Modularity and Geometry

Gert Kadunz
Universitaet Klagenfurt
<gert.kadunz@uni-klu.ac.at>

This report investigates the use of modules/macros with software for dynamic geometry (DGS). Modules will be shown in synthetic and analytic use. The explanation concentrates on ideas and concepts from cognitive science and mathematics education. It will be shown that using modules can be motivated by DGS and, simultaneously, point beyond the borders of DGS. Modules support the learner organizing his own thinking quantitatively and qualitatively.

Remarks on modules in cognitive science

Within cognitive science there has been a long lasting and often controversial debate on "modularity of mind" (Fodor, 1987; Chomsky, 1988; Samuels, 1998; Gerrans, 2002). Most of these researchers see modules as encapsulated, specialized units for handling information.

Philip Gerrans (Gerrans, p. 261) presents three different views on modules. Modules are cognitive devices which define our mind. In this view Gerrans talks of modules as if they are some kind of hardware. Firstly modules seem to be like computer circuits, with each module built to deal with a special task. "Are modules innate?" is a question often asked in this context[1]. Secondly "modules are like algorithms" makes use of a computer metaphor to describe the mind, which then is seen as a computer organized by a large number of algorithms. Finally Gerrans offers a more epistemic view on modules namely that they can be seen as domain specific bodies of innate knowledge.

By investigating the connections between the eye and the brain, biology has shown that modules control the mind even on a physiological level.

> Similar functional specialisation occurs within the neural system on which vision depends. The visual cortex contains individual and suites of neurons specialised for detecting orientation, disparity, wavelength, velocity, direction and length. (Gerrans, p. 263)

It seems convincing that developments from evolution which support successful activities are imitated on the level of thinking. If we can modularize our thinking as well, then we may hope it will be similarly successful. In a previous article I expressed some ideas about collecting data from the visual sense and their effect on the learning of mathematics (Kadunz, 2000). For the following considerations I only record the fact that it seems useful to imagine that many areas of the mind are organized as modules and that students are capable of constructing and using modules.

---

[1] "The conception of modularity first articulated by Noam Chomsky in his explanation of language acquisition treats a module as a body of innately specified prepositional knowledge. We all acquire languages so readily, despite our linguistic environments, because innate knowledge of some very fundamental grammatical principles, such as nouns and verb phrase structure is universal in the human species." (Gerrans, p. 259)

"Evolutionary psychologists defend a massively modular conception of mental architecture which views the mind, … , as composed largely or perhaps even entirely of innate, special-purpose information-processing organs … ." (Samuels, 1998, p.576)

Modules in mathematics education

Papers on mathematics education have dealt with the idea of modularity in many different ways (Dörfler, 1991; Dubinsky, 1991; Sfard, 1994; Tall, 1999; Kadunz & Sträßer, 2001). First I shall concentrate on Dörfler and his view on the use of modules. He focuses his deliberations concerning modules on the use of computers in mathematics education.

In Dörfler's view the thinking of experts is mainly characterized by the use of structured knowledge. A chess champion analyses an arrangement and its relations to other possible arrangements as a whole, in the same way as a mathematician argues using theorems from mathematics or engineers use well known modules when developing their plans. Such experts always use arrangements, theorems or modules. For mathematicians, such arrangements can also be, in addition to standard mathematical constructs, algorithms and concepts, in which mathematics can be found in condensed form.

On the one hand such modules can be the means of solving process-like problems, for example differentiation rules, the Euclidean algorithm or linear mappings. On the other hand we use modules as objects. As mentioned above, in most cases performing a proof needs the use of known theorems.

Both the process and object use of modules can reduce the complexity of a given problem. We notice this reduction in complexity literally (materialized) when we observe students solving mathematical problems, e.g. the "formula for quadratic equations" transforms a more "sophisticated" quadratic equation into two simpler linear equations. On the other hand reducing complexity can also be done in the mind alone. For instance a figure in geometry can often be successfully described by applying a theorem from geometry. To apply a module does not mean the user needs to know all the knowledge condensed "inside" the module. This is similar to the use of modules in computer languages where knowing the module's interfaces and its effect is much more important than knowing the module in detail.

