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SPATIAL GEOMETRY AND SPATIAL ABILITY - HOW TO MAKE SOLID GEOMETRY SOLID?

Abstract:

The guidelines for geometry teaching demand training in spatial abilities to be a principal aim of geometry teaching. Unfortunately, daily teaching very often does not reflect this postulated aim. The major aim of geometry didactics must be to diminish this discrepancy. Experimental activities with a modular construction system should be the innovative guideline for geometry teaching.

Therefore, a new modular construction system is being introduced in this article. There is a description of how the system can easily be produced and handled by the students. The modular construction system allows the production of translucent and aesthetical polyhedra, which are both solid and skeleton models. Finally, there are concrete suggestions how the five elements of spatial ability can effectively be taught by means of the polyhedra.

Spatial ability in maths teaching

Spatial ability is a human qualification that is relevant to a high degree to our lives. Several studies (ETHINGTON & WOLFE 1984; GALLAGHER 1989, TARTRE 1990) and a meta-analysis (KLEIME 1986), show, that in school spatial skills can be used in specific ways for many mathematical tasks. Obviously spatial abilities are used in a wider range than just for solving geometrical exercises. They are also of importance in therapies for dyscalculia (LORENZ 1991) and dyslexia (CRANO & JOHNSON 1991). Even in some other subjects, e.g. chemistry (PRIBYL & BODNER 1987), biology (LORD 1990) and physical education (MEEKER 1991), success bases fundamentally on spatial abilities. As well as in school we also profit from a well developed spatial ability in professional life (STUMPF & FAY 1983). We also need it in daily life to make use of the increasing mobility of our modern life.

Thus, in many countries the development of spatial ability is a major aim in many guidelines for geometry teaching (CLEMENTS & BATTISTA 1992). Although they exist, the curricula unfortunately do not come up to their own guidelines. The topics to geometrical teaching extraordinarily lag behind their own guidelines.

In many curricula, the specific training of spatial ability is of little importance. The main emphasis is put on 2D-geometry, while 3D-geometry rather stays in the background. On the one hand, the teaching of 3D-shapes and drawing in 3D-geometry have been neglected for decades. On the other hand, stereometrical calculations dominate 3D-geometric acitivties. Spatial-visual skills are frequently and extensively avoided, the way that geometrical polyhedra are already projected in oblique parallel perspective. For calculation rectangular triangles are often integrated into these drawings. Thus, the student has only to copy the formula, fill in the measurements and, if necessary, rearrange the formula for calculation. This is a very unsatisfying characteristic of today's geometrical teaching. So the major criticism leads to the following initial thesis:

> *Spatial geometry is still not more than learning mathematical vocabulary, arithmetic and algebra* (ANDELFINGER 1988). Therefore, space geometry education has to be fundamentally reformed.

Elements of spatial ability

Since 1938, there has been a large number of theories of intelligence, which distinguish between different aspects of human intelligence, for example linguistical and reasoning aspects. Many theories consider the aspect of spatial ability as one of the most important. GARDNER (1991), for example, who supports a very comprehensive theory of human intelligence, states that in our competitive society spatial intelligence is invaluable.

Many researchers have proved that spatial ability is very complex. Thus spatial ability is usually divided into three elements (THURSTONE 1950; MICHAEL et al 1957; LINN & PETERSEN 1985). But detailed knowledge of spatial abilities, also for understanding gender differences, needs a specification into more than three elements. So it becomes obvious that some researchers subsume different tasks under equally named elements. Therefore there were many evident misunderstandings and seeming contradictions (MAIER 1994; 1996a). That is why MAIER in his papers distinguishes *five elements of spatial intelligence* as described below, which are based on several theories of intelligence, meta-analyses and a number of studies of spatial ability. In the area of maths didactics this differentiating approach provides valuable insights when analyzing curricula. It was shown that in many grades most of the elements occur hardly or not at all (MAIER 1994). But cultural requirements necessarily demand the training of these five elements because in today's technological world we are faced with varied spatial-visual problems. The relevance of spatial ability leads to the following thesis as the approach for teaching:

> *Based on psychological research findings, all five elements of spatial ability have to be specifically trained.* The present conception of teaching space geometry does not come up to its cultural expectations.

Description of the elements

The first element, the so-called *spatial perception*, must not be mixed up with the identically named preliminary stage in the acquisition of spatial abilities. Spatial perception tests require the location of the horizontal or the vertical in spite of distracting information. Verticality is measured, for example, by the „rod and-frame test“ that asks persons to place a rod vertically while viewing a frame oriented at 22 degrees. Here the person's own spatial position is an essential part of the task. Horizontality is measured by „water-level tasks“, that ask persons to draw or identify a horizontal line into a tilted glass. In these tasks, the person's own spatial position is not part of the problem, because the person is outside the situation. The tasks described above all require *static* mental processes. Static mental processes mean that the relation of the subject to the objects changes, but the spatial relations between the objects themselves do not change.

