Understanding Financial Markets from a General Equilibrium Perspective: Uniqueness of Competitive Equilibria

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Introduction

Chapter 1

Introduction

In the past decade financial markets have experienced substantial growth on the one hand and rapid changes in their constitution on the other. As Eatwell, Milgate, and Newman (1992, p. IX) state "the last twenty years have seen unprecedented change in the world's financial markets". The bigger and more sophisticated markets become, the more important it is to obtain a thorough understanding of the underlyings of the financial system as a whole. Developing a sound and reliable theoretical model to explain market forces is an imperative prerequisite when researchers want to evaluate changes and analyze experienced developments.

The basic idea of financial markets is to provide the conditions for risk sharing and insurance. Changes in the institutional environment, in the sophistication and awareness of market participants as well as in technology are among the most important factors contributing to the past substantial developments of financial markets.

The institutional environment mainly describes the legal background of an economy and the respective financial markets. "Changes in regulation and taxation" initiated a growing variety of financial products (compare Eatwell, Milgate, and Newman (1992, p. IX)) because a substantial amount of capital has been released which is now seeking for investment opportunities. New products have to be designed in a way such that they correspond to the legal framework on the one hand, but on the other hand avoid from investor's point of view "unnecessary" tax payments. It is essential to understand the underlyings of the system as a whole in order to evaluate which effects the introduction of such new products has. In addition, deregulation made it easier for companies to become listed in stock markets and obtain equity capital directly from investors instead of asking for credit at a commercial bank. Therefore, the number of companies listed at stock exchanges increased rapidly.

Competitive pressure made managers of business aware that all their resources need to be used efficiently. This especially includes the management of their cash flows and capital basis. Some — mainly large-sized — companies have even founded their own treasury departments and internal banks. Individual consumers have learnt about the potential gains one can earn on the stock markets and have hence turned to include stocks into their portfolios instead of restricting to plain savings accounts.

Technological progress makes it possible for market participants of all kinds to obtain information about companies, industries, and economies which enhances transparency of markets. In addition, technological progress lowers transaction costs and changes the possibilities and habits of trading, seen for example after the introduction of electronic trading systems like Xetra. Sophisticated computer software enables investors to find promising investment opportunities more easily and quickly. It especially gives the possibility of looking for arbitrage strategies, i.e. for investments which lead to a positive payoff, but cost nothing. This concept of arbitrage is formally defined in the context of general equilibrium models which build the framework I rely on and describe in this thesis. The assumptions for these models exclude arbitrage opportunities and hence lead to the no-arbitrage principle upon which most applied pricing formulas are based. Therefore, it is essential to understand what general equilibrium models can explain and, more importantly, what their limitations are.

A well-suited theoretical foundation for financial markets is given by general equilibrium theory which according to Mas-Colell, Whinston, and Green (1995, p. 511) is based upon two "central features":

- the economy is seen as a "closed and interrelated system" where "equilibrium values of all variables of interest" are "simultaneously determined" and

- the aim is to give a parsimonious modeling of reality.

More substantively, general equilibrium theory is the "determination of equilibrium prices and quantities in a system of perfectly competitive markets".

The roots go back to Adam Smith (1776) who introduced the idea of an "invisible hand" balancing supply and demand in markets. The first mathematical formalization was given by Leon Walras (1874). He used an interdependent system of equations where each of them represents a single market. Kenneth Arrow and Gerard Debreu (1954) put this into a more satisfactory format which now is called Arrow-Debreu model and finds its most elegant presentation in Debreu's "Theory of Value" (1959).

At first glance, the Arrow-Debreu model seems to be well-suited for modeling financial markets because (compare Hens (1995, p. 2))

- the price of one asset depends on the prices of the other assets;
- in general there is a large number of traders acting on a large number of markets;
- traders act as optimizing individuals not taking into account others' welfare; they decide independently of each other;
- financial markets seem to be those markets for which the fiction of price setting by a "Walrasian auctioneer" is most realistic because a lot of trading is administered through auctions.

Originally, the Arrow-Debreu model was formulated in a static framework without uncertainty. Arrow (1953) and Debreu (1959, Chapter 7) however showed that time and uncertainty can be easily included without changing the power of the arguments, though their interpretation is based upon highly stylized and idealized assumptions: decisions and transactions all take place simultaneously, at the same starting-point. They are based on prices which clear future (contingent) markets. What makes these assumptions unrealistic is the fact that markets actually open in a sequential order in the sense that at successive calendar dates spot markets open for those goods available at these dates. In addition, there exist financial markets enabling income transfer across periods and across contingencies, in other words enabling (dis)savings and insurance.

The interpretation of the Arrow-Debreu model in this way is only valid under restrictive assumptions, namely

- a sufficiently rich market structure, called complete market structure and
- a finite time horizon.

In this thesis I consider models referring to these assumptions and try to relate them with focus on the — later on explained — aspect of unique equilibrium.

The attempts to relax the first assumption have led to the so-called theory of general equilibrium with incomplete markets. In Part II I consider the property of a unique equilibrium in this environment. Taking into account this aspect I state parallels and differences to the Arrow-Debreu model.

If one includes uncertainty in the Arrow-Debreu model, but preserves the assumptions of a sufficiently rich market structure and a finite horizon, optimizing behavior of the agents assures that current prices uniquely determine forward and future prices because there cannot be any possibilities of arbitrage. But as soon as one deviates to incomplete markets or an infinite horizon, one has to add assumptions on price expectations, for example the assumption of correctly foreseen future prices. This aspect of perfect foresight is, for example, discussed with respect to the existence of so-called sunspot equilibria where people's expectations about future prices differ even though the fundamentals of the economy are known. The existence of this kind of equilibria is considered in Part III where the focus is again on the coherence to multiple respectively unique equilibria in Arrow-Debreu economies. So-called overlapping generations models can be used to induce an infinite horizon. As is shown in Part IV, special equilibria of such models, namely stationary and periodic ones, can be identified with equilibria in the Arrow-Debreu world. This observation allows to apply the theory of unique equilibria in Arrow-Debreu economies to gain results on the special overlapping generations equilibria.

Financial markets are bounded to be incomplete. One can experience this everyday. Mas-Colell, Whinston, and Green (1995, p. 709) identify potential causes for incompleteness to be

- asymmetric information,
- transaction costs, and
- the problem of enforceability of contracts.

Hens (1995, Part II) points out that extending the Arrow-Debreu model to incomplete markets creates a well-suited foundation for the theory of finance and therefore for financial markets. One reason for this is the existence of equilibria which ensures the matching of all factors in a coherent way.

The beginning of the analysis of general equilibrium models with incomplete markets, for short GEI-models, can be dated back to Arrow (1953) who not only introduced uncertainty into general equilibrium theory, but also made the following far reaching observation: when spot markets are complete, a system of elementary securities for each future state is sufficient to obtain the same equilibrium allocations as are obtained with a complete set of contingent contracts for goods, i.e. contracts for each good in each state. An elementary security — now called Arrow-security — for some future state pays off one unit of account in this state and nothing otherwise.

Arrow (1953) implicitly assumed that agents correctly anticipate future variables which was first explicitly introduced in Radner's (1972) work on "equilibrium of plans, prices, and price expectations". This was followed by a long series of related papers, exhibiting waves of encouraging and disillusioning results. Excellent surveys can be found in Geanakoplos (1990), Magill and Shafer (1991), Magill and Quinzii (1996b), and recently Hens (1998).

Geanakoplos (1990, p. 3) identifies "five fundamental contributions to economic theory":

- GEI-models enable an understanding of asset pricing and the importance of the asset span whenever short sales are allowed;
- they initiated the development of new existence proofs for general equilibrium, leading away from Brouwer's Fixed Point Theorem and not relying on convexity;
- they clarified the existence of decisive differences between real and financial assets;
- they allow to comprehend default and bankruptcy as equilibrium;
- they lead away from the idea of market processes always generating Pareto-efficiency.

Even though the GEI-model allows for a far-reaching generalization of the Arrow-Debreu model, it sticks to an axiomatic foundation and "retains the simplicity, coherence and generality which are the hallmarks of the Arrow-Debreu construction" (Magill and Shafer's (1991, p. 1524)).

In the first and second part of this thesis, I consider the GEI-model under further, but common restrictions:

- I consider a pure exchange economy and exclude any kind of production;[1]
- there are only two periods;[2]
- there is only a single commodity, interpreted as "money" or "total expenditure";
- financial contracts pay off in this underlying commodity.

[1]This restriction is maintained for the whole thesis.
[2]The sunspot economies in Part III also refer to the two-period framework.

A GEI-model using these restrictions is called Finance GEI-model. These assumptions constitute strong limitations, even though they are quite common as I have already mentioned. They are however general enough to get essential insights and allow to include such cornerstones as the Capital Asset Pricing Model (CAPM).[3] I discuss the limitations of this approach in the introduction of the model in Chapter 2.

Incomplete markets economies in general assume a symmetric participation structure, i.e. each consumer is confronted with the same trading restrictions. This is very limiting: one can easily imagine that in reality different people get to know about different investment opportunities and not all investors are able to trade in the same markets. Some part of the literature (for example, Siconolfi (1989)) accounts for these shortcomings and allows for an asymmetric participation structure. I devote Chapter 5 to this aspect.

According to Mas-Colell, Whinston, and Green (1995, p. 590) "for a theorist, the best of all possible worlds is one in which the social situation being analyzed can be formalized in a manner that, on the one hand, is very parsimonious [...] and, on the other hand, manages to predict a unique outcome". Hens (1995, p. 19) in addition remarks that "only if the number of equilibria is small, we can say that the general equilibrium model provides a reasonable 'solution' to the allocation problem" and "when the equilibrium is unique [...] one can say that prices provide a reasonable 'theory of value' of scarce resources" (Hens (1995, p. 29)). Kirman (1989) identifies the lack of uniqueness in general as one of the major criticisms on general equilibrium theory.

After clarifying the existence of equilibria, the next question is if they are unique, first locally, then globally. As indicated by the above quotations, without uniqueness the predictive power of a model is very limited. To illuminate this aspect, take an example from finance: in the CAPM the β-pricing formula enables to determine the valuation of assets. However, this β refers to an underlying equilibrium. As soon as there are multiple equilibria, each of them corresponds to a β and therefore any statement on the valuation of an asset is practically worthless.

Uniqueness of the equilibrium is also a necessary prerequisite for comparative statics. Economists want to understand to what extent exogenous parameters influence the allocation of resources. As soon as the property of uniqueness is violated, it is not clear which equilibria before and after the change of parameters have to be compared, not to mention that even their number might have changed. This would require some mechanisms of selection, for example to keep, as close as possible, to the "old" equilibrium which

[3] Geanakoplos and Shubik (1989), for example, show how to embed the CAPM into a general equilibrium model.

only makes sense when the shocks to the economy are quite small. Hildenbrand and Kirman (1988, pp. 199 ff.) and Arrow and Hahn (1971, p. 207) claim that uniqueness is even a necessary prerequisite for comparative statics. “If, for a given vector of parameters, there is more than one solution to the model,” Kehoe (1985, p. 119) argues that “then the comparative statics method breaks down. Lacking conditions that guarantee uniqueness, we must resort to considerations of historical conditions and dynamic stability, which greatly complicate the analysis.” Arrow and Hahn (1971, p. 207) also point out that uniqueness is helpful when examining “dynamic behavior in general equilibrium models.”

All these explanations show that a sound foundation of finance, i.e. a thorough understanding of financial markets via general equilibrium theory, requires a full understanding of uniqueness. Regrettably, neither in the Arrow-Debreu model nor in the GEI-model global uniqueness of the equilibrium can be expected in general. In the Arrow-Debreu model and in the incomplete markets model considered in this thesis at least local uniqueness holds generically and the equilibrium set is finite. Nevertheless, finiteness can imply anything between one and a million equilibria, i.e. finiteness does not ensure a small number of equilibria. To obtain global uniqueness, additional assumptions on the economies are necessary. Even in the context of the CAPM, Nielsen (1988) gives an example that multiple equilibria may exist whereas in addition Bottazi, Hens, and Löffler (1998) prove that for any given number a CAPM-economy with this number of equilibria can be constructed. Further assumptions on risk aversion however can be found to ensure uniqueness in the CAPM (Dana (1999) and Hens, Laitenberger, and Löffler (2000)).

One can distinguish two different approaches for establishing additional conditions on economies which guarantee a unique equilibrium. Either one puts restrictions on the characteristics of each single individual or one puts restrictions on the society as a whole by some distributional assumptions on the individual characteristics, as it is done by Hildenbrand (1994) and Grandmont (1992). Originally, Grandmont's (1992) results were interpreted with respect to a broad heterogeneity of consumers, but recently Hildenbrand (1998) showed the assumptions there to imply a kind of homogeneity. My focus in this thesis is on the first aspect, i.e. I consider the single individuals. As far as I know, the only one applying distributional assumptions in incomplete markets models is Laitenberger (1998).

In the Arrow-Debreu model, global uniqueness of equilibria has been vastly analyzed and is well-understood. Some good references are Arrow and Hahn (1971, Chapter 9) and Mas-Colell (1985, Chapter 5), (1991). For incomplete markets models there is much room for further research. Not long ago Hens (1995, p. 54) stated that "no condition is known for which GEI-equilibria are unique" where 'no condition' means 'no reasonable condition' since of course, for example, for a single consumer the equilibrium already was known to be unique. From this starting-point, more and more results in this field emerged, starting from Hens (1995) himself and going on with Becker (1995), Bettzüge (1997), (1998), and Detemple and Gottardi (1998). A complete presentation is given in Chapter 4.

Until now, however it is by no means clear to what extent the change of the market structure from a complete to an incomplete system influences uniqueness. This is what my research aims at. I prove that incompleteness of markets can create as well as destroy uniqueness of the equilibrium; I show that the incompleteness of markets eventually improves or worsens the structure of the model. Understanding the reasons for different numbers of equilibria is one part of understanding differences in the Arrow-Debreu model and in the incomplete markets model in general. The second aspect of these considerations is to develop an idea about how to find further sufficient conditions for uniqueness in GEI-models. Besides, I analyze the robustness of the number of equilibria with respect to a change in the asset structure where the changes are restricted to sustain the same dimension of the trading space. This analysis indicates that the beforehand presented conditions which guarantee a unique equilibrium can be weakened since currently many of them refer to special structures of the markets.

The idea of sunspots goes back to Jevons (1884) who tried to determine to what extent unobservable signals like sunspots influence the economic fundamentals and therefore have effects on prices. Today, the sunspot problem has changed to the question to what extent observable signals without any influence on economic fundamentals might have effects on prices. This question can be analyzed within the general equilibrium model: one considers some certain economic characteristics, but in addition there are some extrinsic events only creating uncertainty in people's minds. So-called sunspot equilibria exist whenever this extrinsic uncertainty causes an allocation with consumption varying for the different extrinsic events.[4]

Jevons (1884, p. 203) already observed "despondency, hopefulness, excitement, disappointment and panic" evolved by external events to cause fluctuations in prices. The "influence of mass psychology" is also stated by Keynes (1936, pp. 154/5) who presumes that an investor does not only decide on basis of the real value of an investment, but on the value the market

[4]This is the definition used in the standard framework with strictly risk averse consumers. Under weak risk aversion this seems to be not the right definition. Compare Chapter 6.

will give to it some time later. This value might change "as the result of a sudden fluctuation of opinion due to factors which do not really make much difference to the prospective yield; [...] the market will be subject to waves of optimistic and pessimistic sentiment, which are unreasoning and yet in a sense legitimate where no solid basis exists for a reasonable calculation". Keynes (1936, p. 156) compares professional investment to taking part in a newspaper competition where the participants have to pick the six most beautiful people out of hundred photographs. That participant is winning who is closest to the average opinion. Competitors devote "intelligences to anticipating what average opinion expects the average opinion to be".

Economists mostly agree that mass psychology has as much influence nowhere else than on financial markets. They apply such psychological reasoning to explain excess volatility[5] on the markets which finds much empirical evidence with more and more sophisticated methods of testing.[6] Sunspot theory is used as explanation of overreaction on financial markets (for example Cass (1989, p. 89)). It explains the importance of including expectations into investment decisions: "What is the best strategy for playing the stock market? Should one concentrate on fundamentals or should one instead focus on the psychology of the market? These are interesting questions for those who like to get rich on Wall Street" (Cass and Shell (1983, p. 193)).

Sunspot events are interpreted as states of minds which cannot be verified and therefore a specification of insurance contracts is not possible. It is therefore reasonable to assume that insurance markets are incomplete. This is necessary for the existence of sunspot equilibria and creates a link to incomplete markets models.

In general, as Shell (1987, p. 50) points out, "the absence of sunspot equilibria seems to be the exception rather than the rule". The more important it is to understand conditions excluding the existence of sunspot equilibria: I present such conditions for a special situation which refers to the framework created by Mas-Colell (1992). The interesting point is that in this model the existence of a (non-trivial) sunspot equilibrium depends on the number of commodities. This observation is a further step in clarifying the relationship between multiplicity/uniqueness of competitive equilibria considered in Part II and the existence of sunspot equilibria: whereas Mas-Colell (1992) uses

[5]Excess volatility means that fluctuations in observed "prices are to large to be explained purely by changing views of fundamentals" (compare Blanchard and Fischer (1989, p. 234)).

[6]At first compare Shiller (1979) for bond markets and Shiller (1981) and LeRoy and Porter (1981) for stock markets, and secondly Mankiw, Romer, and Shapiro (1985) as well as Campbell and Shiller (1987) and West (1988) applying better elaborated methods of testing.

multiplicity of Arrow-Debreu equilibria to prove the existence of non-trivial sunspot equilibria, I prove it to be not sufficient.

Sunspot equilibria can be used to explain the rational expectations hypothesis whenever there are multiple equilibria. As Hens (1998, p. 192) points out the extrinsic states serve "as a coordination device of agents' expectations", i.e. in the light of GEI-models this means that people expect different equilibria for tomorrow associated with specific probabilities and financial markets make (restricted) insurance against "bad" realizations possible.

One problem with the financial markets models considered so far is the fact that they are all built on the hypothesis of a finite horizon and therefore include some "doomsday" when the economy stops to exist. This is by no means a realistic assumption and causes strange effects, especially with respect to long-lived assets like money. Nonetheless, it simplifies decision problems to a great extent. An infinite horizon can be modeled either by assuming consumers never to die (compare, for example, Magill and Quinzii (1994), (1996a)) or by accepting a life-cycle hypothesis leading to overlapping generations (for short OLG-) models which are in the focus of Part IV. A complete survey on overlapping generations is given by Geanakoplos and Polemarchakis (1991).

OLG-models are mostly applied to analyze consumers' savings behavior. Saving becomes possible by the introduction of money which clearly is the most important asset. Nevertheless — as illuminated in the beginning — meanwhile people also buy and sell other financial products to obtain risk sharing. This point is taken into account by the introduction of uncertainty and sequentially incomplete markets into OLG-models as, for example, done by Schmachtenberg (1988), Cass, Green, and Spear (1992), and Gottardi (1996). In 1987, Geanakoplos (1987b, p. 778) stated that "the combination of the dynamic effects of the infinite horizon OLG-model with the burgeoning theory of incomplete markets under real uncertainty, [was] already on the agenda for next generation's research".

Gottardi (1996, p. 76) states that the assumption of a single consumer per generation is very restrictive for the application of a model on financial markets because then the asset structure plays a minor role. Much of the OLG-literature, however, refers to this limiting case. In contrast, in Part IV I consider quite general pure exchange overlapping generations models with sequentially complete markets. Understanding these models is the foundation for comprehension of OLG-models with incomplete markets. The focus of Part IV is on the existence and the number of monetary stationary equilibria and cycles. This aspect is related to the multiplicity of equilibria in certain Arrow-Debreu models and things come full circle since this is the beginning of my thesis.

One of the main aspects of this thesis is to clarify that all above mentioned competitive equilibrium models are closely related to each other with respect to uniqueness of equilibria.

Taking conditions for unique equilibria in the Arrow-Debreu model as the starting-point, one observes parallels to conditions for unique equilibria in incomplete markets and restricted participation models. Incompleteness of markets and restricted participation, however, are triggering off sunspot equilibria of which the existence in some parts is related to multiplicity of equilibria in the Arrow-Debreu model. In addition, the number of equilibria in some special kinds of Arrow-Debreu economies allows conclusions on the existence and number of stationary and periodic equilibria in overlapping generations economies which again can be related to the existence of sunspot equilibria.

As already indicated, this thesis is divided into five main parts. The first one, in addition to this introduction, contains a full description of the basic Finance GEI-model (Chapter 2) which lays the foundation for the following part. The model is presented in the introductory part because it can also be easily extended to the basic model of Part III and in addition includes some general definitions[7] and some interpretations. Nevertheless, to maintain all parts self-contained, in each of it a brief, but complete description of the respective model is given. Notations are as close to each other as possible.

Part II is devoted to the question of uniqueness of equilibria in different finite dimensional models. It starts (Chapter 3) with the consideration of unique equilibria in the Arrow-Debreu model which is presented as a special Finance GEI-model. Nearly all known restrictions on individual characteristics are summarized to lay the foundations for understanding uniqueness of equilibria in the Finance GEI-model considered in Chapter 4. I present a complete account of conditions which generate a unique equilibrium in the incomplete markets model and which are obtainable through transfer from the Arrow-Debreu theory. Furthermore, this chapter analyzes the robustness of the number of equilibria against perturbations of the asset structure, the limits of transferability, and the existence of additional results on uniqueness. Chapter 5 extends the results on unique equilibria to asymmetric market participation structures, i.e. to restricted participation economies. Beforehand, however, also the limits of this proceeding are shown.

The third part refers to the existence of sunspot equilibria with focus on finite models. It contains a description of the underlying model. After a

[7]Examples for this are definitions of special utility functions.

presentation of known results (Chapter 6) I prove Mas-Colell's (1992) observation on the existence of so-called non-trivial sunspot equilibria to loose its validity if one considers a two-commodity economy. Mas-Colell (1992) explicitly excludes this because, as he says, he does not know what happens in this case. My result can be interpreted in the sense that even potential multiplicity of Arrow-Debreu equilibria is not sufficient for the existence of non-trivial sunspot equilibria.

Part IV considers the existence of monetary stationary and periodic OLG-equilibria. It starts with a selective presentation of the corresponding literature (Chapter 8) and continues with the description of the OLG-model and specific related Arrow-Debreu economies (Chapter 9). Monetary stationary and cyclical equilibria can be identified with special Arrow-Debreu equilibria. Chapter 10 applies these results to gain knowledge about the existence and the number of these special OLG-equilibria, therefore using the theory of unique Arrow-Debreu equilibria as described in Chapter 3.

In the concluding Part V I consider the OLG-model with incomplete markets introduced by Gottardi (1996): Chapter 11 presents some starting-points for analyzing this model by application of results from Parts II to IV and by this again emphasizes their close relation and interdependences. Chapter 12 summarizes the results of this thesis and gives an outlook for further research.

The Appendix VI contains the mathematical notations, definitions, and theorems (Appendix A) and omitted proofs (Appendix B).

Chapter 2

The Finance GEI-Model and Some Fundamentals

This chapter contains a description of the Finance GEI-model which is underlying the considerations in Chapters 3 and 4 and is extended for the analysis in Chapter 5. Besides it can be seen as a starting-point for the sunspot economies in Part III. In addition, here I give some general definitions and assumptions including their limitations. I broadly explain the underlying intuitions which the reader will probably miss in Parts III and IV and for which he is directed back to look here.

2.1 The Information Structure and the Commodity Space

General equilibrium theory according to Debreu (1959) distinguishes commodities on the one hand by their physical characteristics and their location, on the other hand by time and event of their availability. The GEI-model puts emphasis on this last aspects, i.e. on time and uncertainty.

I consider the standard GEI-model where uncertainty is resolved in a finite number of steps. Uncertainty is modeled by a finite number of "states of the world" à la Savage (1954). The information is described by partitions of the states of the world. A finer partition means more information. In the beginning none of the agents is aware of which state will materialize in the end such that the partition is given by one single set containing all states. At the end of all periods all agents observe which state realizes such that the partition is formed by those sets built up by each single state.

I do not give a formal modeling of this information structure because I restrict attention to the most simple such case in which there are just two periods, $t = 0$ and $t = 1$, which stand for today and tomorrow.

On numbering the states by $s = 1, \ldots, S$, denoting by $s = 0$ the set $\{1, \ldots, S\}$ and by $s = 1$ the set $\{1\}$, by $s = 2$ the set $\{2\}$, etc., the information structure can be represented by the simple decision tree shown in Figure 2.1.

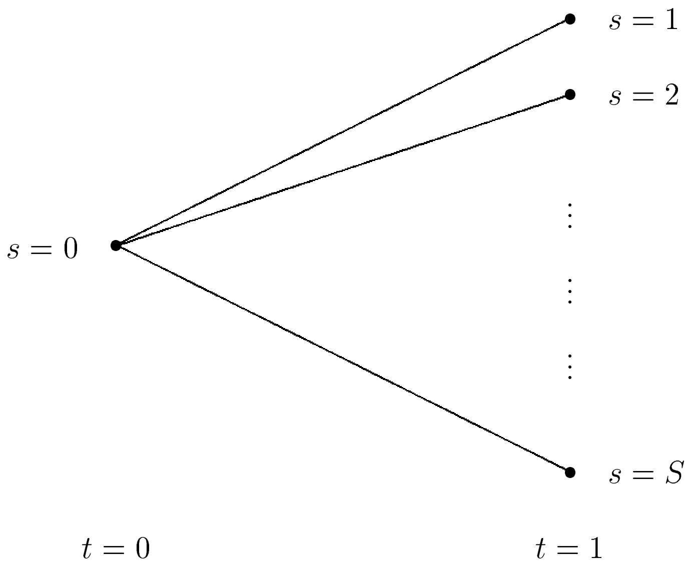

Figure 2.1: The information structure.

The restriction to two periods is simplifying and especially helpful as will become clear later on.

Regrettably, many applications of finance show their full richness only in a multiperiod or continuous trading context. Most assets are traded several times in between their date of issue and the date of terminal payoff. However, the two period model makes it possible to gain basic knowledge for understanding discrete time models with finite horizon as it is demonstrated at full length by Magill and Quinzii (1996b, Chapter 4) or briefly by Hens (1998, pp. 198–202).

I assume that there is only one commodity which is available in each future state $s = 1, \ldots, S$ of the world, but not today. The single commodity should be thought of as the aggregate commodity "expenditure" in that state. This simplification is built on the common assumption in finance that the relative prices of the different commodities in each state have little influence on the financial decisions of the households. A justification is given by the assumption of fixed relative prices (Hicks-Leontief aggregation) or some separability of utility functions (Green-Nataf aggregation). The assumption of a single commodity enables to concentrate on the financial aspects of the model, i.e. on the allocation of income over time and uncertainty (compare Hens (1998)).

Therefore, the commodity space is the finite-dimensional Euclidean space $\mathbb{R}^S$.

Later on, it will be shown that it is more general to assume non-availability of the commodity today than to assume availability: the second case can be modeled by the first one (cf. Section 2.3).

2.2 Characteristics of the Consumers

I restrict the economy to be an exchange economy. Hence, the only units of economic activity like trading and consumption are the consumers, also called households or agents. Having in mind a situation where a large number of traders endowed with certain incomes meet in order to insure among each other and share risk by trading assets, this limitation seems reasonable. If I allowed firms to exist and produce, there would arise many additional issues which would divert from the main purposes.[1]

There is a discrete number $I \in \mathbb{N}$ of consumers $i = 1, \ldots, I$.[2]

The consumers are described by their consumption sets $\mathcal{X}^i \subset \mathbb{R}^S$, their possibly uncertain endowments $\omega^i \in \mathbb{R}^S$ of commodities, and their tastes and risk attitudes represented by utility functions $U^i : \mathcal{X}^i \to \mathbb{R}$.[3] Single consumption plans of a single consumer i are called $x^i \in \mathcal{X}^i$. $x = (x^1, \ldots, x^I) \in$

[1]Nevertheless, Kirman (1989) points out that even in the simple Arrow-Debreu model (which I explain later on as a specific restriction of the model introduced here) the introduction of production might change the outcome with respect to the property of uniqueness. For some conditions guaranteeing uniqueness in production economies compare Kehoe (1985). Besides consider Folkertsma (1991) who examines uniqueness in production economies under the assumption of constant returns to scale and no joint production. Therefore, he applies a method ascribed to Rader (1972) which allows to reduce such a production economy to an exchange economy.

[2]For simplicity I denote the number of agents and the set of agents by the same capital letter. I do the following convention: if there is a variable n which takes values $1, \ldots, N \in \mathbb{N}$, then I define $N := \{1, \ldots N\}$. $\mathbb{N}$ is the set of natural numbers without zero whereas $\mathbb{N}_0$ includes zero.

[3]Note the following fact: upper indices are used for labeling the consumer while lower indices give the state of the world. For example, ω^i_s gives the endowment of consumer i if state s occurs tomorrow. ω^i is equal to the vector $(\omega^i_1, \ldots, \omega^i_S)^T$.

$\times_{i\in I}\mathcal{X}^i$ is an allocation.

The total endowment of the economy is called ω, that means

$$\omega := \sum_{i\in I}\omega^i.$$

A consumption allocation $(x^i)_{i\in I}$ is feasible whenever $\sum_{i\in I} x^i \leq \omega$.

I impose some standard assumptions on the characteristics of the economy. These are not the most general assumptions under which the model works. However, they are strong enough to facilitate the exposition and thus to focus on the aspects arising from the incompleteness of markets.

The following continuity assumptions are made for the whole exposition.

Assumption 2.1 (Continuity Assumptions)
The agents' characteristics satisfy the following properties:

1. $\mathcal{X}^i = \mathbb{R}^S_+$ *for all* $i \in I$*;*
2. $\omega^i \in \mathbb{R}^S_{++}$ *for all* $i \in I$*;*
3. $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *is continuous for all* $i \in I$*;*
4. $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *is strictly monotonic for all* $i \in I$*;*
5. $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *is strictly quasi-concave for all* $i \in I$*.*

These assumptions are called Continuity Assumptions since they guarantee that agents' demand is a continuous function of prices (compare Hens (1991, p. 28)).

Let me explain and comment on these assumptions:

Choosing $\mathcal{X}^i$ as the set of all non-negative commodity bundles is an easy, but standard way to guarantee that the consumption set is closed, convex, and bounded from below. These are typical requirements in the general equilibrium model according to Arrow and Debreu. Since I do not want to complicate the exposition, I take this common restriction.

As every consumer should be able to find a feasible consumption bundle without trading, choosing ω^i as an element of the consumption set seems reasonable. Moreover, I assume that whatever state of the world occurs tomorrow every agent will have some positive amount of resources, i.e. endowments are interior to the consumption sets. Boundary endowments might lead to discontinuities in agents' demand functions which hinder equilibria to exist. Using appropriate restrictions on the structure of asset markets, Gottardi and Hens (1996) demonstrate how to overcome these difficulties in proving existence of competitive equilibria. Since the main focus of this thesis is not on giving the most general existence theorem, but rather on finding conditions under which competitive equilibria are unique, I will not be concerned about boundary endowments. Nevertheless, some of the results on uniqueness of competitive equilibria which I will give later on remain valid under weaker assumptions on the endowments, but consideration of this fact — however complicating the exposition — would not give deeper insights.

What concerns the restriction to continuity, focusing on the uncertainty aspect of commodities, this assumption does not seem too restrictive.

The limitation on strictly monotonic utility functions is also common.

Definition 2.2 (Strict Monotonicity)
A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ is strictly monotonic if $x, y \in \mathbb{R}^S_+$ with $y > x$[4] *implies $U^i(y) > U^i(x)$.*

Hence, the level of utility of a consumer with a strictly monotonic utility function increases as soon as the consumer gets some more consumption in one of the states without getting less consumption in any other state.

In the context of uncertainty, the quasi-concavity has to be interpreted as risk aversion of the consumer: he prefers a mixture of two feasible consumption bundles x and y to each of them.

Definition 2.3 (Strict Quasi-Concavity)
A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ is strictly quasi-concave if for each pair of vectors $x, y \in \mathbb{R}^S_+$ with $x \neq y$ and all $\lambda \in (0,1)$ it is

$$U^i(\lambda x + (1-\lambda)y) > \min\{U^i(x), U^i(y)\}.$$

Figure 2.2 illustrates this notion of risk aversion: consumption in two states is considered. $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ and $\hat{x} = (\hat{x}_1, \hat{x}_2)$ are two risky consumption bundles yielding the same level of utility. Since the utility function is strictly quasi-concave or equivalently the "no worse than" sets are strictly convex, any strict convex combination of $\tilde{x}$ and $\hat{x}$ improves the consumer. In particular, the risk-free consumption bundle $\bar{x}$ for which $\bar{x}_1 = \bar{x}_2$ gives a strictly higher level of utility than the risk-free bundle x^* which by construction delivers the same utility as the risky bundles $\tilde{x}, \hat{x}$. For obvious reasons x^* is called the certainty equivalent. Finally, note that the amount $\bar{x}_1 - x^*_1$ gives a measure for the risk aversion of the agent.

[4] $y > x$ is defined as $y_s \geq x_s$ for all $s \in \{1, \ldots, S\}$ and $y_s > x_s$ for some $s \in \{1, \ldots, S\}$. In contrast, $y \gg x$ means $y_s > x_s$ for all $s \in S$ and $y \geq x$ denotes $y_s \geq x_s$ for all $s \in S$.

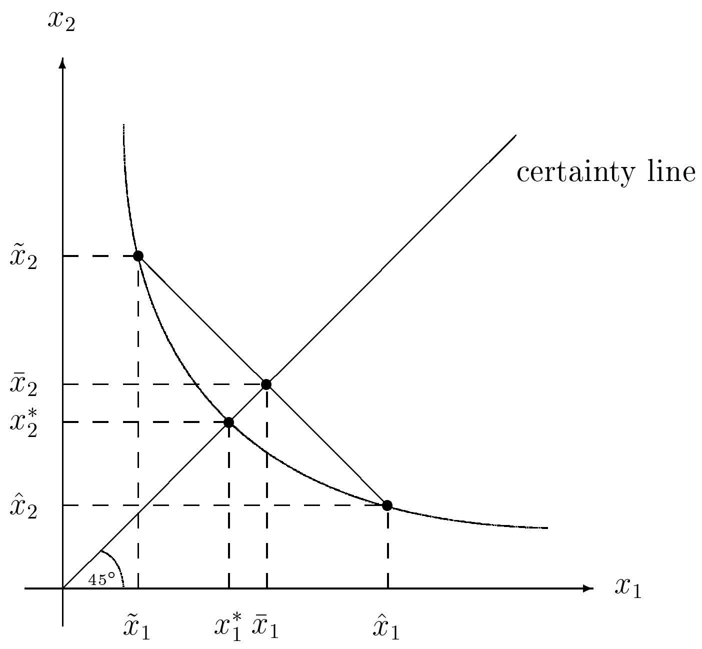

Figure 2.2: Strict quasi-concavity and risk aversion.

Moreover, to obtain a differentiable system of equilibrium allocations Debreu (1972) initiated the following set of Differentiability Assumptions that have to be assumed in addition to the Continuity Assumptions 2.1.

Assumption 2.4 (Differentiability Assumptions)
The agents' characteristics fulfill the Continuity Assumptions 2.1 and additionally the following assumptions:

1. $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *is infinitely often differentiable on* $\mathbb{R}^S_{++}$ *for all* $i \in I$;

2. $\nabla U^i(x) \in \mathbb{R}^{1\times S}_{++}$ *for all* $x \in \mathbb{R}^S_{++}$ *and all* $i \in I$[5];

3. *for all* $i \in I$ *it is* $h \cdot D^2U^i(x)h < 0$ *for all* $x \in \mathbb{R}^S_{++}$ *and all* $h \in \mathbb{R}^S \backslash \{0\}$ *with* $\nabla U^i(x)h = 0$[6];

4. $\{x \in \mathbb{R}^S_+ \mid U^i(x) \geq U^i(\bar{x})\} \subset \mathbb{R}^S_{++}$ *for all* $\bar{x} \in \mathbb{R}^S_{++}$ *and all* $i \in I$.

[5] ∇ denotes the gradient of a real function, i.e. the row vector of the first partial derivatives to each variable.

[6] $D^2U^i(x)$ denotes the Hessian matrix of U^i at x where columns refer to first and rows to second partial derivatives, i.e. $\left(D^2U^i(x)\right)_{s\bar{s}} = \partial_{x_s}\partial_{x_{\bar{s}}}U^i(x)$ with ∂ indicating partial derivatives.

The differentiability of a function is a technical assumption. 2.4.2 implies strict monotonicity of the utility function. 2.4.3 guarantees strict quasi-concavity. 2.4.4 assures that the indifference curves of the consumer through interior consumption bundles do not intersect the boundary of the consumption sets. Together with the strict positivity of the endowments it implies that consumers who maximize their utility choose strictly positive consumption bundles, i.e. consumption bundles in the interior of their consumption sets.

To illustrate Assumption 2.1 and 2.4 a bit more I consider the case of an expected utility function.

Definition 2.5 (Expected Utility Hypothesis)
A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *satisfies the expected utility hypothesis if there exists a probability measure* $\rho^i \in \mathbb{R}^S_+$ *with* $\sum_{s=1}^S \rho^i_s = 1$ *and a von Neumann-Morgenstern utility function* $u^i : \mathbb{R}_+ \to \mathbb{R}$ *such that for all* $x \in \mathbb{R}^S_+$

$$U^i(x) = \sum_{s=1}^{S} \rho^i_s u^i(x_s).$$

Assumption 2.4 requires the following standard assumptions on the von Neumann-Morgenstern utility function underlying an expected utility function.

Continuity and differentiability demand $u^i : \mathbb{R}_+ \to \mathbb{R}$ to be continuous respectively differentiable. Strict monotonicity implies u^i to be strictly increasing as well as to hold the belief that no state s is impossible, i.e. that $\rho^i_s > 0, s = 1, \ldots, S$. Strict quasi-concavity requires u^i to be a strictly concave function.

Definition 2.6 (Strict Concavity)
A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *is strictly concave if for all* $x, y \in \mathbb{R}^S_+$ *with* $x \neq y$ *and* $\lambda \in (0,1)$ *it is*

$$U^i(\lambda x + (1-\lambda)y) > \lambda U^i(x) + (1-\lambda)U^i(y).$$

Besides to assure that indifference curves through interior consumption bundles do not intersect the boundary of the consumption space $\mathbb{R}^S_+$, one requires $\lim_{x_s \to 0} u^{i\prime}(x_s) = \infty$.

According to the expected utility hypothesis an agent's preferences over risky consumption bundles can be conceptually separated into the agent's beliefs over the likelihood of occurrence of the various states and a "utility out of consumption in the states". The expected utility hypothesis can be given an axiomatic foundation in terms of fundamental axioms of choice, like the "sure thing principle" or the "state independence axiom", first introduced by Savage (1954) for the case of a continuous state space. For an axiomatization well-suited for the framework of the standard GEI-model as displayed here see, for example, Wakker (1989) or Hens (1992). In this framework the sure thing principle requires that the choice between two consumption bundles x and y should not depend on the consumption in that subset of states in which x and y are identical. It can be demonstrated that this property holds true if and only if the utility function U^i is additively separable.

Definition 2.7 (Additive Separability)
A utility function $U^i : \mathbb{R}_+^S \to \mathbb{R}$ is additively separable if there exist utility functions $u_s^i : \mathbb{R}_+ \to \mathbb{R}, s = 1, \dots, S$, such that for all $x \in \mathbb{R}_+^S$ it is

$$U^i(x) = \sum_{s=1}^{S} u_s^i(x_s).$$

The state independence axiom requires the preferences for consumption within the states to be identical across the states, i.e. the "state utility functions" $u_s^i, s = 1, \dots, S$, only differ up to some positive monotonic transformations. To complete the characterization of expected utility, note that for an expected utility function the slope of the indifference curves is constant along the certainty line, and it is actually equal to the negative ratio of the likelihood of occurrence of the states according to the agent's beliefs ρ_s^i. Hens (1992) has demonstrated that this property which he calls "weak homotheticity" together with the "sure thing principle" and the "state independence axiom" indeed characterize expected utility in a model with a continuum of consequences and a finite number of states as for example in the standard GEI-model.

The following measures for an agent's risk aversion can be defined.

Definition 2.8 (Risk Aversion)
For an expected utility function $U^i(x) = \sum_{s=1}^{S} \rho_s^i u^i(x_s)$

1. *the absolute risk aversion is defined as*

$$ARA^i(x_s) = -\frac{(u^i)''(x_s)}{(u^i)'(x_s)} \text{ for all } x_s \in \mathbb{R}_{++}$$

2. *and the relative risk aversion is defined as*

$$RRA^i(x_s) = -\frac{x_s \cdot (u^i)''(x_s)}{(u^i)'(x_s)} \text{ for all } x_s \in \mathbb{R}_{++}.$$

Whereas the concept of absolute risk aversion is appropriate to measure the "attitude toward risky projects whose outcomes are absolute gains or losses from current wealth", the measure of relative risk aversion is applied to "evaluate risky projects whose outcomes are percentage gains or losses of current wealth" (Mas-Colell, Whinston, and Green (1995, p. 19)). As theoretically shown by Arrow (1970) and Pratt (1964), reasonable assumptions are a decreasing absolute and an increasing relative risk aversion: assume an investor is faced with a risky and a riskless asset and he has to decide how to divide a given income among the two. A decreasing absolute risk aversion means that with increasing wealth the amount hold of the risky asset increases; increasing relative risk aversion evokes the effect that if wealth as well as the size of the bet are increased proportionally, then the investor is less willing to accept the bet and invest in the risky asset. There is experimental evidence for decreasing absolute risk aversion (Levy (1994)). What concerns the relative risk aversion, Dalal and Arshanapalli (1993) provide empirical support for the assumption of a constant relative risk aversion whereas Levy (1994) has to reject the hypothesis of increasing relative risk aversion. Referring back to Figure 2.2 it is obvious that both notions of risk aversion determine the curvature of the agent's indifference curves.

The definitions of risk aversion for general additively separable utility functions are similar, except for defining ARA^i_s and RRA^i_s in dependence on the corresponding state.

Utility functions which exhibit so-called "hyperbolic absolute risk aversion", abbreviated $HARA$, are of particular importance for the structure of asset demand.

Definition 2.9 (Hyperbolic Absolute Risk Aversion)
An expected utility function $U^i(x) = \sum_{s=1}^S \rho^i_s u^i(x_s)$ satisfies hyperbolic absolute risk aversion, $HARA$, if there exist some $\alpha^i \in \mathbb{R}_+, \beta^i \in \mathbb{R}$ not equal to zero at the same time such that for all $x_s \in \mathbb{R}$ with $\alpha^i + \beta^i x_s > 0$ it is

$$-\frac{(u^i)''(x_s)}{(u^i)'(x_s)} = \frac{1}{\alpha^i + \beta^i x_s}.$$

Hyperbolic absolute risk aversion is defined via a differential equation of the von Neumann-Morgenstern utility function u^i. Integration shows that u^i is then of the form

$$u^i(x_s) = \begin{cases} \frac{(\alpha^i+\beta^i x_s)^{1-1/\beta^i}}{1/\beta^i(1-1/\beta^i)}, & \text{if } \beta^i \neq 0, \beta^i \neq 1, \\ -\alpha^i e^{-x_s/\alpha^i}, & \text{if } \beta^i = 0, \\ \log(\alpha^i + x_s), & \text{if } \beta^i = 1, \end{cases} \tag{2.1}$$

which is defined on the domain $\alpha^i + \beta^i x_s > 0$.

Since $HARA$ utility functions give a nice structure for financial markets models, I would like to include them in my analysis. However, not all the utility functions in the $HARA$ class satisfy the Continuity and Differentiability Assumptions 2.1 and 2.4 without further qualifications.

For $\beta^i = -1$, for example, $u^i(x_s)$ is quadratic, which leads to the mean-variance-criterion for investment underlying the CAPM. The problem is that quadratic utility functions have satiation points; hence, (strict) monotonicity does not hold on the entire non-negative reals. Assuming that aggregate endowments are not sufficient to push any single agent over his satiation point cures this problem.[7] Moreover, with quadratic preferences indifference curves may intersect the boundary of the consumption sets. Hence, differentiability of demand can only be guaranteed if the optimal consumption allocation is interior to the consumption set.

A different example is the case of the Cobb-Douglas utility functions which results for $\alpha^i = 0$ and $\beta^i = 1$. In this case the Differentiability Assumptions 2.4 are satisfied except for the problem that the utility function is not defined on the boundary. Note, however, that with interior endowments for Cobb-Douglas utility functions the set of consumption bundles being at least as good as the initial endowments is strictly included in the interior of the consumption set. Hence, since trade is voluntary and agents act individually rational, again the problem is innocuous for demand.

As indicated by this last example, the class of HARA-functions is closely related to the class of homothetic utility functions which are a helpful device to obtain a unique equilibrium.

Definition 2.10 (Homogenous and Homothetic Utility Functions) [8] *A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ is homogeneous if there is some $k > 0$ such that for all $x \in \mathbb{R}^S_+$ and $\lambda > 0$ it is*

$$U^i(\lambda x) = \lambda^k U^i(x).$$

k is called the degree of homogeneity.

A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ is homothetic if it represents the same preferences as some homogeneous utility function, i.e. there are some continuously differentiable $m^i : \mathbb{R} \to \mathbb{R}$ which is strictly monotonic and some

[7] Such an assumption is made, for example, by Geanakoplos and Shubik (1989) in the CAPM with quadratic utility. It requires that consumption sets are bounded from below, even though the consumption sets in the CAPM are normally unbounded.

[8] At this point I explicitly include monotonic transformations of utility functions because homotheticity is usually defined with respect to preferences instead of utility functions. The definition of homogeneity alone would not include representations $U^i(x) = \sum_{s=1}^S \rho_s \ln x_s$ of Cobb-Douglas preferences, but only $U^i(x) = \prod_{s=1}^S x_s^{\rho_s}$. Later on it will be necessary to distinguish between the two types of utility functions.

homogeneous $V^i : \mathbb{R}^S_+ \to \mathbb{R}$ *such that for all* $x \in \mathbb{R}^S_+$ *it is*

$$U^i(x) = m^i \circ V^i(x)$$

A further class of utility functions will be of special interest later on, namely the class of quasi-linear utility functions.

Definition 2.11 (Quasi-Linear Utility Functions)
A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ *is quasi-linear (in the first state) if there exists some strictly concave function* $\tilde{U}^i : \mathbb{R}^{S-1}_+ \to \mathbb{R}$ *such that*[9]

$$U^i(x) = x_1 + \tilde{U}^i(x_{-1}).$$

$\tilde{U}^i$ *has to satisfy Assumption 2.4 on* $\mathbb{R}^{S-1}_+$.

Obviously, a quasi-linear U^i does not satisfy Assumption 2.4. However, requiring endowments in the first state to be large, is sufficient to circumvent problems arising from this fact.

2.3 The Market Structure

The market structure in the model is a more general, more realistic one than in the standard Arrow-Debreu model: in each future state of the world there is a market for the consumption good, whereas in the Arrow-Debreu model markets open only once, namely today. The price of the commodity in state s is given by $p_s \in \mathbb{R}$. Accordingly, let $p \in \mathbb{R}^S$ denote a spot price vector.

It seems reasonable to assume that the agents can insure themselves against some states of the world. Without any insurance tomorrow agents have to fulfill their budget constraints

$$p_s x^i_s \leq p_s \omega^i_s \text{ for all } s \in S.$$

Since utility functions are strictly monotonic, spot prices have to be positive and consumers choose $x^i_s = \omega^i_s$ for maximizing their utility within their budget constraints, i.e. they always choose the no-trade solution.

However, for example, if a consumer expects a very high endowment in state $\bar{s} \in S$ and a very low endowment in state $\tilde{s} \in S$, it should be able for him to give up something of the commodity in state $\bar{s}$ to receive more in state $\tilde{s}$. To make this possible, J financial contracts $j = 1, \ldots, J$ are introduced which can be traded today and pay off tomorrow. The asset payoffs are

[9] $x_{-1} := (x_2, \ldots, x_S)^T$, i.e. the first coordinate of the vector x is missing. The same definition is used with respect to other coordinates.

described by the matrix $A = (A_{sj})_{s \in S, j \in J}$.[10] $A_{\cdot j} = (A_{1j}, \dots, A_{Sj})^T$ describes asset j, whereas $A_{s\cdot} = (A_{s1}, \dots, A_{sJ})$ gives the payoffs of all assets in state s.

Note that I only allow the existence of so-called real assets, i.e. of assets with payoff given in the underlying commodity. For a thorough discussion of nominal assets or assets with real and nominal components I refer to Magill and Quinzii (1996b, Chapter 7). Equilibria in an economy with nominal assets are equivalent to equilibria in an economy with real assets. There is, however, a dependence on spot price vectors which causes some kind of indeterminacy, but can be handled by the introduction of monetary policy. Therefore, it is legitimate to restrict to real assets.

$q_j \in \mathbb{R}$ is the price one has to pay or gets today for one unit of asset j. $\theta^i_j \in \mathbb{R}$ is the amount of the same asset — depending on the sign — bought or sold by agent i.

If $q = (q_1, \dots, q_J)^T \in \mathbb{R}^J$ is the asset price vector and $\theta^i = (\theta^i_1, \dots, \theta^i_J)^T \in \mathbb{R}^J$ is the portfolio chosen by consumer i, then he has to pay $q \cdot \theta^i$ units of account today to get $A_{s\cdot}\theta^i$ tomorrow if state s occurs.

Considering trade in financial markets, the agents' budget restrictions accordingly become

$$q \cdot \theta^i \leq 0 \tag{2.2}$$

and

$$p_s x^i_s \leq p_s \omega^i_s + p_s A_{s\cdot}\theta^i \text{ for all } s \in S. \tag{2.3}$$

Therefore, portfolios have to be self-financing and expenditures in some state cannot exceed the income of that state altered by the asset payoff. The role of asset trade is to transfer income across the states of period one.

Note that consumers do not have any asset endowment. However, this is implicitly contained in the commodity endowment: if $\bar{\theta}^i$ is some asset endowment, substitute ω^i by $\bar{\omega}^i := \omega^i + A\bar{\theta}^i$.

Since utility functions are strictly monotonic such that consumption in each state is desirable, it is $p_s > 0$ for all $s \in S$ because otherwise all consumers would demand infinitely much of the commodity when the corresponding state is realized. Therefore — because attention is restricted to real assets — p_s can be eliminated from (2.3) to obtain

$$x^i_s \leq \omega^i_s + A_{s\cdot}\theta^i \text{ for all } s \in S. \tag{2.4}$$

[10] The first index marks the state and the second one the asset such that A_{sj} is the number of commodities asset j pays off in state s. In general, if an index whether for a matrix or a vector is given by a whole set, this means a restriction on those coordinates contained in the set. For example $A_{\tilde{S}\tilde{J}}$ with $\tilde{S} \subset S$ and $\tilde{J} \subset J$ is the matrix $(A_{sj})_{s \in \tilde{S}, j \in \tilde{J}}$.

Denote the budget sets by $\mathbb{B}^i(q)$, i.e.

$$\begin{aligned}\mathbb{B}^i(q) &= \{x \in \mathbb{R}^S_+ \mid \exists\, \theta \in \mathbb{R}^J \text{ with } q \cdot \theta \leq 0 \\ &\qquad \wedge\, x_s \leq \omega^i_s + A_s.\theta \text{ for all } s \in S\}.\end{aligned}$$

Note that in particular it is possible to incorporate the case of first period consumption into this model (compare Geanakoplos and Polemarchakis (1986)). In the case of first period consumption the budget constraints are given in the following way:

$$x^i_0 + q \cdot \theta^i \leq \omega^i_0$$

(the worth of today's consumption plus savings is less than the worth of today's endowment) and

$$x^i_s \leq \omega^i_s + A_s.\theta^i \text{ for all } s \in S$$

(the worth of tomorrow's consumption is less than the worth of tomorrow's endowment plus the portfolio payoff). Now choose $\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ and $\tilde{\theta}^i = \begin{pmatrix} x^i_0 - \omega^i_0 \\ \theta^i \end{pmatrix}$. Then the budget constraints change to

$$\begin{pmatrix} 1 \\ q \end{pmatrix} \cdot \tilde{\theta}^i \leq 0$$

and

$$x^i_s \leq \omega^i_s + \tilde{A}_s.\tilde{\theta}^i \text{ for all } s = 0, \ldots, S.$$

Relabeling states shows the equivalence to a GEI-economy without first period-consumption.

Summarizing this gives the following description of a GEI-economy:

$$GEI = \{(U^i, \omega^i)_{i \in I}, A\}.$$

A GEI-model in this form is often called Finance GEI-model. Even though it is based on a variety of assumptions it is sufficiently general to include cornerstones of finance like the static Capital Asset Pricing Model (compare Geanakoplos and Shubik (1989)).

2.4 Financial Markets Equilibria

The market structure defines the agents' choice sets. Before equilibria can be defined, however, the agents still have to be given some objective function.

For consumers the utility function is a natural objective function; that is to say, consumers are assumed to choose demand and portfolios "as if" they were maximizing utility. Now an equilibrium for the general equilibrium model with incomplete markets can be defined.

Definition 2.12 (Financial Markets Equilibrium I)
A tuple $(x^*, \theta^*) \in \mathbb{R}_+^{S \times I} \times \mathbb{R}^{J \times I}$ and a price system $q^* \in \mathbb{R}^J$ constitute a financial markets equilibrium (abbreviated FM-equilibrium) if

1. (x^{*i}, θ^{*i}) *maximizes* $U^i(x)$ *on* $\mathbb{B}^i(q^*)$ *for all* $i = 1, \ldots, I$*;*
2. $\sum_{i=1}^I x^{*i} = \sum_{i=1}^I \omega^i$*;*
3. $\sum_{i=1}^I \theta^{*i} = 0.$

$\mathbb{B}^i(q)$ is defined as a set of consumption plans. For ease of notation, I sometimes use it also as a set of consumption and portfolio plans where the portfolios finance the consumption plans.

2.12.1 requires all agents to be utility maximizers accepting the prices q^*. 2.12.2 and 2.12.3 are the conditions for commodity respectively asset market clearing.

Since U^i is strictly monotonic, it is the best for the consumer to choose x in all components as large as possible. Therefore, tomorrow's budget constraints are always fulfilled with equality. Hence, consumer's maximization problem can be considered in the form

$$(2.5) \qquad \max_{x \in \mathbb{R}_+^S} U^i(x) \quad \text{s.t.} \quad \exists\, \theta \in \mathbb{R}^J \text{ with } q \cdot \theta \leq 0 \\ \wedge\ x_s = \omega_s^i + A_s.\theta \text{ for all } s = 1, \ldots, S.$$

This observation allows to eliminate x and rewrite the maximization problem as

$$(2.6) \qquad \max_{\theta \in \mathbb{R}^J} U^i(\omega^i + A\theta) \quad \text{s.t.} \quad q \cdot \theta \leq 0 \ \wedge\ \omega^i + A\theta \geq 0.$$

Therefore, FM-equilibria can be alternatively defined as only consisting of portfolio allocations and asset price systems.

Definition 2.13 (Financial Markets Equilibrium II)
A tuple $\theta^* \in \mathbb{R}^{J \times I}$ and a price system $q^* \in \mathbb{R}^J$ constitute a financial markets equilibrium if

1. $\theta^{*i} = arg\ \max_{\theta \in \mathbb{R}^J}\ U^i(\omega^i + A\theta)$ *s.t.* $q^* \cdot \theta \leq 0 \ \wedge\ \omega^i + A\theta \geq 0$;
2. $\sum_{i=1}^I \theta^{*i} = 0.$

2.5 No Arbitrage

A trading strategy with a positive payoff in some state which does not require any payments is called an "arbitrage" strategy. Equilibrium asset prices must be such that there are no opportunities for arbitrage. Under the assumptions I made on the model, appropriate formal concepts are as follows:

(NAC) There is no $\theta \in \mathbb{R}^J$ such that $q \cdot \theta \leq 0$ and $A \cdot \theta > 0$.

Sometimes it is helpful to use a stronger no-arbitrage condition than this one:

(SNAC) There is no $\theta \in \mathbb{R}^J$ such that $\binom{-q^T}{A} \cdot \theta > 0$.

This condition is stronger than (NAC) since it excludes not only arbitrage in the sense of "gaining tomorrow without paying today", but also "getting money today without any paying tomorrow".

There is a common assumption guaranteeing the equivalence of (NAC) and (SNAC). It assures that the consumer does always find a portfolio which is desirable in each future state. A consumer would demand infinitely much of such a portfolio if he were not restricted by his budget.

Assumption 2.14 (Existence of a Desirable Portfolio)
There is a $\theta_A \in \mathbb{R}^J$ such that $A\theta_A > 0$.

If θ_A did not cost anything, then — caused by the strict monotonicity of utility U^i — the consumer i would like to have infinitely much of this portfolio.

Lemma 2.15 (Equivalence of No-arbitrage Conditions)
Under Assumption 2.14 (SNAC) and (NAC) are equivalent.

