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PROBLEM SOLVING PROCESSES IN A SPATIAL GEOMETRY ENVIRONMENT[1]

Abstract

This article focuses on the problem solving success and the strategies of students who worked on spatial geometry problems. These problems were presented to some of the subjects in a computer environment and to some in an environment providing concrete means of manipulation. We will argue that individual problem solving strategies as well as errors in the problem solving processes do not differ principally between students assigned to the different learning environments. However students working on a computer are more successful at the beginning of their problem solving and gain faster access to the specifics of the problems.

1. Introduction

Problem solving plays a prominent role in the mathematics classroom. Problems are intended to provide opportunities for applying mathematical knowledge. Moreover, problem solving may broaden the view of what mathematics is all about. It is widely accepted that problems do not constitute mathematical learning processes but have to be examined with respect to the individuals working on them. Therefore, insights into students' problem solving processes are important for mathematics teachers in order to better understand their students and provide challenging as well as adequate problems in their classes. According to the NCTM standards, mathematics instructional programmes should focus on problem solving as part of understanding mathematics. In particular, problem solving may enable all students to develop new mathematical knowledge by working on problems and by applying a wide variety of strategies (NCTM, 2000). It is necessary that such strategies receive instructional attention if students are supposed to learn this kind of knowledge (NCTM, 2000; Schoenfeld, 1992). As a consequence, mathematics teachers should learn more about their students' strategies in order to help them master their problem solving tasks. Complex problems usually demand a sequence of steps in order to get a correct solution. Accordingly, efficient problem solving strategies are characterized by (i) structuring the problem with respect to elements necessary for a solution, (ii) identifying relevant transformations of these elements, and (iii) applying them to the specific situation. From a didactical perspective it is important to learn how these elements of problem solving strategies arise, how to describe developmental aspects in this process and how to identify conditional factors.

1.1 Problem Solving and Spatial Abilities

Not only mathematics educators are concerned with spatial abilities but psychologists as well (Besuden, 1979; Bishop, 1980; Clements & Battista, 1992). In particular there is a large body of research in psychology concerning the basic aspects of spatial problem solving processes. Part of this research is based on specific items of various intelligence tests. Accordingly, it is devoted to general aspects of problem solving. Moreover there is some research which is more closely related to mathematics learning. In the following section, we will discuss briefly some research in psychology which focuses on spatial problem solving.

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Putz-Osterloh and Lüer (1977; 1979) presented spatial problems from a variety of standardized tests (Amthauer, 1953; Horn, 1962; Meili, 1955) to their subjects. All these problems included identifying parts of a cube or a cube as a whole. The problem solving processes were monitored by an eye-tracker. Locus and duration of the fixation were recorded in order to identify the subjects' problem solving strategies. Putz-Osterloh and Lüer revealed that successful problem-solvers did not only take into account the characteristics of the (differently patterned) faces of the cube but also their relationships.

Köller, Rost and Köller (1994) performed a similar experiment, but without eye-tracking equipment. They presented problems to their subjects, which aimed at comparing different perspectives of cubes. The problem solving strategies used by their subjects could roughly be classified as holistic and analytic. In addition to the results of Putz-Osterloh and Lüer (1977; 1979), the experiments revealed that some of the problem solving processes were guided by elaborate analytical strategies based on relational aspects of the problem presented. Moreover, there were some subjects who used guessing strategies. The different strategies were identified by means of a mixed Rasch model and a latent cluster analysis of data from a paper and pencil test. They were then assigned to their specific contents with the help of retrospective interviews. Köller, Rost and Köller (1994) regard the different strategies as personal preferences and do not attribute them to a specific problem.

A study by Leutner and Kretschmar (1988) revealed the influence of different problem solving environments within different presentations of spatial geometry problems. They used four different settings in order to test their research questions. Firstly, the subjects were assigned to a specific group with respect to the visualization of problems in a computer environment or in an environment providing concrete objects. Secondly, they were assigned to a specific subgroup with respect to the teaching method and either guided by a teacher presentation or by the hands-on activities of the students. The results demonstrate that students working in a computer environment succeed as well as their classmates using a manual learning environment. Moreover, spatial abilities proved to be the best predictor for succeeding in a geometry achievement test.