Figure 1 (exercise 1) is a typical „water-level task“ The left picture shows a glass half full of water. Which of the other four glasses show the correct water level with the same filling?

The second element, the so called *visualisation*, comprises the ability to visualise a configuration in which there is movement or displacement among (internal) parts of the configuration. For example, solids are being intersected by a plane, or drawings in oblique parallel persepective are being compared with their nets. These types of tasks require mainly *dynamic* mental processes, which means that spatial relations between the objects are changed. The person's own spatial position is not part of the task. In exercise 2 (figure 1) those nets are to be identified which form the pyramid (regular tetrahedron) shown on the left side.

Another important element, *mental rotation* involves the ability to rapidly and accurately rotate a 2D- or 3D-figure. Nowadays this ability becomes more and more important, because many people work with different graphics software. Just as visualization, these mental processes are mainly *dynamic*. Also the person's own spatial position is not part of the task.

Exercise 3 (figure 1) is a typical psychological test (VANDENBERG 1971). Which of the four responses are identical with the standard figure on the left?

Spatial relations means the ability to comprehend the spatial configuration of objects or parts of an object and their relation to each other. For example, a person has to recognise the identity of an object that is shown in different positions. In contrast to mental rotation, the mental processes of spatial relations are *static*. Nevertheless, the person’s own spatial position is as well an essential part of the problem.

Exercise 4 (figure 1) shows cubes with a different design on each face. The person has to find out, whether the drawings could represent the cube of the standard figure or if they necessarily represent a cube other than in the standard figure.

<table>
<tr><th>spatial position of test person</th><th>dynamic mental processes</th><th colspan="2">static mental processes</th></tr>
<tr><td rowspan="2">outside position</td><td>visualization</td><td colspan="2">spatial relations</td></tr>
<tr><td colspan="2">mental rotation</td><td rowspan="2">spatial perception</td></tr>
<tr><td>inside position</td><td colspan="2">spatial orientation</td></tr>
</table>

Tasks of *spatial orientation* require a person’s own orientation in any particular spatial situation. Spatial orientation is the ability to orient oneself physically or mentally in space. Therefore, the person’s own spatial position is

