On the Arithmetical Flexibility of Primary School Children - Analyses Based on the Example Task 701-698

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This contribution describes the problem-solving behaviour of German primary school children who were repeatedly given, along with other tasks, the task to find the difference 701-698. First of all the design of the study is presented then an analysis is made as to how successfully the children proceeded, which methods of calculation they used and which strategies they applied. Four case studies involving school children of differing levels of ability demonstrate the preceding details in more factual terms. The research results are then put into context through the ideas of the famous German mathematics educator Johannes Kühnel who always purported arithmetical flexibility.

In 1916 Johannes Kühnel published his “Neubau des Rechenunterrichts (New construction of arithmetic teaching)” (1916/1925). On studying Kühnel’s (1869-1928) works it becomes apparent that his views are still of value today and inspire and provide a significant degree of orientation. Of course some of his postulations appear somewhat antiquated and exaggerated by today’s standards, but this does not detract from the fact that already at the beginning of the 20th century Kühnel had recognised many basic problems in arithmetic teaching with amazing clarity and pointed impressively to finding the solutions (Schmidt 1978; 1999).

1. Kühnel’s Criticism of the Standard Method

Kühnel had always spoken out in no uncertain terms against the so-called standard methods of calculation since these destroy the student’s own individual activity and deprive him of responsibility for his own thoughts. Learning standard procedures, he claimed, leads to “drilling and training which lasts as long as the exercise. As soon as this exercise is finished the drilling disappears too … Mathematics lessons which strive for the application of standard methods are hence a pedagogical sin … Through the standard methods ... the children are accustomed to mechanical activity, even where the particularity of an individual case demands specific attention, i.e. the pupils are made into bureaucrats, machines, made to follow suit” (Kühnel 1930, 63, all translations by CS; compare Kühnel 1925a, 41).

Kühnel did not only regard the standard method as being, one may assume, the four standard algorithms of written calculation but also the almost forty rules for calculation which the students at the beginning of the last century all had to learn, An example is the rule for the mental addition of two-digit numbers (compare Kühnel 1930, 60).

According to Kühnel, it is not possible to penetrate into the spirit of mathematical education until *several* procedures have been dealt with. “Then of course none of these methods is the standard method, indeed by comparison it becomes apparent that each has its own advantages and disadvantages; in the one case it is this, in the next case that, which is more suitable” (Kühnel 1930a, 2f.). “That is why we indeed put so much emphasis on the variety of solving methods, on the children’s independent discovery of, and reporting on, such methods, on reciprocal assessment of these methods and above all on the growth of these relationships, which with time leads to the discovery of better and better methods” (Kühnel 1930a, 153).

Elsewhere it is claimed “Independent searching for, discovery and understanding of several methods – it is with this we must replace the old standard methods. These words are really a magic wand: 'Who can do it differently?'" (Kühnel 1930, 69). It is then not a matter of the teacher striving to *show* better methods (and most certainly not that he *puts the child under any obligation*), but that the student should *develop* these gradually. "It is not a question of showing the better and more elegant, in my opinion, but a *majority,* and showing that the majority of methods allows and augments quite considerably the necessary clarification" (Kühnel 1930, 65f., see Kühnel 1925a, 41ff.).

The aim of this text is to contribute to answering the question as to whether the flexibility, as demanded by Kühnel many years ago, exists today in primary schools or whether there is still a strong penchant towards the standard methods. This is a report on the problem-solving behaviour of 3rd and 4th graders (nine-and ten-year-olds), as they deal with three-digit addition and subtraction tasks.

2. Design

The central concern of this research project was elucidating the question "To what degree do students resort to adept calculation strategies?" A further objective was to discover how they changed the way they went about dealing with the task over the course of a year. For this reason the examination, which involved 298 students, took place at three different times:

- in March, before the written standard method was introduced in the third year at school,
- in July, subsequent to its introduction,
- in October, at the beginning of the fourth year at school, some time after the standard method had been dealt with.