Proof:
(SNAC) ⇒ (NAC): Assume that (NAC) is violated. Then, there is $\tilde{\theta} \in \mathbb{R}^J$ such that $q \cdot \tilde{\theta} \leq 0$ and $A \cdot \tilde{\theta} > 0$, i.e. especially $\binom{-q^T}{A}\tilde{\theta} > 0$, which contradicts (SNAC).
(NAC) ⇒ (SNAC): Assume that (SNAC) is violated. Then, either there is a $\tilde{\theta} \in \mathbb{R}^J$ with $A\tilde{\theta} > 0$ and $q \cdot \tilde{\theta} \leq 0$ which contradicts (NAC), or there is $\hat{\theta} \in \mathbb{R}^J$ with $A\hat{\theta} = 0$ and $q \cdot \hat{\theta} < 0$. Since $\theta_A \in \mathbb{R}^J$ exists such that $A\theta_A > 0$, there must be an $\varepsilon > 0$ such that for $\tilde{\theta} = \hat{\theta} + \varepsilon\theta_A$ the inequalities

$$A\tilde{\theta} = \underbrace{A\hat{\theta}}_{=0} + \varepsilon\underbrace{A\theta_A}_{>0} > 0$$

and

$$q \cdot \tilde{\theta} = \underbrace{q \cdot \hat{\theta}}_{<0} + \varepsilon q \cdot \theta_A \leq 0$$

are fulfilled, again contradicting (NAC).

□

If the model is used in the form of first period consumption, i.e.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \breve{A} \end{pmatrix},$$

then Assumption 2.14 is always satisfied. As will become clear later on (compare, for example, Proposition 4.20) including first period consumption into the model is not only helpful because of this aspect. It even seems to support uniqueness of equilibria.

A nice consequence of Assumption 2.14 lies in the fact that all consumers with strictly monotonic preferences exploit today's budget.

Assume the contrary: there is a consumer $i \in I$ who chooses a utility maximizing consumption plan $x^i \in \mathbb{B}^i(q)$, financed by the portfolio θ^i, such that today's budget constraint is not fulfilled with equality, i.e.

$$x^i = arg \max_{x \in \mathbb{R}^S_+} U^i(x) \quad \text{s.t.} \quad \exists\, \theta^i \in \mathbb{R}^J \text{ with } q \cdot \theta^i \leq 0$$
$$\wedge\ x_s = \omega^i_s + A_s.\theta^i \text{ for all } s = 1, \ldots, S,$$

and the corresponding θ^i satisfies

$$q \cdot \theta^i < 0.$$

Because of Assumption 2.14 $\hat{x}^i$, defined by

$$\hat{x}^i = \omega^i + A(\theta^i + \varepsilon\theta_A)$$

is bigger than x^i ($\hat{x}^i > x^i$), but also element of the budget set for $\varepsilon > 0$ such that $q \cdot \theta + \varepsilon q \cdot \theta_A \leq 0$. Since U^i is strictly monotonic, this is a contradiction to the maximizing property of x^i.

This observation together with the assumptions on utility functions (Assumption 2.4) guarantees sufficiency of the first order conditions for solving the maximization problem (2.6), i.e.

$$\nabla U^i(\omega^i + A\theta^i)A - \lambda^i q^T = 0, \tag{2.7}$$
$$q \cdot \theta^i = 0 \tag{2.8}$$

for all $i \in I$ where $\lambda^i > 0$ is the Lagrangian multiplier. Together with the market clearing condition

$$\sum_{i=1}^{I} \theta^i = 0 \tag{2.9}$$

the so-called *extended system* is given which fully describes a financial markets equilibrium $((\theta^i)_{i\in I}, q)$ (Definition 2.13).

A fundamental theorem of linear algebra says that (NAC) is equivalent to the requirement that the asset price vector q lies in the interior of the cone spanned by the rows of A, i.e.

$$\text{There is } \pi \in \mathbb{R}^S_{++} \text{ such that } q = A^T\pi. \tag{2.10}$$

This observation together with its relation to the agent's utility maximization problem are contained in the following theorem.

Theorem 2.16 (Fundamental Theorem of Asset Pricing)
For a GEI-economy $\{(U^i, \omega^i)_{i\in I}, A\}$, satisfying Assumptions 2.1 and 2.14, the following conditions are equivalent:

(i) The consumer's decision problem

$$\max\ U^i(x)\ s.t.\ x \in \mathbb{B}^i(q)$$

has a solution.

(ii) There are no arbitrage opportunities, i.e. (NAC) holds for q.

(iii) There is a vector of state prices $\pi \in \mathbb{R}^S_{++}$ satisfying (2.10) for q.

(iv) The consumer's budget set $\mathbb{B}^i(q)$ is compact.

The proof is the same as in Magill and Quinzii (1996b, Theorem 9.3). One only has to take into account that under Assumption 2.14 the different no-arbitrage conditions (NAC) and (SNAC) are equivalent (compare Lemma 2.15). The necessity of extending the proof comes from the fact that Magill and Quinzii (1996b) prove the theorem only under the assumption of first period consumption. The main idea of the proof consists in the application of a Separation Theorem.

By application of the Implicit Function Theorem on equations (2.7) and (2.8) one obtains a continuously differentiable demand function $\theta^i(q)$ on the set of arbitrage-free asset price systems, i.e. on

$$\mathcal{Q}(A) := \left\{q \in \mathbb{R}^J \mid q = A^T\pi \text{ for some } \pi \in \mathbb{R}^S_{++}\right\}. \tag{2.11}$$

Define $\theta(q) := \sum_{i=1}^{I} \theta^i(q)$ as market asset demand. FM-equilibrium price systems are then defined as zeros of market demand $\theta(q)$.

2.6 No-Arbitrage Equilibria

The no-arbitrage principle characterizes the asset prices in a way that makes it possible to present the financial markets model in a form most similar to the Arrow-Debreu model. In the light of Theorem 2.16 the condition (2.10) can be applied to rewrite consumers' budget sets in a way where the dependence on the asset prices q has changed to dependence on state prices such that for

$$\mathbb{B}^i(\pi) = \{x \in \mathbb{R}^S_+ \,|\, \pi \cdot (x - \omega^i) \leq 0 \text{ and } x - \omega^i \in < A >\}$$

with $q^T = \pi \cdot A$ the equality of $\mathbb{B}^i(\pi)$ and $\mathbb{B}^i(q)$ is known. Note that for notational convenience the same $\mathbb{B}^i$ is used, even though the budget sets are defined in different ways; the context will make clear to which set I refer.

According to these budget sets another equilibrium concept can be defined.

Definition 2.17 (No-Arbitrage Equilibrium)
An allocation $x^* \in \mathbb{R}^{S \times I}_+$ and a state price vector $\pi^* \in \mathbb{R}^S_{++}$ constitute a no-arbitrage equilibrium (abbreviated: NA-equilibrium) if the following conditions hold:

1. *x^{*i} maximizes $U^i(x)$ on $\mathbb{B}^i(\pi^*)$ for all $i = 1, \ldots, I$;*
2. *$\sum_{i=1}^I x^{*i} = \sum_{i=1}^I \omega^i$.*

2.17.1 implies that all consumers show optimizing behavior whereas good market clearing is a consequence of 2.17.2.

Since one knows $q = A^T \pi$ from the Fundamental Theorem of Asset Pricing 2.16 and θ^i is continuous on $\mathcal{Q}(A)$ individual (commodity) demand $x^i(\pi) = \omega^i + A\theta^i(A^T\pi)$ as solution of

$$\max U^i(x) \text{ s.t. } x \in \mathbb{B}^i(\pi) \tag{2.12}$$

is a continuously differentiable function on $\mathbb{R}^S_{++}$. Market demand is defined as $x(\pi) := \sum_{i=1}^I x^i(\pi)$ with corresponding excess demand $z(\pi) = x(\pi) - \omega$. NA-equilibrium price systems are alternatively defined as zeros of market excess demand $z(\pi)$.

Look at the presentation of $\mathbb{B}^i(\pi)$. If $< A >= \mathbb{R}^S$, then obviously the incomplete markets model is equivalent to the Arrow-Debreu model. Therefore, the following distinction is very decisive.

Definition 2.18 (Complete and Incomplete Markets)
If the market subspace $< A >$ has full dimension, i.e. rank $A = S$, then asset markets are said to be complete. If the transfer possibilities are limited, i.e. rank $A < S$, then financial markets are incomplete.

In the case of complete markets the equivalence to the Arrow-Debreu model allows to apply the results gained in that context. Especially, existence and Pareto-efficiency are known under quite general assumptions. Incomplete markets demand a thorough analysis of existence, efficiency, uniqueness, and so forth of equilibria.

In the following the terms Arrow-Debreu model and GEI-model with complete markets are used as synonyms.

The importance of the no-arbitrage equilibrium concept is the coincidence of its allocations with those of the financial markets equilibrium. This is implied by the following theorem.

Theorem 2.19 (Allocational Equivalence)
Let the GEI-economy $\{(U^i, \omega^i)_{i=1}^I, A\}$ satisfy the Assumptions 2.1 and 2.14. Then, the following equivalence holds:

1. *If $(x^*, \theta^*, q^*) \in \mathbb{R}_+^{S\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}^J$ is FM-equilibrium, then (x^*, π^*) is NA-equilibrium for some $\pi^* \in \mathbb{R}_{++}^S$ with $q^* = A^T\pi^*$.*

2. *If $(x^*, \pi^*) \in \mathbb{R}_+^{S\times I} \times \mathbb{R}_{++}^S$ is NA-equilibrium, then there is a $\theta^* \in \mathbb{R}^{J\times I}$ such that with $q^* = A^T\pi^* \in \mathbb{R}^J$ (x^*, θ^*, q^*) is FM-equilibrium.*

For a proof in the case of first period consumption again compare Magill and Quinzii (1996b, Proposition 10.3) which can be easily transferred to the more general case. The main idea consists in the application of the Fundamental Theorem of Asset Pricing (Theorem 2.16) with which one obtains the equality of $\mathbb{B}^i(q^*)$ and $\mathbb{B}^i(\pi^*)$ whenever $q^* = A^T\pi^*$.

At this point it seems reasonable to do some comparative statics: what happens if new assets are introduced? To answer this question, it is essential to distinguish between redundant and non-redundant assets.

Consider first the introduction of redundant assets. In doing this it is especially helpful that π allows to evaluate any payoff stream that is a linear combination of the payoffs of the given assets $j = 1, \ldots, J$. From (2.10) one observes that the price of a payoff stream in the market subspace must be a linear combination of the prices of the duplicating assets. The weights of the assets in the duplicating portfolio define the weights in the linear combination of the prices and (2.10) can be interpreted as a linear pricing rule. Therefore, taking a collection of assets which form a basis of the market subspace and knowing their prices, one is able to value any payoff stream in the market subspace, i.e. any duplicable (redundant) asset.

Corollary 2.20 (Pricing of Redundant Assets)
Let $\{(U^i, \omega^i)_{i=1}^I, A\}$ *be a GEI-economy which satisfies Assumptions 2.1 and 2.14. If one of the assets is redundant, i.e.* $A^k = \sum_{j \in J \setminus \{k\}} \alpha_j A^j$ *for some* $\alpha_j \in \mathbb{R}$, $j \in J \setminus \{k\}$, *then in each FM-equilibrium asset prices* q^* *fulfill*

$$q_k^* = \sum_{j \in J \setminus \{k\}} \alpha_j q_j^*.$$

Since equilibrium price systems have to be arbitrage-free, this corollary is a direct consequence of the Fundamental Theorem of Asset Pricing 2.16. Introducing a redundant asset leaves the no-arbitrage equilibrium unaffected. Trading the new asset is unnecessary, and its price can be determined by the linear pricing rule. Corollary 2.20 has shown how easy redundant assets can be added to any GEI-economy. Therefore, from now on assume that there are no redundant assets, i.e.

Assumption 2.21 (No Redundance of Assets)
There are no redundant assets, i.e. rank $A = J$.

This assumption assures that for each consumption plan x^i satisfying $x^i - \omega^i \in < A >$ there is a unique $\theta^i \in \mathbb{R}^J$ with $x^i = \omega^i + A\theta^i$.

In contrast to Corollary 2.20 the introduction of non-redundant assets leads to drastic changes of the equilibrium. As can be seen later on, even the number of equilibria might change.

The theory on uniqueness of equilibria would be a vacuous theory if such equilibria did not exist. However, the assumptions guarantee existence.

Theorem 2.22 (Existence of FM-Equilibria)
In each GEI-economy $\{(U^i, \omega^i)_{i \in I}, A\}$, *satisfying Assumptions 2.1, 2.14, and 2.21 an FM-equilibrium exists (and therefore also an NA-equilibrium).*

A proof is given, for example, in Magill and Quinzii (1996b, Proposition 10.4) for the model with first period consumption. Their proof — applying the well-known Cass-trick (Cass (1984)) to obtain the existence of an NA-equilibrium — can be easily transferred to the general model. Hens (1991, p. 33) gives an alternative proof, directly referring to FM-equilibria. The basic mathematical tool in both cases is Brouwer's Fixed Point Theorem.

2.7 Constrained Pareto-Efficiency

In the Arrow-Debreu model it is well-known that competitive equilibria are Pareto-efficient in the sense that there does not exist any reallocation of the endowments such that at least one agent is improved and all the others do not become worse than with the equilibrium allocation. Regrettably, but intuitively, this result looses its validity when markets are incomplete.

Theorem 2.23 (Pareto-Inefficiency)
Let the GEI-economy $\{(U^i, \omega^i)_{i\in I}, A\}$ satisfy Assumptions 2.4, 2.14, and 2.21. If there are at least two agents and financial markets are incomplete, i.e. $J < S$, then generically in endowments FM-equilibria are Pareto-inefficient.

If a property is generically satisfied in endowments, it is satisfied for almost all endowments. For an exact definition of genericity compare, for example, Mas-Colell (1985, Chapter 8).

A proof of Theorem 2.23 can be found in Magill and Quinzii (1996b, Corollary 11.8) which again has to be adapted for the model without first period consumption. The main idea is to show the generic distinctness of normalized utility gradients for equilibrium allocations by application of the Parametric Transversality Theorem.

Fortunately, the definition of Pareto-efficiency can be brought into line with the Finance GEI-model.

Definition 2.24 (Constrained Pareto-Efficiency)
Given a GEI-economy $\{(U^i, \omega^i)_{i\in I}, A\}$ a feasible allocation $(x^{*i})_{i\in I} \in \mathbb{R}_+^{S\times I}$ is constrained Pareto-efficient if $x^{*i} - \omega^i \in < A >$ and there is no other feasible allocation $(\tilde{x}^i)_{i\in I} \in \mathbb{R}_+^{S\times I}$ with $\tilde{x}^i - \omega^i \in < A >$ and $U^i(\tilde{x}^i) \geq U^i(x^{*i})$ for all $i \in I$ with at least one strict inequality.

For ease of exposition, sometimes not only consumption, but also portfolio allocations are called constrained Pareto-efficient. This is possible because the assumptions assure a one-to-one-correspondence between consumption and portfolio plans.

Theorem 2.25 (Constrained Pareto-Efficiency)
If $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ satisfies Assumptions 2.4, 2.14, and 2.21, then FM-equilibria are constrained Pareto-efficient.

The proof is a transformation of the corresponding proof in Magill and Quinzii (1996b, Theorem 12.3). It is an indirect proof: Pareto-improvement contradicts market clearing and arbitrage-freeness at the same time.

If $A_{\cdot j} > 0$ for some $j \in J$, a necessary and sufficient condition for $(\theta^i)_{i\in I} \in \mathbb{R}^{J\times I}$ to be constrained Pareto-efficient is to satisfy

$$\frac{\nabla U^1(\omega^1 + A\theta^1)A_{\cdot -j}}{\nabla U^1(\omega^1 + A\theta^1)A_{\cdot j}} = \ldots = \frac{\nabla U^I(\omega^I + A\theta^I)A_{\cdot -j}}{\nabla U^I(\omega^I + A\theta^I)A_{\cdot j}}. \tag{2.13}$$

This can be proved by maximizing one consumer's utility under minimum wealth for the other consumers, market clearing, and attainability by asset trade.

From now on, Assumptions 2.4, 2.14, and 2.21 are always required and therefore not explicitly mentioned each time.

Part II

On Uniqueness of Competitive Equilibria in General Equilibrium Models

Chapter 3

On Uniqueness of Competitive Equilibria in the Arrow-Debreu Model

3.1 Introduction

The purpose of this chapter is to give an overview of various conditions on consumers' characteristics in an Arrow-Debreu economy which imply uniqueness of the equilibrium. Understanding these conditions in the complete markets case of the GEI-model lays the foundations of the results for incomplete markets presented in the next chapter.

The Arrow-Debreu model is one of the most — if not the most — important contributions to general equilibrium theory and has had significant influence on mathematical economics (Geanakoplos (1987a, p. 59)).

The main object of general equilibrium theory is the allocation of resources between individuals, across time, under uncertainty, etc. As Geanakoplos (1987a, p. 60) states "the Arrow-Debreu model studies those allocations which can be achieved through the exchange of commodities at one moment in time."

General equilibrium theory is part of the neoclassical theory which postulates individual rationality, market clearing, and rational expectations. The development of this theory is dated back to the second half of the 19th century, mainly to Walras (1874), even though the idea of an 'invisible hand' which equates demand and supply in an economy had already been existent since Smith (1776). Walras (1874) mathematically formalized this idea by a system of equations, each of it representing the market of a single commodity. This converts the mechanical equilibrium concept described by Poinsot

(1803) into a basic economic concept. The institution of markets, set up by economic agents, is in the focus of general equilibrium theory: one is interested in the "viability and efficiency" which are among "the oldest and most important questions of neoclassical economics" (Geanakoplos (1987a, p. 60)). The agents are only motivated by self-interest.

Already Walras (1874) himself identified the existence of an equilibrium as the major objective of general equilibrium theory. The Arrow-Debreu model was the first general equilibrium model which enabled a rigorous proof of existence. The model is built on a series of axioms of which "the selection was influenced by the possibility of making useful interpretations, but also by the facility with which results can be derived" (McKenzie (1987, p. 501)). Aside from existence of equilibria, optimality and local uniqueness are among the most important results.

As the name indicates, the main contributions for this model can be ascribed to Arrow and Debreu (Arrow (1951), Debreu (1951), Arrow and Debreu (1954)) who obtained the Nobel Price for economics for their work. One of the most accurate presentations of the model is given in Debreu (1959). Nevertheless, the model is sometimes called Arrow-Debreu-McKenzie model because McKenzie (1954), (1959) also axiomatized the model and proved existence of the equilibrium.

When analyzing the Arrow-Debreu model, convexity theory (with the separating hyperplane theorem) and Brouwer's as well as Kakutani's fixed point theorems have been discovered as basic mathematical tools for economics.

At the beginning, in the model commodities were only distinguished with respect to their physical properties such that the model was atemporal and did not allow for any time- and uncertainty aspects. However, more elaborated notions of commodities were developed: goods are distinguished by geographical location, temporal location (Hicks (1939)), its state of nature (Arrow (1953), Debreu (1959), Radner (1968)), and the ultimate consumer (Arrow (1969)). This allows to interpret equilibrium outcomes with respect to transportation costs, optimal allocation of risk, and externalities and public goods (Geanakoplos (1987a, p. 60)).

Considering financial markets as it is done in this thesis, the inclusion of time and uncertainty is imperative. It is realized by the notion of contingent commodities, i.e. commodities which can be traded contingent on time and states of nature (Arrow (1953), Debreu (1959)). An important aspect is to observe that even if the notion of commodities contains time and uncertainty aspects, one remains with a single budget constraint in the Arrow-Debreu model.

As Geanakoplos (1987a, p. 60) states, in reality one would never find a market for each pure Arrow-Debreu commodity, but groups of these commodities are traded together. This leads to the more realistic consideration of incomplete markets which are considered in the next chapter. The importance of the Arrow-Debreu model is to serve as a benchmark.

Arrow (1953) is one of the seminal works about asset economies. He includes uncertainty into his commodity definition (which can be easily extended to also include time). "Optimal allocations of risk bearing" can be realized by what is meanwhile known as contingent commodity trading. But Arrow (1953) observed that to obtain an optimal allocation of risk it is more realistic that people trade securities. He proved that with a set of securities where each security pays off one unit of account if one special state of nature realizes and which are traded before uncertainty is resolved the same equilibrium allocations are obtained as if there were a complete system of contingent commodity markets. Proving this, he implicitly assumes perfect forecast on prices and requires complete spot markets. Consumers first trade the securities, then the state of nature is realized, and spot markets open where agents adjust their consumption bundles, therefore taking into account also their income from asset payoffs. From Arrow (1953) it becomes clear that even to obtain an optimal allocation of risk bearing it is not necessary to have claims on all contingent commodities, but this can be substituted by a reduced system of assets which are now called Arrow-securities. Arrow's (1953) idea experiences different generalizations in the literature, for example Kreps (1968) and Magill and Shafer (1990). Since in this thesis it is assumed that all goods have the same physical properties and they are not distinguished by location, the richness of Arrow's (1953) results goes by the board.

The importance of a unique equilibrium in a general equilibrium model was already pointed out by Walras (1874): the analysis of uniqueness and stability are the most important objectives in general equilibrium theory after the proof of existence and the demonstration of optimality properties. As will be argued later on, uniqueness is helpful for doing comparative statics which Walras (1874) also identified as a major aim.

Walras (1874) opinion is shared by many economists. For example, Kirman (1989, pp. 128/9) states that "the existence of such an equilibrium is a first and crucial test of [the model's] consistency. Yet once this first test is passed we obviously wish to know more." One of the major arising questions is to find out whether one "can [...] hope to proceed and obtain global uniqueness and stability."

Three main reasons for the interest in uniqueness of the equilibrium can be named:

At first, general equilibrium models and therefore the Arrow-Debreu model are designed to explain and determine prices and the allocation of resources. If a model displays multiple equilibria, this aim is only partially obtained. Pricing formulas referring to special equilibria as they are obtained in the CAPM are useless with multiple equilibria (Hens (1995)).

Secondly, meaningful comparative statics require a unique equilibrium. Comparative statics analyze in how far the exogenous parameters influence the equilibrium outcome. Therefore, equilibria for different constellations of the model parameters are compared with each other. Under multiplicity of equilibria it is difficult, if not even impossible, to identify the equilibria which have to be put in relation to each other (Kehoe (1985), Arrow and Hahn (1971)).

Thirdly, uniqueness of the equilibrium is often required for the examination of the dynamic behavior of a model (Arrow and Hahn (1971)).

A quotation of Schumpeter (1954, p. 969) summarizes the importance of uniqueness of the equilibrium in general equilibrium models.

> "Multiple equilibria are not necessarily useless but, from the standpoint of *any* exact science, the existence of a 'uniquely determined equilibrium (set of values)' is, of course, of the utmost importance, even if proof has to be purchased at the price of very restrictive assumptions; without any possibility of proving the existence of uniquely determined equilibrium — or at all events, of a small number of possible equilibria — at however high a level of abstraction, a field of phenomena is really a chaos that is not under analytic control."

The framework of this chapter is the no-arbitrage approach of the Finance GEI-model because as mentioned before this is totally equivalent to the Arrow-Debreu model when markets are complete.

The focus is on a summary of assumptions on the individual utility functions and endowments which are strong enough to guarantee a unique equilibrium and not on formulating all properties of market demand functions which assure uniqueness. Therefore, I only present those of these properties used later on to prove results in the first sense. Besides, I do not give detailed proofs, but only relevant references and sketches of proofs.

After defining uniqueness of the equilibrium in its different forms, I shortly present some well-known results on local uniqueness. Starting from some special properties of market excess demand functions, I give a complete overview of conditions at the individual level of economies which assure a unique equilibrium. At this point, the reader should have his eye on the last section of this chapter where a figure gives a summary.

As one special aspect I consider the fact that assuming the satisfaction of the expected utility hypothesis for some objective probability measure is not sufficient to obtain a unique equilibrium whenever one allows for aggregate risk. This gains some interest because it refers to the aspect of complete risk sharing which is of growing importance.

3.2 Uniqueness of Arrow-Debreu Equilibria

Consider a GEI-economy with complete markets, i.e.

$$< A >= \mathbb{R}^S.$$

For simplification I name an NA-equilibrium under the assumption of complete markets an Arrow-Debreu equilibrium which is legitimate since $\mathbb{B}^i(\pi)$ describes the standard Arrow-Debreu budget set.

Definition 3.1 (Arrow-Debreu Equilibrium)
A tuple $x^* \in \mathbb{R}_+^{S \times I}$ and a (state) price vector $\pi^* \in \mathbb{R}_{++}^S$ constitute an Arrow-Debreu equilibrium if the following conditions hold:

1. $x^{*i} = arg \max U^i(x)$ *s.t.* $\pi^* \cdot (x - \omega^i) \leq 0$ *for all* $i = 1, \dots, I$;
2. $\sum_{i=1}^I x^{*i} = \sum_{i=1}^I \omega^i$.

To express this definition in terms of excess demand: an equilibrium price system π^* is defined as a zero of market excess demand, i.e. $z(\pi^*) = 0$.

Returning to the asset structure A, the corresponding portfolio demands would be uniquely defined by

$$\theta^{*i} = A^{-1}(x^{*i} - \omega^i) \text{ for all } i = 1, \dots, I,$$

and equilibrium asset prices by

$$q^* = A^T \pi^*.$$

A GEI-economy $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ with $< A >= \mathbb{R}^S$ is called Arrow-Debreu economy and is shortly described as

$$GE = \{(U^i, \omega^i)_{i \in I}\}.$$

This is legitimate because for different asset structures commodity allocations and state prices are the same and portfolio allocations can be easily converted.

Considering consumers' maximization problems, one observes homogeneity of degree zero of excess demand in prices, i.e. $z^i(\pi) = z^i(\lambda\pi)$ for all

$\lambda > 0$. Hence, if π^* is an equilibrium price system, this is the same for $\lambda\pi^*$ with $\lambda > 0$. Without any price normalization there is always a continuum of equilibrium price systems! Therefore, a price normalization has to be defined, i.e. the restriction to a set

$$\hat{\mathcal{P}}_{c,n} := \{\pi \in R^S_{++} | \ \pi \cdot n = c\}$$

for some $n \in \mathbb{R}^S_+ \backslash \{0\}$ and $c > 0$.

Definition 3.2 (Local Uniqueness of Arrow-Debreu Equilibria)
Given $GE = \{(U^i, \omega^i)_{i\in I}\}$, an Arrow-Debreu equilibrium $(x^*, \pi^*) \in \mathbb{R}^{S\times I}_+ \times \hat{\mathcal{P}}_{c,n}$ is locally unique if there is some $\epsilon > 0$ such that $\|\pi - \pi^*\| < \epsilon$ and $\pi \in \hat{\mathcal{P}}_{c,n}$ imply $z(\pi) \neq 0$.

In this context $\|.\|$ is an arbitrary vector norm on $\mathbb{R}^S$.

Local uniqueness is a very helpful property, but nevertheless it is much more convenient if equilibria are globally unique.

Definition 3.3 (Global Uniqueness of Arrow-Debreu Equilibria)
Given $GE = \{(U^i, \omega^i)_{i\in I}\}$, an Arrow-Debreu equilibrium $(x^*, \pi^*) \in \mathbb{R}^{S\times I}_+ \times \hat{\mathcal{P}}_{c,n}$ is globally unique if there is no other $\pi \in \hat{\mathcal{P}}_{c,n}$ with $z(\pi) = 0$.

Since the main focus of this thesis is on global uniqueness, the term "uniqueness" always refers to the global aspect.

3.3 Local Uniqueness of Arrow-Debreu Equilibria

Remember that equilibrium price systems have to be zeros of market excess demand, i.e.

$$z(\pi^*) = 0 \tag{3.1}$$

if π^* is an equilibrium price system. Assume the numeraire price normalization $\pi^*_1 = 1$, i.e. $c = 1$ and $n = e^1$ where e^s for $s \in S$ is the sth unit vector. From now on define $\hat{\mathcal{P}}$ as normalization with respect to 1 and the first unit vector. Walras' Law ($\pi \cdot z(\pi) = 0$) allows to reduce the system (3.1) by one equation to

$$z_{-1}(\pi^*) = 0 \tag{3.2}$$

on which the Implicit Function Theorem can be applied and therefore local uniqueness is obtainable whenever the normalized Jacobian has maximal rank, i.e. whenever

$$rank\ \partial_{\pi_{-1}} z_{-1}(\pi^*) = S - 1.$$

Definition 3.4 (Regularity)
The economy $GE = \{(U^i, \omega^i)_{i\in I}\}$ is regular if for all $\pi^* \in \hat{\mathcal{P}}$ with $z(\pi^*) = 0$ it is rank $\partial_{\pi_{-1}} z_{-1}(\pi^*) = S - 1$ (or equivalently rank $D_\pi z(\pi^*) = S - 1$ for all $\pi^* \in \mathbb{R}^S_{++}$ with $z(\pi^*) = 0$).

If one parameterizes Arrow-Debreu economies via endowment vectors whereas one takes utility functions as fixed, the application of the Parametric Transversality Theorem delivers generic regularity as a result of so-called local controllability. This is the main content of Debreu (1970).

Theorem 3.5 (Generic Regularity)
Given $(U^i)_{i\in I}$, for almost every endowment vectors $(\omega^i)_{i\in I} \in \mathbb{R}^{S\times I}_{++}$ the economy $GE = \{(U^i, \omega^i)_{i\in I}\}$ is regular.

This theorem allows generically to apply the Implicit Function Theorem on the system (3.2) and therefore to obtain local uniqueness of the equilibria. If the GE-economy is regular, the index of an Arrow-Debreu equilibrium is defined as

$$index\ (\pi^*) = sign \begin{vmatrix} -D_\pi z(\pi^*) & \pi^* \\ -\pi^{*T} & 0 \end{vmatrix}$$

where $|.|$ denotes the determinant of the included matrix.

The following theorem was given by Dierker (1972).

Theorem 3.6 (Index Theorem)
If $GE = \{(U^i, \omega^i)_{i\in I}\}$ is a regular economy and $\mathcal{E}_{c,n} := \Big\{\pi \in \hat{\mathcal{P}}_{c,n} |\ z(\pi) = 0\Big\}$ is the set of normalized equilibrium price systems, then

$$\sum_{\pi\in\mathcal{E}_{c,n}} index\ (\pi) = +1.$$

Therefore, the number of normalized equilibria in an Arrow-Debreu economy has to be odd and a sufficient condition for uniqueness of the equilibrium is to assure all indices of an economy to be positive.

The proof of the theorem is mainly based on the Poincaré-Hopf Theorem.

3.4 Some Properties of Market Excess Demand

At least since Sonnenschein (1973), Debreu (1974), and Mantel (1974) it is known that in addition to continuity, Walras' Law, and Homogeneity no additional properties are always satisfied by the market excess demand function. Nevertheless, some excess demand functions exhibit special structures assuring, for example, uniqueness. The most basic property is the Weak Axiom of Revealed Preferences.

Definition 3.7 (Weak Axiom of Revealed Preferences)
The excess demand function $z : \mathbb{R}^S_{++} \to \mathbb{R}^S$ satisfies the Weak Axiom of Revealed Preferences (WARP) if

$$\pi \cdot z(\tilde{\pi}) \leq 0 \Rightarrow \tilde{\pi} \cdot z(\pi) > 0 \tag{3.3}$$

for all $\pi, \tilde{\pi} \in \mathbb{R}^S_{++}$ with $z(\pi) \neq z(\tilde{\pi})$.

A sufficient condition for this is given by

$$[\pi \cdot v = 0, z(\pi) \cdot v = 0] \Rightarrow v \cdot D_\pi z(\pi) v < 0 \tag{3.4}$$

for all $v \in \mathbb{R}^S \backslash \{0\}$ and $\pi \in \mathbb{R}^S_{++}$. Allowing that the inequality is weak, the condition is also necessary, i.e.

$$\text{WARP} \Rightarrow v \cdot D_\pi z(\pi) v \leq 0 \tag{3.5}$$

for all $v \in \mathbb{R}^S, \pi \in \mathbb{R}^S_{++}$ with $z(\pi) \cdot v = 0$. For a proof compare Mas-Colell (1985, Proposition 5.7.3).

The definition of the Weak Axiom of Revealed Preferences was chosen under the consideration of the Differentiability Assumptions 2.4.

The idea underlying this property comes from individual excess demand functions which always satisfy this axiom. As soon as a consumer chooses a consumption bundle $x(\pi)$ even though he can afford also $x(\tilde{\pi})$, $x(\pi)$ has to be better for him than $x(\tilde{\pi})$. Since he chooses $x(\tilde{\pi})$ at prices $\tilde{\pi}$, consistency of decisions is only given when he cannot afford $x(\pi)$ at prices $\tilde{\pi}$.

Regrettably, WARP is not an additive property: even though individual excess demand functions satisfy the axiom, this need not to be true for market excess demand (compare Hicks (1956)).

A stronger property for excess demand functions is monotonicity.

Definition 3.8 (Monotonicity with Normalizing Vector)
Let $n \in \mathbb{R}^S_+ \backslash \{0\}$. The excess demand function $z : \mathbb{R}^S_+ \to \mathbb{R}^S$ is monotone with respect to n if

$$(\pi - \tilde{\pi}) \cdot [z(\pi) - z(\tilde{\pi})] < 0 \tag{3.6}$$

for all $\pi, \tilde{\pi} \in \mathbb{R}^S_{++}$ with $(\pi - \tilde{\pi}) \cdot n = 0$ and $z(\pi) \neq z(\tilde{\pi})$.

Note that for distinction I use the term "monotone" in contrast to the term "monotonic" in the sense of Definition 2.2.

A sufficient condition for monotonicity with respect to n is given by the strict form

$$v \cdot n = 0 \Rightarrow v \cdot D_\pi z(\pi) v < 0 \tag{3.7}$$

for all $v \in \mathbb{R}^S \backslash \{0\}$ and $\pi \in \mathbb{R}^S_{++}$. Again a necessary condition is given by

$$v \cdot n = 0 \Rightarrow v \cdot D_\pi z(\pi) v \leq 0 \tag{3.8}$$

for all $v \in \mathbb{R}^S$ and $\pi \in \mathbb{R}^S_{++}$. These statements can also be found in Mas-Colell (1985, Proposition 5.7.3).

In addition, Mas-Colell (1985, p. 216) demonstrates that it is really necessary to restrict price systems π by normalization since otherwise monotonicity would never be fulfilled. In contrast to WARP, monotonicity passes from individual to market excess demand, but only if the underlying normalizing vector is the same for all individuals. The economic interpretation for WARP and monotonicity is that these properties assure demands to move in opposite directions of prices. For WARP this only has to be true for compensated price changes in the sense that $v \cdot z(\pi) = 0$ (compare Mas-Colell (1985, p. 217)).

Proposition 3.9 (Uniqueness for Monotonicity and WARP)
Suppose $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that market excess demand satisfies monotonicity or WARP. If the economy is regular, it has a unique Arrow-Debreu equilibrium.

Given the strong differentiable versions of monotonicity and WARP, Mas-Colell (1985, p. 216) proves a chain of implications:

$$\text{Monotonicity} \Rightarrow \text{WARP} \Rightarrow \text{positive index for all equilibria}$$

which proves uniqueness because of the Index Theorem 3.6.

Given monotonicity and WARP in their weak forms, Mas-Colell (1985, Proposition 5.7.2) proves monotonicity to imply WARP. WARP, however, assures convexity of the normalized equilibrium price set (compare Mas-Colell, Whinston, and Green (1995, p. 609)) which under local uniqueness implies global uniqueness.

A last property helpful when considering uniqueness of the equilibrium is the strict gross substitution property which requires demand for a commodity to be strictly increasing in other commodities' prices and strictly decreasing in its own price.

Definition 3.10 (Strict Gross Substitution)
The excess demand function $z : \mathbb{R}^S_{++} \to \mathbb{R}^S$ satisfies the property of strict gross substitution whenever the Jacobian matrix has strictly positive off-diagonal elements and strictly negative diagonal elements.

The definition of the weak gross substitution property is nearly the same, but allows that elements of the matrix are zero.

Arrow and Hahn (1971) give different proofs for strict gross substitution to imply uniqueness. One idea (compare Arrow and Hahn (1971, Chapter 9, Corollary 7)) is to do an indirect proof: assuming the existence of two equilibrium price systems π^* and π^{**}, they can be normalized such that $\pi^* \leq \pi^{**}$ and they are equal in (at least) one state. This leads to a contradiction with market clearing when applying strict gross substitution for this state.

Proposition 3.11 (Uniqueness for Strict Gross Substitution)
Suppose $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that market excess demand satisfies the property of strict gross substitution. Then the economy has a unique Arrow-Debreu equilibrium.

Monotonicity and gross substitution are equivalent if there are only two states, i.e. $S = 2$. For relations between gross substitution and the Weak Axiom of Revealed Preferences compare Kehoe and Mas-Colell (1982) and Kehoe (1985).

3.5 How to Obtain Uniqueness

When searching for conditions on the consumers' characteristics guaranteeing a unique equilibrium I apply four different approaches. The reader will observe that these overlap in the sense that some proofs can be done in different manners.

The first approach is to derive an explicit pricing formula from equilibrium conditions. There are only quite a few economies which allow to proceed in this way, but they are extremely nice to handle.

The second approach is to prove the existence of a representative consumer, i.e. it is possible to prove the existence of a utility function and endowments such that an Arrow-Debreu economy with a single consumer defined hereby has the same market excess demand function as the original economy. If the characteristics of the representative consumer can be explicitly given, one also obtains an explicit pricing formula, namely the (normalized) gradient of his utility in his endowments.

Since individual excess demand functions satisfy WARP, existence of a representative consumer implies WARP for market excess demand. This is also true if market excess demand is monotone with respect to some normalizing vector. Therefore, the third approach proving uniqueness via monotonicity is linked to the second one.

The last approach is to work with the gross substitution property.

Figure 3.4 at the end of this chapter demonstrates among other things the links between the different approaches.

3.5.1 Existence of Explicit Pricing Formulas

No-Trade Equilibrium

Clearly, the simplest pricing formula is obtainable whenever a priori any trade is excluded. Imagine the situation where all consumers are identical in utility $\bar{U}$ as well as in endowments $\bar{\omega}$. Then, it is no question to state the pricing vector to be collinear to $\nabla\bar{U}(\bar{\omega})$. If the chosen normalization is $\hat{\mathcal{P}}$, the normalized equilibrium price system is

$$\pi^{*T} = \frac{1}{\partial_{x_1}\bar{U}(\bar{\omega})}\nabla\bar{U}(\bar{\omega}).$$

However, this constitutes only a special case of the situation where the endowment distribution is Pareto-efficient.

Proposition 3.12 (Uniqueness for Pareto-Efficient Endowments) *Given $GE = \{(U^i, \omega^i)_{i\in I}\}$ with $(\omega^i)_{i\in I}$ a Pareto-efficient allocation, the no-trade equilibrium is the unique equilibrium with equilibrium price system $\pi^* \in \hat{\mathcal{P}}$ collinear to $\nabla U^i(\omega^i)$ for all $i \in I$.*

The proof is very simple. Mas-Colell, Whinston, and Green (1995, Proposition 17.F.5) show that as soon as $(\omega^i)_{i\in I}$ constitutes an Arrow-Debreu equilibrium allocation, it has to be unique because otherwise the convex combination of this equilibrium with an alternative one would be an improvement. However, if $(\omega^i)_{i\in I}$ is Pareto-efficient it has to be an equilibrium because in an equilibrium each individual i behaves individually rational and no one would choose an alternative plan which worsens his situation in comparison to ω^i. The pricing formula follows from first order conditions for the maximization problem.

Complete Risk Sharing

Risk sharing is a fascinating idea which found many powerful applications in insurance markets. It finds its microeconomic foundation in different economic models. Most of them are based on the expected utility hypothesis (compare, for example, Borch (1962)).

A foundation of phenomena like risk sharing is only sound for a unique competitive equilibrium by which the allocation in the markets is explained. As it is well-known, uniqueness is satisfied for the case of no aggregate risk and complete markets.

A large part of the literature in risk sharing excludes aggregate risk (compare Malinvaud (1972), (1973) and Levine and Zame (1999)) where aggregate risk is defined in different ways. The definition applied here is the most common one (compare, for example, Magill and Quinzii (1996b)) which defines aggregate risk as fluctuations of aggregate endowments.

Proposition 3.13 (Uniqueness with Complete Risk Sharing)
Assume $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that all consumers satisfy the expected utility hypothesis with respect to an objective probability measure $\bar{\rho} \in \hat{\mathcal{P}}_{1,\mathbb{1}}$[1], i.e.

$$U^i(x) = \sum_{s=1}^{S} \bar{\rho}_s u^i(x_s)$$

for some $u^i : \mathbb{R}_+ \to \mathbb{R}$ infinitely often differentiable, strictly monotonic, and strictly concave with $\lim_{x_s \to 0}(u^i)'(x_s) = \infty$. If there is no aggregate risk, i.e.

$$\omega = \bar{\omega}\mathbb{1}$$

for some $\bar{\omega} \in \mathbb{R}_{++}$, the economy is characterized by a unique Arrow-Debreu equilibrium and normalizing price systems to the unit simplex leads to $\pi^* = \bar{\rho}$.

Economists, Cass and Shell (1983, p. 209) for example, ascribe this result to Malinvaud (1972), (1973) who — among other things — considers Pareto-efficient allocations in a similar framework. As soon as an allocation is risky, it could not be Pareto-efficient since averaging over states with weighting by the objective probability measure improves consumers because of strict concavity: each Pareto-efficient allocation has to be totally riskless. $\pi^* = \bar{\rho}$ follows directly from first order conditions for the maximization problems.

Recently the consequences of aggregate risk have become a central question in (financial) economics (compare Tesar (1993), Townsend (1995), Heaton and Lucas (1996), and Crès and Rossi (2000)). Therefore, it is interesting to know whether Proposition 3.13 also holds without the assumption of a riskless aggregate endowment, i.e. without $\omega = \bar{\omega}\mathbb{1}$ for some $\bar{\omega} \in \mathbb{R}_{++}$. In the following it is shown by means of an example that deviating to aggregate risk might lead to multiple equilibria. This is the same for giving up the assumption of complete markets, but this topic is placed in Section 4.4.3.

[1] $\hat{\mathcal{P}}_{1,\mathbb{1}}$ is the interior of the unit simplex.

Consider an Arrow-Debreu economy with $I = 2$ consumers and $S = 2$ states. The consumers are characterized by endowments

$$\omega^1 = \frac{4}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega^2 = \frac{2}{3}\begin{pmatrix} 1 \\ 4 \end{pmatrix}. \tag{3.9}$$

Obviously, *only* the second consumer is directly confronted with uncertainty, whereas uncertainty happens to the first consumer only indirectly via the other's endowment.

Both consumers profit in the *same* way by consumption of the good, i.e. there is a common utility function

$$u : \mathbb{R}_+ \to \mathbb{R} \tag{3.10}$$

with $u^i = u$ for $i = 1, 2$. Additionally to the standard assumptions, I restrict u via some values of its first derivative, namely (see Figure 3.1)

$$\begin{aligned} &u'(\tfrac{5}{6}) = 24, \quad u'(1) = 5, \quad u'(1\tfrac{1}{6}) = 4, \\ &u'(1\tfrac{1}{3}) = 3, \quad u'(2) = 2\tfrac{1}{2}, \quad u'(2\tfrac{2}{3}) = \tfrac{1}{2}. \end{aligned} \tag{3.11}$$

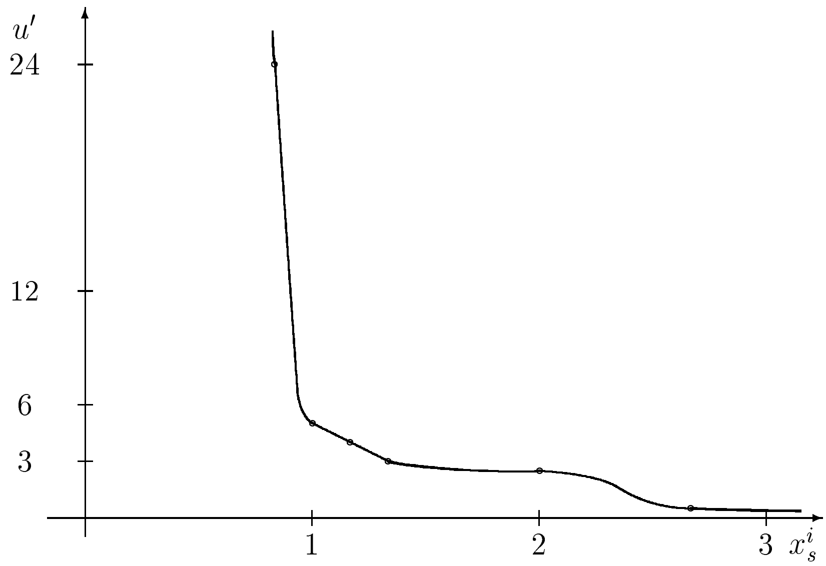

Figure 3.1: A possible form of u'.

Obviously, these values of the derivative are compatible with the Assumption 2.4 on $U^i, i = 1, 2$, and the implied requirements on u.

Both consumers take the two states of the world for equally likely to occur, i.e. both states have the same objective probability $\bar{\rho}_1 = \bar{\rho}_2 = 1/2$.

I claim that the economy defined in this way has at least the following two equilibria:

$$x^{*1} = x^{*2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \pi^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{3.12}$$

$$\hat{x}^1 = \frac{1}{6}\begin{pmatrix} 7 \\ 16 \end{pmatrix}, \hat{x}^{*2} = \frac{1}{6}\begin{pmatrix} 5 \\ 8 \end{pmatrix}, \hat{\pi} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}. \tag{3.13}$$

Market clearing is obvious. To verify the maximization condition, remember first order conditions for $i \in I$:

$$\begin{aligned} \bar{\rho}_1 u'(x_1^i) - \lambda^i \pi_1 &= 0, && (3.14) \\ \bar{\rho}_2 u'(x_2^i) - \lambda^i \pi_2 &= 0, && (3.15) \\ \pi \cdot (x^i - \omega^i) &= 0 && (3.16) \end{aligned}$$

where $\lambda^i > 0$ is the Lagrangian multiplier. Substitute the values of (3.9), (3.11)–(3.13) into (3.14)–(3.16) to obtain the following system of equations:

$$\begin{array}{llll} \frac{1}{2}\cdot 5 - \lambda^{*1}\cdot 2 = 0, & & \frac{1}{2}\cdot 5 - \lambda^{*2}\cdot 2 = 0, \\ \frac{1}{2}\cdot 2\frac{1}{2} - \lambda^{*1}\cdot 1 = 0, & & \frac{1}{2}\cdot 2\frac{1}{2} - \lambda^{*2}\cdot 1 = 0, \\ 2\cdot(-\frac{1}{3}) + 1\cdot\frac{2}{3} = 0, & & 2\cdot\frac{1}{3} + 1\cdot(-\frac{2}{3}) = 0, \\ \\ \frac{1}{2}\cdot 4 - \hat{\lambda}^1\cdot 8 = 0, & & \frac{1}{2}\cdot 24 - \hat{\lambda}^2\cdot 8 = 0, \\ \frac{1}{2}\cdot\frac{1}{2} - \hat{\lambda}^1\cdot 1 = 0, & & \frac{1}{2}\cdot 3 - \hat{\lambda}^2\cdot 1 = 0, \\ 8\cdot(-\frac{1}{6}) + 1\cdot 1\frac{1}{3} = 0, & & 8\cdot\frac{1}{6} + 1\cdot(-1\frac{1}{3}) = 0. \end{array} \tag{3.17}$$

This system is solved for $\lambda^{*1} = \lambda^{*2} = 1\frac{1}{4}$ and $\hat{\lambda}^1 = \frac{1}{4}$, $\hat{\lambda}^2 = 1\frac{1}{2}$. Therefore, the equilibrium properties are proved.

Note that the example is constructed in a way such that there has to be a change in the sign of the third derivative of u. The sign of the third derivative can be interpreted with respect to precautionary saving and prudence (cf. Kimball (1990)). The importance of these concepts in the context of non-insurable idiosyncratic risk was pointed out by Franke et al. (1992) and Weil (1992).

Nevertheless, by deviating from the assumption of identical preferences the example can be modified such that the third derivatives are strictly positive. All the data of the example remain the same, but (3.11) is substituted by (compare Figures 3.2 and 3.3)

$$\begin{array}{llll} (u^1)'(1\frac{1}{6}) = 4, & (u^1)'(1\frac{1}{5}) = 3\frac{1}{2}, & (u^1)'(2\frac{2}{3}) = \frac{1}{2}, & (u^1)'(3) = \frac{14}{50}, \\ (u^2)'(\frac{4}{5}) = 62\frac{1}{2}, & (u^2)'(\frac{5}{6}) = 24, & (u^2)'(1) = 5, & (u^2)'(1\frac{1}{3}) = 3. \end{array}$$

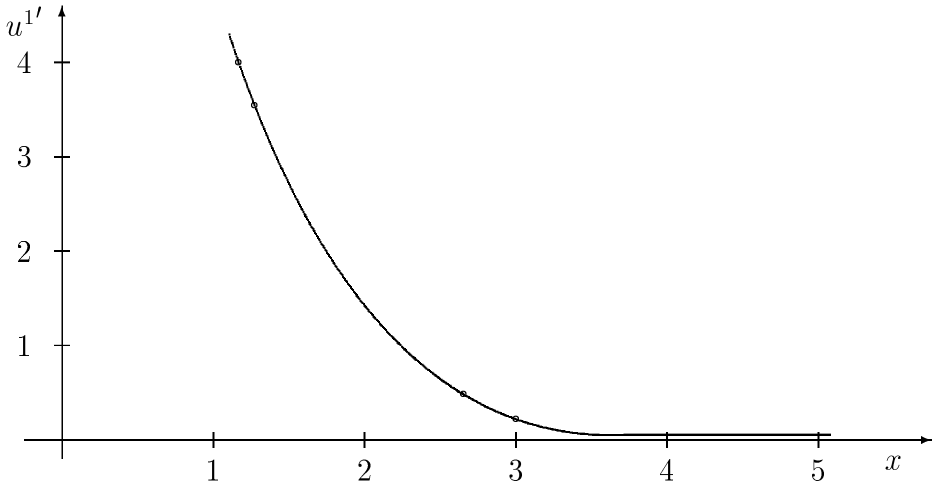

Figure 3.2: A possible form of $(u^1)'$.

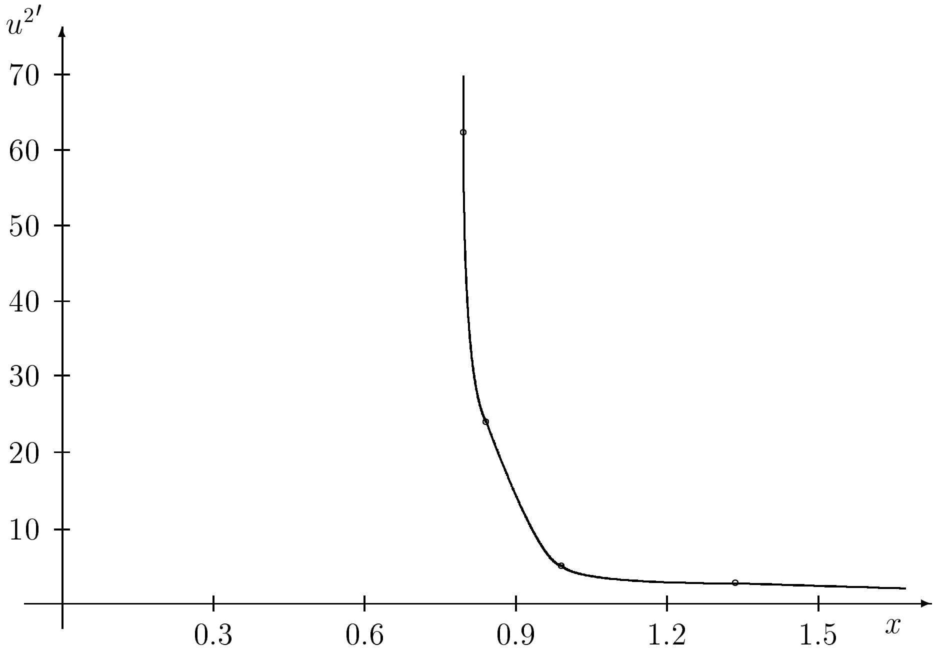

Figure 3.3: A possible form of $(u^2)'$.

The values are compatible with u^1, u^2 satisfying the standard assumptions and having a strictly positive third derivative, presumably even showing decreasing absolute risk aversion in the sense of Arrow and Pratt (Arrow (1970) and Pratt (1964)).

The equilibrium $(\hat{x}^1, \hat{x}^2, \hat{\pi})$ with the Lagrangian multipliers $\hat{\lambda}^1, \hat{\lambda}^2$ is preserved. A second equilibrium is given by

$$\bar{x}^1 = \frac{1}{5}\begin{pmatrix} 6 \\ 15 \end{pmatrix}, \bar{x}^2 = \frac{1}{5}\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \bar{\pi} = \frac{1}{2}\begin{pmatrix} 25 \\ 2 \end{pmatrix}$$

where $\bar{\lambda}^1 = \frac{7}{50}$ and $\bar{\lambda}^2 = 2\frac{1}{2}$.

Multiplicity of competitive equilibria when there is aggregate risk has recently also been demonstrated by Crès and Rossi (2000). Their model is directly based on Malinvaud (1973). Therefore, they allow that different states are realized for different consumers. Consumers do not differ in their characteristics (utility functions are not of the expected utility type), but only in the state realized for them. Besides, multiple commodities exist. Within this context according to Malinvaud (1972), (1973) no aggregate risk means that the proportion of consumers in each state is a priori fixed and endowments for each consumer only depend on his "own" state. Crès and Rossi (2000) however allow that endowments in one special commodity depend also on other consumers' state realizations, nevertheless they assume that this "collective component of risk" vanishes at the aggregate level. Within their framework they implicitly show the multiplicity of equilibria because they demonstrate the possible existence of asymmetric equilibria (in addition to a symmetric equilibrium).

My example can also be given an intertemporal interpretation. Imagine $s = 1$ respectively 2 realizes in the first respectively second period. Then, multiplicity of competitive equilibria can be interpreted as multiplicity of competitive interest rates - a result which can be traced back at least to Fisher (1907).[2]

It would be interesting to know which kind of assumptions may be sufficient to obtain a unique Arrow-Debreu equilibrium even for aggregate risk. One example, as Hens and Löffler (1995) point out, is the assumption of a relative risk aversion less than one (Proposition 3.19). Similarly, new results might be found using the notion of "risk vulnerability" (compare Gollier and Pratt (1996)) or Ross' (1981) conditions on decreasing risk aversion.

[2]I am grateful to John Geanakoplos for pointing out this historical result.

Quadratic Utility

If utility functions do not only satisfy the expected utility hypothesis with respect to an objective probability measure, but even are quadratic, then there is a unique Arrow-Debreu equilibrium despite aggregate risk.

Proposition 3.14 (Uniqueness with Quadratic Utility)
Assume $GE = \{(U^i, \omega^i)_{i \in I}\}$ *such that all consumers satisfy the expected utility hypothesis with respect to an objective probability measure* $\bar{\rho} \in \hat{\mathcal{P}}_{1,\mathbb{I}}$ *and state utility is quadratic, i.e.*

$$U^i(x) = \sum_{s=1}^{S} \bar{\rho}_s (x_s - \frac{1}{2}\gamma^i x_s^2). \tag{3.18}$$

If $\gamma^i \in \left(0, \min_{s \in S} \frac{1}{\omega_s}\right)$*, then the economy is characterized by a unique interior Arrow-Debreu equilibrium and normalizing the first state price to one leads to*

$$\pi^{*T} = \left(1, \frac{\bar{\rho}_2 \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_2\right)}{\bar{\rho}_1 \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_1\right)}, \dots, \frac{\bar{\rho}_S \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_S\right)}{\bar{\rho}_1 \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_1\right)}\right).$$

Note that $x_s - \frac{1}{2}\gamma^i x_s^2$ in (3.18) is a HARA-function which can be easily transferred to the form (2.1).

The assumptions on $(\gamma^i)_{i \in I}$ and ω together with the form of $(U^i)_{i \in I}$ assure strict monotonicity of utility. Strict concavity therefore guarantees the first order conditions to be necessary for utility maximization whenever consumption plans are in the interior of the consumption set. The idea of the proof is to apply individual first order conditions. Normalizing $\pi_1^* = 1$ allows to eliminate the Lagrangian multipliers from these equations. Division by γ^i and summing up over all individuals leads to the pricing formula given in the proposition because one can apply market clearing.

Note that equilibrium prices are totally independent of the distribution of income and risk aversion among consumers. Besides without aggregate risk the equilibrium pricing formula is the same as in the last Proposition 3.13 (except for normalization).

On Cobb-Douglas Utility

Another class of utility functions satisfying the expected utility hypothesis (and even exhibiting hyperbolic absolute risk aversion) is the class of Cobb-Douglas utility functions. In the framework of the Arrow-Debreu model they are widely used because they enable explicit computation of demand functions and equilibrium prices without any further restrictions on probability measures and endowments.

