HERMANN KAUTSCHITSCH, KLAGENFURT

# THE IMPORTANCE OF SCREEN-SPLITTING FOR MATHEMATICAL INFORMATION PROCESSING

**Abstract:**

It is shown how one can use screen-splitting to improve the presentation of mathematical information and support the processing of that information. This technique involves splitting the visual field into several sections in which various mutually linked scenes can be simultaneously observed. This splitting is made possible, and efficient, by advanced new media such as video, computer programs and digital-video-discs (DVD).

## 1. Cognition-psychological foundations.

The suggestions made in this paper are based on the following assumptions:
Information is generally not transferable; it must be constructed in the mind of the learner. Classical channels of thinking are language, systematic perception and imagination as the continuation of perception (AEBLI, p.279ff). Information processing using language is sequential, but in the case of perception it is parallel. In perception various mutually related events happen simultaneously, and imagination reproduces these simultaneous structures.

Pictures have the great advantage over language or text that they allow a quick grasp of each arbitrary point of the visual field. Now, visual perception allows not only a quick focus on each point of the visual field, but **it also offers at any time a great number of pictures as internal models (templates)**, such that a quick comparison with reality, or a new situation, is possible. This is the first essential point that can be aided by screen-splitting. It may be that some pupils do not have sufficient pictures in mind, or that they do not focus on the correct points of their visual field, when they have difficulty following the teacher or the book or even some illustrations. By screen-splitting one can offer the right pictures at the right time or one can focus the pupil's attention on the essential point or region of the picture.

**Hypothesis I:** Screen-splitting can increase the power of perception. Using the support of built in procedures one can provide scenarios as flexible as those in ones imagination, but based on reality. Also, in contrast to human imagination, these scenarios will always be correct.

There is a second advantage to screen-splitting. Human thinking needs illustrative material. Objects that are related to each other by actions and operations must be represented "modal". Only after this step can the human find the relationships between them (AEBLI, p. 319). In mathematics we often use pictures as illustrative material. However, research shows that there is no autonomous visual thought (JAHNKE, KLIEME).

In order to interpret a picture or, more generally, a visual perception, it is necessary to activate conceptual knowledge (KLIEME/RÜPPEL). Since most mathematical concepts are abstractions of actions, they can and should be demonstrated by actions (WITTMANN, p.81). Conversely picture-aided actions, or visualisations, are necessary to apply conceptual knowledge. (KAUTSCHITSCH). Therefore actions, visualisations and concepts are **interdependent** and this can be demonstrated by showing them **simultaneously** using screen-splitting.

**Hypothesis II:** Screen-splitting increases the understanding and applicability of pictures and concepts by simultaneously showing various representation techniques.

There are many other reasons to emphasise **various representation** techniques:

    The understanding of some representations is subjective.

    Each representation cannot express each property or each relation in the same efficient way.

    Some representations are easier for one person to handle than another.

    One representation alone cannot illustrate the whole extent of a relation or concept.

Why **simultaneity** is so useful:

In the traditional teaching of mathematics the various representations follow each other, usually sequentially.This is a conclusion of BRUNER's theory of representational techniques based on examinations of PIAGET. In this way the pupil can build up various subjective micro-worlds in the sense of LAWLER. But these micro-worlds have only an instrumental character: a world of counting, measuring, computing, constructing and acting with various real or geometric objects. In the beginning they are clearly separated. This is a consequence of the sequential representation, so most of the pupils perceive the various representational techniques of the same concept or object as distinct things; the micro-worlds are mutually exclusive and so multiple visualisations do not work, because there is **no pre-symbolic** information processing ( KLIEME ). To remove this mutual exclusivity it is necessary for the pupil to build integrating and co-ordinating micro-worlds (LAWLER), which guide his or her heuristical and logical thinking so that conceptual ( symbolic ) knowledge can be activated.

**Hypothesis III:** Screen-splitting, with its emphasis on simultaneity, supports the generation of integrating and co-ordinating micro-worlds.

## 2. The role of "new" media.

New media such as video, interactive computer programs or digital-video-disc (DVD) offer the following features:

    Execution and saving of actions on real and geometric objects.

    Linking of actions with perpetually updated calculations and measurements.

    Automatic and/or remote execution of actions.

    Technical manipulations on the visual field: zoom, time transformations and screen-splitting.

Computer-aided screen-splitting enables a **montage** of

    temporally and spatially separated pictures and actions.

    real and schematic pictures and actions.

    speeded up, slowed down and fixed pictures.

    optical marks like enlargement and superposition that focus pupils' attention on certain objects and actions.

With such a film-like montage an **expressiveness** is available that cannot be achieved by either simple pictures or by everyday language. However, this expressiveness is, in my opinion, necessary to learn and apply the powerful artificial language of mathematics.

### 3. Some applications of screen-splitting.
### 3.1 Obtaining insight.

To interpret pictures it is usually necessary to remember some groupings of figures or imagine some actions. With screen-splitting one can represent simultaneously the situation before and after a sequence of actions. This enables comfortable and easy comparison and analysis, since simultaneous pictures are more memorable and allow for pattern recognition.

