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METACOGNITION IN MATHEMATICS LESSONS

Abstract

The one who uses the knowledge about his own way of thinking, the one who consciously controls his own way of thinking and learning, i.e. the one who pursues metacognition, creates prerequisites for the improvement of learning effectiveness. The following paper describes possibilities how every day lessons can stimulate pupils to metacognitive activities. The success of metacognition depends on the fact to what extend the individually formed mental models are shown outwardly. There are some tasks that have proved suitable as they cause the pupils to reveal their own cognition.

1. Expectations from metacognition

Imagine a person who asks him-/herself: Am I aware of what I do, what I say, what I think? Do I know what I do, say or think? In case the answer is “yes”, the person has been successful in metacognition, because metacognition means knowing and thinking about the one’s own cognitive system as well as the ability to control and check this system. On the one hand learning, storing, keeping in mind, remembering, understanding, thinking and knowing become subjects of reflection, on the other hand mental activities are consciously planned, controlled and regulated. In short: Cognition of one’s own cognition and regulation of one’s own cognition are the two main components of metacognition.

Metacognition has been an important field of research in cognitive sciences for more than twenty years. The concept “metacognition” goes back to Flavell (Brown 1983, Flavell 1983). Metacognition in a certain sense presupposes a theory of mental models, especially a theory of knowledge representation. There are fundamental papers of Anderson (1983), Johnson-Laird (1983) and Minsky (1985). De Corte (1995) and Boekarts (1996) have summarized the expectations, determination of hierarchy and results as regards metacognition.

Cognitive beginnings in mathematics education go back to Davis & McKnight (1979). Results of research in cognitive structures (Schwank 1993) form the basis for a certain type of mathematics lessons, the Osnabrueck Mathematics Curriculum. There are conceptual considerations to this in Cohors-Fresenborg (1993, 1996). This is further described in two issues of the magazine “Der Mathematikunterricht” (Hoffmann 1993, 2001). Issue 3/1993, “Algorithmisches und axiomatisches Denken”, contains – apart from the cognitive theoretical classification – reports of teachers from mathematics lessons. Issue 1/2001, “Mathematik als Werkzeug zur Wissensrepräsentation”, shows the discursive teaching culture, the type of tasks and the metacognition practiced. Metacognition plays an important role in these mathematics lessons (Sijuts 1999b). The success of these mathematics lessons is, to a considerable extent, to be attributed to the metacognitive activities.

2. Examples for metacognition

The following examples are to show metacognition when learning mathematics. They deal with tasks and a corresponding choice of pupils’ solutions from the mathematics lessons mentioned. These mathematics lessons distinguish from others by a cognitive theoretical

orientation towards their conception, by a constructivism theoretical starting-point towards their execution and by an education and knowledge theoretical argumentation towards their authorization (Sjuts 1999b).

External and internal representations:

A central concept of cognitive science is the concept of “representation”. The question “How do I write it down?” relates to external representations and the question “How do I imagine it?” relates to internal representations.

The following task has been taken from a test (grade 9).

Task: *Elke says: “The number 0.121314 is not a rational number.”*
Martina: “Yes, it is, because it is a periodical decimal point number.”
Antje: “Are you talking about representations or imaginations? I think, you are both right.”
Please comment!

Solutions:

(I) *The number is shown as an irrational number. There are, however, several different ideas how the number could continue. If the number continued with 1213... after the 0.12131415161718191011, the number would be a periodical decimal point number. If you, however, considered the expression, the number would be as follows:*

$$0.1213141516171819110111112...$$

The question is, how does Martina think that the number continues. If she says it is a periodical decimal point number, the periodical line above the figures is missing.

(II) *Elke imagines the number as follows:*

$$0.1213141516171819202122232425...$$

So she is right with her assertion. As the number does not continue periodically it is irrational.

Martina, however, imagines the number as follows:

$$0.\overline{12131415161718191011}$$

She is also right with her assertion. As the number continuously repeats, it is a periodic decimal point number. The expression of the number was not exact enough so that different imaginations could develop.

