Computer Algebra Systems – Old Wine in New Bottles?

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Computer Algebra Systems (CAS) have become indispensable tools in pure and applied mathematics. Due to continuous improvement in terms of application and program handling, as well as availability, CAS are increasingly used in the classroom. Initial experience has been gathered by now in various school projects. One example is the MAPLE project "Mobile Classroom" in the German state of Baden-Wuerttemberg. I will report on my experience as a teacher involved. It stands to reason that the traditional goals of mathematics teaching are still valid when introducing a CAS, but there have been shifts in emphasis. Of course, CAS cannot improve mathematics teaching 'per se'. Some examples will be given which actually prove the contrary. Other examples will demonstrate how a CAS can promote concentration on content-related aspects.

The Old and the New Wine

In the King James Version of the Bible we read in Matthew 9,17 :

> Neither man put new wine into old bottles: else the bottles break, and the wine runneth out, and the bottles perish: but they put new wine into new bottles, and both are preserved.

Computer Algebra Systems (CAS) have become indispensable tools in pure and applied mathematics. Due to continuous improvement in terms of application and program handling, as well as availability, CAS are increasingly used in the classroom. Are they as well the new bottles in which the old wine, i.e. conventional mathematics, can improve in quality? Or do they bring new wine, i.e. new mathematics, into school? Let me begin describing the aims of mathematics teaching:

In an expert's report for the German Conference of the Ministries of Education for teaching at upper secondary level (Borneleit, Danckwerts, Henn, & Weigand, 2001) we took as a basis the three fundamental experiences that, according to Heinrich Winter (Winter, 1995), can help creating stronger links between mathematics education and general education in that they open up specific ways of

1. perceiving and understanding phenomena in the world around us (this principle recognizes the role of mathematics in acquiring important knowledge of our world),
2. learning about and understanding mathematical issues represented in language, symbols, pictures, and formulas as an intellectual, creative act recognizing mathematics as a deductively ordered world of its own (this principle recognizes mathematics as a rigorous science),
3. acquiring problem-solving (heuristic) skills in dealing with tasks extending beyond the domain of mathematics (this principle recognizes mathematics as a school of thought).

The three fundamental experiences are closely related. Deep insight into pure mathematics results comes from application-oriented problems. Conversely, abstract results and methods of pure mathematics prove to be keys for understanding our world. Creative research both in pure and applied mathematics is unthinkable without heuristic competence.

The use of new technologies is important and helpful for all three fundamental experiences: Firstly, CAS are an effective tool to supporting development of modelling and simulation skills, which is the first fundamental experience. Secondly, CAS can positively influence the development of adequate basic concepts of mathematical ideas and results, mainly through dynamic visualisations, which is part of the second fundamental experience. And finally, the computer lends wings to activities for the third basic experience, by possibilities widening the range of heuristic and experimental problem solving.

Hopes, promises and lofty words are many in connection with the use of new media. Bill Gates once said: *The most important use of technology is to improve education.* But sometimes, the euphoria accompanying the accelerated introduction of new media in schools, reminds me of the naivity observed at the time when 'new maths' was praised, in the seventies. The millions of money now spend by ministries for the installation of internet hardware in schools are by no means justified by the results of the few impirical studies which investigated and analysed in detail computer-aided learning of students. There is little information or evidence yet about the beneficial impacts of computer-aided learning on concept formation, so that very subtle approaches are required. Many observations and school experiments judicate that rushing things now may finally not yield the expected results, rather preventing than fostering good and inspiring fundamental experiences, as compared with traditional teaching.

In her paper at the annual conference of the German Society of Didactics of Mathematics in Potsdam in 2000, Michèle Artigue (Artigue, 2000) pointed out that the problems arising with integration of complex technology such as CAS are often underestimated. We know far too little about how epistemological aspects of learning mathematics are influenced by computer-aided learning. In my opinion, the didactical debate on using CAS – promoting more than 1000 entries in the ZDM MATHDI database[1] – places too much emphasis on technical computer hardware and software aspects and on the production of 'nice' teaching examples showing, what can be done with CAS (with emphasis on the tool). "It is not important what and how is served, but what is eaten and digested" Mogens Niss once warned. There is only little information from teaching observations and analyses in this regard. It is an urgent task to performe studies complementing investigations such as those done by Rudolf vom Hofe, on the relations of CAS and calculus teaching. This is a wide area for further research (compare vom Hofe, 1998).