**Example 1:** Theorem of heights in a right-angled triangle.
We split the screen into two parts. This splitting supports the following strategy:
Looking for "nice" constellations (Figure 1) - Changing of position of some partial figures (Figures 2 and 3) - Removing equal parts  of equal figures and recognizing of equalities (Figure 4):
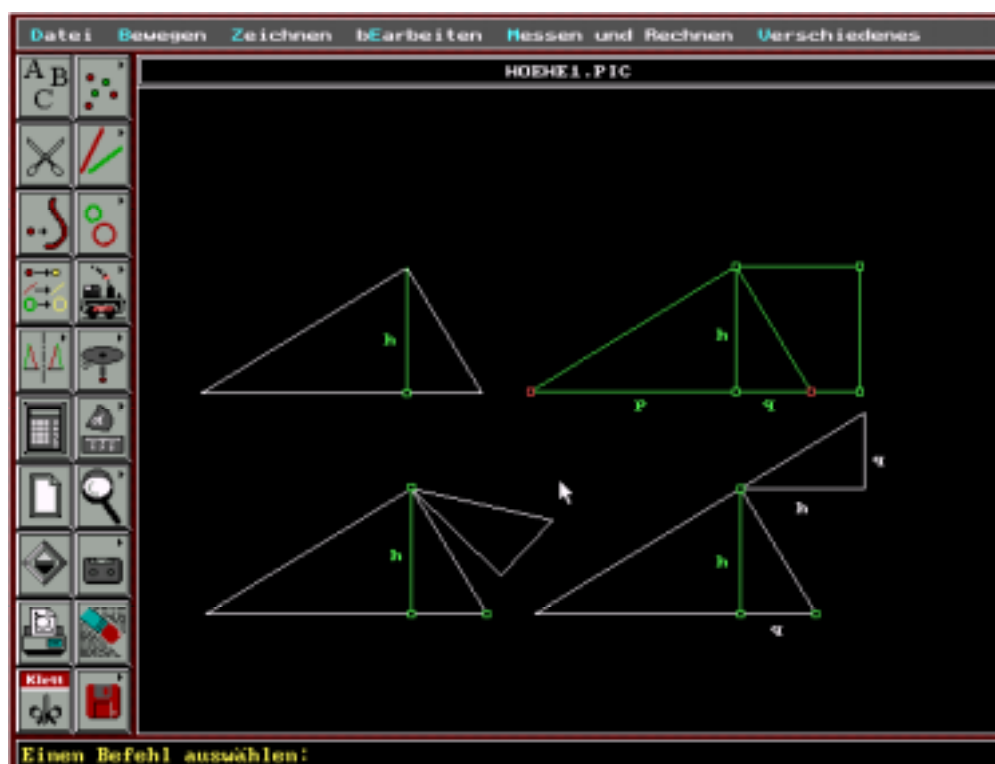


**Figure 1**

The finding of "nice" groupings of figures and changing of positions is aided by movable sections, which are separated out from the whole.  The THALES package, which was developed at the Department of Mathematics in Klagenfurt/Austria (KADUNZ/ KAUT-SCHITSCH), has this ability.

Naturally the above actions are also possible with pencil and paper, but the pupil must have some powers of imagination and construction. The moveable figures on a split screen are simply the workings of human imagination, but taking place outside the mind, with the advantage that pupils can better analyse the figures. These reality-based scenarios can support the results of the above actions and the pupil can discover, and not only understand, such informal proofs. In my opinion it is very difficult to perform this process purely in one's imagination and it is still more difficult to explain it. On a split-screen one can see the