All the school children were presented with 12 tasks in total, each in the form of a test during normal mathematics lessons. The choice of the calculation method was left to the children: mental, informal written or according to the standard method.

In addition, 36 children were selected to perform these tasks once again in the following days during an individual interview. Of these children, 12 were each assessed by their mathematics teacher as being 'good', 'average' or 'poor' performers.

The selection of the six addition and six subtraction tasks was made in such a way that in four cases the use of the adept calculation strategy was quite an obvious, or at least possible, choice whilst its use for the other two tasks did not appear appropriate. The latter tasks were designed so that in each case one task did not contain a 'carry' or contained two 'carries' (see Tab. 1).

Table 1
The twelve problems

problem	possible flexible strategy	'carries'
527+399	auxiliary task (+400; –1)	2
199+198	auxiliary task (for example 199+200; –2)	2
250+279+250	combining (250 and 250)	1
119+120+121	combining (119+121) or balancing (3·120)	1
286+437	no specific one	2
345+634	no specific one	0
845–399	auxiliary task (–400; +1)	2
649–347	auxiliary task (49–47; or adding up: 47+__=49)	0
701–698	adding up (698+__=701) or auxiliary task (–700; +2)	2
610–590	adding up or auxiliary task	1
836–567	no specific one	2
758–515	no specific one	0

Please refer to Selter (2000; 2001) for further information concerning the design and results of the study including details on the type and the frequency of the methods used and of the strategies applied. This contribution confines itself to the basic presentation of the children's methods for solving the most interesting task: 701–698.

3. Success and Results

All in all it was 701-698 which proved to be the most difficult of all the twelve tasks. Table 2 shows that in each case the highest percentage of correct solutions was achieved in October, although still below 50% for the tests and just over 60% for the interviews. Incidentally, in this case, account was taken only of the first answer offered by the children and not of a possibly differing subsequent one.

Table 2
Percentage correct solutions for 701-698

	February	June	October
test	34.8 %	43.8 %	48.1 %
interview	50.0 %	48.4 %	61.8 %

These percentages must be interpreted with caution on account of differing fundamental values: almost 300 children took part in the tests, in the interviews only 36. This however does not alter the general tendency that better results were attained in the interviews. It can be assumed that the request to explain the solution contributed to the slightly better result.

Altogether 84 varying answers were given in the tests alone! Since a separate list for the three sittings does not reveal further significant information, Table 3 merely shows a summary of the relevant values. Similarly, no differentiation is made as to how often the results were arrived at in each case, either by mental, informal written or standard methods.

Table 3
Results and their frequencies

result	frequency	result	frequency	result	frequency	result	frequency
0	2	91	4	183	2	497	1
1	20	93	3	184	1	502	1
2	18	97	11	187	3	503	3
3	*336*	99	2	188	1	510	1
4	1	100	5	190	4	563	1
7	2	101	2	191	1	602	1
8	1	102	3	193	9	604	1
9	3	103	*78*	196	1	613	1
11	2	104	2	197	*126*	687	1
12	3	107	10	198	1	693	1
13	23	109	1	199	10	703	2
17	13	112	1	200	1	761	1
19	5	113	8	203	2	903	2

20	1	114	1	223	1	907	1
28	1	116	1	290	1	997	1
32	1	117	4	302	1	1003	1
63	1	133	1	317	1	1017	2
71	1	147	2	344	1	1393	3
83	2	167	2	364	1	1399	1
84	1	179	1	390	1	1403	1
87	1	182	1	399	3	1903	2

