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COUNTING, CALCULATING, AND SPEAKING BY MENTALLY RETARDED PUPILS

Abstract:

A valid analysis of the individual learner's mathematical thinking is essential for a formulation of an effective approach to learning and teaching. Different aspects of the concept of number, insight into the monetary system, and mastery of basic number combinations were investigated in a three-year longitudinal study that followed two pupils from grade ten to twelve. In addition, in the area of speech, the findings concerning logical-grammatical construction and retrieval of substantives from memory are used as a comparison. It can be shown that the concepts and skills of the two young persons differ sharply from each other and are also in principle different from the thinking of non-retarded pupils. Due to brain dysfunction the intelligent behaviour of the disabled pupils is determined by a particular pattern, which repeatedly appears intra-personally, and is not only reflected in their mathematical thinking, but also in their comprehension and usage of speech.

0. Introduction

The current view of the success of mathematics lessons at schools for mentally retarded is described by the following two quotations. DITTMANN (1982: 283) points out: "The achievements in arithmetic stay nearly completely at a level that does not allow children and youths with Down's syndrome, to make use of the fundamental operations - at the logical-abstract level." Following POHL (1983: 93) this statement can be generalised for "probably all" pupils in schools for the mentally retarded. He continues: "It is common experience that the success of arithmetic lessons is minimal." Accordingly, for mathematics lessons, the listed objectives demand to make accessible the numbers one to ten, at most to twenty, and only in some cases to one hundred or further (REHBEIN, 1979). In two case studies, which are reported here only with few selective examples (for detailed reports see EZAWA, 1996), the special difficulties and strengths concerning the mathematical thinking of two pupils will be investigated and from this we will take a stance on these assumptions.

At first, I will report on a pupil whom I call Anne. She has learned reading, writing and calculating and belongs therefore to the group of pupils with rather mild mental retardation. The findings, which I collect in this first case study, are compared with those of another, also mildly mentally retarded pupil, namely Chris.

1. Anne

Due to cerebral dysfunction as a result of prenatal disorders, she is simultaneously mentally retarded, and physically and speech impaired. She uses a wheelchair; she sees and hears well. The following examples were collected between her seventeenth and nineteenth years of age, when she attended the tenth to twelfth grades; mostly tape-recording was used.

1.1 Counting

In one of my first lessons in her class, Anne did addition and subtraction problems in her notebook. Amongst a large number of correct results I found one mistake:

Calculation in the notebook; 10th grade

66 + 6 = 72	Anne adds and subtracts by counting, partly
81 - 2 = 79	using her fingers;
74 - 4 = 60	subtracts 4 and 10

A couple of weeks later I observed the following result:

Counting up coins; 10th grade

1 DM + 50 Pf =

Anne: 61 Marks	divergence in the assignment to Marks and
Teacher: Why is it 61 Marks?	pennies;
Anne: because 50 and 1 makes 61	adds 10 and 1

The correct solutions show, at first, that Anne can interpret the mathematical symbols correctly, i. e. that she adds and subtracts on the appropriate occasion. In her time at school she also learned to say the number sequence in both directions and to interrupt it at any point. (*breakable*, *bi-directional chain*, FUSON, 1992: 249). Moreover, here she does not count objects, but the number words themselves. With larger addends or subtrahends, she uses her fingers to keep track of the counting accurately. Her procedural knowledge, i. e. her skills in this domain are fairly well developed. She could construct a concept of the sequence of numbers up to one hundred, which, as other examples show, she even could extend to numbers above one hundred.

As the cause of both divergent solutions, the investigation reveals an unconventional number sequence which she uses consistently, e. g.:

68 69 60 71 72 73 74

Anne unusually constructs the series of the ones in the normal way, and also the transition to ten and twenty. When counting over the tens from thirty or forty onwards, and also when she counts over the hundreds from two hundred onwards, she inserts the previous tens respectively the previous hundreds. Intervention in the classroom has the effect that Anne can gradually use the conventional number sequence more often when counting forwards, but not when counting backwards. This divergence can still be seen in the eleventh and twelfth grades.