116

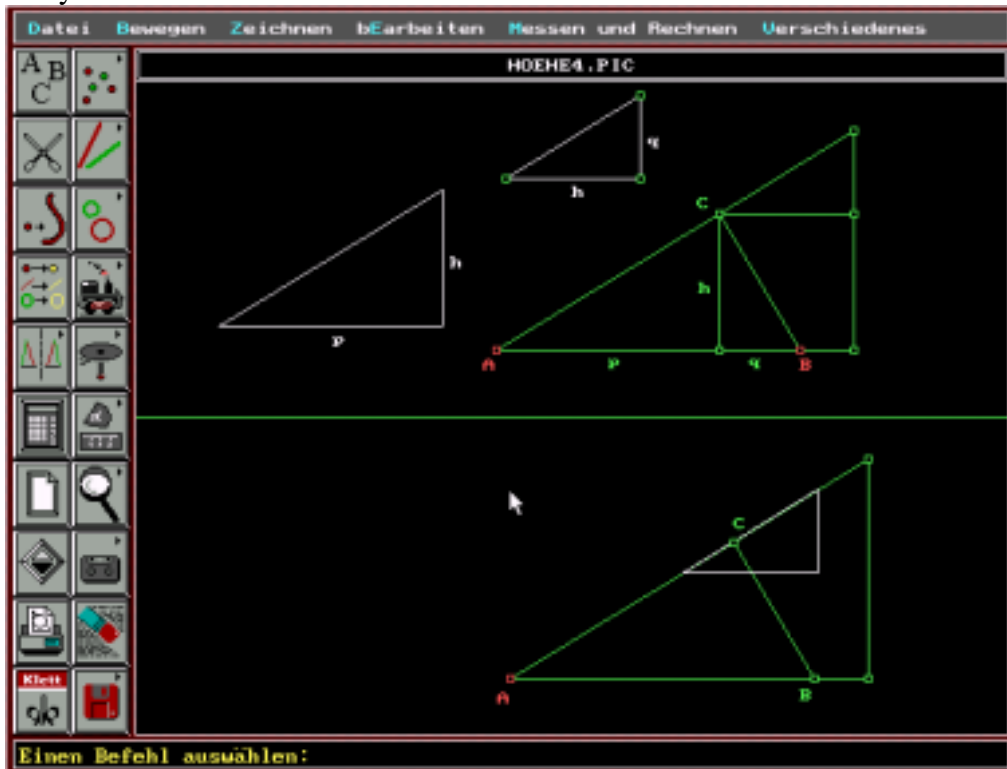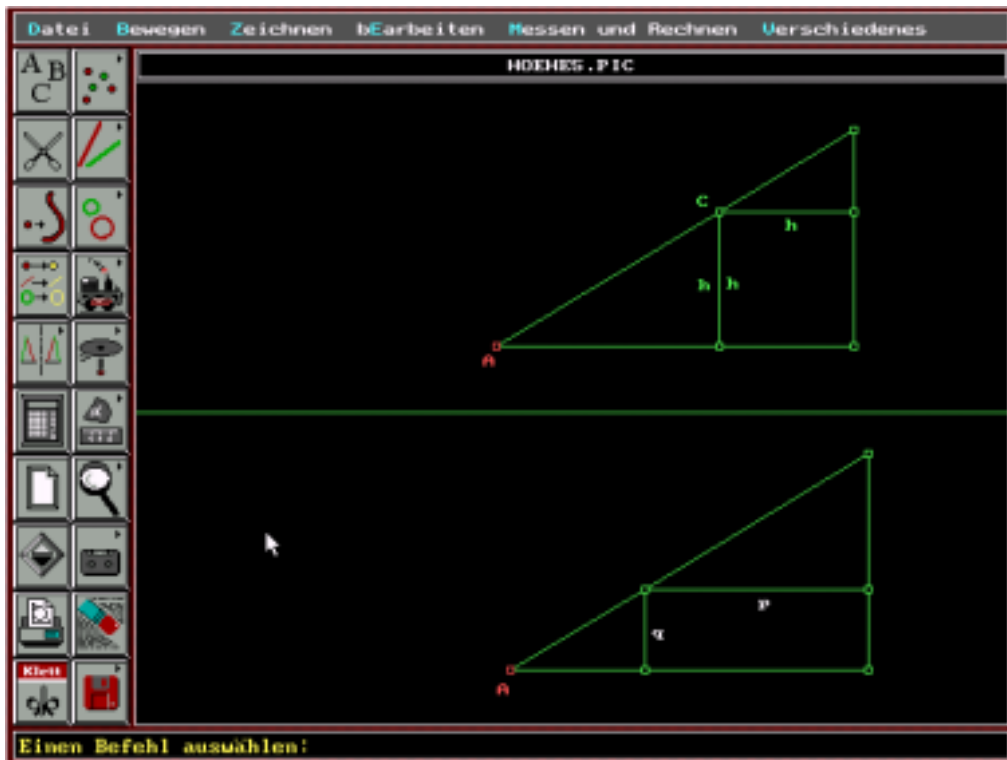relations, because relevant objects and the operations undertaken can be perceived simultaneously.
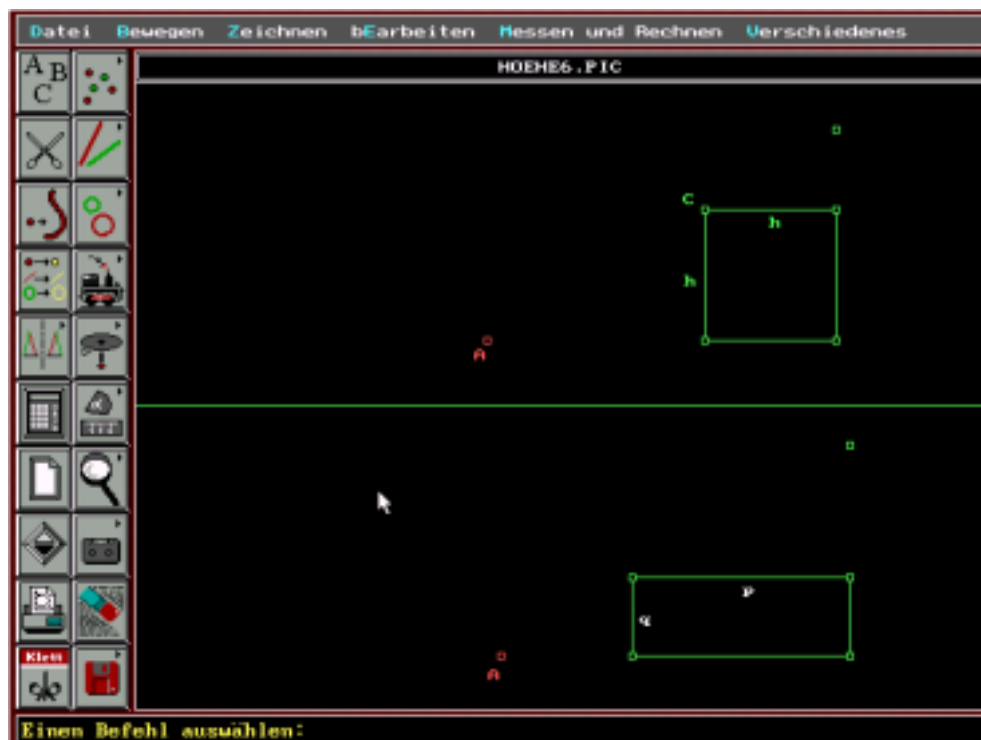


**Figure 2**



**Figure 3**

**Figure 4**

## 3.2 Obtaining conjectures and proofs.

THALES supports the creation of a split screen which contains constructions, measurements, calculations and so called "dependence-graphs". This is possible using the following features of THALES:

> The ability to carry out any kind of movement (not only when drawing but also when carrying out congruence relations such as reflection, rotation and translations). In particular, it is possible to **separate out** parts of a figure and perform actions on the resulting fragments.
>
> Measurement, calculation and construction with measured and calculated sizes.
>
> Duplication of objects and the subsequent implementation of a sequence of operations on only one of the copies. This allows us to show a configuration before, and after, the operations are carried out.

Split screens support a computer-aided, experiment-based, form of the teaching of mathematics, that imitates the activities of natural scientists, such as physicists. The main goal of such activities is the finding of invariants by looking for functional dependences, which are, in general, theorems. This experiment-based form of teaching consists of the following steps:-

(i)     Looking for actions.

> a)     To become familiar with the situation: What can be varied?
>
> b)     Initial recognition of qualitative relationships: What varies and in what manner?

(ii)     Looking for data.

> a)     Collection of data by taking measurements: What values are involved?

b)      Relating the measurements by calculations.

(iii)    Looking for conjectures.

a)      Variation of some conditions and examination of the effects: What happens if?

b)      Induction: **What** is invariable? Use functional interdependence of variables to get relationships.

(iv)    Looking for arguments.

Deduction: **Why** is it invariable? Find a picture-aided action that can be carried out repeatedly.

(v)     Looking for generalisations and specialisations.

