any decomposition

$$N = \sum_{\rho=1}^{r} K^\rho$$

with disjoint sets $K^\rho \subseteq N$ let

$$\mathring{\lambda}^\rho := \mathring{\lambda} \mid_{K^\rho}$$

denote the restriction; we now consider the game

$$\mathring{v} := \bigwedge \left(\mathring{\lambda}^1, ..., \mathring{\lambda}^r \right) \tag{1}$$

or rather the pair $(N, \mathring{v})$.

In order to 'embed' $\overset{\circ}{v}$ into the continuum, let I_i $(i \in N)$ be disjoint intervals of length $\overset{\circ}{\lambda}_i$, i.e.

$$\lambda(I_i) = \overset{\circ}{\lambda}_i \quad (i \in N). \tag{2}$$

If we write

$$C^\rho := \sum_{i \in K^\rho} I^i \ , \lambda_i := \lambda \mid_{I^i} \ , \lambda^\rho := \lambda \mid_{C^\rho} \tag{3}$$

then we have generated a game on the continuum given by

$$v = \bigwedge(\lambda^1, ..., \lambda^r) \ . \tag{4}$$

Definition 5.1. *Let $\overset{\circ}{\lambda}$ be an integer vector and let $\overset{\circ}{v}$ be given by (1). If v is generated via (2), (3), and (4), then v is said to be the (an)* ***embedding*** *of $\overset{\circ}{v}$ into the continuum.*

Now we turn to vNM-Stable Sets. Given the above procedure, the generation of continuous vNM-Stable Sets by means of finite ones is described as follows.

Definition 5.2. *Suppose v is an embedding of $\overset{\circ}{v}$ and let $\mathcal{S}^0$ be a vNM-Stable Set for $\overset{\circ}{v}$. Then the* ***embedding*** *$\mathcal{S}$ of $\overset{\circ}{\mathcal{S}}$ is given by*

$$\mathcal{S} := \{\mu \mid \exists\ m \in \mathcal{S}^0 : \mu = \sum_{i \in N} \frac{m_i}{\overset{\circ}{\lambda}_i} \lambda_i\}. \tag{5}$$

We are now going to exhibit conditions such that the embedding $\mathcal{S}$ of a finite vNM-Stable Set for $\overset{\circ}{v}$ is a vNM-Stable Set for the embedding v of $\overset{\circ}{v}$.

Theorem 5.3 (The Embedding Theorem). *The embedding $\mathcal{S}$ of a vNM-Stable Set $\overset{\circ}{\mathcal{S}}$ is a vNM-Stable Set if and only if there is no $m, x \in \overset{\circ}{\mathcal{S}}$ with the following properties:*

There is an r-tupel $(i_1, ..., i_r) \in K^1 \times \ldots \times K^r$ such that

$$x_{i_\rho} < m_{i_\rho} \quad (\rho = 1, ..., r) \tag{6}$$

$$\sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \leq 1 \tag{7}$$

holds true.

Proof:

The first two steps deal with internal stability:

1^st STEP : Suppose there is $m, x \in \mathcal{S}^0$ satisfying (6) and (7). Choose (via Ljapounoff)

$$S \subseteq \sum_{\rho=1}^{r} I^{i_\rho}$$

such that

$$0 < \lambda(S \cap I^{i_1}) = ... = \lambda(S \cap I^{i_r}) = v(S)$$

holds true. Then we have in view of (7):

$$\begin{aligned} \mu(S) &= \sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \lambda(S \cap I^{i_\rho}) \\ &= \left(\sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \right) v(S) \\ &\leq v(S) \ . \end{aligned} \tag{8}$$

Also, for any $R \subseteq S$ with positive Lebesgue measure we use (6) to check that

$$\begin{aligned} \mu(R) &= \sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \lambda(R \cap I^{i_\rho}) \\ &> \sum_{\rho=1}^{r} \frac{x_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \lambda(R \cap I^{i_\rho}) \\ &= \xi(R) \end{aligned} \tag{9}$$

Now, (8) and (9) show that $\mu \ dom_S \ \xi$ is the case, contradicting the internal stability of $\mathcal{S}$.