David Tall (Tall, 1999) describes construction and use of "cognitive units". He focuses on the use of such cognitive units and defines them as "pieces of cognitive structure that can be held as the focus of attention all at one time" (Tall, 1999, p.226). He stresses that the use of modules in solving mathematical problems gains a kind of "economical" benefit.

Students doing mathematics in school are acquainted with modules. For instance they learn about them when they use software and hardware for doing computer algebra (CAS). Differentiating functions, looking for the zeros of polynomial functions or solving systems of linear equations are literally encapsulated in a button. Edith Schneider investigates and presents a number of CAS examples (Schneider, 2002, p.263)

System for dynamic geometry, DGS

Within the literature on mathematics education DGS describes a class of computer software. With this software we draw figures in geometry and explore them. During the past decade software developers, mathematicians, researchers on the learning of mathematics and mathematics teachers have developed a number of products (Cabri, Geometer's Sketchpad, Cinderella, Thales, Geolog, Euklid). When criteria are required for defining such software, the literature on mathematics education offers three features to describe a system for dynamic geometry namely

- dragmode,
- locus of points,
- the ability to define and use macros (modules).

In the following I shall concentrate on macros (modules). (For deliberations concerning the dragmode or locus of points see Arzarello, 2002; Hölzl, 1999; Jahn, 2002, Sträßer, 2002, Schumann, 1996).

At first glance, when solving problems in geometry, modules appear well suited to the reduction of drawing complexity since manually complex constructions become more manageable using modules. Figure 1 shows a tessellation of the Euclidian plane. The left part of figure 1 was produced with the aid of modules. The drawing on the right shows all the steps of the construction in detail.

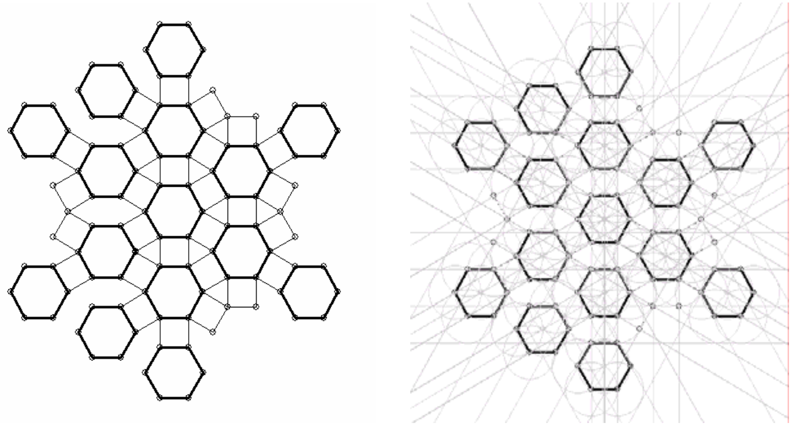

Figure 1.

Even two quite simple modules, namely "square" and "hexagon" commonly used to shorten the construction, show the need for reorganizing mathematics education. The complex construction itself does not become the goal but rather the knowledge about the use of modules, which become geometric tools. The use of modules for the easier and more error free execution of a geometric construction can be addressed as "synthetic". If it is only the "mechanical" implementation of a geometric construction that is required then such synthetic modules are seen from the viewpoint of quantity.

The use of modules points beyond this quantitative aspect, when they are used together with the dragmode in geometrical tasks, to survey concepts of geometry experimentally. This can be observed during the investigation and use of mathematical mappings. When students are examining reflection on a circle (inversion) they can do this step by step, beginning with a point and producing an appropriate module with the software, then using this "point module" to reflect a line or a given circle. Such modules are available within the software and permit observation of the central characteristics of the inversion. With inversion modules curves can also be inverted. If someone wants to observe the inversion image of a conic, then a focus of the given conic should be selected as centre of the inversion circle. Inverting a hyperbola produces a "double point" in which the asymptotes of the hyperbola become tangents to the image of the curve. Here the inversion is appropriate for giving meaning to the difficult concept of the "point at infinity". All variations of the original hyperbola can immediately be observed on the mapped curve. Together with the dragmode, the modules used here can be seen as a source for viewing the construction in a new way. In other words, the construction receives a new quality and in this sense the use of synthetic modules can have a qualitative aspect.