Proposition 3.15 (Uniqueness with Cobb-Douglas Utility)
Assume $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that all consumers satisfy the expected utility hypothesis and state utility is logarithmic, i.e.

$$U^i(x) = \sum_{s=1}^{S} \rho_s^i \ln x_s. \tag{3.19}$$

Then the economy is characterized by a unique Arrow-Debreu equilibrium and normalizing the first state price to one leads to

$$\pi^*_{-1} = -\left[\begin{pmatrix} \sum_{i=1}^I \rho_2^i \omega_{-1}^{iT} \\ \vdots \\ \sum_{i=1}^I \rho_S^i \omega_{-1}^{iT} \end{pmatrix} - \begin{pmatrix} \omega_2 & & \\ & \ddots & \\ & & \omega_S \end{pmatrix}\right]^{-1} \begin{pmatrix} \sum_{i=1}^I \rho_2^i \omega_1^i \\ \vdots \\ \sum_{i=1}^I \rho_S^i \omega_1^i \end{pmatrix}.$$

Proof: The properties of $(U^i)_{i \in I}$ assure first order conditions to be necessary and sufficient for utility maximization under budget constraints. Simple calculations lead to demand functions

$$x_s^i(\pi) = \rho_s^i \frac{\pi \cdot \omega^i}{\pi_s} \text{ for all } i \in I, s \in S.$$

From market clearing one therefore obtains the system of linear equations

$$\sum_{i=1}^{I} \rho_s^i (\pi \cdot \omega^i) - \sum_{i=1}^{I} \pi_s \omega_s^i \quad = \quad 0 \text{ for all } s \in S,$$

which can be rewritten as

$$\left[\begin{pmatrix} \sum_{i=1}^I \rho_1^i \omega^{iT} \\ \vdots \\ \sum_{i=1}^I \rho_S^i \omega^{iT} \end{pmatrix} - \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_S \end{pmatrix}\right] \pi \quad = \quad 0. \tag{3.20}$$

From Theorem 2.22 there has to be some strictly positive solution π^*. Since the entries of the first matrix in (3.20) are strictly positive and the second matrix is a diagonal one with strictly positive diagonal entries, $[\ldots]$ is a strict gross substitute matrix, i.e. has strictly positive off-diagonal and strictly negative diagonal entries. Therefore, eliminating the first column and the first row from [...] results in a matrix which has a row dominant diagonal[3] because (3.20) implies

$$\begin{pmatrix} \sum_{i=1}^{I} \rho_2^i(0, \omega_3^i, \ldots, \omega_S^i) \\ \vdots \\ \sum_{i=1}^{I} \rho_S^i(\omega_2^i, \ldots, \omega_{S-1}^i, 0) \end{pmatrix} \pi_{-1}^*$$
$$= \begin{pmatrix} \pi_2^*(\omega_2 - \sum_{i=1}^{I} \rho_2^i \omega_2^i) \\ \vdots \\ \pi_S^*(\omega_S - \sum_{i=1}^{I} \rho_S^i \omega_S^i) \end{pmatrix} - \pi_1^* \begin{pmatrix} \sum_{i=1}^{I} \rho_2^i \omega_1^i \\ \vdots \\ \sum_{i=1}^{I} \rho_S^i \omega_1^i \end{pmatrix}$$
$$< \begin{pmatrix} \pi_2^*(\omega_2 - \sum_{i=1}^{I} \rho_2^i \omega_2^i) \\ \vdots \\ \pi_S^*(\omega_S - \sum_{i=1}^{I} \rho_S^i \omega_S^i) \end{pmatrix} = \begin{pmatrix} \pi_2^*|\omega_2 - \sum_{i=1}^{I} \rho_2^i \omega_2^i| \\ \vdots \\ \pi_S^*|\omega_S - \sum_{i=1}^{I} \rho_S^i \omega_S^i| \end{pmatrix}.$$

A matrix with a row dominant diagonal is invertible because obviously its transposed has a column dominant diagonal. Following Murata (1977, Theorem 21 and Theorem 24) this transposed is invertible and therefore also the original matrix. Because of Walras' Law the first line of (3.20) is redundant. Therefore, normalizing $\pi_1^* = 1$, it is equivalent to

$$\left[\begin{pmatrix} \sum_{i=1}^{I} \rho_2^i \omega_{-1}^{iT} \\ \vdots \\ \sum_{i=1}^{I} \rho_S^i \omega_{-1}^{iT} \end{pmatrix} - \begin{pmatrix} \omega_2 & & \\ & \ddots & \\ & & \omega_S \end{pmatrix} \right] \pi_{-1}^* \;=\; - \begin{pmatrix} \sum_{i=1}^{I} \rho_2^i \omega_1^i \\ \vdots \\ \sum_{i=1}^{I} \rho_S^i \omega_1^i \end{pmatrix}$$

or

$$\pi_{-1}^* = - \left[\begin{pmatrix} \sum_{i=1}^{I} \rho_2^i \omega_{-1}^{iT} \\ \vdots \\ \sum_{i=1}^{I} \rho_S^i \omega_{-1}^{iT} \end{pmatrix} - \begin{pmatrix} \omega_2 & & \\ & \ddots & \\ & & \omega_S \end{pmatrix} \right]^{-1} \begin{pmatrix} \sum_{i=1}^{I} \rho_2^i \omega_1^i \\ \vdots \\ \sum_{i=1}^{I} \rho_S^i \omega_1^i \end{pmatrix}.$$

□

If one wants to obtain uniqueness and is not interested in an explicit pricing formula, one can stop with equation (3.20): from Theorem 2.22 one knows the existence of a solution for this equation; from Theorem 3.6 one knows generic local uniqueness of equilibria after normalization. Therefore, because the set of solutions of (3.20) is a linear subspace of $\mathbb{R}^S$ and corresponds to the non-normalized equilibrium price set, generically it has to be of dimension one which is equivalent to a unique (normalized) equilibrium price system.

[3]For a definition compare Appendix A.2.1.

Note that Cobb-Douglas utility functions evoke the gross substitution property of individual and therefore market demand such that also Proposition 3.11 can be applied. Besides the functions are homothetic such that under further restrictions uniqueness can be obtained with the Proposition 3.16 in the next section.

3.5.2 On the Existence of a Representative Consumer

The existence of a representative consumer is extremely comfortable because it allows to use his preferences as a measure of social welfare (compare Mas-Colell, Whinston, and Green (1995)). A thorough discussion of the question when market demand is a consumer demand can be found in Shafer and Sonnenschein (1982) who refer for this topic to their predecessors Antonelli (1886), Gorman (1953), Nataf (1953), Eisenberg (1961), Chipman (1974), and Chipman and Moore (1976). To summarize those results important for my interest on uniqueness of Arrow-Debreu equilibria: as soon as consumers preferences are either identical homothetic or homothetic and endowments are collinear[4], there exists a representative consumer.

Therefore, since demand of a single (here the representative) consumer satisfies WARP, uniqueness of the Arrow-Debreu equilibrium is obtained.

Proposition 3.16 (Uniqueness for Homotheticity)
Suppose $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that all $U^i, i = 1, \ldots, I$, are homothetic, then there is a unique Arrow-Debreu equilibrium whenever either all U^i are identical or the endowments $(\omega^i)_{i \in I}$ are collinear, i.e. there is some vector $\delta \in \hat{\mathcal{P}}_{1,\mathbb{I}}$ with $\omega^i = \delta_i \omega$ for all $i \in I$.

The idea of the proofs for the existence of a representative consumer are simple (compare Shafer and Sonnenschein (1982, Theorem 2 and Theorem 4)). For identical homothetic utility functions one applies that all consumers are characterized by the same demand function which is linear in income such that a representative consumer is described by the given utility function and his endowment is the aggregate endowment, i.e. his characteristics are independent of the income distribution. For homothetic preferences and collinear endowments one can prove that a representative consumer is given by utility

$$U(x) = \max_{x \in \mathbb{R}^S_+} \prod_{i \in I} \left(U^i(x^i)\right)^{\delta^i} \text{ s.t. } \sum_{i \in I} x^i = x$$

[4]Collinear endowments imply a price independent distribution of relative wealth in an economy.

and his endowments are the aggregate endowments. The main point of the proof is to show that a consumer with these characteristics chooses the $x^i, i = 1, \ldots, I$, in a way such that they not only satisfy his own budget restrictions, but also those of the single consumers.

Regrettably, it is not clear which are necessary or sufficient conditions for the existence of a representative consumer. From the results of Sonnenschein (1973), Debreu (1974), and Mantel (1974) it is known for a long time that in general market excess demand cannot be represented by a single consumer's excess demand. Mantel (1976) proves that even restricting market excess demand to come from individuals with homothetic preferences (which are not identical) is not sufficient. Kirman and Koch (1986) on the other hand show that also for an economy in which consumers are characterized by identical utility functions and collinear endowments there does not necessarily exist a representative consumer. In other words: neither identical preferences combined with collinear endowments nor homotheticity alone are sufficient to obtain the Weak Axiom of Revealed Preferences to be satisfied by market excess demand.

However, Shafer (1977, p. 1178) points out that homotheticity is not necessary under the assumption of two states where WARP and the so-called Strong Axiom of Revealed Preferences (SARP) are equivalent. For $S \geq 3$ compare Shafer and Sonnenschein (1982, Example 5).

3.5.3 How to Obtain Monotonicity

Quasi-Linear Utility

There is a further class of utility functions which assures a unique Arrow-Debreu equilibrium, namely the class of quasi-linear utility functions. Let me emphasize that I defined quasi-linearity with respect to the first state. As soon as one deviates from this such that linearity is given with respect to different states for different consumers, the uniqueness property might fail as follows from Mas-Colell, Whinston, and Green (1995, Example 15.B.2).

Proposition 3.17 (Uniqueness for Quasi-Linearity)
Assume $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that U^i is quasi-linear for all $i \in I$, i.e. there exist some strictly concave functions $\tilde{U}^i : \mathbb{R}_+^{S-1} \to \mathbb{R}, i = 1, \ldots, I$, such that

$$U^i(x) = x_1 + \tilde{U}^i(x_{-1})$$

and all $\tilde{U}^i$ satisfy Assumption 2.4 on $\mathbb{R}_+^{S-1}$. Then, the economy displays a unique Arrow-Debreu equilibrium if endowments in state one are sufficiently large for all consumers.

The idea of the proof is to show monotonicity of individual demand functions with respect to the normalizing vector e^1. Because the normalizing vector is the same for all consumers, monotonicity is additive and Proposition 3.9 can be applied.

Monotonicity for individual demand can be deferred in the following way: solving budget constraints to state 1 consumption x_1^i, it is possible to eliminate x_1^i from the maximization problem. By application of the Implicit Function Theorem on the corresponding first order conditions one obtains the Jacobian $D_\pi z_{-1}(\pi)$. One observes negative definiteness of $\partial_{\pi_{-1}} z_{-1}(\pi)$ because of strict concavity of $\tilde{U}^i$. First order conditions however are necessary and sufficient: when maximizing without any restrictions on state 1 consumption, the assumptions on $\tilde{U}^i$ and ω^i imply necessity and sufficiency. It can be shown that given large ω_1^i state 1 consumption is always chosen strictly positive such that it does not matter if it is restricted from below by zero.

Quasi-linear utility functions gain special interest for partial equilibrium analysis (compare, for example, Mas-Colell, Whinston, and Green (1995, Chapter 10)). In this context it may be reasonable not to restrict consumption in the state of linearity.

The Mitjushin-Polterovich Coefficient

Given a strictly concave utility function U^i, the Mitjushin-Polterovich coefficient is a measure for the curvature of the indifference curves:

$$MP_{U^i}(x) := -\frac{x^T D^2 U^i(x) x}{\nabla U^i(x) x} \text{ for all } x \in \mathbb{R}^S_{++} \backslash \{0\}.$$

If U^i in addition satisfies the expected utility hypothesis, an upper bound for the Mitjushin-Polterovich coefficient is equivalent to an upper bound for the relative risk aversion (compare Mas-Colell (1991, p. 283)). If U^i is homogeneous of degree k with $k \in (0,1)$, its corresponding Mitjushin-Polterovich coefficient is $1-k$.

Given a strictly concave utility function with a Mitjushin-Polterovich coefficient less than 4, individual demand is monotone with respect to the endowment vector. The original proof is given by Mitjushin and Polterovich (1978), but can be also found in Mas-Colell (1991, Theorem 1) and Hildenbrand (1994, Appendix 4). The main idea is to prove monotonicity of the inverse demand function and not of demand itself. Note that the original proof considers consumers with a fixed income and not with income depending on prices.

Proposition 3.18 (Uniqueness via Mitjushin-Polterovich)
Given $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that all U^i are strictly concave and

$$MP_{U^i}(x) < 4 \text{ for all } x \in \mathbb{R}^S_+ \backslash \{0\}, i \in I,$$

the economy has a unique Arrow-Debreu equilibrium if endowments are collinear, i.e. there is some $\delta \in \hat{\mathcal{P}}_{1,\mathbb{I}}$ with $\omega^i = \delta_i \omega$ for all $i \in I$.

The monotonicity property is additive since the normalizing vectors $(\omega^i)_{i \in I}$ are collinear. Uniqueness follows from Proposition 3.9.

Mas-Colell (1991, Example 1) proves 4 to be the "magic" number in the proposition. Kannai (1989) points out that it is sufficient to choose the representation of preferences which is the least concave.

3.5.4 How to Obtain the Property of Gross Substitution

Another class of Arrow-Debreu economies allows to obtain uniqueness of the equilibrium via gross substitution of market demand. Once again, utility functions which fulfill the expected utility hypothesis are considered.

Hens and Löffler (1995) prove that market demand satisfies the property of strict gross substitution whenever consumers have a constant absolute risk aversion, complete risk sharing in the equilibrium is given (compare p. 50), or the relative risk aversion is less or equal to one. In the first two situations the property of strict gross substitution is directly shown for market demand whereas in the latter case it is demonstrated via individual demand to satisfy this property. Both proofs strongly refer to the Jacobian of the Lagrangian multipliers as functions of prices. The second one even applies strict gross substitution for $\lambda^i(\pi), i \in I$.

Proposition 3.19 (Uniqueness from Gross Substitution)
Given $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that utility functions $(U^i)_{i \in I}$ satisfy the expected utility hypothesis, there is a unique Arrow-Debreu equilibrium if one of the following conditions is satisfied:

1. *consumers have constant absolute risk aversion, i.e. for all $i \in I$ it is*

   $$ARA^i(x_s) = c^i \text{ for all } x_s \in \mathbb{R}_{++}$$

   where $c^i > 0$;

2. *complete risk sharing holds in the equilibrium;*

3. *consumers have a relative risk aversion less or equal to one, i.e. for all $i \in I$ it is*

   $$RRA^i(x_s) \leq 1 \text{ for all } x_s \in \mathbb{R}_{++}.$$

Hens and Löffler (1995) show that the last condition can be relaxed to

$$(x_s - \omega_s^i) ARA^i(x_s) < 1 \text{ for all } i \in I, x_s \in \mathbb{R}_{++} \tag{3.21}$$

such that the size of endowments contributes to the determinacy of the equilibrium. Besides, it is sufficient if utility functions are additively separable and condition (3.21) is satisfied for each state utility function.

Note that when complete risk sharing is given in the equilibrium, Hens and Löffler (1995) only prove the property of strict gross substitution to hold in equilibrium. Even though gross substitution does not hold in general, this is sufficient for a unique equilibrium (compare Mas-Colell (1985, Example 5.7.9)).

3.6 Overview of Conditions for Uniqueness in the Arrow-Debreu Model

Figure 3.4 at the end of this section provides an overview of well-known conditions which guarantee a unique equilibrium in an Arrow-Debreu economy or equivalently in a GEI-economy with complete markets. It has to be read as described below:

- - ->: Following a broken arrow starting at the top, one crosses different conditions which all together guarantee either a special property for market demand (demand can be interpreted as the demand of a representative consumer, monotonicity with respect to a normalizing vector, gross substitution) or an explicit equilibrium pricing formula (which depends on the parameters of the economy) such that the equilibrium is unique.

Example 3.20 *Looking on the right broken arrow one recognizes that as soon as all consumers are characterized by identical utility functions* ***and*** *endowments there exists a representative consumer* ***and*** *an explicit pricing formula. For such an economy the equilibrium price system is unique.*

——>: Non-broken arrows indicate implications which hold in general. They allow to obtain additional conclusions. Some of these arrows are labeled with supplementary information, meaning that the implication holds only under this additional restriction.

Example 3.21 *Cobb-Douglas utility functions are homothetic. Besides for these functions relative risk aversion is defined and (less or) equal to one.*

This in addition implies strict concavity and a Mitjushin-Polterovich coefficient less than four.

Since Cobb-Douglas utility functions are homothetic supplementary to the existence of a pricing formula one knows that a representative consumer exists as soon as the utility functions are identical across all consumers or endowments are collinear. Under collinear endowments monotonicity can also be derived from the size of the Mitjushin-Polterovich coefficient.

Naturally, Figure 3.4 cannot contain *each* implication and conclusion and I do not claim completeness. For example, grouping all U^i to be identical with all ω^i to be identical would imply Pareto-efficiency of the endowment distribution. Besides, utility classes overlap, for example, some homothetic utility functions satisfy the expected utility hypothesis. Including all these details would divert from the essential.

Nevertheless, Figure 3.4 visualizes a great part of the content of this chapter and might be helpful when examining uniqueness of equilibria. Note that I did not include negative results like the fact that with aggregate risk utility functions which satisfy the expected utility hypothesis with respect to an objective probability measure do *not* guarantee a unique equilibrium. There would not even be a chance of partial completeness!

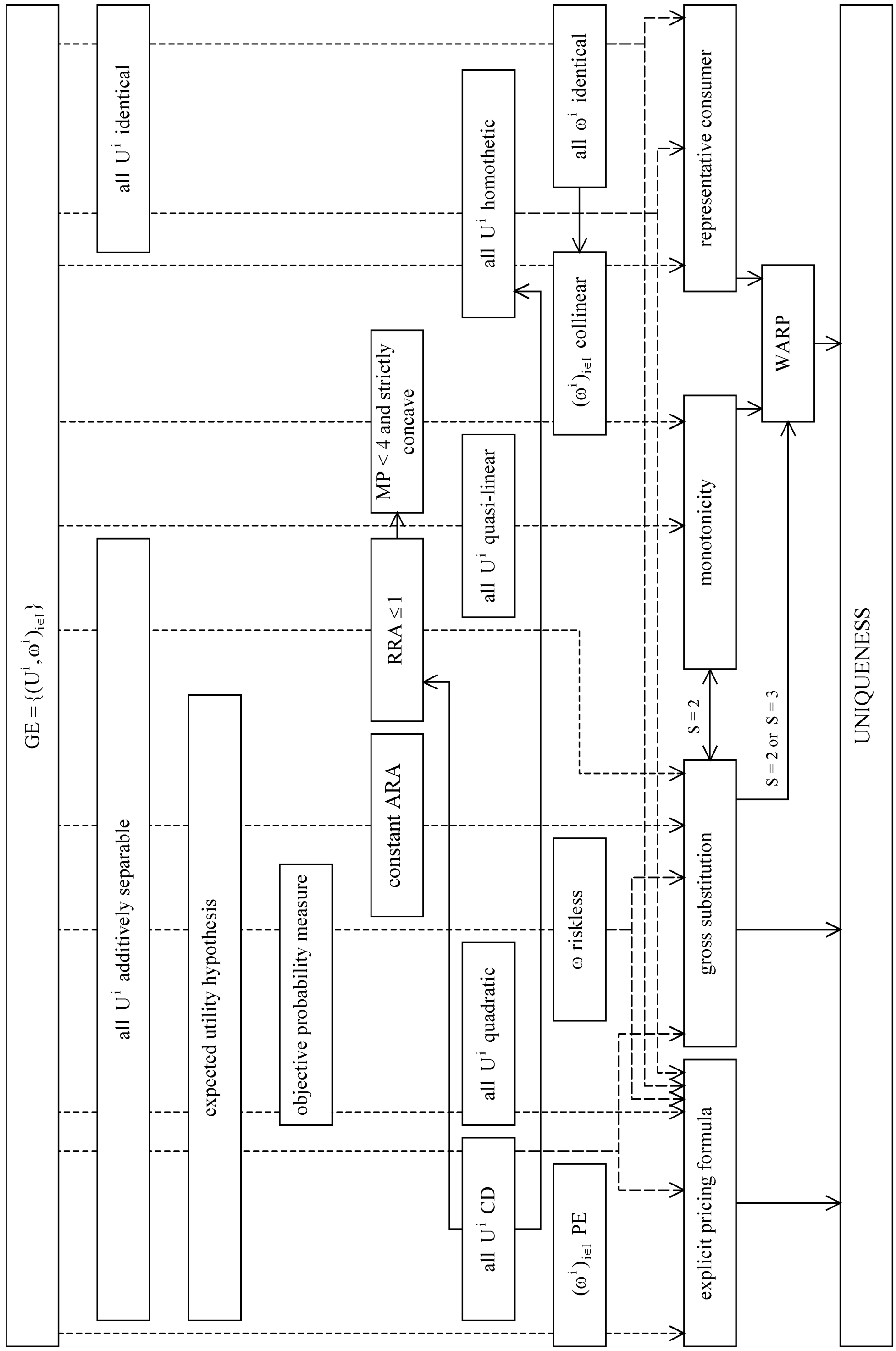

Figure 3.4: Conditions for uniqueness in the Arrow-Debreu model.

Chapter 4

On Uniqueness of Competitive Equilibria in the Finance GEI-Model

4.1 Introduction

As already mentioned before, not only for the Arrow-Debreu model itself, but also for the more general model with incomplete markets it is an important topic to examine uniqueness of the equilibrium. Because the Arrow-Debreu model constitutes a special case of the incomplete markets model, one presumes close relationships between conditions generating a unique equilibrium in both models. However, until now it is not totally clarified what the parallels and differences in the results are. In this chapter I drive on research in this field.

Whereas Arrow-Debreu theory studies the interactions on perfectly competitive commodity markets and the pricing of commodities, general equilibrium theory with incomplete markets (GEI-theory) includes the interactions of perfectly competitive asset markets and asset pricing into its analysis. The main point in GEI-theory is the "division of roles between spot markets for allocating real goods and financial markets for redistributing income and sharing risks" (Magill and Shafer (1991, p. 1576)). Markets are complete if "every agent is able to exchange every good either directly or indirectly with every other agent" (Wilson (1987, p. 759)), otherwise they are incomplete.

General equilibrium theory with incomplete (asset) markets constitutes a framework for the microeconomic analysis of several fields. Take, for example, "financial economics [which] is fundamentally concerned with the pricing of securities, and [...] macroeconomics [which] is fundamentally concerned with the real effects of monetary assets. [...] The flexibility of allowing any asset structure permits the GEI model to incorporate as special cases many of the macroeconomic and financial models that seemed to be orthogonal to the microeconomic tradition as expressed in the Arrow-Debreu model" (Geanakoplos (1990, pp. 1/2)). Magill and Shafer (1991, p. 1524) state that "a microeconomic foundation for macroeconomics may ultimately come from a more concrete version of general equilibrium theory [than the Arrow-Debreu model] in which there is trading on real and financial markets, where nominal contracts and money enter in an essential way, but where the ability to trade into the future is limited by the incompleteness of markets and by the willingness or inability of agents to make more than limited commitments into the future".

Similarly to the Arrow-Debreu model, the general equilibrium model with incomplete markets stands out due to its precise axiomatic foundation. It retains the basic assumptions of optimizing agents, rational expectations, independent taking of decisions, and clearing of perfectly competitive markets. But it allows for missing markets and multiple budget constraints.

The analysis of GEI-models evoked new mathematical tools of which many have to be categorized to differential topology. To name a key word think of 'Grassmanian manifolds'.

Seminal works on asset economies are given by Arrow (1953) and Diamond (1967), even though Geanakoplos (1990, p. 5) states that "the archetype GEI model, with multiple commodities and incomplete asset markets, was first formulated by Radner (1972)." He formally introduced the concept of 'common expectations' which has been implicitly applied by Arrow (1953) already: agents "associate the same (future) prices to the same events" (Radner (1972, p. 289), even though the subjective probabilities for future events might differ. Radner (1972) gives a first proof for the existence of such equilibria, but his equilibrium concept includes short sales constraints on agents' portfolio holdings. When considering an economy with production, he brings up the question of how to define firms' objective functions since in general in GEI-models profit maximization is not unambiguously defined.

Drèze (1974) and Grossman and Hart (1979) offer two different criteria to tackle this problem. Whereas in Drèze's (1974) concept firms take care of the interest of their final shareholders, Grossman and Hart (1979) propose to look on the interest of the original shareholders. Both of them allow transfer payments.

The existence problem is also considered by Hart (1975) who gives an example for non-existence of GEI-equilibria in a multigood model where asset payoffs depend on spot prices. The example is built on the fact that without short sales constraints changes of the spot prices might cause changes in the rank of the payoff matrix which leads to discontinuous demand functions. Bounds on short sales are identified to guarantee the existence of equilibria which however depend on these bounds.

Literature overcomes the problem by two solutions. The first branch of literature restricts attention to those asset structures which cannot change their rank. This is obtained by requiring payoffs to be either in units of account (compare Cass (1984), Werner (1985), Duffie (1987)) or in a numeraire commodity (Geanakoplos and Polemarchakis (1986)). These results suggested "that the GEI model could be put on the same rigorous footing as the Arrow-Debreu model" (Geanakoplos (1990, p. 12)).

The second branch of literature considers general asset structures, but observes discontinuities to be exceptional and therefore proves generic existence of GEI-equilibria (Duffie and Shafer (1985), (1986), Geanakoplos and Shafer (1990), Hirsch, Magill, and Mas-Colell (1990), and Husseini, Lasry, and Magill (1990)). These papers use the mathematical concept of Grassmanian manifolds to circumvent the problems arising form the change of rank of the payoff matrix. The approach is to prove by extensive use of differential topology the existence of so-called pseudo-equilibria which almost always coincide with the GEI-equilibria. The basic idea of the auxiliary concept of pseudo-equilibria is to extend the notion of (no-arbitrage) equilibria by a further variable, namely the space of income transfers where redistribution of income can take place within the so-defined possibilities and to require the asset span to be a subspace of the space of income transfers.

Note that both branches of literature apply the concept of no-arbitrage which is the basic concept of finance.

Aside to the problem of existence of equilibria, it is natural to ask for their efficiency. It suggests itself that GEI-equilibria are in general not Pareto-efficient: a central planner who can redistribute income arbitrarily in general will choose allocations which are not obtainable by trading in the existent markets. At first glance, it seems to be sufficient to adapt the definitions of Pareto-efficiency to constrained Pareto-efficiency where the central planner is restricted to those allocations obtainable via market interactions. But Hart (1975) gives a counterexample of a GEI-economy with multiple commodities where one GEI-equilibrium Pareto-dominates another. Under special restrictions on the number of consumers in comparison to the number of commodities and states, Geanakoplos and Polemarchakis (1986) prove that generically in endowments and utilities a central planner who is allowed to intervene in the first period and induce agents to hold non-maximizing portfolios can give rise to an improvement in comparison to the equilibrium allocation. The intuition is given by the fact that redistribution of assets not only causes a redistribution of wealth, but also a change of relative prices. Mas-Colell (1987) proves that the restriction on the number of consumers is not necessary when allowing transfers between agents in the first period. Geanakoplos et al. (1990) prove generically in endowments constrained Pareto-inefficiency for production economies.

A central observation in general equilibrium theory with incomplete markets is the necessity of distinction between real assets on the one hand and nominal ones on the other. When assets payoff in a numeraire, Geanakoplos and Polemarchakis (1986) prove equilibria to be locally unique such that there is no real indeterminacy. With nominal assets one obtains that generically in endowments there is real indeterminacy (Geanakoplos and Mas-Colell (1989), Balasko and Cass (1989)) which leads to the introduction of money into the model.

In this thesis I restrict attention to the Finance GEI-model, i.e. the GEI-model with the additional restrictions of

- a pure exchange economy,
- two periods,
- one commodity, and
- assets which pay of in the underlying commodity.

Even though this is quite restrictive since several of the above-mentioned results cannot be observed in this framework, it is nevertheless general enough to get basic insights without unnecessarily complicating the analysis. As Magill and Shafer (1991, p. 1607) state, "the one good model can be viewed as the basic equilibrium model of finance." Whereas GEI-theory mainly analyzes the "qualitative properties [...] of equilibrium allocations [...] qualitative properties of the asset prices [are] the domain of finance" (Magill and Shafer (1991, p. 1607). The no-arbitrage principle strongly relates both kinds of analysis.

The study of uniqueness in the Finance GEI-model done in this chapter can be divided into several parts.

The first aim of this chapter is to describe the latest developments in studies on individual conditions guaranteeing a unique equilibrium. Therefore, I start with a thorough definition of local and global uniqueness for the different equilibrium concepts in the GEI-model and some basic results. In correspondence to Chapter 3, I recall some properties of market demand which lead to a unique equilibrium. A new result here is the fact that monotonicity is satisfied independently of the chosen representation of the asset span. Only the considered vector of normalization changes. The presented properties of demand functions have been applied to find conditions on the individual characteristics and asset structures for which equilibria are unique (Subsection 4.2.4). The presentation of the properties of demand functions and their application on uniqueness results in GEI-models is close to the presentation in Hens, Schmedders, and Voß (1999), even though I choose a classification of the uniqueness results which in some parts is different from theirs and I add some points. In the end I give a diagram visualizing all these results which is helpful when checking for a GEI-economy whether it has a unique equilibrium. During this whole section I stay in analogy to the last chapter.

In Section 4.3 I clarify in how far — when the dimension of the asset span remains fixed — it is possible to perturb the asset structure without changing the number of equilibria. This observation is important with regard to the second section because it implies that conditions given there are too restrictive. It also lays the foundations for Sections 4.4 and 4.5. On the one hand I consider the limits of transferability from the Arrow-Debreu model to incomplete markets models: switching from complete to incomplete markets might lead from a unique to multiple equilibria. On the other hand the reverse is also true: incompleteness of markets might create uniqueness of the equilibrium. In the light of Section 4.3, most of the examples used for proofs in Sections 4.4 and 4.5 are known to be not exceptional. These observations on non-equivalence of uniqueness of competitive equilibria in complete and in incomplete markets models give some insight in the consequences of different asset structures on the equilibria which might motivate the reader to think of further extensions of Section 4.2.

4.2 How to transfer the Results on Uniqueness from Complete to Incomplete Markets

4.2.1 Uniqueness of FM-Equilibria

The first point is to start with a definition of a unique competitive equilibrium for GEI-economies. Therefore, it is possible to work with the financial markets or no-arbitrage equilibrium concept. It will become clear that it is more convenient to refer to the first one since the definition there can be done with respect to the price systems.

Obviously, as becomes clear from (2.5) and (2.7)/(2.8) individual and market demand functions are homogeneous of degree zero in asset prices. This means — in correspondence to the Arrow-Debreu model — that as soon as $q^* \in \mathcal{Q}(A)$ constitutes an equilibrium price system, this is also true for λq^*

whenever $\lambda > 0$. An appropriate price normalization has to be defined. This was simple for the Arrow-Debreu model since prices there are always strictly positive. For incomplete markets one has to assure that a normalizing vector $n \in \mathbb{R}^J$ satisfies

$$q \cdot n > 0 \text{ for all } q \in \mathcal{Q}(A), \tag{4.1}$$

i.e. the normalizing vector has to include an acute angle with each arbitrage-free asset price system q. Define

$$\hat{\mathcal{Q}}_{c,n}(A) = \{q \in \mathcal{Q}(A)|\ q \cdot n = c\}$$

where $c > 0$ and $\hat{\mathcal{Q}}(A) := \hat{\mathcal{Q}}_{1,e^1}(A)$ whenever $A_{\cdot 1} > 0$.

In correspondence to the Arrow-Debreu model local and global uniqueness have to be distinguished.

Definition 4.1 (Local Uniqueness of FM-Equilibria)
Given $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$, an FM-equilibrium $(\theta^*, q^*) \in \mathbb{R}^{J\times I} \times \hat{\mathcal{Q}}_{c,n}(A)$ is locally unique if there is some $\epsilon > 0$ such that $\|q - q^*\| < \epsilon$ and $q \in \hat{\mathcal{Q}}_{c,n}(A)$ imply $\theta(q) \neq 0$.

$\|.\|$ is an arbitrary vector norm on $\mathbb{R}^J$.

Definition 4.2 (Global Uniqueness of FM-Equilibria)
Given $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$, an FM-equilibrium $(\theta^*, q^*) \in \mathbb{R}^{J\times I} \times \hat{\mathcal{Q}}_{c,n}(A)$ is globally unique if there is no other $q \in \hat{\mathcal{Q}}_{c,n}(A)$ with $\theta(q) = 0$.

Again, the term "uniqueness" alone refers to the global aspect.

The number of FM-equilibria can be defined via normalized equilibrium price systems, i.e. for given $c \in \mathbb{R}_{++}$ and $n \in \mathbb{R}^J$ satisfying condition (4.1) the number of FM-equilibria is

$$\sharp\{q \in \hat{\mathcal{Q}}_{c,n}(A)|\ \theta(q) = 0\}.$$

Because of the allocational equivalence given in Theorem 2.19 the careful reader becomes aware of two further aspects.

Proposition 4.3 (Independence of the Representation)
Consider two economies $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ and $GEI = \{(U^i, \omega^i)_{i\in I}, \tilde{A}\}$ with $< A >=< \tilde{A} >$. Then, both economies are characterized by the same number of equilibria.

The importance of the proposition comes from the fact that it allows to choose always that representation of the asset span which is most suitable. For example, under Assumption 2.14 it is possible to use a representation $\tilde{A} \in \mathbb{R}^{S \times J}$ with $< A >=< \tilde{A} >$ and $\tilde{A}_{\cdot 1} > 0$ which allows application of the normalizing vector e^1 (compare condition (4.1)). Proposition 4.3 follows directly from the equivalence of uniqueness between FM- and NA-equilibria such that not only the number of equilibria is the same, but even the consumption allocations at the equilibria.

Proposition 4.4 (Same Equilibrium Number)
$GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ has the same number of FM-equilibria as of NA-equilibria.

The number of NA-equilibria has to be defined in terms of consumption allocations, i.e. it is the number of feasible allocations which together with some state price system $\pi \in \mathbb{R}^S_{++}$ form an NA-equilibrium. If this number is equal to one, there is a unique NA-equilibrium. I omit a formal definition because it would be notationally costly, but not more informative. It is not possible to define uniqueness of NA-equilibrium in terms of equilibrium price systems normalized in the standard way, because for the same commodity allocation there may exist multiple state price vectors because $x^i(\pi^*) = x^i(\tilde{\pi})$ for some $\pi^*, \tilde{\pi} \in \mathbb{R}^S_{++}$ which satisfy $A^T \pi^* = A^T \tilde{\pi}$, i.e. which generate the same asset price system.

The last proposition can be directly derived from the allocational equivalence in Theorem 2.19.

Let me finish with the remark that — as will become clear later on — uniqueness sometimes can be proven only with respect to some special normalization, but as soon as it holds for this special one it holds for all.

4.2.2 Local Uniqueness of FM-Equilibria

For GEI-economies generically local uniqueness of FM-equilibria as well as the Index Theorem hold. Even though the idea of the proof is nearly the same as in the Arrow-Debreu model (compare Section 3.3), one has to circumvent one problem: asset demand does not satisfy the "inward pointing vectorfield" property used for application of the Poincaré-Hopf Theorem when markets are complete. An "equivalent inward pointing vectorfield", however, is given in Hens (1991, Lemma 5).

Definition 4.5 (Regularity)
The economy $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ *is regular if for all* $q^* \in \mathcal{Q}(A)$ *with* $\theta(q^*) = 0$ *it is*

$$rank\ D_q\theta(q^*) = J - 1. \tag{4.2}$$

If the price normalization e^1 satisfies condition (4.1), (4.2) can be substituted by

$$rank\ \partial_{q_{-1}}\theta_{-1}(q^*) = J - 1 \tag{4.3}$$

for all $q^* \in \hat{\mathcal{Q}}(A)$ with $\theta(q^*) = 0$.

Theorem 4.6 (Generic Regularity)
Given $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$, *let* $\tilde{\omega}^{\bar{i}}$ *be the rest of* $\omega^{\bar{i}}$ *left after projecting* $\omega^{\bar{i}}$ *on* $< A >$ *for some consumer* $\bar{i} \in I$.[1] *Then, for almost every* $\bar{\theta} \in \{\theta \in \mathbb{R}^J \mid \tilde{\omega}^{\bar{i}} + A\theta \gg 0\}$ *the economy is regular.*

A complete proof via the Parametric Transversality Theorem is given in Hens (1991, Proposition 4). This result together with the Implicit Function Theorem leads to generic local uniqueness of FM-equilibria. Defining the index of an FM-equilibrium as

$$index\ (q^*) = sign \begin{vmatrix} -D_q\theta(q^*) & q^* \\ -q^{*T} & 0 \end{vmatrix}$$

one obtains the Index Theorem for incomplete markets.

Theorem 4.7 (Index Theorem)
Given the regular economy $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ *and the normalized equilibrium price set* $\mathcal{E}_{c,n}(A) = \{q \in \hat{\mathcal{Q}}_{c,n}(A) \mid \theta(q) = 0\}$ *for some* $c > 0$ *and* $n \in \mathbb{R}^J$ *which satisfy (4.1), it is*

$$\sum_{q \in \hat{\mathcal{E}}_{c,n}(A)} index\ (q) = +1.$$

For a proof via Poincaré-Hopf Theorem compare Hens (1991, Theorem 2).

[1] There is some $\tilde{\theta}^{\bar{i}} \in \mathbb{R}^J$ such that $\omega^{\bar{i}} = \tilde{\omega}^{\bar{i}} + A\tilde{\theta}^{\bar{i}}$.

4.2.3 Some Properties of Market Demand for Assets

Not only for the Arrow-Debreu model, but also for the GEI-model it is known that in general market demand (for assets) does not satisfy any other properties than continuity, homogeneity, and Walras' Law. However, in line with the Arrow-Debreu model it is possible to define WARP, monotonicity, and the property of gross substitution. Nevertheless, at some points caution is advisable.

A property satisfied by all individual demand functions for assets is the Weak Axiom of Revealed Preferences.

Definition 4.8 (Weak Axiom of Revealed Preferences)
The demand function $\theta : \mathcal{Q}(A) \to \mathbb{R}^J$ satisfies the Weak Axiom of Revealed Preferences (WARP) if

$$q \cdot \theta(\tilde{q}) \leq 0 \Rightarrow \tilde{q} \cdot \theta(q) > 0 \tag{4.4}$$

for all $q, \tilde{q} \in \mathcal{Q}(A)$ with $\theta(q) \neq \theta(\tilde{q})$.

This property implies negative semidefiniteness of the Jacobian matrix of θ on the orthogonal space with respect to θ, i.e.

$$v \cdot D_q\theta(q)v \leq 0 \tag{4.5}$$

for all $v \in \mathbb{R}^J$ and $q \in \mathcal{Q}(A)$ such that $\theta(q) \cdot v = 0$. Requiring the inequality to be strict on the orthogonal space with respect to q and $\theta(q)$ it is a sufficient condition, i.e.

$$\Big([q \cdot v = 0, \theta(q) \cdot v = 0, v \neq 0] \Rightarrow v \cdot D_q\theta(q)v < 0\Big) \Rightarrow WARP. \tag{4.6}$$

The proof is analogue to Mas-Colell, Whinston, and Green (1995, Sections 2.F and 17.F).

Since it is known that this property not necessarily transfers from individual to market demand, consider the stronger property of monotonicity.

Definition 4.9 (Monotonicity with Normalizing Vector)
Let $n \in \mathbb{R}^J$ such that $q \cdot n > 0$ for all $q \in \mathcal{Q}(A)$. The asset demand function $\theta : \mathcal{Q}(A) \to \mathbb{R}^J$ is monotone with respect to n if

$$(q - \tilde{q}) \cdot [\theta(q) - \theta(\tilde{q})] < 0 \tag{4.7}$$

for all $q, \tilde{q} \in \mathcal{Q}(A)$ with $(q - \tilde{q}) \cdot n = 0$ and $\theta(q) \neq \theta(\tilde{q})$.

Considering the differential version of monotonicity, a necessary condition is given by

$$v \cdot n = 0 \Rightarrow v \cdot D_q\theta(q)v \leq 0 \tag{4.8}$$

for all $v \in \mathbb{R}^J$ and $q \in \mathcal{Q}(A)$ which is also sufficient in its strong version, i.e.

$$v \cdot n = 0 \Rightarrow v \cdot D_q\theta(q)v < 0 \tag{4.9}$$

for all $v \in \mathbb{R}^J \backslash \{0\}$ and $q \in \mathcal{Q}(A)$. For a proof transfer Mas-Colell (1985, Proposition 5.7.3 (iii)) to the GEI-model.

The interpretation of these properties is the same as with complete markets, i.e. they assure demands to move in opposite directions of prices, for WARP with the restriction to compensated price changes. Note that monotonicity behaves nice in the sense that it is an additive property whenever the normalizing vector is the same for all individuals.

An interesting observation is given by the fact that if monotonicity is satisfied by market demand for some asset matrix, it is satisfied independently of the chosen representation of the asset span. Only the normalizing vector changes.

Proposition 4.10 (Monotonicity and Representation)
Consider two economies $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ and $GEI = \{(U^i, \omega^i)_{i\in I}, \hat{A}\}$ where $< A >=< \hat{A} >$ with corresponding market demand functions $\theta : \mathcal{Q}(A) \to \mathbb{R}^J$ and $\hat{\theta} : \mathcal{Q}(\hat{A}) \to \mathbb{R}^J$. Then, θ is monotone with respect to $n \in \mathbb{R}^J$ if and only if $\hat{\theta}$ is monotone with respect to $\hat{n} = M^{-1}n$ with $M \in \mathbb{R}^{J\times J}$ being the invertible matrix such that $AM = \hat{A}$.

Proof: The dependence between individual demand functions for the two economies is given by

$$\begin{aligned}
\theta^i(q) &= arg\max_{\theta\in\mathbb{R}^J} U^i(\omega^i + A\theta) \text{ s.t. } q\cdot\theta = 0 \ \wedge \ \omega^i + A\theta \geq 0 \\
&= arg\max_{\theta\in\mathbb{R}^J} U^i(\omega^i + AMM^{-1}\theta) \text{ s.t. } q\cdot MM^{-1}\theta = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge \ \omega^i + AMM^{-1}\theta \geq 0 \\
&= M\left[arg\max_{\hat{\theta}\in\mathbb{R}^J} U^i(\omega^i + \hat{A}\hat{\theta}) \text{ s.t. } (q\cdot M)\hat{\theta} = 0 \ \wedge \ \omega^i + \hat{A}\hat{\theta} \geq 0\right] \\
&= M\hat{\theta}^i(M^Tq)
\end{aligned}$$

which transfers to market demand. Besides

$$\begin{aligned}
M^T\mathcal{Q}(A) &= M^T\{q\in\mathbb{R}^J \mid q = A^T\pi \text{ for some } \pi\in\mathbb{R}^S_{++}\} \\
&= M^T\{q\in\mathbb{R}^J \mid M^Tq = M^TA^T\pi \text{ for some } \pi\in\mathbb{R}^S_{++}\} \\
&= M^T\{(M^T)^{-1}\hat{q}\in\mathbb{R}^J \mid \hat{q} = \hat{A}^T\pi \text{ for some } \pi\in\mathbb{R}^S_{++}\} \\
&= \mathcal{Q}(\hat{A})
\end{aligned}$$

where the product of a matrix and a set is defined as the set consisting of all elements equal to an element of the original set multiplied with the matrix. Now observe

$$\begin{aligned}
& (q-\tilde{q}) \cdot [\theta(q)-\theta(\tilde{q})] < 0 \ \forall \ q, \tilde{q} \in \mathcal{Q}(A) \\
& \text{s.t. } (q-\tilde{q}) \cdot n = 0 \ \wedge \ \theta(q) \neq \theta(\tilde{q}) \\
\Leftrightarrow \quad & (q-\tilde{q}) \cdot M \left[M^{-1}\theta(q) - M^{-1}\theta(\tilde{q})\right] < 0 \ \forall \ M^T q, M^T \tilde{q} \in M^T \mathcal{Q}(A) \\
& \text{s.t. } (q-\tilde{q}) \cdot M M^{-1} n = 0 \ \wedge \ M^{-1}\theta(q) \neq M^{-1}\theta(\tilde{q}) \\
\Leftrightarrow \quad & (M^T q - M^T \tilde{q}) \cdot \left[\hat{\theta}(M^T q) - \hat{\theta}(M^T \tilde{q})\right] < 0 \ \forall \ M^T q, M^T \tilde{q} \in \mathcal{Q}(\hat{A}) \\
& \text{s.t. } (M^T q - M^T \tilde{q}) \cdot \hat{n} = 0 \ \wedge \ \hat{\theta}(M^T q) \neq \hat{\theta}(M^T \tilde{q}) \\
\Leftrightarrow \quad & (\hat{q} - \hat{\tilde{q}}) \cdot \left[\hat{\theta}(\hat{q}) - \hat{\theta}(\hat{\tilde{q}})\right] < 0 \ \forall \ \hat{q}, \hat{\tilde{q}} \in \mathcal{Q}(\hat{A}) \\
& \text{s.t. } (\hat{q} - \hat{\tilde{q}}) \cdot \hat{n} = 0 \ \wedge \ \hat{\theta}(\hat{q}) \neq \hat{\theta}(\hat{\tilde{q}}).
\end{aligned}$$

□

In analogy to the Arrow-Debreu model, monotonicity with respect to some normalizing vector implies WARP, and it does not make any difference if the original versions are taken or the differentiable ones. The proofs are simple transformations of Mas-Colell (1985, Propositions 5.7.3 (i) and 5.7.1). WARP implies uniqueness of the FM-equilibrium (under the additional requirement of regularity). This follows by application of the Index Theorem 4.7 or from convexity of the normalized equilibrium price set.

Proposition 4.11 (Uniqueness for Monotonicity and WARP)
Suppose $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ such that market demand for assets satisfies monotonicity or WARP. If the economy is regular, it has a unique FM-equilibrium.

The last property to consider is the property of strict gross substitution. Regrettably, for incomplete markets models it looses much of its power with respect to uniqueness of FM-equilibria.

Definition 4.12 (Strict Gross Substitution)
The demand function $\theta : \mathcal{Q}(A) \to \mathbb{R}^J$ satisfies the property of strict gross substitution whenever the Jacobian matrix has strictly positive off-diagonal elements and strictly negative diagonal elements.

Again the property of weak gross substitution is the same except for allowing that entries of the matrix are zero.

The gross substitution property depends on the chosen representation of the asset span. For an example assume there are two states $S = 2$ and two assets $J = 2$. Now consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which generate the same asset span. Since

$$A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{A},$$

one knows from the proof of Proposition 4.10 that

$$\theta_A(q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \theta_{\hat{A}}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} q) \tag{4.10}$$

and if $q \in \mathcal{Q}(A)$, then $\hat{q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} q = \begin{pmatrix} q_1 \\ -q_2 \end{pmatrix} \in \mathcal{Q}(\hat{A})$. From differentiation of (4.10) with respect to q one obtains

$$D\theta_A(q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D\theta_{\hat{A}}(\hat{q}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D\theta_A(q) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D\theta_{\hat{A}}(\hat{q}).$$

Now assume $D\theta_A(q)$ has strictly negative diagonal and strictly positive off-diagonal entries. Then the last equation implies that $D\theta_{\hat{A}}(\hat{q})$ has only strictly negative entries which contradicts the gross substitution property for demand $\theta_{\hat{A}}$.

To obtain uniqueness in the same way as in the Arrow-Debreu model, it has to be assured that $\mathcal{Q}(A) \subseteq \mathbb{R}^S_{++}$ which requires $A > 0$. Even under consideration of Proposition 4.3 this is a very strong restriction as becomes clear from such a simple example like (compare Hens, Schmedders, and Voß (1999, p. 174))

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Proposition 4.13 (Uniqueness for Strict Gross Substitution)
Suppose $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ with $A > 0$ such that market demand satisfies the property of strict gross substitution. Then the economy has a unique FM-equilibrium.

The proof is similar to the one with complete markets, given by Arrow and Hahn (1971, Chapter 7, Corollary 9). However, it needs some further considerations about the underlying finite sequence of prices because one has to take care that it has to stay within the no-arbitrage price set $\mathcal{Q}(A)$. For a thorough treatment see Voß (1997).

For $J = 2$ and $A > 0$ asset demand is monotone with respect to first and second unit vectors if and only if it satisfies the property of gross substitution. This follows because both properties require negative diagonal elements and therefore because of Walras' Law also strictly positive off-diagonal elements.

4.2.4 How to Obtain Uniqueness

Existence of Explicit Pricing Formulas

No-Trade Equilibrium
Obviously, not only for complete markets, but also for incomplete markets the simplest pricing formula is given when each equilibrium has to be no-trade equilibrium.

This is clear if there does not exist any asset or only one. With no asset there cannot be any trade because there is nothing to exchange. For a single asset Assumption 2.14 assures either all consumers to demand or to supply the asset.

These two cases as well as the one of identical consumers constitute economies with constrained Pareto-efficient endowments. More generally, the following proposition holds.

Proposition 4.14 (Uniqueness for Constrained Pareto-Efficiency)
Given $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ with $(\omega^i)_{i \in I}$ a constrained Pareto-efficient allocation, the no-trade equilibrium is the unique equilibrium with equilibrium price system $q^* \in \mathcal{Q}(A)$ collinear to $(\nabla U^i(\omega^i)A)^T$ for all $i \in I$.

Proof: Because of Assumption 2.14 and Proposition 4.3 assume without loss of generality $A_{\cdot 1} > 0$. Then for $q^T = \left(1, \frac{\nabla U^1(\omega^1)A_{\cdot 2}}{\nabla U^1(\omega^1)A_{\cdot 1}}, \ldots, \frac{\nabla U^1(\omega^1)A_{\cdot J}}{\nabla U^1(\omega^1)A_{\cdot 1}}\right)$ the equilibrium properties are obvious.

Assume, there is another equilibrium allocation $(\tilde{\theta}^i)_{i=1,\ldots,I}$ with corresponding price system $\tilde{q}$, then the no-trade allocation $(\theta^i = 0)_{i=1,\ldots,I}$ also satisfies the budget constraints. Therefore, it is

$$U^i(\omega^i + A\tilde{\theta}^i) \geq U^i(\omega^i) \text{ for all } i = 1, \ldots, I. \tag{4.11}$$

Since the no-trade allocation is constrained Pareto-efficient, this implies

$$U^i(\omega^i + A\tilde{\theta}^i) = U^i(\omega^i) \text{ for all } i = 1, \ldots, I. \tag{4.12}$$

Therefore, together with $rank\ A = J$ and strict quasi-concavity of $U^i, i = 1, \ldots, I$, one obtains a contradiction for $\tilde{\theta}^i \neq 0$ because each strict convex combination of $\omega^i + A\tilde{\theta}^i$ and $\omega^i = \omega^i + A\theta^i$ (or equivalently of $\tilde{\theta}^i$ and θ^i) would improve the consumer.

□

A nice property with respect to constrained Pareto-efficiency for an asset structure A admits conclusions on constrained Pareto-efficiency for asset structures which allow trade in a subspace of the original one.

Lemma 4.15 (Implications of Constrained Pareto-Efficiency)
If $(\theta^i)_{i=1,\ldots,I} \in \mathbb{R}^{J \times I}$ with $\theta^i = 0$ for all $i = 1, \ldots, I$, i.e. the no-trade allocation, is constrained Pareto-efficient in the GEI-economy $\{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ where $A \in \mathbb{R}^{S \times J}$ and $A_{\cdot 1} > 0$, then this is also true for all $\{(U^i, \omega^i)_{i=1,\ldots,I}, \tilde{A}\}$ with $\tilde{A} \in \mathbb{R}^{S \times \tilde{J}}$ where $\tilde{J} \leq J$, $\tilde{A}_{\cdot 1} > 0$[2], $< \tilde{A} > \subseteq < A >$ and the no-trade allocation is $(\tilde{\theta}^i)_{i=1,\ldots,I} \in \mathbb{R}^{\tilde{J} \times I}$ with $\tilde{\theta}^i = 0$ for all $i = 1, \ldots, I$.

Proof: Since $< \tilde{A} > \subseteq < A >$, for $\tilde{A}_{\cdot k}, k = 1, \ldots, \tilde{J}$, there is some $c_{\cdot k} \in \mathbb{R}^J$ with $\tilde{A}_{\cdot k} = \sum_{j=1}^J c_{jk} A_{\cdot j}$. Therefore, for all $k = 1, \ldots, \tilde{J}$ one gets

$$\begin{aligned} \frac{\nabla U^1(\omega^1)\tilde{A}_{\cdot k}}{\nabla U^1(\omega^1)\tilde{A}_{\cdot 1}} &= \frac{\sum_{j=1}^J c_{jk} \nabla U^1(\omega^1) A_{\cdot j}}{\sum_{j=1}^J c_{j1} \nabla U^1(\omega^1) A_{\cdot j}} \\ &= \frac{\sum_{j=1}^J c_{jk} \frac{\nabla U^i(\omega^i)A_{\cdot j} \cdot \nabla U^1(\omega^1)A_{\cdot 1}}{\nabla U^i(\omega^i)A_{\cdot 1}}}{\sum_{j=1}^J c_{j1} \frac{\nabla U^i(\omega^i)A_{\cdot j} \cdot \nabla U^1(\omega^1)A_{\cdot 1}}{\nabla U^i(\omega^i)A_{\cdot 1}}} \\ &= \frac{\sum_{j=1}^J c_{jk} \nabla U^i(\omega^i) A_{\cdot j}}{\sum_{j=1}^J c_{j1} \nabla U^i(\omega^i) A_{\cdot j}} \\ &= \frac{\nabla U^i(\omega^i)\tilde{A}_{\cdot k}}{\nabla U^i(\omega^i)\tilde{A}_{\cdot 1}} \text{ for all } i = 2, \ldots, I \end{aligned} \tag{4.13}$$

where the second equality follows from constrained Pareto-efficiency in the economy $\{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$.

□

[2] It would be sufficient to demand $\tilde{A}\tilde{\theta} > 0$ for some $\tilde{\theta} \in \mathbb{R}^{\tilde{J}}$.

Combining Proposition 4.14 and Lemma 4.15 leads to a further result on uniqueness of FM-equilibria.

Proposition 4.16 (Inclusion Property)
If $GEI = \{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ *with* $A \in \mathbb{R}^{S\times J}$ *and* $A_{\cdot 1} > 0$ *is characterized by the unique (constrained Pareto-efficient) no-trade equilibrium, this is also true for all economies* $GEI = \{(U^i, \omega^i)_{i=1,\ldots,I}, \tilde{A}\}, \tilde{A} \in \mathbb{R}^{S\times \tilde{J}}$ *with* $\tilde{J} \leq J$, $\tilde{A}_{\cdot 1} > 0$ *and* $< \tilde{A} > \subseteq < A >$.

Constrained Pareto-efficiency might induce a reduction of the number of equilibria for incomplete markets in comparison to complete markets. Compare Section 4.5.

Complete Risk Sharing
As pointed out in Section 3.5.1 the idea of risk sharing in equilibrium models is recently extended from complete to various incomplete markets models. Regrettably, as soon as trading possibilities are restricted, it becomes much more difficult to obtain complete risk sharing in equilibrium. In addition to the requirements when markets are complete (satisfaction of the expected utility hypothesis and no aggregate risk) the commodity endowments have to be substituted by asset endowments, i.e. commodity endowments have to be elements of the asset span. What happens without this additional assumption is examined in Section 4.4.3.

Proposition 4.17 (Complete Risk Sharing)
Let an economy $GEI = \{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ *with* $A_{\cdot 1} > 0$ *be given such that for all* $i = 1, \ldots, I$ *the expected utility hypothesis is satisfied, i.e.*

$$U^i(x) = \sum_{s=1}^{S} \bar{\rho}_s u^i(x_s),$$

where u^i *satisfies standard assumptions and* $\bar{\rho} \in \hat{\mathcal{P}}_{1,\mathbb{1}}$ *is an objective probability measure. Endowments are spanned, i.e.*

$$\omega^i = A\theta^i_\omega \text{ for some } \theta^i_\omega \in \mathbb{R}^J.$$

If ω *is riskless (i.e.* $\omega = \bar{\omega}\mathbb{1}$ *for some* $\bar{\omega} \in \mathbb{R}_{++}$*), the economy has a unique FM-equilibrium where normalizing the first asset price to one, one obtains the pricing formula*

$$q^{*T} = \left(1, \frac{\sum_{s=1}^S \bar{\rho}_s A_{s2}}{\sum_{s=1}^S \bar{\rho}_s A_{s1}}, \ldots, \frac{\sum_{s=1}^S \bar{\rho}_s A_{sJ}}{\sum_{s=1}^S \bar{\rho}_s A_{s1}}\right).$$

Proof: There is exactly one normalized equilibrium price system q^* such that the corresponding allocation is riskless. This follows because then first order conditions (2.7) determine price-ratios to be

$$\frac{q_j}{q_1} = \frac{\sum_{s=1}^{S} \bar{\rho}_s A_{sj}}{\sum_{s=1}^{S} \bar{\rho}_s A_{s1}}, j \in \{1, \ldots, J\}. \tag{4.14}$$

Note that the assumptions on the economy imply that all equilibrium allocations are riskless. Suppose not. Let $(\hat{x}^i)_{i=1,\ldots,I}$ be the corresponding allocation with $\hat{x}^i = A(\theta^i_\omega + \hat{\theta}^i)$ for some $\hat{\theta}^i \in \mathbb{R}^J, i = 1, \ldots, I$, with $\sum_{i=1}^{I} \hat{\theta}^i = 0$. Strict concavity of $u^i, i = 1, \ldots, I$, assures that $x^{*i} = \left(\sum_{s=1}^{S} \bar{\rho}_s \hat{x}^i_s\right) \mathbb{1}, i = 1, \ldots, I$, improves all consumers and at least one of them strictly. Since

$$\sum_{i=1}^{I} x^{*i} = \sum_{i=1}^{I} \left(\sum_{s=1}^{S} \bar{\rho}_s \hat{x}^i_s\right) \mathbb{1} = \sum_{s=1}^{S} \bar{\rho}_s \left(\sum_{i=1}^{I} \hat{x}^i_s\right) \mathbb{1} = \sum_{s=1}^{S} \bar{\rho}_s \omega_s \mathbb{1} = \bar{\omega}\mathbb{1}$$

and

$$x^{*i} = \sum_{s=1}^{S} \bar{\rho}_s A_{s\cdot}(\theta^i_\omega + \hat{\theta}^i)\mathbb{1} = \omega^i + A\theta^{*i}$$

with $\theta^{*i} = \frac{1}{\bar{\omega}} \sum_{s=1}^{S} \bar{\rho}_s A_{s\cdot}(\theta^i_\omega + \hat{\theta}^i) \left(\sum_{i=1}^{I} \theta^i_\omega\right) - \theta^i_\omega$, this contradicts the constrained Pareto-efficiency of equilibria.