The experiments by Putz-Osterloh and Lüer (1977; 1979) as well as by Köller, Rost and Köller (1994) are restricted to two-dimensional plane geometry problems. Their results suggest an expansion to three-dimensional spatial geometry problems. In particular, it would be interesting to identify strategies used in spatial problem solving processes. The experiment performed by Leutner and Kretschmar (1988) indicates that a specific visualization of a problem is in general not correlated to the problem solving success. In addition it might be interesting to identify a possible correlation between spatial abilities and problem solving strategies.

1.2 Methodological Aspects of Research on Cognitive Processes

Identifying and describing cognitive processes with respect to mathematics is an important issue in mathematics education research. Most of this research is devoted to counting and numbers (e.g. Riley & Greeno, 1980; Greeno, Riley, & Gelman, 1984), to addition and subtraction (e.g. Riley, Green, & Heller, 1983; Behr, Greeno, Leinhardt, Resnick, & Rabinowitz, 1985), and to ratio and fractions (e.g. Hart, 1981; Vergnaud, 1983; Behr, Wachsmuth, Post, & Lesh, 1984; Viet & Kurth, 1989). The methods used in these studies are manifold which is particularly due to the fact that cognitive processes cannot be viewed directly. They may only be described by analyzing and interpreting the output of a specific action, for example a specific problem solving process of an individual. As a consequence it is usually difficult to justify a certain result. Some researchers approach this conflict by modelling the problem solving behaviour of an individual on a computer.

This is an important aspect of research on cognitive processes. It has its foundations in the cooperation of researchers from cognitive psychology and computer science, in particular artificial intelligence (Mandl & Spada, 1988). Modelling cognitive processes provides information on probable aspects of an individual's problem solving. In particular, modelling a problem solving process may be apt in verifying theories on specific human problem solving processes. Modelling cognitive processes is based on a research paradigm which regards human problem solving as information processing (Dörner, 1976). Cognitive processes are knowledge-based acts of symbolic processing (Opwis, 1992). This paradigm has also been accepted by some mathematics educators and their research led to computer models of students' problem solving strategies (Greeno, 1983; Wachsmuth, 1985; Haussmann & Reiss, 1989a, 1989b). Computer modelling is not only an adequate tool for dealing with problems on numbers and counting, but has also been used in geometry (Anderson, Boyle, & Yots, 1985).

2. Method

Problem solving strategies in a spatial environment were investigated in a research experiment with 60 seventh-graders. The students participated in intensive interviews, consisting of a 45-minute problem solving session and a 45-minute concept mapping session. In the first part of the interview the subjects were supposed to work on a spatial geometry problem. All students were assigned to one of two groups and worked in either a computer-based environment or in an environment with real means of manipulation. All sessions were videotaped and transcripts of the sessions included all verbal contributions of the subjects and all their movements in the problem solving environment. The experiment aimed at answering the following research questions:

- What kind of strategies are used by children in a spatial problem solving environment?
- Are there typical errors in the students' problem solving processes?
- Are there differences between successful and less successful problem solvers with respect to the strategies used in their problem solving?
- Do problem solvers use different strategies in computer simulated environments respectively in manual environments?

The students were presented with the front view, top view, and side view of cubes or cube-like solids, which consisted of 8 small cubes or 27 small cubes and prisms. They were asked to construct the corresponding solid either on the computer screen or by direct manipulation.