Are there any other configurations for which the **same** action can be carried out?
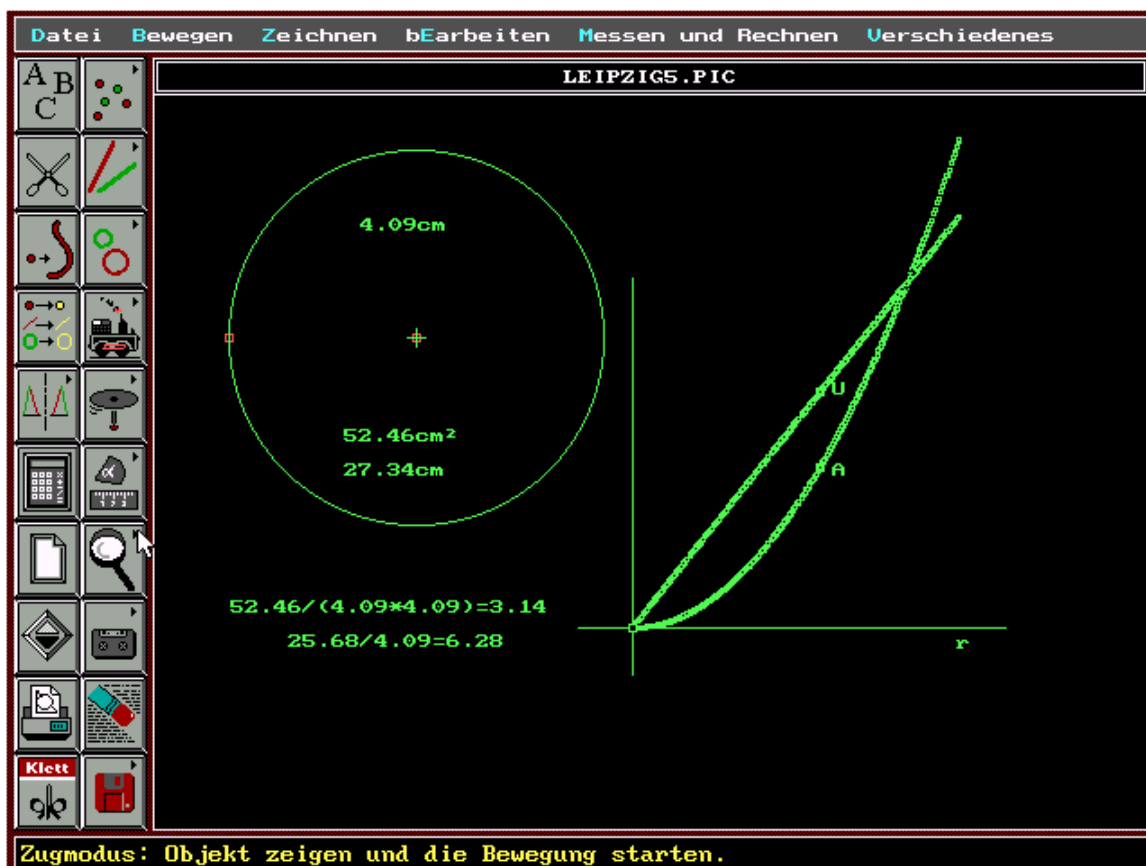
**Example 2:** Area and circumference of a circle.



**Figure 5**

The circumference, $U$, and area, $A$, are determined approximately using polygons (NEIDHARDT/WURM). To find the invariants we link a dependence-graph of $U$ and of $A$ with respect to the radius of the circle $r$, which is simultaneously varied (Figure 5). The usefulness of the split screen is now shown by the fact that the form of the graph shows the appropriate direction for further thought. The straight line graph shows that there is a linear relationship between $U$ and $r$ while the parabolic graph shows that there is a quadratic relationship between $r$ and $A$. The continually updated and visible values of $A/r^2$ and $A/r$ confirm the relationships. A proof cannot be given and it is clear that for such an experimental

approach the pupil must be familiar with the interpretation of graphs. It is also clear that one can do the same with paper and pencil, though it will be more tedious and two such operations cannot be carried out simultaneously. But the simultaneous nature of the split-screen process shows how one and the same content (relationship between area and radius) appears in three forms: picture, graph and constant numerical value.