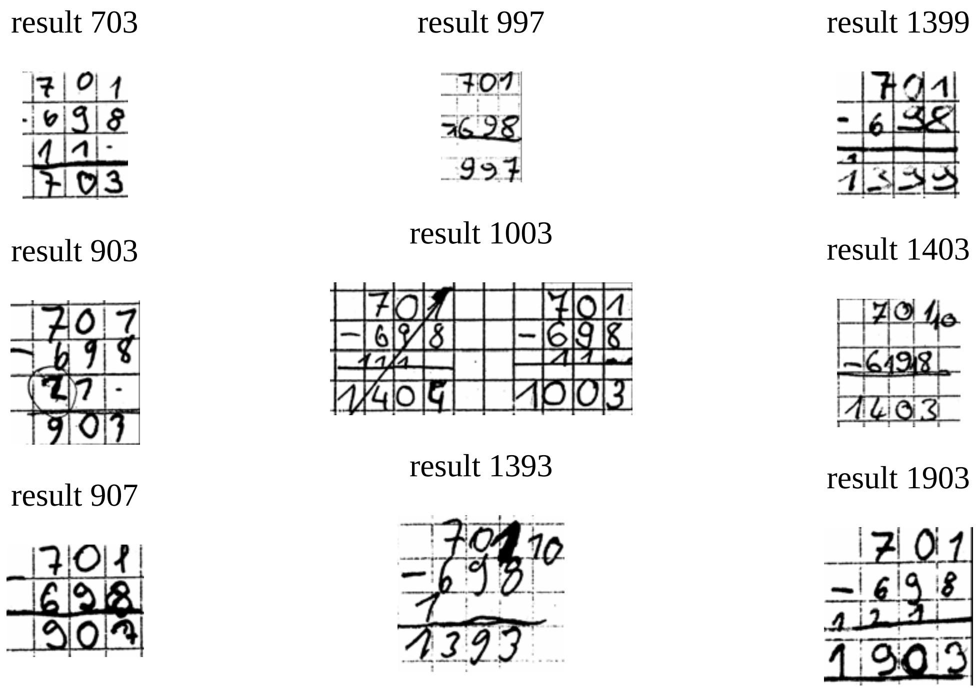

Figure 1. Some solutions

In this context it is interesting to note that the greatest result from the informal written or mental calculation was 604 whilst a series of solutions which were calculated by the standard method showed a result greater than the number to be subtracted from. Figure 1 shows a selection.

4. Methods and Strategies

As can be seen from Table 4 a relatively large proportion of the school children used the standard procedure to find the solution both for the tests (bold) and for the interviews, once this had been introduced in March.

Table 4

Methods used for 701–698 in %

	February	June	October
Standard Algorithm	**6.5** / 6.3	**61.9** / 61.3	**53.4** / 47.1
Informal written	**35.2** / 62.5	**6.0** / 6.5	**8.6** / 8.8
Mental	**52.9** / 28.1	**29.1** / 29.0	**33.1** / 41.2
Mixture	**5.4** / 3.1	**3.0** / 3.2	**4.9** / 2.9

This may seem surprising when one considers that the task 701-698 can be solved comparatively easily mentally by completion. It shows however that for many school children the choice of calculation method here, and for the other subtraction tasks, did not depend on the numbers involved (compare Selter 2000, 236 ff).

Noteworthy discrepancies between the results in the interviews and tests could be established only in February where, in the interviews, significantly more was calculated using informal written strategies whereas mental calculations were only very seldom used. A less significant shift could then be observed from written to mental calculation in October.

On analysing the test papers not only did it become very apparent that many students were performing written calculations but it also became evident that those children who had calculated by *informal writing* in most cases applied the main strategies 'step-by-step' and 'hundreds-tens-units (htu, see table 5)' (without the latter having yet been dealt with in class), or a mixture of these two methods (compare Selter 2000, 2001). It remains unclear which method the students who did their calculations *mentally* had chosen. It could be assumed that they mostly applied adept calculation strategies but for obvious reasons did not note these down. Where a child for example solved 701-698 by completion or by adding up from 698 to 701, there seemed to be no necessity to document this in writing also.

Within the scope of the interviews it was possible to ask the school children as to their procedure. This hypothesis could not however be confirmed. The children's mental strategies were on the whole continuations of their informal written 'standard procedure' which they applied non-specifically to the task.

