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HOW DO MATHEMATICAL SYMBOLS ACQUIRE THEIR MEANING? – THE METHODOLOGY OF THE EPISTEMOLOGY-BASED INTERACTION RESEARCH

Abstract

Mathematics education research – as well as the didactics of other disciplines – takes place in the tension field between constructive didactic work (i.e. when developing learning environments or when didactically analyzing specific mathematical content matter) and the analysis of mathematical instruction and communication processes (by means of quantitative as well as qualitative methods from different related disciplines). In this contribution I would like to present exemplarily the epistemology-based interaction research which analyses the epistemological constraints of the social constitution of mathematical meaning in processes of communication.

1. Introduction

The social construction of new mathematical knowledge in teaching and learning processes depends on two important conditions: The special character of instructional communication and the specific epistemological nature of mathematical knowledge. In mathematics teaching at the primary level new knowledge cannot be constructed in a formal manner by a kind of preview technique, i.e. using algebra or formulas, but this construction is characteristically bound to the children's situated contexts of learning and of experience. The young students have to learn – and they are able to do so with their personal means – to see the general in the particular. To better understand this problem is an important inquiry of the research project „Social and epistemological constraints of constructing new knowledge in the mathematics classroom" (funded by the German Research Society, DFG; cf. Steinbring 2000). In which ways are students in elementary grades able to grasp the new, general mathematical knowledge with their own conceptions and to describe it with their own words? And what factors support or hinder this generalizing interactive knowledge construction?

2. What is the Specific Nature of Mathematical Concepts?

Mathematical concepts and mathematical knowledge are not a priori given in the „external" reality, neither as concrete, material objects, nor as independently existing (platonic) ideas. Mathematical concepts are „mental objects" for the individual cognitive agent (Changeux & Connes 1995; Dehaene 1997); in the course of communication mathematical concepts are constituted as „social facts" (Searle 1997) or as „cultural objects" (Hersh 1997) From an evolutionary point of view mathematical concepts develop as cognitive and social theoretical knowledge objects in confrontation with the material and social environment.

In contrast to objects constructed by humans as a chair, a table, a knife or a screw-driver one cannot deduce the meaning of social facts, as for instance money, time or the number concept neither from their form nor from their material. There are no direct insights about the corresponding mathematical object when regarding the „material" or the functional form of number signs as $\sqrt{2}$, -3.17 or π. The meaning of these theoretical social or mental objects has to be constructed by the individual in interaction with experience-based and abstract reference contexts. In this way mathematical concepts can be generally conceived as „symbolized, operational relations“ between their formal codings and certain socially intended interpretations.

Mathematical knowledge can be looked at in two complementary ways: On the one side every mathematical knowledge domain represents a consistent structural wickerwork, in which all elements are linked in an equivalent logical manner. On the other side new concepts raising new question and problems can be constructed in every mathematical knowledge structure, concepts that are not yet embedded in the actual logical structure, and in this way produce new insights.

This distinction between the *logical structure* and the *mathematical objects* in accordance with the distinction is made in philosophy between a subjective ontology of reality the subject independent structure of the world. „… the ontology of the world is created by the cognitive agent, the structure of the world depends on the mind-independent external reality. In this way, the experiential world can be seen as both created and mind-independent at once. As there cannot be a structure without an ontology, it is the cognitive agent's act of creating an ontology that endows external reality with a structure“ (Indurkhya 1994, p. 106).

The mathematical necessity or the „logical coherence“ and therefore the „unique generativity“ of mathematical knowledge is often taken as an irrefutable „proof“ for the independent existence of mathematical knowledge from any cognitive agent (Changeux & Connes 1995, p. 12); but also this specific property – a specific, epistemological mechanism for the autopoietic development of mathematical knowledge – needs for its unfolding the cognitive as well as the social environment of the cognitive agent.