**Example 3:** Theorem of chords and secants in a circle.
We choose an arbitrary point, *P,* within a circle, *k,* and construct the chord *AB* through *P*. Then we rotate the chord about *P*. What can we say about *PA* and *PB*? In this case the split screen may contain two instructions to begin the process of activating conceptual knowledge.
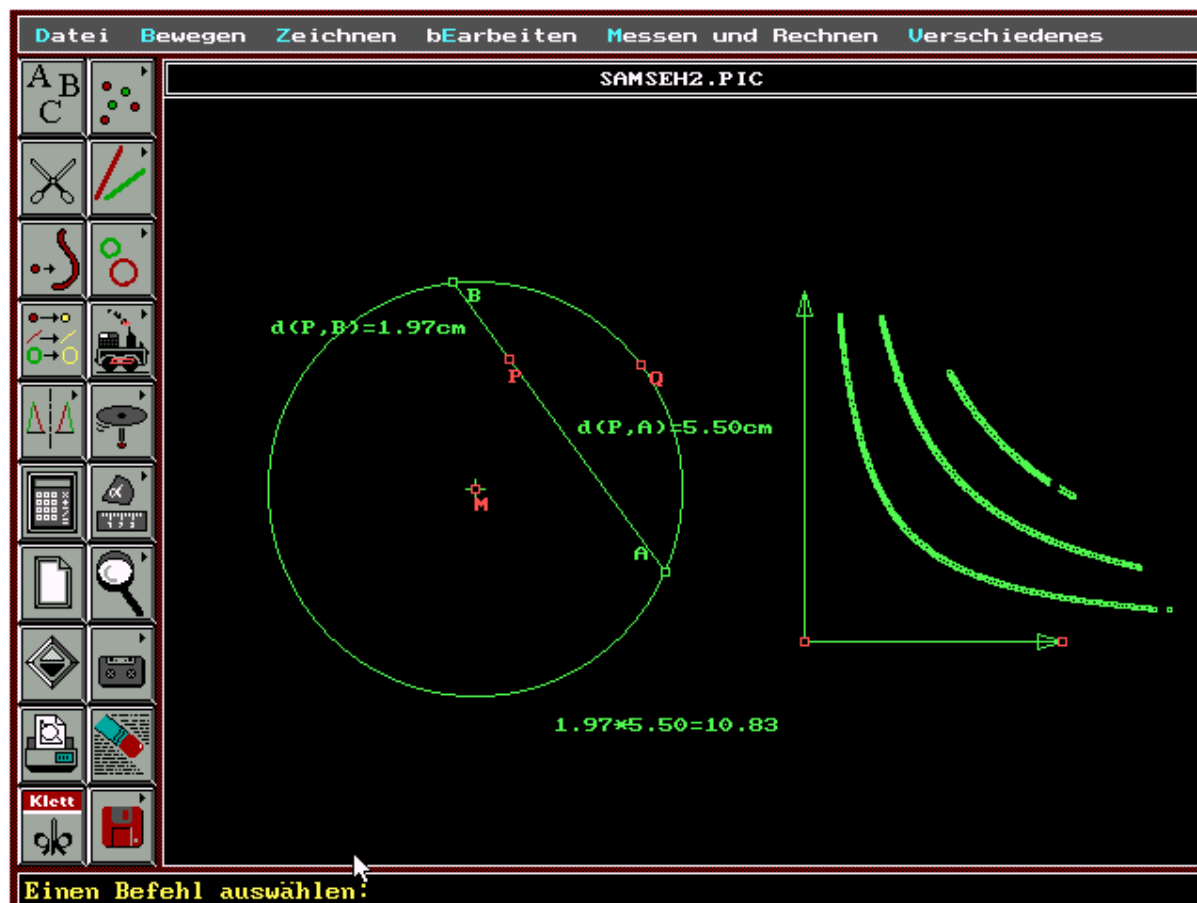


**Figure 6**

1. The dependence-graph of *PA* against *PB* in order to obtain a relationship (Figure 6). Since the graph seems to be a hyperbola, one can conjecture that the product of *PA* and *PB* is constant. (Again familiarity with the interpretation of graphs is necessary). Measurements confirm this conjecture.

2. The figure showing similar triangles (Figure 7). It contains constant products and similar triangles arise by connecting two states of the rotating chord. These similar triangles give the direction for further thought. Using the features of THALES one can move one triangle to the other such that we get the figure of STRAHLENSATZ. The matching processes can be explained by congruence theorems, so we get a proof that not only proves but also explains, because it is an action that can always be carried out. **Proving = Looking for actions that are always possible.**

Remark 1: The same can be done, if one moves the point *P* outside the circle or on to the circumference of the circle. This can be easily observed on the same split screen!

Remark 2: There are only a few strategies like dependence-graph and similar triangles that one can use to explore geometry. With the help of split-screens they can be employed at the right moment. But it is necessary that concepts and theorems are introduced by picture-aided actions from the beginning (see also section 4).
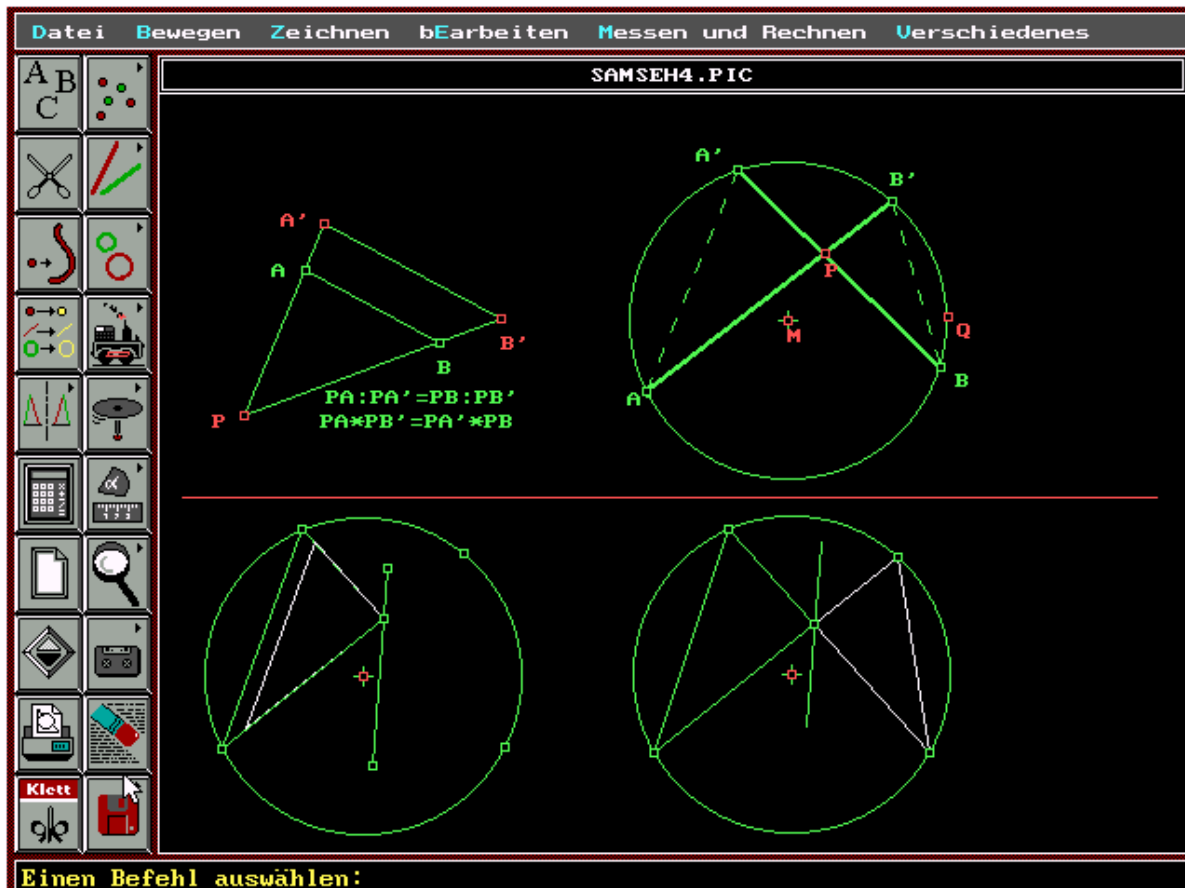