A short history of CAS

I became acquainted with the first calculator in 1965. The price was more than the monthly salary of an engineer and it was able to do the four rules with an accuracy of 8 digits. At ICME 1984 in Adelaide, I saw the first CAS MUMATH, which is antediluvian from today's perspective. MUMATH was one of the first CAS developed in the sixties and was further developed into DERIVE, which is one of the most important CAS for school use today. The second generation emerged in 1985, with first versions of MAPLE and MATHEMATICA. Today, we have a third generation, represented by AXIOM and MUPAD. Implemented algorithms are based on practically all mathematical fields, CAS are used in all mathematical fields as well. *Modern Computer Algebra* (Gathen, 1999) is the standard textbook on

[1] Database of the *Zentralblatt für Didaktik der Mathematik* (International Reviews on Mathematical Education); internet-access via the web page *http:www.emis.de* (restricted access, contact for full access Gerhard König via email: gk@fiz-karlsruhe.de)

algorithms of symbolic mathematics. I think that the authors, Joachim von zur Gathen and Jürgen Gerhard, will take away the word 'Modern' and thus follow the example of van der Waerdens *Modern Algebra*. Booth books contain so much of best mathematics and should not give the impression that they would follow any fashion fads, such as the fata morgana of 'new mathematics'.

Interesting for schools is the fact that CAS are already available on small pocket computers. The best known calculator is the TI-89 by Texas Instruments, the 'little brother' of the successful TI-92, and the two Casio calculators FX-2.0 and Cassiopeia (refer to Henn, 2000). The price developments give reason to hope that such calculators will soon be in the hands of each student, as formerly was the slide rule.

PIMOKL: The *Mobile Classroom* pilot project

In the meantime, much teaching experience has been obtained and some reliable empirical evidence on the use of CAS in mathematics teaching. From among the many DERIVE projects now continued mainly as TI-92 projects especially the national projects in Austria are of particular interest (ACDCA, 2001). In many federal states in Germany, there were and still are pilot projects, and I took part in one of them, the MAPLE project "Mobile Classroom" in my home state of Baden-Wuerttemberg (Henn, 2001). In Baden-Wuerttemberg, there are also some DERIVE projects. Under both projects, the Abitur papers (final examination at the end of upper secondary level) were written using CAS. A review of the situation in Germany is given in the report of Wolfram Koepf on our conference *Computer Algebra in Vocational and Further Education* (Koepf, 2000)

In the pilot project 'Mobile Classroom' mentioned above (in German: **Pi**lot-Projekt ***Mo**biles **Kl**assenzimmer*, abbreviated as PIMOKL) students of 5 classes in different towns of Baden-Wuerttemberg were provided with an own laptop with the CAS MAPLE for the three years at upper secondary level, and it was used for the Abitur examination, as well, especially designed for these project classes. For further information I invite you to read our sourcebook of materials (Henn, Jock, Koller, & Reimer, 1998) and reports and data provided on www.fh–karlsruhe.de/semgym.

In the following section I would like to discuss some main aspects that have emerged during my work with PIMOKL – chances, but also problems, constraints and risks. This is characteristic of all tools – not the tool itself, but how it is used is the decisive factor determing success or failure.

Equations and there solutions

Solving equations is one of the primary tasks of mathematics. Why is it that so much importance is attended especially to formula for solving quadratic equations, and is practiced schematically for weeks (in Germany, the formula is called "midnight formula", because students should be so familiar with it that they are able to repeat it when woken up in the middle of the night). Students gain the impression that other algebraic equations cannot be solved. At least this is what I often hear – "equations of grade 5 and more cannot be solved at all".