A special characteristic distinguishes modules in synthetic use since they are features of construction within a DGS. They can be called by name from a menu system, possibly in combination with icons. Thus modules are a substantial feature of DGS, which can be used in various ways and with which numerous steps of a geometric construction can be combined into a single step. This situation appears to be different, if in a complex question, e.g. a non-trivial theorem, students have to give a proof by finding references to other theorems from

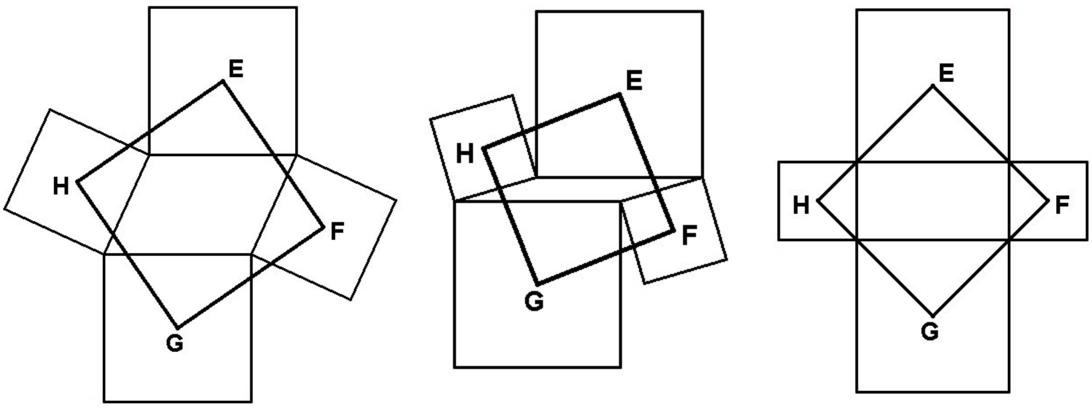

Figure 2.

geometry, as presented in the following example.

Theorem: The centres of the (outside oriented) squares over the sides of a parallelogram themselves form a square.

We can also offer this theorem to students in the following formulation: Draw a parallelogram and all the squares beyond its sides, arranged outwards. What can be assumed about the centres of these squares?

If the parallelogram is varied (dragmode) once the construction has been accomplished, the four centres E, F, G and H can be seen as corners of a rectangle, which, with each change of the parallelogram, always possesses the "Gestalt" of a square (Figure 2). We can say that, with such a use of the four centres, these points are used in a certain way. This new figurative use is also seen as an example of a geometric module. We can say that with the use of the dragmode we see modules "into" a geometric drawing. This will be presented in detail in an example in the last section of this article.

During the realization of variations, i.e. during the execution of experiments with DGS, the user, via the dragmode, accesses those relations which exist between the individual parts of the geometric construction. He sees them as a module. One could say that the relational aspect of a construction supports the modular view on the construction.

How can the development of such modules and thus a modular view on geometrical constructions be imagined?

Looking for modules

We described briefly how students use modules in analytic use. A complex construction can be reconstructed and divided in its structure by the use of modules. Here the dragmode, which supports the execution of experiments, plays a substantial part in the construction and the reconstruction, the "re-cognition" of theorems from geometry. As presented above in the parallelogram example, by the use of variation a "Gestalt" can be seen in a larger configuration. The user reconstructs different known configurations "into" a geometric construction, for instance the "configuration of homothecy", or "the configuration of Thales' theorem".

