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"YOU CAN USE THE EMPTY NUMBERLINE LIKE A RULER, BUT IT'S NOT AS PRECISE."[1]
ANALYSIS OF MATHEMATICS LESSONS ON THE USE OF THE EMPTY NUMBERLINE

Abstract:

Within the scope of a qualitative research project we video taped some mathematics lessons on the introduction and the use of the empty numberline in grades 2-4. The epistemological analysis of some transcribed teaching episodes shows how the children by themselves – departing from the introduced conventions and rules for this reproduction of the standard numberline which is not faithful to the scale but which observes the order of numbers – develop a specific interpretation for this new means of representation of the natural numbers. In a constructive process the children learn to shape actively the structure and the mode of use for the mathematical symbol "empty numberline" and to adjust them to specific tasks in a flexible manner.

1. The Research Project

In the following I would like to present some findings and results of our research project "Visual based argumentation in mathematical instruction of primary school" in the area of interpretive and qualitative classroom research.[2] The aim of this project is to analyse interactive patterns of mathematical justifcation in primary grades and to explore the ways in which children learn different forms of argumentation in everyday mathematics teaching.

During summer 1996 we observed and video taped about 50 mathematics lessons in grades 2-4 of primary school. By analysing selected transcribed teaching episodes we are now investigating questions related to the capacities of young children to interpret symbolic or structured contexts and graphical means of representation and to use them for mathematical reasoning and justification.

The introduction to symbols, symbolic diagrams, pictures and the reading, and interpretation of symbols is of great importance for the formation of every culture (WAGNER, 1981, 1986). And mathematics as well as the teaching of mathematics can be seen as a particular culture in which symbols play a central role:

> "Mathematics deals *per se* with signs, symbols, symbolic connections, abstract diagrams and relations. To understand the specific features of a culture, it is important to become aware of how members of the culture are being introduced into the interpretation and the use of cultural rites and cultural symbols. The use of the symbols in the mathematics classroom culture is constituted in a specific way, giving social and communicative meaning to letters, signs and diagrams during the course of ritualized procedures of negotiation.

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1 This sentence is a translation of a student's statement as recorded in our lessons transcripts. In German he said: "Den Rechenstrich könnte man wie so ein Lineal gebrauchen, aber nicht so genau."

2 The German title of our project is "Anschauungsgestütztes Argumentieren im Mathematikunterricht der Grundschule". It is conducted by Heinz Steinbring and Petra Scherer and financially supported by the University of Dortmund.

Social interaction constitutes a specific teaching culture based on school-mathematical symbols that are interpreted according to particular conventions and methodical rules." (STEINBRING 1997, 50)

In the following analysis of some short teaching episodes the example of the symbol "empty numberline" will be used to illustrate the childrens' constructive achievements in the comprehension or better the production of a symbol and in the interactive processes of fixing the meaning of such a symbol.

2. The Empty Numberline

The empty numberline (terminology adopted from TREFFERS 1991) consists of a horizontal line, on which one can mark numbers and also number operations, the latter by means of arcs above the line.[3]

Up to this point there is no distinction between the numberline and the empty numberline. But they differ in one central point. In contrast to the standard numberline there are neither a scale nor any other pre-given objective landmarks on the empty numberline. And in the case of the empty numberline there is no rule which would require, for example, the same spatial distance between the marks which correspond to two pairs of numbers having an equal arithmetical distance. The empty numberline therefore is a reproduction of the normal numberline that is not faithful to the scale but which respects the order of numbers. Thus one can see the empty numberline as a self-made sketch that helps to elucidate important considerations about the order of numbers, and also promotes the development and – a bit later – the reflection of half-written strategies for addition and subtraction (WITTMANN et. al. 1996b, 60).

For a person, who is accustomed to dealing with the numberline and who has made experiences with that or a similar model for the numbers and their operations for several years, the empty numberline – at first glance – doesn't offer any surprise. At best, it seems to be a simplification of the normal numberline. When a teacher introduces this means of visualization in his or her mathematics lesson in primary school, he or she easily runs the risk of underestimating the problems which can arise for the young students in their first encounter with this mathematical symbol.[4] Our video tapes and lesson observations demonstrate that the children have to work out ways of reading and using such an apparently simply structured symbol in a constructive process.

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3 German authors use the term "Rechenstreifen" (perhaps to be translated as "calculating stripe"), see HÖHTKER & SELTER 1995, SUNDERMANN & SELTER 1995, or "Rechenstrich" (= "calculating line"), see WITTMANN et al. 1996a, p. 8, which also was the current name used by the teachers and the students in the lessons we observed in our project.

4 It is also possible, of course, that the complexity and the potentialities of the empty numberline are underestimated by the teachers.