The transition from linear to quadratic equations has to aspects:

- For the first time there is more than one solution.
- There is a solution formula including square roots.

For the last 2000 years the second aspect was emphasized in school. Solution formulas for equations however rarely exist. There is a deep misunderstanding of the difference between the existence of solutions and existence of formulae for their solution. This was often also not clear in our PIMOKL project. One colleague prepared a MAPLE worksheet for his students on the MAPLE command “solve”, which solves equations. Main parts of it are provided and explained in the following boxes.

General equations eq2, eq3, and eq4 of grades 2, 3, and 4 are defined (as shown in figure 1; all input of the user is given in normal format in MAPLE, the answers of the system appear in italics).

Definition of the equations:

```
> eq2:=x^2+a*x+b=0;
> eq3:=x^3+a*x^2+b*x+c=0;
> eq4:=x^4+a*x^3+b*x^2+c*x+d=0;
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$$eq2 := x^2 + a\,x + b = 0$$

$$eq3 := x^3 + a\,x^2 + b\,x + c = 0$$

$$eq4 := x^4 + a\,x^3 + b\,x^2 + c\,x + d = 0$$

Figure 1. MAPLE-Definition of the equations of degree 2, 3, and 4.

The command “solve(eq,x)” asks MAPLE to solve the equation eq for the variable x. This is done in Figure 2:

Solving the equations:

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> solve(eq2,x);
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$$-\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4\,b},\ -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4\,b}$$

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> solve(eq3,x);
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$$\frac{1}{6}(36\,a\,b - 108\,c - 8\,a^3 + 12\sqrt{12\,b^3 - 3\,b^2\,a^2 - 54\,a\,b\,c + 81\,c^2 + 12\,c\,a^3})^{(1/3)}$$

Figure 2. Solutions of the equations of degree 2, and 3.

Solutions of the general equation of grade 2 are given by MAPLE in the usual presentation. Commentary of the teacher: “Aha, MAPLE knows the midnight formula.” Solutions of the general equation of grade 3 are provided using the Cardano formulae. This is rather confusing. Given here is only the first line with many more to follow. Commentary of the teacher: “Do you see now why the general solution of equations of grade 3 was not required up to now? – but that will change!” Such a commentary leads to a completely wrong impression of mathematics. Firstly, the existence of solution formulae for algebraic equations of grade 4 and less is extremely important within mathematics – the underlying reason for the existence up to grade 4 and the non-existence from grade 5 up is given by the Galois theory and the theory of simple groups. Developing the formulae for grade 3 is not much more difficult than for quadratics, for grade 4 it becomes more complicated. But for concrete solutions, these formulae are not relevant today or in the future, with or without MAPLE.

In the next step of the worksheet (shown in Figure 3) we try to solve the equation of grade 4:

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> solve(eq4,x);
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$$\mathrm{RootOf}(\_Z^4 + a\,\_Z^3 + \_Z^2\,b + \_Z\,c + d)$$

Figure 3. Equation of degree 4.

It makes sense that MAPLE does not respond with even larger root expressions but simply says “RootOf(…)”, i.e. “root of the equation …”. Now again, the teacher’s comment is very misleading: “In case of an equation of order four even MAPLE seems to be helpless”. Such comments inevitably build up wrong concepts of mathematics, such as: this version of MAPLE will solve equations of grade 3, the next version will even be able to solve equations of grade 4, and so on ….