**Figure 7**

### 3.3 Operative concept formation.

A spreadsheet with its columns and rows is also a split screen. The content of the columns and/or rows is linked together and can be made permanently available. The dependence of some columns and/or rows can again be seen using a dependence graph and the form of this graph again gives further directions for thought. The numbers in the columns are numerical records of actions, the titles of the columns are instructions for actions in a symbolic form. Further actions can be defined and recorded by inserting further columns with their titles. Such a procedure is particularly suitable for an action-orientated approach to a concept. At least one obtains many mutually linked schedules, graphs and formulas.

**Example 4:** Differentiability of a function.
In the example the following actions (see also J.v.d.STEINEN) to gain the concept of the derivation of $x^2$ at the point $x=0$ are recorded numerically and graphically. All columns and

graphs together should give a deep understanding of the concept of the differentiability of a function $f(x)$ at a certain point $x=a$. The same can be done for various functions.

1. Experiment: Examine $f(x) = x^2$ on small neighbourhoods of $x = a$.

Observation: For small increments of $x$ the values of $f(x)$ increase by approximately the same amount, so we get, in small neighbourhoods of $x$, a linear growth and the graph of $f(x)$ is nearly a straight line (Figure 8).

2. Experiment: Examine the error for various linear function $g(x) = k(x-a)+f(a)$ through the point $(a,f(a))$ which should be an approximation of $f(x)$.

Observation and first disappointment: Many values of $k$ are suitable, the "best" approximation one gets may be for $k = [(f(a+h)-f(a)]/h$.
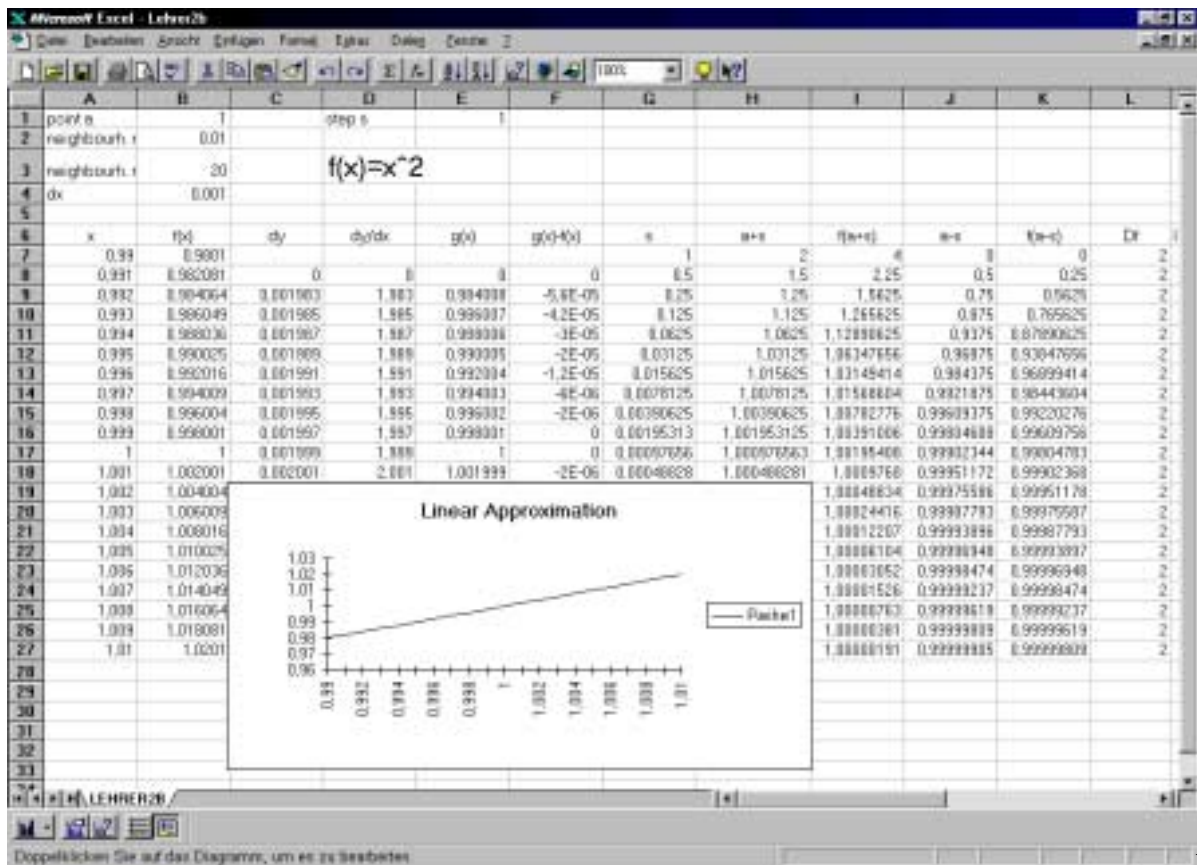


**Figure 8**

3. Experiment: Improvement by using symmetric values about $x = a$: $k = [f(a+d)-f(a-d)]/2d$, where $d$ runs through a zero-sequence.