Counting backwards from 25 and 53; 11th grade

Anne: 25 . 24 . 23 . 22 . 21 . 20 . . . [1]	ends with the correct ten
Anne: 53 . . 53 . 52 . 51 . 40 . . .	subtracts 1 and 10

Counting backwards from 100 and 605; 12th grade

Anne: 99 . 98 . 97 . 96 . 95 . 94 . 94 . . 93 . 92 . 91 . 80.	subtracts 1 and 10
Anne: 605 604 603 hm . 602 601 500 hm	subtracts 1 and 100

The transition to the first two tens is always correct which means that the numbers up to twenty are treated differently from the larger ones. BAROODY (1990: 282) points out that they are more concrete. In addition the numbers up to about thirteen are memorised in a rote fashion. That is probably also true for those to about twenty, which most of the children already know when they start school (SCHMIDT/WEISER, 1982), that means, before they get systematical instruction on the place-value numeration system. Also linguistically, the first number words are own forms, which, different from multidigit numbers (e. g. twenty-three, three hun-

[1] . short stop of about one sec.; . . stop of about 2 secs; . . . stop of more than 2 secs.

dred), stand unrelatedly next to one another. Three and four for instance have linguistically nothing in common.

To understand our number system, children have to master the place-value concept. It combines the number words into a *higher* unit, the number-word sequence. When counting from ten on, and using two-digit numerals, the number sequence is no longer a simple sequence where number words are placed merely adjacent one after the other as it is for the ones. In learning to count, children do not only learn the *succession of* number words, but rather they have to conceive and construct their *hierarchical* system. The concept of the conventional base-ten place-value system develops in children only gradually, beginning with the interpretation of a multidigit numeral as the whole number it represents, through the simple distinguishing of *tens* and *ones,* to the use of the ten-for-one trade rule (e. g. RESNICK, 1983: 125 ff.; COBB/WHEATLEY, 1988; ROSS, 1990; FUSON, 1992). When Anne ends the series of the ones *68, 69* with *60* and then continues with the new series 71, 72, she uses her own rule. With this construction she does not only create a different order of the numbers, but also a different meaning of the tens and of the relationship between the tens and the ones. It is based on the assumption, that one ten consists of ten ones, 9 + 1 ones are a ten. Thus Anne replaces the conventional number sequence with its defining features, the iterative combination of single elements into a hierarchical organised sequence, by a string with a merely concatenated structure.

1.2 Reading, writing, and magnitude comparison of multidigit numerals

The number words themselves are composed additively of multiples of units which are again hierarchically ordered: ones, tens, hundreds and so on. In numeral-reading and -writing as well as in magnitude comparison, Anne has special difficulties which correspond to the findings described above. She learned reading and writing multidigit numerals slowly. For three-digit numbers in the tenth grade she still wrote down the place-name designation used in speech:

Writing of numerals after dictation; addition; 10th grade

1005 instead of 105	writes the place-name designation *hundred*
40020 instead of 420	
2301 instead of 231	writes the place-name designation -*ty*
200 + 1 = 2010	in addition the two last digits are reversed

In the eleventh grade Anne can transfer the named values of spoken three-digit numbers into the positional system of the written figures, but she cannot generalise the rules for four-digit numerals she had learned up to this point.

Reading and magnitude comparison of numerals; 12th grade

Problem	*Read*	*Comparison*
199 - 236	correct	correct
307 - 703	correct	correct
459 - 453	correct	incorrect
761 - 729	correct	correct
510 - 129	correct	incorrect, not corrected when re-examined with the help of base-ten blocks
631 - 816	incorrect	correct
201 - 184	correct	correct

In German the ones are always spelled before the tens. This reverse of the order of the tens and ones when reading and writing numerals still causes lots of difficulties for Anne in the tenth grade (see 200 + 1 = 2010). However, in the twelfth grade, she is usually successful, as the examples show. When making mistakes, she can now correct herself:

631 - 816

Anne (*reads*): 631 and 861 Teacher: Could you read them once more? Anne: 816 and 631	reads first numeral correctly, reverses the tens and ones in the second, corrects herself on request

Special difficulties occur when Anne counts over a hundred and reads and writes the following numbers where zero is used as a place holder for the tens. These numbers are in general less well-known and are learned later than those actually consisting of hundreds, tens, and ones:

Dictation of numbers; 12th grade; 98 to 104

Anne (*writes*): 98, 99, (90)[2] , 100, 101, (12)[2] 120, 130, 140 Anne (*reads*): 98, 99, 100, 101, 102, 103, 104	reverses tens and ones when reading and writing

Here she uses the rule which is correct for two-digit numerals, namely, that for the last two digits the order is changed, which is incorrect for numerals consisting of hundred and ones.

Ordering numbers according to magnitude; 12th grade

As the above examples show, Anne can compare three-digit numbers in many, but not in all cases correctly.

459 - 453

Anne: four . hundred fifty three . four hundred fifty nine Teacher: And who has more? Anne: hm . . that is hard to say . . that is the same . that is the same	reads numerals correctly, thinks numbers to be equal, does not pay attention to the ones

According to SINCLAIR/SCHEUER (1993) often even six-year-olds can decide correctly over more or less when given two- and more-digit written numbers. However, brain-injured adults can, on account of certain lesions, lose the ability to evaluate the magnitude of numbers (see MOSES, 1984: 8; LURIA, 1970: 200). They decide only referring to the constitutive components, the digits, and are no longer able to refer to the content.

In fact, with multidigit numerals, it is in principle possible to decide on more or less without recourse to the content, and when base-ten blocks are used, without taking into consideration the total quantity of the cubes. Instead, the comparison can be executed with the help of an algorithm, that is to say, with the help of a sequential system of rules (see also DEHAENE, 1992; DELOCHE/SERON, 1987). When doing such work, the corresponding digits and multiple units are compared step by step from left to right and the decision is made according to the first occurring difference. Anne proceeds appropriately according to the described method, but often in an incomplete manner. As other examples show (see EZAWA, 1996), she can find the corresponding digits, she always assigns the appropriate names as well, *hundreds*, *tens* or *ones*, but she is not always successful in finding the critical digit.

Thus she interprets the three-digit numbers as single-digit numbers placed adjacently to each other. They are merely concatenated sequentially, without referring simultaneously to the un-

[2] () corrects herself.

derlying relationships and meaning of the whole number. She even treats base-ten blocks in the same way, too. Here as well she cannot grasp the whole situation. Therefore, the embodiment does not lead to a different solution and even when using concrete material exactly the same errors occur.

Each number of the base-ten place-value system can be broken down into powers of ten. For two-digit numerals one gets:
$a = q_2 \cdot T + q_1 \cdot O \ , \ 0 \le q_m < 10$ [3]
where the units *ten* and *one* stand in certain, mathematically defined relationships to each other:
$T = 10 \cdot O$ and therefore
$a = q_2 \cdot 10 \cdot O + q_1 \cdot O$
Formally Anne's construction resembles this construction
$a = q_2 \cdot U_2 + q_1 \cdot U_1 \ , \ 0 \le q_m < 10$
where the U_n represent the units, with relationships to each other which are yet not defined. That is to say: even in the construction of the number words itself, Anne's system differs from the conventional one. She replaces the hierarchical rules, without relying on the content, again by a mere concatenation of the elements. Rules are over-generalised, and used incorrectly.

1.3 Money

A diverging pattern of that kind can also be found elsewhere in Anne's mathematical thinking and especially affects the dealings with measurement. This is particularly important for everyday life, as for instance calculating with money. Examples from the 10th and 11th grade:

Comparison of amounts of money
0.76 2.51 11.00 4.15 0.19 1.01
Problem: Which is the smallest amount?
Anne: *eine Mark eins . weil . das ist weniger . das ist Einer und Einer*
(one Mark one . because . that is less . that is one and one)
To choose the most expensive from a list: 2.19 1.79 1.19 3.58 0.78
Anne chooses 1.79

Assignment of Mark and penny

See the above example: *1 DM + 50 Pf* = 61 Mark and the omission of the assignment of the units Mark and penny in most of the examples.