3. The Epistemological and Communicative Function of Mathematical Signs

3.1 The Communicative Dimension

The sociologist Niklas Luhmann characterizes »communication« as the constitutive concept of sociology: „… when communication shall come about, … an autopoietic system has to be activated, that is a social system that reproduces communications by communications and does nothing else but this“ (Luhmann 1996 p. 279).

The concept of „autopoietic system“ has been introduced by Maturana and Varela (cf. i. e. 1987); it characterizes self-referential systems, that exist and develop autonomously on the basis of this self-referential relation. These systems consist of components that are permanently reproduced within the system for its maintenance. With the concept of „autopoietic system“ not only biologic processes are investigated but it is also applied to social and psychic processes.

What is the essential difference between a social and a psychic system? The psychic system is based on consciousness and the social system is based on communication. „A social system cannot think, a psychological system cannot communicate. Nevertheless, from a causal point of view there are immense, highly complex interdependencies“ (Luhmann 1997, p. 28). How can these interdependencies be understood? „Communication systems and psychic systems (or consciousness) form two clearly separated autopoietic domains; … But these two kinds of systems are linked in an especially narrow relation and they reciprocally form a »portion of necessary environment«: Without the participation of consciousness systems there is no communication, and without the participation of communication there is no development of consciousness“ (Baraldi, Corsi & Esposito 1997, p. 86).

Language is a central „linking means“ between communication and consciousness. Within language, one has to distinguish between »sound« and »sense«; accordingly, within written language one has to distinguish between »sign« (more exactly »signifier«) and »sense«. This distinction between sign and intended meaning is the starting point – the take off (Luhmann 1977, p. 208) – for the autopoiesis of communicative systems.

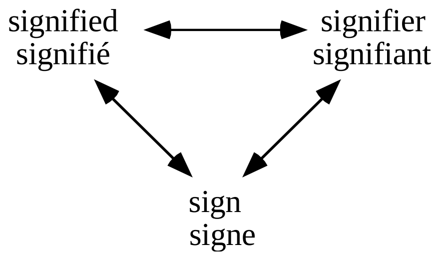


Fig. 1: The semiotic triangle

For the analysis of the conditions of the autopoiesis Luhmann refers among other things to the work of de Saussure, who made the following distinction between signifier (signifiant), signified (signifié) and sign (signe). Luhmann (1997, p. 208f.) writes: „Signs are also forms, that means marked distinctions. They distinguish, according to Saussure, the signified (signifiant) from the signifier (signifié). In the form of the sign, that means in the relation between signifier and signified, there are referents: The signifier signifies the signified. But the form itself (and only this should be named sign) has no reference; it functions only as a distinction , and that only when it is actually used as such“ (Luhmann 1997, p. 208f.).

How is the autopoiesis of the social, of communication possible? According to Luhmann in the course of interaction or in the communication system the participants provide mutually with their *„conveyances“* (or communicative actions) „signifiers“ which may signify certain „information“ (signifieds). „Decisive might be…, that speaking (and this imitating gestures) elucidates an intention of the speaker, hence forces a distinction between information and conveyance with likewise linguistic means“ (Luhmann 1997, p. 85).

The conveyor only can convey a signifier, but the signified intended by the conveyor, which alone could lead to an understandable sign, remains open and relatively uncertain; in principle it can only be constructed by the receiver of the conveyance, in a way that he himself articulates a new signified. Luhmann explains this in the following way: „We do not start with the speech action, which will only happen when one expects it to be expected and understood, but we start with the situation of the receiver of the conveyance, hence the person who observes the conveyor and who ascribes to him the conveyance, *but not the information*. The receiver of the conveyance has to observe the conveyance as the designation of an information, hence both together as a sign (as a form of the distinction between signifier and signified) ….“ (Luhmann 1997, p. 210). The receiver must not ascribe the possible signified strictly to the conveyor of the conveyance but he has to construct the signified himself; the signified and hence the sign are constituted within the process of communication.

