Knowledge Acquisition in Students' Argumentation and Proof Processes

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Expert problem solving may be regarded as a process of understanding and modelling real world phenomena. Inductive thinking, empirical observations and deductive reasoning are crucial parts of this process. Experts and students differ in this respect, but they often show similarities in their problem-solving behaviour. Our research aims at identifying similarities and differences between experts and students in their mathematical problem solving with respect to argumentation and proof at the upper secondary level. Moreover, we will argue that adequate, as well as inadequate scientific models guiding the students' argumentation are influenced by the practices in the mathematics classroom.

Proof and Scientific Reasoning

In the last few years there has been an intense discussion in mathematics education research on students' concepts of argumentation and proof. Both aspects are regarded as important for the understanding and application of mathematics. This positive attitude towards argumentation and proof is the result of an important debate among mathematics educators. It was Freudenthal who argued against geometrical proofs, particularly those in the form of classical Euclidean proofs. Accordingly, for many years proofs were regarded as superfluous in the mathematics classroom. It was conjectured that Euclidean proofs were far from providing any kind of mathematical insight, but were a means of initiation into a highly standardised and schematised type of argumentation cultivated only in school mathematics. Since that time, Hanna and other authors (Hanna, 1990; Hanna & Jahnke, 1993; Hersh, 1993; Moore, 1994; Hoyles, 1997; Harel & Sowder, 1998) have rehabilitated mathematical justification and proof (including Euclidean proofs) in the classroom, pointing out that in mathematical research as well as in school instruction, "proving" spans a broad range of formal and informal arguments and that being able to understand or generate such proofs is an essential component of mathematical competence. In constructivist mathematics instruction in particular, the critical exchange of arguments and elements of proof has achieved new significance.

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) call for a "focus on learning to reason and construct proofs as part of understanding mathematics so that all students

- recognize reasoning and proof as essential and powerful parts of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use types of reasoning and methods of proof as appropriate."

These goals are applied to all stages of education from pre-school to grade 12. In the early years informal inductive elements are emphasized whereas the formal deductive elements become more important for older students.

In recent research on mathematical proof there is a broad range of approaches to this topic in order to better understand students' ideas of argumentation and proof. In particular there is a considerable number of empirical studies on students' proving abilities (e.g. Senk, 1985;

Usiskin, 1987; Healy & Hoyles, 1998; Lin, 2000; Reiss, Hellmich & Thomas, 2002), which have mostly revealed wide gaps in respondents' understanding of proofs. Healy and Hoyles (1998) made a significant contribution to the field with their recent systematic investigation of students' understanding of proofs, their ability to construct proofs, and their views on the role of proof. Their empirical study was conducted in various types of schools spread across England and Wales. Almost 2,500 tenth grade students, nearly all of them high-attaining students in mathematics, participated in the study. The results show that even these high-attaining students had great difficulties in generating proofs. The students were far from proficient in constructing mathematical proofs and were more likely to rely on empirical verification. However, most of them were well aware that once a statement has been proved it holds for all cases within its domain of validity. Moreover, they were frequently able to recognize a correct proof, though their choices were influenced by factors other than correctness, such as perceived teacher preference. Students considered that their teachers would be more likely to accept formally presented proofs, though they were personally more likely to construct proofs which they assumed to have an explanatory character. In all domains, students with higher levels of mathematical competence outperformed less able students. Reiss, Klieme and Heinze (2001) were able to identify aspects of geometrical competence. Based on empirical data from upper secondary schools, they found that high-level geometrical competence is specifically influenced by spatial ability, declarative knowledge, and methodological knowledge. Moreover, these research results suggest that students are often able to remember mathematical facts but are not able to combine these facts in a concise mathematical argument or even a mathematical proof. Reiss, Hellmich, and Thomas (2002) confirmed this result for students at the lower secondary level. In addition, they discussed aspects which influence students' performance in argumentation and proof at that level. As a result they provide a proficiency scale of mathematical proving, which comprises three levels of underlying competencies.

Greeno's taxonomy (1980) describes the knowledge of facts as an important part of an individual's knowledge structure. He distinguishes between propositions and rules, visual patterns and strategic principles. Obviously, *propositions and rules* are basic elements of mathematics and students often regard them as the most important aspect of the subject. Mathematical rules seem to be the kernel of the discipline, but mathematics educators and mathematicians emphasize that facts and rules must be applied in order to demonstrate mathematical understanding. *Visual patterns* or, more generally, the ability to visualize, support the understanding of mathematics. In particular, the role of visualization in mathematical performance has been identified in a number of studies. Jöckel and Reiss (1999), Reiss and Heinze (2000), and Reiss, Klieme and Heinze (2001) gave empirical evidence that spatial abilities and mathematical problem-solving performance are significantly correlated. This supports the hypothesis that mathematical performance is influenced by the ability to visualize mathematical ideas. *Strategic principles* comprise not only heuristic knowledge or the knowledge of strategies for the solution of a problem but also general ideas about the characteristic properties and the use of mathematics. These three aspects of the knowledge structure play an important role in mathematical proving and performing a proof requires knowledge of the facts and rules. It is supported by adequate visualizations of the problem, and it relies on specific methods, which can be regarded as strategic principles.