Modules in analytic use, with which students describe a geometrical configuration and

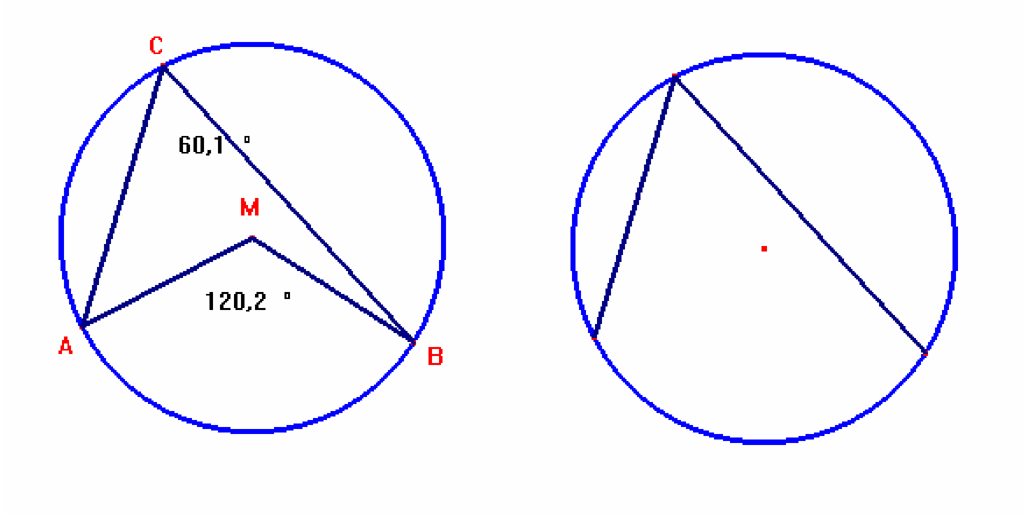

Figure 3.

thus structure this configuration, do not always have a direct correspondence in the software however. Modules in synthetic use are called by their names from a constantly growing menu system. They look like formulas, since they are defined by an exactly specified number of input parameters and an exact quantity of output elements determined by the module's developer. This need not be the case with modules which are used analytically. They remind us rather of well-known constructions, which are overlaid on the given construction. Put simply one could say that the synthetic use of modules takes place "on the paper", while the analytic use of modules "happens in the head". If an analytically used module fits[2], then the module can be used e.g. in an argumentation of a proof. Now the statement of "theorem of the circumferential angle" (CFA) or the characteristic of the similarity of triangles is not part of a

[2]One could say that a module fits if the student using this module succeeds in his analysis of the submitted problem.

menu system of a DGS. In this sense analytic modules do not have to be parts of the software, although the software is involved in the construction of the subjective understanding of such modules. Let us look at the following example.

The CFA can be explored by use of the dragmode. Its statement is used in different elementary geometrical questions, as well as in construction tasks and in proofs. If one observes the invariance of the angles measured during the movement of the angle vertex, the student can see different geometrical features in the surrounding geometrical configuration. For instance there are the points on the circle which specify the angle field, the circle and the sides of the angle. If one focuses in this configuration on the statement of the CFA, then geometrical configuration, having far fewer design features than the more varied construction picture, comes into play. Subsequently this “poorer” representation needs no labelling and can in the end be seen as one “whole”, i.e. as a module. The movement of the vertex (C) has resolved the moved elements (vertices and sides) and all other elements (circle and points on the circle) having determined them directly from the original figure, and now justifying their condensation into a configuration. This condensed configuration can be addressed as a “figure of the theorem of the circumferential angle” (see Figure 3). It thereby receives a name and is in this sense a module which could be stored as a DGS module but this is not appropriate. Furthermore such modules can be used primarily to describe geometrical constructions, rather than to do these constructions. The naming connected with the forming of the module enriches the students’ “geometrical language” similar to the use of synthetic modules. As a consequence they can build up[3] modules for analytic use. We should offer students such a stock in two modes: as “geometric construction” with points, lines and circles and in linguistic (written) form. One puts at their disposal two representations for geometrical circumstances. Students can access both representations and use each of them in specific situations. The geometrical representation can serve to recognize the module in a geometrical construction, and linguistic representation can be used to formulate the geometrical relations. The use of such modules which can possess a complex genesis with which the student need not be familiar, leads to the next orientation in the description of learning geometry, as mentioned above. From the student’s view the geometrical construction which is varied in the experiment changes its quality. Changing the quality of the view means reconstructing modules using the dragmode, for instance describing the configuration with the names and characteristics of these modules and seeing a simpler geometric construction. This means that during this process of change a student arranges their thinking in a new way.