In fact, RootOf(..) is a MAPLE subroutine which will only provide all solutions of the equation when command “Allvalues” is given, and then, if possible, algebraically, but mainly numerically. Of course, MAPLE will do it for grade 4 algebraically, too. The solution of our equation of grade 4 will fill several screens and is completely confusing and unfathomable. More interesting however is the fact that MAPLE will calculate *all* solutions of an algebraic equation of any grade with concrete numerical coefficients with given accuracy.

However, the main point from a mathematical perspective is: Are there any solutions at all? What are solution formulae? How can you be certain to have found all solutions? Which graphical and numerical methods are there to find them? How do I get them with given accuracy? Questions like these are often neglected. Exactly for this kind of questions CAS provides an ideal tool. It helps to develop the adequate algebraic structure and discuss and analyze the solutions.

Let us think about solution formulae: If writing down $\sqrt{2}$ as the one solution to $x^2 = 2$, we invent a new name for a number. We are convinced that this number exists, we know where to draw it on the number line, but we cannot describe it any better, because it is irrational. This es exactly the aspect of letters or variables described by Malle as the “Gegen-standsaspekt” (aspect of a concrete object) (Malle, 1993). Expressions such as $\sqrt{2}$ can easily be manipulated. But we cannot write it down as concrete decimal number because it has infinitilly many digits. We use similar ‘tricks’ when we call other important numbers e or π. Also, when we denote the five existing, unique and real roots of the equation $x^5 -5x^3+4x+1 = 0$ by a_1, a_2, a_3, a_4, a_5 according to size we do the same thing. All these numbers are actually ‘unknown’, have been given a name, and now we can use them for calculations (which is exactly what is done in algebraic number theory). But if we need $\sqrt{2}$ or any other of these numbers in a concrete application, for example for building a bridge, it has to be suitably approximated. In any case, numerical aspects are very important when solving equations. From the point of view of numerical analysis a number “a” possesses indefinitely many digits which are hidden behind a curtain. By using our appropriate algorithm we can withdraw that curtain in order to see an arbitrary number of digits.

In pre-computer times extremely artificial methods have been invented which were all directed towards a more or less special type of equation. For example, differential equations resulting from modelling a concrete problem were not discussed generally, but reality was forced into the ‘Procrustean bed’ of one of the few types of differential equations (linear, separable, etc.) for which there was an explicit solution formula. “A process of self-selection started, whereby equations which could not be solved explicitely were of less interest than those which could. Naturally, textbooks for the new generation contained only methods for solvable problems.” (Steffen, 1994).

Half as good is not on the mark

Numerical aspects deserve to be discussed in detail. Many years ago Arthur Engel urged already to put more emphasis on numerical and algorithmic aspects in mathematics teaching. Unfortunately, even in new curricula the analytic-algebraic perspective of mathematics dominates, and numerical aspects are hardly mentioned. Mathematicians are to be blamed for the fact that fundamental mathematical ideas like iteration are written on the banner of computer scientists, that numerical aspects have been dreadfully neglected and have done their bit to building a one-sided picture of mathematics. Numerical methods are not a poor substitute, as many mathematicians still see it today, but normal practise in all applications of mathematics. Numerical methods have not yet found their way into school. New technologies without numerical methods are not possible. Using a CAS these aspects have to be emphasized, but on the other hand, it is easy to introduce the basic concepts already at lower secondary level.