2^nd STEP : On the other hand, assume that there is some coalition S such that

$$\mu \ dom_S \ \xi$$

holds true. Define

$$J^\rho := \{i \in K^\rho | \lambda(I^i \cap S) > 0\} \ (\rho = 1, ..., r)$$

then it must necessarily follow that

$$m_i > x_i \quad (i \in \sum_{\rho=1}^{r} J^\rho) \tag{10}$$

is the case. We may as well assume

$$\begin{aligned} \lambda(C^1 \cap S) &= \sum_{i \in J^1} \lambda \ \ (I^i \cap S) = ... \\ &= \sum_{i \in J^r} \lambda \ \ (I^i \cap S) = \lambda(C^r \cap S) \\ &= v(S) \end{aligned} \tag{11}$$

- otherwise diminish some $C^\rho \cap S$ appropriately.

Now choose, for every $\rho = 1, ..., r$ some $i_\rho \in J^\rho$ such that

$$\frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} = \min_{i \in J^\rho} \frac{m_i}{\overset{\circ}{\lambda}_i} \tag{12}$$

is satisfied. Then, in view of $\mu \ dom_S \ \xi$ we have

$$\begin{aligned} v(S) &= \mu(S) \\ &= \sum_{\rho=1}^{r} \sum_{i \in J^\rho} \frac{m_i}{\overset{\circ}{\lambda}_i} \lambda(I^i \cap S) \\ &\geq \sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \sum_{i \in J^\rho} \lambda(I^i \cap S) \\ &= \sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \lambda(C^\rho \cap S) \\ &= \left(\sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \right) \ v(S) \ ; \end{aligned}$$

here we have employed (10) and (12). But then it is clear that

$$\sum_{\rho=1}^{r} \frac{m_{i_\rho}}{\overset{\circ}{\lambda}_\rho} \leq 1$$

holds true, i.e., (7) is satisfied.

Similarly, as $\mu(R) > \xi(R)$ for all $R \subseteq S$ follows from μ dom_S ξ, we may choose $R \subseteq I^{i_\rho}$ with positive Lebesgue measure, obtaining

$$\begin{aligned}\frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}}\lambda(R \cap I^{i_\rho}) = \mu(R) > \xi(R) \\ = \frac{x_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}}\lambda(R \cap I^{i_\rho}) \ .\end{aligned}$$

That is, we have

$$\frac{m_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} > \frac{x_{i_\rho}}{\overset{\circ}{\lambda}_{i_\rho}} \quad (\rho = 1, ..., r) \ ,$$

meaning that (6) is satisfied as well,

3$^{\text{rd}}$STEP : Within this last step we deal with external stability, this part is much easier.

Indeed, assume that $\overset{\circ}{\mathcal{S}}$ is externally stable. We shall prove external stability of $\mathcal{S}$. Let $\eta \notin \mathcal{S}$ be an imputation.

If $\overset{\bullet}{\eta}$ is constant on each I_i $\ (i \in N)$, then we are done.

Otherwise, define an imputation for $\overset{\circ}{v}$ by

$$(13) \qquad y_i \ := \ \frac{\eta(I_i)}{\overset{\circ}{\lambda}_i} \quad (i \in N).$$

.

Also, for $i \in N$, define

$$(14) \qquad F_i \ := \ \left\{ \eta < \frac{\eta(I_i)}{\overset{\circ}{\lambda}_i} \right\} \cap I_i \ = \ \{\eta < y_i\} \cap I_i$$

and

$$(15) \qquad J := \{i \in N \mid \lambda(F_i) > 0\} \neq \emptyset.$$

Pick small positive constants α and β such that

$$(16) \qquad E_i \ := \ \{\eta < y_i - \alpha\} \cap I_i \subseteq F_i$$

still yields $\lambda(E_i) > 0$ $(i \in J)$ and

$$z \;:=\; (y-\alpha)1_J + (y+\beta)1_{N-J} \tag{17}$$

satisfies $z(N) < \mathring{v}(N)$. Then $z \notin \mathring{\mathcal{S}}$ and, therefore, we find $x \in \mathring{\mathcal{S}}$ and $\mathring{S} \subseteq N$ with

$$x \; dom_{\mathring{S}} \; z. \tag{18}$$

Now, for some $\delta > 0$ sufficiently small we may find $S^i \subseteq E_i \quad (i \in J)$ and $S^i \subseteq I_i \quad (i \in N-J)$ such that

$$\lambda(S^i) = \delta\lambda(I_i) \quad (i \in N) \tag{19}$$

holds true. Clearly, by (17) and (18) we may infer the inequalities

$$\begin{aligned} x_i &> y_i - \alpha \quad (i \in \mathring{S} \cap J) \\ x_i &> y_i + \beta \quad (i \in \mathring{S} \cap (N-J)), \end{aligned} \tag{20}$$