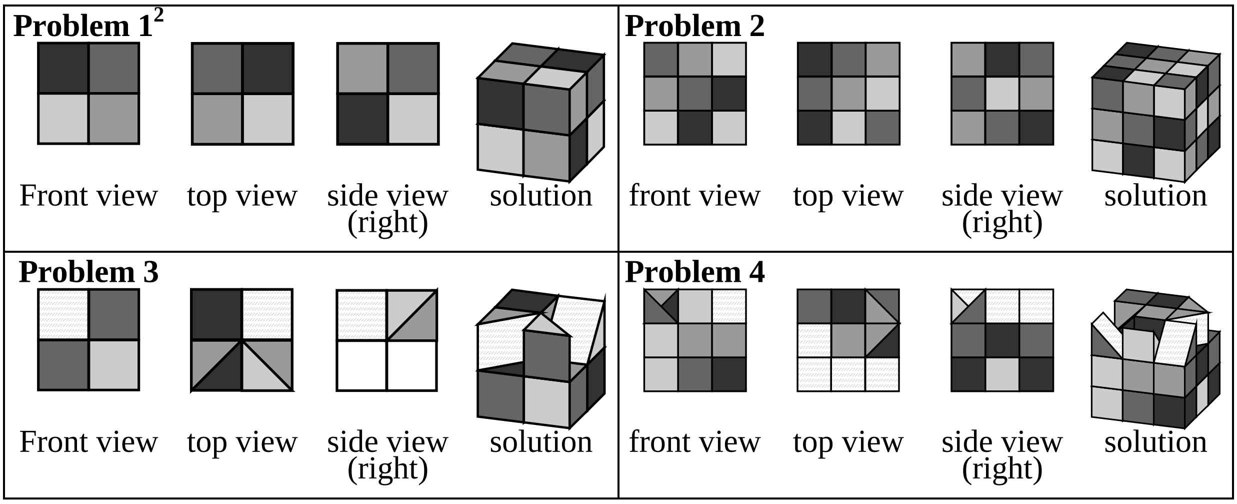

Figure 1: Spatial-geometry problems: views and corresponding solid

3. Results

The students applied a wide range of problem solving strategies. In particular, the description of their strategies as spatial, relational, or plane strategies, as suggested by Köller, Rost, and Köller (1994) did not match the students' solutions. Their work showed a multitude of distinct personal as well as intra-individual differences. As a consequence we described characteristic stages of the problem solving processes with respect to the specific context. The characteristic properties of the problem solving environment may be described in terms of the position of a specific cube or prism, its colours, the arrangement and shape of neighbouring cubes or prisms and the orientation of cubes or prisms. Successful problem solving presupposes consideration of all these characteristics which can be arranged in a sequence underlying the problem solving process (Pospeschill & Reiss, 1999).

3.1 Errors in the students' problem solving processes

If students do not regard the characteristics of the problem solving environment, e.g. position, colour, arrangement, shape, and orientation of blocks (cubes or prisms), they will probably perform specific errors in their problem solving processes. These errors were identified by analyzing the transcripts of the problem solving sessions. These errors can be assigned to specific classes of errors to be discussed in the following section and will be described via typical examples.

Position

An error concerning the position of a block is straightforward. An example of this kind of error is shown in Figure 2.

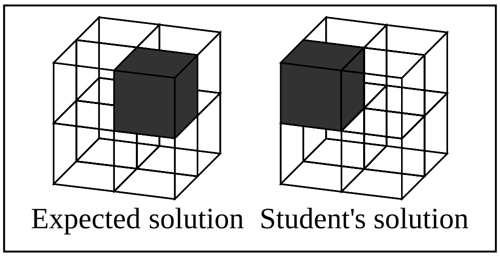

Figure 2: Error of Position

[2] Due to black and white printing, colours were changed to different shades of grey. This table shows the relation:

Yellow (light Grey)	Green (middle Grey)	Red (dark Grey)	Blue (Black)	Sloping surface

Colour

An error concerning the colour is encountered if the problem solver chooses a wrong colour with respect to one or more faces of a block. Figure 3 shows various errors of colour with respect to the four cubes involved in the arrangement.

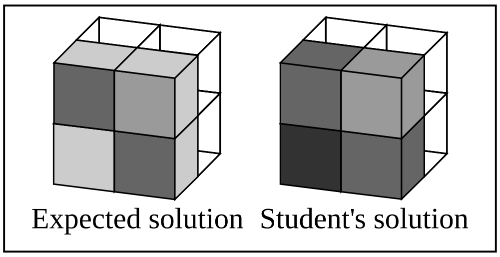

Figure 3: Error of Colour

Arrangement

Errors of arrangement are higher order errors, which will occur in complex problem solving environments. A typical situation is demonstrated in Figure 4. In the front view and the side view there are squares split into different coloured triangles. They provide information on different layers of the block. Accordingly, a problem solver may mix up the positions in the front and in the back.

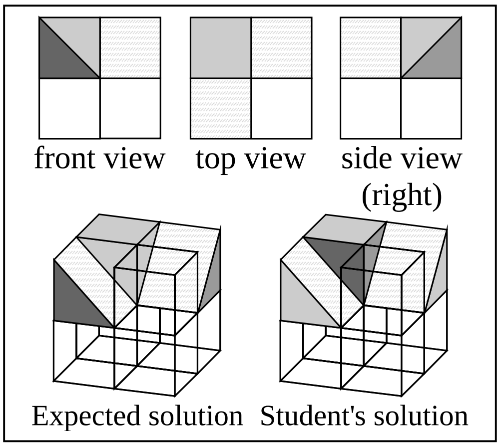

Figure 4: Error of Arrangement

Shape

This category includes two different kinds of error. They have in common the fact that the shape of the correct block differs from the choice of the problem solver.