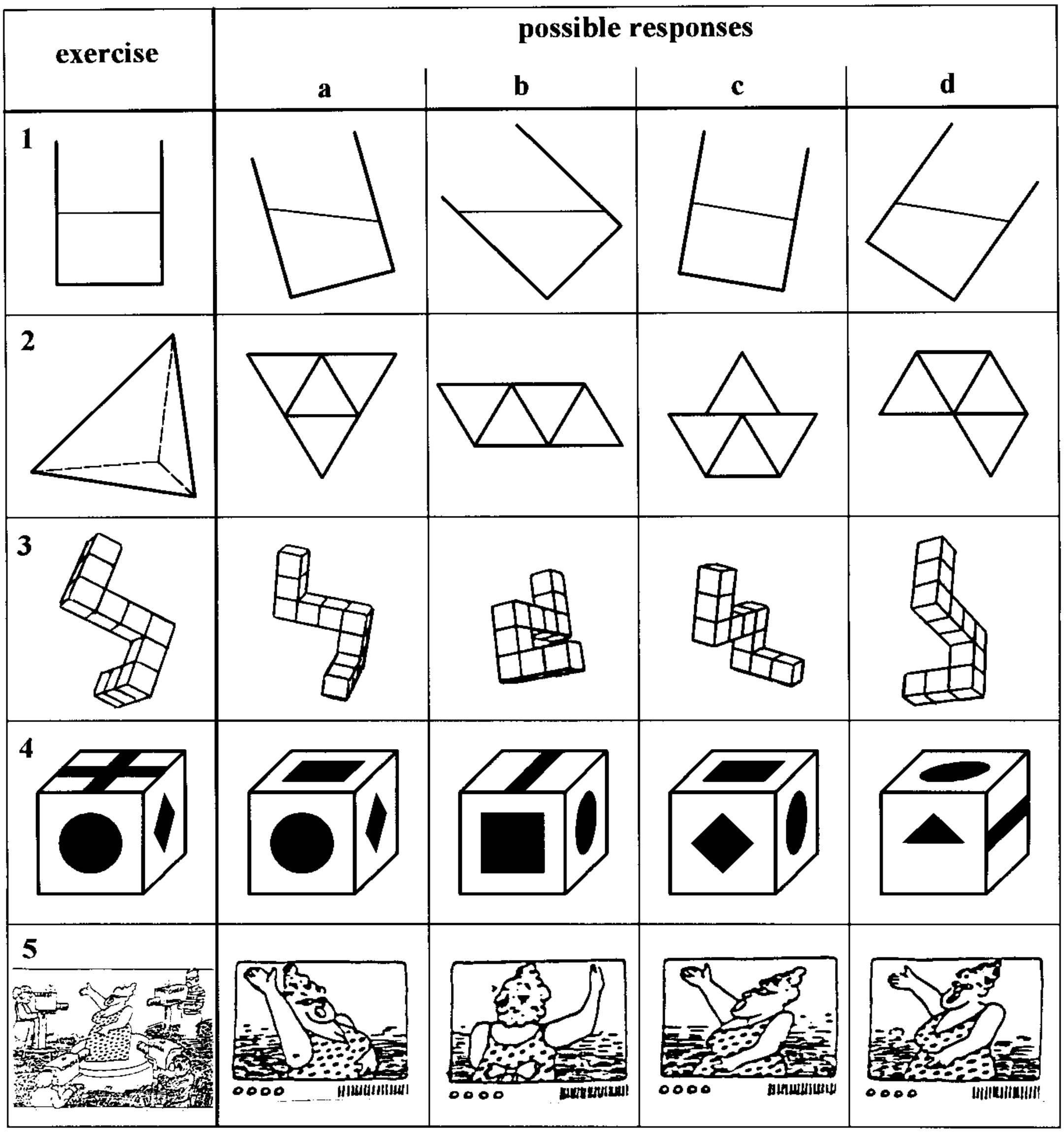

Figure 1: exercises of the five elements of spatial ability

necessarily an essential part of the task. The mental processes are mainly dynamic.

DE LANGE (1984) gives an example for a task for children of about ten years of age (exercise 5, figure 1). The question is: „Which camera displays the picture shown?“

Table 1 is a survey of the elements of spatial ability. Please notice that one cannot distinguish strictly between the five elements (MAIER 1994): In reality there are interrelations between these elements which is shown in the suggestions for teaching described below.

Table 1: survey of the elements of spatial ability

Innovative guideline for geometry teaching

Spatial ability is without a doubt one of the best investigated aspects of human intelligence. During the last 15 years international research on spatial ability has increased, also in terms of gender differences. Recently, extensive results of great importance for maths didactics have been published. Many studies undoubtedly prove that spatial ability of persons of different age can be trained (BAENNINGER & NEWCOMBE 1989; BATTISTA, WHEATLEY & TALSMA 1982; BEN-CHAIM, LAPPAN & HOUANG 1985; 1989; EMBRETSON 1987; and additional studies in MAIER 1994)

Some earlier studies describe ineffective training methods and therefore do *not* prove a significant improvement of spatial ability (BROWN 1954; MYERS 1958; SEDGWICK 1961; SERPELL & DEREGOWSKI 1972; THOMAS, JAIMISON & HIUMME 1973; and further studies in MAIER 1994). Nowadays we know, for example, that formal proofs should not prevail. Furthermore, training methods with tasks that can be solved just by means of logical reasoning are not successful. Neither the mechanical drawing nor the mere naming of new geometrical shapes should dominate over the learning of the spatial properties. Furthermore, the main emphasis should not be put on numerical measures and operations, while the aspects of teaching of 3D-shapes and drawing are neglected. Additionally, the period of time for spatial training should be longer than just a few hours, and 2D-geometry must not prevail over 3D-geometry.

Not surprisingly, these former methods could not improve the performance in spatial ability. Nevertheless, geometry teaching still makes use of these unsuccessful methods! In spite of our didactic efforts we are obviously not able to learn from these faults.

Most training approaches are still relying only on visual models. There are only a few studies of training methods that are based on work with physical models to improve spatial skills. But especially action oriented training methods that work with real models have always shown good or even very good results in the improvement of spatial ability. Hence, experimental activities with solid models are extremely successful. In my opinion, the maths education community should fully recognise the desirability of such a spatial training. This is exactly the way maths didactics should work to achieve the improvement of spatial ability in geometry teaching! Stereometrical calculation must no longer be in the centre of teaching. We need a new structure of spatial geometry teaching, a structure that is based on recent psychological knowledge.

> *Innovative guideline for geometry education: The teaching of 3D-shapes must be in the centre of geometry teaching.* This is the way genuine geometry problems should be dealt with.

Activities with 3D-shapes are exclusively preparatory dealt with. They are almost only found in primary school didactics. As a consequence, there is an enormous lack of polyhedra in secondary school didactics today - in most cases only prisms and pyramids occur. Thus, by this restriction a great didactical potential is being wasted.

One example with a valuable didactic potential are the convex face-regular polyhedra, with regular polygons as faces. Best known are the regular polyhedra, also known as Platonic solids. A polyhedron is regular if it has only one type of regular polygon as its faces, and if all its vertices are congruent. There are only five regular polyhedra: The tetrahedron (4 equilateral triangles), octahedron (8 equilateral triangles), icosahedron (20 equilateral triangles), cube (6 squares) and the dodecahedron (12 equilateral pentagons; see figure 2).