The aim is not to indulge in detailed calculations of error but to develop a principle understandig of the limits of modelling. First of all this involves a sensible use of numbers. We have the ‘ideal’ numbers of mathematics, where 2 = 2.0 = 2.00 is naturally true, and the ‘real’ numbers of daily life, which often mean intervals and where 2 is certainly not equal 2.0. Mostly, i.e. when measuring, intervals, not exact numbers are an adequate model of the situation. Intervals, however, lead to error proliferation in continued model calculations. If this is not kept in mind, results become arbitrary. In addition we have today the ‘computer’ numbers which follow their own rules, too. The speed of processors has been rapidly increased, but the error analysis of the implemented floating-point arithmetic has been neglected in an irresponsible way. A CAS can help to demonstrate the hidden error proliferation of recursive calculations. Following a suggestion of the Karlsruhe mathematician Ulrich Kulisch (Kulisch, 1998) 2000 elements of the point sequence $P_n(x_n/y_n)$ with $P_0(0/0)$ and

$$x_{n+1} = y_n - \mathrm{sgn}(x_n) \cdot \sqrt{|3 \cdot x_n - 36|}, \quad y_{n+1} = 11 - x_n \ \text{ for } n \geq 0,$$

were calculated with the help of MAPLE. Using the ‘Digits = m’ command the number of digits can be chosen in MAPLE. The results are given in Figure 4 for accuracy m = 5, 10, 15, and 20. The result speaks volumes, the relation with the beautiful, computer-generated pictures of chaos theory is obvious. The presentation is even more convincing when the colour of the points is changed after every 500 points.

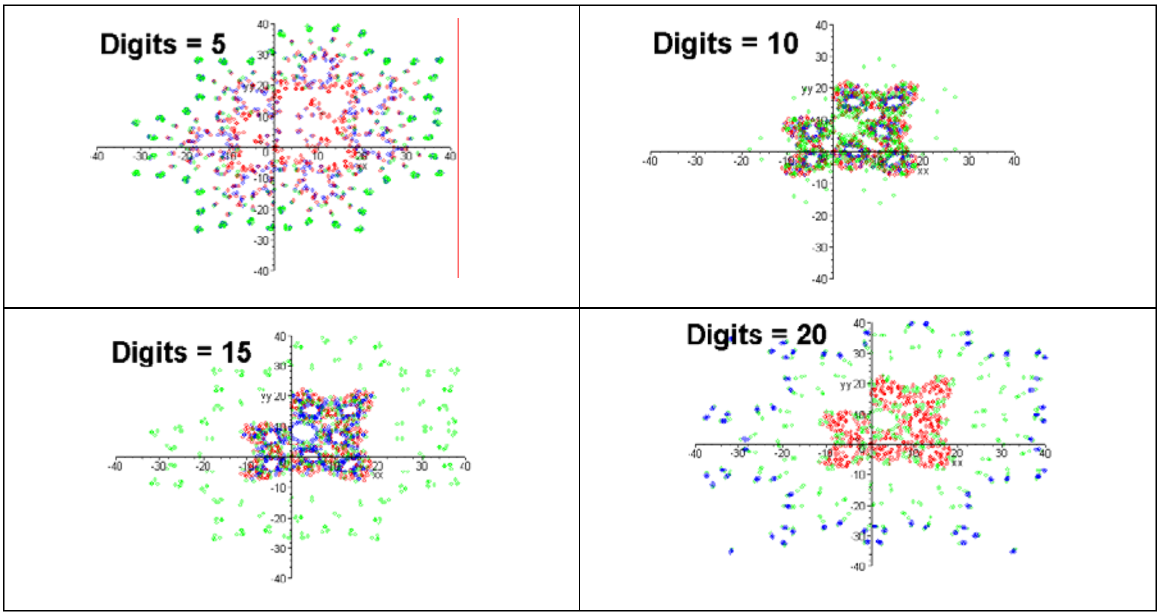

Figure 4. The point sequence with accuracy digits = 5, 10, 15, and 20.

In daily life the calculation-oriented treatment of numbers plays a dominating role, numerical aspects are neglected. Of course, pocket calculator or CAS alone do not provide a better understanding. Absence of clear teaching concepts will lead to incorrect basic concepts being built in various ways:

On the one hand, expressions such as $\frac{1}{3}$, $(\sqrt{3})^2$, $\frac{\pi}{2}+2+3\cdot\frac{\pi}{4}$ or cos(arcsin(0)) are not seen as well defined numbers which can be simplified, but rather as recipes or algorithms to be followed. Even among future teachers who have studied mathematics, I have encountered this misunderstanding: They sincerely discussed whether $4 + \sqrt{3}$ was one or two numbers!

On the other hand, fundamental numbers such as $\sqrt{2}$, e and π are reduced to finite decimal numbers. In replay to my question of whether $\sqrt{2}$ was rational, a glimpse on the calculator, I prompted the answer “no”, because the calculator did not show a number period.

The “two-facedness” is always important: *Result* as well as *process*, for example $\sqrt{2}$ as a number and as the process of numerical aproximation.