(III) *I also think that they are both right, but only have different imaginations. Elke thinks the number continues as follows:*

$$...15161718191011112...,$$

so that it is irrational, and Martina thinks the number continues as follows:

$$...1516171819101112131415...,$$

so that it would be rational and that the numbers would repeat in the same order after a certain time.

(IV) When Elke says the number $0.121314\dots$ is not a rational number, she is right, as the number does not break off and is not periodical. Consequently it is an irrational number.

If Martina, however, thought that the number $0.121314\dots$ started from its beginning after the 4, Martina would be also right as the number would then be a periodical decimal point number.

This is how the two think the number will look like:

Elke: $0.1213141516171819\dots$

Martina: $0.121314121314121314\dots = 0.\overline{121314}$

The number Elke imagines is an irrational one, because it is not periodical or does not break off.

The number Martina imagines is periodical and therefore a rational one.

(I) concludes from the expression that it has to be an irrational number. Consequently Elke is right. For (I) the principle of construction is therefore unambiguous, as the expression itself shows an unambiguous principle. It is possible to have “different ideas how the number could continue”, but it is important that this is then clearly expressed (e.g. by a periodical line above the figures).

(II), (III) and (IV) consider Elke’s and Martina’s interpretations as equal. They mention, without exception, concrete possibilities of continuation. They realize the problem concerning the three dots “behind” the number. The dots do not stipulate the same continuation for everybody. If suitable continuations are found for each case, Elke and Martina can be both right. It is remarkable to what extent the pupils’ solutions and ideas arise and on what high level the pupils formulate their thoughts.

Tasks of this kind stress the mental work, not necessarily the respective contents or subjects of mathematics and this without interfering with their significance or demand. This task has got a cognitive constructivism design. It activates previous knowledge, consequently provokes revisions and consolidations, opens a wide field for tests and gives rise to ideas. It picks out the interplay between internal and external representations as a central theme.

Metamemory:

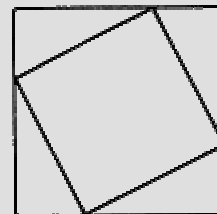
Tasks which ask for putting down one’s thoughts in writing promote the reading comprehension of unknown formulations as well as the gradual clarification of one’s own formulations. Through this a metamemory is developed which enables a conscious use of knowledge storage and call up.

Task: Pupils of grade 9 were asked: "How do you remember the Pythagoras Theorem?"

Following answers were given:

Anna: "When I think of the Pythagoras Theorem

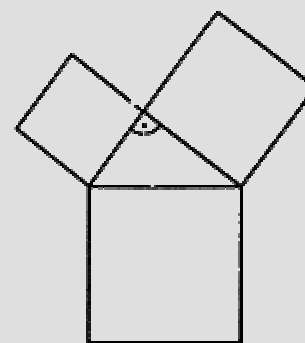
I imagine the following picture:"



Tim: "I first think of the numbers 3, 4 and 5 as $3^2 + 4^2 = 5^2$."

Wiebke: "I immediately think of $a^2 + b^2 = c^2$."

Simon: "This picture is most helpful for me:"



Please comment!

Solutions:

(I) Anna thinks of the proof for the Pythagoras Theorem by means of the complementary method. Her idea is right, as in this square $a^2 + b^2 = c^2$ is valid, if a and b are the short sides of the rectangular, complementary triangles and c the hypotenuse, i.e. the side of the square.

Tim's idea is also right, as $9 + 16 = 25$.

Wiebke, however, as all the others, should know the conditions of the theorem as well, i.e. the triangle has to be right-angled and a and b are the short sides and c the hypotenuse. Apart from that her idea is right, too.

Simon thinks of the geometric meaning. The two squares of the short sides and the hypotenuse square are equal in area. Simon's thoughts in mind, the Pythagoras Theorem can be remembered best.

(II) Anna first thinks of a proof for the Pythagoras Theorem, i.e. the complementary method.

Tim thinks of a pythagorean 3-tuple.