Proof is a central aspect of mathematics as a scientific discipline, and mathematics is regarded as a *proving science* (Heintz, 2000). Mathematical results do not rely on empirical knowledge but are obtained by deductive reasoning based on axioms and definitions. In this respect mathematics is unique and, moreover, different from any other scientific discipline.

This uniqueness causes specific difficulties with respect to the mathematical understanding of students.

It is not easy for children learning to reason and argue in any scientific discipline. In particular, with respect to the natural sciences, there is some research which suggests specific problems encountered by young students in their learning processes. The scientific basis of the natural sciences is characterized by empirical evidence ("the sun can be seen during the day and it disappears at night"), which is the starting point for a (not necessarily correct) theory ("the sun moves around the earth"). Kuhn, Amsel and O'Loughlin (1988) regard as the most important aspect in scientific reasoning the fact that theory and (empirical) evidence become separated. Regarding theory and evidence as different aspects of science may lead to new models and will provide the possibility of casting doubts about a theory. This principle holds for the scientific knowledge of adults as well as for that of children, but children's and students' argumentation and thinking processes are characterized by certain constraints (Tschirigi, 1980; Kuhn, 1989; Bullock & Ziegler, 1994; Thomas, 1997). Some of these constraints may even survive at the adult level.

- Students (even at the secondary level) may not be able to generate evidence which contradicts their own assumptions; accordingly, the falsification of an argument is a methodological tool of scientific work which these students are not able to use in their problem-solving processes.
- Empirical observations are rarely used in order to test a theory but rather to illustrate a theory. Students do not understand the idea that a negative result of an empirical observation may lead to a revised theory.
- Contradictions between a theory and empirical evidence may cause a student to modify the evidence and interpret it in a new way but will not lead to changing the theory.
- Students are hardly able to develop an experimental design which is suitable for testing a theory. In particular, it will not be taken into account that variables may be confused.
- Students tend to accept hypotheses very quickly even if the alternatives cannot be fully rejected at that time.
- Students do not accept the idea of a single counter example. They look for more empirical evidence which supports their theories, even if a counter example has been identified.

Flavell (1977) argues that this more or less empirical thinking, which is guided by induction, is typical for the state of concrete operations in the sense of Piaget:

> The formal operational thinker inspects the problem data, hypothesizes that such and such a theory or explanation might be the correct one, deduces from it that so and so empirical phenomena ought logically to occur or not occur in reality ... it is also called hypothetico-deductive thinking, and it contrasts sharply with the much more nontheoretical and nonspeculative empirico-inductive reasoning of concrete-operational thinkers (p. 103f).

Some research studies suggest that the plausibility of a hypothesis is of principal importance for problem-solving processes (Oswald, 1993; Thomas & Schillig, 1996). Problem solvers tend to test plausible hypotheses first (Klayman & Ha, 1987). Accordingly, if a problem solver finds plausible arguments for a solution he or she will probably accept a solution without testing alternatives (McDonald, 1990). Moreover, there are some results which suggest that even adults prefer verifying their theories rather than looking for contradicting arguments (Klahr, Fay, & Dunbar, 1993; McDonald & Stenger, 1993).

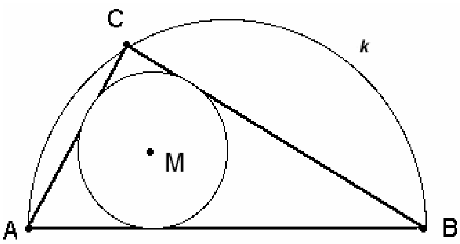
The scientific understanding of children and its development might be different for different sciences though there are similarities between mathematical argumentation processes and reasoning in the natural sciences. Research in mathematics education suggests that these constraints may at least partly apply to mathematics as they apply similarly to the natural sciences (Reiss & Thomas, 2000). Accordingly our investigation focuses on these constraints with respect to students at the upper secondary level. We were interested in how their mathematical problem-solving and argumentation processes had developed at the end of their secondary school career and only a few months before entering a university. Our research is primarily aimed at identifying aspects of implicit scientific theories which might hinder students from understanding mathematics as a proving science and which might influence their abilities to perform correct and concise mathematical proofs.