Assignment of place values

Anne counts up: 50 Pf + 10 Pf + 2 Pf = 6.2

Understanding the relative value of the coins

In the tenth grade Anne was able to learn the *relative value* of coins, e. g. she knows then that a 10 Pf coin is worth more than a 5 Pf coin. She also understands the *unit value* that a 10 Pf coin is worth the same as ten single pennies. In the eleventh grade she can add up two 50 Pf coins to one Mark, but in spite of that she does not know the *equivalence*, that one Mark may be replaced by two 50 Pf coins.

These different examples show Anne's special difficulties in grasping the *hierarchical system* of money with its firmly defined relationships existing between the units Mark and penny.

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[3] T tens; O ones; U units.

1.4 Addition and Subtraction

Anne has a well developed understanding of the operations addition and subtraction, even with three-digit numbers. To solve problems, she uses different strategies, e. g. retrieval from memory and counting. She also knows and applies patterns and relationships to the number combinations, especially commutativity and the N + 1 rule. She can also apply the addition-subtraction complement principle which means she comprehends the part-whole schema, saying that the whole is the sum of its parts (RESNICK, 1983).

Retrieval from memory is rather stable for the doubles up to seven; the number combinations with solutions up to about eleven; problems with zero as addend or subtrahend and combinations of tens and ones. Anne also adds and subtracts pure multiples of hundreds or tens and she solves easily other problems where the solution can be found in analogy to known single-digit number combinations and without explicit knowledge of place-value. The key concepts and strategies can be used with numbers up to one thousand, but calculation with single-digit numbers is by far better developed than that with multidigit numbers. Especially when handling numbers over twenty, the number of correct solutions goes down rapidly. She manages the great number of components she has to deal with simultaneously and their hierarchical order only in a very limited way.

1.5 Speaking

A pronounced impairment of speech is characterised by non-fluent verbal utterances and reduced intonation. Anne's understanding is less limited as far as it concerns sounds and words. However, it is restricted quite obviously when dealing with complex facts and logical-grammatical constructions depending on a system of components which refer to one another in a particular manner, as for instance in the following task (BLN-K, 71):

Showing objects; 12th grade

Show the pencil with the key. Show the key with the pencil.	Anne shows on both occasions both objects.

Thus the reduction of the intonation can be explained, as it organises word-overlapping connections and transmits the communicative intention of the speaker with the help of stress, pause and pitch. When composing words or sentences, Anne's constructions are diminished. In speaking spontaneously, words endowed with purely grammatical functions, such as conjunctions, prepositions, pronouns, and articles, are missing. Their function is to convey the relationships of speaker, listener and object and of the different objects the sentence deals with.

Repeat, H-S-E-T, subtest 3

Es sitzt der kleine Vogel im Gebüsch.	Anne: *Der Vogel sitzt im Gebüsch.*
(There is sitting the small bird in the bush.)	(The bird is sitting in the bush.)
Die Tante, die weit weg wohnt, kommt zu Besuch.	Anne: *Die Tante wohnt weit weg.*
(The aunt, who lives far away, is coming to visit us.)	(The aunt lives far away.)

Correspondingly she manages here to arrange the components simply one after the other, but she cannot repeat the construction with *es* serving as place-holder and the subordination of the different parts of the sentence. The elements of her sentences are predominantly ordered sequentially and not hierarchically (JAKOBSON, 1971).

To sum up, it can be said that Anne learns knowledge only tediously which represents a *complete system of knowledge* (LURIA, 1977: 42), a structured, hierarchically organised whole. This was shown here for the number sequence, numbers, money and sentences. Yet, in the selective retrieval of the number combinations from memory, she is more successful. Thus the solution 74 – 4 = 60 which was observed in the classroom was no coincidence, but an expression of her diverging thinking. Its predominant feature is the use of mere concatenation instead of hierarchical order, i. e. the sequential instead of the simultaneous mode.