Table 5

Calculation Strategies for 701-698

Strategy	Feb	Jun	Oct
Hundreds, tens & units 701-698= 3 700-600=100 1- 98= 97 100- 97= 3	14	2	4
Step-by-step 701-698= 3 701-600=101 101- 90= 11 11 - 8= 3	8	4	4
Mixture 701-698= 3 700-600=100 101- 90= 11 11- 8= 3	5	5	7
Auxillary task 701-698= 3 700-698= 2 2+ 1= 3	1	1	1
Completion 701-698= 3 698+ 2=700 700+ 1=701	0	1	2
Simplification 701-698= 3 700-600=100 100- 97= 3	3	1	0

Table 5 shows which strategies were used for calculating 701-698 (informal written or mental) during the interviews and with which frequency. Comparable values were attained also for the other tasks.

5. Case Studies

In the following, the methods of solving the task 701-698 which four school children demonstrated during the interviews are described and analysed (compare Meseth 2000, 45ff.). First of all the solutions of Philipp are described – a child good at mathematics who solved almost all the tasks correctly and in the process selected his calculation strategies in a flexible and task-specific manner. Then Nicole is introduced, a girl who experienced problems both with the informal writing and also the standard method (compare Meseth & Selter, 2002).

Finally the methods are presented as used by Andreas who worked with informal writing in each case during the interviews and by Miriam who, from the June interview on, only calculated by the standard method and even went on to say in the October interview that she no longer knew exactly how else to calculate. These four children, with their individual approaches, represent the heterogeneity of the methods which could be observed during the interviews. Their methods represent very well the range within which children of the third and fourth grades find themselves.

5.1 Philipp

In February Philipp first of all calculated by informally writing the difference between the two hundreds (700-600) and added the 1 from 701 to the 100. Having attained 101 he then subtracted first of all 1 and then 97 so achieving the result 3.

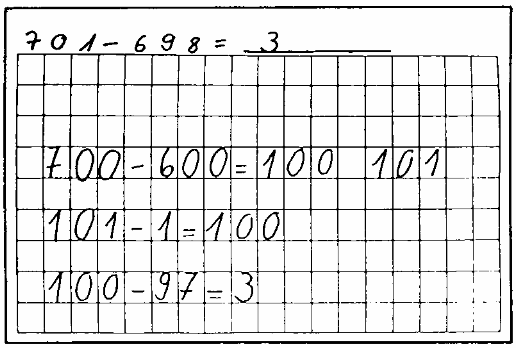

Figure 2. Philipp’s solution in February

In June and October, once the standard method had been introduced, Philipp solved this task again according to the previous method, although this time in his head. When asked, he admitted in the last interview to another, second, way of calculation: he could add 2 to 698, then that would make 700. 701 minus 700 is 1, that makes 3. When asked why he had not calculated 701-698 using the standard method, he said: “It’s takes more time to write things down than to work out in your head”. For other tasks where mental calculation was not so obvious he resorted to the standard form.

5.2 Nicole

In February Nicole first of all wrote down the task correctly, drew a line underneath it and wrote under this 701-690. She then calculated the difference between 700 and 600, wrote the result 100 as a note at the bottom of the sheet and crossed the two hundred figures through. Then she noted the 90 and 1 as further sub-totals.

She could not explain later how she came to these two figures. Indeed she subtracted the smaller figure in each case from the greater one. Then Nicole added all three figures noted and wrote down 191 as the result.

When the interviewer asked her to explain the calculation she said she had first subtracted 600 from 700. Then she told of a method which she had not used: “Then I counted them (98 and 1) together, and that makes 99”. She then altered the result accordingly.