Wiebke: If $a^2 + b^2 = c^2$, then a triangle is right-angled [...]. This is the inversion of the Pythagoras Theorem. She mustn't mix this up with the Pythagoras Theorem!

Simon imagines the formula $a^2 + b^2 = c^2$ as a picture. The squares of the short sides of a rectangular triangle a^2 and b^2 and the square of the hypotenuse c^2 are equal in area.

(III) Anna remembers the Pythagoras Theorem thinking back to how she proved it. This is the complementary method.

Tim thinks of the numbers 3, 4 and 5, because a triangle for which $a^2 + b^2 = c^2$ is valid, is right-angled. This is, however, the inversion of the Pythagoras Theorem! This means he will always be very involved in “getting” to the actual Pythagoras Theorem.

Wiebke forgets that the triangle has to be right-angled, in order to meet the conditions of the equation. Probably she has got this in mind, but just didn’t mention it.

Simon just thinks of his sketch, as a triangle in it is right-angled, i.e. the two upper squares together and the lower one are equal in area.

(IV) Anna thinks of the complementary method.

$$(a + b)^2 - 4 ab \frac{1}{2} = c^2$$

$$a^2 + 2ab + b^2 - 2ab = c^2$$

$$a^2 + b^2 = c^2$$

The complementary method is used to calculate the area of the inner square. If you simplify the equation you will get $a^2 + b^2 = c^2$. a , b and c are the side lengths of the complementary triangles.

Tim thinks of the numbers 3, 4 and 5, because if you draw a triangle with side lengths of 3, 4 and 5, it is right-angled according to the inversion of the Pythagoras Theorem. Tim does not really think of the Pythagoras Theorem, but of its inversion.

Wiebke thinks of the equation $a^2 + b^2 = c^2$. She has, however, forgotten that the triangle has to be right-angled.

Simon thinks of a geometric depiction of the Pythagoras Theorem. The two small squares together and the big one are equal in area.

Everybody associates Anna’s picture with the proof (called “complementary method”). (IV) shows the corresponding calculating steps. Tim’s memory refers to a pythagorean triple of numbers. The comments of (III) and (IV) are worth mentioning. They show extremely well the insufficiency in remembering the Pythagoras Theorem by means of pythagorean number triples. Wiebke presents the Pythagoras Theorem in its standard version $a^2 + b^2 = c^2$. All solutions reveal the weakness that the theorem’s prerequisite, namely the triangle’s being rectangular, is missing. (See (III): “*Probably she has got this in mind, but just didn’t mention it.*”) Or else, the equation is understood as a premise. ((II): “*This is the inversion of the Pythagoras Theorem. She mustn’t mix it up with the Pythagoras Theorem!*”) Simon’s thoughts are completed by the formulation of the geometric interpretation. It is remarkable that (I) describes the picture that Simon has got in his mind as the imagination that can be remembered best.

The structuring of such a task is often based on different ideas, which, in relation to a person indicated by name, introduce the task and become the subject of a discussion. The solutions then contain a reconstruction of the thoughts mentioned including their prerequisites. Such a

knowledge psychological analysis moreover uncovers one's own cognitive structure and can lead to a specific use of one's own metamemory.

Misconceptions:

In school mathematics term and equation algebra are extremely error-prone. It is therefore obvious that a closer analysis of mistakes and misconceptions can be very useful here.

Task: *Please comment each of the following steps:*

$$\begin{aligned} & 2 \cdot (x-1) = 2 \\ \Leftrightarrow & 2 \cdot x - 1 = 2 & (1) \\ \Leftrightarrow & x \cdot 2 - 1 = 2 & (2) \\ \Leftrightarrow & x = 2 & (3) \end{aligned}$$

If you check the equation $2 \cdot (2-1) = 2$, it proves to be correct.

The following solutions of two pupils contain correct analyses.