Sample

The sample consisted of 26 students at the end of their upper secondary education (in Germany grade 13, which is the last grade of upper secondary education). These students were asked to solve geometrical problems including proofs, which were adopted from the TIMS/III study (cf. Baumert, Bos, & Lehmann, 2000). We restricted our investigation to geometry problems, because geometry is that part of school mathematics where students usually encounter proofs for the first time in their mathematical instruction. All problems presented to them required some basic geometrical knowledge (e.g. know the sum of the angles of a triangle, know that the base angles in an isosceles triangle are of identical size) and also some methodological knowledge with respect to mathematical argumentation (e.g. knowing how to build up a logical chain of two or more consecutive arguments). The following problem may serve as an example of the type of tasks which were presented to the students (see figure below).

The solution requires geometrical knowledge, which a student is supposed to acquire at the lower secondary level. The student has to remember that the angles in a triangle add up to 180°, that $\angle$ACB is a right angle, and that the bisectors of the angles intersect in the centre of the inner circle of a triangle. Nonetheless, the results of the TIMS study showed that this was one of the more difficult items with a difficulty index of 741 (m = 500) and a solution rate of 24% in the German population. According to Klieme (2000) this item requires the highest level of competency as well as abilities of argumentation and higher order problem solving.

The students solved the problems in individual interviews. They were asked to verbalize their problem-solving steps and were again encouraged to do so if they did not speak loudly for some time, but the interviewer did not intervene in their problem-solving processes. Only after they told the interviewer that they were finished did he ask specific questions concerning errors which he had observed during their work on the problem. All sessions were videotaped and transcriptions of each session were made.



$\overline{AB}$ is the diameter of a semicircle *k*. C is an arbitrary point on the semicircle (other than A and B), and M is the centre of the circle inscribed into ΔABC. Then the measure of

A. the size of $\angle$AMB changes as C moves on *k*.

B. $\angle$AMB is the same for all positions of C but cannot be determined without knowing the radius.

C. $\angle AMB = 135°$ for all C,

D. $\angle AMB = 150°$ for all C.

Results

The individual interviews revealed how students perform at the end of the upper secondary level (after nearly 13 years of mathematics instruction) especially with respect to their scientific thinking and their mathematical argumentation. In the following we will provide typical examples of these argumentations. We refer mostly to the problem described above in which the size of an angle with respect to the inner circle has to be determined. In our argumentation we follow the constraints identified and investigated by Tschirigi (1980), Kuhn (1989), Bullock and Ziegler (1994), Ewert, Thomas, and Schumann-Hengsteler (1994), Dunbar and Klahr (1995) and Thomas (1997). For a better orientation we will summarize these constraints and then we will describe examples of specific student solutions of the problem, which illustrate the constraints.

(1) Students (even at the secondary level) may not be able to generate evidence which contradicts their own assumptions; accordingly, the falsification of an argument is a methodological tool of scientific work, which these students are not able use in their problem-solving processes.

(2) Contradictions between a theory and empirical evidence may cause the students to modify the evidence but not the theory.

(3) Students do not accept the idea of a single counter example. They look for more empirical evidence which supports their theories, even if a counter example has been identified.

These constraints may be illustrated in the problem-solving process of LUCIA. She reads the problem and misunderstands it while reading the text for the first time. She presumes that she has to find the angle at C, which is obviously a right angle. She argues that the angle which she is supposed to identify "must be a 90 degree angle according to Pythagoras". She adds that answer A cannot be correct because the angle remains unchanged if C moves on k. During her problem-solving process she is probably aware of the contradiction that the multiple choice answers provided in the task and her own solution do not match. In spite of that she argues: "Let us take answer B. This is the only possible answer. It is not bad to choose this answer because the angle remains unchanged. It cannot be calculated, but we probably know that it is a 90 degree angle."

The student is not concerned that her own assumptions and the solutions provided in the text of the problem contradict each other. Her statements sound like finding a compromise between these conflicting issues. On one hand, she is quite sure about her own solution; on the other hand she trusts the validity of the answers provided by the authors of the problem. Apparently she has hardly any doubt that both aspects may apply simultaneously. With respect to her erroneous understanding of the problem she gives a correct solution, which is exclusively based on her declarative knowledge. It may not be regarded as a severe mathematical problem that she uses the name of Pythagoras instead of the name of Thales, but this supports the evidence that her geometrical knowledge is far from being profound. Accordingly, she has more or less tacit or inert knowledge (cf. Whitehead, 1929; Renkl, 1998), which cannot be used independently of specific situational contexts.

(4) Students are barely able to develop an experimental design which is suitable for testing a theory.

For many students, the problem-solving processes may be characterized as trial and error activities. There is rarely a systematic analysis of the variables involved and of the methods which might be used. The students recognize some elements of a probable solution (for example the 90° angle at C) but are not able to use their knowledge in their problem-solving processes.