□

Note that asset prices refer to the expected asset payoffs and are independent of aggregate endowments as well as of the endowment distribution.

Quadratic Utility
Even though with incomplete markets utility functions which satisfy the expected utility hypothesis with respect to an objective probability measure do not allow any more an explicit pricing formula in spite of no-aggregate risk (compare Section 4.4.3), under the restriction to quadratic utility an explicit pricing formula remains to exists. This is important because quadratic utility can be interpreted as a special form of mean-variance utility in the framework of the CAPM.

Proposition 4.18 (Uniqueness with Quadratic Utility)
Assume $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ with $A_{\cdot 1} > 0$ such that all consumers satisfy the expected utility hypothesis with respect to an objective probability measure $\bar{\rho} \in \hat{\mathcal{P}}_{1,\mathbb{1}}$ and state utility is quadratic, i.e.

$$U^i(x) = \sum_{s=1}^{S} \bar{\rho}_s (x_s - \frac{1}{2}\gamma^i x_s^2). \tag{4.15}$$

If $\gamma^i \in \left(0, \min_{s\in S} \frac{1}{\omega_s}\right)$*, then the economy is characterized by a unique interior FM-equilibrium and normalizing the first asset price to one leads to*

$$q^{*T} = \left(1, \frac{\sum_{s=1}^{S} \bar{\rho}_s A_{s2} \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_s\right)}{\sum_{s=1}^{S} \bar{\rho}_s A_{s1} \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_s\right)}, \dots, \frac{\sum_{s=1}^{S} \bar{\rho}_s A_{sJ} \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_s\right)}{\sum_{s=1}^{S} \bar{\rho}_s A_{s1} \left(\sum_{i=1}^{I} \frac{1}{\gamma^i} - \omega_s\right)}\right).$$

The combined restrictions on $(\gamma^i)_{i\in I}$ and ω again assure strict monotonicity of utility functions in the relevant domain. Together with strict concavity this leads to sufficiency of first order conditions for the maximization property of interior FM-equilibria. Normalizing the first asset price to one and using this to eliminate the Lagrangian multipliers from first order conditions, with the market clearing property one obtains the pricing formula after some divisions and summations.

The equilibrium prices here are also independent of the distribution of endowments and risk aversion. Note that without aggregate risk prices are equal to the expected asset payoff (after renormalization) which corresponds to Proposition 4.17.

Existence of a Representative Consumer

To assure the existence of a representative consumer, in the GEI-model apart from homothetic preferences additional restrictions on consumers' endowments are necessary. First results in this field are given by Detemple and Gottardi (1998) who prove the transferability of the Arrow-Debreu results to incomplete markets models whenever utility functions are identical and endowment vectors $(\omega^i)_{i\in I}$ are elements of the asset span, i.e. commodity endowments are substituted by asset endowments. A further possibility to obtain a representative consumer is to assume identical homothetic utility functions for all consumers and collinearity of the vectors $(\omega^i)_{i\in I}$. Only considering homotheticity without the restriction to identical utility functions requires spanning and collinearity of the endowment vectors.

Proposition 4.19 (Uniqueness for Homotheticity)
Suppose the economy $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ *such that all* $U^i, i = 1, \dots, I$*, are homothetic, then there is a unique FM-equilibrium whenever one of the following conditions is fulfilled:*

1. *all* $U^i, i = 1, \dots, I$*, are identical and* $\omega^i \in < A >$ *for all* $i \in I$*;*
2. *all* $U^i, i = 1, \dots, I$*, are identical and endowments* $(\omega^i)_{i\in I}$ *are collinear, i.e. there is some vector* $\delta \in \hat{\mathcal{P}}_{1,\mathbb{I}}$ *with* $\omega^i = \delta_i \omega$ *for all* $i \in I$*;*

3. *endowments* $(\omega^i)_{i\in I}$ *are spanned and collinear, i.e.* $\omega^i \in < A >$ *for all* $i \in I$ *and there is some vector* $\delta \in \hat{\mathcal{P}}_{1,\mathbb{I}}$ *with* $\omega^i = \delta_i \omega$ *for all* $i \in I$.

Under the first and the second conditions the existence of a representative consumer can be proved by simple transformations of the maximization problems of the consumers. For the third condition in addition the application of the results from Chipman (1974) on homotheticity and representative consumers in the Arrow-Debreu model is necessary. Therefore, WARP is satisfied. Compare Detemple and Gottardi (1998) and Voß (1997).

How to Obtain Monotonicity

Quasi-Linear Utility
Modeling first period consumption, quasi-linearity of utility with respect to today's consumption assures a unique equilibrium. It does not seem too strange to give utility of today's consumption this special form.

In Section 4.4.1 it will be demonstrated that deviating from first period consumption might destroy uniqueness of the equilibrium.

Proposition 4.20 (Uniqueness for Quasi-Linearity)
Assume $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ *such that* U^i *is quasi-linear for all* $i \in I$*, i.e. there exists some strictly concave function* $\tilde{U}^i : \mathbb{R}_+^{S-1} \to \mathbb{R}$ *with*

$$U^i(x) = x_1 + \tilde{U}^i(x_{-1})$$

and $\tilde{U}^i$ *satisfies Assumption 2.4 on* $\mathbb{R}_+^{S-1}$*. If* A *models first period consumption, i.e.*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \check{A} \end{pmatrix}$$

for some $\check{A} \in \mathbb{R}^{(S-1)\times(J-1)}$ *with rank* $\check{A} = J-1$*, the economy displays a unique FM-equilibrium whenever today's endowments are sufficiently large.*

Under these special restrictions on the asset structure and endowments individual demand functions are monotone with respect to the first unit vector e^1, i.e. when normalizing the price of today's consumption to be one: the normalized Jacobian of asset demand is negative definite. Because of strict concavity of $\tilde{U}^i$, this can be proved by the Implicit Function Theorem from first order conditions (2.7) after eliminating demand for the first asset, i.e. for today's commodity excess demand, with the help of (2.8). Endowments today have to be sufficiently large to assure necessity of first order conditions for maximization. For details compare Voß (1997, Satz 4.17).

The Mitjushin-Polterovich Coefficient
For complete markets it is possible to obtain monotonicity of individual demand functions by restricting the risk aversion of each consumer with help of the so-called Mitjushin-Polterovich coefficient. At least under the assumption of spanned endowments one should expect this result to hold also with incomplete markets because this additional assumption closely relates the GEI-model to the Arrow-Debreu model via some transformation of the underlying space.

Proposition 4.21 (Uniqueness via Mitjushin-Polterovich I)
Given $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ such that all U^i are strictly concave and

$$MP_{U^i}(x) < 4 \text{ for all } x \in \mathbb{R}^S_{++} \setminus \{0\}, i \in I,$$

the economy has a unique FM-equilibrium if endowments are elements of the asset span and collinear, i.e. if there is some $\delta \in \hat{\mathcal{P}}_{1,\mathbb{I}}$ with $\omega^i = \delta_i \omega$ for all $i \in I$ and $\omega \in < A >$.

A proof is given in Hens (1995, p. 168). Spanned endowments can be understood as asset endowments $(\bar{\theta}^i)_{i \in I}$. Therefore, the idea is to transform consumers' maximization problems (2.6) into

$$\max_{\hat{\theta} \in \mathbb{R}^J} W^i(\hat{\theta}) \text{ s.t. } q \cdot \hat{\theta} = q \cdot \bar{\theta}^i \ \wedge A\hat{\theta} \geq 0$$

where $W^i(\hat{\theta}) = U^i(A\hat{\theta})$. This new function is characterized by the same Mitjushin-Polterovich coefficient as the original utility U^i. Even though within the new framework not all requirements of the Mitjushin-Polterovich Theorem are fulfilled, the proof via monotonicity of the inverse demand function can be transferred (compare Voß (1997, Satz 4.2.5)). Collinearity of $(\omega^i)_{i \in I}$ respectively $(\bar{\theta}^i)_{i \in I}$ therefore guarantees monotonicity of market demand.

As soon as utility satisfies the expected utility hypothesis, Bettzüge (1998) proves the theorem to hold without the restriction to spanned endowments. He requires the existence of a so-called fundamental set, i.e. a set $R \subseteq S$ with $\sharp R = J$ such that $A_{R\cdot}$ is invertible and $A_{S \setminus R \cdot}(A_{R\cdot})^{-1} \geq 0$. Whenever

$$\omega^i_s - A_{s\cdot}(A_{R\cdot})^{-1}\omega^i_R \geq 0 \tag{4.16}$$

for all $s \in S \setminus R$ and $i \in I$, consumers' maximization problems can be reduced: they only have to decide about consumption in states $s \in R$ and the "new" utility is a function of these variables. Demand functions defined hereby are monotone if and only if this is true for original asset demand. However, on the reduced decision problem, the Mitjushin-Polterovich Theorem can be applied: the expected utility structure of U^i assures the "new" utility to have a sufficiently small Mitjushin-Polterovich coefficient. Because of collinear endowments monotonicity is additive.

Proposition 4.22 (Uniqueness via Mitjushin-Polterovich II)
Given $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ such that all U^i satisfy the expected utility hypothesis and

$$MP_{U^i}(x) < 4 \text{ for all } x \in \mathbb{R}^S_{++} \backslash \{0\}, i \in I,$$

the economy has a unique FM-equilibrium if there exists some fundamental set $R \subseteq S$ satisfying (4.16) and endowments are collinear, i.e. there is some $\delta \in \hat{\mathcal{P}}_{1,\mathbb{1}}$ with $\omega^i = \delta_i \omega$ for all $i \in I$.

Let me remark that collinearity of $(\omega^i)_{i \in I}$ can be weakened to collinearity of $((\omega^i_s)_{s \in R})_{i \in I}$ which is an interesting point because this requirement is weaker than for complete markets. Besides the expected utility hypothesis can be replaced by additive separability.

Small Relative Risk Aversion and Two Assets
For complete markets a relative risk aversion less or equal to one is sufficient to obtain that market demand satisfies the property of strict gross substitution and therefore that there is a unique equilibrium. Regrettably, for incomplete markets this property fails. Nevertheless, when there are only two assets, it is possible to choose the approach of monotonicity.

Proposition 4.23 (Uniqueness with $RRA^i \leq 1$ and 2 Assets)
Assume $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ with only two assets ($J = 2$) and $A_{\cdot 1} > 0$. If all U^i satisfy the expected utility hypothesis with

$$RRA^i(x_s) \leq 1 \text{ for all } x_s \in \mathbb{R}_{++},$$

the economy has a unique FM-equilibrium.

Correspondingly to complete markets, the assumption of expected utility can be relaxed to additive separability. Besides the condition on the relative risk aversion can be relaxed to a condition on absolute risk aversion influenced by the size of endowments (compare (3.21)).

A proof of the proposition was initiated by Becker (1995) and extended by Bettzüge (1997). The idea is to normalize the price of the first asset to one, use the budget constraint to eliminate demand for the first asset, and then consider the first order conditions. Application of the Implicit Function

Theorem leads to monotonicity of individual demand for the second asset whenever the relative risk aversion is less or equal to one. This transfers to market demand.

Moreover, under the restriction to $A > 0$ the property of strict gross substitution can be proved for individual and market demand by direct computation of the Jacobian (compare Voß (1997, Satz 4.4.3)).

How to Obtain the Property of Gross Substitution

Under a small relative risk aversion for more than two assets, until now it has only been possible to formulate those extremely restrictive conditions on the asset structure given in the next proposition such that a unique equilibrium is guaranteed. As will be presented in Section 4.4.2 even with Cobb-Douglas utility functions not only the property of gross substitution, but even uniqueness might fail.

Consider so-called weakly separating matrices, i.e. asset structures where for each state $s \in S$ there is exactly one asset $j \in J$ which has a payoff unequal to zero.

Proposition 4.24 (Uniqueness with $RRA^i \leq 1$ and Special A)
Assume $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$ such that A is weakly separating or $J = 2$. If A is non-negative and all U^i satisfy the expected utility hypothesis with

$$RRA^i(x_s) \leq 1 \text{ for all } x_s \in \mathbb{R}_{++},$$

the economy has a unique FM-equilibrium.

The special structure of the economy enables direct computation of the Jacobians of individual asset demands. With the restrictions on relative risk aversions it is possible to fix the signs of the different contained terms (compare Voß (1997, Satz 4.3.3 and Satz 4.4.3)). Again it is possible to relax the assumption on the utility functions, but since the restrictions on the asset structure are so strong, I do not want to go further into details. Remember that Proposition 4.3 allows to weaken the assumptions on A a little bit.

4.2.5 Overview of Conditions for Uniqueness in the Finance GEI-Model

Figure 4.1 provides an overview of conditions which guarantee a unique FM- and therefore also NA-equilibrium in the GEI-model. It has to be read in the same way as the corresponding Figure 3.4 for complete markets (compare Section 3.6). Besides it is subjected to the same limitations.

Since complete markets constitute a special case for the GEI-model, actually Figure 3.4 would have to be contained in Figure 4.1 given here. However, I omitted those conditions which are only valid with complete markets to preserve clarity.

Besides I decided to assume $A_{\cdot 1}$ to be positive because otherwise it would become quite difficult to handle price normalization. In addition, the reader has to take care of the fact that it is sufficient if one of the representations of the asset span fits in the scheme (compare Proposition 4.3).

4.3 Robustness of the Number of Competitive Equilibria against Perturbations of the Asset Structure

In the next section I give examples where although consumers' characteristics remain the same there might be a change from a unique to multiple equilibria when switching from complete to incomplete markets, i.e. when the dimension of the asset span changes. As well, for different incomplete asset structures of the same dimension the number of equilibria might vary. Section 4.5 proves by means of an example that changing from complete to (non-trivial) incomplete markets might also cause to switch from multiple to a unique equilibrium.

All these examples would not be very interesting if they were exceptional: but this section proves the examples to be robust against some changes of the asset structure. The general question is to what extent changes of the asset structure are possible without changing the number of equilibria. One reason why I attach importance to this question consists in the fact that it allows to get some understanding for the influence of the asset structure on the number of equilibria: the long-term aim is to use this understanding for establishing more general sufficient conditions for a unique equilibrium than given in the second section of this chapter, i.e. than those results known up to now in this topic; most of the propositions require some special structure for the asset matrix whether this is in combination with restrictions on endowments (compare, for example, Proposition 4.21) or not (compare, for example, Proposition 4.24). The robustness result of this chapter clarifies that one can weaken these assumptions.

The main outcome of this section is the following one: starting with a regular Finance GEI-economy, one can allow that the asset structure is perturbed a little bit without changing the number of equilibria or — even stronger — such that equilibrium prices move continuously! The only re-

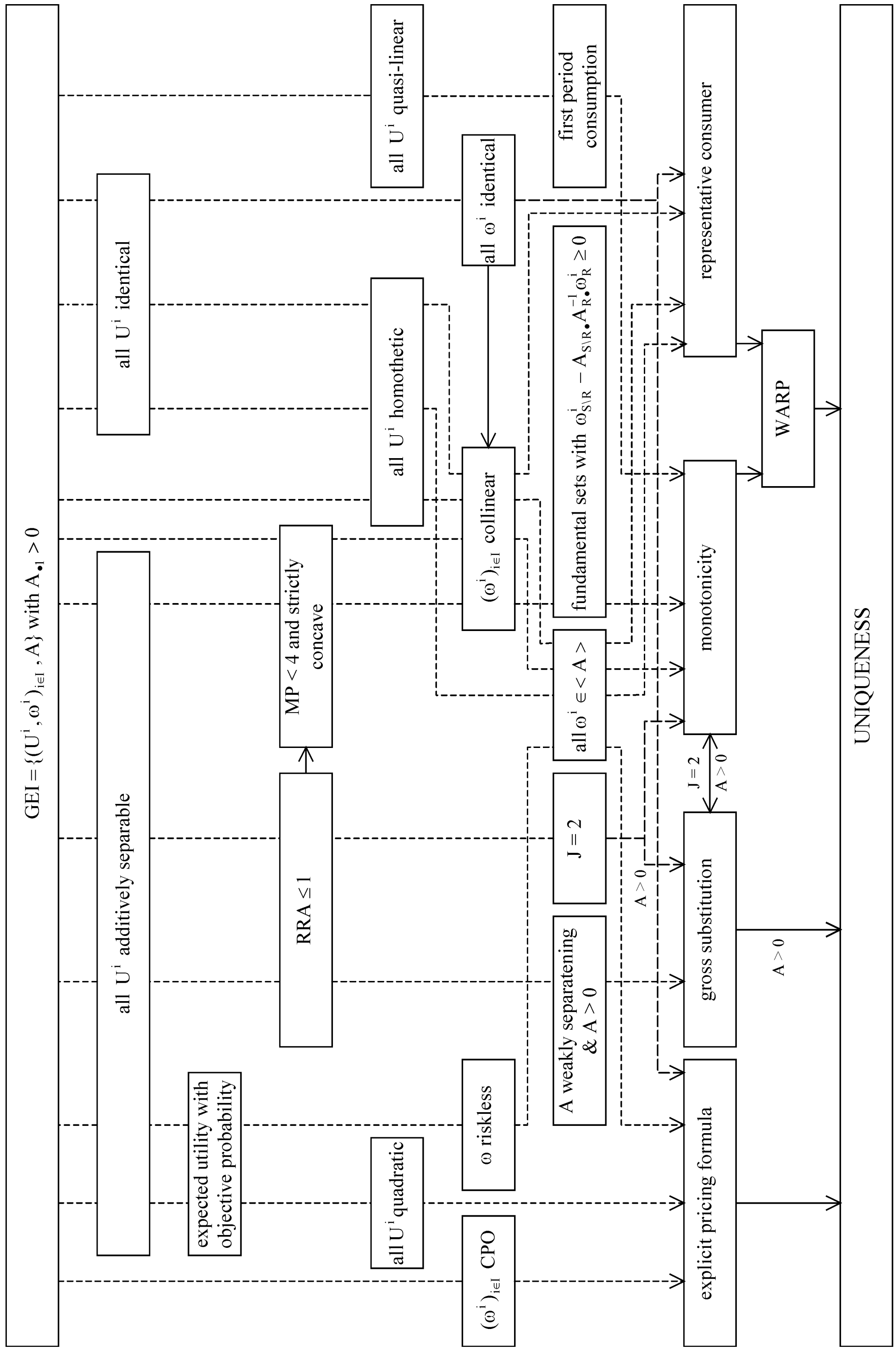

Figure 4.1: Conditions for uniqueness in the Finance GEI-model.

strictions on the direction of perturbation are to remain with a desirable portfolio and with the same rank of the asset matrix, i.e. the same number of assets. This fact corresponds to intuition if the closure of the set of normalized arbitrage-free asset prices constitutes a continuous correspondence of the asset matrix, but I prove it to be even true with discontinuous price-correspondences. Up to my knowledge even for the first case there is no general proof of this fact in the literature.

One interim result in the proof gives a kind of "continuous differentiability" of market asset demand as a function of the asset matrix and asset prices which is a simple conclusion from an application of the Implicit Function Theorem.[3] A second important observation on the way to the main result is given by the following fact: although arbitrage-free normalized price-sets are open and sometimes unbounded, for equilibrium considerations it is possible to restrict attention to some compact subset which is even continuous in the asset structure.

This section is organized as follows: in the next subsection I describe some extensions of notation for the underlying Finance GEI-model. Within this framework I thoroughly investigate the normalized arbitrage-free price set and its dependence on the asset structure. Afterwards I give a short proof of "continuous differentiability" of individual and market asset demand with respect to the asset matrix and asset prices. The third subsection uses this property to get continuity of equilibrium prices in "regular" asset structures because one can restrict attention to a compact subset of arbitrage-free asset prices which allows to apply uniform continuity.

4.3.1 Further Properties of the Set of Arbitrage-Free Prices

Recall the system of first order conditions given as necessary and sufficient conditions for consumers' maximization problems, i.e.

$$\begin{aligned} (2.7) \qquad \nabla U^i(\omega^i + A\theta^i)A - \lambda^i q^T &= 0, \\ (2.8) \qquad q \cdot \theta^i &= 0 \end{aligned}$$

for all $i \in I$ where $\lambda^i > 0$ is the Lagrangian multiplier. Up to now, I considered a solution of this system $(\theta^i(q), \lambda^i(q)) \in \mathbb{R}^J \times \mathbb{R}_+$ as a function on $\mathcal{Q}(A)$ and assumed A to be fixed. However, since I want to clarify the

[3]One has to be careful: the range of definition of this function is neither open nor connected such that for some elements the derivatives have to be restricted in some directions and continuous differentiability in the common sense is obtained only on small open subsets.

dependence on the asset structure, from now on it is sometimes convenient to write $(\theta^i(q,A), \lambda^i(q,A)) \in \mathbb{R}^J \times \mathbb{R}_+$. I do not use different notations because the context clarifies whether the asset structure A is fixed or not, i.e. whether $(\theta^i(q), \lambda^i(q))$ or $(\theta^i(q,A), \lambda^i(q,A))$ is the right notation.

If there is some $\bar{j} \in \{1, \ldots, J\}$ with $A_{\cdot\bar{j}} > 0$, a possible price normalization is given by $q_{\bar{j}} = 1$ such that asset j is numeraire asset. Because of Proposition 4.3, under Assumption 2.14 it is no restriction to assume $A_{\cdot\bar{j}} > 0$ since for each A satisfying this assumption one can find an asset matrix $\tilde{A}$ satisfying $rank\ \tilde{A} = J$ and $\tilde{A}_{\cdot\bar{j}} > 0$ with $< A >=< \tilde{A} >$. Therefore Assumption 2.14 can be substituted by

Assumption 4.25 (Some Asset with Positive Payoff)
$A \in \mathbb{R}^{S\times J}$ *satisfies* $A_{\cdot 1} > 0$.

This means that the first asset is always desirable because its payoff is positive.

Under this assumption one can restrict attention to the following projection of the normalized arbitrage-free asset price set:

$$\mathcal{Q}_{norm}(A) := \{\hat{q} \in \mathbb{R}^{J-1} |\ \begin{pmatrix} 1 \\ \hat{q} \end{pmatrix} = A^T\pi \text{ for some } \pi \in \mathbb{R}^S_{++}\}.$$

$\mathcal{Q}_{norm}(A)$ is the projection of $\hat{\mathcal{Q}}(A)$ on $\mathbb{R}^{J-1}$.

Later on, it will become clear that — when investigating perturbations of the asset matrix — I have to distinguish three different classes of matrices:

(I) $A \in \mathbb{R}^{S\times J}$, satisfying Assumption 4.25 with $A_{\cdot 1} \gg 0$;

(II) $A = \begin{pmatrix} 1 & 0 \\ 0 & \breve{A} \end{pmatrix} \in \mathbb{R}^{S\times J}$ where $\breve{A} \in \mathbb{R}^{(S-1)\times(J-1)}$ with $rank\ \breve{A} = J - 1$, i.e. A models first period consumption;[4]

(III) $A \in \mathbb{R}^{S\times J}$, satisfying Assumption 4.25 with $A_{\cdot 1} > 0$, but not $A_{\cdot 1} \gg 0$.

Case (III) includes case (II). However, to remain within the framework of first period consumption later on I will have to define perturbations in case (II) different from those in case (III).

Define

$$\begin{aligned} \mathcal{S}^+(A_{\cdot 1}) &:= \{s \in \{1, \ldots, S\} |\ A_{s1} > 0\}, \\ \mathcal{S}^0(A_{\cdot 1}) &:= \{s \in \{1, \ldots, S\} |\ A_{s1} = 0\} \end{aligned}$$

[4]For simplicity I numerate the rows of $\breve{A}$ by $2, \ldots, S$ and the columns by $2, \ldots, J$.

such that $\mathcal{S}^+(A_{\cdot 1}) \cup \mathcal{S}^0(A_{\cdot 1}) = \{1, \ldots, S\}$.

With these definitions alternative representations of the normalized arbitrage-free asset price sets can be given which highlight the structure of these sets. Distinguish the classes of matrices:

(I)

$$\begin{aligned}
&\mathcal{Q}_{norm}(A) \\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \begin{pmatrix} 1 \\ \hat{q} \end{pmatrix} = A^T \pi \text{ for some } \pi \in \mathbb{R}^S_{++}\} \\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \hat{q} = A^T_{\cdot(-1)} \pi \ \wedge 1 = \sum_{s=1}^{S} \underbrace{A_{s1}\pi_s}_{=:\mu_s} \text{ for some } \pi \in \mathbb{R}^S_{++}\} \\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \hat{q} = \sum_{s=1}^{S} \frac{\mu_s}{A_{s1}} A^T_{s(-1)} \text{ for some } \mu \in \hat{\mathcal{P}}_{1,\mathbb{1}}\} \\
&= int \left(conv \ \{\frac{1}{A_{11}} A^T_{1(-1)}, \ldots, \frac{1}{A_{S1}} A^T_{S(-1)}\} \right)
\end{aligned}$$

where *int* defines the interior of a set[5] and *conv* denotes the convex hull, i.e. the smallest convex set containing $\frac{1}{A_{11}} A^T_{1(-1)}, \ldots, \frac{1}{A_{S1}} A^T_{S(-1)}$. Obviously, $\mathcal{Q}_{norm}(A)$ is the interior of a convex polytop.

(II)

$$\begin{aligned}
&\mathcal{Q}_{norm}(A) \\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \begin{pmatrix} 1 \\ \hat{q} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \breve{A}^T \end{pmatrix} \pi \text{ for some } \pi \in \mathbb{R}^S_{++}\} \\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \hat{q} = \breve{A}^T \tilde{\pi} \text{ for some } \tilde{\pi} \in \mathbb{R}^{S-1}_{++}\} \\
&= \sum_{s=2}^{S} \{\mu_s \breve{A}^T_{s\cdot} |\ \mu_s \in \mathbb{R}_{++}\},
\end{aligned}$$

i.e. $\mathcal{Q}_{norm}(A)$ is either the interior of a convex polyeder with $\hat{q} = 0$ being element of the boundary or the whole space $\mathbb{R}^{S-1}$.[6]

[5] In contrast *cl* denotes the closure of a set.

[6] The sum of two sets $\mathcal{R}, \mathcal{W} \subseteq \mathbb{R}^N$ is defined as

$$\mathcal{R} + \mathcal{W} \ := \ \{x \in \mathbb{R}^N \mid \exists\, r \in \mathcal{R}, w \in \mathcal{W} \text{ with } x = r + w\}.$$

(III)

$$
\begin{aligned}
&\mathcal{Q}_{norm}(A)\\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \begin{pmatrix} 1 \\ \hat{q} \end{pmatrix} = A^T\pi \text{ for some } \pi \in \mathbb{R}^S_{++}\}\\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \hat{q} = A^T_{\cdot(-1)}\pi\ \wedge 1 = \sum_{s \in \mathcal{S}^+(A_{\cdot 1})} A_{s1}\pi_s \text{ for some } \pi \in \mathbb{R}^S_{++}\}\\
&= \{\hat{q} \in \mathbb{R}^{J-1} |\ \hat{q} = \sum_{s \in \mathcal{S}^+(A_{\cdot 1})} \frac{\mu_s}{A_{s1}} A^T_{s(-1)} + \sum_{s \in \mathcal{S}^0(A_{\cdot 1})} \mu_s A^T_{s(-1)}\\
&\qquad \text{for some } \mu \in \mathbb{R}^S_{++} \text{ with } \sum_{s \in \mathcal{S}^+(A_{\cdot 1})} \mu_s = 1\}\\
&= int\left(conv \left\{ \left(\frac{1}{A_{s1}} A^T_{s(-1)} \right)_{s \in \mathcal{S}^+(A_{\cdot 1})} \right\} \right)\\
&\qquad + \sum_{s \in \mathcal{S}^0(A_{\cdot 1})} \{\mu_s A^T_{s(-1)} |\ \mu_s \in \mathbb{R}_{++}\},
\end{aligned}
$$

i.e. $\mathcal{Q}_{norm}(A)$ is either the interior of a convex polyeder where the extremal directions are contained in $\{A^T_{s(-1)} |\ s \in \mathcal{S}^0(A_{\cdot 1})\}$, or it is the whole space $\mathbb{R}^{J-1}$.

Define

$$
\begin{aligned}
\mathcal{E}_{norm}(A) \ :=\ & \{\hat{q}^* \in \mathcal{Q}_{norm}(A) |\ \exists\ (\theta^{*i})_{i=1,\dots,I} \text{ with}\\
& \left((\theta^{*i})_{i=1,\dots,I}, \begin{pmatrix} 1 \\ \hat{q}^* \end{pmatrix} \right) \text{ being an FM-equilibrium}\}
\end{aligned}
$$

as the set of normalized equilibrium price vectors. Remember that the GEI-economy exhibits a *unique* normalized financial markets equilibrium if $\sharp\mathcal{E}_{norm}(A) = 1$.

Note that the notation does not take into account the dependence on utility functions and endowments. This is because here I am only interested in the effect of changing the asset structure while consumers' characteristics remain fixed.

At first I have to specify the idea of perturbation of the asset matrix. Therefore, I define the maximum norm for a matrix $M \in \mathbb{R}^{N \times L}$ by

$$
\|M\|_{\max} := \max \{|M_{nl}| \mid n \in \{1, \dots, N\}, l \in \{1, \dots, L\}\}.
$$

The same notation is used for all $N, L \in \mathbb{N}$. Besides, notice that for $L = 1$ or $N = 1$ one obtains a vector norm.

An ϵ-*neighborhood* of some $\bar{M} \in \mathbb{R}^{N \times L}$ is defined as

$$\mathcal{V}_\epsilon(\bar{M}) := \{M \in \mathbb{R}^{N \times L} \mid \|\bar{M} - M\|_{\max} < \epsilon\}.$$

Let $\bar{A} \in \mathbb{R}^{S \times J}$ and $\epsilon > 0$. An asset matrix $A \in \mathbb{R}^{S \times J}$ is an ϵ-*perturbation* of $\bar{A}$ if and only if $\|\bar{A} - A\| < \epsilon$ or equivalently $A \in \mathcal{V}_\epsilon(\bar{A})$, i.e. A differs in each entry by less than ϵ from the corresponding entry of $\bar{A}$. Therefore, the set of all ϵ-perturbations of $\bar{A}$ is given by the ϵ-neighborhood of $\bar{A}$.

Before I start with a detailed proof of the main theorem of this section (Theorem 4.34) let me give a brief outline.

To obtain robustness of the equilibrium price set against perturbations of the asset matrix, at first these perturbations are restricted such that the perturbed matrices display the same rank as the "original" one and their span remains to contain a desirable portfolio.[7] Then, the Implicit Function Theorem is applied to obtain an asset demand function on the — beforehand defined — reduced set of asset matrices and their no-arbitrage price systems which is continuously differentiable.[8] The same theorem is used a second time to get local uniqueness and lower hemi-continuity of equilibrium price sets under a further restriction of perturbations.[9] Besides, it is possible to reduce no-arbitrage price sets to a compact subset which has to contain all additional equilibrium price systems, but none of the "old" ones.[10] The compact price set allows to apply the Maximum Theorem on the function defined by the maximum norm of asset demand and restrict perturbations[11] such that this function is bounded away from zero and takes its minimum.

To summarize, the proof of Theorem 4.34 can be divided into three main steps:

1. the proof of "continuous differentiability" of asset demand on some subset of its whole range;

2. the proof of lower hemi-continuity of the equilibrium price correspondence, i.e. perturbations of the asset matrix can be restricted such that for each equilibrium price systems of the "original" matrix there is some "nearby" equilibrium price system of the "new" matrix;

3. the proof of upper hemi-continuity of the equilibrium price correspondence, i.e. perturbations can be restricted further such that in addition

[7]This induces the definitions of ϵ_1 and ϵ_2 in the following proof.

[8]Within this context ϵ_4 and ϵ_5 will be defined.

[9]Compare ϵ_6.

[10]Compare the definitions of ϵ_7, ϵ_8, ϵ_9, and ϵ_{10} of which some are only used in the proofs presented in the appendix.

[11]Compare definition of ϵ_{11}.

to the equilibrium price systems obtained by the second step no other equilibrium price systems arise.

One problem in the consideration of ϵ-perturbations of some matrix $\bar{A}$ satisfying 2.14 respectively 4.25 consists in the fact that the set of all asset matrices satisfying these assumptions is not an open subset of $\mathbb{R}^{S\times J}$. Therefore, it might happen that for given $\bar{A}$ one cannot specify an $\epsilon > 0$ such that for all $A \in \mathcal{V}_\epsilon(\bar{A})$ Assumption 2.14 respectively 4.25 is satisfied.

Example 4.26
Take the asset matrix $\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ *which satisfies Assumption 2.14 as well as Assumption 4.25. For all* $\epsilon > 0$ *one observes* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{\epsilon}{2} & -\frac{\epsilon}{2} \end{pmatrix} \in \mathcal{V}_\epsilon(\bar{A})$ *even though 2.14 respectively 4.25 are violated.*

Assume $\bar{A}$ satisfies Assumptions 2.21 and 4.25. Let me first start with the assumption of a maximal rank. If $rank\ \bar{A} = J$, there has to be some subset $G \subseteq \{1, \ldots, S\}$ with $\sharp G = J$ such that $rank\ \bar{A}_{G\cdot} = J$ or equivalently that the determinant of this matrix is unequal to zero. Therefore, one can define an ϵ_1-neighborhood of $\bar{A}_{G\cdot}$ such that the determinant of all $J \times J$-dimensional matrices in this neighborhood is unequal to zero, i.e. the matrices satisfy $rank = J$. This implies for all $A \in \mathcal{V}_{\epsilon_1}(\bar{A})$ the property $rank\ A = J$.

Now consider Assumption 4.25, i.e. $A_{\cdot 1} > 0$. Here one has to distinguish the cases (I) to (III) for $\bar{A}$.

(I) Choose $\epsilon_2 \in (0, \min\{\epsilon_1, \bar{A}_{11}, \ldots, \bar{A}_{S1}\})$. Then all $A \in \mathcal{V}_{\epsilon_2}(\bar{A})$ satisfy Assumptions 2.21 and 4.25 and ϵ-perturbations do not have to be restricted.

(II) Modeling first period consumption only those ϵ-perturbations of $\bar{A}$ seem to make sense which make it possible to stay within this framework, i.e. one only perturbs the matrix $\check{A}$ and all the other entries of $\bar{A}$ remain fixed. Choose $\epsilon_2 > 0$ such that for all $\dot{A} \in \mathcal{V}_{\epsilon_2}(\check{A})$ it is $rank\ \dot{A} = J - 1$. The existence of such an $\epsilon_2 > 0$ follows from the explanations given above for Assumption 2.21.

Define the set of all restricted ϵ-perturbations as

$$\{A \in \mathbb{R}^{S\times J} \mid A = \begin{pmatrix} 1 & 0 \\ 0 & \dot{A} \end{pmatrix} \text{ with } \dot{A} \in \mathcal{V}_\epsilon(\check{A})\}.$$

Choosing $\epsilon = \epsilon_2$ all restricted ϵ-perturbations of $\bar{A}$ model first period consumption and satisfy Assumptions 2.21 and 4.25.

(III) Choose $\epsilon_2 \in (0, \min\{\epsilon_1, \min\{A_{s1} |\ s \in \mathcal{S}^+(\bar{A}_{\cdot 1})\}\}$ and define the set of all restricted ϵ-perturbations of $\bar{A}$ as

$$\{A \in \mathbb{R}^{S \times J} |\ A_{s1} = 0 \ \forall\ s \in \mathcal{S}^0(\bar{A}_{\cdot 1}) \ \wedge\ \|\bar{A} - A\|_{\max} < \epsilon\}.$$

With $\epsilon = \epsilon_2$ all restricted ϵ-perturbations satisfy Assumptions 2.21 and 4.25.

In the general Finance GEI-model an asset matrix $\bar{A}$ does not satisfy Assumption 4.25, but only Assumption 2.14. Assume $\bar{A}\theta_{\bar{A}} > 0$ where without loss of generality $\theta_{\bar{A}1} \neq 0$. Then for

$$\tilde{A}(\bar{A}) := \bar{A}\left(\theta_{\bar{A}}, e^2, \ldots, e^J\right). \tag{4.17}$$

the Assumption 4.25 is satisfied and since $< \bar{A} >=< \tilde{A}(\bar{A}) >$ for the corresponding GEI-economies there is a one-to-one correspondence between the equilibria (compare Proposition 4.3 and Theorem 2.19). Therefore — since attention will be restricted to Assumption 4.25 and the above-defined (restricted) perturbations of matrices satisfying this assumption — the following point has to be clarified:

Lemma 4.27
Having defined a set of (restricted) $\tilde{\epsilon}$-perturbations of $\tilde{A}(\bar{A})$, there is an ϵ and restricted ϵ-perturbations of $\bar{A}$ such that the attainable matrices A satisfy $\tilde{A}(A)$ to be (restricted) $\tilde{\epsilon}$-perturbations of $\tilde{A}(\bar{A})$.

Proof: The proof which requires distinction of the three types of matrices is deferred to Appendix B.1.

Now assume a GEI-economy $\left\{(U^i, \omega^i)_{i=1,\ldots,I}, \bar{A}\right\}$ is given where $\bar{A}$ satisfies Assumption 4.25. The long-term aim is to investigate whether one is able to define some set of (restricted) ϵ-perturbations of $\bar{A}$ such that all elements A satisfy

$$\sharp\mathcal{E}_{norm}(A) = \sharp\mathcal{E}_{norm}(\bar{A}),$$

i.e. the number of normalized equilibria for the perturbed asset structure is the same as for the original one.

To avoid the necessity of distinction of cases (I) to (III) all the time, I define

$$\bar{C} := \begin{cases} \bar{A}, & \text{if } \bar{A} \text{ satisfies case (I);} \\ \check{A}, & \text{if } \bar{A} \text{ satisfies case (II);} \\ (\bar{A}_{sj})_{(s,j)\in\{1,\ldots,S\}\times\{1,\ldots,J\}\setminus\{(s,1)|\ s\in\mathcal{S}^0(\bar{A}_{\cdot 1})\}}, & \text{if } \bar{A} \text{ satisfies case (III);} \end{cases}$$

and

$$\mathcal{R} := \begin{cases} \mathbb{R}^{S\times J}, & \text{if } \bar{A} \text{ satisfies case (I);} \\ \mathbb{R}^{(S-1)\times(J-1)}, & \text{if } \bar{A} \text{ satisfies case (II);} \\ \mathbb{R}^{\sharp\mathcal{S}^+(\bar{A}_{\cdot 1})} \times \mathbb{R}^{S\times(J-1)}, & \text{if } \bar{A} \text{ satisfies case (III).} \end{cases}$$

Notice that for case (III) $\bar{C}$ does not have the form of a matrix, but looks as follows:

$$\bar{C} = (\bar{C}_{\cdot 1}, \bar{C}_{\cdot(-1)})$$

where $\bar{C}_{\cdot(-1)} = \bar{A}_{\cdot(-1)}$ is an $S \times (J-1)$ matrix and $\bar{C}_{\cdot 1} = (\bar{A}_{s1})_{s\in\mathcal{S}^+(\bar{A}_{\cdot 1})}$ is a column vector with $\sharp\mathcal{S}^+(\bar{A}_{\cdot 1}) < S$ entries. For simplification I transfer indexation from A to C.

With this notation all (restricted) ϵ_2-perturbations in the different cases can be summarized by

$$\mathcal{V}_{\epsilon_2}(\bar{C}) := \{C \in \mathcal{R} | \; \|\bar{C} - C\|_{\max} < \epsilon_2\}$$

where $\epsilon_2 > 0$ was specified before. Analogously, I define the reverse direction, i.e.

(I)

$$A_{sj}(C) := C_{sj} \; \forall \; (s,j) \in \{1,\dots,S\} \times \{1,\dots,J\};$$

(II)

$$A_{sj}(C) := \begin{cases} 1, & \text{if } (s,j) = (1,1), \\ 0, & \text{if } (s,j) \in (\{1\}\times\{2,\dots,J\}) \cup (\{2,\dots,S\}\times\{1\}), \\ C_{sj}, & \text{else;} \end{cases}$$

(III)

$$A_{sj}(C) := \begin{cases} C_{sj}, \text{if } (s,j) \in \{1,\dots,S\}\times\{1,\dots,J\} \backslash \{(s,1) | \; s \in \mathcal{S}^0(\bar{A}_{\cdot 1})\}, \\ 0, \quad \text{else;} \end{cases}$$

where $A(\bar{C}) = \bar{A}$ and $A(\mathcal{V}_{\epsilon_2}(\bar{C}))$ is the set of all (restricted) ϵ_2-perturbations of $\bar{A}$. From now on take asset demand as a function of $\hat{q}$ and C, i.e.

$$\tilde{\theta}(\hat{q}, C) := \theta(1, \hat{q}, A(C)).$$

Correspondingly, it is

$$\tilde{\mathcal{Q}}_{norm}(C) := \mathcal{Q}_{norm}(A(C))$$

such that $\tilde{\theta}$ is defined on $\mathcal{D}_{\epsilon_2}$ where

$$\mathcal{D}_{\epsilon_2} := \{(\hat{q}, C) \in \mathbb{R}^{J-1} \times \mathcal{R} | \; \|\bar{C} - C\|_{\max} < \epsilon_2 \; \wedge \; \hat{q} \in \tilde{\mathcal{Q}}_{norm}(C)\}.$$

$\mathcal{D}_{\epsilon_2}$ is an open connected set in $\mathbb{R}^{J-1} \times \mathcal{R}$. This is a consequence of the following lemma because $\tilde{\mathcal{Q}}_{norm}(C)$ is an open convex set in $\mathbb{R}^{J-1}$ for all $C \in \mathcal{V}_{\epsilon_2}(\bar{C})$.

Lemma 4.28
Let $\mathcal{K} \subseteq \tilde{\mathcal{Q}}_{norm}(\bar{C})$ be a compact set. Then, there is an $\epsilon_3 \in (0, \epsilon_2]$ such that for all $C \in \mathcal{V}_{\epsilon_3}(\bar{C})$ it is $\mathcal{K} \subseteq \tilde{\mathcal{Q}}_{norm}(C)$.

Proof: See Appendix B.1.

Take, for example, case (II) with $S = 4$ and $J = 3$ such that

$$\tilde{\mathcal{Q}}_{norm}(\bar{C}) = \{\hat{q} \in \mathbb{R}^2 |\ \hat{q} = \bar{C}^T \pi \text{ for some } \pi \in \mathbb{R}^3_{++}\}.$$

The intuition for the lemma is given by Figure 4.2 where

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \text{ or } \bar{C}^T = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix}.$$

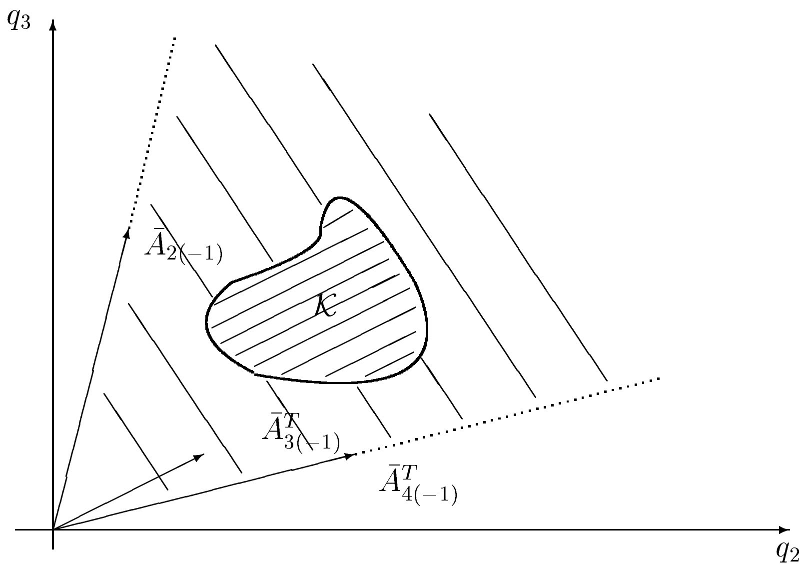

Figure 4.2: Example for Lemma 4.28.

The shaded area defines $\tilde{\mathcal{Q}}_{norm}(\bar{C})$. It is obvious that small perturbations of $\bar{C}$ are possible such that $\mathcal{K}$ remains within the area defined by convex combinations of positive multiples of the row vectors of C.

Note that in case (II) and (III) the correspondence $cl\left(\tilde{\mathcal{Q}}_{norm}(C)\right)$ need not be continuous in $\bar{C}$.[12]

Example 4.29

Again take $\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Therefore it is* $\bar{C}_{case(II)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\bar{C}_{case(III)} = \left(1, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$. *Observe* $cl\left(\tilde{\mathcal{Q}}_{norm}(\bar{C}_{case(II)}\right) = cl\left(\tilde{\mathcal{Q}}_{norm}(\bar{C}_{case(III)}\right) = \mathbb{R}_+$. *On the other hand for* $C_{case(II)} = \binom{1}{-\epsilon}$ *and* $C_{case(III)} = \left(1, \begin{pmatrix} 0 \\ 1 \\ -\epsilon \end{pmatrix}\right)$ *it is* $cl\left(\tilde{\mathcal{Q}}_{norm}(C_{case(II)})\right) = cl\left(\tilde{\mathcal{Q}}_{norm}(C_{case(III)})\right) = \mathbb{R}$ *for* $\epsilon > 0$ *arbitrarily small.*

Note that in case (I) the correspondence $cl\left(\tilde{\mathcal{Q}}_{norm}(C)\right)$ is continuous in $\bar{C}$ because the convex hull of a finite number of points is continuous in the coordinates of these points. For cases (II) and (III) one observes that even if one excludes such strange matrices with $\bar{A}_{s\cdot} = 0$ for some $s \in \{1, \ldots, S\}$ the correspondence $cl\left(\tilde{\mathcal{Q}}_{norm}(C)\right)$ is not continuous in $\bar{C}$ because the set is unbounded and changing $\bar{C}$ to C might change the extremal directions such that points on such an extremal ray are arbitrarily far away from $cl\left(\tilde{\mathcal{Q}}_{norm}(\bar{C})\right)$ if their distance to the origin is sufficiently large.

4.3.2 Continuous Differentiability of Asset Demand

With respect to the first order conditions (2.7) and (2.8) define

$$F^i(\theta^i, \lambda^i, \hat{q}, C) := \begin{pmatrix} \nabla U^i(\omega^i + A(C)\theta^i)A(C) - \lambda^i \binom{1}{\hat{q}} \\ (1, \hat{q}^T)\ \theta^i \end{pmatrix}$$

where one knows (compare, for example, Hens (1991)) that for all $(\breve{\hat{q}}, \breve{C}) \in \mathcal{D}_{\epsilon_2}(\bar{C})$ there is some $\breve{\theta}^i \in \mathbb{R}^J$ and $\breve{\lambda}^i \in \mathbb{R}_{++}$ solving

$$F^i(\theta^i, \lambda^i, \breve{\hat{q}}, \breve{C}) = 0. \tag{4.18}$$

[12]For the concept of continuous correspondences I refer to Takayama (1974, p. 151). Compare the Appendix A.2.2 for a summary.

Since $\mathcal{D}_{\epsilon_2}(\bar{C})$ is an open set in $\mathbb{R}^{J-1} \times \mathcal{R}$, it is obvious that for all $(\breve{\theta}^i, \breve{\lambda}^i, \breve{q}, \breve{C})$ with $F^i(\breve{\theta}^i, \breve{\lambda}^i, \breve{q}, \breve{C}) = 0$ there is an open neighborhood on which F^i is continuously differentiable. Therefore, the Implicit Function Theorem can be applied: for all $(\breve{q}, \breve{C}) \in \mathcal{D}_{\epsilon_2}(\bar{C})$ there exist $\epsilon_4, \epsilon_5 > 0$ such that there is a continuously differentiable function

$$(\tilde{\theta}^i, \tilde{\lambda}^i) : \mathcal{V}_{\epsilon_4}(\breve{q}, \breve{C}) \rightarrow \mathcal{V}_{\epsilon_5}(\breve{\theta}^i, \breve{\lambda}^i)$$

with

$$F^i(\tilde{\theta}^i(\hat{q}, C), \tilde{\lambda}^i(\hat{q}, C), \hat{q}, C) = 0.$$

One only has to prove

$$\partial_{\theta^i, \lambda^i} F^i(\breve{\theta}^i, \breve{\lambda}^i, \breve{q}, \breve{C}) := \begin{pmatrix} A^T(\breve{C}) D^2 U^i(\omega^i + A(\breve{C})\breve{\theta}^i) A(\breve{C}) & -\binom{1}{\breve{q}} \\ \left(1, \breve{q}^T\right) & 0 \end{pmatrix}$$

to be invertible which follows from strict quasi-concavity of U^i and the assumption of non-redundant assets (Assumption 2.21).

Therefore, since $\mathcal{D}_{\epsilon_2}$ is connected, first order conditions (2.7) and (2.8) define a continuously differentiable function

$$\tilde{\theta}^i : \mathcal{D}_{\epsilon_2}(\bar{C}) \rightarrow \mathbb{R}^J.$$

Continuous differentiability is an additive property such that $\tilde{\theta}(\hat{q}, C) := \sum_{i=1}^{I} \tilde{\theta}^i(\hat{q}, C)$ is also continuous differentiable on $\mathcal{D}_{\epsilon_2}(\bar{C})$.

Proposition 4.30 (Continuous Differentiability of Asset Demand)
For each GEI-economy $\{(U^i, \omega^i)_{i=1,\ldots,I}, \bar{A}\}$ where $\bar{A}$ satisfies the Assumption 4.25, there is an $\epsilon_2 > 0$ such that market demand for assets

$$\tilde{\theta} : \mathcal{D}_{\epsilon_2}(\bar{C}) \rightarrow \mathbb{R}^J$$

is continuously differentiable.

Remark 4.31
It would be nice to have the differentiability of asset demand $\theta(\hat{q}, A)$ on the whole set

$$\mathcal{D} := \{(\hat{q}, A) \in \mathbb{R}^{J-1} \times \mathbb{R}^{S \times J} \,|\, \mathit{rank}\ A = J, A_{\cdot 1} > 0 \ \wedge \ \hat{q} \in \mathcal{Q}_{norm}(A)\}.$$

However, as already mentioned, this set is not open. Therefore, aiming at application of the Implicit Function Theorem it is at least necessary to restrict to the set

$$\tilde{\mathcal{D}} := \{(\hat{q}, A) \in \mathbb{R}^{J-1} \times \mathbb{R}^{S \times J} \,|\, \mathit{rank}\ A = J, A_{\cdot 1} \gg 0 \ \wedge \ \hat{q} \in \mathcal{Q}_{norm}(A)\}.$$

Even this restriction is not sufficient to obtain a continuous function $\theta(\hat{q}, A)$ *defined on* $\tilde{\mathcal{D}}$ *because*

$$\tilde{\mathcal{A}} := \{A \in \mathbb{R}^{S\times J} \mid rank\ A = J\ \wedge A_{\cdot 1} \gg 0\}$$

is not connected as can be proved with a simple example: assume $S = J$ *and take two different matrices* $\hat{A}, \check{A} \in \tilde{\mathcal{A}}$ *with* $\det(\hat{A}) > 0$ *and* $\det(\check{A}) < 0$. *Since the determinant function is continuous in the entries of the matrix, for each continuous path connecting* $\hat{A}$ *and* $\check{A}$ *there has to be some element of the path with determinant value equal to zero. This contradicts the assumption on the rank.*

4.3.3 Robustness of the Number of Equilibria

Under the normalization used in this section regularity of asset demand requires

$$rank\ \partial_{\hat{q}}\theta_{-1}(1, \hat{q}, \bar{A}) = J - 1$$

for all $\hat{q} \in \mathcal{E}_{norm}(\bar{A})$. For given regularity the set of normalized equilibrium price systems is finite, i.e. $\mathcal{E}_{norm}(\bar{A}) = \{\hat{q}^{*1}, \ldots, \hat{q}^{*N}\}$ for some $N \in \mathbb{N}$.

Because of Walras' Law, i.e. $(1, \hat{q}^T)\tilde{\theta}(\hat{q}, A(C)) = 0$, to investigate equilibrium properties of a price system it is sufficient to test for the clearing of $J - 1$ asset markets, i.e.

$$\tilde{F}(\hat{q}, C) := \tilde{\theta}_{-1}(\hat{q}, C) = 0. \tag{4.19}$$

This function $\tilde{F} : \mathcal{D}_{\epsilon_2} \to \mathbb{R}^{J-1}$ is continuously differentiable and satisfies $\tilde{F}(\hat{q}^{*1}, \bar{C}) = \ldots = \tilde{F}(\hat{q}^{*N}, \bar{C}) = 0$. Additionally — because of regularity — the matrices

$$\partial_{\hat{q}}\tilde{F}(\hat{q}^{*n}, \bar{C}) = \partial_{\hat{q}}\tilde{\theta}_{-1}(\hat{q}^{*n}, \bar{C}) = \partial_{\hat{q}}\theta_{-1}(1, \hat{q}^{*n}, \bar{A}), n = 1, \ldots, N,$$

are invertible which allows to apply the Implicit Function Theorem on equation (4.19) for $\bar{C}$ and all $\hat{q}^{*n}, n = 1, \ldots, N$: there exist $\delta^n > 0$ and $\epsilon^n \in (0, \epsilon_2)$ such that there are continuously differentiable equilibrium price functions

$$\hat{q}^n_* : \mathcal{V}_{\epsilon^n}(\bar{C}) \to \mathcal{V}_{\delta^n}(\hat{q}^{*n})$$

with $\tilde{F}(\hat{q}^n_*(C), C) = \tilde{\theta}_{-1}(\hat{q}^n_*(C), C) = 0$ for all $C \in \mathcal{V}_{\epsilon^n}(\bar{C})$ and for given $C \in \mathcal{V}_{\epsilon^n}(\bar{C})$ the price system $\hat{q}^n_*(C)$ is the only element of $\mathcal{V}_{\delta^n}(\hat{q}^{*n})$ which satisfies equation (4.19).

With $\epsilon_6 := \min\{\epsilon^n \mid n = 1, \ldots, N\}$ this implies

$$\sharp\mathcal{E}_{norm}(\bar{A}) \leq \sharp\mathcal{E}_{norm}(A(C))$$

for all $C \in \mathcal{V}_{\epsilon_6}(\bar{C})$, i.e. for all (restricted) ϵ_6-perturbations of $\bar{A}$ the corresponding GEI-economy is characterized by at least the same number of equilibria as the starting economy. Even stronger: there exist continuously differentiable equilibrium price paths!

Theorem 4.32 (Lower Hemi-Continuity of Equilibrium Prices)[13]
For all regular GEI-economies $\{(U^i, \omega^i)_{i=1,\ldots,I}, \bar{A}\}$ which satisfy Assumption 4.25 there exists some $\epsilon > 0$ such that all GEI-economies $\{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ with A (restricted) ϵ-perturbation of $\bar{A}$ ($C \in \mathcal{V}_\epsilon(\bar{C})$) satisfy

$$\sharp\mathcal{E}_{norm}(\bar{A}) \leq \sharp\mathcal{E}_{norm}(A).$$

If $\mathcal{E}_{norm}(\bar{A}) = \{\hat{q}^{*1}, \ldots, \hat{q}^{*N}\}$, there are continuously differentiable functions

$$\hat{q}_*^n : \mathcal{V}_\epsilon(\bar{C}) \to \mathcal{V}_\delta(\hat{q}^{*n}), n = 1, \ldots, N,$$

for some $\delta > 0$ with $\{\hat{q}_*^1(C), \ldots, \hat{q}_*^N(C)\} \subseteq \mathcal{E}_{norm}(A(C))$ whenever $C \in \mathcal{V}_\epsilon(\bar{C})$.

In other words: the set $\mathcal{E}_{norm}(A(C))$ is a lower hemi-continuous correspondence in $\bar{C}$. At this point it seems to be interesting to know whether $\mathcal{E}_{norm}(A(C))$ is also upper hemi-continuous and therefore continuous in $\bar{C}$. It will be shown that this is even true for those market structures for which the set $cl(\tilde{\mathcal{Q}}_{norm}(C))$ is not upper hemi-continuous.