3. The Childrens’ Interpretations: Ways of Using the Empty Numberline

3.1. Spontaneous Views – First Approaches to a New Introduced Means of Representation

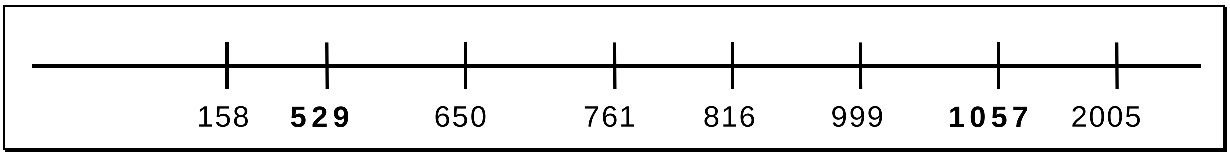
Within our project we observed and documented several lesson sequences that concentrated on the empty numberline in 3rd and 4th grade classes of primary school. In these lessons the ways, in which the teachers introduced the empty numberline, differed from each other. One commonality in these differing procedures was that the greatest amount of attention in the first lesson was given to the orientation on the empty numberline. The representation of addition and subtraction tasks was only dealt with in subsequent lessons.

In the discussion and resolution of a problem, that was given to a 4th grade class as a lead-in, one can see very clearly with what kind of pre-comprehension and pre-knowledge the children approach the new symbol to be learned and what references they have at their disposal for interpreting and understanding it.

The teacher has drawn a straight line on the blackboard and she opens the lesson as follows:

T: “Ok, there is a line on the blackboard. We call it empty numberline. … *(she writes “empty numberline” on the board)* And on the empty numberline … five hundred and twenty-nine could be here, and here, one thousand and fifty-seven … *(she writes onto the board)*. Enter a number on the empty numberline. Diana, please.”[5]

Successively several children come to the blackboard, each of them chooses a number and enters it on the empty numberline. Finally they have produced the following image on the blackboard:



[5] It is a very difficult task to translate a transcribed statement of a teacher or a student from German to English without losing some subtleties in the meaning of what was originally said. Often the translation can’t avoid being already, to a certain extent, an interpretation. Therefore I will always add the original German text in a footnote, when I cite from our transcripts. In this case the original text is the following:

“So, ein Strich ist da an der Tafel. Wir nennen ihn Rechenstrich. … (*schreibt “Rechenstrich” an die Tafel*) Und auf dem Rechenstrich … könnte hier die fünfhundertneunundzwanzig sein, und hier steht die tausendsiebenundfünfzig …(*trägt ein*). Trage mal eine Zahl auf dem Rechenstrich ein. Diana.”

During the ensueing discussion it becomes very clear that the children in their first approach to the symbol “empty numberline” take their bearings from the standard numberline with which they have already become acquainted. From their point of view it therefore seems to be a bit strange that, in this case, the spatial distances of the marks on the numberline don’t correspond to the arithmetical distances of the numbers. One child remarks:

S: “Yes, that’s funny, because, because, suddenly, first one hundred and fifty-eight, and then it makes a gigantic jump to five hundred and twenty-nine.”[6]

The children also criticize the absence of certain landmarks that eventually could serve as an orientation for the accurate division of the empty numberline. Another student says:

V: “We have, if you count like this, then, it is like, um, five, six, seven, eight, nine, ten; but – uh – the one hundred and fifty-eight, somehow two, three, and four should be in between them.”[7]

Obviously she speaks about the missing hundred numbers. It is clear from the following short intection that the background for this statement is shaped by the conception which is familiar to the children from the standard numberline and its scale:

Teacher: “You know other lines with numbers on them. I suppose you compare this one with those. What did they look like in the past, those ones we also know from the mathematics book?”

M: “Um, there are always little lines drawn in between, where you then write in the numbers everywhere.”

D: “And they are always, um, for example, placed apart in the same distance instead of being randomly placed.”[8]

But in this lesson, as well as in the other lessons, in which the empty numberline is introduced, the discussion finally culminates in the agreement that one has to keep the arithmetical order of the numbers when entering numbers on the empty numberline, but need not observe the spatial distances (as one must do in the case of the standard numberline). One student expresses this as follows:

S: “Yeah, they are actually jumbled numbers, you can’t put them in any sequence. And – mmm, uh, mmm, but somehow they are ordered according to their size.”[9]

One child also cites another reference context which is situated outside of mathematical instruction:

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6 “Ja. Das ist komisch, weil, weil auf einmal erst hundertachtundfünfzig, und dann macht es einen Riesensprung zu fünfhundertneunundzwanzig.”