(I) There is an equation which has to be solved. (1) The brackets were eliminated, but there is no paragraph which allows that. (2) The variable x and 2 were exchanged according to § K^ . (3) The result was achieved. As $2 - 1 = 1$ and $x \cdot 1 = x$, the result is correct. But here the subtraction was made before the multiplication and this is not allowed. The steps are wrong. It should be:*

$$\begin{aligned} \Leftrightarrow & 2 \cdot (x-1) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} & \text{§ } G^* \\ \Leftrightarrow & x - 1 = 1 \\ & 2 \cdot (x-1) = 2 \\ \Leftrightarrow & x - 1 + 1 = 1 + 1 & \text{§ } G^+ \\ \Leftrightarrow & x = 2 \end{aligned}$$

The checking also proves to be right.

(II) § D has been used incorrectly with the first equivalence sign. In the second step § K^ has been correctly used. The third step does not take the correct order of multiplication before subtraction into consideration. The person solving the equation was lucky. He/she did not keep to the rules, but in spite of that came to the correct result. The checking just proves that the result is correct, not the steps which led to the solution.*

The following comments reveal several misconceptions; an intensive correction is necessary here:

(III) First the brackets have been eliminated, then § K^ has been applied to $x \cdot 2$. In the end the result was 2.*

(IV) (1) § D has been used here, in order to calculate the brackets. (2) § K* has been used so that “-1” can be subtracted from “ $x \cdot 2$ ”. (3) The paragraphs N* and * have been used, so that you get to the result mentioned.

(V) In the first step the brackets have been eliminated. In the second the variables term has been turned back to front, in order to solve the task more easily. In the last step $2 - 1$ was calculated and as $1 \cdot x = x$ each step of the solution is correct.

Especially the distribution law and the rule “multiplication/division before addition/subtraction” cause difficulties. The final and consecutive sentences appearing give further information. Purpose, intention and effect govern the way of thinking; everything that serves it is made into a justification. The use of the conjunctions “in order to” and “so that” and the phrase “*in order to solve the task more easily*” show that a simplification is aimed at and the legitimation is adapted to its purpose.

The task is constructed in such a way that typical mistakes are included in a fictitious solution. By the way, the teachers as well have to accept responsibility for some of the misconceptions. In this context the so-called “elimination of brackets” has to be mentioned. It can be gathered from the pupils’ comments that an outward description alters the understanding (in this case the distribution law).

The task itself does not come from own thinking. But dealing with it shows that this task is able to reveal mental models. The capability of analyzing other people’s way of thinking, of putting oneself in somebody else’s position or identifying and correcting mistakes can also be seen. The analysis of mistakes is only a part of metacognition, but surely an essential part as a prerequisite for correction and conscious checks.

Remediation:

If an analysis of mistakes and misconceptions is executed, help or remediation can follow.

Task : *A typical pupil’s mistake is to think that “-a is negative”. How would you help?*

Solutions:

(I) *I would first of all make it clear to the pupils that $-a$ is not always negative $-a$ can be positive, negative or neither: 0.*

(II) *The mistake is that the pupils take “-“ always as the minus sign for a figure. It can also be an arithmetic sign: $b - a$. Furthermore “-a” is only a variable and can of course be a positive number, as well as “a” can be a negative number.*

(III) *First of all I would clarify what type of sign the “-“ is It cannot be a minus sign as “a” is not a figure. It is the inversion sign.*

(IV) I would say: There is not only one meaning for the “-“. It can be the inversion sign, in another task it can be an arithmetic sign or in another task perhaps the sign by which it is expressed that the number is negative. You have to look at the task closely in order to realize which “-“ is meant.

(V) It has got to be clear to oneself that “a” can be negative. If you put the counter-number sign in front of it, -a is positive.

(VI) -a is not always negative, because it is not known whether “a” is a positive or a negative number. Consequently the minus sign has to be an inversion sign: If $a = 8$, you get $-a = -8$. If $a = -8$, you get $-a = 8$.

(VII) -a is not always negative, as -a can have three different meanings:

1. “-“ can be the arithmetic sign which demands “a” to be subtracted from another number.
2. It can be the minus sign, which shows that -a is a negative number.
3. It can demand the formation of the counter-number.