“How much manipulation of expressions is necessary” is the well-known question formulated by Wilfried Herget in the GDM-workshop “mathematics teaching and computer science” (Hischer, 1993, p 128), which still is unresolved. I am convinced that computional skills help in the sensible handling of numbers and terms. Manipulating expressions should not serve as a ritualized end in itself. Here, a CAS can help to build up sensible standards. Often, expressions should be transformed into a simpler final form. But what is “as simple as possible”? For rational numbers a standardized final form, for instance as reduced fraction, makes sense. However, it is a mere convention to look at the equivalent expressions

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and pick out the right hand expression to be the “better” one, or even, to give less points for the left hand expression. Making the denominator to a rational number becomes a senseless ritual. Why then can $\frac{1}{\pi}$ be left as it is? Both terms above are equivalent and acceptable as an end result. When approximate values are to be calculated mentally, the right-hand term is better. If you want to find an approximate value using a computer, the left term is often superior. Hans Humenberger and Hans-Christian Reichel describe a very convincing example (Humenberger & Reichel, 1995, p 107). The issue becomes very delicate when the simplify command, which is provided by every CAS, is used to simplify expressions. It is often very surprising to see what for CAS is a "simple" solution. The problem is very deep-rooted: There is no algorithm which can decide if two given expressions are algebraically equivalent. We will always have to accept this "shortcoming" of a CAS! The more important it is therefore to make students familiar with this problem.

A very interesting and well-known example showing many aspects is the approximation of the circle by inscribed regular polygons, a topic, which is covered at grade 5 pre-formally and then again at grade 10. The idea to approximate the unit circle by inscribed polygons with $n = 3{\cdot}2, 3{\cdot}2^2, 3{\cdot}2^3, \ldots$ sides (is it clear for all students why we do not use 3-, 4-, 5-, … sided polygons?) leads through elementary geometrical argumentation as shown in figure 5 to a simple recursion formula for the side length s_n of the n-sided polygon.

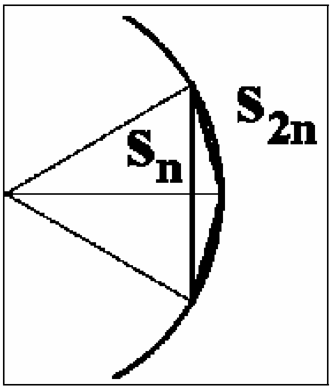

Figure 5. The side length of the inscribed n-sided polygon.

From this well-known formula $s_{2n} = f(s_n)$ we can easily deduct an analogous recursion formula for half the circumference p_n of the polygon in consideration, which leads to an approximation of π. We use this less well-known recursion formula for p_n here, because it makes the numercal problems very obvious. Starting with n = 6 for the unit circle the side length $s_6 = 1$ and the first approximation for π becomes $p_6 = 3$. The easily verified result for the recursion is

$$p_{2n} = n\sqrt{2-2\sqrt{1-\left(\frac{p_n}{n}\right)^2}}\ .$$

This nice, simple formula can be "made worse" by

$$p_{2n} = \frac{2p_n}{\sqrt{2+2\sqrt{1-\left(\frac{p_n}{n}\right)^2}}}\ .$$

Both formulae are absolutely equivalent and should therefore, theoretically, lead to an increasingly better and better approximation of π. The approximation is possible only numerically, which is usually not done at school. But with a CAS it is as simple as a child's play. The computional accuracy used for the results in figure 6 was five digits.

1$^{\text{th}}$ recursion formula	**2$^{\text{d}}$ recursion formula**
n = 1, π = 3.0	*n = 1, π = 3.0*
n = 2, π = 3.1055	*n = 2, π = 3.1058*
n = 3, π = 3.1315	*n = 3, π = 3.1326*
n = 4, π= 3.1385	*n = 4, π = 3.1394*
n = 5, π = 3.1476	*n = 5, π = 3.1412*
n = 6, π = 3.1839	*n = 6, π = 3.1416*
n = 7, π= 3.3256	*n = 7, π = 3.1418*
n = 8, π= 3.8400	*n = 8, π = 3.1418*
n = 9, π = 0	*n = 9, π = 3.1418*
n = 10, π = 0	*n = 10, π = 3.1418*

Figure 6. Results of the two recursion formulae.