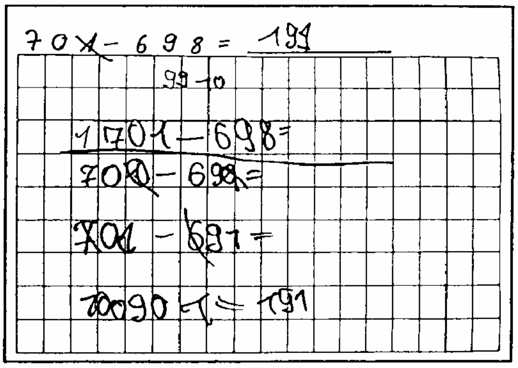

Figure 3. Nicole’s first solution in February

Since Nicole was not concentrating sufficiently this first time when she was confronted with the task, the work was interrupted. When it was presented again, Nicole dealt with the task in a very similar manner, making different notes however. First of all she crossed through both hundred figures and noted the number 100. Then she crossed out the zero in 701. When asked how she came to 90 she answered: “Yes, there’s (pointing at the zero in 701) no number there”. Then she added both the units, noted the sum of 9 and came to the result 199. When the interviewer pointed out that this was still a subtraction exercise so why had she then worked out ‘plus’, she wrote 8 – 1 = 7 and altered her result to 197.

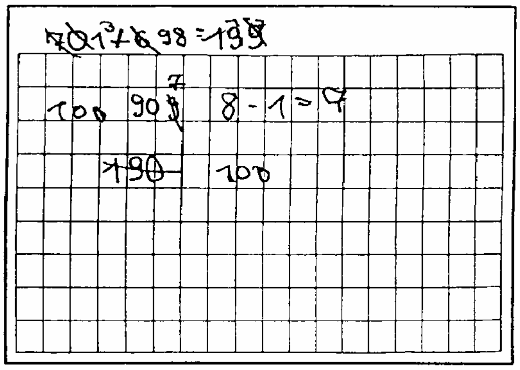

Figure 4. Nicole’s second solution in February

The interviewer then encouraged her to count from 698 to 701, to which Nicole replied: “3 missing!”. She was unable however to associate adding up with the task of subtraction. She therefore did not write the 3 in the space for results between the 701 and 698 and changed the minus to a plus sign.

When asked by the interviewer what would result if 701 was added to 698, Nicole said “more than 1000”. Then quite unexpectedly she declared the result of the original task would be 1, as “700 minus 600 makes 100. And now I have simply added them (1 and 98, CS) together, 99 minus 100”. No further questioning was able to convince her otherwise: 1 was the correct result.

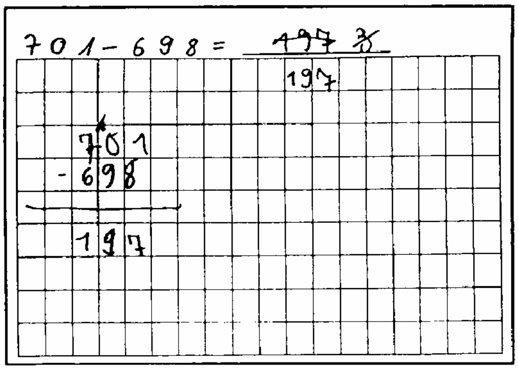

Figure 5. Nicole’s solution in June

In June Nicole solved the task mentally after consistently subtracting the smaller from the greater number in each case and then adding the sub-totals to reach the result 197. At the interviewer’s request Nicole then counted from 698 to 701. Then she answered ‘You have to count three times to 701’. Again she was unable to associate between this finding and the task. For a short while she decided that 3 was indeed the result, but then revoked this assessment again by re-calculating the task by the standard method and once again came to the result 197. When asked why she had decided that 3 was not right she answered: “Because when you do it like this - one under the other - it’s always right”.

It is, incidentally, seldom that children see completion as a suitable method for solving subtraction tasks. This in my opinion would indicate that the idea of completion is neglected in the textbooks used in classes.