An examination of these pupils’ solutions does not only lead to the assessment that there are obviously clear concepts and descriptions, but also to the conclusion that something has been developed in the pupils’ heads, which had not been mentioned in the preceding lessons in that way. One example is the point of view that “a is a variable” (see (II)), although -a is actually a term. It has, however, to be pointed out that the help demanded by the task is not always complete. Only answer (I) explicitly mentions that the number “0” has to be considered as well. The fact that -a can be positive is only explicitly mentioned in answers (I), (II), (V) and (VI). Furthermore explanation (VI) gives a definite example number.

Mistakes or lack of knowledge can be worked on by remedial tasks, consequently remediation occurs in every pupil him-/herself. The teacher is not the only correcting authority. This results in a more acceptable and more productive relation to mistakes. Self-remediation is definitely an attainable objective.

Self-regulation:

Self-regulation requires the ability of separating cognitive and metacognitive activities.

The following passage has been taken from a pupil’s exercise book (grade 9).

<p>2500 DM are to be distributed among 4 brothers and sisters in such a way that every following child gets 50 DM more than the older one. How much does every single child get?</p> <p>x: amount for the oldest child. $x + 50$: amount for the younger child $x + 100$: amount for the next younger child $x + 150$: amount for the youngest child</p> $x+x+50+x+100+x+150 = 2500$ $4x + 300 = 2500$	<p>This task deals with a problem, the solution of which has to be examined. The text is transferred to a different description.</p> <p>The text has to be translated.</p> <p>This is the transferred equation.</p> <p>Simplification.</p>
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<p>$4x = 2200$</p> <p>$x = 550$</p> <p>Check:</p> <p>$550+550+50+550+100+550+150 = 2500$</p> <p>1st child: 550 2nd child: 600</p> <p>3rd child: 650 4th child: 700</p> <p>Check:</p> <p>$550+600+650+700 = 2500$</p> <p>The oldest child gets 550 DM, the 2nd child gets 600 DM, the 3rd child 650 DM and the 4th child 700 DM.</p>	<p>§ G^+ is used to separate the number terms from the variable terms.</p> <p>§ G^+ is used to get to the solution for x.</p> <p>The result has to be checked.</p> <p>It is correct.</p> <p>The results have to be translated back.</p> <p>The result has to be checked. Correct.</p> <p>This is the final sentence.</p>
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The pupil sticks to the arrangement to divide the page of the exercise book into two columns; the left column is for the task and its solution, the right one for the comments. It could even be said that the middle line separates cognitive and metacognitive activities. The methodical procedure is obvious, i.e. to look for the solutions of a problem, to translate the text, to refer to paragraphs in order to solve the equation, to check the result, to translate the result back and to check the solution, as well as the thoughts which serve monitoring and self-checking, and finally the checks, in this case a double-check, first the check of the calculation and second the check of the result with reference to the text.

Obviously the cognitive activities are supremely well dealt with. The single steps of the model formulation can be exactly identified: The first one is the reduction of the text to those parts that have to be formalized (variable terms, equation), the second one the mathematical procedure (solving of the equation, check), the third one the transfer back of the number results to the given situation and the fourth one the check which has to be executed there and which is limited to check the calculations and the final sentence.

The parallel description of cognitive and metacognitive procedures always requires a lot of time and energy. It does, however, almost conclusively result in the fact that metacognitive activities are carried out. One has to ask oneself: "What is it about?" One has to pause, make sure of what one does, one has to check. The profit is obvious.

The shown method of separating the page of the exercise book into an "executing column" and an "annotating column" can be practised in many fields of school mathematics. The metacognition, which is being carried out at the same time, controls the cognitive processes. Literature distinguishes such a way of controlling between planning, monitoring and checking. Such a type of self-regulation can undoubtedly be seen in the example mentioned above. The attribute "metacognitive" is in so far appropriate as the control activities refer to the own person, as they are consciously executed and as they do not occur by chance.

Introspection:

Introspection is an essential part of metacognition. It means the ability to comprehend and describe mathematical concepts and ideas.