3.2 The Epistemological Dimension

The possible detachment of the information belonging to the conveyance from the conveyor is the starting point of the autopoiesis of the communicative system. Together with this „mechanism“ that describes the autopoietic functioning of the communicative system as an ongoing conveyance of signifiers which are then transformed into signs by the contrasting conveyance of other, new signifiers general properties of the functioning of mathematical communication are explained as well.

The peculiar interrelation between „Signs / Symbols“ and „Objects / Reference context“ is central for the description and analysis of mathematical teaching as a specific culture. This relation also represents the core item of the epistemology-based interaction analysis. All mathematical knowledge needs certain *systems of signs or symbols* in order to grasp and code the knowledge in question. These signs themselves do not have an isolated meaning; their meaning must be constructed by the learning child. In a general sense, to endow mathematical signs with meaning, one needs an adequate *reference context*. Meanings of mathematical concepts emerge in the interplay between sign/symbol systems and objects/reference contexts.

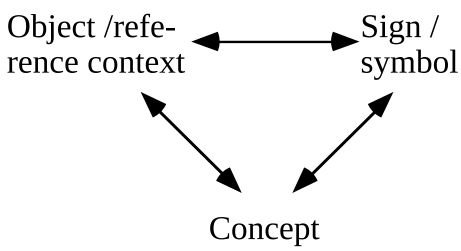


Fig. 2: The epistemological triangle

The interrelation between coding signs of knowledge and reference contexts can be structured in the *epistemological triangle* (cf. Maier & Steinbring 1998; Steinbring 1989; 1991; 1998). The links between the corners in this epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported and balanced system. In the course of further developing knowledge, the interpretation of signs systems and their accompanying reference contexts will be modified and generalized accordingly.

Similar triangular schemes have been introduced in philosophy of mathematics, in linguistics and philosophy of language for analyzing the semiotic problem of the relation between symbol and referent (Frege 1969; Ogden & Richards 1923).

Mathematical concepts are constructed as symbolic relational structures and are coded by means of *signs and symbols* that can be combined logically in mathematical operations. With regard to the analysis of conditions for the construction of new mathematical knowledge in classroom interaction, mathematical signs and symbols are the central connecting links between the epistemological and the communicative dimension of interactive construction processes; on the one hand signs and symbols are the *carriers of mathematical knowledge*, and on the other hand they contain the *information of the mathematical communication* at the same time.

As opposed to the semiotic triangle, a different marking was chosen for the epistemological triangle: „sign / symbol, reference context, concept“. This terminology considers the fact that mathematical signifiers are always signs themselves, and that, in mathematics, the mathematical concept is of central epistemological importance beyond the function of being a „sign / symbol“ (and therefore it takes the place of the signe).

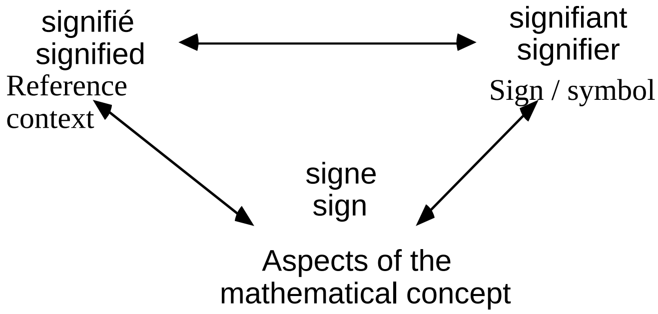


Fig. 3: Semiotic triangle and epistemological triangle

The superposition of the semiotic triangle by the epistemological triangle shown above makes clear that the general communicative elements of the epistemological analysis of mathematical interactions are given by Luhmann’s conception of communication as an autopoietic system. It is tried to do justice to the particular conditions of mathematical knowledge with the help of the conceptual markings in the epistemological triangle. The perspective of the semiotic triangle aims at the general conditions of social communication; the perspective of the epistemological triangle reflects the special status of mathematical knowledge as a socially – constructed fact which is characterized especially by the symbolization of *relations*.