Lemma 4.33
There is some $\epsilon_{10} \in (0, \epsilon_6)$ and a compact set $\mathcal{Q}^H \subset \mathbb{R}^{J-1}$ such that whenever $C \in cl(\mathcal{V}_{\epsilon_{10}}(\bar{C}))$

- $\mathcal{Q}^H \subseteq \tilde{\mathcal{Q}}_{norm}(C) \cap \tilde{\mathcal{Q}}_{norm}(\bar{C})$,
- *for all equilibrium price systems $\hat{q}^*(C)$ with respect to C which satisfy $\hat{q}^*(C) \notin \{\hat{q}_*^1(C), \ldots, \hat{q}_*^N(C)\}$ it is*
$$\hat{q}^*(C) \in \mathcal{Q}^H,$$
- *and $\tilde{\theta}_{-1}(\hat{q}, \bar{C}) \neq 0$ for all $\hat{q} \in \mathcal{Q}^H$.*

Proof: See Appendix B.1.

Since $\tilde{\theta}_{-1} : \mathcal{D}_{\epsilon_2} \to \mathbb{R}^{J-1}$ is continuous, this is also true for $\|\tilde{\theta}_{-1}(.,.)\|_{\max}$ on $\mathcal{Q}^H \times cl(\mathcal{V}_{\epsilon_{10}}(\bar{C}))$ because $\mathcal{Q}^H \subseteq \tilde{\mathcal{Q}}_{norm}(C)$ when $C \in cl(\mathcal{V}_{\epsilon_{10}}(\bar{C}))$. Defining

$$m := \min_{\hat{q} \in \mathcal{Q}^H} \|\tilde{\theta}_{-1}(\hat{q}, \bar{C})\|_{\max}$$

[13]By application of Lemma 4.27 this theorem can be transferred for more general matrices satisfying Assumption 2.14.

it is $m > 0$ because $\|\tilde{\theta}_{-1}(\hat{q}, \bar{C})\|_{\max}$ is a continuous function on a compact set which takes its minimum, but is unequal to zero. Since $\|\tilde{\theta}_{-1}(.,.)\|_{\max}$ is a continuous function on a compact set, there is some $\epsilon_{11} > 0$ such that $C \in \mathcal{V}_{\epsilon_{11}}(\bar{C})$ implies

$$\left| \|\tilde{\theta}_{-1}(\hat{q}, C)\|_{\max} - \|\tilde{\theta}_{-1}(\hat{q}, \bar{C})\|_{\max} \right| < m$$

for all $\hat{q} \in \mathcal{Q}^H$ and therefore $\|\tilde{\theta}_{-1}(\hat{q}, C)\|_{\max} \neq 0$ for all $\hat{q} \in \mathcal{Q}^H$. Together with the last theorem one obtains:

Theorem 4.34 (Robustness of the Number of Equilibria)[14]
For all regular GEI-economies $\{(U^i, \omega^i)_{i=1,\dots,I}, \bar{A}\}$ satisfying Assumption 4.25 there exists some $\epsilon > 0$ such that all GEI-economies $\{(U^i, \omega^i)_{i=1,\dots,I}, A\}$ with A (restricted) ϵ-perturbation of $\bar{A}$ ($C \in \mathcal{V}_\epsilon(\bar{C})$) satisfy

$$\sharp\mathcal{E}_{norm}(\bar{A}) = \sharp\mathcal{E}_{norm}(A).$$

One could think about doing the proof of this theorem not in the way I do it, but via the so-called Grassmanian manifolds which are used in different proofs of existence of equilibria in GEI-models because they allow to circumvent the problem of a change in the rank of the matrix of income transfers.[15] I chose another approach for different reasons.

The first one is that the concept of Grassmanians takes into account changing rank of the matrix of income transfers, but not the existence of a desirable portfolio. This is why existence proofs using Grassmanians either assume first period consumption when a parameterization of the economy by consumers' characteristics is given or — when the economy is defined via a market demand function — assume Walras' Law which implicitly requires the existence of a desirable portfolio.

The second argument against the usage of Grassmanians is that for considerations of uniqueness of equilibria it is more appropriate to work with the concept of financial markets equilibrium instead of the no-arbitrage equilibrium. But application of the concept of Grassmanians requires the no-arbitrage equilibrium concept. This is especially because a no-arbitrage equilibrium is independent of the asset matrix itself, but only depends on the space of income transfers, and the considered Grassmanians are sets of spaces of income transfers with given dimension. In contrast, for financial

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[14] By application of Lemma 4.27 this theorem can be transferred for more general matrices satisfying Assumption 2.14.

[15] The Grassmanian manifold $\mathcal{G}_{K,N}$ is the set of all N-dimensional linear subspaces of $\mathbb{R}^K$.

markets equilibria asset demand and asset prices strongly depend on the representation of the space of income transfers.

It would be more interesting to examine the consequences of perturbations allowing changes of the dimension of the trading space than to investigate perturbations within the set of asset structures with constant dimension. The main problem within this context consists in defining a suitable concept of small perturbations: the examples in Sections 4.4 and 4.5 demonstrate that as soon as the market structure changes from $S-1$ assets to complete markets and vice versa the number of equilibria might change. It does not matter how large the entries of the additional asset are, the decisive point is the introduction of a *new* direction and it does not play any role which new (non-redundant) asset is introduced. This question leaves much scope for further research.[16]

4.4 Limits of Transferability

When examining uniqueness of the equilibrium in incomplete markets models, one point is to check the limits of transferability from complete markets. In this section I refer to three different aspects.

The first one is quasi-linearity of utility. From Proposition 4.20 it is known that quasi-linearity is sufficient for a unique equilibrium whenever first period consumption is modeled. The example I give here proves non-sufficiency of quasi-linearity for uniqueness of the equilibrium with incomplete markets in general. It stands out due to its simplicity because it is built on Kehoe's (1991, p. 2066) well-established example on multiple Arrow-Debreu equilibria. I explicitly give consumers' characteristics and the asset structure.

The second aspect of this section is to check what happens with Cobb-Douglas utility when markets are incomplete. I refer to an example given in Hens, Schmedders, and Voß (1999) where a computer algorithm is developed to compute "approximate" characteristics of an economy with multiple equilibria.

In the third part I demonstrate that even though all consumers satisfy the expected utility hypothesis with respect to an objective probability measure

[16]Cass and Citanna (1998) introduce a concept of "small" perturbations of the asset structure which allows changes in the rank of the matrix. They start with the introduction of either a redundant asset or one which is not traded and then vary the yield of this asset such that it becomes non-redundant respectively is traded. But Cass and Citanna's (1998) considerations strongly depend on a sufficient incompleteness of markets such that their concept also does not seem to be suitable for general situations, including the one with incompleteness of only one degree.

and there is no aggregate risk, there might exist multiple equilibria. The reason is that restricted trading hinders consumers from total risk sharing.

These examples examine restrictions on consumers' characteristics which are established to guarantee a unique Arrow-Debreu equilibrium. Since the only change is made for the asset structure while consumers' characteristics remain the same, the examples stress the decisive influence which the market structure has on uniqueness and multiplicity of equilibria. Regrettably, the results are quite destructive because they give the impression that it is much more difficult to obtain a unique equilibrium when markets are incomplete than when they are complete.

This section and the next one show that actual incompleteness of markets may evoke effects which are contrary to the well-known results for complete markets. They demonstrate once more how extremely cautious one has to be about transferring results between complete and incomplete markets models, even if one restricts attention to the simplest model.

Kehoe (1991) proposes his example to be quite helpful in testing computational algorithms for Arrow-Debreu equilibria. I emphasize that the examples presented here are quite helpful when testing computational algorithms for incomplete markets models: many of these algorithms are based on algorithms for the Arrow-Debreu model. Therefore, one should test the undertaken changes with examples stressing the differences between the models.

4.4.1 Limits with Quasi-Linearity

In this section I construct a GEI-economy with two consumers ($I = 2$) and three states ($S = 3$) which proves non-sufficiency of quasi-linearity for a unique equilibrium even though this follows in the Arrow-Debreu model.

Consumers are symmetrical to each other. Whereas utility is linear in state 1-consumption, for the other two states it is of CES-type, i.e.

$$U^i(x) = x_1 + \sum_{s=2}^{3} a_s^i \frac{(x_s^{b^i} - 1)}{b^i}$$

with $a_2^1 = a_3^2 = 1024$, $a_3^1 = a_2^2 = 1$, and $b^1 = b^2 = -4$. Consumers' endowments are given by

$$\omega^1 = (14, 12, 1)^T \quad \wedge \quad \omega^2 = (14, 1, 12)^T.$$

When markets are complete, from Proposition 3.17 one knows the existence of a unique Arrow-Debreu equilibrium because in this case market demand is monotone.

Now consider the incomplete asset structure

$$\bar{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Substitution of tomorrow's budget constraints $x = \omega^i + A\theta$ and substitution of $\hat{\theta}_1 := \omega_2^i + \theta_1$ and $\hat{\theta}_2 := \omega_3^i + \theta_2$ transform consumers' maximization problems to

$$(4.20) \qquad \max_{\hat{\theta} \in \mathbb{R}^2_+} \ \omega_1^i + \left[\sum_{s=2}^{3} \quad a_s^i \frac{(\hat{\theta}_{s-1}^{b^i} - 1)}{b^i} \right] \quad s.t. \quad q \cdot \hat{\theta} = q \cdot \binom{\omega_2^i}{\omega_3^i}.$$

This is equivalent to the consumers' maximization problems in Kehoe's (1991, p. 2066) multiplicity-example. Moreover, market clearing conditions are equivalent as well. Therefore, the number of equilibria is also the same. According to Kehoe (1991, p. 2066) the equilibria are defined by

$$\begin{aligned}
\hat{\theta}^{a1} &= \begin{pmatrix} 8.631 \\ 1.429 \end{pmatrix}, \hat{\theta}^{a2} = \begin{pmatrix} 4.369 \\ 11.571 \end{pmatrix}, q^a = \begin{pmatrix} 0.1129 \\ 0.8871 \end{pmatrix}; \\
\hat{\theta}^{b1} &= \begin{pmatrix} 10.4 \\ 2.6 \end{pmatrix}, \hat{\theta}^{b2} = \begin{pmatrix} 2.6 \\ 10.4 \end{pmatrix}, q^b = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}; \\
\hat{\theta}^{c1} &= \begin{pmatrix} 11.571 \\ 4.369 \end{pmatrix}, \hat{\theta}^{c2} = \begin{pmatrix} 1.429 \\ 8.631 \end{pmatrix}, q^c = \begin{pmatrix} 0.8871 \\ 0.1129 \end{pmatrix}.
\end{aligned}$$

Hence, an example has been found where the incompleteness of markets causes multiplicity of equilibria. However, the example would be very weak if multiplicity did hinge on the zero-row of the asset matrix for exactly that state for which utility functions are quasi-linear with respect to the corresponding consumption.

With reference to Section 4.3 the asset structure $\bar{A}$ is of type (III) and allows only restricted perturbations. If the three known equilibria of the economy $GEI = \{(U^i, \omega^i)_{i \in I}, \bar{A}\}$ are regular[17], Theorem 4.32[18] implies that the result is robust against restricted perturbations of $\bar{A}$, i.e. there is some $\epsilon > 0$ such that for all

$$A \in \{A \in \mathbb{R}^{S \times J} \mid A_{11} = A_{31} = 0 \ \wedge \|A - \bar{A}\|_{\max} < \epsilon\}$$

[17] Regularity of a single equilibrium price system is defined as the determinant of the Jacobian of market demand having maximal rank for this price system.

[18] To be more precise, this theorem has to be adapted in the following way: for a set of regular equilibria with respect to $\bar{A}$ an $\epsilon > 0$ can be defined such that for (restricted) ϵ-perturbations of $\bar{A}$ the corresponding economies have at least the number of equilibria as are contained in the set.

the economy $GEI = \{(U^i, \omega^i)_{i\in I}, A\}$ has multiple equilibria even though $GE = \{(U^i, \omega^i)_{i\in I}\}$ is characterized by a unique Arrow-Debreu equilibrium. Regularity of the three equilibria in $GEI = \{(U^i, \omega^i)_{i\in I}, \bar{A}\}$ however follows directly from Kehoe's example (1991, p. 2066) since he proves "his" second equilibrium to be regular and it is straightforward to show regularity of the other two equilibria.

Remark 4.35

1. *It can be shown that endowments* $\omega_1^i = 14, i = 1, 2,$ *are sufficiently large to guarantee* $x_1^i \gg 0, i = 1, 2,$ *in the relevant domain of feasible allocations for complete as well as for incomplete markets. Therefore, no problems arise because of corner solutions.*

2. *Since it is not clarified whether* $GEI = \{(U^i, \omega^i)_{i\in I}, \bar{A}\}$ *admits more than the three established equilibria and hence it is not known whether further non-regular equilibria exist, Theorem 4.34 cannot be applied: the number of equilibria might change as a consequence of perturbing the asset structure. The only thing to be sure is the fact that there are at least three equilibria, i.e. multiplicity is guaranteed.*

3. *The example is also robust against perturbations of consumers' characteristics* $a_s^i, b^i, i = 1, 2, s = 2, 3.$

The intuition behind the example is very simple: whereas individual asset demand functions are monotone with respect to the same normalizing vector when markets are complete, this property fails to hold for the considered incomplete asset structure.

To illuminate a little bit to what extent different asset structures effect the number of equilibria restrict attention to matrices of the form

$$A(\beta) := \begin{pmatrix} 0 & \beta \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with arbitrary $\beta \in \mathbb{R}$. For considerations of the number of equilibria this is equivalent to varying the second asset in

$$A = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, A_{\cdot 2} \right).$$

since the number of equilibria does not depend on the asset matrix itself, but on its span (Proposition 4.3). Because of symmetry of the example with respect to states 2 and 3 the same can be done for varying the first asset in

$$A = \left(A_{\cdot 1}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

From the first considerations in this section one knows that for β close or equal to zero there are multiple equilibria which excludes market asset demand to be monotone for any normalizing vectors. For $\beta \to \infty$ one observes collinearity between

$$\lim_{\beta \to \infty} A_{\cdot 2}(\beta) \text{ and } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For the asset matrix

$$\bar{A}(\infty) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

however the function $\theta_1(q_1, 1, \bar{A}(\infty))$[19] is strictly decreasing in q_1 (eliminate state 3 from the maximization problem and then apply Proposition 4.20[20]) which implies uniqueness of the equilibrium. Since equilibria for $\bar{A}(\infty)$ are regular, because of Proposition 4.3 and Theorem 4.34 for large $\beta \in \mathbb{R}$ equilibria are unique. Besides the Implicit Function Theorem does not only assure continuity of the demand functions in prices and parameter $\beta \in \mathbb{R}$ for β large, but also continuous differentiability. Therefore, demand for the first asset is strictly monotone in its own price for large $\beta \in \mathbb{R}$.[21]

Let me finish the considerations of the first example with some further remarks:

Remark 4.36

1. *Obviously, the monotonicity behavior of market demand for given $A(\beta)$ depends on the size of β.*

2. *If the market demand function for given $A(\beta)$ is monotone with respect to some normalizing vector, then the market demand for other asset*

[19] Here, the price of the second asset is normalized to one.

[20] Note that the proposition is formulated for an economy with $\left(\bar{A}_{(-3)2}(\infty), \bar{A}_{(-3)1}(\infty)\right)$, but it is intuitive how to reformulate it to be appropriate at this point.

[21] One can apply the same reasoning for $\beta \to -\infty$, but another price normalization has to be chosen.

matrices with the same span is also monotone. Merely the normalizing vector changes (compare Proposition 4.10).

3. *For $\beta = -244/243$ the allocation (ω^1, ω^2) is constrained Pareto-efficient with respect to $A(\beta)$ such that the equilibrium, namely the no-trade equilibrium, is unique (compare Section 2.7 and Proposition 4.14).*

The example given in this section is one of the easiest to imagine which proves the possibility of multiple equilibria for incomplete markets in spite of a unique Arrow-Debreu equilibrium. However, it would be interesting to know what happens under the expected utility hypothesis. Therefore, I refer to the following two subsections. Regrettably, examples given there violate Assumption 2.14 (Subsection 4.4.2) respectively are built on utility functions which are only implicitly defined by fixing some values of their first derivatives (Subsection 4.4.3).

4.4.2 Limits with Cobb-Douglas Utility

One possibility to prove multiplicity of equilibria is to compute an equilibrium with a negative index because the Index Theorem 3.6/4.7 requires the existence of at least one equilibrium with a positive index. However, standard computational methods are bound to compute equilibria with a positive index (compare Mas-Colell (1985)).

The example in Hens, Schmedders, and Voß (1999) is constructed by a new method to compute multiple equilibria. The main interest is to find out whether some class of economies, namely economies with Cobb-Douglas utility functions, admits multiple equilibria. The trick is to take the data of two prospective equilibria as given and use them to compute by means of a homotopy algorithm characteristics of an economy displaying these two equilibria. For a description of the algorithm compare Hens, Schmedders, and Voß (1999, Section 5). The homotopy path is defined via transition from the first to the second equilibrium. Notice that Cobb-Douglas preferences allow to simplify the extended system (2.7), (2.8), and (2.9) which constitutes the basis for the algorithm.

As pointed out in Section 3.5.1, when markets are complete an explicit pricing formula is given. Besides, market demand satisfies the property of strict gross substitution (Section 3.5.4). For incomplete markets this property fails and there does not exist an explicit pricing formula. As the following example (compare Hens, Schmedders, and Voß (1999, Example 4.2)) demonstrates, even the uniqueness of the equilibrium breaks down, i.e. all the merits of Cobb-Douglas preferences get lost when switching from complete to incomplete markets.

Assume two agents are characterized by utility functions

$$\begin{aligned} U^1(x^1) &= 0.1\ln x_1^1 + 0.2\ln x_2^1 + 0.3\ln x_3^1 + 0.1\ln x_4^1 + 0.3\ln x_5^1, \\ U^2(x^1) &= 0.3\ln x_1^2 + 0.2\ln x_2^2 + 0.1\ln x_3^2 + 0.3\ln x_4^2 + 0.1\ln x_5^2, \end{aligned}$$

and endowments

$$\omega^1 = \begin{pmatrix} 1.5942379526 \cdot 10^{-2} \\ 2.9374491276 \cdot 10^{-3} \\ 6.1116956313 \cdot 10^{-2} \\ 2.6125379370 \cdot 10^{-1} \\ 1.3112543254 \cdot 10^{-1} \end{pmatrix}, \omega^2 = \begin{pmatrix} 6.5484744962 \cdot 10^{-3} \\ 6.1159308802 \cdot 10^{-2} \\ 4.6047779430 \cdot 10^{-3} \\ 7.1874620630 \cdot 10^{-1} \\ 7.6887456746 \cdot 10^{-1} \end{pmatrix}.$$

The asset structure is given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5.7551080261 & -11.211070031 \\ -49.735960347 & -14.564562225 & 1.4276396811 \end{pmatrix}.$$

This economy has at least the three equilibria:

$$(\bar{x}^1, \bar{x}^2, \bar{\theta}^1, \bar{\theta}^2, \bar{q}) = \left(\begin{pmatrix} 3.1569068369 \cdot 10^{-3} \\ 2.0847535004 \cdot 10^{-2} \\ 5.6794692704 \cdot 10^{-2} \\ 4.0 \cdot 10^{-1} \\ 5.0 \cdot 10^{-1} \end{pmatrix}, \begin{pmatrix} 1.9333947185 \cdot 10^{-2} \\ 4.3249222926 \cdot 10^{-2} \\ 8.9270415519 \cdot 10^{-3} \\ 5.8 \cdot 10^{-1} \\ 4.0 \cdot 10^{-1} \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1.2785472689 \cdot 10^{-2} \\ 1.7910085876 \cdot 10^{-2} \\ -4.3222636089 \cdot 10^{-3} \end{pmatrix}, \begin{pmatrix} 1.2785472689 \cdot 10^{-2} \\ -1.7910085876 \cdot 10^{-2} \\ 4.3222636089 \cdot 10^{-3} \end{pmatrix},$$
$$\left. \begin{pmatrix} 1 \\ 1.1 \\ 1.6 \end{pmatrix} \right);$$

$$(\hat{x}^1, \hat{x}^2, \hat{\theta}^1, \hat{\theta}^2, \hat{q}) = \left(\begin{pmatrix} 3.2964558373 \cdot 10^{-3} \\ 1.6548013477 \cdot 10^{-2} \\ 5.2816066068 \cdot 10^{-2} \\ 4.2 \cdot 10^{-1} \\ 5.5 \cdot 10^{-1} \end{pmatrix}, \begin{pmatrix} 1.9194398185 \cdot 10^{-2} \\ 4.7548744453 \cdot 10^{-2} \\ 1.2905668188 \cdot 10^{-3} \\ 5.6 \cdot 10^{-1} \\ 3.5 \cdot 10^{-1} \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1.2645923689 \cdot 10^{-2} \\ 1.3610564349 \cdot 10^{-2} \\ -8.3008902451 \cdot 10^{-3} \end{pmatrix}, \begin{pmatrix} +1.2645923689 \cdot 10^{-2} \\ -1.3610564349 \cdot 10^{-2} \\ +8.3008902451 \cdot 10^{-3} \end{pmatrix},$$

$$\left.\begin{pmatrix} 1 \\ 1.6 \\ 1.1 \end{pmatrix}\right);$$

$$(\tilde{x}^1, \tilde{x}^2, \tilde{\theta}^1, \tilde{\theta}^2, \tilde{q}) = \left(\begin{pmatrix} 5.53593779 \cdot 10^{-3} \\ 2.98288576 \cdot 10^{-3} \\ 4.72273200 \cdot 10^{-2} \\ 4.06826530 \cdot 10^{-1} \\ 6.28208646 \cdot 10^{-1} \end{pmatrix}, \begin{pmatrix} 1.69549162 \cdot 10^{-2} \\ 6.11138722 \cdot 10^{-2} \\ 1.84944143 \cdot 10^{-2} \\ 5.73173470 \cdot 10^{-1} \\ 2.71791354 \cdot 10^{-1} \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1.04064417 \cdot 10^{-2} \\ 4.54366309 \cdot 10^{-5} \\ -1.38896363 \cdot 10^{-2} \end{pmatrix}, \begin{pmatrix} 1.04064417 \cdot 10^{-2} \\ -4.54366309 \cdot 10^{-5} \\ 1.38896363 \cdot 10^{-2} \end{pmatrix},$$
$$\left. \begin{pmatrix} -1 \\ 11.3031102 \\ 0.786198913 \end{pmatrix} \right).$$

Note that the asset matrix does not satisfy the assumption of a desirable portfolio, i.e. Assumption 2.14. However, this “defect” is innocuous since all consumers still satisfy their budget constraints with equality if at least one of the assets has a strictly positive price (compare Hens, Schmedders, and Voß (1999, Lemma 5.1)). This allows to work with the first order conditions (2.7) and (2.8). Otherwise one would have to apply the Kuhn-Tucker conditions (compare Mas-Colell, Whinston, and Green (1995, Theorem M.K.2)).

The Lagrangian multipliers are given by

$$\begin{array}{rclrcl} \bar{\lambda}^1 & = & 3.6, & \bar{\lambda}^2 & = & 2.085, \\ \hat{\lambda}^1 & = & 3.445, & \hat{\lambda}^2 & = & 1.955, \\ \tilde{\lambda}^1 & = & 5.4417381, & \tilde{\lambda}^2 & = & 8.19317872 \cdot 10^{-2}. \end{array}$$

Computation of the determinants of the Jacobians at the equilibria shows regularity and the first and third equilibrium to have a positive index while the second one has a negative index. Since neither Assumption 4.25 nor the weaker Assumption 2.14 is satisfied, it is by no means sure whether Theorem 4.32 can be applied to show robustness of this example against perturbations of the asset structure.

The characteristics of the economy here are only approximations: the iteration via the algorithm was used until a maximal error of $2.84 \cdot 10^{-14}$ for fulfillment of the extended system was obtained.

Up to now it could not be clarified what is the intuition for the failure of uniqueness with Cobb-Douglas utility when markets are incomplete. It was conspicuous that many starting-points for the algorithm (each consisting of two FM-equilibria) failed to deliver a Cobb-Douglas economy with multiple equilibria which gives a hint that Propositions 4.23 and 4.24 can be extended to more general cases.

4.4.3 Limited Risk Sharing

Since incomplete markets are much more realistic than complete markets, it seems reasonable to try to extend the idea of complete risk sharing from the Arrow-Debreu model to GEI-economies. However, as expected, a limitation of trading possibilities stops complete risk sharing when utility satisfies the expected utility hypothesis with respect to an objective probability measure and there is no aggregate risk. Even worse, the following example demonstrates that deviating from complete to incomplete markets might cause multiplicity of equilibria. This can be explained by the fact that with incomplete markets constrained Pareto-efficient allocations need not be co-monotone.[22]

Look at an economy with $I = 2$ consumers, $S = 3$ states, and $J = 2$ assets. Consumers are now characterized by endowments

$$\omega^1 = \frac{1}{4}\begin{pmatrix} 4 \\ 23 \\ 6 \end{pmatrix} \quad \text{and} \quad \omega^2 = \frac{1}{4}\begin{pmatrix} 36 \\ 17 \\ 34 \end{pmatrix}. \tag{4.21}$$

such that aggregate endowment $\omega = (10, 10, 10)^T$ is riskless.

The objective probability measure is given by

$$\bar{\rho} = \frac{1}{9}\begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} \tag{4.22}$$

and the von Neumann-Morgenstern utility functions satisfy the standard assumptions. Furthermore, fix the following values of the first derivatives (see Figures 4.3 and 4.4):

$$\begin{array}{llllll} (u^1)'(3) = 20, & (u^1)'(3\frac{1}{2}) = 4, & (u^1)'(3\frac{3}{4}) = 3\frac{2}{3}, \\ (u^1)'(4\frac{3}{4}) = 3\frac{1}{3}, & (u^1)'(5) = 1, & (u^1)'(5\frac{1}{2}) = \frac{1}{2}. \end{array} \tag{4.23}$$

$$\begin{array}{llllll} (u^2)'(4\frac{1}{2}) = 3\frac{1}{4}, & (u^2)'(5) = 3, & (u^2)'(5\frac{1}{4}) = 2\frac{5}{6}, \\ (u^2)'(6\frac{1}{4}) = 2\frac{1}{4}, & (u^2)'(6\frac{1}{2}) = 1\frac{1}{4}, & (u^2)'(7) = \frac{1}{2}. \end{array} \tag{4.24}$$

[22] An allocation $(\bar{x}^i)_{i\in I} \in \mathbb{R}_+^{S\times I}$ is co-monotone if for each tuple $s, s' \in S$ whenever $\bar{x}_s^i \geq \bar{x}_{s'}^i$ for some consumer $i \in I$, then it is $\bar{x}_s^h \geq \bar{x}_{s'}^h$ for all $h \in I$. This implies $\omega_s \geq \omega_{s'}$ such that without aggregate risk it has to be $\bar{x}_s^h = \bar{x}_{s'}^h$ for all $s, s' \in S$. Since Pareto-efficient and hence Arrow-Debreu equilibrium allocations are co-monotone (cf. Borch (1962)), they have to be riskless and therefore first order conditions uniquely determine the equilibrium price system with complete markets.

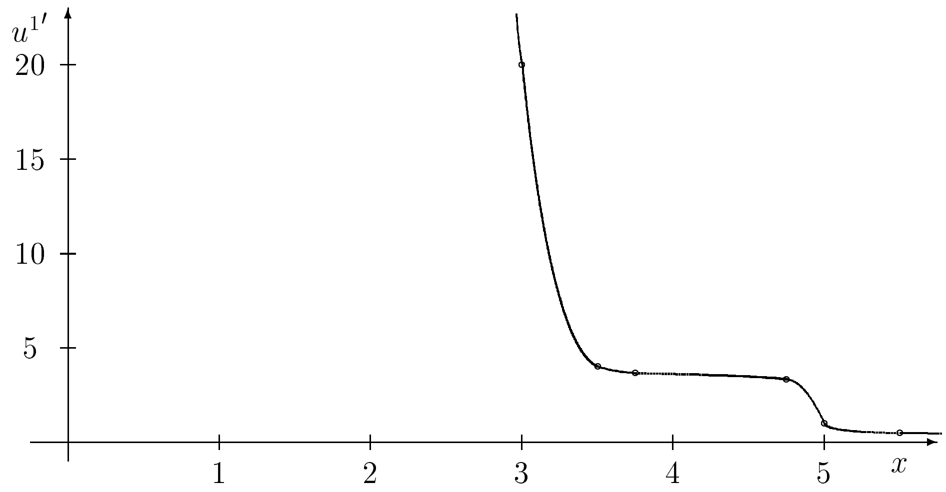

Figure 4.3: A possible form of $(u^1)'$.

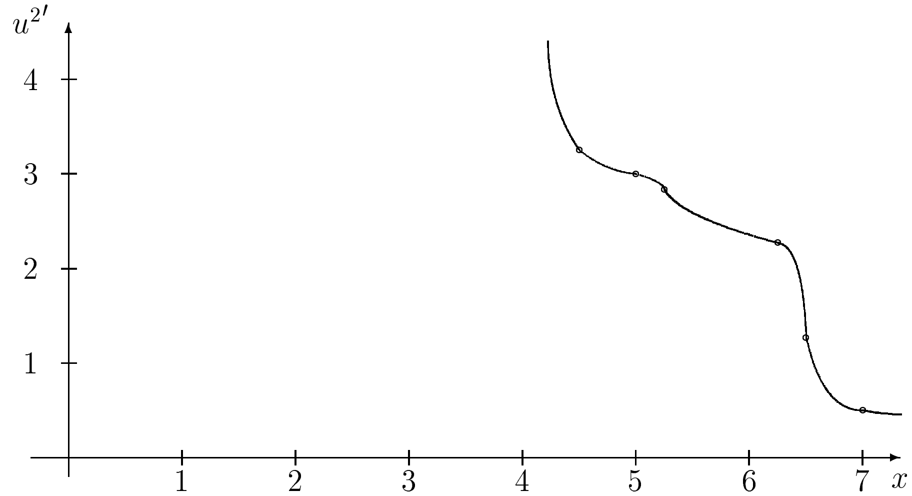

Figure 4.4: A possible form of $(u^2)'$.

Observe that these values are compatible with the standard assumptions on $u^i, i = 1, 2$. Again (compare Section 3.5.1) there is a change in the sign of the third derivative of u^i.

Trade in this economy is restricted by the asset structure

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}, \tag{4.25}$$

i.e. asset markets are incomplete.

If an economy is characterized by these data, there are at least the following two equilibria:

$$x^{*1} = \frac{1}{4}\begin{pmatrix} 12 \\ 15 \\ 22 \end{pmatrix}, x^{*2} = \frac{1}{4}\begin{pmatrix} 28 \\ 25 \\ 18 \end{pmatrix}, \theta^{*1} = -\theta^{*2} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \text{ and } q^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.26}$$

$$\hat{x}^1 = \frac{1}{4}\begin{pmatrix} 20 \\ 19 \\ 14 \end{pmatrix}, \hat{x}^2 = \frac{1}{4}\begin{pmatrix} 20 \\ 21 \\ 26 \end{pmatrix}, \hat{\theta}^1 = -\hat{\theta}^2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } \hat{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4.27}$$

where the first asset price is normalized to one.

Market clearing properties are obvious and one easily computes $x^i = \omega^i + A\theta^i, i = 1, 2$, for the *- and the $\hat{}$-equilibrium. To check the maximization condition substitute the values of (4.21)–(4.27) into the first order conditions (2.7) and (2.8) to obtain the following system of equations:

$i = 1$, *-equilibrium:

$$\begin{aligned} 1 \cdot 20 \cdot 3 - \lambda^{*1} \cdot 1 &= 0, \\ 1 \cdot 20 \cdot 2 + 6 \cdot 3\frac{2}{3} \cdot 1 + 2 \cdot \frac{1}{2} \cdot (-2) - \lambda^{*1} \cdot 1 &= 0, \\ 1 \cdot 2 + 1 \cdot (-2) &= 0; \end{aligned}$$

$i = 1$, $\hat{}$-equilibrium:

$$\begin{aligned} 1 \cdot 1 \cdot 3 - \hat{\lambda}^1 \cdot 1 &= 0, \\ 1 \cdot 1 \cdot 2 + 6 \cdot 3\frac{1}{3} \cdot 1 + 2 \cdot 4 \cdot (-2) - \hat{\lambda}^1 \cdot 2 &= 0, \\ 1 \cdot 2 + 2 \cdot (-1) &= 0; \end{aligned}$$

$i = 2$, *-equilibrium:

$$\begin{aligned} 1 \cdot \frac{1}{2} \cdot 3 - \lambda^{*2} \cdot 1 &= 0, \\ 1 \cdot \frac{1}{2} \cdot 2 + 6 \cdot 2\frac{1}{4} \cdot 1 + 2 \cdot 3\frac{1}{4} \cdot (-2) - \lambda^{*2} \cdot 1 &= 0, \\ 1 \cdot (-2) + 1 \cdot 2 &= 0; \end{aligned}$$

$i = 2$, $\hat{}$-equilibrium:

$$\begin{aligned} 1 \cdot 3 \cdot 3 - \hat{\lambda}^2 \cdot 1 &= 0, \\ 1 \cdot 3 \cdot 2 + 6 \cdot 2\frac{5}{6} \cdot 1 + 2 \cdot 1\frac{1}{4} \cdot (-2) - \hat{\lambda}^2 \cdot 2 &= 0, \\ 1 \cdot (-2) + 2 \cdot 1 &= 0. \end{aligned}$$

This system is solved for $\lambda^{*1} = 60$, $\hat{\lambda}^1 = 3$, $\lambda^{*2} = 1\frac{1}{2}$, and $\hat{\lambda}^2 = 9$. Therefore, the equilibrium properties are proved.

Note that nothing else than the derivatives (4.23) and (4.24) is fixed for von Neumann-Morgenstern utility functions. Therefore, it is not possible to compute the Jacobians of market demand and check their determinants for equilibrium price systems. Therefore, no statement about regularity is possible. However, since such a few restrictions are put on utility functions, it should be possible for each endowment allocation $(\omega^i)_{i \in I}$ to find a utility function exhibiting these derivative values such that the given two equilibria are regular. This allows to obtain robustness of the multiplicity against (restricted) perturbations of the asset structure (Theorem 4.32).

4.5 Additional Results on Uniqueness with Incomplete Markets

Section 4.4 contains negative results with respect to incomplete markets because it demonstrates that conditions on utility functions like quasi-linearity, Cobb-Douglas preferences, or the expected utility hypothesis under objectivity together with no aggregate risk alone are not sufficient for a unique FM-equilibrium. This section takes the other way around: there exist economies such that the equilibrium with incomplete markets is unique though there are multiple Arrow-Debreu equilibria. This is obvious when the asset matrix contains none or only one asset because assets cannot be exchanged between consumers a priori. To avoid this trivial case, the following examples are based on more than one asset.

The idea is to exploit the power of constrained Pareto-efficiency (compare Section 2.7). I start with a leading example which again is built on Kehoe's

(1991, p. 2066) well-established example of multiple Arrow-Debreu equilibria (Subsection 4.5.1). This is generalized in Subsection 4.5.2.

4.5.1 A Leading Example

Consider an economy with $I = 2$ agents and $S = 3$ states. Consumers are characterized by CES-utility functions

$$U^i(x) = \sum_{s=1}^{3} a_s^i \frac{(x_s^{b^i} - 1)}{b^i}$$

with $a^1 = (1024, 1/2, 1/2)$, $\omega^1 = (12, 1, 1)^T$, $b^1 = -4$ and $a^2 = (1, 512, 512)$, $\omega^2 = (1, 12, 12)^T$, $b^2 = -4$. In contrast to Kehoe (1991) there is a third good which for each consumer has the same value as the second one. The second and the third good sum up in weight to Kehoe's (1991) second one.

Hence, for complete markets, one gets multiple equilibria in the same manner as in Kehoe (1991, p. 2066) since the second and the third state can be merged. The equilibria are the following ones:[23]

$$\hat{x}^1 = \begin{pmatrix} 8.631 \\ 1.429 \\ 1.429 \end{pmatrix}, \hat{x}^2 = \begin{pmatrix} 4.369 \\ 11.571 \\ 11.571 \end{pmatrix}, \hat{\pi} = \begin{pmatrix} 0.1129 \\ 0.44355 \\ 0.44355 \end{pmatrix};$$

$$x^{*1} = \begin{pmatrix} 10.4 \\ 2.6 \\ 2.6 \end{pmatrix}, x^{*2} = \begin{pmatrix} 2.6 \\ 10.4 \\ 10.4 \end{pmatrix}, \pi^* = \begin{pmatrix} 0.5 \\ 0.25 \\ 0.25 \end{pmatrix};$$

$$\bar{x}^1 = \begin{pmatrix} 11.571 \\ 4.369 \\ 4.369 \end{pmatrix}, \bar{x}^2 = \begin{pmatrix} 1.429 \\ 8.631 \\ 8.631 \end{pmatrix}, \bar{\pi} = \begin{pmatrix} 0.8871 \\ 0.05645 \\ 0.05645 \end{pmatrix}.$$

From now on, consider incomplete markets. In the first step, focus on

$$\bar{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which obviously shows also a kind of symmetry with respect to states 2 and 3.

Hence, even though there are two different assets, the situation is the same as with a single asset because — as I will show — there is no trade. The

[23] Apply the price normalization $\pi_1 + \pi_2 + \pi_3 = 1$, i.e. $\pi \in \hat{\mathcal{P}}_{1,\mathbb{1}}$.

reason for this is that both assets allow very similar trading opportunities. Therefore, after proving uniqueness of equilibria for $\bar{A}$, I apply Theorem 4.34 to consider perturbations of $\bar{A}$.

Assume the asset structure $\bar{A}$. The only possible equilibrium is the no-trade equilibrium $\theta^{*1} = \theta^{*2} = 0$ with $q_1^* = q_2^*$: symmetry of both consumers' utility functions and endowments according to states 2 and 3 and the particular asset structure $\bar{A}$ imply that for $q_1 > q_2$ both consumers demand for the second and supply the first one and vice versa for $q_2 > q_1$.

The intuition behind this example is very simple: the incomplete asset structure

$$\bar{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is chosen such that the endowment distribution is constrained Pareto-efficient (but not Pareto-efficient!) which causes uniqueness of the equilibrium (compare Proposition 4.14). One easily computes that for all $\bar{\alpha} \in \mathbb{R}_+$ the asset structure

$$A = \begin{pmatrix} \bar{\alpha} & \bar{\alpha} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

causes the same effect.

$\bar{A}$ is an asset matrix which enables a portfolio with a strictly positive payoff. Therefore, instead of $\bar{A}$ one can consider the matrix

$$\tilde{A} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Defining a set of ϵ-perturbations of $\tilde{A}$, one can define a set of $\bar{\epsilon}$-perturbations of $\bar{A}$ of which the elements can be represented as ϵ-perturbations of $\tilde{A}$ (compare Lemma 4.27).

Because of Proposition 4.3 not only $\bar{A}$, but also $\tilde{A}$ evokes the unique no-trade equilibrium. No-trade equilibria, however, are regular (compare Lemma 4.37 in the next section) such that Theorem 4.34 assures the existence of an $\epsilon > 0$ with all ϵ-perturbations of $\tilde{A}$ and therefore all $\bar{\epsilon}$-perturbations of $\bar{A}$ being characterized by a unique equilibrium.

The example is robust against perturbations of the asset matrix even though the endowment distribution then stops to be constrained Pareto-efficient. The intuition for uniqueness of the equilibrium to remain is the fact that the allocation (ω^1, ω^2) is very "close" to the set of constrained Pareto-efficient allocations which causes a "small" chance for the existence of multiple equilibria.

4.5.2 Generalization of the Leading Example

When generalizing the idea of the leading example, the number of consumers has to be less than the number of states which might give a hint that one has to put attention on this relation when looking for sufficient conditions for a unique equilibrium. I state that the no-trade equilibrium is always regular such that the results from Theorem 4.34 can be applied: for each fixed number of assets smaller than or equal to one plus the difference between number of states and consumers there exists a non-exceptional subset of all allowed asset matrices where the corresponding GEI-economy is described by a unique equilibrium. A remarkable point consists in the fact that — no matter which utility functions and endowments are assumed — for each strictly positive asset an asset matrix can be constructed containing this asset and generating a non-trivial GEI-economy with unique equilibrium.

For this subsection the fulfillment of Assumption 4.25 is required. Therefore, price normalization with respect to the first unit vector, i.e. restricting prices to be elements of $\hat{\mathcal{Q}}(A)$, is possible such that the first asset is numeraire asset.

Now, assume there is an Arrow-Debreu-economy $\{(U^i, \omega^i)_{i=1,\ldots,I}\}$ with multiple equilibria. According to Proposition 3.12 this implies that the no-trade allocation cannot be Pareto-efficient, i.e. there have to be two consumers $i, h \in \{1, \ldots, I\}$ with

$$\nabla U^i(\omega^i) \not\parallel \nabla U^h(\omega^h) \tag{4.28}$$

where without loss of generality $i = 1$. Now define $A_{\cdot 1}^T = (1, \ldots, 1) \gg 0$. Assume $\nabla U^i(\omega^i) A_{\cdot 1} = 1$ for all $i = 1, \ldots, I$ without loss of generality. Otherwise multiplication of U^i with a positive constant is possible. According to equation (2.13) a sufficient condition for $(\theta^i)_{i=1,\ldots,I}, \theta^i = 0$ for all $i = 1, \ldots, I$ to be constrained Pareto-efficient for some matrix $A = (A_{\cdot 1}, A_{\cdot(-1)})$ is given by

$$\nabla U^1(\omega^1) A_{\cdot(-1)} = \ldots = \nabla U^I(\omega^I) A_{\cdot(-1)} \tag{4.29}$$

or equivalently

$$\underbrace{\begin{bmatrix} \nabla U^1(\omega^1) - \nabla U^2(\omega^2) \\ \vdots \\ \nabla U^1(\omega^1) - \nabla U^I(\omega^I) \end{bmatrix}}_{:=H} A_{\cdot(-1)} = 0. \tag{4.30}$$

Together with Proposition 4.14 this implies a unique (no-trade) equilibrium for $GEI = \{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$.

It has to be clarified under which conditions a solution $A_{\cdot(-1)} \in \mathbb{R}^{S\times(J-1)}$ for equation (4.30) exists. Obviously, each $A_{\cdot j}, j = 2, \ldots, J$ has to be in the null space $\mathcal{C}$ of H: $\mathcal{C} := \{a \in \mathbb{R}^S |\ Ha = 0\}$. Therefore, considering that $A_{\cdot 1}$ is in $\mathcal{C}$, it is possible to find a solution for equation (4.30) with $rank\ A = J$ if and only if $dim\ \mathcal{C} \geq J$. Because of condition (4.28) $dim\ \mathcal{C} < S$. For $I = 2$ one observes $dim\ \mathcal{C} = S - 1$, i.e. it would be possible to construct a counterexample for all $J < S$! In general it is

$$dim\ \mathcal{C} \geq \max\{1, S - (I-1)\}, \tag{4.31}$$

i.e. if $S - (I-1) \geq 2$ or equivalently $S > I$, then one is able to construct a non-trivial counterexample for all $J \in \{2, \ldots, S - I + 1\}$.[24]

Note that for $S \leq I$ any solution unequal to $A_{\cdot 1}$ only exists in the exceptional case where $I - S + 1$ consumers can be eliminated from equation (4.30) because they are characterized by the same marginal utility for their endowment as some other consumer.

The examples would not be very interesting if they only depend on constrained Pareto-efficiency of the endowment. However, from Theorem 4.34 it follows that as soon as one has found a GEI-economy $\{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ with unique no-trade equilibrium, then A can be disturbed a little bit without changing the uniqueness property, but leaving the no-trade equilibrium. Therefore, one needs the following lemma.

Lemma 4.37 (Regularity of No-Trade Equilibria)
If $GEI = \{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ is characterized by the unique no-trade equilibrium, then this equilibrium is regular, i.e. the normalized Jacobian of market demand is regular.

Proof: The Jacobian of an individual demand function can be split into two matrices for the substitution respectively income effect. Since there is no trade, the income effect is zero such that only the Slutzky-matrix remains which is negative definite on the orthogonal space for q, $q^{\perp} := \left\{v \in \mathbb{R}^J |\ q^T v = 0\right\}$. Adding up these individual Jacobians, one ends up with a negative definite Jacobian of market demand on $q^{\perp}$.

□

Now, application of Theorem 4.34 for regular GEI-economies with $A_{\cdot 1} \gg 0$ delivers:

[24] For $J = 1$ it is trivial because there cannot be any trade a priori.

Proposition 4.38 (Limits of Transferability)
Let $\{(U^i, \omega^i)_{i=1,\ldots,I}\}$ be an Arrow-Debreu economy where $S > I$[25] and which displays multiple equilibria, then there exists a non-exceptional set of J-dimensional asset matrices, $2 \leq J \leq S - I + 1$, which enable a portfolio with strictly positive payoff and for which the GEI-economy $\{(U^i, \omega^i)_{i=1,\ldots,I}, A\}$ displays a unique equilibrium.

Non-exceptional means that small arbitrary changes in arbitrary entries of the asset matrix do not change the uniqueness property.[26]

Having a further look on the proof of this result, especially the construction method of the counterexamples, one observes that instead of $A_{\cdot 1}^T = (1, \ldots, 1)^T$ it is possible to take *any* other strictly positive vector[27], which means that each strictly positive asset can be part of a matrix generating a counterexample! This observation is quite important because it emphasizes that not the single assets are important, but their interaction. Having constructed a counterexample with a specified matrix $A \in \mathbb{R}^{S \times J}$ where $J < S$ it is always possible to define a vector $\tilde{A}_{\cdot 1} \notin < A >$, but being strictly positive such that for an $\tilde{A}$ which is constructed by starting with $\tilde{A}_{\cdot 1}$ it is $< \tilde{A} > \neq < A >$.

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[25] This is a quite restrictive assumption, but it becomes weaker if one interprets $i = 1, \ldots, I$ not as single consumers, but types of consumers.

[26] In other words: for a given Arrow-Debreu economy $\{(U^i, \omega^i)_{i=1,\ldots,I}\}$ with multiple equilibria the set of $S \times J$ matrices generating corresponding GEI-economies with multiple equilibria is not generic in the set of all $S \times J$ matrices with $rank = J$ and the first asset being positive.

[27] If one does not want to admit disturbances in any direction, it is even sufficient to demand $A_{\cdot 1} > 0$.

Chapter 5

On Uniqueness of Competitive Equilibria in Restricted Participation Economies

5.1 Introduction

In this chapter it will be clarified in how far the results on a unique equilibrium can be transferred from the Finance-GEI to restricted participation economies. This is the natural step after extending Arrow-Debreu results to incomplete markets.

Restricted participation (RPGEI-) economies are a generalization of the incomplete markets economies. The basic situation is the same: the economy is a multiperiod economy where in the beginning it is unknown which states of nature will be realized in the future. Consumers can trade without any restrictions on spot markets, but trade on financial markets — which allows transfers between periods and states — is restricted. Whereas in the incomplete markets economies all consumers are subjected to the same constraints, in restricted participation economies the constraints might be asymmetric among consumers. Polemarchakis and Siconolfi (1997, p. 289) explain that "participation is restricted when some reallocations of revenue are not attainable by some individuals at some commodity prices." The Arrow-Debreu and GEI-models are special RPGEI-models.

Allowing for asymmetric constraints is motivated by the observation that "households may face different opportunities to transfer wealth between spot markets" (Siconolfi (1989, p. 273). Location problems and asymmetric information experience a better modeling than with symmetric constraints.

Two seminal papers have to be named in the context of restricted participation which arose nearly at the same time: Siconolfi (1989) and Balasko, Cass, and Siconolfi (1990).

Siconolfi (1989) presents a quite general proof of existence for equilibria in a two-period restricted participation economy where assets pay off in units of account. In addition to standard assumptions on preferences and endowments he requires nothing else than some regularity assumptions on the trading restrictions. The paper is a generalization of Siconolfi (1986). Siconolfi (1989, p. 274) himself states that his result can be easily extended to arbitrary multiperiod models if the payments of assets which are traded at different dates do not overlap.

Balasko, Cass, and Siconolfi (1990) also consider two-period economies with asset payoffs denominated in units of account. They put linear homogeneous constraints on asset trading of the agents to obtain that generically in endowments the restricted participation economies display real indeterminacy as has been already observed for incomplete markets economies (Geanakoplos and Mas-Colell (1989), Balasko and Cass (1989)). An example of a three-period economy is given with assets, traded at different dates and having overlapping payoffs, for which no equilibrium exists.

Generic existence of equilibria under restricted participation is proved by Polemarchakis and Siconolfi (1997) for economies where assets pay off in commodities. Similarly to incomplete markets economies, within this framework existence problems arise because of discontinuities in the attainable reallocations. But asymmetric constraints in addition cause these discontinuities to differ among individuals. Nevertheless, Polemarchakis and Siconolfi (1997) are able to transfer the concept of pseudo-equilibria and proof their existence. Generically in endowments, assets, and trading constraints these pseudo-equilibria coincide with restricted participation equilibria.

Benveniste (1987) defines an economy for which in spite of restricted participation he characterizes when financial markets are essentially complete such that Arrow-Debreu equilibrium allocations are attainable.

Some part of the incomplete markets literature also considers economies where asymmetric information structures across consumers are modeled explicitly. As Polemarchakis and Siconolfi (1993) and Cass and Villanacci (2000) prove, the existence of fully non-revealing and partially revealing equilibria in such economies can be traced back to the existence of equilibria in restricted participation economies.

After a short presentation of the model including some considerations about constrained Pareto-efficiency I show by means of two simple examples that in general symmetric and asymmetric participation structures influence the number of equilibria. Nevertheless I am able to formulate sufficient conditions for a unique restricted participation equilibrium which allow asymmetric market participation (compare the overview given in Figure 5.1 at the end of this chapter). Note that for restricted participation economies I change the order of presentation in comparison to Arrow-Debreu and GEI-economies. This is because the limitations of transferability in Section 5.3 give some insight in problems of price normalization arising in the RPGEI-framework which is an important point when considering uniqueness of equilibria.

5.2 Restricted Participation Extension of the Finance GEI-Model

The model I consider is a special version of the equilibrium model with asymmetric constraints on portfolio holdings and incomplete financial markets for which Siconolfi (1989) proves the existence of an equilibrium. In contrast to Siconolfi (1989) I have to restrict attention to one-commodity economies. This seems reasonable because all results on uniqueness in the models with symmetric participation refer to this special case (compare Chapter 4). Siconolfi (1989) considers assets which pay off in units of account, but this transfers to real assets if there is only a single commodity. The model which I consider is narrower than Siconolfi's (1989) since I only look on special linear subspaces of the trading space as individual restrictions: consumers are allowed to trade arbitrary quantities of assets contained in a subset of all assets. Otherwise, it would not be possible to apply first order conditions as necessary conditions for maximization which nearly all known results on uniqueness in complete and incomplete markets models require.

In one point I am more general than Siconolfi (1989): I do not require consumption today. With the same arguments as on page 27 for the Finance GEI-model consumption today can be included in this model. I am sure that Siconolfi's (1989) result on existence remains valid in this model. Nevertheless, note that all the results can be reformulated for the model with consumption today. Merely the second example in Section 5.3 does not fit to that framework.[1]

In contrast to the GEI-model, for each agent $i \in I$ a trading set $J^i \subseteq J$ is defined. Consumers are not allowed to trade all existing assets, but only those contained in J^i. This means that they have to choose a portfolio plan

[1]Polemarchakis and Siconolfi (1997, pp. 305/6) state that existence might fail without first period consumption, but their example does not satisfy the assumptions I establish below.

$\theta^i \in \mathbb{T}^i := \{\theta \in \mathbb{R}^J \mid \theta_j = 0 \; \forall \; j \in J \backslash J^i\}$. $(J^1, \ldots, J^I) \in \times_{i \in I} \mathcal{P}(J)$ is called the participation structure. Here $\mathcal{P}(J)$ denotes the power set of J, i.e. the set of all subsets of J. If $J^i = J^h$ for all $i, h \in I$ the model is equivalent to the Finance GEI-model and the participation structure is called symmetric, otherwise asymmetric.

To summarize, a restricted participation economy is defined by

$$RPGEI = \{(U^i, \omega^i, J^i)_{i \in I}, A\}.$$

In addition to the Differentiability Assumptions 2.4 I impose the following assumptions on the asset matrix and the participation structure.

Assumption 5.1 (RPGEI-Assumptions)
For a restricted participation economy $RPGEI = \{(U^i, \omega^i, J^i)_{i \in I}, A\}$ the following properties are fulfilled:

1. *For all $j \in J$ there exists some $i \in I$ such that $j \in J^i$.*
2. *There is no partition $K^1, K^2 \subseteq J$ with $K^1 \cup K^2 = J$ and $K^1 \cap K^2 = \emptyset$ such that for all $i \in I$ either $J^i \subseteq K^1$ or $J^i \subseteq K^2$.*
3. *For all $i \in I$ there is some $\theta \in \mathbb{T}^i$ such that $A\theta > 0$.*
4. *For all $i \in I$ it is rank $A^i = \sharp J^i$ where $A^i := (A_{sj})_{s \in S, j \in J^i}$.*

The first assumption assures that all assets are tradable by at least one consumer. Otherwise, the price of such an asset would not be determined. One could redefine each economy such that this assumption is satisfied.[2]

The second assumption prevents the economy from falling into subeconomies. If is not satisfied, one can work with each subeconomy on its own.[3]

The third assumption is needed to assure non-satiation of individual asset demand which follows together with strict monotonicity of the utility functions.[4] It generalizes Assumption 2.14 to RPGEI-economies.

The last assumption is imposed to uniquely determine individual asset demand. Otherwise in equilibrium consumers would be indifferent between different portfolios.[5] [6] It is the generalization of Assumption 2.21 according to the RPGEI-extensions of the model.

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[2]This assumption corresponds to Siconolfi's (1989) Assumption A4.

[3]In Siconolfi (1989) this assumption is always satisfied because his model includes first period consumption.

[4]Again, in Siconolfi (1989) this assumption follows from first period consumption.

[5]Compare Assumption A5. in Siconolfi (1989).

[6]Balasko, Cass, and Siconolfi (1990) show how to reduce a restricted participation economy which does not satisfy this assumption to one which does and displays the same equilibrium spot prices and allocations.

Under these assumptions each consumer's decision problem in this new context is given by

$$\max_{x\in\mathbb{R}^S_+,\theta\in\mathbb{T}^i} U^i(x) \text{ s.t. } q\cdot\theta = 0 \ \wedge \ x = \omega^i + A\theta \tag{5.1}$$

which can be rewritten as

$$\max_{\theta\in\mathbb{T}^i} U^i(\omega^i + A\theta) \text{ s.t. } q\cdot\theta = 0 \ \wedge \ \omega^i + A\theta \geq 0. \tag{5.2}$$

Its solution is a continuous function $\theta^i(q)$ on the set of individually arbitrage-free asset prices[7]

$$\mathcal{Q}^i(A) := \{q \in \mathbb{R}^J \mid \ \nexists\, \tilde{\theta}^i \in \mathbb{T}^i \text{ s.t. } \begin{pmatrix} -q^T \\ A \end{pmatrix} \tilde{\theta}^i > 0\}.$$

As becomes clear from the considerations in Siconolfi (1989) on the set of no-arbitrage prices, under the Assumption 5.1 the no-arbitrage set in the Finance GEI-model is a subset of the one here, i.e.

$$\begin{aligned}
\mathcal{Q}(A) &= \left\{q \in \mathbb{R}^J \mid q = A^T\pi \text{ for some } \pi \in \mathbb{R}^S_{++}\right\} \\
&\subseteq \bigcap_{i=1}^{I} \left\{q \in \mathbb{R}^J \mid q_{J^i} = A^{iT}\pi^i \text{ for some } \pi^i \in \mathbb{R}^S_{++}\right\} \\
&= \bigcap_{i=1}^{I} \left\{q \in \mathbb{R}^J \mid \ \nexists\, \tilde{\theta}^i \in \mathbb{R}^{\sharp J^i} \text{ with } \begin{pmatrix} -q^T_{J^i} \\ A^i \end{pmatrix} \tilde{\theta}^i > 0\right\} \\
&:= \mathcal{Q}^R(A)
\end{aligned}$$

where the penultimate equality follows on the individual level in the same way as in the Finance GEI-model.

To simplify notation define

$$\hat{\theta}^i(q_{J^i}) := \arg\max_{\theta\in\mathbb{R}^{\sharp J^i}} U^i(\omega^i + A^i\theta) \text{ s.t. } q_{J^i}\cdot\theta = 0 \ \wedge \ \omega^i + A^i\theta \geq 0.$$

The coordinates of $\hat{\theta}^i$ are denoted by $j \in J^i$.

Obviously it is

$$\theta^i_j(q) \quad = \quad \begin{cases} 0, & \text{if } j \in J\backslash J^i, \\ \hat{\theta}^i_j(q_{J^i}), & \text{if } j \in J^i. \end{cases}$$

[7] The proof is analogue to Hens (1991, Proposition 2).

Definition 5.2 (RPGEI-Equilibrium I)
A tuple $\left((x^{*i}, \theta^{*i})_{i \in I}\right) \in \bigtimes_{i \in I} \left(\mathbb{R}^S_+ \times \mathbb{T}^i\right)$ *and a price system* $q^* \in \mathbb{R}^J$ *constitute an RPGEI-equilibrium if*

1. (x^{*i}, θ^{*i}) *solves the maximization problem (5.1) with* $q = q^*$ *for all* $i = 1, \ldots, I$*;*
2. $\sum_{i=1}^I x^{*i} = \sum_{i=1}^I \omega^i$*;*
3. $\sum_{i=1}^I \theta^{*i} = 0$.