[3] H. Kautschitsch uses the term “optical encyclopaedia” (Kautschitsch, 1987, p. 223).

An image of the orthocentre

An example from elementary geometry is given, in which the orthocentre of a triangle is reflected at the sides of the triangle. The images of the orthocentre lie on the circumscribed circle of the triangle. In school this is usually handled by means of linear algebra. Figure 4 shows the orthocentre H and the points Ha, Hb and Hc on the circumscribed circle. That at least is how the DGS picture appears. If one varies the triangle, this feature can be noted in all configurations. How can this observation be introduced into a geometrical context? Because the orthocentre is reflected in the same way on all three sides, it is sufficient to consider and

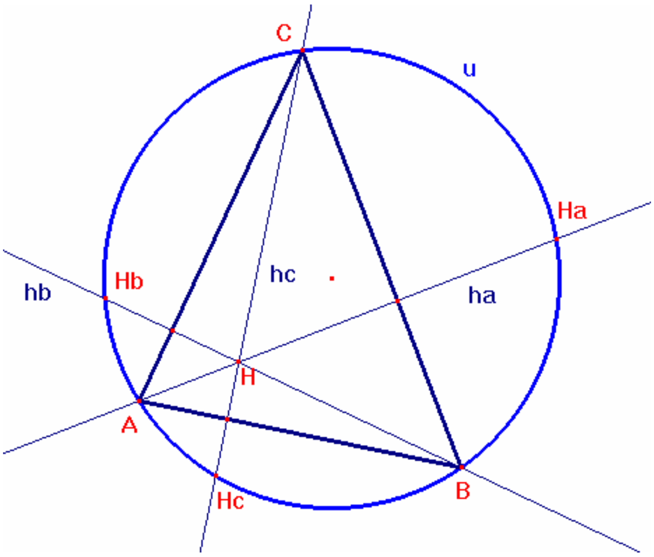

Figure 4.

discuss only one of these reflected points, for instance Ha. In doing so, the configuration gains in clarity so that the points H, Ha and C can be seen as vertices of a triangle. However, one does not know whether Ha lies on the circumscribed circle. In this situation the student can construct intersection points of the circumscribed circle with the height ha. How can we be sure that one of these points is the reflection point of H, reflected on the side of the triangle BC? If we intersect the altitude through A with a circumscribed circle, then the two points are A and Ha’, which must therefore lie on the circumscribed circle. If we can demonstrate that the points Ha’ and Ha are the same, then the claim is proved. The first geometrical figure of assistance to the student is the triangle HHa’C. Is this triangle an isosceles triangle? In an isosceles triangle angles at the base are of equal magnitude. In order to gain an idea of how to examine the triangle’s angles, the triangle can again be varied using the dragmode. The triangle HHa’C is expressly observed (Figure 5). If the vertex B is moved, then Ha’ remains on the circumscribed circle and, more precisely, Ha’ remains on an arc of this circle. A geometrical figure connecting angle and arc is the figure of the theorem of the circumferential angle, which students can access like a formula from their construction store (see footnote 2). If one views Figure 6, then we see the vertex B together with point Ha’ on the arc over AC. By the theorem of the circumferential angle ABC = AHa’C follows. To prove that the angles CHF_a and ABC are equal we can use right triangles, which can be seen by using the altitudes of the construction. Thus the statement is proven.