4. Interactive Interpretation of Abstract Signs in Mathematical Learning Environments – Analysis of Exemplary Teaching Episodes

In the following, interactive patterns are analyzed in the course of constructing and justifying new mathematical knowledge within a typical teaching episode. The episode is part of a teaching unit about special number squares, where the new mathematical signs have to be interpreted with the help of *arithmetically structured* reference contexts.

4.1 Kim and Eva argue with an Unknown Number in the Number Square

During this lesson in a mixed class of grade 3 and 4 the children had to work on the following problem: How could one recover a lost number in a certain number square, in such a way that this number reproduces the old arithmetical structure? (cf. Fig. 4). The special number squares as used in this class can be constructed in the following way: First one adds some given numbers in the border row and border column of a table (cf. Fig. 5). The squares thus created have the following property: You can choose (circle) in a $(3 \cdot 3)$ number square three arbitrary numbers such that in every row and in every column there is one and only one circled number. The sum of these three chosen numbers is always constant – independent of its choice (cf. Fig. 6). Such squares are called „crossing out number squares“, because when circling a certain number in the square, all other numbers in the same row and in the same column have to be crossed out. The children called this square »magic square« and the constant sum the

»magic number«. During this episode the children reproduced the lost number with three different strategies.

15	16	17
14	15	16
13	14	

Fig. 4

+	5	6	7
10	15	16	17
9	14	15	16
8	13	14	

Fig. 5

~~15~~	~~16~~	17
~~14~~	15	~~16~~
13	14	

Fig. 6

4.1.1 Kim's Reasoning with the Magic Number

Earlier in the course of this lesson, Kim has already sketched her idea. Later she explains her plan in detail. First she calculates the magic number 45 by adding the numbers 13, 15 and 17 in the diagonal. With this proposal she expresses that one can determine the magical number in an incomplete magical square. Then her argumentation starts.

147 K And then one could already do it this w[*points at 15 in the first row*] and <u>this</u> fifteen [*points at 15 in the second row*] and adds it up. And then one still calculates, how much there must be up to forty-five.

The signifier „One circles the fifteen and <u>this</u> fifteen and adds it up." denotes the intention to apply the known procedure for calculating the magical number to two numbers in the diagonal. The second signifier „And then one still calculates, how much there must be up to forty-five." could be understood in this way: One has to calculate how much is left from the sum of 15 + 15 up to 45 (one has to calculate the difference); this number seemingly has to be placed into the empty field.

At this moment several classmates object that nothing could be really calculated here. „Well, that really leads nowhere … Where would you like to calculate up to? … Exactly. After all, you do not at all know which number is the result here!" (152, 153).

Kim formulates further explanations.

161 K First one calculates, one first calculates these numbers, that I have, which are there, what is their result. And then …, and then one calculates …

165 K These three, oh, yes, this, this and then afterwards one calculates fifteen [*circles 15 in the second row*], one takes this way. Cross out that, and that. And cross out that and that [*crosses the other numbers in the same column and the same row*]. Then one also takes the fifteen [*circles 15 in the first row*]. Cross out then the seventeen and the thirteen [*crosses out the still uncrossed numbers in the same column and the same row*]. And then one circles this here, this here [*circles the empty field*]. And then one has to calculate, fifteen and fifteen this makes thirty, how much is left up to forty-five.

Kim repeats that the calculation of the magic number should be dealt with first, i. e. as it was done in the addition exercise (cf. Fig. 7). Then Kim starts with the concrete application of the „crossing out algorithm" to the two 15 (in the first and second row).

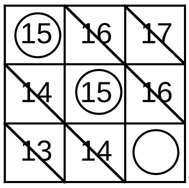
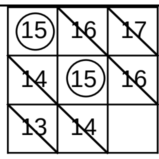


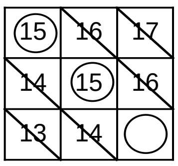
Fig. 7

Kim’s last remark can be represented in the scheme of communicated analysis in the following manner:

signifier signi- fiant	Kim:  13 + 15 + 17 = 45
sign signe	
signified signifier signifié signi- fiant	Kim: And then one circles this here, this here [*circles the empty field*]
sign signe	
signified signifier signifié signi- fiant	Kim: And then one circles this here, this here [*circles the empty field*]. And then one has to calculate, fifteen and fifteen this makes thirty, how much is left up to forty-five.