7 “Wir haben die, wenn man so zählt, dann, ist das so – ähm – fünf, sechs, sieben, acht, neun, zehn, aber – ähm – die einhundertachtundfünfzig, dann müßte irgendwie noch so zwei, drei und vier zwischen.”

8 L: “Ihr kennt andere Streifen mit Zahlen dran. Ich glaube, damit vergleicht ihr das. Wie sahen die denn in der Vergangenheit aus, die wir kennen, auch aus dem Mathebuch?”

M: “Äh, da sind immer auch so Striche zwischengezogen, wo man dann überall die Zahlen einträgt.”

D: “Die sind auch, ähm, z.B. immer gleich auseinandergesetzt anstatt immer so kreuz und quer.”

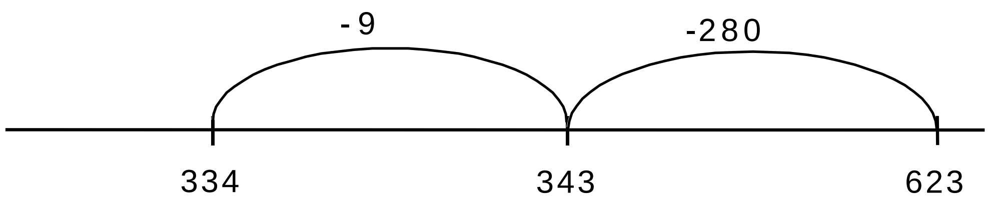
9 “Ja, das sind zwar durcheinandere Zahlen, die kann man nicht in eine Reihenfolge irgendwie tun. Und – mh, äh, mh, – aber die sind irgendwie nach der Größe geordnet.”

C: “That’s also like, that’s like an alphabet. They are ordered alphabetically. It could be.”[10]

3.2. The Approved Rules, Interpretations, and Modes of Use are relativated in Applying the Empty Numberline

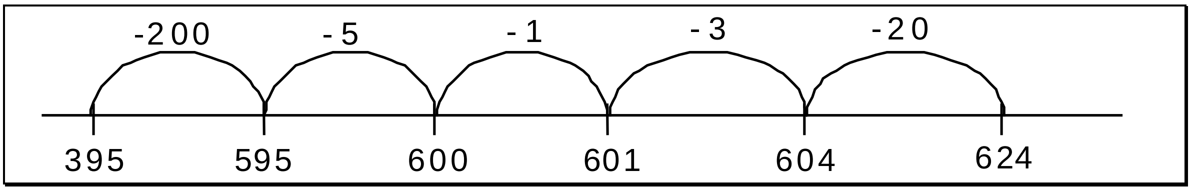
At this point, the strong connection to the standard numberline in the interpretation of the empty numberline is already loosened a bit. By operating with this new means of representation in the subsequent lessons the children discover more and more the peculiarities of the new symbol. They experience that the empty numberline doesn’t provide a fixed structure – as is suggested by the conventional use of the standard numberline – and that they themselves in their own testing activities and interpretations can or must give a flexible structure and interpretation to the symbol “empty numberline”.

For example, the expectation, formulated in the beginning, that the arithmetical difference between the numbers should be translated to the spatial distance between the marks, is relativated when the children themselves try to represent addition and subtraction tasks. In this context they now produce diagrams like the following one:



A student realizes: “Yes, one simply has to – um – think, imagine by oneself what the distance is.”[11] And another child remarks: “Indeed, it’s not a map. Maps have to be exact.”[12]

The following way of calculating and representing the task “624 – 229”, developed on the blackboard by several students in team-work, demonstrates very well that the children begin to test the potentialities, which the empty numberline offers, and shows how they now start to play with them:

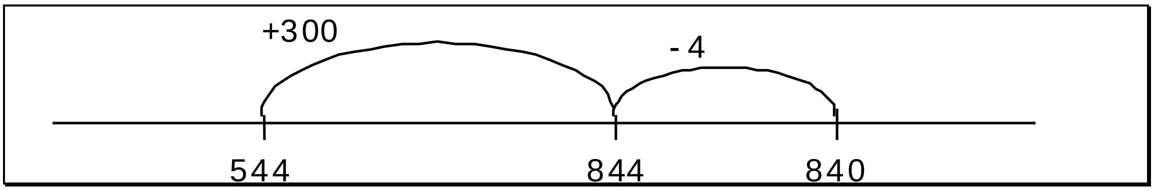


10 “Das ist ja auch so, das ist wie ein Alphabet. So alphabetisch geordnet sind die. Könnte sein.”