2. Chris

Chris shows severe motor deficiencies because of cerebral dysfunction from birth onwards: he can neither sit on his own nor write by hand. He hears well, but his sight is severely limited. He has learned very little reading and writing. At the time of the investigation, he was like Anne, between sixteen and eighteen years old, and in his tenth to twelfth grade.

Because of his severe motor disabilities Chris often has considerable pain. Therefore it was not possible to exam his mathematical thinking exhaustively in individual sittings. Instead his utterances in the classroom lessons were noted. At the request of his classmates, no tapes were made.

2.1 Counting and Calculating

Up to his seventeenth year Chris mostly practised addition and subtraction of single-digit combinations in his maths classes, primarily with results under ten, with attempts to proceed to twenty. According to the curriculum for German elementary schools, the plan was to make numbers over twenty accessible only after the stabilization of the basic number combinations.

However, when he was sixteen, in his tenth grade, the curriculum was changed. The focus in arithmetic was now on the learning of two- and three-digit numbers. Chris learned them within one year.

Oral counting up to one thousand; end of grade 10; teaching diary

No mistake in oral counting by Chris, correct when going to tens, hundreds and when going from 109 to 111. Only once a slight hesitation when counting backwards: 701, 700, 699.

Mental addition up to twenty and for the first time also further; end of grade 10

2 + 3 = : 9	incorrect
5 + 4 = : 10	incorrect
Teacher: And 5 + 5?	
Chris: No, is not possible, 5 + 4 = 9.	correct; uses relationship between number combinations
The doubles with solution up to ten are all correct.	
9 + 8 = : 19	incorrect
6 + 7 = :	no solution
5 + 10 = : 15	correct
7 + 40 = : 47	correct
47 + 20 = : 67	correct; needs a lot of time
3 + 100 = : 103	correct
103 + 200 = : 303	correct; is very happy

Chris’ great difficulties with single-digit combinations did not change, even with solutions up to ten. However, he can solve problems with the help of his concept of the number system, i. e. with the help of his knowledge of place-value, which is fairly highly developed. He noticed this difference himself. He said that it seemed to him curious that he was unable to solve simple problems but had no difficulties with the big ones.

Mental calculation*; *11 grade

41 + 10 = : 500, nonsense, 52	incorrect
32 + 10 = : 42	correct
45 + 10 = : 55	correct
21 - 10 = : 19, don’t know	no solution; needs a lot of time
42 - 10 = : 32	correct
35 - 10 = : 25	correct
21 - 10 = : don’t know	no solution; needs a lot of time
24 + 10 = : 34	correct
61 + 10 = : 71	correct
21 - 10 = : don’t know	no solution; needs a lot of time
82 + 10 = : 92	correct
41 - 10 = : 31	correct
55 - 10 = : 45	correct
21 - 10 = :	no solution

Even here, nearly half a year later, it can be seen that Chris is able to solve problems most correctly when he can find the result with the help of his place-value concept. Moreover, he manages the shift from addition to subtraction. Yet, for numbers up to just over twenty, where the place-value concept is hardly present, Chris cannot find a solution.

His answer to the question on the quantity of numbers was then: ”It does not stop, because if you take a number, you can always add one more”. Without being asked, he also pointed out that out of two different numbers one is always bigger and the other one smaller. That means he is conscious of the order in the set of natural numbers.

2.2 Money

There were some tenth grade lessons on coin recognition. That was extremely difficult for Chris because of his disability, his limitations in grasping feeling and also seeing. In eleventh grade the lessons were continued with counting out sums of money. The receipts from shopping trips we took together were used in the classroom. They were read, the amounts were paid with real money, the prices were compared and so on.