With the signifier „And then you circle this here, here.“ the crossing out algorithm is applied intentionally to an (unknown) third number, the missing number in the empty field. In the second signifier „then you have to calculate, fifteen and fifteen makes thirty, how much is missing up to forty-five.“, the calculation of the magic number with the help of the three circled numbers is intended on the one hand: 15 and 15 make 30. Still, it is not possible to continue calculating with the third circled field. On the other hand, it is to be calculated „reversely“ with the unknown field now, in a way that this field will be occupied by the number that represents the difference between 30 and 45 (the magic number). The „scheme“ of the calculation of the magic number is transferred to the empty field, and thereby a new sign is intended and iconically represented.

With the help of the teacher, Kim notes the addition exercise as a completion exercise (cf. Fig. 8). The signifier „15 + 15 + __ = 45“ intentionally refers to the crossing out algorithm for three numbers and to the „unknown“ number in the empty field at the same time; by this means a new (open) sign is created and written down in mathematical symbols.



13 + 15 + 17 = 45

15 + 15 + __ = 45

Fig. 8

The epistemological analysis verifies that Kim constructs really new knowledge when including the empty field into the mathematical operation to determine the magic number. She argues that one cannot only calculate with concrete numbers, but the algorithm for the magical number also can be extended to arbitrary fields – with or without numbers.

The newly-constructed mathematical knowledge in Kim's argumentation can be described with the help of the epistemological triangle (cf. Fig. 9). The new relation (or „unknown number" or „variable") is symbolized in two ways; once as a „circled number" and then as a missing term in the addition task. In this domain of representation and of mathematical operation we can observe how Kim works with the „unknown number" in a specific situated manner. Kim places the unknown number into a new mathematical relation with other numbers and in this way she constructs new knowledge; the new mathematical object is created as a relation in the extended and generalized operational structure of the number square.

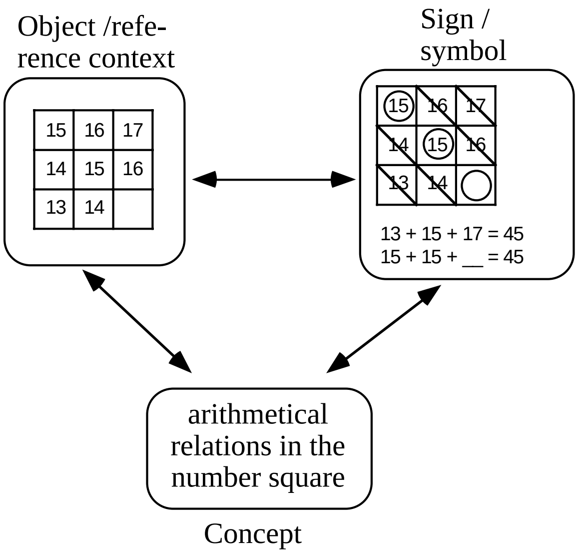


Fig. 9 The epistemological triangle representing Kim's and Eva's explanation

4.1.2 Eva Repeats the Strategy Using the Magic Number

During the discussion of Kim's strategy the teacher encourages other children to explain how this „trick" works. Eva argues in the following way.

193 E Well. Ohm, one has to take three numbers out of the magic square. Add them up. But not the empty field, there is not much to calculate, right?

196 E Yes. And then you get the magic number. Then one must, ohm, that, ohm take again three numbers, but now also the empty field must be therein. And then you have to … from this number, you get then … from these two, one has still to go to forty-five.

With the first signifier Eva intends the calculation of the magic number taking three concrete arbitrary numbers from the square. She adds: „But not the empty field, there is not much to calculate, right?“. This field cannot and should not be used for the calculation of the magic number.