Figure 2: dodecahedron

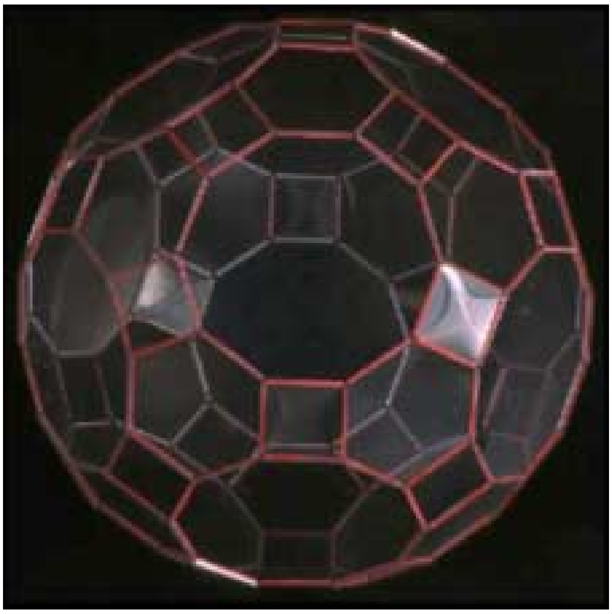

Figure 3: great rhombicosidodecahedron

Modifications of Platonic solids lead to the Archimedean polyhedra. Today thirteen (according to definition fourteen or even sixteen) of these semi-regular polyhedra are known (for example figure 3: great rhombicosidodecahedron). Their faces are all regular polygons, but they are of two or more kinds, and their vertices are identical. Furthermore it has been proved, that apart from the regular and semi-regular polyhedra and the regular prisms and anti-prisms, there are 92 convex polyhedra with regular faces (ZALGALLER 1969). Among these are the eight convex polyhedra, where faces are all equilateral triangles. They are called convex deltahedra; three of the Platonic solids mentioned above also belong to this group. Many of these solids are of fascinating beauty and have occupied outstanding thinkers through all centuries.

The Modular Construction System

We now know that spatial ability can be trained. The training with polyhedra asks for a suitable medium that facilitates real spatial visual perceptions and is a basis for the acquisition of spatial abilities. One has to be able literally to grasp this medium. Abstract perceptions that are conveyed by computer programs, for example, are inappropriate at this early stage of learning.

For the purpose of a successful realisation of psychological insights a modular construction system for students has been developed. Therefore, the easy system of joining polygons with rubber bands to make polyhedra (e.g. CAMPBELL 1983) has been substantially improved. A polyhedron is constructed by laying prepared panels side-by-side and by placing rubber bands over the folds of all adjacent panels (figure 4).

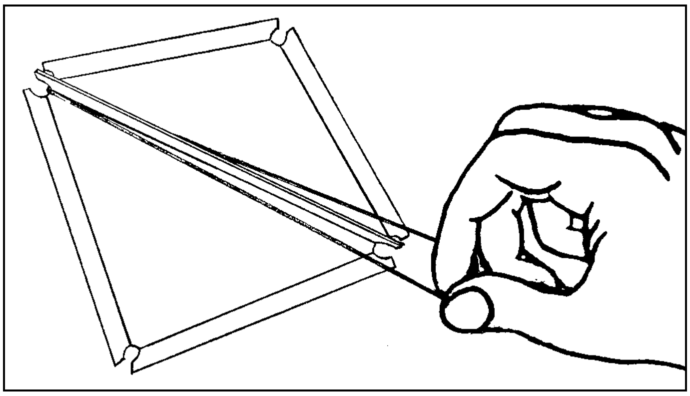

Figure 4: connecting the panels

The work with the old system was sometimes very discouraging. Very often the panels, that were produced with effort, were torn too easily while working with them and so became unusable. Although they had broad and therefore unsightly folds, larger polyhedra were not stable enough. The use of shorter rubber bands for higher stability led to deformed edges of the panels and made the whole system useless.

These disadvantages do not occur when using the further developed modular construction system: Now the panels are made of a rigid and translucence film instead of cardboard. This film is inexpensive and as easily processed as thin cardboard. With this film *all* polyhedra described above can be built without a problem (see examples in figures 2 and 3). The panels can easily be produced with the help of the patterns shown in figure 5. The edges of the polygons are 8 cm in length. The holes for fastening the rubber bands can easily be punched out of the film using a hand paper punch.
The following table gives a survey of all steps that are necessary to produce the modular