Since strict monotonicity and Assumption 5.1 assure a one-to-one-correspondence between consumption and portfolio plans, most of the time I omit consumption plans and consider the maximization problem in the form (5.2).

Definition 5.3 (RPGEI-Equilibrium II)
A tuple $(\theta^{*i})_{i \in I} \in \bigtimes_{i \in I} \mathbb{T}^i$ *and a price system* $q^* \in \mathbb{R}^J$ *constitute an RPGEI-equilibrium if*

1. (θ^{*i}) *solves the maximization problem (5.2) with* $q = q^*$ *for all* $i = 1, \ldots, I$*;*
2. $\sum_{i=1}^I \theta^{*i} = 0$.

Individual and market demand are obviously homogeneous of degree zero in asset prices. Therefore, if q^* is an equilibrium price system, this is also true for λq^* with $\lambda > 0$. Within the restricted participation framework price normalization becomes more difficult. Therefore, let me start with the following definition of a unique RPGEI-equilibrium.

Definition 5.4 (Global Uniqueness of RPGEI-Equilibria)
Given the economy $RPGEI = \{(U^i, \omega^i, J^i)_{i \in I}, A\}$*, an equilibrium* $(\theta^*, q^*) \in (\bigtimes_{i \in I} \mathbb{T}^i) \times \mathbb{R}^J$ *is globally unique if there is no other portfolio allocation* $\bar{\theta} \in \bigtimes_{i \in I} \mathbb{T}^i$ *with* $\sum_{i=1}^I \bar{\theta}^i = 0$ *such that there exists a common* $\bar{q} \in \mathbb{R}^J$ *for which each consumer solves (5.2) with* $\bar{\theta}^i$*.*

If Assumption 5.1.4 is violated, the definition has to be done with respect to consumption allocations.

Concerning price systems, one has to be cautious: not in all restricted participation economies a conventional price normalization is possible (compare the second example in the next section). However, under some additional assumptions on the asset and the participation structure the normalization

$$\hat{\mathcal{Q}}^R_{1,e^j}(A) := \{q \in \mathcal{Q}^R(A) \mid q \cdot e^j = q_j = 1\}$$

can be applied. It is defined with respect to some numeraire asset $j \in J$, for example if $A > 0$. Moreover, one can take $\hat{\mathcal{Q}}^R_{1,e^{\bar{j}}}(A)$ if there is some $\bar{j} \in J$ such that $A_{\cdot\bar{j}} > 0$ and $\bar{j} \in J^i$ for all $i \in I$. Under these additional assumptions a unique equilibrium allocation is equivalent to the existence of a unique normalized equilibrium price system.

The definitions of the Weak Axiom of Revealed Preferences and monotonicity with respect to some normalizing vector basically remain the same as in the GEI-model and also the implications of these properties for the uniqueness of the equilibria. Merely, the domain changes: for individual demand functions it is $\mathcal{Q}^i(A)$, for market demand $\mathcal{Q}^R(A)$. These sets are also open and convex. Obviously, individual demand functions only satisfy the properties in their weak versions whenever $J^i \subset J$.

In Section 5.4 it is demonstrated which strong restrictions on the participation structures are necessary to obtain WARP and monotonicity for market demand whenever the participation structures are asymmetric.

In the same way as for the Finance GEI-model the notion of Pareto-efficiency has to be adapted to the restricted participation framework: reallocations should only be allowed within the possibilities given by the trading sets $(J^i)_{i\in I}$.

Definition 5.5 (Constrained Pareto-Efficiency)
Given an economy $RPGEI = \{(U^i, \omega^i, J^i)_{i\in I}, A\}$, a feasible allocation $x^* \in \mathbb{R}^{S\times I}_+$ is constrained Pareto-efficient if for all $i \in I$ it is $x^{*i} - \omega^i = A\theta^{*i}$ for some $\theta^{*i} \in \mathbb{T}^i$ with $\sum_{i=1}^I \theta^{*i} \leq 0$ and there is no other feasible allocation $\check{x} \in \mathbb{R}^{S\times I}_+$ with $\check{x}^i - \omega^i = A\check{\theta}^i$ for some $\check{\theta}^i \in \mathbb{T}^i$ with $\sum_{i=1}^I \check{\theta}^i \leq 0$ such that $U^i(\check{x}^i) \geq U^i(x^{*i})$ for all $i \in I$ with at least one strict inequality.

The corresponding portfolio allocations are also called constrained Pareto-efficient. Under some further restrictions one can prove constrained Pareto-efficiency of RPGEI-equilibria.

Theorem 5.6 (Constr. Pareto-Efficiency of RPGEI-Equilibria)
Given an economy $RPGEI = \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ with $J^1 = J$ or $A_{\cdot\bar{j}} > 0$ for some $\bar{j} \in \bigcap_{i\in I} J^i$, the RPGEI-equilibria are constrained Pareto-efficient.

The proof of the GEI-model via the no-arbitrage condition (compare Magill and Quinzii (1996b, Theorem 12.3) cannot be transferred since price systems stop to be arbitrage-free and one only knows that they are individually arbitrage-free. An alternative proof is possible when the assumptions for differentiability are satisfied (compare Magill and Quinzii (1996b, p. 105)). Therefore, the following two lemmas are helpful which also establish easily verifiable conditions for constrained Pareto-efficiency and can be applied for the proofs of Lemma 5.11 and Proposition 5.12.

Lemma 5.7 (Alternative Def. of Constr. Pareto-Efficiency)
Given RPGEI $= \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ *with either* $J^1 = J$ *or* $A_{\cdot\bar{j}} > 0$ *for some* $\bar{j} \in \bigcap_{i\in I} J^i$ *or* $A > 0$, *an allocation* $\bar{x} = (\bar{x}^i)_{i\in I} \in \mathbb{R}_+^{S\times I}$ *is constrained Pareto-efficient if and only if there is some tuple* $(\bar{k}^i)_{i=2,\ldots,I} \in \mathbb{R}_{++}^{I-1}$ *such that* $\bar{x}$ *solves*

$$(5.3)\qquad \max_{x\in\mathbb{R}_+^{S\times I}} \left\{ U^1(x^1) \;\middle|\; \begin{array}{l} U^i(x^i) \geq \bar{k}^i \text{ for all } i = 2, \ldots, I; \\ x^i = \omega^i + A\theta^i \text{ for some } \theta^i \in \mathbb{T}^i \text{ for all } i \in I; \\ \sum_{i=1}^I \theta^i = 0 \end{array} \right\}.$$

Proof:
$\Rightarrow$: Define $\bar{k}^i = U^i(\bar{x}^i)$ for all $i = 2, \ldots, I$. Since $(\bar{x}^i)_{i\in I}$ is constrained Pareto-efficient, there exist $\bar{\theta}^i \in \mathbb{T}^i$ with $\bar{x}^i = \omega^i + A\bar{\theta}^i$ and $\sum_{i\in I} \bar{\theta}^i = 0$ because of Assumption 5.1 and strict monotonicity of $U^i, i = 1, \ldots, I$. Now, assume $\bar{x}$ is not maximizer of (5.3), i.e. there is $\hat{x} \in \mathbb{R}_+^{S\times I}$ satisfying the constraints of (5.3) with

$$U^1(\hat{x}^1) > U^1(\bar{x}^1).$$

Then $\hat{x}$ is a reallocation of $\bar{x}$ with at least the same utility for all $i = 2, \ldots, I$, but higher utility for $i = 1$ which contradicts constraint Pareto-efficiency.

$\Leftarrow$: Assume $\bar{x}$ solves the maximization problem (5.3) for $(\bar{k}^2, \ldots, \bar{k}^I)$, but is not constrained Pareto-efficient. Let $\hat{x} \in \mathbb{R}_+^{S\times I}$ be the constrained Pareto-improvement within the trading sets where

$$(5.4)\qquad U^h(\hat{x}^h) > U^h(\bar{x}^h) \text{ for some } h \in I,$$

$$(5.5)\qquad U^i(\hat{x}^i) \geq U^i(\bar{x}^i) \text{ for all } i \in I.$$

If (5.4) is satisfied for $h = 1$, $\bar{x}$ is not maximizer of (5.3) because (5.5) implies $U^i(\hat{x}^i) \geq U^i(\bar{x}^i) \geq \bar{k}^i$ for $i = 2, \ldots, I$ and the other constraints in (5.3) are part of the definition of constrained Pareto-efficiency. Therefore assume $U^1(\hat{x}^1) = U^1(\bar{x}^1)$.

1. $J^1 = J$: Because of Assumption 5.1.3 there is some $\bar{\theta}^h \in \mathbb{T}^h$ with $A\bar{\theta}^h > 0$. Choose $\epsilon > 0$ sufficiently small such that $\hat{x}^h - \epsilon A\bar{\theta}^h \gg 0$[8] and $U^h(\hat{x}^h - \epsilon A\bar{\theta}^h) > U^h(\bar{x}^h) \geq \bar{k}^h$. Then, $\tilde{x} \in \mathbb{R}_+^{S\times I}$ with $\tilde{x}^1 = \hat{x}^1 + \epsilon A\bar{\theta}^h, \tilde{x}^h = \hat{x}^h - \epsilon A\bar{\theta}^h, \tilde{x}^i = \hat{x}^i$ for all $i \neq 1, h$ satisfies the constraints of (5.3), but assures $U^1(\tilde{x}^1) > U^1(\bar{x}^1)$ which contradicts the maximization property.

[8] The assumptions on U^h and ω^h assure $\hat{x}^h \gg 0$.

2. $A_{\cdot\bar{j}} > 0$ for some $\bar{j} \in \bigcap_{i\in I} J^i$: This can be proved analogously to the first case because $\bar{\theta}^h = e^{\bar{j}} \in \mathbb{T}^1 \cap \mathbb{T}^h$ and $Ae^{\bar{j}} = A_{\cdot\bar{j}} > 0$.

3. $A > 0$: There exists a chain of consumers $i_1, \ldots, i_N, N \in \mathbb{N}$, such that $i_1 = h, i_2, \ldots, i_{N-1} \in I\backslash\{1\}, i_N = 1$ and there is some $\bar{j}_n \in J^{i_n} \cap J^{i_{n+1}}$ for all $n \in \{1, \ldots, N-1\}$. Therefore, the idea of the first case can be extended in the sense that consumer i_n gives a little bit of the positive asset $\bar{j}_n$ to consumer i_{n+1} with receiving at least utility $\bar{k}^{i_n}$ for $n < N$ and the last consumer $i_N = 1$ receiving more than $U^1(\bar{x}^1)$. This contradicts the maximization property.

$\square$

This lemma is used to prove the following one.

Lemma 5.8 (Alternative Def. for Constr. Pareto-Efficiency)

1. *Given RPGEI* $= \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ *with* $A_{\cdot\bar{j}} > 0$ *for some* $\bar{j} \in \bigcap_{i\in I} J^i$, *an allocation* $\bar{x} = (\bar{x}^i)_{i\in I} \in \mathbb{R}_+^{S\times I}$ *is constrained Pareto-efficient if and only if* $\bar{x}^i = \omega^i + A\bar{\theta}^i$ *for some* $\bar{\theta}^i \in \mathbb{T}^i, i = 1, \ldots, I$, *where* $\sum_{i\in I} \bar{\theta}^i = 0$ *and*

$$\frac{\nabla U^i(\omega^i + A\theta^i)A_{\cdot j}}{\nabla U^i(\omega^i + A\theta^i)A_{\cdot\bar{j}}} = \frac{\nabla U^h(\omega^h + A\theta^h)A_{\cdot j}}{\nabla U^h(\omega^h + A\theta^h)A_{\cdot\bar{j}}} \tag{5.6}$$

for all $i, h \in I \times I$ *and* $j \in J^i \cap J^h$.

2. *Given RPGEI* $= \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ *with* $J^1 = J$, *an allocation* $\bar{x} = (\bar{x}^i)_{i\in I} \in \mathbb{R}_+^{S\times I}$ *is constrained Pareto-efficient if and only if* $\bar{x}^i = \omega^i + A\bar{\theta}^i$ *for some* $\bar{\theta}^i \in \mathbb{T}^i, i = 1, \ldots, I$, *where* $\sum_{i\in I} \bar{\theta}^i = 0$ *and*

$$\frac{\nabla U^1(\omega^1 + A\theta^1)A_{\cdot j}}{\nabla U^1(\omega^1 + A\theta^1)A\tilde{\theta}^i} = \frac{\nabla U^i(\omega^i + A\theta^i)A_{\cdot j}}{\nabla U^i(\omega^i + A\theta^i)A\tilde{\theta}^i} \tag{5.7}$$

for all $i \in I\backslash\{1\}$ *and* $j \in J^i$ *where* $\tilde{\theta}^i \in \mathbb{T}^i$ *such that Assumption 5.1.3, i.e.* $A\tilde{\theta}^i > 0$, *is satisfied.*

Proof: Instead of constrained Pareto-efficiency consider the maximization condition (5.3) of the last lemma. From the proof of the lemma it becomes clear that the maximizer will satisfy all constraints with equality. The corresponding Lagrangian function is

$$
\begin{aligned}
&L(x^1,\ldots,x^I,\theta^1_{J^1},\ldots,\theta^I_{J^I},\tilde{\alpha}^2,\ldots,\tilde{\alpha}^I,\pi^1,\ldots,\pi^I,q_1,\ldots,q_J) = \\
&\quad U^1(x^1) - \sum_{i=2}^{I} \tilde{\alpha}^i \left(U^i(x^i) - \bar{k}^i\right) - \sum_{s=1}^{S}\sum_{i=1}^{I} \pi^i_s \left(x^i_s - \omega^i_s - A_{sJ^i}\theta^i_{J^i}\right) \\
&\quad - \sum_{j=1}^{J} q_j \left(\sum_{i:j\in J^i} \theta^i_j\right)
\end{aligned}
$$

where $\tilde{\alpha}^i, \pi^i_s, q_j$ are the Lagrangian multipliers. Because of strict quasi-concavity of $U^i, i \in I$, and the other assumptions on the economy the maximization property is equivalent to fulfillment of the first order conditions:

$$
\begin{aligned}
&(5.8) & \nabla U^1(x^1) - \pi^{1T} &= 0; \\
&(5.9) & \alpha^i \nabla U^i(x^i) - \pi^{iT} &= 0, i = 2,\ldots,I; \\
&(5.10) & \sum_{s\in S} \pi^i_s A_{sj} - q_j &= 0, i \in I, j \in J^i; \\
&(5.11) & U^i(x^i) - \bar{k}^i &= 0, i = 2,\ldots,I; \\
&(5.12) & x^i - \omega^i - A^i\theta^i_{J^i} &= 0, i = 1,\ldots,I; \\
&(5.13) & \sum_{i:j\in J^i} \theta^i_j &= 0, j = 1,\ldots,J.
\end{aligned}
$$

Here is $\alpha^i = -\tilde{\alpha}^i$ for $i \neq 1$.

$\Rightarrow$: Substitute (5.8) and (5.9) in (5.10) to obtain

$$\alpha^i \nabla U^i(x^i) A_{\cdot j} \quad = \quad q_j \text{ for all } i \in I, j \in J^i$$

where $\alpha^1 = 1$. This leads to the following implications for $\bar{x} = (\bar{x}^i)_{i\in I}$:

1. $A_{\cdot \bar{j}} > 0$ for some $\bar{j} \in \bigcap_{i\in I} J^i$:

$$\frac{\nabla U^i(\bar{x}^i) A_{\cdot j}}{\nabla U^i(\bar{x}^i) A_{\cdot \bar{j}}} \quad = \quad \frac{q_j}{q_{\bar{j}}}$$

   whenever $j \in J^i$;

2. $J^1 = J$:

$$\frac{\nabla U^i(\bar{x}^i) A_{\cdot j}}{\nabla U^i(\bar{x}^i) A\tilde{\theta}^i} \quad = \quad \frac{q_j}{q\cdot\tilde{\theta}^i}$$

   whenever $i \in I\backslash\{1\}$ and $j \in J^i$. Note $q\cdot\tilde{\theta}^i \neq 0$ because $A\tilde{\theta}^i > 0$. Besides concerning $i = 1$ it follows for $\hat{\theta} \in \mathbb{R}^J$ with $q\cdot\hat{\theta} \neq 0$ (i.e. especially for $\tilde{\theta}^i, i = 2,\ldots,I$) that

$$\frac{\nabla U^1(\bar{x}^1) A_{\cdot j}}{\nabla U^1(\bar{x}^1) A\hat{\theta}} \quad = \quad \frac{q_j}{q\cdot\hat{\theta}}.$$

The remaining points are obvious.
$\Leftarrow$: Define $\bar{k}^i = U^i(\bar{x}^i)$ for $i = 2, \ldots, I$. Therefore, $\bar{x} = (\bar{x}^i)_{i\in I}$ and $(\bar{\theta}^i_{J^i})_{i\in I}$ satisfy equations (5.11)–(5.13) such that it remains to prove (5.8)–(5.10).

1. $A_{\cdot\bar{j}} > 0$ for some $\bar{j} \in \bigcap_{i\in I} J^i$: Define $q_j = \nabla U^1(\bar{x}^1)A_{\cdot j}$ for all $j \in J^1$, $\alpha^1 = 1$ and $\alpha^i = \frac{\nabla U^1(\bar{x}^1)A_{\cdot\bar{j}}}{\nabla U^i(\bar{x}^i)A_{\cdot\bar{j}}}$ for all $i \in I\backslash\{1\}$. Now choose inductively for all $i \in I\backslash\{1\}$

$$q_j = \alpha^i \nabla U^i(\bar{x}^i)A_{\cdot j} \text{ for all } j \in J^i \backslash \bigcup_{h=1}^{i-1} J^h.$$

   With $\pi^{iT} = \alpha^i \nabla U^i(\bar{x}^i)$ for all $i \in I$ equations (5.8)–(5.10) follow from (5.6).

2. $J^1 = J$: Define $\pi^{1T} = \nabla U^1(\bar{x}^1)$. For $j \in J^1 = J$ and $i \in I\backslash\{1\}$ take

$$\begin{aligned} q_j &= \nabla U^1(\bar{x}^1)A_{\cdot j}, \\ \alpha^i &= \frac{q \cdot \tilde{\theta}^i}{\nabla U^i(\bar{x}^i)A\tilde{\theta}^i}, \\ \pi^{iT} &= \alpha^i \nabla U^i(\bar{x}^i). \end{aligned}$$

   Equations (5.8) and (5.9) are obvious as well as (5.10) for $i = 1$. Besides for $i \neq 1$ and $j \in J^i$

$$\begin{aligned} \pi^{iT} A_{\cdot j} &= \alpha^i \nabla U^i(\bar{x}^i)A_{\cdot j} \\ &= \frac{q \cdot \tilde{\theta}^i}{\nabla U^i(\bar{x}^i)A\tilde{\theta}^i} \nabla U^i(\bar{x}^i)A_{\cdot j} \\ &= \frac{q \cdot \tilde{\theta}^i \cdot \nabla U^1(\bar{x}^1)A_{\cdot j}}{\nabla U^1(\bar{x}^1)A\tilde{\theta}^i} \\ &= \nabla U^1(\bar{x}^1)A_{\cdot j} \\ &= q_j \end{aligned}$$

   where the third equality follows from (5.7) and the fourth and the fifth one from definition of q_j.

□

This lemma allows to prove Theorem 5.6: first order conditions for consumers' maximization problems implying conditions (5.6) respectively (5.7) together with market clearing allow to apply Lemma 5.8.

5.3 Limits of Transferability from GEI-Economies to RPGEI-Economies

In this section I give two simple examples which have the following somewhat surprising features:

1. There are multiple equilibria in the restricted participation economy with asymmetric participation, but there is no symmetric participation structure with multiple equilibria.

2. There is a unique equilibrium in the restricted participation economy with asymmetric participation, but none of the economies with a symmetric participation structure equal to one of the individual participation structures or the union of all individual participation structures has to have a unique equilibrium.

The examples I present in the following are really quite simple. Nevertheless, they illuminate the effects of asymmetric participation on the uniqueness of equilibria.

To prove the first point, consider a restricted participation economy with $I = 4$ consumers and $S = 2$ states.

Consumers $i = 1$ and $i = 2$ are characterized by utility functions $U : \mathbb{R}^2_+ \to \mathbb{R}$ respectively $V : \mathbb{R}^2_+ \to \mathbb{R}$ satisfying the expected utility hypothesis with respect to an objective probability measure. Their endowments ω^1 and $\omega^2 \in \mathbb{R}^2_{++}$ are chosen such that the corresponding Arrow-Debreu economy displays multiple equilibria.[9]

Consumers $i = 3$ and $i = 4$ are characterized by the same utility functions $U : \mathbb{R}^2_+ \to \mathbb{R}$ respectively $V : \mathbb{R}^2_+ \to \mathbb{R}$, but their endowments are permuted such that $\omega^3 = (\omega^1_2, \omega^1_1)^T$ and $\omega^4 = (\omega^2_2, \omega^2_1)^T$. Now define

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then, for all symmetric participation structures (i.e. there is some $\tilde{J} \subseteq \{1, 2, 3\}$ such that for all $i \in I$ it is $J^i = \tilde{J}$) there is a unique equilibrium:

If $\tilde{J} \in \{\{1, 2\}, \{1, 3\}\}$, uniqueness follows because the model is equivalent to an Arrow-Debreu economy with a riskless aggregate endowment where consumers are characterized by expected utility functions with respect to the same objective probability measure (compare Proposition 3.13).

[9]For an example of such an economy compare page 51 ff.

If $|\tilde{J}| \leq 1$, then because of $A > 0$ and individual arbitrage-freeness each kind of trade is excluded such that the no-trade equilibrium is the unique equilibrium.

If $\tilde{J} = \{1, 2, 3\}$, the symmetric restricted participation structure does not satisfy Assumption 5.1.4. Nevertheless with the same argument as in the first case, the equilibrium commodity allocation is unique (not the portfolio allocation!).

If $\tilde{J} = \{2, 3\}$, this is equivalent to symmetric participation structures $\tilde{J} = \{2\}$ or $\tilde{J} = \{3\}$ because of asset redundancy.

However, for the asymmetric participation structure given by $J^1 = J^2 = \{1, 2\}$ and $J^3 = J^4 = \{1, 3\}$ there are multiple equilibria in $\hat{\mathcal{Q}}^R_{1,e^1}(A)$: let (x^{*1}, x^{*2}, π^*) and $(\hat{x}^1, \hat{x}^2, \hat{\pi})$ with $\pi^*_1 = \hat{\pi}_1 = 1$ be two different equilibria of the Arrow-Debreu economy defined by the first two consumers and let $(\bar{x}^3, \bar{x}^4, \bar{\pi})$ with $\bar{\pi}_1 = 1$ be an equilibrium of the Arrow-Debreu-economy defined by consumers 3 and 4. Then, $(x^{*1}, x^{*2}, \bar{x}^3, \bar{x}^4)$ with the corresponding portfolio allocations and $q^* = (1, \pi^*_2, 2\bar{\pi}_2)^T$ constitute an equilibrium with respect to normalization $\hat{\mathcal{Q}}^R_{1,e^1}(A)$ as well as $(\hat{x}^1, \hat{x}^2, \bar{x}^3, \bar{x}^4)$ with the corresponding portfolio allocations and $q^* = (1, \hat{\pi}_2, 2\bar{\pi}_2)^T$. The intuition is that for these two equilibria the RPGEI-economy falls into two subeconomies which are related by the first normalized state.

Now, consider the reverse direction: even though there is a unique equilibrium for an asymmetric participation structure, *all* symmetric participation structures defined by one of the individual trading sets evoke multiple equilibria.

Let $(U^i, \omega^i)_{i=1,2}$ be an Arrow-Debreu-economy with three different normalized equilibria

$$(x^{*1}, x^{*2}, \pi^*); (\hat{x}^1, \hat{x}^2, \hat{\pi}); (\bar{x}^1, \bar{x}^2, \bar{\pi})$$

with $\pi^*_1 = \hat{\pi}_1 = \bar{\pi}_1 = 1$ and $x^{*1} < \hat{x}^1 < \bar{x}^1$.[10] Note that I have not specified the number of states/commodities. Define

$$A_{\cdot 1} = \omega^1 - x^{*1}; A_{\cdot 2} = \omega^1 - \hat{x}^1; A_{\cdot 3} = \omega^1 - \bar{x}^1.$$

The assumptions on the equilibrium allocations assure that for participation structures with $J^i \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ for all $i \in I$ the Assumptions 5.1.2–5.1.4 are satisfied. Especially the existence of multiple equilibria implies

[10] For example, this is satisfied by the example given in Kehoe (1991, p. 2066). Hens and Pilgrim (2001, Lemma 1) note that this is true for each two-consumer Arrow-Debreu economy with multiple equilibria where consumers are characterized by additively separable utility functions.

that there has to be trade in the equilibria and equilibrium allocations are unequal to the endowments. This implies $A_{\cdot j} \neq 0$ for all $j \in J$.

If $J^1 = \{1, 2\}$ and $J^2 = \{1, 3\}$ Assumption 5.1.1 is satisfied and the only asset which can be traded in the restricted participation economy is the first asset. Therefore, there is a unique equilibrium with commodity allocations given by the first Arrow-Debreu equilibrium, portfolio plans $\theta^1 = -\theta^2 = (-1, 0, 0)^T$ where the price of the first asset is zero and the other asset prices do not matter. Note that this is not the no-trade equilibrium!

However, for the symmetric participation structures where all consumers are characterized by the same trading set $\tilde{J} \in \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$[11] one easily computes the existence of the *- and the $\hat{}$- respectively the *- and the $\bar{}$- respectively the *-, the $\hat{}$- and the $\bar{}$-equilibrium allocations, corresponding portfolio allocations, and asset price systems q defined by $q_j = \pi \cdot A_{\cdot j}$ for $j \in \tilde{J}$.[12]

5.4 How to Transfer the Results on Uniqueness from GEI- to RPGEI-Economies

In this section I give sufficient conditions to guarantee a unique restricted participation equilibrium. Attention is directed to asymmetric participation structures. For symmetric participation structures the results from Chapter 4 can be applied because then the RPGEI-economy is equivalent to a GEI-economy. This is contained in the following proposition which requires restrictive properties of the participation structure.

Proposition 5.9 (Coherency to the GEI-Model)
Given the economy $RPGEI = \{(U^i, \omega^i, J^i)_{i \in I}, A\}$ such that there is some $\bar{J} \subseteq J$ with

$$J^i \cap \left(\bigcup_{h \in I \setminus \{i\}} J^h \right) = \bar{J}$$

for all $i \in I$ and $A_{\cdot \bar{j}} > 0$ for some $\bar{j} \in \bar{J}$, the restricted participation economy has a unique equilibrium if and only if the Finance GEI-economy given by $\{(U^i, \omega^i)_{i \in I}, (A_{\cdot j})_{j \in \bar{J}}\}$ has a unique equilibrium.

---

[11] Clearly for $\tilde{J} = \{1, 2\}$ respectively $\tilde{J} = \{1, 3\}$ Assumption 5.1.1 is not satisfied, but here I am interested in the corresponding GEI-economies.

[12] For $\tilde{J} = \{1, 2, 3\}$ the portfolio demands are not necessarily uniquely determined because $A_{\cdot 1}, A_{\cdot 2}, A_{\cdot 3}$ might be linearly dependent.

$A_{\cdot\bar{j}} > 0$ is no strong restriction since $\bar{J} \subseteq J^i$ for all $i \in I$ allows to substitute $A_{\cdot\bar{J}}$ by $A_{\cdot\bar{J}}M$ for some invertible $\bar{J} \times \bar{J}$-matrix M without changing the number of equilibria. This is the same as with Proposition 4.3.

The condition on the participation structure requires the existence of a set of assets which can be traded by *all* consumers and all the other existing assets can be traded by exactly one consumer. Therefore, a priori it is clear that there cannot be any trade of these additional assets.

The result is quite obvious. Therefore I only give a proof for the case where it is sufficient to look at first order conditions to solve consumers' maximization problems.[13]

Proof: It is possible to apply the price normalization $q_{\bar{j}} = 1$ for the restricted participation as well as the Finance GEI-economy.

For the restricted participation economy $\left((\theta^{iR})_{i\in I}, q^R\right) \in \mathbb{R}^{J\times I} \times \mathbb{R}^J$ is equilibrium if and only if

$$\nabla U^i(\omega^i + A\theta^{iR})A_{\cdot j} - \lambda^{iR}q_j^R = 0 \text{ for all } (i,j) \in I \times J \text{ with } j \in J^i, \tag{5.14}$$

$$q^R \cdot \theta^{iR} = 0 \text{ for all } i \in I, \tag{5.15}$$

$$\theta_j^{iR} = 0 \text{ for all } (i,j) \in I \times J \text{ with } j \in J \backslash J^i, \tag{5.16}$$

$$\sum_{i\in I} \theta_j^{iR} = 0 \text{ for all } j \in J, \tag{5.17}$$

where $\lambda^{iR}, i = 1, \ldots, I$, are the Lagrangian multipliers. Because of the restriction on the participation structure (5.16) and (5.17) can be substituted by

$$\theta_j^{iR} = 0 \text{ for all } (i,j) \in I \times J \text{ with } j \in J \backslash \bar{J}, \tag{5.18}$$

$$\sum_{i\in I} \theta_j^{iR} = 0 \text{ for all } j \in \bar{J}. \tag{5.19}$$

(5.18) allows the substitution of (5.14) and (5.15) by

$$\nabla U^i(\omega^i + A_{\cdot\bar{J}}\theta_{\bar{J}}^{iR})A_{\cdot j} - \lambda^{iR}q_j^R = 0 \text{ for all } (i,j) \in I \times J \text{ with } j \in J^i, \tag{5.20}$$

$$q_{\bar{J}}^R \cdot \theta_{\bar{J}}^{iR} = 0 \text{ for all } i \in I, \tag{5.21}$$

where the subscript $\bar{J}$ denotes the restriction to $\bar{J}$.

For the Finance GEI-economy $\left((\theta^{iF})_{i\in I}, q^F\right) \in \mathbb{R}^{\sharp\bar{J}\times I} \times \mathbb{R}^{\sharp\bar{J}}$[14] is equilibrium if and only if

---

[13] All the results on uniqueness in the Finance GEI-model given in Chapter 4 are formulated under these conditions.

[14] For simplicity the coordinates are denoted by the elements of $\bar{J}$.

$$\nabla U^i(\omega^i + A_{\cdot\bar{J}}\theta^{iF})A_{\cdot j} - \lambda^{iF} q_j^F = 0 \text{ for all } (i,j) \in I \times \bar{J}, \tag{5.22}$$

$$q^F \cdot \theta^{iF} = 0 \text{ for all } i \in I, \tag{5.23}$$

$$\sum_{i \in I} \theta_j^{iF} = 0 \text{ for all } j \in \bar{J}, \tag{5.24}$$

for Lagrangian multipliers $\lambda^{iF}, i = 1, \ldots, I$. With

$$\begin{aligned}
\theta^{iR} &= \begin{cases} \theta_j^{iF} \text{ for } j \in \bar{J}, \\ 0 \text{ for } j \in J \backslash \bar{J}, \end{cases} \\
\lambda^{iR} &= \lambda^{iF}, \\
q_j^R &= \begin{cases} q_j^F \text{ for } j \in \bar{J}, \\ \frac{1}{\lambda^{iF}} \nabla U^i(\omega^i + A_{\cdot\bar{J}}\theta^{iF})A_{\cdot j} \text{ for } j \in \big(J \backslash \bar{J}\big) \cap J^i, \end{cases}
\end{aligned}$$

the content of the proposition follows.

□

In addition, there are results on the uniqueness of restricted participation equilibria which are not directly built on the GEI-results.

5.4.1 Existence of Explicit Pricing Formulas

No-Trade Equilibrium

The simplest case to imagine is the one where all U^i and ω^i are identical such that there is no incentive for trading. This can be seen as special case of constrained Pareto-efficient endowments. According to Lemma 5.8 a necessary and sufficient condition when $A_{\cdot\bar{j}} > 0$ for some $\bar{j} \in \bigcap_{i \in I} J^i$ is given by

$$\frac{\nabla U^i(\omega^i)A_{\cdot j}}{\nabla U^i(\omega^i)A_{\cdot\bar{j}}} = \frac{\nabla U^h(\omega^h)A_{\cdot j}}{\nabla U^h(\omega^h)A_{\cdot\bar{j}}} \tag{5.25}$$

for all $(i,h) \in I \times I$ and $j \in J^i \cap J^h$ or when $J^1 = J$ by

$$\frac{\nabla U^1(\omega^1)A_{\cdot j}}{\nabla U^1(\omega^1)A\tilde{\theta}^i} = \frac{\nabla U^i(\omega^i)A_{\cdot j}}{\nabla U^i(\omega^i)A\tilde{\theta}^i} \tag{5.26}$$

for all $i \in I \backslash \{1\}$ and $j \in J^i$ where $\tilde{\theta}^i \in \mathbb{T}^i$ such that Assumption 5.1.3, i.e. $A^i\tilde{\theta}^i_{J^i} > 0$, is satisfied. The chance for fulfillment of these equations is bigger than the one for the corresponding equations for $GEI = \{(U^i, \omega^i)_{i \in I}, A\}$.

Proposition 5.10 (Uniqueness for Constrained Pareto-Efficiency)
Given the economy RPGEI $= \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ *with either* $J^1 = J$ *or* $A_{\cdot\bar{j}} > 0$ *for some* $\bar{j} \in \bigcap_{i\in I} J^i$ *and* $(\omega^i)_{i\in I}$ *constrained Pareto-efficient, there is a unique RPGEI-equilibrium, namely the no-trade equilibrium. For* $J^1 = J$ *normalization* $q \cdot \tilde{\theta}^1 = 1$ *delivers*

$$q_j^* = \frac{\nabla U^1(\omega^1) A_{\cdot j}}{\nabla U^1(\omega^1) A \tilde{\theta}^1}$$

and for the second case normalization $q_{\bar{j}}^* = 1$ *delivers*

$$q_j^* = \frac{\nabla U^i(\omega^i) A_{\cdot j}}{\nabla U^i(\omega^i) A_{\cdot\bar{j}}}$$

for all $i \in I$ *with* $j \in J^i$.

The proof is the same as the one for Proposition 4.14 because trading sets $\mathbb{T}^i$ are linear subspaces of $\mathbb{R}^J$ and contain $\theta^i = 0$.

The analogue to Lemma 4.15 is given in the following way. Note that one cannot switch to alternative representations of the asset span $< A >$ because the participation structure depends on A.

Lemma 5.11 (Implications of Constrained Pareto-Efficiency)
If the no-trade allocation $(\omega^i)_{i\in I}$ *is constrained Pareto-efficient in RPGEI* $= \{(U^i, \omega^i, J^i)_{i\in I}, A\}$, *this also true for* $\widetilde{RPGEI} = \{(U^i, \omega^i, \tilde{J}^i)_{i\in I}, A_{\cdot\tilde{J}}\}$ *where* $\tilde{J} \subseteq J$ *and for all* $i \in I$ *it is* $\tilde{J}^i \subseteq J^i \cap \tilde{J}$. *RPGEI and* $\widetilde{RPGEI}$ *both have to satisfy Assumption 5.1 and either* $J^1 = J$ *respectively* $\tilde{J}^1 = \tilde{J}$ *or* $A_{\cdot\bar{j}} > 0$ *for some* $\bar{j} \in \bigcap_{i\in I} J^i$ *respectively* $\tilde{A}_{\cdot\tilde{j}} > 0$ *for some* $\tilde{j} \in \bigcap_{i\in I} \tilde{J}^i$.

This follows because the possibilities of reallocation in $\widetilde{RPGEI}$ constitute a subset of those in *RPGEI*. Apply Lemma 5.8 and transfer the proof of Lemma 4.15.

Proposition 5.12 (Inclusion Property)
If the economy RPGEI $= \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ *is characterized by the unique (constrained Pareto-efficient) no-trade RPGEI-equilibrium, this is also true for* $\widetilde{RPGEI} = \{(U^i, \omega^i, \tilde{J}^i)_{i\in I}, A_{\cdot\tilde{J}}\}$ *whenever RPGEI and* $\widetilde{RPGEI}$ *satisfy the conditions given in the previous lemma.*

Complete Risk Sharing

To obtain complete risk sharing in equilibria, in addition to the conditions for the Arrow-Debreu model in RPGEI-economies one needs the assumption of "individually spanned" endowments.

Proposition 5.13 (Complete Risk Sharing)
Let an economy $RPGEI = \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ *be given such that for all* $i = 1, \ldots, I$ *the expected utility hypothesis is satisfied, i.e.*

$$U^i(x) = \sum_{s=1}^{S} \bar{\rho}_s u^i(x_s),$$

where u^i *satisfies standard assumptions and* $\bar{\rho}$ *is an objective probability measure. Endowments are individually spanned, i.e.*

$$\omega^i = A\theta^i_\omega \text{ for some } \theta^i_\omega \in \mathbb{T}^i.$$

If ω *is riskless (i.e.* $\omega = \bar{\omega}1\!\!1$ *for some* $\bar{\omega} \in \mathbb{R}_{++}$*) and* $\sum_{i=1}^I \theta^i_\omega \in \mathbb{T}^i$ *for all* $i \in I$*, the equilibrium is unique. With price normalization* $q \cdot \sum_{i=1}^I \theta^i_\omega = \bar{\omega}$ *one obtains the pricing formula*

$$q^{*T} = \left(\sum_{s=1}^{S} \bar{\rho}_s A_{s1}, \ldots, \sum_{s=1}^{S} \bar{\rho}_s A_{sJ} \right).$$

Note that the pricing formula is independent of the participation structure and therefore equal to the GEI-formula.

Proof: There is exactly one normalized equilibrium price system q^* such that the corresponding allocation is riskless. This follows because in this case first order conditions for (5.2) determine prices to satisfy

$$\text{(5.27)} \qquad \frac{q_j}{\bar{\omega}} = \frac{q_j}{q \cdot \sum_{i=1}^I \theta^i_\omega} = \frac{\sum_{s=1}^S \bar{\rho}_s A_{sj}}{\sum_{s=1}^S \bar{\rho}_s A_{s\cdot} \sum_{i=1}^I \theta^i_\omega} = \frac{\sum_{s=1}^S \bar{\rho}_s A_{sj}}{\bar{\omega}}$$

whenever $j \in J^i$ for some $i \in I$. Then Assumption 5.1 implies this relation for arbitrary $j \in J$. The assumptions on the economy imply that all equilibrium allocations are riskless. Suppose not. Let $(\hat{x}^i)_{i=1,\ldots,I}$ be the corresponding allocation with $\hat{x}^i = A(\theta^i_\omega + \hat{\theta}^i)$ for some $\hat{\theta}^i \in \mathbb{T}^i, i = 1, \ldots, I$, with $\sum_{i=1}^I \hat{\theta}^i = 0$. Strict concavity of $u^i, i = 1, \ldots, I$, assures that $x^{*i} = \left(\sum_{s=1}^S \bar{\rho}_s \hat{x}^i_s\right) 1\!\!1, i = 1, \ldots, I$, improves all consumers and at least one of them strictly. Since

$$\sum_{i=1}^{I} x^{*i} = \sum_{i=1}^{I} \left(\sum_{s=1}^{S} \bar{\rho}_s \hat{x}^i_s \right) 1\!\!1 = \sum_{s=1}^{S} \bar{\rho}_s \left(\sum_{i=1}^{I} \hat{x}^i_s \right) 1\!\!1 = \sum_{s=1}^{S} \bar{\rho}_s \omega_s 1\!\!1 = \bar{\omega} 1\!\!1$$

and

$$x^{*i} = \sum_{s=1}^{S} \bar{\rho}_s A_{s\cdot}(\theta^i_\omega + \hat{\theta}^i) 1\!\!1 = \omega^i + A\theta^{*i}$$

with $\theta^{*i} = \frac{1}{\bar{\omega}} \sum_{s=1}^{S} \bar{\rho}_s A_{s\cdot}(\theta^i_\omega + \hat{\theta}^i) \sum_{i=1}^{I} \theta^i_\omega - \theta^i_\omega$, this contradicts the constrained Pareto-efficiency of RPGEI-equilibria because $\theta^{*i} \in \mathbb{T}^i$ for all $i \in I$ and $\sum_{i=1}^{I} \theta^{*i} = 0$.

□

The reasoning on uniqueness of FM-equilibria via quadratic utility or homothetic preferences (existence of a representative consumer) cannot be transferred to RPGEI-economies with asymmetric participation structures. Therefore, I go on with the property of monotonicity.

5.4.2 How to Obtain Monotonicity

Those results on uniqueness in the GEI-model which are built on the Mitjushin-Polterovich coefficient do no longer hold in the RPGEI-framework with asymmetric participation structures. Even the additional assumption of individually spanned endowments is not helpful, because it implies monotonicity of individual demand $\hat{\theta}^i(q_{J^i})$ with respect to the spanning vector $(\bar{\theta}^i_\omega)_{J^i}$, but for asymmetric participation structures collinearity of $(\bar{\theta}^i_\omega)_{i\in I}$ never holds and therefore market demand cannot be proved to be monotone. Also under the expected utility hypothesis — even for the existence of fundamental sets with respect to individual trading sets — monotonicity of market demand cannot be shown because individual demand functions are not monotone with respect to the same normalizing vector. Therefore, it remains to check what happens for quasi-linear utility or two assets and a small relative risk aversion.

Proposition 5.14 (Uniqueness for Quasi-Linearity)
Assume $RPGEI = \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ such that U^i is quasi-linear for all $i \in I$, i.e. there exists some strictly concave function $\tilde{U}^i : \mathbb{R}^{S-1}_+ \to \mathbb{R}$ such that

$$U^i(x) = x_1 + \tilde{U}^i(x_{-1})$$

and $\tilde{U}^i$ satisfies Assumption 2.4 on $\mathbb{R}^{S-1}_+$. If A models first period consumption, i.e.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \breve{A} \end{pmatrix}$$

for some $\breve{A} \in \mathbb{R}^{(S-1)\times(J-1)}$ and for all consumers it is $1 \in J^i$, the economy exhibits a unique RPGEI-equilibrium whenever today's endowments are sufficiently large.

Proof[15]: Since $1 \in J^i$ for all $i \in I$ and $A_{\cdot 1} > 0$ the first asset can be taken as a numeraire and price normalization $\hat{\mathcal{Q}}^R_{1,e^1}(A)$ can be applied. Therefore, each consumers maximization problem can be rewritten as

$$\begin{aligned} \max_{\theta_{-1} \in \mathbb{R}^{J-1}} \; & \omega^i_1 - q_{-1} \cdot \theta_{-1} + \tilde{U}^i(\omega^i_{-1} + \check{A}\theta_{-1}) \\ \text{s.t.} \;\; & \omega^i_1 - q_{-1} \cdot \theta_{-1} \geq 0, \\ \wedge \;\; & \omega^i_{-1} + \check{A}\theta_{-1} \geq 0, \\ \wedge \;\; & \theta_j = 0 \text{ for } j \in J \backslash J^i. \end{aligned}$$

Assumptions 5.1 and 2.4 assure that tomorrow's consumption plans are chosen to be strictly greater than 0. Therefore, w.l.o.g. the restriction $\omega^i_{-1} + \check{A}\theta_{-1} \geq 0$ can be omitted.

At first consider the maximization problem also without the restriction $\omega^i_1 - q_{-1} \cdot \theta_{-1} \geq 0$. Strict concavity of $\tilde{U}^i$ then assures sufficiency of the following (first order) conditions for maximization (together with $\theta^i_j = 0$ for $j \in J \backslash J^i$):

$$q_j = \nabla \tilde{U}^i(\omega^i_{-1} + \check{A}\theta^i_{-1})\check{A}_{\cdot j} \text{ for all } j \in J^i \backslash \{1\}.$$

Since $\theta^i_j, j \in J \backslash J^i$ is fixed to zero, the differential form is for $\check{A}^i = (\check{A}_{\cdot j})_{j \in J^i \backslash \{1\}}$ given by

$$dq_{J^i \backslash \{1\}} = (\check{A}^i)^T D^2 \tilde{U}^i(x^i_{-1}) \check{A}^i d\hat{\theta}^i_{-1}(\begin{pmatrix} 1 \\ q_{J^i \backslash \{1\}} \end{pmatrix})$$

where $H^i := (\check{A}^i)^T D^2 \tilde{U}^i(x^i_{-1}) \check{A}^i$ is negative definite because $\tilde{U}^i$ is strictly concave and there are no redundancies in $\check{A}^i$. Therefore, H^i has a negative definite inverse $(H^i)^{-1}$ and it is

$$\partial_{q_{J^i \backslash \{1\}}} \hat{\theta}^i_{-1}(\begin{pmatrix} 1 \\ q_{J^i \backslash \{1\}} \end{pmatrix}) = (H^i)^{-1}$$

which implies negative semi-definiteness of $\partial_{q_{-1}} \theta^i_{-1}(\binom{1}{q_{-1}})$. To prove that

$$\theta(\begin{pmatrix} 1 \\ q_{-1} \end{pmatrix}) = \sum_{i \in I} \theta^i(\begin{pmatrix} 1 \\ q_{-1} \end{pmatrix}) = 0$$

has a unique solution q_{-1}, it is sufficient to prove the negative definiteness of $\partial_{q_{-1}} \theta_{-1}(\binom{1}{q_{-1}}) = \sum_{i \in I} \partial_{q_{-1}} \theta^i_{-1}(\binom{1}{q_{-1}})$. Therefore, take some arbitrary $v \in$

[15] This proof is a transformation of the corresponding proof for GEI-economies given in Voß (1997, Satz 4.1.7).

$\mathbb{R}^{J-1}\backslash\{0\}$. Since $v \neq 0$ and each $j \in J$ can be traded by at least one consumer, there has to be some $\bar{h} \in I$ with

$$v^T \partial_{q_{-1}} \theta^{\bar{h}}_{-1}(\begin{pmatrix} 1 \\ q_{-1} \end{pmatrix})v < 0.$$

Together with the negative semidefiniteness of the normalized Jacobian of all other individual demand functions this proves negative definiteness of the normalized Jacobian of market demand.

In addition, a sufficiently large first period endowment for each consumer allows to restrict ex post first period consumption to be non-negative without changing the equilibria.

From first order conditions and budget constraints one obtains

$$\begin{aligned} x^i_1 &= \omega^i_1 - q_{-1} \cdot \theta^i_{-1} \\ &= \omega^i_1 - q_{J^i\backslash\{1\}} \cdot \theta^i_{J^i\backslash\{1\}} \\ &= \omega^i_1 - \nabla \tilde{U}^i(x^i_{-1}) \check{A}^i \theta^i_{J^i\backslash\{1\}} \\ &= \omega^i_1 - \nabla \tilde{U}^i(x^i_{-1})(x^i_{-1} - \omega^i_{-1}). \end{aligned}$$

Since $x^i_{-1} \geq 0$ for all $i \in I$, one knows for all attainable commodity allocations

$$0 \leq x^i_{-1} \leq \omega_{-1} := \sum_{i \in I} \omega^i_{-1}.$$

Therefore, together with strict monotonicity of $\tilde{U}^i$ this implies

$$x^i_1 \geq \omega^i_1 - \nabla \tilde{U}^i(x^i_{-1}) \sum_{h \in I\backslash\{i\}} \omega^h_{-1}. \tag{5.28}$$

If $\mathcal{I}(\omega^i_{-1})$ is consumer i's indifference curve through $\begin{pmatrix} 0 \\ \omega^i_{-1} \end{pmatrix}$ and $\mathcal{S}(\omega^i_{-1}) := \mathcal{I}(\omega^i_{-1}) \cap (\{0\} \times \mathbb{R}^{S-1}_+)$, the assumptions on $\tilde{U}^i$ guarantee $\mathcal{S}(\omega^i_{-1}) \subseteq \{0\} \times \mathbb{R}^{S-1}_{++}$. The set $\mathcal{H}(\omega^i_{-1}) := \left\{y \in \mathbb{R}^{S-1}_+ \mid \exists\, z \in \mathcal{S}(\omega^i_{-1}) \text{ with } z_{-1} \leq y \leq \omega_{-1}\right\}$ obviously is compact such that in this set there is a maximizer $\bar{m}^i$ of the continuous function $\nabla \tilde{U}^i(y) \sum_{h \in I\backslash\{i\}} \omega^h_{-1}$. Now assume

$$\nabla \tilde{U}^i(\bar{m}^i) \sum_{h \in I\backslash\{i\}} \omega^h_{-1} < \omega^i_1, \tag{5.29}$$

i.e. first period endowment is sufficiently large. If x^i_1 were negative, strict monotonicity of U^i and $\omega^i_1 > 0$ together with attainability would imply $x^i_{-1} \in$

$\mathcal{H}(\omega^i_{-1})$. Then (5.28) and (5.29) assure $x^i_1 > 0$ which is a contradiction.

□

Other results on the uniqueness of restricted participation equilibria are obtainable in economies where consumers are characterized by expected utility functions. Here, the results are very close to the results on uniqueness under symmetric participation. The reason is that the proof in the Finance GEI-model either applies monotonicity with respect to the normalizing vector e^1 which is independent of consumers' endowments or the property of strict gross substitution for individual demand functions which implies the weak gross substitution property of market demand in the restricted participation economy. A priori this is not sufficient for a unique restricted participation equilibrium. However, since each asset is traded by at least one consumer, uniqueness can nonetheless be proved.

Proposition 5.15 (Uniqueness with $RRA^i \leq 1$ and 2 Assets)
Assume $RPGEI = \{(U^i, \omega^i, J^i)_{i\in I}, A\}$ where $\sharp J^i \leq 2$ for all $i \in I$ and there is some $\bar{j} \in J$ with $A_{\cdot\bar{j}} > 0$ and $\bar{j} \in \bigcap_{i\in I} J^i$. If $(U^i)_{i\in I}$ satisfy the expected utility hypothesis with

$$RRA^i(x_s) \leq 1 \text{ for all } x_s \in \mathbb{R}_{++},$$

the economy has a unique RPGEI-equilibrium.

Within this framework the assumption of expected utility can also be weakened to additive separability. Besides notice the possibility of restrictions on absolute risk aversion (compare (3.21)).

This proposition especially includes the case of first period consumption. Proof: W.l.o.g. $\bar{j} = 1$. Then, from the Finance GEI-model one obtains for all $i \in I$ with $\sharp J^i = 2$ (compare Bettzüge (1997))

$$\partial_{q_j} \theta^i_j(q) < 0 \text{ for } j \in J^i \backslash \{1\}$$

and all the other entries in the normalized Jacobian of individual demand $\partial_{q_{-1}} \theta^i_{-1}(q)$ are equal to zero. If there is some $i \in I$ with $\sharp J^i = 1$, the normalized Jacobian obviously only has entries equal to zero. Since for all $j \in J$ there is some $i \in I$ such that $j \in J^i$, the normalized Jacobian of market demand $\partial_{q_{-1}} \theta_{-1}(q) = \sum_{i\in I} \partial_{q_{-1}} \theta^i_{-1}(q)$ is a diagonal matrix with strictly negative diagonal elements which is negative definite. This is a sufficient condition for a unique normalized RPGEI-equilibrium price system.

□

5.4.3 How to Obtain the Property of Gross Substitution

Working with the property of gross substitution whenever the relative risk aversion is sufficiently small, does not necessarily require restrictions on the participation structure if one imposes some strong assumptions on the asset structure.

Proposition 5.16 (Uniqueness with $RRA^i \leq 1$ and Special A)
Assume $RPGEI = \{(U^i, \omega^i, J^i)_{i \in I}, A\}$ such that for all $i \in I$ U^i satisfies the expected utility hypothesis with

$$RRA^i(x_s) \leq 1 \text{ for all } x_s \in \mathbb{R}_{++}.$$

There is a unique RPGEI-equilibrium if one of the following conditions is satisfied:.

1. $\sharp J^i \leq 2$ *and* $A > 0$*;*
2. $A > 0$ *is weakly separating, i.e. for each* $s \in S$ *there is exactly one* $j \in J$ *such that* $A_{sj} > 0$*.*

Proof: From results in the Finance GEI-model (compare Voß (1997)) one knows that for all $i \in I$ the matrix $\partial_{q_{J^i}} \theta^i_{J^i}(q)$ is a matrix with strictly negative diagonal and strictly positive off-diagonal elements. Therefore, for the Jacobian of market demand it is

$$\partial_{q_k} \theta_j \begin{cases} < 0 & \text{for } j = k; \\ > 0 & \text{for } j \neq k \text{ and there is some } i \in I \text{ such that } j, k \in J^i; \\ = 0 & \text{for } j \neq k \text{ and there is no } i \in I \text{ such that } j, k \in J^i. \end{cases}$$

Now, assume there are two different equilibrium price systems[16] $\tilde{q}$ and q^*. Then, since $A > 0$, it is possible to normalize $\tilde{q}$ and q^* such that $0 \ll q^* < \tilde{q}$ and the smallest entry for both vectors is 1. W.l.o.g. let $q^*_1 = \tilde{q}_1 = 1$; otherwise renumerate the assets and redefine the participation structure. Now define

$$\tilde{K} := \{k \in J \mid q^*_k = \tilde{q}_k\}$$

and

$$\bar{k} := \min\{k \in \tilde{K} \mid \exists\, j \in J \backslash \tilde{K} \text{ with } j, k \in J^i \text{ for some } i \in I\}.$$

Assumption 5.1.2 guarantees the existence of $\bar{k}$: there is no disjoint partition $\tilde{K}, J \backslash \tilde{K}$ of J such that for all $i \in I$ either $J^i \subseteq \tilde{K}$ or $J^i \subseteq J \backslash \tilde{K}$. Define

$$\bar{j} := \min\{j \in J \backslash \tilde{K} \mid \bar{k}, j \in J^i \text{ for some } i \in I\}.$$

[16] The assumptions on the economy allow to define a unique equilibrium via asset prices.

Take a finite sequence of price systems $p^1, \ldots, p^N$ in the set $\hat{\mathcal{Q}}^R_{1,e^1}(A)$ such that $p^1 = q^*$, $p^N = \tilde{q}$ and $p^n \leq p^{n+1}$ for all $n \in \{1, \ldots, N-1\}$ and p^n and p^{n+1} are different in one coordinate. If $A > 0$ and A is weakly separating define $(p^n)_{n \in N}$ by coordinatewise rising of q^* to $\tilde{q}$. If $A > 0$, but not weakly separating, it might be that one has to choose a sequence with more elements to assure to stay in the set of individually arbitrage-free price systems. This is possible because according to Siconolfi (1989) the set is open and convex.[17] Because of the definition of $\bar{j}$ there has to be some $n \in N$ such that $p^n_{\bar{j}} < p^{n+1}_{\bar{j}}$ and $p^n_j = p^{n+1}_j$ for all $j \in J \backslash \{\bar{j}\}$.

Let $p^{\bar{n}}$ be an element of the sequence $(p^n)_{n \in N}$ which satisfies these properties. Then because of $\bar{k} \in \tilde{K}$ and $\bar{j}, \bar{k} \in J^i$ for some $i \in I$ and $\bar{j} \neq \bar{k}$

$$0 = \theta_{\bar{k}}(p^1) \leq \theta_{\bar{k}}(p^{\bar{n}}) < \theta_{\bar{k}}(p^{\bar{n}+1}) \leq \theta_{\bar{k}}(p^N) = 0$$

which gives a contradiction.

□

5.5 Overview of Conditions for Uniqueness in the RPGEI-Model

The following figure presents in analogy to Figures 3.4 and 4.1 conditions on the economies' characteristics guaranteeing a unique RPGEI-equilibrium. However, it does not take into account Proposition 5.9: for symmetric participation structures I refer to Figures 3.4 and 4.1. The additional economies covered by Proposition 5.9 are identical to GEI-economies except for the existence of further assets where each of them is only traded by exactly one consumer. Therefore, I also refer to the figures mentioned before.

[17] Compare Voß (1997) for a thorough discussion in the Finance GEI-model.