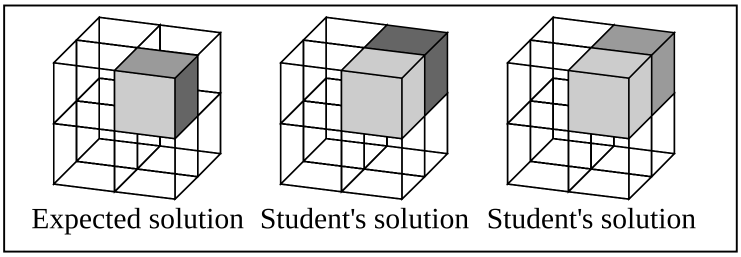

Figure 5: Error of Shape C (Cube)

The first kind of error is typical during the early stages of working on the problems. It occurs if children are not able to combine two pieces of information on colours in a single cube. As a consequence they choose different unicoloured cubes and arrange them in a sequence (Figure 5). This phenomenon has been described by Wollring (1994) in spatial geometry environments and is regarded as sequential coding of the spatial depth.

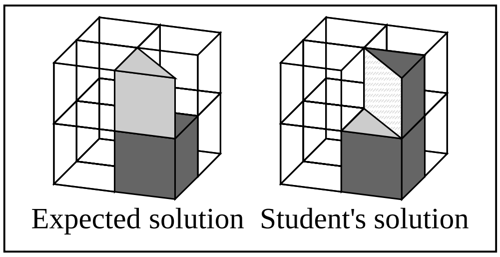

Figure 6: Error of Shape P (Prism)

The second kind of errors of this type is equivalent to errors of arrangement. It is associated with views providing information on more than one layer. In this case, the triangles are interpreted correctly with respect to their arrangement but problem solvers disregard information from other views resulting in the choice of a cube instead of a prism (Figure 6).

Orientation

Students performing errors of orientation choose a correct block but turn it around. Figure 7 and Figure 8 provide examples of errors of orientation.

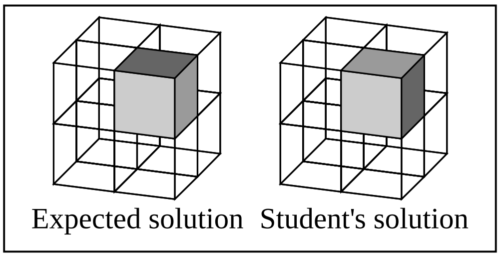

Figure 7: Error of Orientation C (Cube)

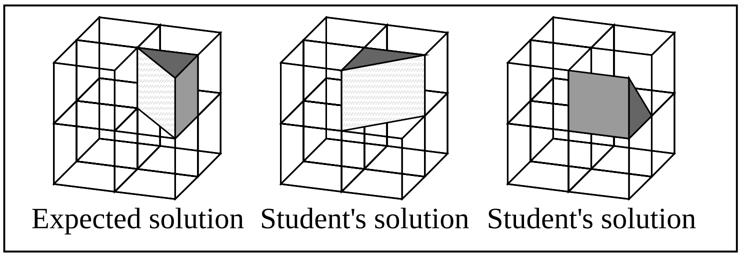

Figure 8: Error of Orientation P (Prism)

Mysterious Errors

Mysterious errors are those errors which cannot be explained in terms of any of the classes described above.

3.2 Analysis of Data from the Students' Problem Solving Processes

The classes of errors identified in the students' problem solving processes were implemented in a computer program. All data from our subjects were analyzed with respect to these categories. In particular, the implementation provides data from students working in the computer environment and data from students working with direct means of manipulation.

Comparing the overall number of moves performed by 60 children with respect to four problems in the computer environment and the manual environment reveals significant differences (see Table 1). There is also a highly significant difference between the two groups with respect to the percentage of incorrect moves ($\chi^2(1)=22.31$; $p<.001$).

Moves	Computer environment	Manual environment
Number of moves	2231	4068
Number of incorrect moves	26.8%	29.4%

Table 1: Number of moves performed by the students

The individual data with respect to the four problems reveal a significant difference between the computer environment group and the manual environment group. Tables 2 and 3 provide statistical information on the number of moves performed by the students and their problem solving success.