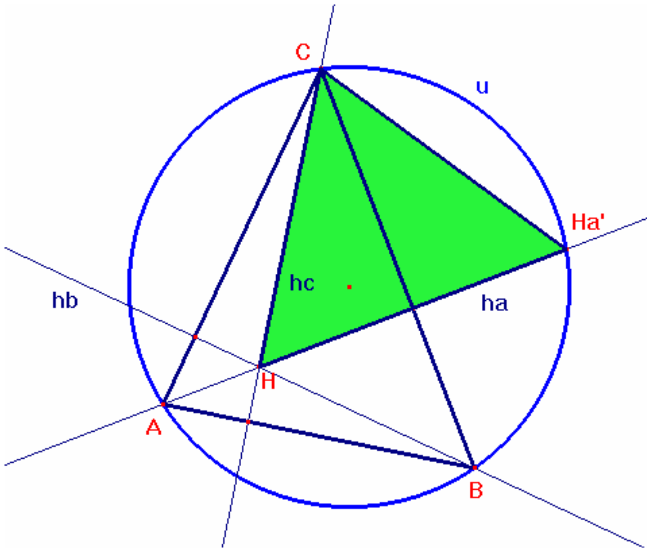

Figure 5.

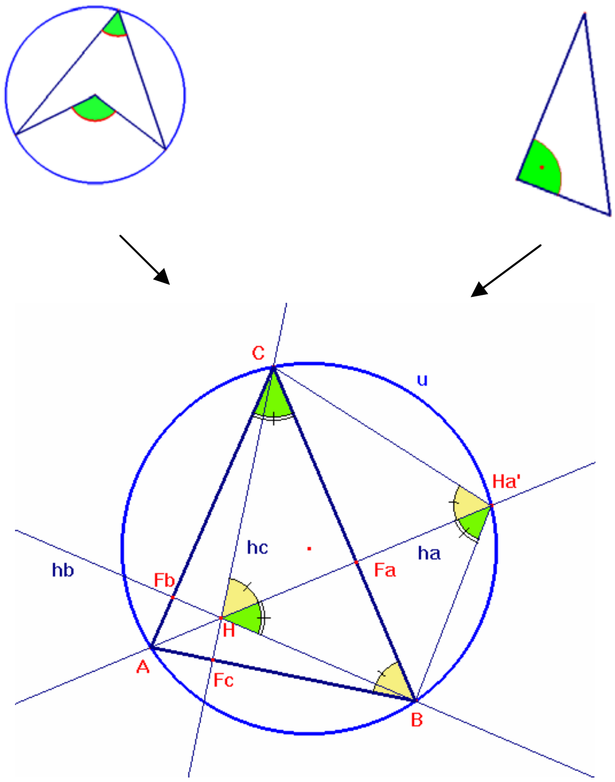

Figure 6.

The interplay between dragmode and module use is what characterises the treatment of this example. The variation of the construction produces situations which can remind students of special theorems from geometry. H_a'CH reminds them of an isosceles triangle, while moving H_a' on the circumscribed circle together with the vertices A and C reminds them of the theorem of the circumferential angle. The description of the construction is directed by the use of these theorems and a proof is carried out. The complexity of the configuration is reduced by the use of theorems from geometry, while the dragmode makes the figure so mobile that further geometrical theorems can be seen. Students consider the construction from new points of view. They emphasize relations when they recognize theorems from geometry, and they vary the construction, when they are looking for a theorem as a module. These changing perspectives are also characteristic of a specific view of visualization (Kadunz, 2000).

Summary

In the discussion above the significance of modules in the use of DGS was presented. From the use of modules, criteria for the description of modules for the learning of geometry can be established. On the one hand geometrical constructions can be implemented more economically with modules (synthetic use of modules). On the other hand in a complex geometrical construction a geometrical fact still unknown to the learner can be discovered through module use. Thus we see both a quantitative use and a qualitative use of DGS modules.

The significance of modules in proofs is a separate issue. It was argued that in this case students must leave the DGS to do a proof. We can also offer students these theorems in special forms (optical encyclopaedia, stock of modules). This was described as the analytic use of modules. Both uses of modules, the synthetic and the analytic, are indispensable for learning geometry.