MERLE is another female student who shows difficulties while solving the problems. She tells the interviewer that she does not like mathematical proofs at all and that she does not know how to build up a logical chain. In her problem-solving process she identifies separated pieces of geometrical knowledge which might be useful in this specific context. She is not able to combine even a few of these pieces in consecutive arguments. While making loud comments about her problem-solving process, MERLE tells the interviewer that her steps are not correctly intertwined. Accordingly, she has a vague idea what mathematical proving is all about but she is not able to perform a correct proof on her own.

KONSTANZE shows an argumentation which is based on her convictions. She tells the interviewer that she is sure that the angle may be determined by calculations but that she is not able to do these calculations. Apparently she has no idea how to deal with this specific problem-solving context. Nonetheless she excludes the probable answers A ("I roll away the circle, therefore the angle remains unchanged") and B ("I am sure that the angle can be determined by calculations"). She does not provide any theoretical ideas of how the alternatives C and D could be tested.

(5) Students tend to accept hypotheses very quickly even if the alternatives cannot be fully rejected at that time.

LEA reads the text and chooses A as her solution spontaneously. Afterwards she argues that B cannot be correct because "everything can be calculated". Moreover, she states that C is not correct because the angle is too large, and D cannot be correct because the angle is even larger than in answer C. She does not spend more than two minutes on this task but is convinced that she has the correct solution. LEA is an example of a problem solver who sticks to an intuitive solution. Moreover, she accepts her first hypothesis without even testing it and without testing other alternatives.

KONSTANZE argues that "most probably, you are able to calculate the angle". She does not give any reason for this statement. LUCIA assumes that the angle has the size of 90°. She argues: “Pythagoras said, it is a 90° angle. You cannot calculate it, but we know most probably that it is a 90° angle.”

Discussion

Successful problem solvers as well as their low-achieving counterparts are generally unable to deal systematically with hypotheses and logical arguments. This result is true for many of these grade 13 students, who are in their last year of secondary education. In particular our data reveal severe problems with a basic understanding of the principles of mathematical reasoning. The students are barely aware that mathematical reasoning is not guided by plausibility arguments. Moreover, they are not able to explore the problem-solving situation and discuss arguments which might possibly lead to a solution. Mathematical principles are mostly not used during problem solving. Reasoning on mathematical problems and their solutions, as well as proving mathematical theorems, seem to be activities which have not been important aspects in the mathematics classrooms. In particular, the nature of mathematical reasoning seems to be a topic which is hardly known to most of them.

Our findings suggest that students’ difficulties with mathematics may have one origin in their inadequate model of mathematical argumentation. The nature of mathematics as a scientific discipline is quite unfamiliar to students even at the end of their upper secondary education. Rules of mathematical argumentation processes are hardly known to these students and will not be applied in their mathematical problem solving. Their mathematical argumentation processes include empirical arguments as well as the application of intuitive thinking. Therefore, problem-solving processes are not guided by systematic investigations but may better be characterized as trial and error processes.

Argumentation and proof are regarded as important aspects of the mathematics classroom. Obviously it is not easy to reach the goal of fostering students’ understanding of the specific nature of mathematics. The TIMS study (Baumert, Bos, & Lehmann, 2000) has shown that German students at the upper secondary level perform below average in an international comparison. The problems presented in the TIMS study at the upper secondary level have been criticised as representing mainly the lower secondary level with respect to their mathematical contents. As a consequence it was hypothesised that the students probably had forgotten the underlying mathematics. The results of the study presented here contradict this assumption. We were able to show evidence for the fact that students’ difficulties with mathematical problems cannot be attributed to a lack of basic, declarative knowledge but to a lack of methodological knowledge. The constraints of mathematical argumentation and in particular mathematical proof are unfamiliar to the students and they obviously feel uncomfortable during mathematical problem-solving activities which ask for argumentation and proof.

According to Schoenfeld (1997), there are five aspects of mathematics cognition, namely a knowledge base, problem-solving strategies, self-regulation or monitoring, beliefs, and practices. At the end of their education, students have built up a (more or less sufficient) knowledge base, which mainly consists of probably unrelated mathematical facts. There is a severe lack of knowledge of problem-solving strategies. Moreover, self-regulation and monitoring is essentially unavailable to the students. With respect to this specific study, their beliefs or their scientific reasoning is inadequate for mathematics. The data support findings which identified specific problems in German mathematics classrooms. Mathematics instruction in Germany is typically concerned with problem solving in small steps thus hindering students from developing a holistic view of the problem (e.g. Baumert et al., 1997). It is most probable that the deficiencies described above are not deficiencies in individuals but rather can be attributed to the specific mathematics classroom. Problem solving has been regarded as a goal of mathematics instruction for decades, but obviously we are not succeeding in implementing this goal.

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