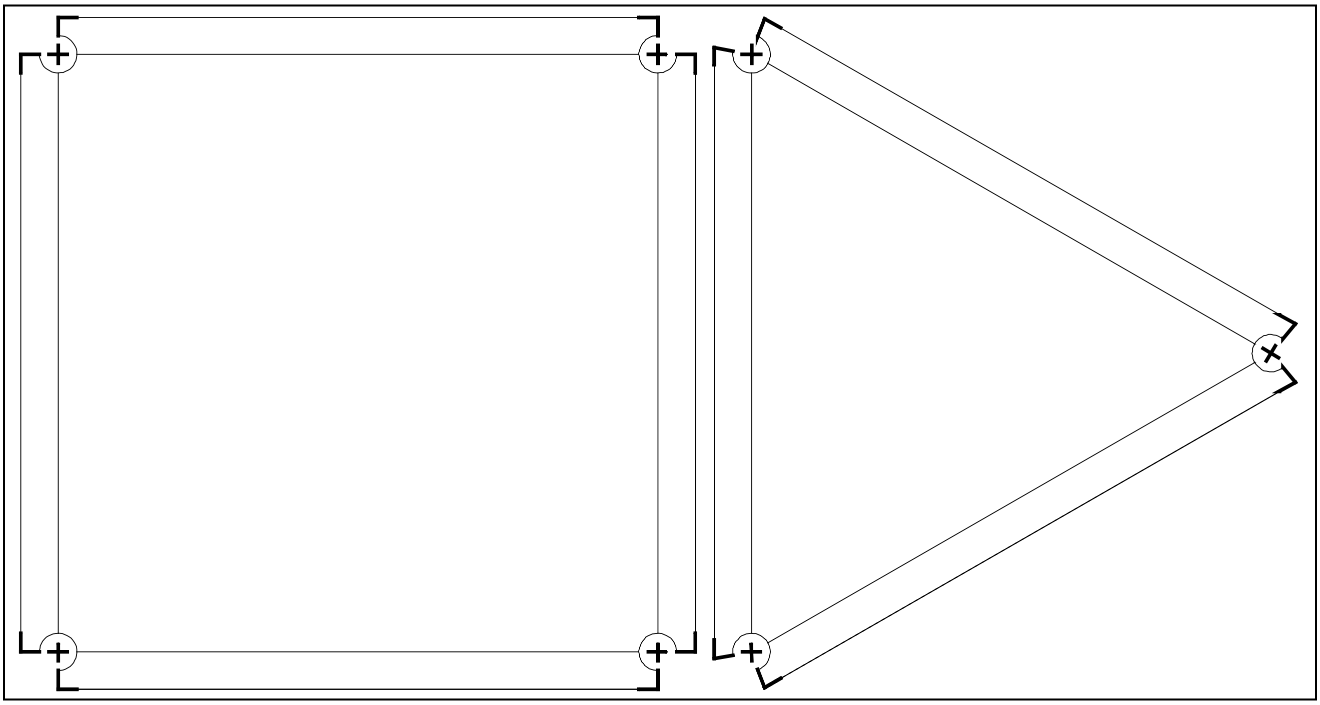

Figure 5: patterns for squares and equilateral triangles

construction system. (For further information about the production of the modular construction system and templates of different patterns - mostly in the form of tessellations - please write to the address below). As a film use a rigid-PVC film which is 0.5 mm thick. Rubber bands having a diameter of 2 cm (or 1 inch) work very well.

Proceeding	implementation	notes
preparation	• distribute the patterns • cut out film strips of about 30 cm	It can be useful to build templates out of the pattern above
marking	• put film strips on the template (blue tack can be useful against slipping) • mark the polygons incl. their folds on the film (using a OH-pen) • mark the centres of the circles	When using a cutter and a bending device (see description below) it is sufficient just to mark the broad lines of the patterns
cutting	• cut out the panels (faces incl. their folds) using a cutter or scissors • punch in the holes where circles are marked: for doing so open the hand paper puncher and turn it upside down to see the marked centres of the circles	The more accurately you work, the better the faces fit when building a solid
bending	• bend the panels at their folds (see proceeding a or b) a) bending with bending-device: put the fold into the slit and bend the panel as far as possible (use bare hands or ruler) b) bending without bending-device: put the fold between the edge of a table and a ruler and bend the panel	The angle between folds and faces should be about 110°
cleaning	• remove the marks	

Table 1: steps to produce the modular construction system

A bending device made out of hardwood simplifies and speeds up work. (For the measures see figure 6: Please note that the slit of the bending-device should be 5 mm deep and about 1 mm wide.). For bending put the fold into the slit of the bending-device and bend the panels over the bevel. (You get the angle of 110° between faces and their folds because the film is rather rigid.) The lateral bevels are useful for bending triangles, because this way they do not interfere with the top bevel.

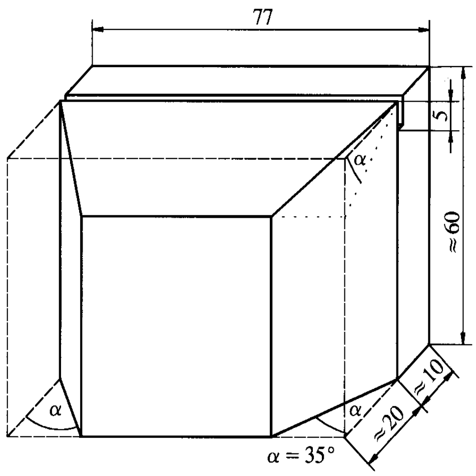

Figure 6: bending-device

Elements of spatial ability in teaching

After a short period of practice, the students can put together all face-regular polyhedra, especially all Platonic and Archimedean polyhedra, by means of the new modular construction system without a problem. The translucence of the polyhedra is of great didactic advantage. The construction and the geometrical characteristics become literally transparent.