Observation and second disappointment: One gets a quick numerical convergence but this happens for example also for the function $f(x) = Abs(x)$, which obviously is not differentiable at $x = a$ (see figure 10, column L). This observation helps to understand the necessity of the condition "for all sequences..." in the definition of differentiability. We simulate this condition by random numbers:

4. Experiment: Improvement by using $z$ and $w$ of an arbitrary random zero-sequence:
$k = [f(a+z)-f(a-w)] / (z+w)$

Observation: There is a quick numerical convergence, but not for *f(x) = Abs (x)*, so *k* can serve as an appropriate approximation for *f'(0)*. By the features of EXCEL one can collect the derivatives at many points, such that the character of *f'(x)* can be conjectured.

So we have explored the equivalence of differentiability and linear approximation by a sequence of actions with nearly arbitrary small numbers.

The split screen for differentiability is a combination of mutually linked tables (the columns), which are the records of suitable mutually linked actions, and graphs. By the splitting any change of inputs and its effects can be observed simultaneously in any column or graph. Again we get directions of further thought.
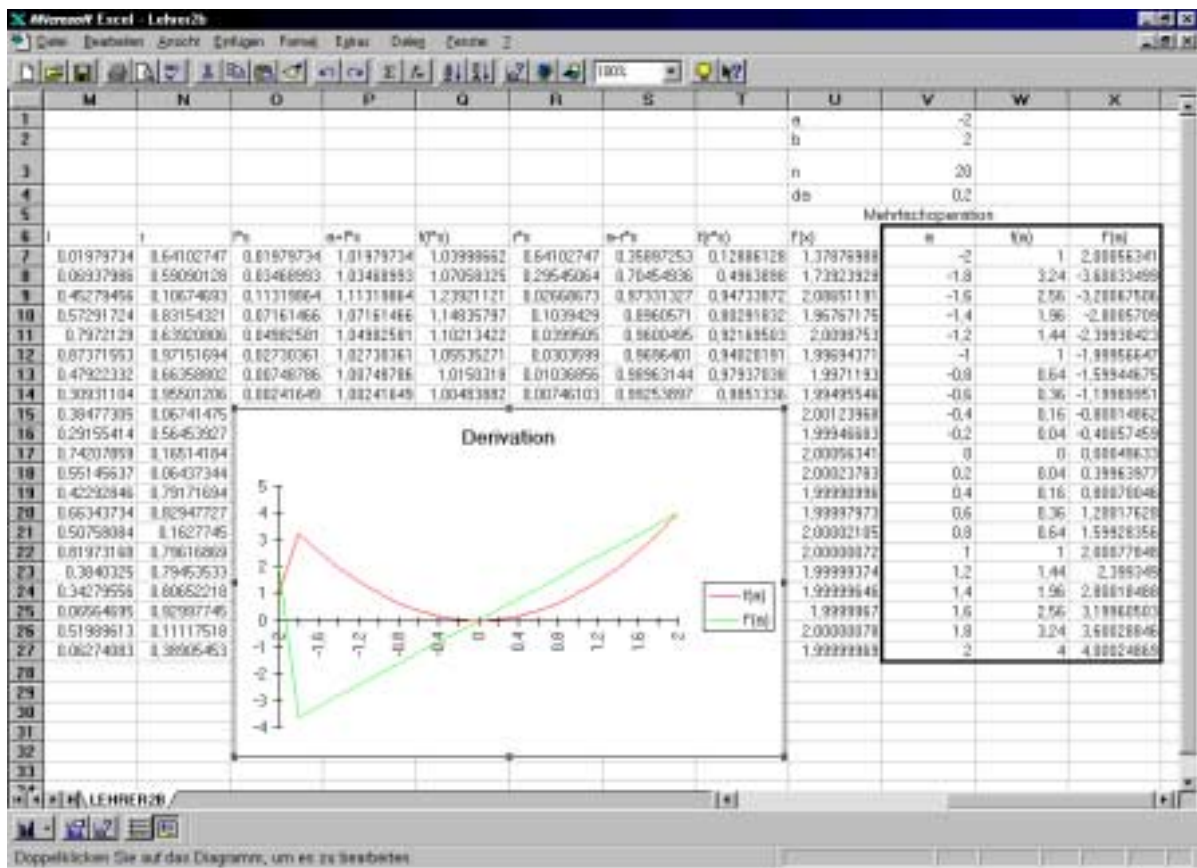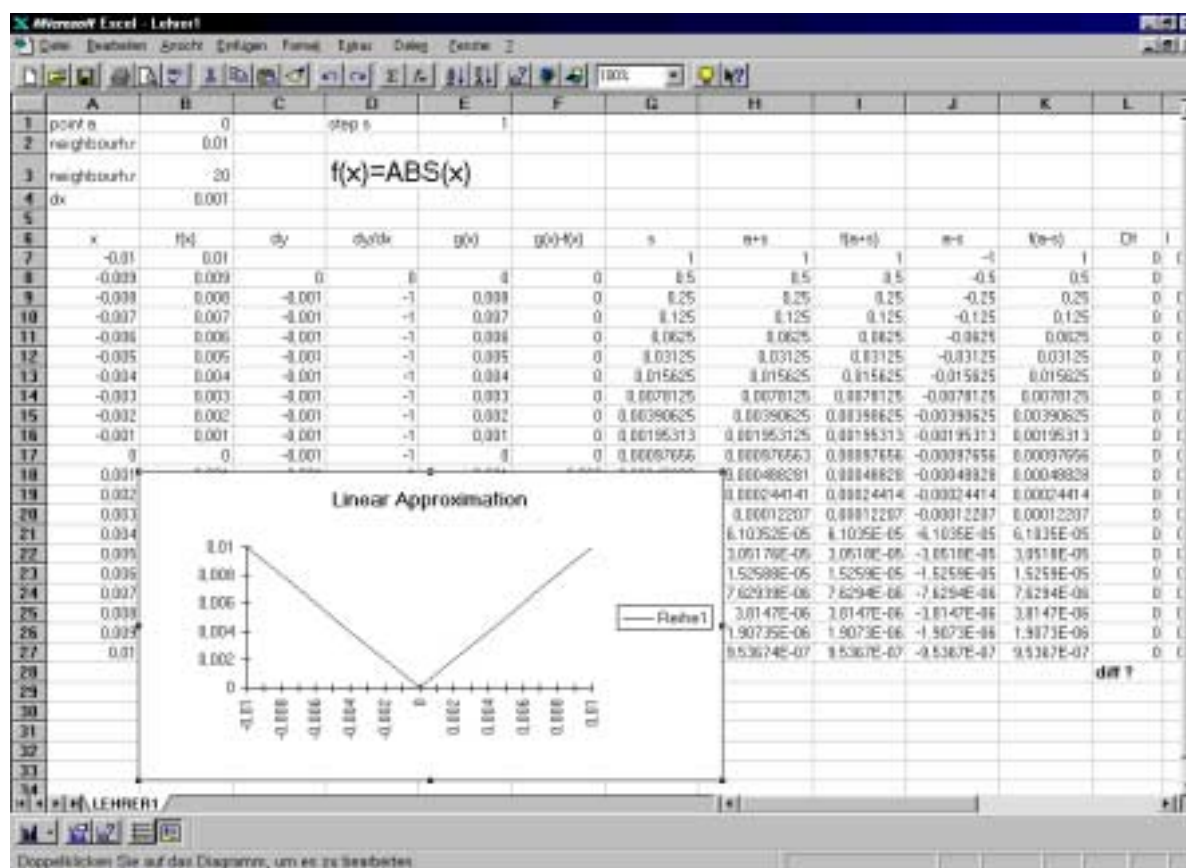