After a short introduction Chris makes no mistakes in

– coin recognition
– understanding the relative value of the coins and amounts, i. e. deciding on the one hand over *more* or *less*, and on the other hand recognising a specific equivalence, i. e. amount which has been made up with different combinations of coins (e. g. a 10 Pf coin is worth the same as two 5 Pf coins or five 2 Pf coins)
– assigning Marks and pennies to place-values and units
– counting out pieces of money
– reading and writing amounts
– paying amounts.

In his eleventh grade, Chris manages even simple adding up problems (*having 13.90 you need another 10 Pf to get 14 DM)*, and also simple rough estimation (*0,118 kg that's about 100 g; if one kilo costs 16.80 you have to pay about 1.70*).

All together he grasped the concept of the named values of spoken multidigit numbers and the positional base-ten system of the written numbers, as well as the decimal system of money within a rather short time. His understanding of the hierarchical relationships existing between units and digits, number words and the value of coins and amounts is highly developed.

2.3 Speaking

His family and his educators do not find Chris to be speech disabled. There are no syntactical divergences, and relationships between elements are expressed appropriately, e. g. by inflection or prepositions (father's brother, the pencil beside the key). However, in general, it is conspicuous that he often does not come to the point, *he beats around the bush.*

Explaining words (BLN-K, 65); 11th grade; tape-recording
Katze, Fratze, Tatze (cat, grimace, paw)
Chris: also . (?)[4] bei Katze . das ist ein Raubtier . aber auch ein ehm . . und die Fratze . die man (?) praktisch . . ehm wenn man jemandem ein komisches Gesicht macht das ist dann eine Fratze . und dann ehm . (Lautsprecher) und unter der Tatze versteht man . ehm . . ehm eine Tatze ist praktisch ist praktisch der Fuß von ner Katze
(well . (?) for cat . that is a predator . but also an hm . . and grimace . which (?) one practically . . hm if you make an odd face to somebody that is a grimace . and then hm . (loudspeaker) and for a paw one understands practically that is practically the foot of a cat)

the substantive, domestic animal, lacks; each time several attempts to find the noun; sophisticated syntax, complex sentence

This test shows that Chris has acquired a well developed syntax, the rule-system of the language, but the retrieval of words, especially of substantives, causes difficulties for him. Substantives (lat. substantivus, to exist on its own) are the most distinctive class of words, consisting of those which are the least redundant, which rely the least on context. As opposed to this, other words, for instance articles and prepositions, refer much more to context, i. e. the given situation. Yet the most distinctive of the number words are the ones. The multidigit numbers by comparison are connected with each other by diverse relationships, bound into the context of the system of numbers, e. g. forty-three, hundred and three, or three hundred.

3. Conclusions

3.1 The thinking of disabled pupils is characterised by specific patterns, which are intrapersonally consistent.

In a similar way, each of the pupils performs according to a specific pattern when mastering the number concept and the monetary system, addition and subtraction problems, and also when speaking. Their ability to gain insight into the system of number corresponds to the observed development of the syntactical system when speaking, as does the voluntary and selective retrieval of number combinations to the recall of nouns from memory. For each of them

[4] (?) indistinct word, might be hawking.

speaking and mathematical thinking as behaviour form a functional entity. They both adhere to the same rules, because they depend on the same mental activities each with characteristic strengths and weaknesses.

Given the current view that brain dysfunction is a concomitant of mental retardation, it stands to reason to expect neurostructural impairment to be associated with mentally retarded people. This leads to a dysfunction of diverse components of behaviour in a way that, so to say, one-sided persons develop. Therefore, it cannot be assumed that ”gaps, holes or aberrations in the course of development” (MELJAC 1992: 325), occurred, but rather a deep change in the perception and the processing of information. It is not about an accidental assemblage of deficits, but of an internally coherent combination of manners of behaviour.

This homogeneity of thinking makes diagnosis much easier. In particular, by observing speech, which depends less on classroom instruction than mathematical thinking, conclusions can be drawn about underlying processes of counting and calculating. Problem-solving behaviour for novel situations becomes predictable, and more appropriate help can be offered. It can, therefore, be supposed that Chris, with his well developed ability to grasp a system and to use relationships, might be able to learn the basic number combinations more easily if the lesson relied more on structured exercises instead of formal drill. And for Anne, a better use of her well developed sequential performing might help her solve problems, like the comparison of numbers.