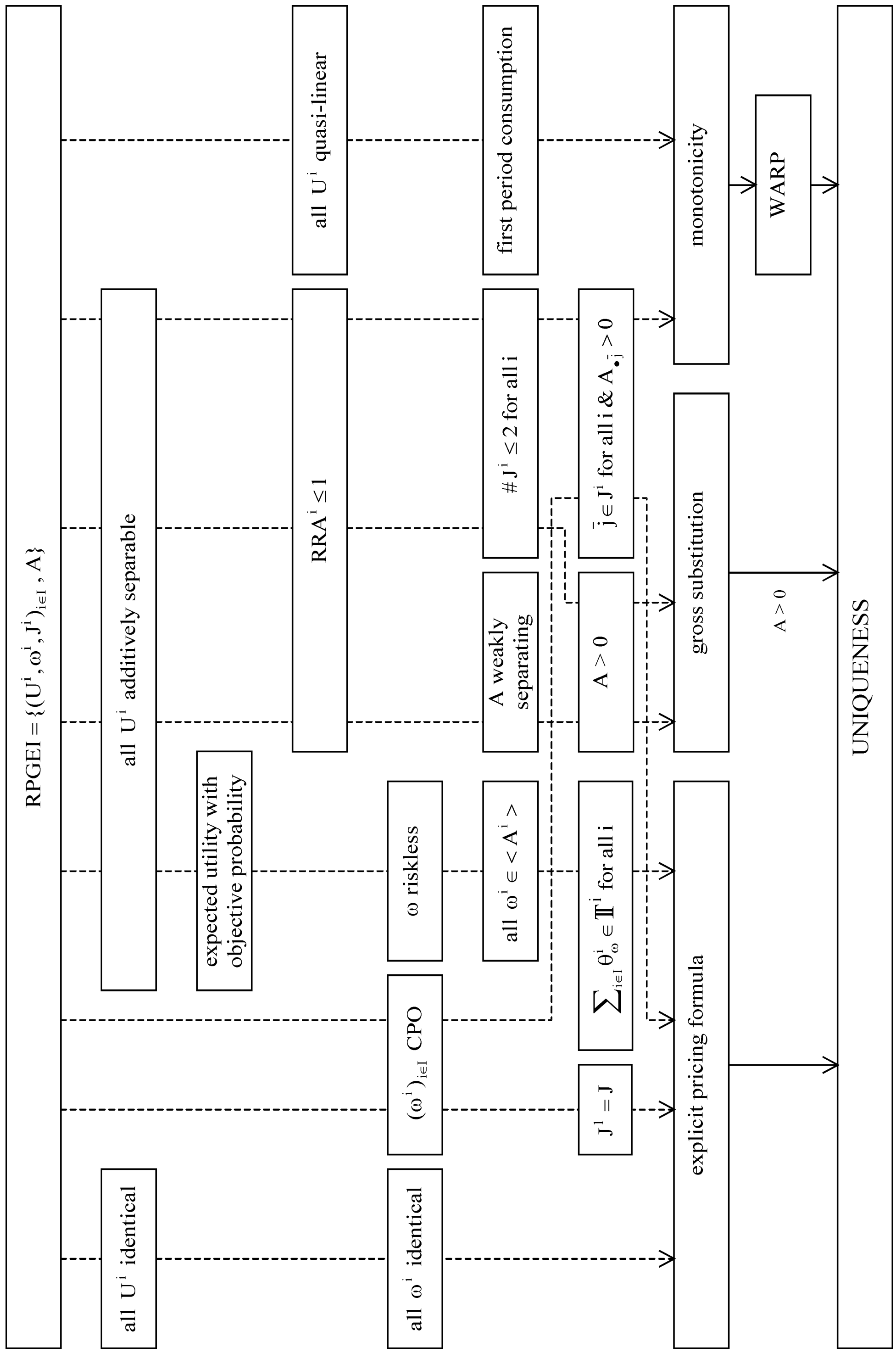

Figure 5.1: Conditions for uniqueness in the RPGEI-model.

Part III

Multiplicity of Competitive Equilibria and the Existence of Sunspot Equilibria

Chapter 6

Introduction

The purpose of this part is to clarify the relationship between multiplicity of competitive equilibria and the existence of sunspot equilibria. The analysis is built up in the framework given in Mas-Colell's (1992) famous paper which relates multiplicity of Arrow-Debreu equilibria to the existence of sunspots in a two-period market economy extended for extrinsic uncertainty. His methodology allows to prove the existence of non-trivial sunspot equilibria under the assumption of three or more commodities. In this part I prove that under additive separability of utility functions his result does no longer hold when considering only two commodities. As will become clear later on, this observation puts some light on the fact that even potential multiplicity of competitive equilibria might not be sufficient for the existence of sunspot equilibria. This aspect of "potential multiplicity" has been illuminated recently by Gottardi and Kajii (1999) who prove a kind of generic existence of sunspot equilibria, but who in contrast to Mas-Colell (1992) — and therefore also to me — consider a sunspot economy with first period consumption.

Economic fluctuations can be partly, but not totally explained by changes in the economic fundamentals. Even extensions of equilibrium models by, for example, irrational expectations or asymmetric information are not sufficient to account for the volatility of prices observed "in reality". At this point, the analysis of sunspot equilibria allows to search for further reasons, namely a kind of psychological ones.

Take an economy evolving over time in which uncertainty affects the economic fundamentals like agents' preferences and endowments. This kind of uncertainty is considered as *intrinsic* uncertainty and clearly causes fluctuations in economic activity. If there are additional exogenous random events which do not have any influence on the economic fundamentals, this kind of uncertainty is called *extrinsic*. Such events are commonly referred to as sunspot events. The sunspot theory takes these events to explain additional fluctuations in economic activity.

If all agents believed only in the economic fundamentals and knew all the others to share this opinion, this approach would be unfounded. However, if everybody takes care of the sunspot events, this behavior can be shown to be self-enforcing. The "sunspot explanation is that not only fundamentals matter because everybody might believe that not only fundamentals matter" (Hens (1991, p. 62)).

The literature distinguishes various kinds of general equilibrium models with sunspots, whether with finite or infinite horizon, whether with complete or incomplete markets and so on. A complete overview is given by Chiappori and Guesnerie (1991) who however mainly consider an infinite horizon framework. For a survey on two-period economies as they are examined in this part compare Hens (1998, Section 9).

I restrict my attention to a finite, namely two-period model with no intrinsic, but only extrinsic uncertainty. Besides I mainly consider real assets and only touch on nominal assets. In the model which I apply there is no first period consumption such that a priori efficient allocations are obtainable without trading the assets.

After a presentation of the underlying model, I give an overview of the literature on the existence of sunspot equilibria in frameworks related to the one defined by Mas-Colell (1992) and therefore to my own. The main point thereafter is to prove the non-existence of sunspot equilibria for some special economies built up on Arrow-Debreu economies displaying arbitrary many equilibria.

6.1 The Model

It would be possible to describe the model underlying this part as a sunspot extension of the basic Finance GEI-model described in Chapter 2. However, to maintain this part essentially independent and separate from the rest, I give a complete description of the model.

Consider a two-period competitive exchange economy. Uncertainty arises because tomorrow one of a finite number of states $s \in \mathbb{S} := \{1, \ldots, S\}$[1], $(S \geq 2)$ — so-called sunspot events — realizes with a given publicly known probability $\bar{\rho}_s$ where $\bar{\rho} \in \hat{\mathcal{P}}_{1,\mathbb{1}} := \{\rho \in \mathbb{R}^S_{++} |\ \sum_{s=1}^{S} \rho_s = 1\}$. Uncertainty is only extrinsic, i.e. economic fundamentals are independent of these states.

[1]Correspondingly to the notations introduced in the other parts, I define a set $\mathbb{N} := \{1, \ldots, N\}$ if there is a variable n taking values $1, \ldots, N$ where $N \in \mathbb{N} := \{1, 2, \ldots\}$.

There are L physically distinguishable commodities $l \in L := \{1, \dots, L\}$ in each future spot. Assume $L \geq 2$.[2] In each future spot there is a complete system of commodity markets. Price systems for each of these market systems are called $p_s \in \mathbb{R}^L_{++}$ where $p_s := (p_{s1}, \dots, p_{sL})^T$ and the collection of all spot price systems is $p^T := (p_1^T, \dots, p_S^T) \in \mathbb{R}^{SL}_{++}$.

A finite number of households/consumers $i \in I := \{1, \dots, I\}$ is active on these markets. These households are characterized by their consumption sets $\mathbb{R}^{SL}_+$, their utility function $\hat{U}^i : \mathbb{R}^{SL}_+ \to \mathbb{R}$, and endowments $\hat{\omega}^i \in \mathbb{R}^{SL}_+$ where these characteristics satisfy Assumption 2.4, i.e. utility functions are infinitely often continuously differentiable, strictly monotonic, strictly quasi-concave, and indifference curves do not intersect the boundary of $\mathbb{R}^{SL}_+$, and endowments are strictly positive. What makes this economy a sunspot economy with only extrinsic uncertainty, are the following assumptions.

Assumption 6.1 (Sunspot Assumptions)

1. *Utility functions* $\hat{U}^i : \mathbb{R}^{SL}_+ \to \mathbb{R}, i = 1, \dots, I$, *are of the expected utility type with respect to* $\bar{\rho}$, *i.e. there are some strictly concave* $U^i : \mathbb{R}^L_+ \to \mathbb{R}, i = 1, \dots, I$, *satisfying Assumption 2.4 with*

$$\hat{U}^i(x) = \sum_{s=1}^{S} \bar{\rho}_s U^i(x_s).$$

2. *For all* $i \in I$ *there is some* $\omega^i \in \mathbb{R}^L_{++}$ *such that* $\hat{\omega}^{iT} = (\omega^{iT}, \dots, \omega^{iT})$.

In other words: endowments are independent of state realization and utility is the same for each spot. Allowing not only the existence of extrinsic, but also of intrinsic uncertainty, one would introduce a further level of distinction of commodities between the level of extrinsic uncertainty and the level of physical properties of the commodities.

As before, individual consumption plans are denoted by $x^i \in \mathbb{R}^{SL}_+$. The notational convention is $x^{iT} := (x_1^{iT}, \dots, x_S^{iT})$ with $x^i_s \in \mathbb{R}^L_+$ for all $s \in S$ and $x_s^{iT} := (x^i_{s1}, \dots, x^i_{sL})$ where $x^i_{sl} \in \mathbb{R}_+$ for $(s, l) \in S \times L$. The same indexation is used for $\hat{\omega}^i$.

The Arrow-Debreu economy $GE = \{(U^i, \omega^i)_{i \in I}\}$ underlying this sunspot economy is called the corresponding *certainty economy*. Observe that the model chosen here assumes complete markets for the underlying economy.

In addition to future spot markets for commodities, today there open markets for some assets $j \in J := \{1, \dots, J\}$. These assets pay off tomorrow

[2] For $L = 1$ one obtains a special form of the Finance GEI-model given in Chapter 2.

where their payoff is described by $V(p) \in \mathbb{R}^{S\times J}$.[3] What concerns the asset matrix, the literature on sunspots can be distinguished along the following lines:

- There is some matrix $A \in \mathbb{R}^{L\times J}$ such that $V_{s\cdot}(p) = p_s \cdot A$ for all $s \in S$ and $p \in \mathbb{R}^{SL}_{++}$, i.e. there exists some sunspot-independent real asset structure.
- There is some matrix $\tilde{A} \in \mathbb{R}^{SL\times J}$ such that $V_{s\cdot}(p) = p_s \cdot \tilde{A}_s$ for all $s \in S$ and $p \in \mathbb{R}^{SL}_{++}$ where $\tilde{A}_s := (\tilde{A}_{(s-1)L+l,j})_{l\in L, j\in J}$, i.e. there exists some real asset structure which might be sunspot-dependent.
- $V(p)$ models nominal assets, i.e. there is some $\bar{V} \in \mathbb{R}^{S\times J}$ such that $V(p) = \bar{V}$ for all $p \in \mathbb{R}^{SL}_{++}$.

For simplicity put the following assumption on the asset structure of which the importance becomes clear by the considerations in Part II.

Assumption 6.2 (Sunspot Asset Structure)
For all $p \in \mathbb{R}^{SL}_{++}$ the matrix $V(p)$ satisfies $V(p)\bar{\theta} > 0$ for some $\bar{\theta} \in \mathbb{R}^J$.

In correspondence with the Finance GEI-model asset demand of individual i is denoted by $\theta^i \in \mathbb{R}^J$ with $\theta^i := (\theta^i_1, \ldots, \theta^i_J)^T$. Asset prices are called $q := (q_1, \ldots, q_J)^T$.

To summarize, a sunspot economy is defined as

$$SE := \{(\hat{U}^i, \hat{\omega}^i)_{i\in I}, V\}$$

where individual characteristics satisfy Assumption 6.1 and the asset structure Assumption 6.2.

Given spot prices $p_s \in \mathbb{R}^L_{++}, s = 1, \ldots, S$,[4] and asset price system $q \in \mathbb{R}^J$, each consumer i is confronted with the decision problem

$$(6.1) \quad \max_{x\in\mathbb{R}^{SL}_{+}, \theta\in\mathbb{R}^J} \quad \textstyle\sum_{s=1}^S \bar{\rho}_s U^i(x_s) \quad s.t. \ q\cdot\theta = 0$$
$$\wedge \ p_s \cdot x_s = p_s \cdot \omega^i + V_{s\cdot}(p)\theta, s = 1, \ldots, S$$

where equalities in the constraints follow from the assumptions on utility functions, endowments, and the asset structure.

With respect to this maximization problem different types of equilibria are defined in a sunspot economy.

[3] For simplicity I sometimes write V.
[4] As before $p^T := (p_1^T, \ldots, p_S^T) \in \mathbb{R}^{SL}_{++}$.

Definition 6.3 (Equilibria)
Take a sunspot economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i\in I}, V\}$ which satisfies Assumptions 6.1 and 6.2.

1. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a competitive equilibrium if*
   (a) *for all $i \in I$ (x^{*i}, θ^{*i}) solves (6.1) with respect to (p^*, q^*);*
   (b) $\sum_{i=1}^I x_s^{*i} = \sum_{i=1}^I \omega^i$ *for all $s \in S$;*
   (c) $\sum_{i=1}^I \theta^{*i} = 0$.
2. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a sunspot equilibrium if*
   (a) *it is a competitive equilibrium and*
   (b) *there is some $i \in I$ such that $x_{s'}^i \neq x_{s''}^i$ for some $(s', s'') \in S \times S$.*
3. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a non-sunspot equilibrium if*
   (a) *it is a competitive equilibrium and*
   (b) *for all $i \in I$ and all $(s', s'') \in S \times S$ it is $x_{s'}^i = x_{s''}^i$.*
4. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a trivial equilibrium if*
   (a) *it is a competitive equilibrium and*
   (b) *for all $s \in S$ (x_s^*, p_s^*) is Arrow-Debreu equilibrium for the certainty economy $GE = \{(U^i, \omega^i)_{i\in I}\}$.*
5. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a non-trivial equilibrium if*
   (a) *it is a competitive equilibrium and*
   (b) *there is some $s \in S$ such that (x_s^*, p_s^*) is not Arrow-Debreu equilibrium for the certainty economy $GE = \{(U^i, \omega^i)_{i\in I}\}$.*
6. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a randomization equilibrium if*
   (a) *it is a competitive equilibrium and*
   (b) *there is some Arrow-Debreu economy such that for all $s \in S$ (x_s^*, p_s^*) is Arrow-Debreu equilibrium for this economy.*

7. $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}_+^{SL\times I} \times \mathbb{R}^{J\times I} \times \mathbb{R}_{++}^{SL} \times \mathbb{R}^J$ *is a non-randomization equilibrium if*

   (a) *it is a competitive equilibrium and*

   (b) *there is no Arrow-Debreu economy such that for all* $s \in S$ (x_s^*, p_s^*) *is Arrow-Debreu equilibrium for this economy.*

Definition 6.3.2 is the standard definition for sunspot equilibria (compare, for example, Cass and Shell (1983)). Excluding first period consumption, the assumptions on consumers' characteristics and the asset structure assure that this is equivalent to defining a sunspot equilibrium as a competitive equilibrium with sunspot-contingent commodity prices or ex-post utilities.

But Barnett and Fisher (2000) consider a sunspot economy where consumers are only weakly risk averse. In this framework one easily constructs an example where sunspots matter in the sense of Definition 6.3.2, i.e. equilibrium allocations are sunspot-dependent, but generate the same ex ante-utility as some sunspot-independent equilibrium allocation and are ex ante-Pareto-efficient. This trivial example motivates an alternative definition of sunspot equilibria as a competitive equilibrium which is not ex ante-Pareto-efficient.[5] In the standard framework this definition is equivalent to Definition 6.3.2, but one should have in mind the example given by Guesnerie and Laffont (1990): allowing utility functions to be non-concave, there exists a sunspot economy with a unique competitive equilibrium which displays sunspot-dependent allocations and Pareto-dominates all sunspot-independent allocations. Therefore, according to the standard definition sunspots would matter, but the new definition would imply sunspots not to matter, i.e. without risk-aversion the new definition would not be appropriate.

The existence of a competitive equilibrium for a given sunspot economy is closely related to the existence of competitive equilibria in a general equilibrium model with incomplete markets. For a survey on this see Hens (1998, Section 5). However, the main point of this part of my thesis is the existence of sunspot equilibria. This is in many respects related to multiplicity of Arrow-Debreu equilibria via the equilibrium concepts 4. and 6. However, it will become clear that for the existence of sunspot equilibria the approach via multiple Arrow-Debreu equilibria is not always the best one.

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[5]If there is first period consumption (compare the end of this section) this definition has to be adapted in the sense that a sunspot equilibrium is a competitive equilibrium where no (ex ante-) reallocation of *future* consumption plans exists which improves at least one consumer and worsens no-one.

Some part of the sunspot literature refers to the case of first period consumption. Therefore, commodity and consumption spaces are of dimension $(S+1)L$. Spot $s=0$ refers to today. Assumption 6.1 is adapted in the following way.

Assumption 6.4 (Sunspot Ass. with First Period Consumption)

1. *Utility functions* $\hat{U}^i : \mathbb{R}_+^{(S+1)L} \to \mathbb{R}, i=1,\dots,I,$ *are given such that there are some* $\bar{\rho} \in \mathbb{R}_{++}^S$ *with* $\sum_{s\in S} \bar{\rho}_s = 1$ *and some strictly concave* $U^i : \mathbb{R}_+^{2L} \to \mathbb{R}, i=1,\dots,I,$ *which satisfy Assumption 2.4, with*

$$\hat{U}^i(x) = \sum_{s=1}^{S} \bar{\rho}_s U^i(x_0, x_s),$$

i.e. utilities are no longer additively separable with respect to consumption today. One has a kind of adjusted expected utilities.

2. *For all* $i \in I$ *there are some* $\omega_0^i \in \mathbb{R}_{++}^L$ *and some* $\omega^i \in \mathbb{R}_{++}^L$ *such that* $\hat{\omega}^{iT} = (\omega_0^{iT}, \omega^{iT}, \dots, \omega^{iT})$, *i.e. endowments today may differ from those of tomorrow.*

Under first period consumption the certainty economy is defined as

$$GE = \left\{ \left(U^i, (\omega_0^i, \omega^i)\right)_{i\in I} \right\}$$

where the underlying consumption space is $2L$-dimensional. The adaptation of the equilibrium definitions (Definition 6.3) is straightforward.

As soon as one introduces first period consumption into the sunspot economy, generically in (certainty) endowments $(\omega_0^i, \omega^i)_{i\in I}$ it is not possible to obtain a Pareto-efficient allocation in the sunspot economy without trading assets. This kind of inefficiency induces consumers to use those assets which are designed for insurance against extrinsic uncertainty about tomorrow for income transfer between today and tomorrow and therefore reduce inefficiency. The asset trade itself creates sunspot-dependence of the allocations (which is obvious when assets' payoffs are sunspot-dependent).

It will become clear later on, especially from comparison of the results given by Gottardi and Kajii (1999) with my own, that the introduction of first period consumption might influence the existence of sunspot equilibria. The basic idea that first period consumption generically creates some activity of (dis)saving was developed by Cass (1989), (1992).

This observation motivates the following definition.

Definition 6.5 (Efficient Spot Markets)
Distinguish whether the sunspot economy models first period consumption or not.

1. *A sunspot economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}, V\}$ which satisfies Assumptions 6.1 has efficient spot markets if there is some Pareto-efficient allocation $(\hat{x}^i)_{i \in I} \in \mathbb{R}_+^{SL \times I}$ which is attainable by trading only on spot markets and not on asset markets, i.e. there is some $\hat{p} \in \mathbb{R}_{++}^{SL}$ such that*

   (a) *for all $i \in I$ it is*

   $$\hat{x}^i = arg \max_{x \in \mathbb{R}_+^{SL}} \hat{U}^i(x) \ s.t. \ \hat{p}_s \cdot x_s = \hat{p}_s \cdot \omega^i \ \forall \ s \in S,$$

   (b)
   $$\sum_{i=1}^{I} \hat{x}^i = \sum_{i=1}^{I} \omega^i.$$

2. *A sunspot economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}, V\}$ which satisfies Assumptions 6.4 has efficient spot markets if there is some Pareto-efficient allocation $(\hat{x}^i)_{i \in I} \in \mathbb{R}_+^{(S+1)L \times I}$ which is attainable by trading only on spot markets and not on asset markets, i.e. there is some $\hat{p} \in \mathbb{R}_{++}^{(S+1)L}$ such that*

   (a) *for all $i \in I$ it is*

   $$\hat{x}^i = arg \max_{x \in \mathbb{R}_+^{(S+1)L}} \hat{U}^i(x) \ s.t. \ \hat{p}_0 \cdot x_0 = \hat{p}_0 \cdot \omega_0^i \wedge \hat{p}_s \cdot x_s = \hat{p}_s \cdot \omega^i \ \forall \ s \in S,$$

   (b)
   $$\sum_{i=1}^{I} \hat{x}^i = \sum_{i=1}^{I} \omega^i.$$

Otherwise the economies have inefficient spot markets.

The existence of sunspot equilibria in a sunspot economy with incomplete markets is a much stronger result when spot markets are efficient than when they are not. This is because in the first case extrinsic uncertainty can be identified as the only reason for asset trading and therefore for sunspots to matter whereas in the second case assets are traded for income transfer (i.e. because of inefficient spot markets) and insurance against extrinsic uncertainty such that it is not clear which is the main driving force.

Under efficient spot markets the existence of sunspot equilibria is much more problematic from a social point of view since theoretically society could end up with a Pareto-efficient allocation on its own, but people's expectations create Pareto-inefficiency.

6.2 Multiplicity of Arrow-Debreu Equilibria and the Existence of Sunspots — Related Literature

The existence of sunspot equilibria is a widely examined question in the literature. Different starting-points have to be distinguished: on the one hand there are the two extreme cases with no asset respectively full insurance against intrinsic and extrinsic uncertainty. On the other hand economists consider the more interesting intermediate case of incomplete markets where some asset trade is possible. When assets are nominal, the existence of sunspot equilibria is closely related to the indeterminacy of equilibria in standard incomplete markets models; when assets are real, economists want to clarify the relation to multiplicity of Arrow-Debreu equilibria.

6.2.1 Two Extreme Cases

Let me first start with those sunspot economies where there are no assets which allow to insure against intrinsic and extrinsic uncertainty. This assumption is quite unrealistic, but nevertheless should be mentioned. Whenever $J = 0$, a sunspot economy falls into Arrow-Debreu economies and the equilibria of the sunspot economy consist of a combination of the corresponding Arrow-Debreu equilibria. Therefore, within the framework considered in this part, that is when only extrinsic uncertainty is considered, this gives the following result.

Proposition 6.6 (No Assets and the Existence of Sunspot Equil.) *Given a sunspot economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}\}$ satisfying Assumption 6.1, the competitive equilibria can all be decomposed into Arrow-Debreu equilibria of the underlying certainty economy $GE = \{(U^i, \omega^i)_{i \in I}\}$ such that they are all trivial randomization equilibria. Sunspot equilibria therefore exist if and only if the certainty economy GE is characterized by multiple Arrow-Debreu equilibria.*

This proposition clarifies a general problem arising under sequential market economies: GE- and GEI-models assume all consumers to have common future price expectations. However, as soon as there exist multiple equilibria, a priori this is no longer justified. According to Hens (1998, p. 192), this can be solved by using "extrinsic, publicly observed states [...] as a coordination device of agents' expectations".

Now take the other polar case: assume that markets are complete, i.e. complete insurance against intrinsic as well as extrinsic uncertainty is possible. This property assures competitive equilibria to be Pareto-efficient. As is known from Cass and Shell (1983) with the underlying objective probability measure assumed in sunspot economies, these equilibrium allocations have to be sunspot-independent. Otherwise — because of strict concavity of utility functions — a Pareto-improvement would be possible by giving each consumer in each state the same probability weighted mixture of the different sunspot-dependent consumption plans. The same idea is underlying Proposition 3.13.

Proposition 6.7 (Cass and Shell (1983))
Given an economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}, V\}$ satisfying Assumption 6.1 such that $< V(p) >= \mathbb{R}^S$ for all equilibrium price systems $p \in R^{SL}_{++}$, the economy does not display any sunspot equilibrium.

Cass and Shell (1983, Proposition 3) only consider the case of two sunspot states. This can be easily extended to the more general case of an arbitrary finite number of sunspot states.

The importance of an objective probability measure is stressed by Cass and Shell (1983, p. 208) and Shell (1987, p. 549).

A large part of Cass and Shell's (1983) results considers a complementary situation where markets are also complete, but market participation of some consumers is restricted. Under such assumptions it is relatively simple to prove the existence of sunspot equilibria, for example if all consumers are excluded from asset trade (compare the first polar case) respectively — to obtain an asymmetric participation structure — all except for a single one are excluded from asset trade (compare Cass and Shell (1983, Observations 1 and 2)).

Cass and Shell (1983) also demonstrate for the case of complete markets that under restricted participation multiplicity of equilibria in the underlying certainty economy is not necessary for the existence of sunspot equilibria, i.e. sunspot equilibria do not necessarily have to be trivial randomization equilibria. The basic idea of their example is to exclude one single consumer from participation in asset markets. Starting from non-proportional spot price systems they construct equilibrium allocations and endowments which allow the existence of an objective probability measure and of additively separable utility functions which satisfy a condition on absolute risk aversion similar to (3.21). This condition assures a unique Arrow-Debreu equilibrium in the certainty economy.

6.2.2 Sunspot Equilibria under First Period Consumption

When considering incomplete markets models, one clearly is not interested in these polar cases, but in some intermediate ones which allow to clarify the following point: with symmetric market participation for all consumers not only multiplicity of Arrow-Debreu equilibria in the certainty economy is a driving force for sunspot equilibria, but also the incompleteness of markets. Two frameworks have to be distinguished: either there is first period consumption in the economy or it is excluded. As explained before, to clarify the effects of incomplete insurance against uncertainty on the existence of sunspot equilibria, the second approach is more appropriate than the first one, even though research started with including today's consumption. The first contributions assumed nominal assets and then later on also real assets were examined.

Nominal Assets

Let me start with the consideration of nominal assets: the earliest result was given by Cass' (1989) "leading example". He assumes a very simple framework including first period consumption. Cass (1989) excludes intrinsic uncertainty and restricts extrinsic uncertainty to two sunspot states.[6] Only two consumers are alive who can trade a single commodity and a single asset. The asset is a nominal bond with positive payoff (also called inside money), i.e. $V(p) = \mathbb{1}$[7]. The relation between income from asset payoff and from commodity trading in sunspot states strongly depends on the respective level of spot prices. Therefore, different levels of spot prices generate different allocations in the different sunspot states and hence sunspot equilibria. Cass' (1989) takes an inverse point of view: he fixes the current level of prices and varies the asset payoff in the different sunspot states. A necessary requirement for Cass' (1989) proof is an incentive for saving/dissaving by consumers, i.e.

$$\frac{\partial_{x_0} U^1(\omega^1)}{\partial_{x_1} U^1(\omega^1)} \neq \frac{\partial_{x_0} U^2(\omega^2)}{\partial_{x_1} U^2(\omega^2)}. \tag{6.2}$$

[6] As soon as this allows the existence of sunspot equilibria, this can be easily extended to an arbitrary finite number of sunspots.

[7] As beforehand, the vector $\mathbb{1}$ is defined by $\mathbb{1} := (1, \dots, 1)^T$.

This condition assures that the sunspot economy has inefficient spot markets.

Proposition 6.8 (Cass (1989))
Assume an economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}, V\}$ *satisfying Assumption 6.4 with* $S = I = 2$, $L = 1$, *and* $V(p) = \mathbb{1}$ *where inequation (6.2) is fulfilled. Then, there exist some sunspot equilibria.*

With first period consumption it is sufficient to assume the existence of a single commodity because it can be exchanged with respect to time.

Before Cass (1989) proves the result in the way described above, he shows the equivalence between non-sunspot equilibria of this economy and monetary equilibria in a monetary economy which is equivalent to a two-commodity Arrow-Debreu economy.

Cass' (1989) result can be interpreted in the sense that generically with respect to endowment distributions of the certainty economy the set of equilibrium allocations contains a one parameter family of allocations.

Siconolfi (1991) extends Cass' (1989) statement for an arbitrary finite number of extrinsic states of uncertainty, of commodities, and of assets as well as of consumers. Therefore, at first he proves under the assumption of one single asset — inside money — and two extrinsic states that generically in endowments (i.e. whenever endowments are Pareto-inefficient in the certainty economy and the sunspot economy has inefficient spot markets) the asset is traded. This allows the application of Cass' (1989) results, even for more than a single commodity. The equilibrium allocations generated in this way can be embedded into Siconolfi's (1991) more general sunspot economy. The only requirements for this are a nominal asset structure $\bar{V}$ with $J < S$, $\bar{V}\bar{\theta} \gg 0$ for some $\bar{\theta} \in \mathbb{R}^J$ where $\bar{V}$ is in general position, i.e. each set of J arbitrary rows is linearly independent.

Cass (1992) himself and Suda, Tallon, and Villanacci (1992) use the same idea, namely the indeterminacy of equilibria with nominal assets. The aim of these two papers was to come closer to the results of Geanakoplos and Mas-Colell (1989) respectively Balasko and Cass (1989) who prove the indeterminacy of equilibria for incomplete markets models with nominal assets, but in comparison to Cass (1989) and Siconolfi (1991) a more severe real indeterminacy of dimension $S - 1$ respectively $S - J$. The decisive point for the sunspot economy is the fact that endowments cannot be perturbed arbitrarily because Assumptions 6.1/6.4 have to remain satisfied. Whereas Cass (1992) himself adjusts endowment perturbations to this context, Suda, Tallon, and Villanacci (1992) deviate to perturbations of the utility functions.

Consider the results from Cass (1992). In his sunspot economy with first period consumption the numbers of states for extrinsic uncertainty, commodities, and consumers are arbitrary, but finite. Cass (1992) now allows the existence of more than one nominal asset, but requires to have inside money, i.e. $\mathbb{1} \in \bar{V}$, and excludes redundant assets, i.e. *rank* $\bar{V} = J$.

The basis for Cass' (1992) result is built by two points:

- The Cass' (1984) trick is applied where one consumer is allowed to trade without any restrictions and is only implicitly restricted by the restrictions of the other consumers. This trick was already applied in Cass' (1989) "leading example".
- In analogy to the "leading example", instead of considering different spot price levels Cass (1992) fixes one level and varies the asset payoff: description of the equilibria by the extended system, i.e. the system of first order conditions and market clearing equations, gives price systems/price levels in dependence on the Lagrangian multipliers. Cass (1992) switches from this dependence to dependence of asset payoff on the Lagrangian multipliers.

Cass' (1992) proof can be divided into three steps: the first one is to identify non-sunspot equilibria with the Arrow-Debreu equilibria of the corresponding certainty economy. The second point is to show that generically in endowments of the certainty economy non-sunspot equilibria use asset markets (since the economy has inefficient spot markets) and are regular. These results allow to prove that generically in endowments of the certainty economy equilibrium allocations are indeterminate with degree $S - J$.

Proposition 6.9 (Cass (1992))
Assume some economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}, \bar{V}\}$ satisfying Assumption 6.4 with nominal asset structure $\bar{V} \in \mathbb{R}^{S \times J}$ where $J < S$, rank $\bar{V} = J$, and $\bar{V}_{\cdot 1} = \mathbb{1}$. Generically in the endowments of the certainty economy $(\omega_0^i, \omega^i)_{i \in I}$, there exists a sunspot equilibrium. The dimension of the set of sunspot equilibrium allocations is equal to $S - J$.

The dimension of indeterminacy is — as in Balasko and Cass (1989) — equal to the deficiency of markets.

Besides the result is in line with Siconolfi and Villanacci (1991) who do not consider a sunspot economy, but an economy facing the same problems: they assume an Arrow-Debreu economy without aggregate risk such that $\sum_{i \in I} \omega_s^i = \sum_{i \in I} \omega_{\bar{s}}^i$ for all $s, \bar{s} \in S$. Within this context similar problems arise as within a sunspot economy since endowments cannot be perturbed arbitrarily.

Suda, Tallon, and Villanacci (1992) also consider generic real indeterminacy of equilibrium allocations in sunspot models. However, since they work with the space of utility functions, they obtain a more severe result than

Cass (1992) and especially than Cass (1989) respectively Siconolfi (1991), namely an indeterminacy of degree $S-1$ which is independent of the degree of deficiency.

Proposition 6.10 (Suda, Tallon, and Villanacci (1992))
Assume an economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i \in I}, \bar{V}\}$ *satisfying the Sunspot Assumptions with First Period Consumption 6.4. If the nominal asset structure models only inside money, i.e.* $\bar{V} = \mathbb{1}$*, generically in utility functions there exists a sunspot equilibrium. The dimension of the set of sunspot equilibrium allocations is* $S-1$.

The idea of perturbing utility functions uses a technique developed by Kajii (1997). This idea is also used by Gottardi and Kajii (1999) when proving the generic existence of sunspot equilibria with real assets. Therefore, there are some parallels between the different cases.

Suda, Tallon, and Villanacci (1992) prove the equivalence between the competitive equilibria of their sunspot economy and the equilibria of some numeraire asset model where — in analogy to Cass (1992) — the basic idea is to work with different levels of prices for the spot markets. Since the second step is to prove generically in utility functions trading of the numeraire asset (i.e. generic inefficiency of spot markets), equilibrium allocations have to vary in the different spot markets and the competitive equilibria have to be sunspot equilibria. The proof of generic indeterminacy is done within the numeraire asset model.

Perturbing utility functions is also applied by Pietra (1992) when proving real indeterminacy of sunspot equilibrium allocations. However, like Cass (1992) he only obtains a degree $S-J$, i.e. the degree of deficiency of markets.

Real Assets

Now let me turn to the case of real assets. The existence of sunspot equilibria for given nominal asset structures in principle follows from the indeterminacy of the equilibria such that as soon as one wants to check the consequences of multiplicity, the important considerations are for real asset structures. Nevertheless, the results of Gottardi and Kajii (1999) show some parallels between the basic ideas to prove the existence of sunspot equilibria for nominal and real assets.

Gottardi and Kajii (1999) consider the dependence between multiplicity of Arrow-Debreu equilibria and the existence of sunspot equilibria in an economy with real sunspot-independent asset structures. Their main point is to prove that generically in utility functions and endowments there exists an open set of real sunspot-independent assets which generate regular sunspot equilibria. This implies multiple non-sunspot equilibria not to be necessary for the existence of sunspot equilibria. Especially the corresponding certainty economy must not have multiple equilibria for the sunspot economy to have sunspot equilibria.

Intrinsic uncertainty is excluded by Gottardi and Kajii (1999). The first commodity is taken as a numeraire.

Proposition 6.11 (Gottardi and Kajii (1999))
In the framework of sunspot economies satisfying Assumption 6.4 with $S = 2$, $J = 1$, and $L \geq 2$, for a generic choice of utility functions and endowments there exists an open set of asset vectors $A \in \mathbb{R}^L$ such that the generated sunspot economy with $V_{s1}(p) = p_s \cdot A$ displays regular sunspot equilibria.[8]

One should be conscious that in the same way as in Cass (1992) first period consumption is the driving force for the existence of sunspot equilibria: the proof is built on certainty economies for which equilibria include income transfer between today and tomorrow such that spot markets are inefficient.

The restriction $S = 2$ can be taken without loss of generality since the sunspot equilibria obviously can be extended to sunspot equilibria of an economy with more sunspot states.

Considering the space of utility functions, endowments, *and* asset structures the result breaks down: sunspot equilibria no longer have to exist generically in this space. This point is clarified with the help of Pareto-efficient endowments.

To illuminate the non-necessity of multiple Arrow-Debreu equilibria in the certainty economy, let me formulate the content of the proposition in another way: their are no robust conditions on the certainty economy which exclude sunspot equilibria. However, since certainty economies with a unique equilibrium constitute an open set in the set of all certainty economies, these economies have to exhibit sunspot equilibria generically for suitable asset choice. Therefore, multiplicity of certainty equilibria is not necessary for the existence of sunspot equilibria. This reasoning has to be put in its true light: one could not have expected the multiplicity of Arrow-Debreu equilibria in the certainty economy to be the decisive point for the existence of sunspot equilibria under the assumption of first period consumption: the sunspot economy has inefficient spot markets such that without asset trade the Arrow-Debreu equilibria (or to be more precise: their extensions to non-sunspot equilibria) are not obtainable and sunspot equilibria cannot be trivial randomization equilibria. In my opinion the main point of Gottardi and Kajii (1999) is the fact that they transfer the argument made by Cass (1989), (1992) from nominal to real asset structures.

[8]Regularity refers to the maximal dimension of the Jacobian of excess demand.

However, it is no question that those Arrow-Debreu economies with the same utility as in the certainty economy, but endowments equal to the sum of original endowment and returns from asset holding must display multiple equilibria. To say it with Gottardi and Kajii (1999, p. 715) "that is, there have to be multiple future spot equilibria *after* some reallocation of resources and the sunspot equilibrium is 'randomizing' over them". They call this property "potential multiplicity".

The idea of the proof for Proposition 6.11 is to work with some auxiliary economy which is equivalent to the original one except for asset payoff. The asset now pays off in the numeraire commodity in dependence on the sunspot state. As long as this asset is traded[9], income and equilibrium allocations depend on sunspot states. This is the same for price systems if some special kinds of utility functions are excluded. Now, it is possible to define some commodity bundle A such that $p_s \cdot A, s = 1, 2$, generates the asset payoff of the auxiliary economy. Since the equilibria of the auxiliary economy and the competitive equilibria of the sunspot economy correspond to each other, generating sunspot-dependent equilibria in the auxiliary economy leads to sunspot equilibria in the sunspot economy. The last step is to prove regularity of the sunspot equilibria.

The proof is built on perturbations of utility functions developed by Kajii (1997).

First period consumption has to be interpreted similarly to intrinsic uncertainty since both aspects can be seen as reason for trading assets which are designed for insurance against extrinsic uncertainty and therefore as one driving force for the mattering of sunspots.

The most important contribution concerning intrinsic uncertainty is done by Guesnerie and Laffont (1990) who consider a relatively simple underlying economy with only two consumers, two commodities, and two states of intrinsic uncertainty. Both consumers are characterized by expected utility with an objective probability measure. Since Guesnerie and Laffont (1990) only allow that a single asset exists which models a kind of future contract on the first commodity (uncontingent delivery of commodity one), each kind of asset trade is prevented and the economy falls into two separate Arrow-Debreu economies. Guesnerie and Laffont (1990) assume at least one of these economies to have multiple Arrow-Debreu equilibria and therefore the incomplete markets economy to have multiple equilibria. Until now, there is only intrinsic uncertainty.

Now they extend the economy by two states of extrinsic uncertainty and assure the possibility of complete insurance against extrinsic uncertainty by

[9] This point was also decisive when considering nominal assets.

allowing to trade the asset contingent on extrinsic states. The beforehand constructed incomplete markets equilibria can be easily extended to competitive non-sunspot equilibria. Guesnerie and Laffont (1990) prove all non-sunspot equilibria — assume there are N where one knows $N > 1$ — to have a positive index and therefore together with the Poincaré-Hopf Theorem and the Index Theorem the existence of further $N - 1$ sunspot equilibria follows.

6.2.3 Sunspot Equilibria without Consumption in the First Period

Without first period consumption and intrinsic uncertainty it is much more difficult to construct non-trivial sunspot equilibria, at least when assets pay off in commodities. As soon as the certainty economy displays multiple Arrow-Debreu equilibria, these can be easily extended to trivial randomization sunspot equilibria.

Let me start with sunspot-contingent assets since for sunspot-independent asset structures simple argumentation makes clear that as soon as the certainty economy exhibits a unique equilibrium and this remains true for each redistribution of endowments, the existence of sunspot equilibria is prevented. The question is whether under the assumption of unique spot market equilibria the existence of sunspot equilibria can be proved when sunspot-contingent asset structures are allowed. As Mas-Colell (1992) and Hens and Pilgrim (2001) point out, this would demonstrate that not all sunspot equilibria can be interpreted as coordinating expectations about spot prices.

Hens and Pilgrim (2001) prove under admittance of sunspot-dependence the following result:

Proposition 6.12 (Hens and Pilgrim (2001))
Let $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i\in I}, V\}$ be a sunspot economy satisfying Assumptions 6.1 and 6.2. If there are $I = 2$ consumers being characterized by additively separable utility functions (or some strictly monotonic, concave transformations of them) and a real (sunspot-dependent or not) asset structure such that for all endowment distributions attainable through asset trading spot market equilibria are unique, the economy does not have any sunspot equilibrium.

The idea of the proof is to establish relations between the existence of sunspot equilibria and the occurrence of (different versions of) the transfer paradox by the examination of "order crossing" of the Lagrangian multipliers for the different spot market equilibria. The transfer paradox itself is already related to the multiplicity of equilibria (compare, for example, Trannoy (1986)).

While Hens and Pilgrim (2001) give a framework with sunspot-contingent asset structures where multiplicity of spot market equilibria for some attainable endowment distributions is a necessary condition for the existence of sunspot equilibria, I prove it to be not sufficient in a framework with sunspot-independent asset structures.[10]

However, Mas-Colell (1992) demonstrates for a sunspot economy without first period consumption that under the assumption of "potential multiplicity" an asset structure and an endowment distribution can be found such that some sunspot equilibrium exists randomizing over those multiple equilibria.

Proposition 6.13 (Mas-Colell (1992))
Given an Arrow-Debreu economy $GE = \{(U^i, \omega^i)_{i \in I}\}$ where $I = 2$ with at least three (and therefore multiple) equilibria where the corresponding price systems are linearly independent, under the assumption $S > 2$ there exists some redistribution of endowments and two positive real assets $A_{\cdot 1}, A_{\cdot 2} \in \mathbb{R}^L_+$ such that the sunspot economy characterized by these "new" endowments, utility functions $(U^i)_{i \in I}$, and the asset structure V defined by $V_{s\cdot}(p) = p_s \cdot (A_{\cdot 1}, A_{\cdot 2})$ has a non-trivial sunspot equilibrium.

With the requirement of three linearly independent price systems this proposition implicitly assumes $L > 2$. I consider the case $L = 2$ in the next chapter where I prove that it behaves differently.

Without loss of generality Mas-Colell (1992) assumes that there are three extrinsic states of uncertainty $S = 3$ which arise with equal probability $\bar{\rho}_s = \frac{1}{3}, s = 1, 2, 3$. Since there are two consumers and three equilibria, six Lagrangian multipliers $\lambda^i_s, i = 1, 2, s = 1, 2, 3$[11], are defined for the budget constraints in the Arrow-Debreu economy. Taking a vector $a \in \mathbb{R}^3$ such that

$$\sum_{s=1}^{3} \lambda^i_s a_s = 0, i = 1, 2, \tag{6.3}$$

because of linear independence of the equilibrium price systems there exists some $r \in \mathbb{R}^L \backslash \{0\}$ with

$$p_s \cdot r = a_s, s = 1, 2, 3. \tag{6.4}$$

This vector r can be written as difference of two positive vectors $A_{\cdot 1}, A_{\cdot 2} \in \mathbb{R}^L_+$. Now take $\hat{\omega}^1, \hat{\omega}^2$ with $\hat{\omega}^1_s = \omega^1 + r, \hat{\omega}^2_s = \omega^2 - r, s = 1, 2, 3.$[12] For the

[10] The reader should be conscious that it is never possible to obtain multiplicity for all endowment distributions. For example, take Pareto-efficient allocations which always exist (compare Proposition 3.12).

[11] Notice that at first s is used for numbering the equilibria.

[12] The vectors a and therefore r can be chosen sufficiently small in their entries to guarantee these endowment vectors to be positive.

equilibrium prices $p^{*T} = (p_1^T, p_2^T, p_3^T)$ and $q^{*T} = (1,1)$, the generated sunspot economy with real sunspot-independent asset structure defined by $(A_{\cdot 1}, A_{\cdot 2})$ evokes first order conditions mainly equivalent to the collection of the Arrow-Debreu first order conditions. Choose $\theta_1^1 = -\theta_2^1 = -\theta_1^2 = \theta_2^2 = -1$. The only difference is given by an additional equation which is by (6.4) equivalent to (6.3).

Roughly spoken, a sunspot economy is constructed such that the endowment distribution induced by asset trading is the one of the starting Arrow-Debreu economy.

Mas-Colell's (1992) result has to be interpreted as follows: if consumers take care of sunspot events and insurance against these events is restricted, then this behavior is self-enforcing and leads to a from a social point of view unwelcome situation which could be improved by some central authority. In this framework people's expectations in combination with incomplete insurance can be identified as the only reason for the non-desirable outcome.

Chapter 7

On Mas-Colell's (1992) First Observation on Sunspots

Gottardi and Kajii (1999, p. 723) state that there is little relation between the existence of sunspot equilibria and the multiplicity of non-sunspot equilibria, or the multiplicity of spot market equilibria. However, as explained in the last section, Gottardi and Kajii's (1999) results on the (in)dependency of the existence of sunspot equilibria on the multiplicity of Arrow-Debreu equilibria have to be assessed in the knowledge that their framework includes first period consumption. Nonetheless, they explain "potential multiplicity" for some distribution of initial endowments to be a necessary condition for the existence of sunspot equilibria and prove this to be generically satisfied within their framework. They realize that "no other condition [than the lack of potential multiplicity] on the agents' preferences and endowments can be found which ensures the economy [to be] immune from sunspots."

This is my starting-point: I give conditions built up in Mas-Colell's (1992) sunspot framework which allow "potential multiplicity", but leave the economy immune from sunspots.

Mas-Colell (1992, p. 469) himself states that he does not know what happens with his result in a two-commodity framework: "We do not know if there is an example with $l = 2$." I take his model and prove that his result breaks down with only two goods and additively separable utility functions displaying non-decreasing relative risk aversion, even if one allows the existence of more than two consumers. I think it is very interesting to observe this point because on the one hand it contradicts intuition and on the other hand my assumptions include famous examples of multiple Arrow-Debreu equilibria (for example, Kehoe (1991)) which one would otherwise expect to be useful for constructing explicit examples of non-trivial sunspot equilibria. It is well-known that in many economic models the two-commodity case behaves in another way than the "more than two"-commodity case. Mas-Colell's (1992) positive existence results are built up in a two-person framework. My non-existence result however is valid for an arbitrary number of consumers. I stress this fact because there are examples of two-commodity economies where the two-person case evokes special effects (cf. the transfer paradox in Balasko (1978) and Polemarchakis (1983)). If I assumed that there are only two consumers, it would be sufficient to restrict utility by additive separability and nothing else.

At first glance my result seems to contradict the one of Gottardi and Kajii (1999). However, one observes

- that additive separability is no robust property in the set of utility functions and
- that — following the idea developed by Cass (1989), (1992) — Gottardi and Kajii (1999) consider sunspot economies including first period consumption such that there is some incentive for saving/dissaving which might be stronger than the negative consequences of the inefficiency caused by sunspot-dependent allocations.

I exactly consider the framework of Mas-Colell (1992), except for the restriction to $L = 2$ commodities and allowing the existence of an arbitrary finite number of consumers. This means I assume an economy with no intrinsic uncertainty where restricted insurance against extrinsic uncertainty is possible with the help of a real sunspot-independent asset structure.

Proposition 7.1 *Assume a sunspot economy $SE = \{(\hat{U}^i, \hat{\omega}^i)_{i\in I}, V\}$ satisfying Assumption 6.1 with exactly $L = 2$ commodities and a real sunspot-independent asset structure, i.e. there is some $A \in \mathbb{R}^{2\times J}$ with $A\bar{\theta} > 0$ for some $\bar{\theta} \in \mathbb{R}^J$ such that $V_{s\cdot}(p) = p_s \cdot A$. If the state utility functions $(U^i)_{i\in I}$ are additively separable and display non-decreasing relative risk aversion, then there is no non-trivial sunspot equilibrium.*

Proof: Let $\omega \in \mathbb{R}^2_{++}$ be some fixed total endowment and

$$U^i(x) \quad = \quad u^i_1(x_1) + u^i_2(x_2), i = 1, \ldots, I,$$

some utility functions where u^i_l exhibits non-decreasing relative risk aversion. Assume there is an endowment distribution $\hat{\omega}^1, \ldots, \hat{\omega}^I \in \mathbb{R}^{S2}_{++}$ satisfying the sunspot assumptions with $\sum_{i=1}^I \omega^i = \omega$ and an asset structure $A \in \mathbb{R}^{2\times J}$ with $A\bar{\theta} > 0$ for some $\bar{\theta} \in \mathbb{R}^J$ which admit a non-trivial sunspot equilibrium $(x^*, \theta^*, p^*, q^*)$. Then, the Arrow-Debreu economy $(U^i, \tilde{\omega}^i)_{i=1,\ldots,I}$ with

$$\tilde{\omega}^i = \omega^i + A\theta^{*i}, i = 1, \ldots, I,$$

has the following S equilibria. Without loss of generality allocations in each state differ; otherwise merge the different states:

$$\begin{gathered}(x_1^{*1}, \ldots, x_1^{*I}, p_1^*);\\ \ldots;\\ (x_S^{*1}, \ldots, x_S^{*I}, p_S^*).\end{gathered}$$

The sunspot economy has to include at least 3 sunspot states. If $S = 2$, to avoid the well-known case of extrinsic completeness (compare Proposition 6.7), the asset structure would have to be restricted to a single asset (or collinear assets) which — under the assumption of no first period consumption — excludes any asset trade and therefore only allows the existence of trivial sunspot equilibria.

Normalize $p_{s2}^* = 1$ for all $s \in S$. The Lagrangian multipliers which correspond to the equilibria in the Arrow-Debreu economy are called $\lambda_s^i, s \in S, i \in I$.

Non-triviality implies $A\theta^{*i} \neq 0$ for some $i \in \{1, \ldots, I\}$.

Normalize $q_1^* = 1$.[1] Then, $q^* \cdot \theta^{*i} = 0$ implies $\theta_1^{*i} = -q_{-1}^* \theta_{-1}^{*i}$ and therefore $A\theta^{*i} = (A_{\cdot(-1)} - \hat{A}(q^*))\theta_{-1}^{*i}$ where $A_{\cdot(-1)} = (A_{\cdot 2}, \ldots, A_{\cdot J})$ and $\hat{A}(q^*) = (q_2^* A_{\cdot 1}, \ldots, q_J^* A_{\cdot 1})$. Because of non-triviality of the sunspot equilibrium there has to be some $\bar{j} \in \{2, \ldots, J\}$ such that $r := A_{\cdot\bar{j}} - q_{\bar{j}}^* A_{\cdot 1} \neq 0$.

The first order conditions for the Arrow-Debreu equilibria are of the following form: for all $s \in S$ it is

$$\begin{aligned} {u_1^i}'(x_{s1}^{*i}) - \lambda_s^i p_{s1}^* &= 0,\\ {u_2^i}'(x_{s2}^{*i}) - \lambda_s^i &= 0,\\ p_s^* \cdot (x_s^{*i} - \tilde{\omega}^i) &= 0. \end{aligned}$$

Without loss of generality it is $p_1^* < \ldots < p_S^*$. Assume, there are two consumers $\bar{h}, \hat{h} \in \{1, \ldots, I\}$ such that

$$x_{12}^{*\bar{h}} < \ldots < x_{S2}^{*\bar{h}}, \tag{7.1a}$$

$$x_{12}^{*\hat{h}} > \ldots > x_{S2}^{*\hat{h}}, \tag{7.1b}$$

which because of strict concavity of U^i implies

$$\lambda_1^{\bar{h}} > \ldots > \lambda_S^{\bar{h}}, \tag{7.2a}$$

$$\lambda_1^{\hat{h}} < \ldots < \lambda_S^{\hat{h}}. \tag{7.2b}$$

[1] This is possible without loss of generality because in the equilibrium one of the asset prices has to be unequal to zero. If necessary, renumerate the assets and switch from A to $-A$.

The first order conditions for consumer i's maximization problem in the sunspot economy are given by

$$
\begin{aligned}
&(7.3a) & \bar{\rho}_s u_1^{i\prime}(x_{s1}^{*i}) - \hat{\lambda}_s^i p_{s1}^* &= 0, s = 1, \ldots, S;\\
&(7.3b) & \bar{\rho}_s u_2^{i\prime}(x_{s2}^{*i}) - \hat{\lambda}_s^i &= 0, s = 1, \ldots, S;\\
&(7.3c) & p_s^* \cdot (x_s^{*i} - \omega^i - (A_{\cdot(-1)} - \hat{A}(q^*))\theta_{-1}^{*i}) &= 0, s = 1, \ldots, S;\\
&(7.3d) & \sum_{s=1}^{S} \hat{\lambda}_s^i p_s^* \cdot (A_{\cdot j} - \hat{A}(q^*)_{\cdot j}) &= 0, j = 2, \ldots, J.
\end{aligned}
$$

Obviously $\hat{\lambda}_s^i = \bar{\rho}_s \lambda_s^i$ for all $s \in S$ and $i = 1, \ldots, I$. Then, equation (7.3d) for $i = \bar{h}, \hat{h}$ and $\bar{j}$ can be rewritten as

$$
(7.4) \quad \begin{pmatrix} \lambda_1^{\bar{h}} & \ldots & \lambda_S^{\bar{h}} \\ \lambda_1^{\hat{h}} & \ldots & \lambda_S^{\hat{h}} \end{pmatrix} \begin{pmatrix} \bar{\rho}_1 & & \\ & \ddots & \\ & & \bar{\rho}_S \end{pmatrix} \begin{pmatrix} p_{11}^* & 1 \\ \vdots & \\ p_{S1}^* & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0
$$

where $r := A_{\cdot\bar{j}} - \hat{A}(q^*)_{\cdot\bar{j}}$. Only r_1 and r_2 can take negative values. Therefore, since $r \neq 0$, this equation implies $r_1 \cdot r_2 < 0$. W.l.o.g. $r_2 < 0, r_1 > 0$. Then

$$p_{11}^* r_1 + r_2 < \ldots < p_{S1}^* r_1 + r_2.$$

Take $\bar{s} \in S$ such that

$$p_{11}^* r_1 + r_2 < \ldots < p_{\bar{s}1}^* r_1 + r_2 < 0 \leq p_{(\bar{s}+1)1}^* r_1 + r_2 < p_{S1}^* r_1 + r_2.$$

Define a_s by

$$
\begin{aligned}
a_s &:= -\bar{\rho}_s(p_{s1}^* r_1 + r_2) \quad \text{for} \quad s \leq \bar{s},\\
a_s &:= \bar{\rho}_s(p_{s1}^* r_1 + r_2) \quad \text{for} \quad s > \bar{s},
\end{aligned}
$$

where $(a_1, \ldots, a_{\bar{s}})^T > 0$ and $(a_{\bar{s}+1}, \ldots, a_S)^T \geq 0$. Then, equation (7.4) is equivalent to

$$
(7.5) \quad \begin{pmatrix} \lambda_1^{\bar{h}} & \ldots & \lambda_{\bar{s}}^{\bar{h}} \\ \lambda_1^{\hat{h}} & \ldots & \lambda_{\bar{s}}^{\hat{h}} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{\bar{s}} \end{pmatrix} = \begin{pmatrix} \lambda_{\bar{s}+1}^{\bar{h}} & \ldots & \lambda_S^{\bar{h}} \\ \lambda_{\bar{s}+1}^{\hat{h}} & \ldots & \lambda_{\bar{S}}^{\hat{h}} \end{pmatrix} \begin{pmatrix} a_{\bar{s}+1} \\ \vdots \\ a_S \end{pmatrix}
$$

where without loss of generality $\lambda_{\bar{s}+1}^{\bar{h}} = \lambda_{\bar{s}+1}^{\hat{h}}$ because otherwise consumer $\hat{h}$'s utility can be multiplied with a positive constant. Substracting these two equations together with conditions (7.2a) and (7.2b) implies

$$
(7.6) \quad \sum_{s=1}^{\bar{s}} \underbrace{(\lambda_s^{\bar{h}} - \lambda_s^{\hat{h}})}_{>0} \cdot \underbrace{a_s}_{>0} = \sum_{s=\bar{s}+1}^{S} \underbrace{(\lambda_s^{\bar{h}} - \lambda_s^{\hat{h}})}_{\leq 0} \cdot \underbrace{a_s}_{\geq 0},
$$

which displays a contradiction. Therefore, there cannot exist any vector $r \neq 0$ satisfying (7.4), i.e. there is no asset structure $A \in \mathbb{R}^{2\times J}$ which admits a non-trivial sunspot equilibrium.