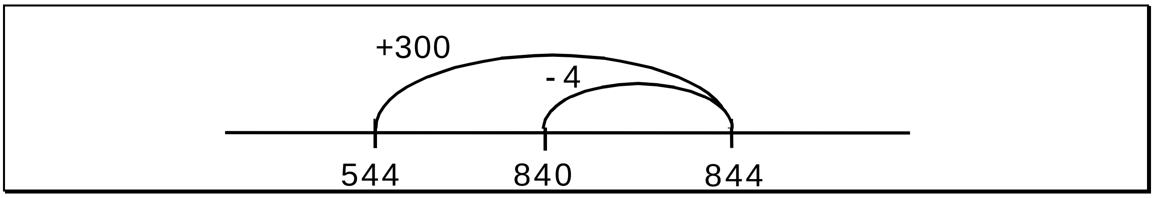
11 “Ja, man muß sich halt des, ähm, denken, vorstellen, was der Abstand ist.”

12 “Es ist ja keine Landkarte. Landkarten müssen genau sein.” It was in this context that one student said the the sentence cited in the title: “You can use the empty numberline like a ruler, but not as exactly.”

Working on certain concrete tasks the children temporarily – when it seems to be suitable – even disregard the rule, that when entering numbers on the empty numberline one must respect their arithmetical order. A student, for instance, who simplified the task “544 + 296” to “544 + 300 - 4”, represents his way of calculating in the following diagram:



At first the other children are not disturbed by this irregularity. Only when the teacher questions the presented diagram do they notice the violation of the main rule of the empty numberline. And it is under guidance of the teacher that they then produce a representation of the suggested way of calculating “544 + 296”, which oberserves this rule:



4. Summary

This last example demonstrates very well that in the progress of the lessons on the empty numberline the children in the interpretation of the new symbol get loose from their initial ideas and conceptions, which were linked very strongly with the standard numberline. Due to the experiences they have made in their everyday mathematics instruction the children see the numberline – and many other mathematical symbols, too – as an accomplished and pregiven structure, of which the steady interpretation as well as the correct mode of use are communicated by the teacher (STEINBRING 1996). But when the children work with the empty numberline it turns out, that the structure and the interpretation for this symbol can only be developed in an active process of entering numbers and calculations. The children learn that they themselves are responsible for the production of the symbolic character of the empty numberline.

To a very large degree the construction and the interpretation of the mathematical symbol “empty numberline” demands an activity of one’s own. It doesn’t require a mere accurate imitation, but a flexible, methaphorical, and generalizing perceiving, handling and imitating. In this context of using the empty numberline, it turns out very clearly, that

> mathematical symbols don’t exist absolutely independently nor beyond any social and subjective knowing agents. The symbolic character of mathematical diagrams is closely related to the way in which the structure lying in the symbol is interpreted, created, and formed actively by the knowing agent. (translated from STEINBRING, 1996)

References

HÖHTKER, B. & SELTER, C. (1995): *Von der Hunderterkette zum leeren Zahlenstrahl. Orientierungsübungen im Hunderterraum,* in: G. N. Müller & E. Ch. Wittmann (Hrsg.), Mit Kindern rechnen, Arbeitskreis Grundschule, Frankfurt am Main, 122-137

STEINBRING, H. (1996): *Die mathematische Unterrichtskultur als eine spezifische Interpretation von Zeichen, Symbolen und Diagrammen - Analyse von Unterrichtsstunden zur Einführung und zum Gebrauch des Rechenstrichs,* unpublished paper

STEINBRING, H. (1997): *Epistemological Investigation of Classroom Interaction in Elementary Mathematics Teaching,* in: Educational Studies in Mathematics (32), 49-92

STEINBRING, H. & SCHERER, P. (1998): Strategien und Begründungen an Veranschaulichungen – statische und dynamische Deutungen, to appear in: Grundschulzeitschrift, July 1998

SUNDERMANN, B. & SELTER, C. (1995): *Halbschriftliche Addition und Subtraktion im Tausenderraum (I): Die Rechenmethoden der Schüler,* in: Grundschulunterricht (42), Heft 1, 22-25

TREFFERS, A. (1991): *Didactical background of a mathematics program for primary education,* in: L. Streefland (Hg.), Realistic mathematics education in primary school, Freudenthal Institute, Utrecht, 21-56

WAGNER, R. (1981). *The invention of culture*, University of Chicago Press, Chicago

WAGNER, R. (1986). *Symbols that stand for themselves*, University of Chicago Press, Chicago

WITTMANN, E. CH. et al. (1996a): *Das Zahlenbuch. Mathematik im 3. Schuljahr,* Ernst Klett Grundschulverlag, Düsseldorf u.a.

WITTMANN, E. CH. et al. (1996b): *Das Zahlenbuch. Mathematik im 3. Schuljahr, Lehrerband,* Ernst Klett Grundschulverlag, Düsseldorf u.a.

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