With the help of the system the following proposals for teaching become really useful - the students get insights conveyed by their senses. Furthermore, the students are proud of their work. The aesthetics of the polyhedra is motivating for many and facilitates eventually the access to space geometry especially for female students.

We choose the regular octahedron as a representative for the various polyhedra for the following lesson proposals.

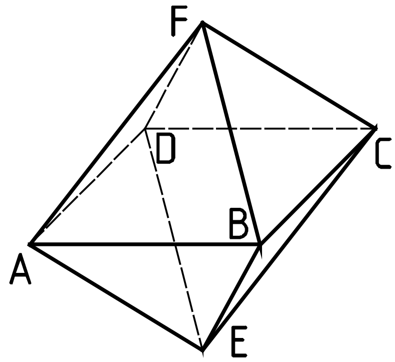

Figure 7: octahedron put on a vertex

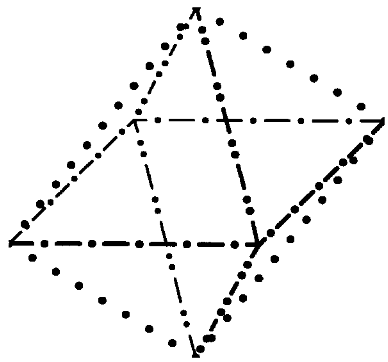
Figure 8: squares of the octahedron

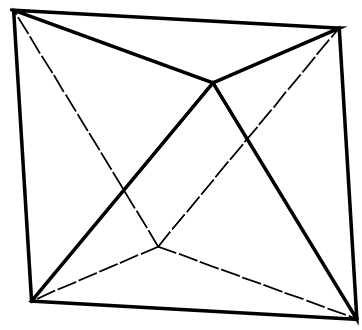
Figure 9: octahedron put on a face

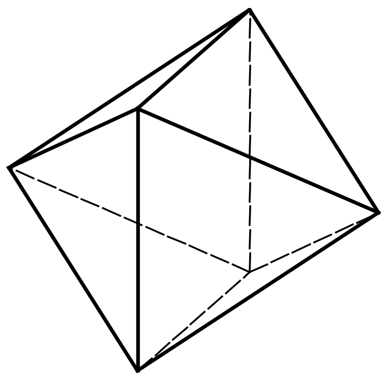
Figure 10: octahedron put on an edge

Spatial perception **activities**

We put together an octahedron and examine it in more detail: First everything seems to be crooked. And we also notice that the octahedron has a tendency to stretch in one direction: Regardless of which vertex points to the top, it always seems to be bigger in the vertical than in the horizontal direction.

Eventually we find out the octahedron is regular. It looks best when it is put on its vertex (figure 7). In doing so we discover a horizontal plane of symmetry. Obviously it is possible to divide the octahedron into two congruent pyramids with a square base: Consequently the octahedron is a double-pyramid. What is its height? Due to the special shape of the vertices of the panels, we can work out the height: We just put in a thin stick (e.g. a long match for lightning pipes or a skewer) through the octahedron.

The symmetrical face is a square. How many squares does an octahedron have? We discover three such planes in our „world of right angles“ and mark those edges with different coloured rubber bands (figure 8).

Now we put the octahedron to one face (figure 9) and find out that the opposite planes are parallel. When we look directly from above, we see that the top area is just rotated: Thus the octahedron is also a regular anti-prism.

Then we put the octahedron with an edge on the table (figure 10). Now the square symmetrical plane is perpendicular from which we are able to see its shape very well.

Finally we imagine our octahedron to be half filled with water. We show and describe the position of the surface of the water as we view the octahedron from different angles (figures 7-10).

***Visualisation* activities**

What shape does the water surface of a half filled octahedron have? We can tell when we use a relatively fine grain such as rice or millet and fill up half of the octahedron. (Use longer rubber bands to fasten the octahedron, so that the vertices of the model are covered with the rubber bands.) Because our octahedron is translucent we are able to see a square, a rhombus or a regular hexagon in the three standard positions. How long are the edges and of what size are the angles of the rhombus? This stereometrical task does indeed require spatial ability. We can also explain why the hexagon is regular: This is because the curved surface of our anti-prism is regular and all edges of the hexagon are the same length, that is half as long as the edges of the octahedron.

Now we ask ourselves how the shape of the (cross-)sections will change when we reduce the material inside. For this case we punch three semicircular holes with regular distances from each other into the folds of the faces (figure 11). In these holes we span a rubber band which exemplifies one possible cross-section (figure 12). Because our model is translucent it is easily imaginable. Did we also stretch the rubber bands so that there is a plane cross-section? If we regard the cross-section from all sides the rubber bands should be taut.