**Figure 9**

**Figure 10**

## 4. E-I-S pictures.

We have seen that screen-splitting together with computer procedures allow a combination of movable real and workable geometrical figures, graphs, symbols, measurements and calculations, which are mutually linked. So enactively, iconically and symbolically represented knowledge is stored in **one** picture. We therefore call it an E-I-S-picture. It has the possibility of activating conceptual knowledge in the appropriate representation, which, as mentioned in section 1, is very important for information processing. An E-I-S-picture may serve as an errorless imagination outside of the mind and cognitive activities can be executed outside of the mind, such that in mind other cognitive activities can be controlled or even initiated.

Simultaneity of the representational techniques in E-I-S pictures enables further the generation of co-ordinating and integrating micro-worlds, which help a person to understand and interpret simple pictures.

Whether a picture has E-I-S-character depends on the way it is imparted. Sometimes even handmade pictures can have E-I-S-character, but it takes a lot of time and a certain amount of imagination and constructional power to create such pictures. Whether computer aided or not, the following steps should be taken into account (didactics of E-I-S-pictures):

1. Establishment and adaptation of the conceptual, iconic and enactive pre-knowledge.

2. Each E-I-S-picture is built up from other E-I-S-pictures. Therefore it is necessary to use from the beginning E-I-S-pictures for each concept and theorem, a revival of a rich geometrical knowledge is therefore necessary.

3. A rich stock of picture-aided actions, or actions with tables, must be made available (visual

 mathematics library).

4. Reduction to a few strategies (looking for invariants, elementary configurations, lines,
 similar triangles ...).

5. Practice and acknowledgement of picture- and action-orientated solutions and arguments.

It can be seen that "functional- and visual-thinking" are key concepts in teaching with screen-splittings, which implies that graphs of functions and geometry should play an important and early role in the curriculum. It is not only concepts and theorems that are imparted efficiently using this form of teaching, but also graphical techniques, creativity and the power of learning.

## References

AEBLI, H.: 1981, *Denken: Das Ordnen des Tuns*, Bd 2: *Denkprozesse,* Klett-Cotta, Stuttgart.

JAHNKE, H.N.: 1984, 'Anschauung und Begründung in der Schulmathematik', in *Beiträge zum Mathematikunterricht,* Franzbecker, Bad Salzdetfurth, 32-41.

KADUNZ, G. and KAUTSCHITSCH, H.: 1993, *Thales. Software zur experimentellen Geometrie,* Ernst Klett, Stuttgart.

KAUTSCHITSCH, H.: 1988, 'Bild-unterstützte Abstraktion und Verallgemeinerung', in W.Dörfler (ed), *Kognitive Aspekte mathematischer Begriffsentwicklung*, B.G. Teubner, Stuttgart, Hölder-Pichler-Tempsky, Wien, 191-259.

KLIEME, E. and RÜPPEL, H.: 1983, 'Die Bedeutung und Förderung des bildlichen Denkens im Unterricht', in L. Kötter and H. Mandl (eds.), *Jahrbuch für Empirische Erziehungswissenschaft*, Schwann, Düsseldorf, 109-138.

KLIEME, E.: 1987, 'Bildliches Denken. Kognitionspsychologische Modelle und didaktische Strategien', in H. Kautschitsch and W.Metzler (eds.), *Medien zur Veranschaulichung von Mathematik*, B.G.Teubner, Stuttgart, Hölder-Pichler-Tempsky, HPT Wien, 43-68.

LAWLER, W.: 1980, 'The Progressive Construction of Mind', *Cognitive Science* **5**, 1-30.

NEIDHARDT, W. and WURM, Ch.: 1997, *Arbeitsbuch Thales. Geometrie mit dem Computer,* Ferd. Dümmlers Verlag, Bonn.

STEINEN, J.v.d.: 1977, 'Experimentelle Analysis', MNU 30, 23-29.

WITTMANN, E.: 1981, *Grundfragen des Mathematikunterrichts,* 6.Auflage, Vieweg, Braunschweig, Wiesbaden.

Mag.Prof. Dr. Hermann Kautschitsch
Universität Klagenfurt
Institut für Mathematik,Statistik und Didaktik der Mathematik
Universitätsstr. 65-67
A-9020 Klagenfurt