5.3 Andreas

In February Andreas calculated all subtraction tasks using the same method. First of all he split the figures to solve the task in stages (700-600=100; 98-1 = 97), always subtracting the smaller from the greater figure. Then he subtracted both sub-totals from each other to achieve the correct result. The two systematic mistakes he made here, and which evened each other out in this task 701-698, became obvious for example in the task 649-347 for which he determined 298 to be the result (300-2).

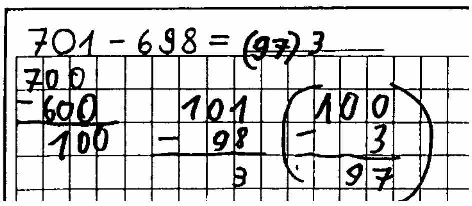

Figure 6. Andreas' solution in June

Later in June he no longer made the first mistake – the systematic subtraction of the smaller from the greater sum. He continued however to make the second mistake – the false link between the sub-totals.

For 701-698 he first subtracted the hundreds from each other and then established that 98 is greater than 1 and therefore cannot be taken away. He then added another hundred to it, subtracted the 98 from 101 and got the result 3. Then he took away 3 again from the first sub-total 100 and got 97. He was however not convinced that this step was correct since he had already considered the 100 in the second part of the calculation. Eventually he came to the conclusion that the last calculation was superfluous, put brackets round the 97 and noted 3 as the final result.

In October Andreas calculated – as in June – all tasks by informal writing, even when he sometimes wrote the subtraction tasks under each other. He achieved a very good success rate. The mistakes which he had made in February and June did not appear again. Hence he calculated 701-698 by writing down 700-600 and 101-98 under each other but calculated the results in his head however (101-90=11; 11-8=3). It is to be assumed that Andreas had not understood standard subtraction at all because on questioning he could neither apply it nor explain it.

5.4 Miriam

In February Miriam first of all achieved 027 as the result for the task 701-698 and explained this with: “First of all I took 8 minus 1 and then 9 minus 0”. Then she became unsure and corrected her result to 17. She had established that 100-90 = 10, explaining this however as follows: “The last number is 100, which has 3 numbers. And then it gets less and less”.

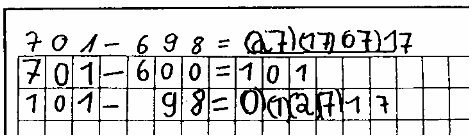

Figure 7: Miriam’s solution in February

Then she thought again and wrote down the new result 07, explaining this thus: “Because 9 take away 0 is 0”. She also did not consider that 8 had to be subtracted from 11 not 8 take away 1. As 100-90=10, she then corrected the 7 to a 17 because a 10 was missing.

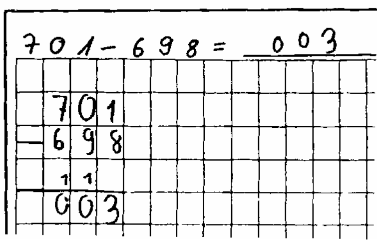

Figure 8: Miriam’s solution in June

In June and October Miriam solved all the subtraction tasks by the standard method and with the correct result.

6. Closing Comments

On analysing the students' methods for solving the task 701-698 it would appear appropriate to pick up on Kühnel's introductory explanations on the flexibility of calculating as a guiding idea for inclusion in an up-to-date mathematics text book. Although some children did indeed proceed according to the relevant task, for example completion from 698 to 701, as the analysis of the methods also showed for the other tasks, most of the students did apply their standard methods of standard written, informal written or mental calculations relatively consistently. For most of the children interviewed it was unusual form them to have to think about the methods of calculation with regard to the task.

The fact that Kühnel's explanations on flexible calculation methods have not lost any of their relevancy lies, amongst other things, in the difficulty in fulfilling the demands made by such methods in the classroom. It is encouraging, in my opinion, that a series of new teaching books and teachers' hand books universally address or encourage flexibility. It is also to be hoped that the 'adept look' at calculating tasks can continue to develop through stimulation in the classroom, because flexibility in approach is relevant for much more than merely adding and subtracting with three-digit numbers.

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