The left column shows the results of the "nice", the first formula. At first, the approximation of π improves, but from step 7 on the approximation declines and from step 9 on the approximation value is always zero! A higher accuracy only defers the catastrophy by a few steps. Using the second formula, π is continuously better approximated, from step 7 on the recursion stabilizes.

Looking at both formulae helps to understand what happens (figure 7):

$$p_{2n} = \underbrace{n}_{\to\infty}\sqrt{\underbrace{2-2\sqrt{1-\left(\frac{p_n}{n}\right)^2}}_{\to 0}}$$

"catastrophy for large n"

$$p_{2n} = \frac{2p_n}{\sqrt{2+2\underbrace{\sqrt{1-\left(\frac{p_n}{n}\right)^2}}_{\to 1}}}.$$

"$p_{2n} \approx p_n$ for large n"

Figure 7. Analysis of the two recursion formulae.

Because computers can only compute with finite accuracy, very small values become zero, therefore the first recursion must fail. The root of the problem lies much deeper: Unfortunately, there is no possibility to decide for an arbitrary number whether it is zero or not; Luitzen Egbertus Jan Brouwer has constructed such numbers. However, if we use the second formula which is termed the bad formula, the problem is solved, the recursion is stable, and the approximation is much better. Archimedes has principally used this second recursion formula to calculate his famous approximate values

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

The case of Geometry

For years, we have been observing a decline of geometry in school curricula; at the upper secondary level geometry is often twisted into some basics of linear algebra. In our already mentioned expertise of mathematics teaching at upper secondary level you will not find the word "linear algebra". This was a purposeful political decision to which I answer with my own head. As my mathematical background is in algebraic number theory and algebraic geometry I know very well the value of linear algebra. At upper secondary level, the students choose in mathematics (and in other subjects) between advanced-level courses (*Leistungskurse*) and basic-level courses (*Grundkurse*). Referring to this, we write in our expertise (Borneleit, Danckwerts, Henn, & Weigand, 2001): “There is no other topic revealing as plainly as geometry at upper secondary level does that it is completely off the subject to conceive basic-level courses as ‘light versions’ of advanced-level courses, which themselves are ‘light versions’ of lectures in pure mathematics at the beginning of university level”. I have come to know linear algebra in school as words without meaning, as “general abstract nonsens”. Out of pure enthusiasm for abstract terms, the really thrilling ideas of geometry are often forgotten. One uses puffed-up methods only for boring objects such as straight lines and planes, methods which are not really needed there. Instead, the richness of solid geometry can bring valuable insights for students by emphasizing content and not mere calculations, as for example Hans Schupp has proved in many articles (Schupp, 2000).

Unfortunately, the new draft curriculum for my home state of Baden-Wuerttemberg includes the words like “linear dependent” and “linear independent”, but teaching of the obligatory geometrical content ends with planes, even spheres are not part of future mathematics courses. A speciality of geometry teaching at lower secondary level is the exploration of certain convex plane figures, such as triangles and quadriangles. The natural generalisation to convex solid figures is out of scope, the only convex figure in our curriculum is the point.