References

Arzarello, F., F. Olivero, et al. (2002). A cognitive analysis of dragging practises in Cabri environment. *Zentralblatt Didaktik der Mathematik* 34(3), 66-72.

Chomsky, Noam. 1988. *Language and Problems of Knowledge*. Cambridge MA: MIT Press.

Dörfler, Willi. 1991. Der Computer als kognitives Werkzeug und kognitives Medium. Pp. 51-76 in *Computer - Mensch - Mathematik*, edited by P. W. Dörfler W., Schneider E., Wegenkittl K. Wien, Stuttgart: Hölder-Pichler-Tempsky; B.G.Teubner.

Dubinsky, Ed. 1991. Reflective Abstraction in Advanced Mathematical Thinking. Pp. 95-123 in *Advanced Mathematical Thinking*, edited by D. Tall. Dordrecht: Kluwer.

Fodor, Jerry A. 1987. *The Modularity of Mind*. Cambridge, MA, MIT Press.

Gerrans, Philip. 2002. "Modularity reconsidered." *Language & Communication* 22, 259-268.

Graumann, Günter, Reinhard Hölzl, Konrad Krainer, Michael Neubrand, and Horst Struve. 1996. "Tendenzen der Geometriedidaktik der letzten 20 Jahre." *Journal für Mathematikdidaktik* 17, 163 - 237.

Hölzl, Reinhard (1999). Qualitative Unterrichtsstudien zur Verwendung dynamischer Geometrie-Software. Augsburg, Wißner-Verlag.

Jahn, Ana, P. (2002). "Locus" and "Trace" in Cabri-géometrè: relatationships between geometric and functional aspects in a study of transformations. *Zentralblatt Didaktik der Mathematik* 34(3), 78-84.

Kadunz, Gert. 2000. "Visualisierung, Bild und Metapher." *Journal für Mathematik-Didaktik* 21, 280-302.

Kadunz, Gert & Rudolf Sträßer. 2001. Visualisation in geometry: multiple linked representations? Pp. 201-208 in *25th Conference of the International Group for the Psychology of Mathematics Education.*, edited by M. v. d. Heuvel-Panhuizen. Utrecht: Freudenthal Institute, Utrecht University.

Kautschitsch, Hermann. 1987. Ein 'optisches Lexikon' für die Mathematik. Pp. 223-232 in *Medien zur Veranschaulichung von Mathematik*, edited by Kautschitsch H. and Metzler W. Wien: Hölder - Pichler - Tempsky.

Samuels, Richard. 1998. Evolutionary Psychology and the Massive Modularity Hypothesis. *The British Journal for the Philosophy of Science* 49, 575-602.

Schneider, Edith. 2002. *Computeralgebrasysteme in einem allgemeinbildenden Mathematikunterricht. Didaktische Orientierungen - Praktische Erfahrungen*, vol. 2. München, Wien: Profil.

Schumann, Heinz (1996). The influence of interactive tools in geometry learning. *Intelligent learning environments. The case of geometry*. J. M. Laborde. Berlin, Springer**,** 157-187.

Sfard, Anna. 1994. Reification as the Birth of Metaphor. in *for the learning of mathematics* 14, 44-55.

Sträßer, Rudolf (2002). Research on Dynamic Geomtry Software (DGS) - an introduction. *Zentralblatt Didaktik der Mathematik* 34(3), 65.

Weth, Thomas. 1992. "Computerunterstütztes modulares Konstruieren im Geometrieunterricht." *Zentralblatt für Didaktik der Mathematik* 24(4),148-153.

Tall, David & Lillie Crowley. 1999. The Roles of Cognitive Units, Connections and Procedures in achieving Goals in College Algebra. Pp. 225-232 in *23. Conference of the International Group for the Psychology of Mathematics Education*, vol. 2, edited by O. Zaslavsky. Haifa: Technion, Israel Institute of Technology.