and hence (16) shows that

$$\xi \;:=\; \sum_{i \in N} \frac{x_i}{\mathring{\lambda}_i}\,\lambda_i \;\in \mathcal{S}$$

yields

$$\xi > \eta \;\text{ on }\; S \;:=\; \sum_{i \in \mathring{S}} S^i. \tag{21}$$

On the other hand it is seen that S in view of (19) satisfies

$$\xi(S) \;=\; \delta x(\mathring{S}), \quad v(S) \;=\; \delta\mathring{v}(\mathring{S}), \tag{22}$$

from which it follows at once that we have also

$$\xi(S) \leq v(S) \tag{23}$$

in view of (18). Now, equations (21) and (23) show that $\xi \; dom_S \; \eta$ holds true,

qed.

The converse is not true in the general case.

Theorem 5.4. *Suppose that the embedding* $\mathcal{S}$ *of a set of imputations* $\overset{\circ}{\mathcal{S}}$ *is a vNM-Stable Set. If* $\overset{\circ}{\lambda}$ *is the uniform distribution (that is,* $\overset{\circ}{\lambda}^{\rho} = (1, \ldots, 1)$, $(\rho = 1, \ldots, r)$*), then* $\overset{\circ}{\mathcal{S}}$ *is a vNM-Stable Set.*

Proof: Internal stability is rather straightforward. All we have to prove is external stability. To this end, let x be an imputation for $\overset{\circ}{v}$ which is no element of $\overset{\circ}{\mathcal{S}}$. Extend x to some ξ in the usual fashion, clearly ξ is no element of $\mathcal{S}$.

Hence, there is $\mu \in \mathcal{S}$, piecewise constant, such that $\mu \ dom_S \ \xi$ holds true with suitable S. Now repeat the procedure offered in the **2nd STEP** of the proof of Theorem 5.3 in order to find a suitable coalition $\overset{\circ}{S}$ such that $m \ dom_{\overset{\circ}{S}} \ x$ holds true. **qed.**

Remark 5.5. *Note that condition (7) is a close relative of the general density property as discussed in Remark 4.10.*

6 Finite Games versus Large Games

Within this section we are going to combine the results of the previous sections in order to draw some conclusions regarding the theory of vNM-stable sets for finite games. Surprisingly, the characterization of all convex stable sets as obtained within sections 3 and 4 allows also to characterize stable sets of finite games in view of the embedding theorem provided in section 5 and its converse.

Again, let $N = \{1, , ..., n\}$ be the set of players in the finite context and let $\mathring{\lambda} = (\mathring{\lambda}_1, ..., \mathring{\lambda}_n) = (1, \ldots , 1)$ be an integer vector representing uniform distribution. For any decomposition

$$N = \sum_{\rho=1}^{r} K^\rho$$

with disjoint sets $K^\rho \subseteq N$ let

$$\mathring{\lambda}^\rho := \mathring{\lambda} \mid_{K^\rho}$$

as previously denote the restriction; we again consider the game

$$\mathring{v} := \bigwedge \left(\mathring{\lambda}^1, ..., \mathring{\lambda}^r\right) \tag{1}$$

or the pair $(N, \mathring{v})$.

Theorem 6.1 (The Theorem of Characterization in the Finite Case). *Let $\mathring{v}$ be given by (1) and assume that $\mathring{\lambda}$ is the uniform distribution. Let $\mathring{\mathcal{S}}$ be the convex hull of r orthogonal imputations, each of which is majorized by some $\mathring{\lambda}^\rho$. Then $\mathring{\mathcal{S}}$ is a vNM-Stable Set (Of course we call this a **standard solution**). On the other hand, let $\mathring{\mathcal{S}}$ be a polyhedral vNM-Stable Set for $\mathring{v}$. Then, there are imputations $m^1, \ldots m^r$ satisfying $m^\rho \leq \lambda^\rho \quad (\rho = 1, \ldots , r)$ such that $\mathring{\mathcal{S}} = \text{ConvH}\{m^1, \ldots m^r\}$. That is, **every polyhedral solution is standard.***

Proof: Consider the embedding $\mathcal{S}$ of $\mathring{\mathcal{S}}$. By Theorem 3.2 we know that $\mathcal{S}$ is a vNM-stable Set for the embedding v of $\mathring{v}$. Therefore, by Theorem 5.4 we conclude that $\mathring{\mathcal{S}}$ is a vNM-Stable Set for $\mathring{v}$.