Computer environment	Mean	Standard deviation	Percentage of correct solutions
Problem 1	13.23	11.57	96.7%
Problem 2	25.03	5.20	80.0%
Problem 3	17.97	8.68	16.7%
Problem 4	21.76	10.18	16.7%

Table 2: Summary statistics for each problem in the computer environment

Manual environment	Mean	Standard deviation	Percentage of correct solutions
Problem 1	30.33	22.88	66.7%
Problem 2	52.37	26.23	60.0%
Problem 3	26.00	18.07	3.3%
Problem 4	31.04	10.21	6.7%

Table 3: Summary statistics for each problem in the manual environment

The mean number of moves differs significantly for all problems with respect to the different environments. This difference is highly significant for problem 1 (p=.001) and problem 2 (p<.000) and significant for problem 3 (p=.032) and problem 4 (p=.002).

Moreover, the computer environment group and the manual environment group show significant differences with respect to their problem solving success for problem 1 ($\chi^2(1)=7.124$; p=.008; continuity correction). It is not possible to report on significant differences for problem 2 ($\chi^2(1)=1.984$; p=.159), problem 3 (p=.097) and problem 4 (p=.193).

All data were assigned to the specific types of errors. In Table 4, the data are presented for both the computer environment and the manual environment. They suggest a similar tendency in the types of errors with respect to the different environments.

Type of error	Frequency computer environment (%)	Frequency manual environment(%)
Position	19.2	15.9
Colour	56.9	51.6
Arrangement	1.0	0.1
Shape C	4.5	2.2
Shape P	0.7	0.6
Orientation C	6.7	15.8
Orientation P	8.0	7.9
Mysterious	3.0	5.9

Table 4: Percentage of specific errors in the different environments

There is only one remarkable difference between the computer and the manual environments concerning errors of orientation with respect to cubes. The fit between suggested types of errors and the data is supported by the low number of moves regarded as mysterious moves.

The data have been analyzed with respect to probable errors. Table 5 shows the results for the colour and shape of a specific block (mean m_C and standard deviation std_C in the computer environment and m_M and std_M in the manual environment respectively).

Problems and characteristics	m_C	std_C	m_M	std_M	F	df	p
Problem 1 colour	0.17	0.91	1.03	1.65	6.336	1.58	.015
Problem 2 colour	0.67	2.12	1.13	2.22	0.691	1.58	.409
Problem 3 shape	2.57	2.43	4.00	2.18	5.778	1.58	.019
Problem 3 colour	1.77	1.25	1.80	1.32	0.010	1.58	.920
Problem 4 shape	6.24	7.32	6.31	4.25	0.002	1.49	.968
Problem 4 colour	2.52	2.24	3.31	2.26	1.564	1.49	.217

Table 5: Colour and shape as characteristics of the problem solving environments

The problem solving success of a student was not assessed by the number of correct blocks, but by the number of correct views of the solution block. Accordingly, problem 1 and problem 3 were assigned a maximum of 3*4=12 points, and problem 2 and problem 4 were assigned a maximum of 3*9=27 points. The data do not suggest differences between children working on the computer and children working manipulatively. With respect to the easier problems 1 and 2 we have m_C=38.167 (std_C =2.984) for the computer environment and m_M=36.833 (std_M =3.630) for the manual environment (p=.1256). The high scores reveal that most children finally succeeded in their problem solving processes. With respect to problems 3 and 4 which involved prisms we find m_C=22.867 (std_C =10.963) for the computer environment and m_M=21.267 (std_M =9.108) for the manual environment (p=.5411).

3.3 Sequence of Errors in the Students' Problem Solving Processes

According to the model of characteristics in the problem solving process (Pospeschill & Reiss, 1999), the sequence of students' errors should follow a common pattern. With regard to this model, students are supposed to first master identifying the correct position of a block. The characteristics of colour, arrangement, shape, and orientation are supposed to represent a sequence of increasing difficulties to master.

In order to identify a sequence of types, all errors were assigned to the specific classes with respect to each student. The number of moves performed until a specific error occurred was regarded as variable. For the different classes of errors the median was identified with respect to the number of moves performed.

Two of the problems presented to the students did not involve prisms. Accordingly, the types of possible errors were restricted to position, colour, shape (C), and orientation (C). The medians and their sequences with respect to this type of errors and their sequence were identified for the computer environment and the manual environment respectively. The medians for errors of shape (C) are set in parentheses as they were infrequently performed.