In order to learn more about our octahedron, we want to experiment and discover various nets. For this reason we just take off one rubber band after another of our octahedron. A net is found once most rubber bands have been taken off without losing the complex of the eight faces. With the help of an overhead projector we are able to present the results. How many different nets did the students discover? Which of those have an axis of symmetry, which have rotational symmetry?

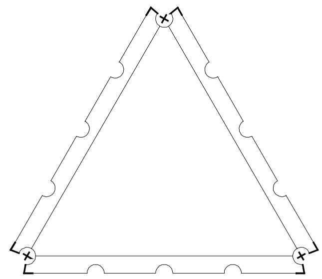

Figure 11: modified template

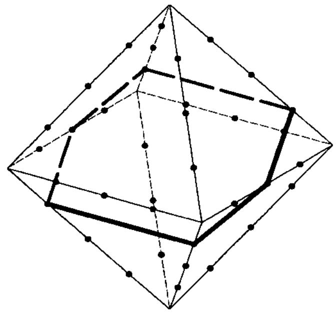

Figure 12: Spanning a rubber band

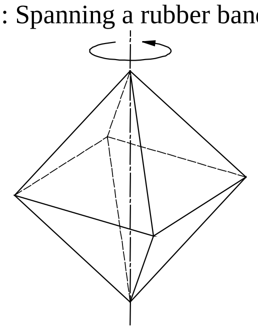

Figure 13: rotational symmetry of order 4

***Mental rotation* activities**

Which features of rotational symmetry does the octahedron incorporate? The easiest way is a rotation about the line which runs through the opposite apexes of the octahedron (figure 13). We see the rotational symmetry of order 4, that means after four rotations of the same angle, the octahedron is back to its original position. Altogether we can discover three such axes of rotational symmetry of order 4.

We discover further axes of rotation: The connection line between the middle of the opposite faces are also possible axes of rotation. There are four such axes of rotation, each of order 3 (figure 14). Finally the six connecting lines of the opposite edges (connection of the middle of each edge) are axes of rotational symmetry of order 2 (figure 15).

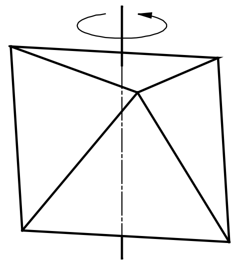

Figure 14: rotational symmetry of order 3

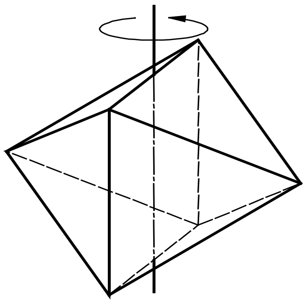

Figure 15: rotational symmetry of order 2

Now we construct a cube and examine its symmetrical characteristics. We are surprised to see that the cube
rotational symmetry as the octahedron. Likewise it has three rotational symmetries of order 4, four times of order
of order 2. We will refer to this later on.

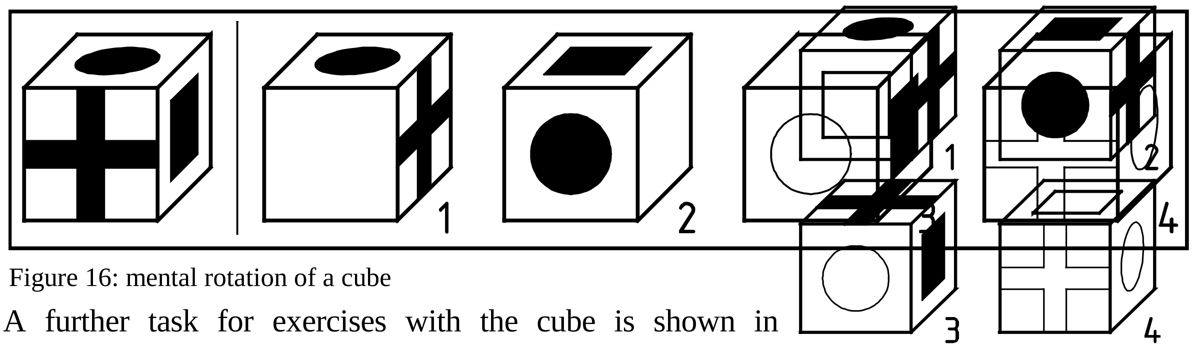

Figure 16: mental rotation of a cube

Figure 17: solutions of the mental rotation task

A further task for exercises with the cube is shown in figure 16. Which symbols are to be added to the blank face when referring to the drawing left? It is presumed that the symbols on the parallel planes of the cube have the same shape. However the same symbol appears once filled in and once just outlined.

First we try to complete the task mentally. After that we check our assumptions with our models, by drawing the symbols using a (water soluble) OH-pen. Now we can see that the first and the fourth cube have an outlined square and the second and the third a filled in cross (figure 17).

Further additional tasks with increasing difficulty - and also various tasks concerning other elements of spati
provided in MAIER (1996b) for photo copying. Once we are capable of doing these, we can give easier tasks where
can be used instead of a cube.