3.2 The mathematical thinking of different pupils is interpersonally divergent.

In all domains of the investigated behaviour, there are clear differences between both pupils. Their thinking cannot be characterised by a unity of features or attributes, or just as being underdeveloped, and even less so as *the* thinking of mentally retarded pupils. Instead, it should be explained along with individual features, i. e. with the specific strengths and weaknesses of behaviour in combination with an intelligence which is, in general, affected.

Many investigations of grown-ups as well as of children with brain dysfunction were performed in the last hundred years. They reveal that it is, in principle, possible to distinguish different patterns of thinking, different *syndromes.* Two groups or subtypes of pupils with impaired arithmetic performances have also been found by ROURKE and his collaborators (e. g. ROURKE, 1982; HARNADEK/ROURKE, 1994). The first group exhibits profound difficulties in reading and writing and also in auditory perception, recall of information and word definitions (group R-S, reading & spelling syndrome); the other difficulties in visual-spatial-organizational and psychomotor skills, but better performance in the area of speech (group NLD, nonverbal learning disabilities syndrome). Their profile of performance is seen as analogous to the syndrome first described by GERSTMANN (1927). However, the linguistic manifestations of the NLD-type are recently characterised by HARNADEK/ROURKE (1994: 146) as ”of a repetitive, straightforward rote nature” with ”poor psycholinguistic pragmatics”… ”misspellings almost exclusively of the phonetically accurate variety” and ”little or no speech prosody”. That means these pupils are thoroughly limited, with primarily sequential processing and other symptoms as they occur in efferent motor aphasia (see LURIA, 1970).

Referring to the prevalence of the different patterns of arithmetic disabilities VON ASTER (1992: 158) states that the type with predominantly sequential performance, as in the first case report, is by far the more seldom. This corresponds to my findings with the pupils I educated at a school for the physically impaired. For pupils with concurrent mild mental retardation or rather severe learning disability, I observed impairment of hierarchical thinking, as in the case of Anne, in about one quarter of them. For the others, there were different patterns of disabilities with limitations of selective processing, as was shown here in the second case study.

Distinguishing different subtypes of intellectual, especially arithmetical behaviour on account of the underlying mental processes, implies the possibility of developing specific models for the teaching of mathematics for both groups. Thus, tested concepts can be used for other pupils with similar patterns of thinking.

3.3 The mathematical thinking of pupils with disability differs from that of pupils who are not disabled.

People with mental retardation think differently. As research in our century, especially by STERN (1928) and Piaget, has shown, children are not small grown-ups, they think in another way. Moreover, the thinking of pupils with mental retardation cannot be equated with that of children, and also not described in terms of Piagetian stages (see INHELDER, 1963; SCHMITZ, 1989). On the whole, to be different can no longer be understood as not performing according to a standard or diverging from certain norms, but as an expression of *own* rules.

In classes for the mentally retarded one cannot find one's bearings by teaching subjects as they are meant for children, just distributing the content of the curriculum at elementary school over the school time. Instead, the instructional methods should be consistent with the ways in which the pupils with disabilities think and learn, and the school mathematics curriculum should be consistent with their needs. That is to say, unique concepts have to be developed based not only on their individual state of knowledge, but also on the underlying processes of thinking.

Here, it is especially important to use and to extend the better developed skills and abilities. Particularly, most pupils with mental retardation can easily gain insight into the system of numbers and measurement, as well as into spatial problems. All together their conceptual and procedural thinking is quite highly developed. However, others encounter difficulties just in this area and therefore need special help, consistent with their divergent thinking. Therefore it does not make sense to just restrict teaching to the basic number combinations and the small numbers, or to the learning of numerical skills. Other subject-matters can and should be introduced quite early. There is need for particular guidelines that help our instructional decision making for pupils with different kinds of disabilities. Further research has to create the required foundations.

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