The second signifier indicates that the scheme of the algorithm for the magic number shall be transferred to three other numbers: „… take again three numbers, but now also the empty field must be therein.“ Here the empty field is in a way identified with a number; Eva characterizes a mathematical unknown in a situation specific manner. When transferring the algorithm the following calculation cannot be done in the usual direct manner, because the third term is missing; consequently Eva modifies the calculation procedure: „And then …from these two, one has still to go to forty-five“. This is how she explains the calculation of the missing number.

From an epistemological point of view, Eva’s presentation shows that she primarily describes a general procedure to determine a missing number in the number square (by using the magic number) step by step.

(1) Applying the crossing out algorithm to three (circled) numbers and determining the magic number;

(2) Applying the crossing out algorithm to two numbers and the empty field;

(3) Determining the missing number out of the magic number and the partial task of the second addition exercise.

Eva shifts between general concepts and exemplary references; i. e. three arbitrary numbers out of a number square and the concrete number 45 as a characterization of the general term *magic number.*

Yet, in her presentation, she distinguishes clearly between the impossibility to calculate a sum with an empty field as a term of a sum (193) and the possibility to calculate with the empty field (a pre-notion of mathematical unknowns) as with (known) numbers, if one already knows the result (the magic number) (197). This aspect of the interpretation of the empty field as a „number“ in Eva’s proof of Kim’s trick can be comprised in the epistemological triangle (cf. also Fig. 9). The construction of the new knowledge „calculating with unknown numbers“ originally appears in the statement: „take once again three numbers, but now this empty field has to be in there, too“, altogether, Eva wants to represent the procedure to determine the unknown number as general as possible – partly referring to concrete numbers – when repeating Kim’s arguments. So in her presentation, the logical, structural part outweighs the constructive part of constructing new mathematical knowledge.

5. The Particularity of the Dependence on Contexts When Generalizing Mathematical Knowledge

The problem of interactive developing and generalizing mathematical knowledge is discussed in literature as the contrast between *situatedness* and *universal validity*. In a simplified way, one proceeds on the assumption that mathematical knowledge is bound to concrete experience contexts – one particularly believes that this interpretation applies to primary teaching – and that universal mathematical knowledge could then develop independently from any context by increasing abstraction from the concrete qualities of the experience context. According to our research conception, the fundamental context–dependence of mathematical knowledge is not questioned. The tender point in this relation is the *kind* of context-dependence: Is the epistemological interpretation of mathematical knowledge directly bound to the *finished*, empirical qualities of the situation respectively is it derived from the concrete qualities or does the situation with its structure serve as an *open* reference context, which has to be interpreted first and which always allows new interpretations?

This point of view understands the situatedness of mathematical knowledge as a relation between signs / symbols and reference context. In this relation, the found context does not directly explain the knowledge, but the situated context can be given a new interpretation to and it can be used as an embodiment of structural connections which allow to construct new mathematical knowledge. The construction and the proof of the new knowledge relating to the arithmetical context in the example episode take place as follows: the interpretation of the empty field as a „general" number is not derived from the empirical, arithmetical qualities of the number square, but this interpretation is read into the arithmetical structure by Kim (and then by Eva).

A central problem of the construction and proof of new mathematical knowledge is that neither the (concrete, situated, and structural) reference contexts nor the sign and symbol systems alone directly contain the mathematical concept. Signs and reference contexts form a possible basis for construction in so far as structural connections and relations can be indicated with their help which are interpretable as embodiments of concept relations. This means that mathematical signs and reference contexts do not directly and suddenly reflect the newly – constructed knowledge, but are used as unalterable, iconic carriers of knowledge in the sense of reference to other structural relations of the concept. So students in math class face the particular interpretation problem of partly keeping their distance from the concreteness of the situation when dealing with mathematical signs and the accompanying reference contexts and of seeing, interpreting or recognizing „something else", another structure into it. In didactic literature, this problem is discussed under the caption „Seeing the general in the particular"(cf. Mason & Pimm 1984).

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