Spatial relations activities

Figure 18 shows four diagrams each representing the same cube. The letters A to F are written on each of the f
determine which letters are on opposite faces.

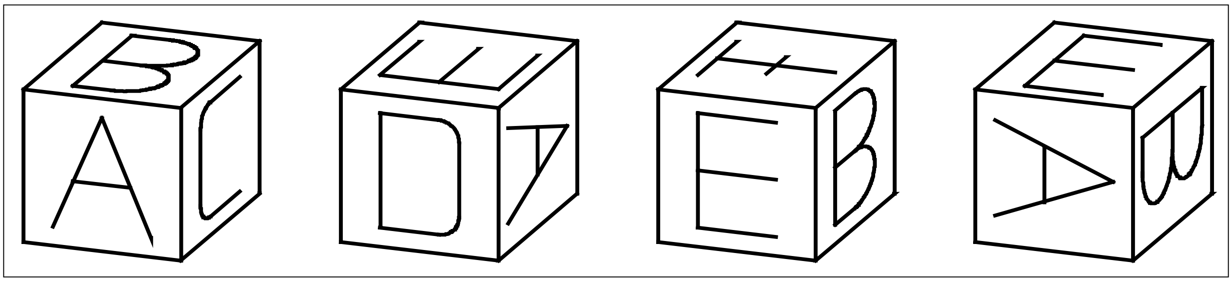

Figure 18: spatial relations of cubes

One explanation of a result may be like this: „Because B in picture 1 lies above A and D in picture 2 lies below A, B and D must be on opposite faces. Besides C in picture 1 lies to the right of A, whereas E in picture 2 lies to the left of A. That is why C and E are on opposite faces. A and F are left over and therefore must also be parallel to each other“ (figure 19).

A	opposite	F
B	opposite	D
C	opposite	E

Figure 19: solutions of the spatial relations task

What makes it interesting is the fact that we only needed the first two of the four pictures to find a solution. This is no coincidence. The task is created in such way that at most 3 pictures are needed to

find a solution.
Let us go back to our octahedron in the shape of a triangular anti-prism. Is it also possible to construct a square anti-prism? As we can see in figure 20 it is possible and we compare it with the triangular anti-prism. Now we can also imagine pentagonal, hexagonal, etc. anti-prisms.
But are we also able to imagine an anti-prism with no triangular base but instead having „biangles“, containing just lines? We try to construct such a solid and are amazed to see the result: We are holding a tetrahedron in our hands (figure 21).
Finally we take a cube and an octahedron: Which polyhedra is formed by joining the centres of adjacent faces? The result is a socalled „dual polyhedron“. Each vertex corresponds to a face of the original, each face of the new polyhedron to an original vertex, and the edges match, one for one. We see that the cube and the regular octahedron are dual to each other (figure 22 and 23) and we now understand why they both have the same symmetrical characteristics.

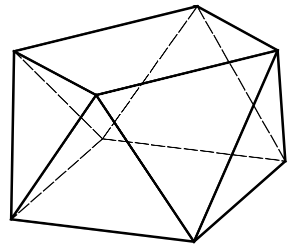
Figure 20: square antiprism

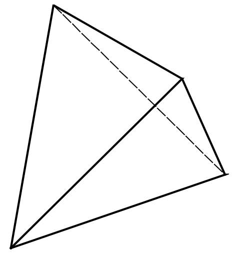
Figure 21: tetrahedron

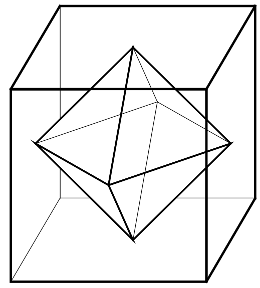
Figure 22: octahedron in a cube

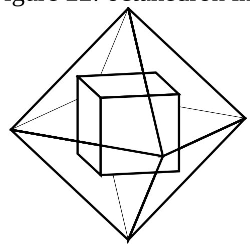
Figure 23: cube in an octahedron

Spatial orientation activities

Finally we let our imagination go along the edges of a large octahedron. In doing so we can look at our model or at a drawing (figure 7) or we do not use a model at all: „Starting at the point A (see figure 7) we turn right, then go up, then turn left to the back and finally we go forward. Where are we now?“ That’s right, we have come back to our starting point A. We carry on. Who would like to suggest the next „spatial orientation walk“?

Teaching experiences

Teaching experiences in grades 5 to 10 have shown that the students definitely like experimental activities with the modular construction system. From grade 7 onwards the students consider the production of the system and the building of the polyhedra mostly as „very easy“. At least in grade 5 and 6 a short period of experimental handling of the system is recommended for the beginning. At this stage the students, as well as older students and university students, like to put their solids together to make houses, towers etc. Teachers are often astonished about how creative their students work with that material - evidently the system provokes creative working. Qualitative analyses confirm these observations: The students appreciate that „phantasy plays a role in maths teaching“ and that „one gets beautiful results to be proud of“.

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