0.317—0.429—(0.585)—0.727 (Problem 1; computer environment)

0.593—0.454—(0.578)—0.630 (Problem 2; computer environment)

0.329—0.429—(0.586)—0.600 (Problem 1; manual environment)

0.271—0.349—(0.187)—0.375 (Problem 2; manual environment)

These sequences of data are, in most aspects, a good fit to the model. Moreover the two environments do not differ significantly with respect to the sequences of their medians. An exception is the sequence of the medians for position and colour in the second row. A possible explanation is based on the specifics of the computer environment. Initially students sometimes had difficulties controlling the computer program. Accordingly, blocks were moved to wrong positions but this error was corrected immediately. The other gap between the model and the data is found in the fourth row. This difference might be due to the low number of errors of shape, only nine in all, made by all subjects working in the manual environment.

It is not possible to provide a similar sequence of data for the more difficult problems 3 and 4. On the one hand students worked on these problems after solving the simpler problems hence there are training effects which manifest themselves in a lower rate of errors. Moreover, there is a tendency that errors arise later in the course of problem solving; in particular they occur when students try to integrate prisms into their problem solving. Part of the characteristics is already integrated due to the training provided by the simpler problems, while part is new to the students and has to be integrated into their problem solving.

There are some qualitative results which might reveal interesting aspects of spatial problem solving processes:

- Students who performed errors of shape (C) in problem 3 or 4 were not able to solve any of these. In particular, these children tend to perform lower than average.
- Errors of position are mostly performed by students with scores above average.
- Errors of shape (P) are most frequently found among students with high scores.

It has to be taken into account that there is only a small database for these findings. They should only be regarded as tendencies and further research is needed.

4. Discussion

The empirical data from students' problem solving processes support the model and thereby the characteristics for spatial problem solving (Pospeschill & Reiss, 1999). In particular, they demonstrate the validity of this model in a computer environment as well as in a manual environment. Students tend to make similar errors in both environments.

With respect to the environments the experiment reveals that the most important difference is the number of moves performed. Children working manually perform significantly more moves than their classmates working in a computer environment. Additionally, at the beginning of the problem solving sessions, children working manipulatively tend to be less successful than children working with a computer program. Solving the first problem is much easier for the computer group. These differences decrease when children get more used to the specific problems. Differences in the problem solving behaviour with respect to problems 2, 3, and 4 cannot be reported. A similar result is supported by the data on differences between characteristic errors. Whereas there are significant differences in regarding specific characteristics when they are first introduced, no differences can be identified in the further course of problem solving.

The findings suggest that children progress faster when in the computer problem solving environment. Handling concrete objects is more difficult in the beginning, but the differences between the environments tend to play a less important role in the course of problem solving. Accordingly, it may be assumed that the different environments do not influence the problem solving processes significantly after children have become acquainted with a specific environment.

The manual environment differs from the computer environment in a number of important features (see Table 6) which may cause the difficulties described.

Manual environment	Computer environment
Manipulations are bound to physical laws, e.g. gravity forces the problem solver to begin with the bottom layer.	Physical laws do not hinder the choice of a specific problem solving strategy. There is no need to place hidden blocks.
Objects have three dimensions. They may be turned around and may be used differently.	Blocks cannot be rotated. Therefore the mental image of the corresponding three-dimensional object is more important.
Objects have characteristic properties like volume, weight, and structure. Tactile experiences are possible.	Objects have only visual properties like form, colour, or position.
There are no restrictions on direct placement of a block in the problem solving environment.	Movements of elements are restricted to predefined manipulation of elements on the computer screen.
The problem solver is responsible for the order of the stack of blocks.	The working environment is predefined and cannot be changed by the problem solver.
The size of blocks does not change.	Due to restrictions in the size of the computer screen, blocks are represented in two different sizes.

Table 6: Differences between manual and computer environments

The results suggest that the model and the identified characteristics of the problem solving environment may be used as a basis for categories of errors. The analysis reveals that about 95% of the errors performed by the students may be assigned to specific classes of errors. Even with respect to these categories, differences between the computer environment and the manual environment have not been found. This supports the idea that children construct a mental model of the two-dimensional presentation on the computer screen. They are then capable of successful problem solving with respect to their mental models.

Our research suggests a revised view of children's work with computers in the mathematics classroom. We argued that the principal strategies used by students in spatial problem solving do not differ within either a computer or a manual environment. Moreover, the kinds of errors students perform are probably basically identical. The most important differences between children working on a computer and children working manually may be found in their access to the problem solving environment. In our study, the students had less problems in the computer environment and succeeded faster. It would be an interesting research question to examine students' motivation for problem solving with regard to the different problem solving environments and their success therein.

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