Now to the converse direction: suppose $\overset{\circ}{\mathcal{S}}$ is a polyhedral vNM-Stable Set for $\overset{\circ}{v}$. By the embedding Theorem 5.3 is it clear that $\mathcal{S}$ is a polyhedral vNM-Stable Set for v. Note that the conditions (6) and (7) are automatically satisfied by the internal stability of $\overset{\circ}{\mathcal{S}}$ and by the fact that we have uniform distribution at hand. Apply now the Main Theorem of Characterization 4.9 to obtain the result that $\mathcal{S}$ is the convex hull of r orthogonal probabilities each of which is majorized by the Lebesgue measure restricted to the appropriate C^ρ. Turning back to $\overset{\circ}{\mathcal{S}}$, we obtain the desired result. **qed.**

Finally we would like to present an example which shows that there might be nonconvex vNM-Stable Sets both, for the continuous and the finite version of a game.

Example 6.2. Let $N = \{1, \dots, 6\}$ and decompose N into two sets $K^1 := \{1,2\}$ and $K^2 := \{3,4,5,6\}$. Let $\overset{\circ}{\lambda} := (1,1,1,1,1,1)$. For $0 \leq p \leq 1$ define a vector x^p by

$$
\begin{aligned}
x_1^p &:= \tfrac{2p^2}{1+p} \\
x_2^p &:= \tfrac{2p}{1+p} \\
x_i^p &:= \tfrac{1-p}{2}
\end{aligned}
\tag{2}
$$

for $i = 3,4,5,6$.

Let

$$
\overset{\circ}{\mathcal{S}} := \{x^p \mid 0 \leq p \leq 1\}, \tag{3}
$$

we claim that $\overset{\circ}{\mathcal{S}}$ is a vNM-Stable Set for $\overset{\circ}{v}$ which is clearly nonconvex. Moreover, its embedding $\mathcal{S}$ is as well a vNM-Stable Set for the embedding v of $\overset{\circ}{v}$ - of course also nonconvex.

Proof: The example is constructed in the spirit of the class given by Shapley ([SHA59]).

Internal stability follows from the fact that x_1^p and x_2^p are increasing in p while the last four coordinates obviously decrease in p.

External stability runs as follows: Pick an imputation x which is no element of $\overset{\circ}{\mathcal{S}}$. Define p to be the average of the first two coordinates of x. First of all consider the case that $x_2 < x_2^p$ holds true, that is, assume $x_2 < \frac{2p}{1+p}$.

Now $x_3 + x_4 + x_5 + x_6 = 2 - 2p$, hence all x_i $(i \in \{3,4,5,6\}$ are equal to x_i^p or else one of them is smaller. In the latter case, say if $x_i < x_i^p$, we see immediately that x^p $dom_{\{2,i\}}$ x because $x_2^p + x_i^p = \frac{2p}{1+p} + \frac{1-p}{2} \leq 1$ can be checked. In the first case, if all x_i $(i \in \{3,4,5,6\})$ are equal, replace x^p by $x^{p-\varepsilon}$ for sufficiently small $\varepsilon > 0$ and repeat the argument *mutatis mutandis.*

Similarly, the case that $x_1 < x_1^p$ is dealt with. So it remains to consider the case that $x_j \geq x_j^p$ $(j \in \{1,2\})$. But by the definition of p, this means $x_j = x_j^p$ $(j \in \{1,2\})$. Since $x \neq x^p$ there is $i \in \{3,4,5,6\}$ such that $x_i < x_i^p$. Now replace x^p by $x^{p+\delta}$ for sufficiently small $\delta > 0$ and repeat the previous argument *mutatis mutandis.*

Now to the continuous case. The fact that the embedding $\mathcal{S}$ is a vNM-Stable Set follows from Theorem 5.4 because we are dealing with uniform distribution. **qed.**

Remark 6.3. *For convex polyhedral solutions of finite games with uniform distribution involved, we have the 'general density property' which says that, for every element x of the vNM-Stable Set we have*

$$x_{i_1} + \ldots + x_{i_r} \leq 1 \quad (i_1, \ldots, i_r) \in K^1 \times \ldots \times K^r. \tag{4}$$

Remark 6.4. *For every standard solution $\overset{\circ}{\mathcal{S}}$ the distribution on the short side of the market coincides with the corresponding λ^ρ, that is, everyone obtains utility proportional to the one of his initial endowment (equal treatment) in any element of the standard solution $\overset{\circ}{\mathcal{S}}$. This economically appealing property may fail for nonconvex vNM-Stable Sets, as is shown by the above example.*

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