The geometry of the space around us should be one focus of mathematics teaching, which is supported by the powerful methods of analytic geometry at secondary level. A CAS is extremely well suited for visualization, and as a calculation tool. However, to use the advantages of a CAS depends on well-founded 3D-experiences of students with real objects, starting, for example, with a systematic exploration of the cube in grade 5. Rightly, Michael Schmitz (2000) points out that the visualizations of 3D-objects with a CAS, which are so clear for us, are only understandable for students if they have well-founded basic concepts of space and its analytic description. It is necessary that students possess the ability to choose a coordinate system which is adequate with respect to the situation and therefore to support the development of the fundamental idea of coordinatization. Through the use of an adequate coordinate system objects are described by equations as simple as possible, geometric transformations by matrices. This situation then calls for a CAS to solve the equations and to present their solutions in three dimensions. This in fact means not to treat calculus and analytic geometry as disjunct topics any more, but to discuss their interrelations. Then functions in several variables are not treated as functions in one variable with parameters giving rise to much computation, but describe – introduced and discussed qualitatively – graphs as surfaces and curves, and thus help also to better understand scientific modelling (Klika, 2000). Especially the beautiful old conics can experience a rebirth with the help of a CAS. The following example (figure 8) explicitely shows the interrelations between calculus and analytic geometry, showing two ways leading to saddle surfaces. Figure 9 shows two real examples of saddle surfaces, the left one is a work of art in the port of Lausanne/Switzerland, the right one is a bus stop in Offenburg/Germany.

A concrete and very nice starting point for the case of the family of curves is paper folding. For further suggestions compare Henn (1998) and Meyer (2000).

Calculus	Analytic geometry
Families of curves	**Metric issues**
Discuss the family of curves $f(x,t) = tx + t^2$ (parameter $t \in \mathbb{R}$)	Describe all points in the plane having the same distance to two straight lines
Discuss the family of parabolas $f(x,t) = tx + t^2$ (parameter $x \in \mathbb{R}$)	Same question in spatial geometry
Discuss $f(x,t) = tx + t^2$ as a function ot two variables	Investigate the new case of skew straight lines (suitable coordinate system)
Saddle surfaces are be found often in the real world! The tomography of saddle surfaces leads to straight lines, parabolas, hyperbolas, and many interesting questions!	

Figure 8. Two ways to saddle surfaces.

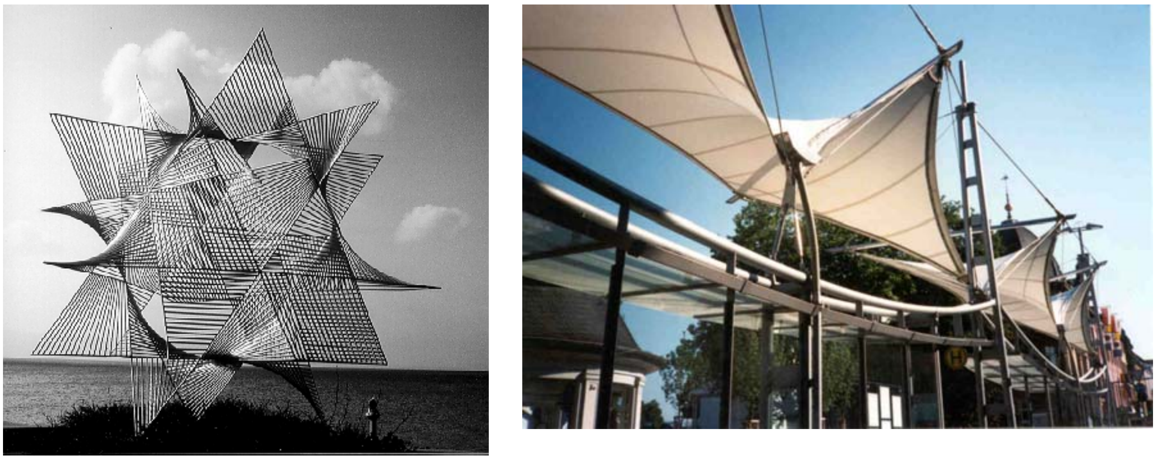

Figure 9. Two examples of saddle surfaces.

CAS – challenge and chance

Important is the wine, not the barrel or the skin in which it is stored. Our barrel is a CAS, be it MAPLE, DERIVE or any other. It is a further supplement for the traditional media but cannot substitute them. Most of the work for design of a problem has to be done with head, paper, and pencil. The MAPLE-publicity slogan *"You can forget paper and pencil"* is nonsense. Many student's comments such as "You cannot think in front of a computer screen" confirm this. In detail, we have learned in our PIMOKL project that the sensible handling of a computer, which is such a fast tool with respect to calculations, needs the contrary, namely a culture of slowness. The time and leasure which is needed for thinking, the determined inductive search and experimenting has nothing to do with losing or wasting time. Taking your time means that you will progress faster in the end. Let your students discover this creative slowness!

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