Task: *What idea do I have of the function which is described by $n(x) = 0 = x - x = 0 \cdot x$*

Solutions:

(I) $x - x$ or $0 \cdot x$ helps me best, as I know exactly what the machine will calculate. If there were only “0”, I would first have to think about the steps which lead to the solution, which is made by the function/ machine.

(II) I like the description $0 \cdot x$ best, as the number x , which you put in, comes out “0”-times.

(III) I like $x - x$ better, because for me it is easier to imagine where the x goes. With $0 \cdot x$ it seems to me that the x just disappears.

(IV) This function is easiest for me to imagine as $n(x) = x - x$, because this means that a number x is put into the machine and immediately after subtracted.

(V) To me $n(x) = 0$ makes the most sense, as the number x disappears and the result is 0. x is not needed in the function. The machine could also deliver the correct result without any number x .

(VI) $n(x) = x - x$. This is my idea of it, as $x - x = 0$ and I think there should be an “ x ” in a function.

Obviously there are different mental models for the use of functions. (I), (II), (III) and (IV) prefer the concept of a function as a machine with input and output after processing. (V), a boy, has also got the idea of a machine, i.e. a deleting one. This is expressed in his first sentence (“*as the number x disappears and the result is 0*”). This pupil is in a form which has been taught according to the book “Einführung in die Computerwelt mit Registermaschinen” (Introduction into the World of Computers with Register Machines) (Cohors-Fresenborg & Kaune & Griep 1995). The next sentence (“ *x is not needed in the function*”) reminds of the analysis in Cohors-Fresenborg (1993, p. 243). According to Schwank (1993) this pupil is to be assigned to the functional cognitive structure. With his last sentence (“*The machine could also deliver the correct result without any number x .*”) the pupil draws the conclusion that it is an 0-digit function. (VI), a girl, refers to the description level, not to the effect of the function. According to Schwank (1993) this corresponds to a predicative cognitive structure. A foresighted qualification can arise from introspection, namely the consideration of the manner in which knowledge has been worked out in one’s own head in a new situation. If knowledge about the own cognitive strengths and weaknesses on someone’s own mental behaviour is applied successfully, both main components of metacognition, i.e. cognition and regulation of someone’s own cognition, are combined.

3. Conditions for the possibility of metacognition

The examples mentioned have been taken from mathematics lessons in which the two views “Learning mathematics as an individual social-interactive construction of knowledge” (Ernest

1994, Wittmann 1996) and “Mathematics as a tool of the representation of knowledge” (Cohors-Fresenborg 1996) characterize basic understanding. This makes it possible to go into the following questions: Is metacognition ascertainable or not? Is it possible or not? Is it necessary or not?

A possible answer could be: Metacognition cannot be independent when learning mathematics. There cannot be metacognition on its own, but only parallel to cognition, not subordinate, not superordinate, but co-ordinate as a necessary accompaniment and supplement. Metacognition is *necessary* when learning mathematics, as mistakes, errors and insufficiencies occur in the development of mathematical ideas.

In a certain conception of school mathematics metacognition is possible. The examples mentioned above prove this. They illustrate an apparatus for cognitive and metacognitive activities according to age. At the same time they refer to the central condition of the possibility of metacognition. Metacognition in learning mathematics is *possible* if productive mental tools for the representation of knowledge are available (Sjuts 1999a).

Investigations have shown that neither parents nor teachers spend a lot of time or effort in the promotion of metacognition (Hasselhorn 1998). Obviously the opinion prevails that metacognition can be more or less obtained by oneself through daily learning experience. Furthermore the existence of assumed metacognitive abilities is overestimated and the pupils’ use of metacognition is underestimated. Without having to criticize or correct the theories as regards metacognition in literature, the mentioned tasks and learning situations show that metacognition does not have to be mysterious, that metacognition can be comprehensible and *ascertainable*. The importance is empirically safeguarded (Spada & Wichmann 1996). Subsequently the following statement is valid: A high degree of knowledge of the subject goes along with effective methods of planning, monitoring and regulation of learning and thinking processes. The contrary proves true as well: The more exactly the course of acquisition and use of knowledge can be assessed, the better this competence can be used for thinking and learning.

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