It remains to prove the existence of two consumers $\bar{h}, \hat{h}$ for whom (7.1a) and (7.1b) are satisfied. Therefore, define excess demand functions

$$z^i(p_{s1}) := \arg \max_{z\in\mathbb{R}^2_+ - \omega^i} U^i(\omega^i + z) \text{ s.t. } p_{s1}z_1 + z_2 = 0.$$

Then, (7.1a) and (7.1b) are equivalent to

$$z_2^{\bar{h}}(p_{11}) < \ldots < z_2^{\bar{h}}(p_{S1}), \tag{7.7a}$$

$$z_2^{\hat{h}}(p_{11}) > \ldots > z_2^{\hat{h}}(p_{S1}). \tag{7.7b}$$

Grandmont's (1985, Lemma 1.2) results assure[2]

$$z_1^{i\,\prime}(p_{s1}) < 0, \quad \text{if } z_2^i(p_{s1}) \le 0, \tag{7.8a}$$

$$z_2^{i\,\prime}(p_{s1}) > 0, \quad \text{if } z_1^i(p_{s1}) \le 0. \tag{7.8b}$$

This follows from

$$\begin{aligned} z_1^{i\,\prime}(p_{s1}) &= +\frac{u_2^{i\,\prime}(\omega_2^i + z_2^i(p_{s1})) + z_2^i(p_{s1}) \cdot u_2^{i\,\prime\prime}(\omega_2^i + z_2^i(p_{s1}))}{u_1^{i\,\prime\prime}(\omega_1^i + z_1^i(p_{s1})) + (p_{s1})^2 u_2^{i\,\prime\prime}(\omega_2^i + z_2^i(p_{s1}))}, \\ z_2^{i\,\prime}(p_{s1}) &= -\frac{u_1^{i\,\prime}(\omega_1^i + z_1^i(p_{s1})) + z_1^i(p_{s1}) \cdot u_1^{i\,\prime\prime}(\omega_1^i + z_1^i(p_{s1}))}{u_1^{i\,\prime\prime}(\omega_1^i + z_1^i(p_{s1})) + (p_{s1})^2 u_2^{i\,\prime\prime}(\omega_2^i + z_2^i(p_{s1}))}. \end{aligned}$$

Besides it is

$$z_2^i(p_{s1}) \le 0 \quad \Leftrightarrow \quad z_1^i(p_{s1}) \ge 0. \tag{7.9}$$

Observe the following equivalences when $z_1^i(p_{s1}) > 0$:

$$\begin{aligned} & z_2^{i\,\prime}(p_{s1}) && = \text{(resp. } <, >) \ 0 \\ \Leftrightarrow \ & u_1^{i\,\prime}(\omega_1^i + z_1^i(p_{s1})) + z_1^i(p_{s1}) \cdot u_1^{i\,\prime\prime}(\omega_1^i + z_1^i(p_{s1})) && \\ & && = \text{(resp. } <, >) \ 0 \\ \Leftrightarrow \ & -\frac{u_1^{i\,\prime\prime}(\omega_1^i + z_1^i(p_{s1}))}{u_1^{i\,\prime}(\omega_1^i + z_1^i(p_{s1}))}(\omega_1^i + z_1^i(p_{s1})) && = \text{(resp. } >, <) \ \frac{\omega_1^i + z_1^i(p_{s1})}{z_1^i(p_{s1})} \\ \Leftrightarrow \ & \mathrm{RRA}_1^i(\omega_1^i + z_1^i(p_{s1})) && = \text{(resp. } >, <) \ \frac{\omega_1^i + z_1^i(p_{s1})}{z_1^i(p_{s1})} \end{aligned} \tag{7.10}$$

[2]Grandmont (1985) restricts z_1^i to be less or equal to zero, but nevertheless his results can be transferred to the context here.

where RRA_1^i is the relative risk aversion of consumer i with respect to good 1. The left-hand side of equation (7.10) is non-decreasing in $\omega_1^i + z_1^i(p_{s1})$ or equivalently in $z_1^i(p_{s1})$ whereas the right-hand side is strictly decreasing in $z_1^i(p_{s1})$ to the lower bound 1. Therefore, there is either a unique value $\bar{z}_1^i$ such that ${z_2^i}'(p_{s1}) = 0$ if $z_1^i(p_{s1}) = \bar{z}_1^i$ and ${z_2^i}'(p_{s1}) < 0$ if $z_1^i(p_{s1}) > \bar{z}_1^i$ or ${z_2^i}'(p_{s1})$ never gets zero which implies strict monotonicity of $z_2^i(p_{s1})$.

Now define $\bar{p}_1^i$ by $z_1^i(\bar{p}_1^i) = \bar{z}_1^i$. $\bar{p}_1^i$ is uniquely determined if $z_1^i(p_{s1})$ is strictly decreasing for $z_1^i(p_{s1}) \geq 0$. This follows from (7.9) and (7.8a). Therefore, it is

$$\text{(7.11a)} \qquad {z_2^i}'(p_{s1}) > 0 \quad \text{for } p_{s1} > \bar{p}_1^i,$$

$$\text{(7.11b)} \qquad {z_2^i}'(p_{s1}) < 0 \quad \text{for } p_{s1} < \bar{p}_1^i.$$

Because of market clearing in the equilibria, there has to be some $\bar{h} \in \{1, \ldots, I\}$ with

$$z_2^{\bar{h}}(p_{11}) < z_2^{\bar{h}}(p_{21}).$$

Either $\bar{p}_1^{\bar{h}}$ does not exist, then $z_2^{\bar{h}}(p_{s1})$ is increasing and therefore (7.7a). Or since it is (7.11a) and (7.11b), this implies $p_{21} > \bar{p}_1^{\bar{h}}$ and therefore $p_{s1} > \bar{p}_1^{\bar{h}}$ for all $s \in \{2, \ldots, S\}$. Then, from (7.11a) it follows (7.7a).

On the other hand — again as a consequence of market clearing — there exists some $\hat{h} \in \{1, \ldots, I\}$ with

$$z_2^{\hat{h}}(p_{(S-1)1}) > z_2^{\hat{h}}(p_{S1}).$$

Because of (7.11a) and (7.11b) this implies $p_{(S-1)1} < \bar{p}_1^{\hat{h}}$ and therefore $p_{s1} < \bar{p}_1^{\hat{h}}$ for all $s \in \{1, \ldots, S-1\}$.[3] Then (7.11b) assures (7.7b).

□

Remark 7.2

1. *The main idea of the proof is to show the inconsistency of some existing ordering of the Lagrangian multipliers in Arrow-Debreu economies (exhibiting a kind of "order crossing") under additive separability of utility functions on the one hand with the special restrictions on the ordering of Lagrangian multipliers at a non-trivial sunspot equilibrium on the other hand.*

2. *For $I = 2$ the additional restriction to non-decreasing relative risk aversion is not necessary because in an Edgeworth-box economy (two-consumer two-commodity Arrow-Debreu exchange economy) where consumers are characterized by additively separable utility functions the*

[3] $\bar{p}_1^{\hat{h}}$ has to exist because otherwise $\hat{h}$'s demand for good 2 has to be strictly increasing.

ordering of the Lagrangian multipliers corresponding to the different equilibria is always inverse, i.e.

$$\lambda_1^{3-i} > \ldots > \lambda_S^{3-i},$$
$$\lambda_1^i < \ldots < \lambda_S^i$$

for $i = 1$ or $i = 2$.

3. *For $I = 2$ it is obvious that this result remains valid if consumers' utility functions are strictly monotonic, concave transformations of additively separable utilities because considering Pareto-efficiency in the Edgeworth-box clarifies the equilibrium allocations of both commodities to move in the same direction.*

4. *For a theoretical justification of non-decreasing relative risk aversion in the context of uncertainty compare Arrow (1970, Chapter 3), for empirical evidence see Dalal and Arshanapalli (1993). For example, the important class of CES-utility satisfies this assumption.*

5. *The proof applies results from Grandmont (1985) which he developed in the framework of overlapping generations models.*

6. *If the relative risk aversion of a consumer is less or equal to one, equation (7.10) does not have any solution. Therefore, this consumer's demand for good 2 is strictly increasing. Obviously, all individual demand functions are strictly monotone if relative risk aversion is bounded above by one for all consumers. This, however, contradicts multiplicity of equilibria in the Arrow-Debreu economy $(U^i, \tilde{\omega}^i)_{i=1,\ldots,I}$ such that at least for one consumer relative risk aversion has to exceed one.*[4]

Regrettably, until now, I have not been able to find out how important the additive separability of utility functions is. I strongly conjecture that this assumption can at least be weakened to the assumption of normal goods. However, I have not been successful in proving the inverse ordering of Lagrangian multipliers of different consumers when deviating from the assumption of additive separability. Nevertheless, I conjecture that my result does not hold without any restrictions on utility functions in addition to the standard ones.

Let me summarize and comment on results on the existence of sunspot equilibria whenever the asset structure is real:

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[4]For results on the uniqueness of equilibria as a consequence of a relative risk aversion less or equal to one compare Proposition 3.19.

1. Under first period consumption it is known from Gottardi and Kajii (1999) that with a sunspot-independent asset structure sunspot equilibria might exist even though the underlying certainty economy has a unique equilibrium, i.e. multiplicity of Arrow-Debreu equilibria in the certainty economy is no necessary condition for the existence of sunspot equilibria. But one should notice that their proof would not work without the assumption of first period consumption and it is not clear whether the result remains true without first period consumption. Nevertheless, I presume that it can be verified because also Cass and Shell (1983) in their two period-framework with restricted participation obtain the existence of a sunspot equilibrium, even though the certainty economy has a unique equilibrium.

2. For sunspot-independent asset structures "potential multiplicity" is a necessary condition for the existence of sunspot equilibria. Without first period consumption for economies with two consumers and more than two commodities Mas-Colell (1992) shows how to apply this condition to obtain the existence of non-trivial sunspot equilibria. Under first period consumption "potential multiplicity" is generically in utility functions and endowments satisfied and admits an open set of assets generating sunspot equilibria (Gottardi and Kajii (1999)).

3. However, as I proved in this section, for sunspot-independent asset structures in general "potential multiplicity" is no sufficient condition for the existence of any sunspot equilibrium. Therefore, take a two-commodity sunspot economy with an underlying certainty economy where the utility functions are additively separable and show non-decreasing relative risk aversion. If the certainty economy has a unique equilibrium, the sunspot economy neither allows the existence of trivial nor — according to Proposition 7.1 — of non-trivial sunspot equilibria, even though there might be endowment distributions for which multiple equilibria exist.

   Starting with a Pareto-efficient distribution of endowments, sunspot equilibria obviously never exist. However, this is a trivial case. My result is more general since it gives conditions such that for all starting distributions no non-trivial sunspot equilibria exist.

Part IV

How to use Arrow-Debreu Theory for OLG-Theory

Chapter 8

Introduction

In this part I identify competitive Arrow-Debreu equilibria of certain symmetric Arrow-Debreu economies with monetary stationary and cyclical equilibria of OLG-models. This allows to apply the theory of uniqueness in Arrow-Debreu models — summarized in Chapter 3 — to gain results on the existence and the number of this special kind of OLG-equilibria. The OLG-model is a simple model which allows an infinite time horizon by establishing life-cycles.

8.1 Introductory Remarks

The overlapping generations model — abbreviated OLG-model — is often described as a workhorse of macroeconomics (for example Mas-Colell, Whinston, and Green (1995, p. 769) and Blanchard and Fischer (1989, p. XI)). It serves as a basis for monetary theory and public finance.[1]

The identification of Arrow-Debreu equilibria of certain symmetric Arrow-Debreu economies with monetary stationary and cyclical equilibria of OLG-models is one further step in connecting micro- with macroeconomics as well as static with dynamical models. Many economists (for example Mas-Colell (1991)) presume close relations between unique Arrow-Debreu equilibria in static Arrow-Debreu economies on the one hand and stationary equilibria in dynamic economies on the other hand. This gives rise to great hopes that Arrow-Debreu theory can be applied to obtain results in dynamic models.

When elaborating this part, I only had knowledge of one result in this topic, namely Balasko and Ghiglino (1995) who identified two-cycles in the

[1] According to Balasko and Shell (1980, pp. 281/2) the reason is that "the overlapping-generations model is the only natural model in which money is permitted to serve as a store of value".

simplest overlapping generations model (one consumer who lives for two periods and one commodity in each period) with the Arrow-Debreu equilibria of a symmetric two-consumer two-commodity Arrow-Debreu economy. I generalize their results for arbitrary cycles in general overlapping generations models, that is in OLG-models without any other restriction than finiteness on the number of consumers, periods of life, and commodities. This allows to apply the theory of uniqueness of Arrow-Debreu equilibria presented in Chapter 3 to obtain a series of statements on the existence and number of monetary stationary and periodic OLG-equilibria. Some of these results are new, some of them are already well-known, but undergo a new simple proof.

Simultaneously and independently Tuinstra and Weddepohl (1999) have developed the same generalization of Balasko and Ghiglino (1995). However, their main focus is on finding OLG-models with two respectively three generations displaying different kinds of equilibrium cycles. In my survey on related literature I present their results in detail and contrast them to my own ones.

This part is built up as follows: after finishing this short introduction by pointing out the importance of OLG-models for economics and explaining the interest in cyclical equilibria, the next section is devoted to already existing literature. Since there is a plenty of this, I have to make a very selective subjective choice. I decided to start with the seminal paper by Samuelson (1958), to go on with the path-breaking results on the existence of cycles given by Gale (1973), Benhabib and Day (1982), and Grandmont (1985), and to continue with *some* results by Kehoe, Levine, and their co-authors (1984a), (1984b), (1985), (1990), (1991) who did important work on general overlapping generations models with some results close to my own ones. Besides I mention Ghiglino and Tvede (1995) because I extend one of their results on Pareto-efficient endowments. Of course, the results by Balasko and Ghiglino (1995) and Tuinstra and Weddepohl (1999) should not be missed.[2] Chapter 9 contains a description of the underlying OLG-model and the corresponding Arrow-Debreu exchange economies. The periodic OLG-equilibria and the Arrow-Debreu equilibria are related to each other and the structure of the set of Arrow-Debreu equilibria is illuminated. Chapter 10 applies these ideas to obtain results on the existence and the number of stationary and periodic OLG-equilibria with the help of Chapter 3. To maintain a self-contained train of thought, a large part of the proofs is deferred to Appendix B.2.

The basic structure of OLG-models consists in the fact that at any time individuals of different generations are alive and can trade with each other. Therefore, each generation trades with different generations in different peri-

[2] Even for these few papers it is not possible to give their whole contents!

ods of its life while there are unborn generations with preferences not taken care of during the current trading. Blanchard and Fischer (1989, p. 91) call this "life-cycles hypothesis".

A large part of the literature refers to the simple OLG-model where agents are only alive for two periods and where there is only one commodity per period. One reason is computational simplicity. Conceptually the generalization does not impose any problems, but computational simplicity gets lost. Hence, one advantageous aspect of my approach is to enable computation of stationary and cyclical OLG-equilibria by computation of Arrow-Debreu equilibria.

An important aspect of the OLG-model lies in the fact that it allows to extend the economic system indefinitely into the future while it does not require the agents to have perfect foresight up to infinity and does not force "the doomsday problem onto the agents in the economy or onto some representative individual" (Benhabib and Day (1982, p. 37)). The model allows a double infinity of consumers and commodities without making individual decision problems difficult to handle. It stands in contrast to those models where agents live infinitely long.[3]

One main purpose of macroeconomics and therefore of the application of OLG-models is to understand the existence and persistence of economic fluctuations and explain output, employment, and price movements. A systematic study of business cycles should lead to a clarification of lengths, contractions, expansions as well as the amplitude of fluctuations in the behavior of economic variables. One has to distinguish different kinds of cycles: those caused by exogenous shocks and those coming into being without any exogenous influence. Following Grandmont (1985) these last ones to which I restrict my attention are called endogenous cycles. Their existence is important because it explains observed economic fluctuations without any "*ad hoc* assumptions [on] cycles [...] due to exogenous shocks" (Grandmont (1985, p. 996)). According to Grandmont (1985) these cycles give the government the chance of some countercyclical policies which stabilize the economy. Since there is a large dimension of indeterminacy of OLG-equilibria[4] (compare, for example, Kehoe and Levine (1990)), economists value stationary and periodic OLG-equilibria as means to reduce the dimension of indeterminacy (for example, Blanchard and Fischer (1989, p. 261)) and the periodic OLG-

[3]Compare the introductory description of Ramsey economies given in Blanchard and Fischer (1989, Chapter 2).

[4]Existence of OLG-equilibria itself was shown much earlier. For example, a quite general proof — at least in OLG-exchange economies which I consider — is given by Balasko, Cass, and Shell (1980) who work with so-called "truncated economies", that is they cut the OLG-economy at some time t and let t go to infinity.

equilibria as a generalization of steady states (for example, Azariadis (1993, p. 85)).

8.2 Related Literature

Samuelson's (1958) paper can be seen as the seminal work on OLG-models. It initiates a lot of discussion and asks for extensions aiming at different aspects, but most of it in line with Samuelson (1958) himself on optimality of stationary (or steady state)[5] equilibria.[6] Even though Samuelson (1958) does not refer to cycles on which I focus, I think it is important to recall his results because his paper is the basis for all following ones on overlapping generations.

Samuelson (1958) considers similar problems in a two- and a three-period OLG-framework with one representative consumer for each generation. The model is a simple production economy where consumers produce exactly one unit of the non-storable single commodity when young (and middle-aged). This can be reinterpreted as a pure exchange economy where consumers get an endowment when young (and middle-aged), i.e. during the working time, but get nothing when old, i.e. after retirement. Since agents want to consume during their whole life, they would like to transfer income between the different periods.

Samuelson (1958) assumes a stationary economy except for population growth and only considers those OLG-equilibria with constant interest rates. Allowing exchange between present and future goods, i.e. under the introduction of fiat money, his first point is to prove equality of one equilibrium interest rate and the growth rate which even constitutes a socially optimal solution. Samuelson (1958) observes in the three-period model that middle-aged not necessarily pay to the young generation which will pay for them during retirement, but to the old one. The analogue in the two-period framework is that for a young generation there is nobody to bribe to make provision for the time of retirement. An alternative is the autarky equilibrium where everybody consumes immediately his income. Samuelson (1958) states the surprising fact that even when everybody is allowed to trade on each market, the social optimum is generically not obtained because without money the

[5]This means that individual consumption plans are constant over time. When there is no population growth, stationary equilibria are those where prices are also constant over time whereas under population growth price movements reflect this change.

[6]To mention some of this literature take Lerner (1959), Cass and Yaari (1966), and Shell (1971). Diamond (1965) extends Samuelson's (1958) model to include non-trivial production.

double coincidence of wants is missing. An example demonstrates that there is even no local convergence against the social optimum.

For Samuelson (1958, p. 479) the function "of money [is] a social compact that can provide optimal old age social security". He states that even though money cannot be directly consumed, it is "something to hold and to carry over into retirement age" (Samuelson (1958, p. 482)).

Gale (1973) is one of the first economists who discovers the existence of endogenous cycles in OLG-models. The model he considers is an n-period generalization of Samuelson's (1958), but directly formulated as pure exchange economy. In some parts it includes bequests.

Gale (1973) proves the general existence of two kinds of steady states: the so-called "golden rule paths" where the interest rate equals the growth rate of population and the so-called "balanced paths" where indebtedness of the society is zero.[7] An important insight by Gale (1973) is the necessity of distinction between the "classical" and the "Samuelson" case, that means whether society as a whole is debtor or creditor at the golden rule steady state.

In the two-period framework Gale (1973) proves under separable or homothetic utility functions that in the classical case the forward dynamic system is determinate in the sense that consumption of the current generation is determined from consumption of the past generations.[8] Gale (1973) conjectures that in the classical case the golden rule steady state is stable whereas in the Samuelson case this is true for the balanced steady state. This result is proved for the two-period framework. The golden rule steady state is often called monetary steady state because except for non-generic situations it requires income transfer between different generations enabled by the introduction of fiat money.

Gale (1973) considers three two-period examples, one of them with quadratic utility (with different factors of risk aversion for the different periods of consumption) for which he demonstrates the existence of a two-cycle. He identifies this property to be independent of expectations and to be caused by the equilibrium price mechanism.

Benhabib and Day (1982) consider a two-period OLG-model with one consumer type and one commodity in each period. They establish conditions leading to "erratic dynamics" or chaos which means that there exist trajectories (sequences of equilibrium consumption bundles) which are bounded, but

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[7]This corresponds to Samuelson's (1958) autarky equilibrium whereas the golden rule path is his social optimum.

[8]Compare Assumptions Gale 9.4 later on.

never converge to a stationary or periodic equilibrium.

Following Gale (1973), Benhabib and Day (1982) assume separable or homothetic utility functions such that in the classical case forward perfect foresight dynamics are uniquely determined, i.e. consumption of the young in period t is a function f of consumption of the young in period $t-1$.

Benhabib and Day (1982) apply a theorem[9] by Li and Yorke (1975, Theorem 1)[10] which under special conditions on the function f guarantees erratic behavior as well as the existence of cycles of arbitrary length. Benhabib and Day's (1982) main success is to establish conditions on the "constrained marginal rate of substitution"[11] which assure f to satisfy the assumptions of Li and Yorke's (1975) theorem. This result is applied to construct three different robust examples of two-period OLG-economies displaying chaos as well as cycles of arbitrary length.[12]

Grandmont (1985) is the first who systematically studies endogenous business cycles. The underlying model is a simple overlapping generations model without population growth, i.e. there is a single consumer living for two periods and a single commodity in each period. Grandmont (1985) himself reformulates the simple production economy in terms of a pure exchange economy. Requiring an additively separable utility function which satisfies standard assumptions and a strictly positive endowment vector such that the quotient of first and second life-period marginal utility at the endowment point is less than one (this value is called $\bar{\theta}$)[13], he assures to be in the Samuelson case and systematically studies individuals' offer curves. This enables him to characterize the dynamics in the OLG-model by a function ϕ (compare Figure 8.1) in some θ which is the quotient between today's price and the one expected for tomorrow. $\theta - 1$ is the expected real interest rate.

$$\theta_{t-1} = \phi(\theta_t)$$

defines uniquely a backward perfect foresight dynamic.

The graph of ϕ is closely related to the relative risk aversion with respect to utility of consumption when old.[14]

[9] Benhabib and Day (1982) call it Chaos Theorem.

[10] An analogue for higher dimensional systems, for example necessary when considering more periods of life, can be found in Diamond (1976).

[11] This is derived from utility functions and endowments.

[12] Notice that all utility functions are linear in the consumption when old.

[13] For these assumptions compare Assumptions Grandmont 9.3.

[14] One should be conscious that interpreting the quotient of second and first derivative of a utility function as a measure of risk aversion is only possible when there is uncertainty (compare Arrow (1970) and Pratt (1964)). Nevertheless, Grandmont (1985) himself ap-

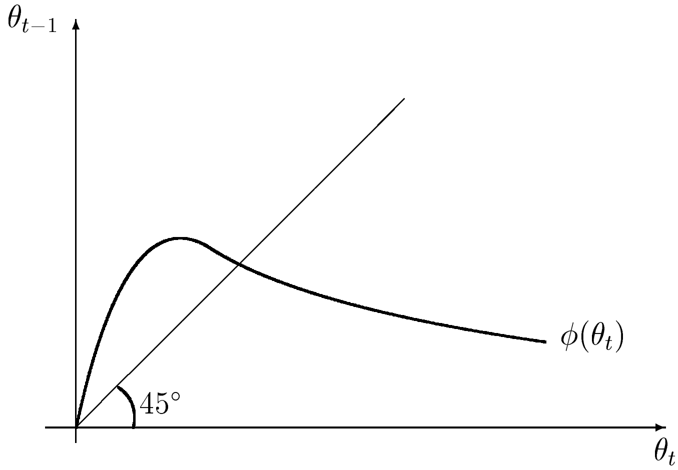

Figure 8.1: A possible form of ϕ.

Most important for my considerations are Grandmont's (1985) results on the existence of periodic equilibria. According to him, the main reason for the existence of cycles is the conflict between the wealth effect on the one hand and the intertemporal substitution effect on the other hand, both caused by changes in the interest rate. Having a look on Figure 8.1, this requires a significant hump of ϕ. If the "old" relative risk aversion RRA_1 is less than one, a hump and therefore cycles are excluded. If the supremum of RRA_1 is bigger than 1 and the risk aversion itself is non-decreasing, a hump exists. A necessary condition for the existence of two-cycles in this last case is given by $\phi'(1) < 0$[15], a sufficient one by $\phi'(1) < -1$. This last condition is satisfied if the relative risk aversion at stationary consumption when old ($RRA_1(c_1(1))$) is bigger than two times the relative risk aversion at stationary consumption when young ($RRA_0(c_0(1))$) multiplied by the quotient $\frac{c_1(1)}{c_1(1)-\omega_1}$ where $c_1(1)$ is the consumption and ω_1 the endowment when old. The intuition is that the old must have a sufficiently high risk aversion.

The coexistence of cycles of different lengths follows from the application of Sarkowski's Theorem (compare, for example, Grandmont (1986, Theorem 1)) on ϕ. The main point consists in the fact that ϕ can be restricted to map from a real interval into itself.

Sarkowski's ordering of all positive integers first contains all odd integers from 3 onwards, then the powers of 2 times these odd integers, and at last

plies the term of "risk aversion" even though he states that it should be interpreted as a measure of degree of concavity.

[15] 1 corresponds to the monetary stationary equilibrium.

the powers of 2 backwards, i.e.

$$\begin{array}{lllllllll} & 3 & >_S & 5 & >_S & 7 & >_S & \dots & \\ >_S & 2\cdot 3 & >_S & 2\cdot 5 & >_S & 2\cdot 7 & >_S & \dots & \\ \dots & & & & & & & & \\ >_S & 2^n\cdot 3 & >_S & 2^n\cdot 5 & >_S & 2^n\cdot 7 & >_S & \dots & \\ \dots & & & & & & & & \\ >_S & \dots & >_S & 2^m & >_S & \dots & >_S & 2 & >_S \ 1. \end{array}$$

Sarkowski's Theorem implies that as soon as the simple OLG-model (or equivalently ϕ) displays a cycle of length k, it also displays cycles of length k' for all k' with $k >_S k'$. This result determines the order with which cycles of different length appear and especially assures that as soon as a three-cycle exists cycles of arbitrary length coexist.[16]

Sufficient conditions for three-cycles are given by $\phi^3(\theta) \leq \theta$ for some $\theta \in (\bar{\theta}, 1)$ or $\phi^3(\theta) \geq \theta$ for some $\theta > 1$. Besides Grandmont establishes that generically there is an even number of three-cycles.

Under non-decreasing relative risk aversion RRA_1 and *sup* $RRA_1 > 1$, Grandmont (1985) gives conditions on marginal utilities and endowments which assure the existence of three-cycles. This is applied to prove that for constant elasticity, $0 < \omega_1 < 1$, and $\omega_0 + \omega_1 > 1/\omega_1$ a sufficiently high risk aversion for "old" consumption implies the existence of a three-cycle.

That observation leads Grandmont (1985) to a computer experiment on a constant elasticity example where the "old" relative risk aversion is continuously increased which evokes "traders' offer curves [to] become[...] more and more humpy [and] more and more cycles appear" (Grandmont (1985, p. 1029)). Underlying this experiment is the idea of bifurcation which is an important tool for the examination of dynamical systems.[17]

Grandmont (1985) also obtains interesting statements on the stability of cycles which — no question — is the next step after understanding the conditions for their existence. Nevertheless, since my own contribution concentrates only on existence, I omit Grandmont's (1985) work on this topic.

Kehoe and Levine (1984a), (1984b), (1985), (1990) did enormous work on the generalized overlapping generations model where all these papers are

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[16] Devaney (1989, p. 62) states the additional implication that as soon as there exists some periodic point of an order not equal to a power of two, then there have to exist infinitely many periodic points. The converse result also holds: if there is a finite number of periodic points, they all have to be of order equal to a power of two.

[17] The idea is to make the function ϕ describing the dynamical system dependent on a parameter λ which models some exogenous influences. A bifurcation arises at some point whenever the dynamic behavior changes, i.e. for example if a new cycle appears. For an introductory description compare Devaney (1989, § 1.12) or Azariadis (1993, Chapter 8).

closely related to each other. Even though most of this work is on the (in)determinacy of OLG-equilibria, some of their results overlap with my own ones. To summarize briefly their results on indeterminacy they prove that in general for balanced as well as for monetary steady states OLG-economies might exhibit large degrees of indeterminacy (Kehoe and Levine (1984b), (1985), (1990)), but under further restrictions like intertemporal separability (Kehoe and Levine (1984a)) this breaks down to determinacy of balanced steady states and one dimension of indeterminacy for monetary steady states which is the analogue to the simple overlapping generations model. One degree of indeterminacy of the monetary steady state (the only one in the simple OLG-model) can be eliminated by price normalization or equivalently by fixing the value of money. Kehoe and Levine's considerations on equilibrium paths near steady states are mainly based on linearizations of equilibrium conditions at the steady states.

Most of the work allows the consideration of many goods and many consumers. Large parts put no other restriction on life-periods than finiteness or at least — if the number of periods is restricted to be two — Kehoe and Levine prove how to generalize their results. In Kehoe and Levine (1984b) a method applicable for most purposes is described where an $n+1$-period model is reduced to a two-period model by grouping n periods into a single one. However, this proceeding requires to have no restrictions on the number of agents and commodities. Besides, it is not applicable for the consideration of stationary equilibria and cycles. What distinguishes Kehoe and Levine's work from other approaches, is the fact that consumers are not characterized by utility functions and endowments, but whole generations are described by two excess demand functions for the different periods.

The main reason why I mention Kehoe and Levine's work is the paper in which they develop some regularity theory for OLG-models (Kehoe and Levine (1984b)) which in some parts contains results for which my approach gives another view. They prove for general overlapping generations models the existence of a monetary and a balanced steady state. In both cases the way of proof is to put OLG-excess demand functions at the steady states down to excess demand functions of a static economy which allows application of Brouwer's Fixed Point Theorem to obtain both kinds of steady states. Kehoe and Levine (1984b) formulate some regularity assumptions based on these static economies (which they prove to be generically satisfied in the topological space of excess demand functions), allowing the application of the Index Theorem and therefore generically leading to odd numbers of monetary and balanced steady states.

Besides Kehoe, Levine, and their coauthors (Kehoe et al. (1991)) consider the consequences of gross substitution of excess demand functions in so-called "large-square economies"[18] and especially in general overlapping generations models. For these last models they demonstrate balanced steady states to be unique and for monetary states they prove uniqueness respectively one dimension of indeterminacy depending on whether one fixes the nominal debt that passes between generations or not. Besides, their results on convergence against steady states imply the non-existence of equilibrium cycles whenever demand satisfies the property of gross substitution.

Ghiglino and Tvede (1995) consider a general overlapping generations pure exchange economy, but restricted to two periods. Within this framework they prove the following result: generically in the set of economies described by utility functions and endowments and displaying an optimal no-trade equilibrium, there exists a neighborhood of the endowments such that all economies with the same utility functions and endowments out of this neighborhood are characterized by a globally unique balanced and a globally unique golden rule steady state. The line of proof is to show coincidence of balanced and golden rule steady state when endowments are optimal, then demonstrate local uniqueness for sufficiently small neighborhoods of the endowments, and use some results on convergence to obtain the global property. The restriction to a generic set is necessary because of the consideration of balanced steady states.

To the best of my knowledge, Balasko and Ghiglino (1995) are the first contributing to the identification of periodic OLG-equilibria and Arrow-Debreu equilibria. The model they consider is the simple overlapping generations model with one commodity and a single two-period living consumer without any population growth. They show that whenever the wealth expansion path defined for equal prices, i.e. the image of the Engel function for equal prices[19], displays at least one point with a slope less than one, there is an open set of endowments for which the OLG-economy has a two-cycle. Balasko and Ghiglino (1995, p. 566) interpret this property as "mild form of impatience" and contrast it to requirements like high relative risk aversion for the old (compare Grandmont (1985)). The basic idea is to describe OLG-dynamics by young people's excess supply of which the properties are related to those of the offer curve. This allows to prove the above mentioned

[18] These are general exchange economies with no restrictions on the number of consumers and commodities.

[19] The Engel function for equal prices is consumer's demand as a function of income when prices are fixed to $(1, 1)$. Balasko and Ghiglino (1995) themselves verbally require the slope condition for the Engel curve, but actually prove their result for the slope condition with respect to the wealth expansion path.

result for endowments not restricted to be non-negative. However, endowments can be translated to be positive and utility functions can be redefined such that the result remains to hold. This is true even if utility is of the discounted utility type: for each given discount rate $\delta < 1$ there exists an open set of utility functions and an open set of positive endowments such that the corresponding simple OLG-economy exhibits two-cycles.

The relationship between young consumer's supply and the offer curve leads Balasko and Ghiglino (1995) to the construction of a two-consumer two-good symmetric Arrow-Debreu economy where the first agent is characterized by the given OLG-utility and -endowments whereas the second has the inverse characteristics. Mainly by geometric arguments Balasko and Ghiglino (1995) conclude that there is a bijection between the two-cycles of the OLG-model and multiple equilibria of the Arrow-Debreu economy.

Tuinstra and Weddepohl (1999) study monetary stationary and cyclical OLG-equilibria with the help of a special Arrow-Debreu economy of which the construction is made along the lines of Balasko and Ghiglino (1995), i.e. they take the same starting-point as I do in this part. At first modeling a single-consumer single-commodity OLG-economy cycles of length bigger than or equal to the length of the life-cycle are reproduced in an Arrow-Debreu economy with as many consumers and goods as the length of the equilibrium cycle where preferences and endowments undergo a right round shift. This way of construction is used to find three different examples of OLG-economies displaying equilibrium cycles:

1. There is a three-period pure exchange OLG-model where consumers are characterized by CES-utility which exhibits two-, three- and four-cycles.

2. There is a two-period pure exchange OLG-model with CARA-utility functions defining consumers which allows the existence of three-cycles and hence — because of Sarkowski's Theorem — of cycles of arbitrary length. This result is closely related to Grandmont's (1985) statements on this class of utility functions except for the fact that he considers the Samuelson case whereas Tuinstra and Weddepohl's (1999) example is for the classical one.

3. Besides a two-period OLG-economy with capital along the lines of Reichlin (1986)[20] is constructed for which also via the Arrow-Debreu

[20] Reichlin (1986), Farmer (1986), Jullien (1988), and Benhabib and Laroque (1988) study the emergence of endogenous cycles and chaos in OLG-economies with production.

approach the existence of a three-cycle and therefore of cycles of arbitrary length can be proved.

However, Tuinstra and Weddepohl (1999) do not make any claims on the general existence of (multiple) stationary equilibria and cycles. In contrast to them, I apply the same approach not to construct specific examples, but to gain some general results on these kind of OLG-equilibria. In my eyes this exploits the power of this new identification result much more, even though it surely leaves much room for further research. My considerations force me to a deeper study of the equilibrium set in the symmetric exchange economies.

In general, Tuinstra and Weddepohl (1999) restrict their considerations to a single agent and a single commodity, but do a short digression on generalizing each of these aspects separately. In contrast I directly work on the most general model except for including population growth which Tuinstra and Weddepohl (1999) also consider in their digression.

Chapter 9

The Models and Their Parallels

The OLG-model itself as well as particularly the special exchange economies described in the next section are related to the model described in Chapter 2. Nevertheless, essential differences prevent me from extending the model given there to this new framework. Instead I provide a self-contained description. Of course, the reader should have an eye on Chapter 2 for interpretations and some basic definitions. I leave this part understandable on its own, but nevertheless choose notations as close as possible to the other parts.

9.1 The OLG-Model

The analyzed stationary overlapping generations model is a quite general pure exchange economy without population growth.

There are I different types of consumers, $i = 1, \ldots, I$[1] [2], who live for $N+1$, $N \in \mathbb{N}$[3] periods. In each period, there are L commodities, $l = 1, \ldots, L$. I distinguish consumers by the period in which they are born, i.e. generation $t \in \mathbb{Z}$[4] consists of I different types of consumers all born at date t and living until date $t + N$.[5] At each date the same number of consumers is born. At

[1]As before for ease of notation I always denote the set as well as the number of its elements with the same capital letter, i.e. $H := \{1, \ldots, H\}$. If 0 is included, I write $\tilde{H} := \{0, \ldots, H\}$.

[2]The first introducing heterogeneity of consumers within OLG-generations were Cass, Okuno, and Zilcha (1979) who consider Pareto-optimality of stationary equilibria in this new framework.

[3]$\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, \ldots\}$, and $\mathbb{N}_{>m} := \{n \in \mathbb{N} | n > m\}$.

[4]This means that time extends indefinitely into the past and into the future. In contrast, it is often assumed that the economy starts at some time $t = 0$. For a discussion of the relations between the different approaches compare Geanakoplos (1987b).

[5]Most of the literature on overlapping generations applies such discrete time models, but for example Blanchard (1985) considers a continuous time model where people die

date t there are $N+1$ generations alive.

Consumers are denoted $(i,t) \in I \times \mathbb{Z}$ where i describes their type and t the date of birth. Consumer (i,t) is characterized by time-independent endowments

$$\omega^i = \begin{pmatrix} \omega_0^i \\ \vdots \\ \omega_N^i \end{pmatrix} \in \mathbb{R}_{++}^{(N+1)L}$$

and utility functions

$$\begin{array}{rccc} U^i: & \underbrace{\mathbb{R}_+^L \times \ldots \times \mathbb{R}_+^L}_{N+1} & \to & \mathbb{R} \\ & (c_0, \ldots, c_N) & \mapsto & U^i(c_0, \ldots, c_N) \end{array}$$

which are infinitely often differentiable, strictly monotonic, strictly quasi-concave, and the indifferent sets are bounded away from the boundary of the consumption set, i.e. $\partial\mathbb{R}_+^{(N+1)L}$. Utility functions therefore satisfy Assumption 2.4. $c_0^i, \ldots, c_N^i \in \mathbb{R}_+^L$ denote consumer i's consumption plans for the respective periods of his life. Whereas lower indices label the periods of life (and commodities), upper indices refer to the type of consumer and the date of birth. The whole economy is described by[6]

$$OLG = \left\{I, N, L; (U^i, \omega^i)_{i \in I}\right\}.$$

Notice that attention is restricted to a pure exchange economy even though this is a serious, but nevertheless well-accepted simplification.[7]

Let $p^t \in \mathbb{R}_{++}^L$ denote the commodities' prices at date t. Under the assumption of perfect foresight a consumer (i,t) then has to solve the following maximization problem:

$$\begin{aligned} (9.1) \qquad & \max_{c_0, \ldots, c_N \in \mathbb{R}_+^L} U^i(c_0, \ldots, c_N) \\ \text{s.t.} \quad & p^t \cdot c_0 + \ldots + p^{t+N} \cdot c_N = p^t \cdot \omega_0^i + \ldots + p^{t+N} \cdot \omega_N^i. \end{aligned}$$

The assumption of perfect foresight is common in the OLG-framework. Nevertheless, there is OLG-literature deviating from this assumption and incorporating some learning behavior (compare, for example, Grandmont (1985) and Fuchs (1976),(1977), (1979a), and (1979b)).

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with a constant probability. This model is quite flexible because it even allows to include the infinite horizon for each individual, but the life-cycle aspect gets lost.

[6]In contrast to previous notation I here explicitly mention the number of agents and goods. Later on the importance of these numbers for my results will become clear.

[7]Grandmont (1985) demonstrates how to interpret the exchange economy as a very simple production economy when $L = 1$.

The solution of the maximization problem defines a continuously differentiable demand function

$$c^i(p^t,\ldots,p^{t+N}) = \begin{pmatrix} c^i_0(p^t,\ldots,p^{t+N}) \\ \vdots \\ c^i_N(p^t,\ldots,p^{t+N}) \end{pmatrix}$$

which satisfies standard properties like the budget identity.

Definition 9.1 (OLG-Equilibrium)
An overlapping generations (OLG-) equilibrium (with perfect foresight) is a tuple of prices $(\ldots,p^{*-1},p^{*0},p^{*1},\ldots)$ *with* $p^{*t}\in\mathbb{R}^L_{++}$ *and commodity allocations* $({c^*}^{i,t}_0,\ldots,{c^*}^{i,t}_N)_{i\in I,t\in\mathbb{Z}}$ *with* ${c^*}^{i,t}_n\in\mathbb{R}^L_+$ *such that*

1. $({c^*}^{i,t}_0,\ldots,{c^*}^{i,t}_N)^T = c^i(p^{*t},\ldots,p^{*t+N})$ *for all* $(i,t)\in I\times\mathbb{Z}$,
2. $\sum_{n\in\tilde{N}}\sum_{i\in I}{c^*}^{i,t-n}_n = \sum_{n\in\tilde{N}}\sum_{i\in I}\omega^i_n$ *for all* $t\in\mathbb{Z}$.

I am especially interested in periodic and monetary stationary equilibria.

Definition 9.2 (OLG-Equilibrium of Period k)
An OLG-equilibrium $\big(({c^*}^{i,t}_0,\ldots,{c^*}^{i,t}_N)_{i\in I,t\in\mathbb{Z}},(p^{*t})_{t\in\mathbb{Z}}\big)$ *is*

1. *of period* k, $k\in\mathbb{N}$, *if for all* $t\in\mathbb{Z}$ *it is* $p^{*t}=p^{*t+k}$,
2. *of exact period* k, $k\in\mathbb{N}$, *if it is of period* k *and not of period* k' *for all* $k'\in\{1,\ldots,k-1\}$,
3. *a monetary stationary equilibrium if for all* $t\in\mathbb{Z}$ $p^{*t}=p^{*t+1}$.

Tuinstra and Weddepohl (1999) do not distinguish the two kinds of periodic equilibria. I call periodic equilibria also cycles.

Instead of using price systems, this definition could have been given alternatively in terms of allocations.

Until now, I have not assumed anything to assure so-called well-defined backward or forward perfect foresight dynamics, which would mean that as soon as consumption of generation $t,t+1,\ldots$ resp. $\ldots,t-1,t$ and prices $p^t,p^{t+1},\ldots$ resp. $\ldots,p^{t-1},p^t$ are known, prices in period $t-1$ resp. $t+1$ and consumption of the corresponding generation are uniquely determined.[8] As indicated before, Grandmont (1985) formulates assumptions which guarantee well-defined backward perfect foresight dynamics whereas Gale (1973) gives conditions for well-defined forward perfect foresight dynamics.

[8]For a thorough discussion compare, for example, Grandmont (1985).

Assumption 9.3 (Assumptions Grandmont (1985))

1. $I = N = L = 1$.

2. $U^1(c_0, c_1) = V_0(c_0) + V_1(c_1)$ *where* $V_\tau, \tau = 0, 1$, *are continuous on* $\mathbb{R}_+$, *twice continuously differentiable on* $\mathbb{R}_{++}$ *with* $V'_\tau(x_\tau) > 0, V''_\tau(x_\tau) < 0$, *and* $\lim_{x_\tau \to 0} V'_\tau(x_\tau) = \infty$.

3. $V'_0(\omega_0^1) < V'_1(\omega_1^1)$.

Under these assumptions Grandmont (1985) proves that for given p^t, p^{t+1} the system

$$c_0^1(p^t, p^{t+1}) + c_1^1(p^{t-1}, p^t) = \omega_0^1 + \omega_1^1$$

has a unique solution p^{t-1}. This is called backward perfect foresight dynamics since it is assumed that agents have perfect foresight and prices are determined by backward induction.

Now, take on the other hand the following assumptions.

Assumption 9.4 (Assumptions Gale (1973))

1. $I = N = L = 1$.

2. $U^1 : \mathbb{R}_+^2 \to \mathbb{R}$ *is either separable or homothetic and continuous on* $\mathbb{R}_+^2$, *strictly monotonic, strictly quasi-concave, and indifference curves do not intersect* $\partial \mathbb{R}_+^2$.

3. $(\bar{c}_0^1, \bar{c}_1^1) = \textit{arg max}_{c_0, c_1 \in \mathbb{R}_+} U^1(c_0, c_1)$ *s.t.* $c_0 + c_1 = \omega_0^1 + \omega_1^1$ *satisfy* $\bar{c}_0^1 > \omega_0^1$, *i.e. one is in the so-called classical case.*

Under these assumptions Gale (1973) proves that for given p^{t-1}, p^t the system

$$c_0^1(p^t, p^{t+1}) + c_1^1(p^{t-1}, p^t) = \omega_0^1 + \omega_1^1$$

has a unique solution p^{t+1}. This is called forward perfect foresight dynamics since it is assumed that agents have perfect foresight and prices are determined by forward induction.

For separable utility functions in the form given by Assumption 9.3.2 Assumption 9.4.3 follows from $V'_0(\omega_0^1) > V'_1(\omega_1^1)$.

9.2 The $(K + 1)$-Modulo Exchange Economy

A modulo exchange economy is defined with respect to a fixed OLG-economy $OLG = \{I, N, L; (U^i, \omega^i)_{i \in I}\}$. As Tuinstra and Weddepohl (1999, p. 187) state in the OLG-economies consumers of different generations are characterized by the same utility and endowments, but face different prices, whereas in the modulo exchange economy consumers face the same prices, but have different characteristics.

Before I define the modulo exchange economy in its general form which refers to some general $OLG = \{I, N, L; (U^i, \omega^i)_{i\in I}\}$ I consider the special case referring to a simple OLG-economy with $I = N = L = 1$, i.e. to $\overline{OLG} = \{1, 1, 1; (U^1, \omega^1)\}$. I present that modulo exchange economy — later on called two-modulo exchange economy — which can be used to examine monetary stationary and two-periodic equilibria in this kind of OLG-model. This special example creates the intuition for the general construction.

Assume $\overline{OLG}$ satisfies Assumptions Grandmont 9.3 such that the OLG-economy is exactly that one to which Balasko and Ghiglino (1995) refer. The modulo exchange economy I present in the following is the one with which they initialized my own work as well as that of Tuinstra and Weddepohl (1999).

The representative OLG-consumer is defined by

$$U^1 : \mathbb{R}^2_+ \to \mathbb{R} \text{ and } \omega^1 \in \mathbb{R}^2_{++}.$$

Under Assumptions Grandmont 9.3 the offer curve has the shape as given in Figure 9.1.

The modulo exchange economy used to examine monetary stationary equilibria and two-cycles is characterized by the two-dimensional commodity space $\mathbb{R}^2_+$ and two consumers.

The first consumer 1_0 is described by the same characteristics as the representative consumer in the OLG-economy, i.e.

$$U^{1_0} = U^1 \text{ and } \omega^{1_0} = \omega^1.$$

The second consumer 1_1 is defined by "permuted" characteristics

$$U^{1_1}(x_0, x_1) = U^1(x_1, x_0) \text{ and } \omega^{1_1} = \begin{pmatrix} \omega^1_1 \\ \omega^1_0 \end{pmatrix}$$

where $x = (x_0, x_1) \in \mathbb{R}^2_+$ denotes consumption plans. This consumer's demand function is obtained as reflected image of the first consumer's demand with respect to the angle bisector of the first quadrant.

The intuition behind this construction is as follows: given a two-cycle the consumer born in period t exchanges the current (t) consumption commodity with the consumer born in $t-1$ and tomorrow's $(t+1)$ consumption good with the consumer born in $t+1$. However, in a two-cycle consumer $t-1$ and consumer $t+1$ are confronted with the same prices $((p^*_{t-1}, p^*_t) = (p^*_{t+1}, p^*_{t+2}))$ and

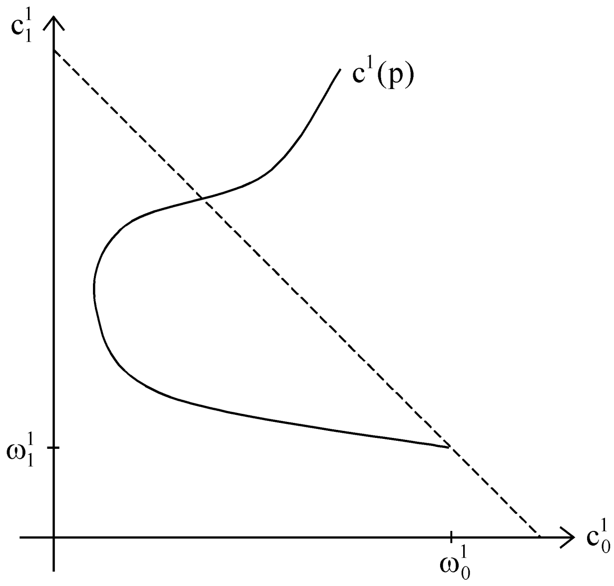

Figure 9.1: Offer curve in the simple OLG-economy under Assumptions Grandmont 9.3.

obtain the same consumption bundles $((c{*}_0^{1,t-1}, c{*}_1^{1,t-1}) = (c{*}_0^{1,t+1}, c{*}_1^{1,t+1}))$, except for the time aspect. Therefore, in the modulo exchange economy the time aspect is eliminated such that the commodity in $t+1$ is identified with the one in $t-1$.

The Edgeworth-box of the modulo exchange economy is given as in Figure 9.2.

This exchange economy displays three equilibria: one with equal prices and two others. The equilibrium with equal prices on the diagonal corresponds to the monetary stationary equilibrium, the two other equilibria to a two-cycle in the OLG-economy.

Using the gained intuition one is now able to define a general modulo exchange economy with respect to $OLG = \{I, N, L; (U^i, \omega^i)_{i \in I}\}$.

The $(K+1)$-modulo exchange economy is defined via a right round shift on the set $\tilde{K} = \{0, 1, \ldots, K\}$ which itself is given by a modulo function.

The right round shift for a fixed set $\tilde{K} = \{0, 1, \ldots, K\}$ with $K \geq N$ is defined as

$$\begin{array}{rccl} s_K : & \tilde{K} & \to & \tilde{K} \\ & k & \mapsto & s_K(k) = mod_{K+1}(k+1) \end{array}$$

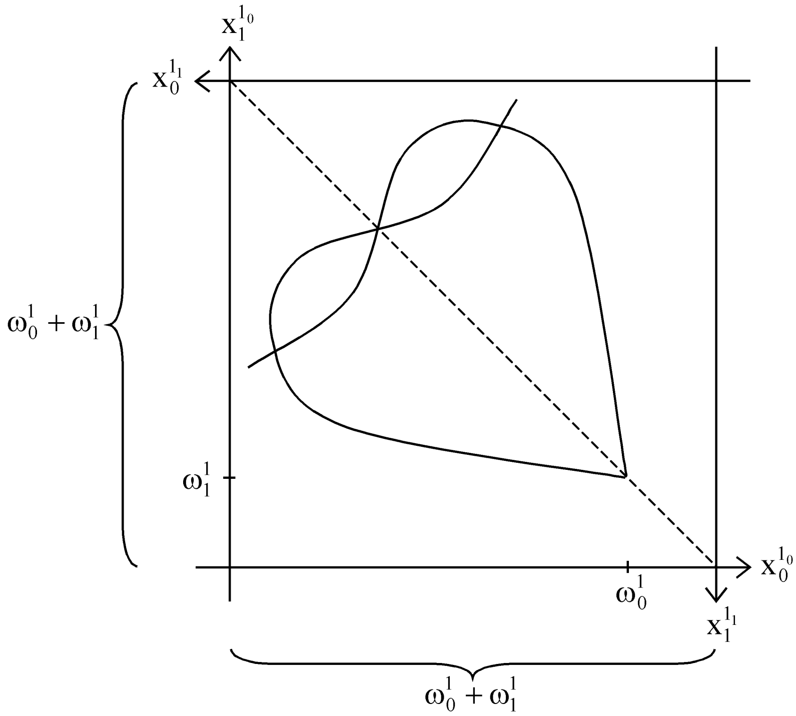

Figure 9.2: The (two-) modulo exchange economy for the simple OLG-model (compare Balasko and Ghiglino (1995, p. 576)).

where

$$
\begin{array}{rccl}
mod_N: & \mathbb{N}_0 & \to & \{0,1,\ldots,N-1\} \\
& n & \mapsto & mod_N(n) \\
& & & = \min\left\{r \in \{0,1,\ldots,N-1\} \mid n = \lfloor \frac{n}{N} \rfloor \cdot N + r\right\},
\end{array}
$$

i.e. with mod_N a natural number is mapped on what remains after division with remainder.

s_K shifts the last element in $\tilde{K}$ to the first position and all the others one position forward.

With respect to a fixed type $i \in I$ in the OLG-economy define $K+1$ different consumers by

$$
\tag{9.2}
\begin{array}{rccl}
\tilde{U}^{i_0}: & \underbrace{\mathbb{R}_+^L \times \ldots \times \mathbb{R}_+^L}_{K+1} & \to & \mathbb{R} \\
& (x_0,\ldots,x_K) & \mapsto & \tilde{U}^{i_0}(x_0,\ldots,x_K) := U^i(x_0,\ldots,x_N),
\end{array}
$$

$$\tilde{\omega}^{i_0} = \begin{pmatrix} \tilde{\omega}_0^{i_0} \\ \vdots \\ \tilde{\omega}_K^{i_0} \end{pmatrix} := \begin{pmatrix} \omega_0^i \\ \vdots \\ \omega_N^i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}_+^{(K+1)L}, \tag{9.3}$$

and in general for $k = 1, \ldots, K$

$$\begin{array}{rcl} \tilde{U}^{i_k}: \underbrace{\mathbb{R}_+^L \times \ldots \times \mathbb{R}_+^L}_{K+1} & \to & \mathbb{R} \\ (x_0, \ldots, x_K) & \mapsto & \tilde{U}^{i_k}(x_0, \ldots, x_K) := \\ & & U^i(x_{s_K^k(0)}, \ldots, x_{s_K^k(N)}), \end{array} \tag{9.4}$$

$$\tilde{\omega}^{i_k} = \begin{pmatrix} \tilde{\omega}^{i_0}_{s_K^{-k}(0)} \\ \vdots \\ \tilde{\omega}^{i_0}_{s_K^{-k}(K)} \end{pmatrix} \in \mathbb{R}_+^{(K+1)L} \tag{9.5}$$

where

$$s_K^0(n) = n$$

is the identity,

$$s_K^k(n) = \underbrace{s_K \circ \ldots \circ s_K}_{k-\text{times}}(n)$$

a k-fold right round shift and

$$s_K^{-k}(n) = \underbrace{s_K^{-1} \circ \ldots \circ s_K^{-1}}_{k-\text{times}}(n)$$

a k-fold left round shift.

These definitions transform each single type of OLG-consumer to $K+1$ Arrow-Debreu consumers.

Definition 9.5 ($(K+1)$-Modulo Exchange Economy)
The $(K+1)$-modulo exchange economy[9] *with respect to the OLG-economy $OLG = \{I, N, L; (U^i, \omega^i)_{i \in I}\}$ where $K \geq N$ is the $(K+1)I$-consumer $(K+1)L$-commodity Arrow-Debreu economy*

$$GE = \{(K+1)I, (K+1)L; (\tilde{U}^{i_k}, \tilde{\omega}^{i_k})_{k \in \tilde{K}, i \in I}\}$$

[9] Tuinstra and Weddepohl (1999) call these economies "cyclical exchange economies".

where $\tilde{U}^{i_k}$ and $\tilde{\omega}^{i_k}$ are defined by (9.2)–(9.5).[10]

The expression of a modulo exchange economy is appropriate since it reflects the construction of these economies: an infinite economy — the OLG-economy — is mapped on a finite economy — the modulo exchange economy — in a similar way as the modulo function maps the infinite set of natural numbers on a finite subset of the natural numbers.

Prices in GE are denoted by[11]

$$\tilde{p} = \begin{pmatrix} \tilde{p}_0 \\ \vdots \\ \tilde{p}_K \end{pmatrix} \in \mathbb{R}_{++}^{(K+1)L}.$$

Therefore, a consumer $(k, i) \in \tilde{K} \times I$ is confronted with the following maximization problem:

$$\max_{x \in \mathbb{R}_+^{(K+1)L}} \tilde{U}^{i_k}(x) \quad \text{s.t.} \quad \tilde{p} \cdot x = \tilde{p} \cdot \omega^{i_k}. \tag{9.6}$$

The solution is a function $x^{i_k}(\tilde{p})$. Note that x^{i_k} is a function which maps from $\mathbb{R}_{++}^{(K+1)L}$ to $\mathbb{R}_+^{(K+1)L}$.

Lemma 9.6
For the demand function in the $(K+1)$-modulo exchange economy for all $i \in I, k \in \tilde{K}, g \in \tilde{K}$ one gets:

$$\begin{aligned} (9.7)\quad & x_g^{i_k}(\tilde{p}_0, \ldots, \tilde{p}_K) \\ &= \begin{cases} 0, & \textit{if} \quad g \notin \{s_K^k(0), \ldots, s_K^k(N)\}, \\ x^i_{s_K^{-k}(g)}(\tilde{p}_{s_K^k(0)}, \ldots, \tilde{p}_{s_K^k(N)}), & \textit{otherwise,} \end{cases} \end{aligned}$$

where $x^i(p_0, \ldots, p_N) = \quad \textit{arg max}_{x \in \mathbb{R}_+^{(K+1)L}} U^i(x) \quad \textit{s.t.} \quad p \cdot x = p \cdot \omega^i;$

$$x_g^{i_k}(\tilde{p}_{s_K(0)}, \ldots, \tilde{p}_{s_K(K)}) \;=\; x_{s_K(g)}^{i_{s_K(k)}}(\tilde{p}_0, \ldots, \tilde{p}_K), \tag{9.8}$$

which reflects the construction of the modulo exchange economy.

Proof: See Appendix B.2.[12]

An equilibrium in the $(K+1)$-modulo exchange economy is defined as usual.

---

[10] Note that in contrast to previous parts I explicitly include the numbers of consumers and commodities.

[11] Beforehand, Arrow-Debreu price systems were denoted by π, but since later on I will get some strong relationship to the OLG-price systems I choose here the other name.

[12] Compare Tuinstra and Weddepohl (1999, Proposition 1 for $I = L = 1$ and Section 2.6 for $L = 1$ or $I = 1$).

Definition 9.7 (Equilibrium of the Modulo Exchange Economy)
An (Arrow-Debreu) equilibrium in the $(K+1)$-modulo exchange economy GE is a price system $\tilde{p}^* \in \mathbb{R}_{++}^{(K+1)L}$ and an allocation $(x^{*i_k})_{k\in\tilde{K},i\in I}$ such that

1. $x^{*i_k} = x^{i_k}(\tilde{p}^*)$ *for all* $k \in \tilde{K}, i \in I$,

2. $$\sum_{k\in\tilde{K}}\sum_{i\in I} x^{*i_k} = \sum_{k\in\tilde{K}}\sum_{i\in I}\tilde{\omega}^{i_k} = \begin{pmatrix} \sum_{n\in\tilde{N}}\sum_{i\in I}\omega_n^i \\ \vdots \\ \sum_{n\in\tilde{N}}\sum_{i\in I}\omega_n^i \end{pmatrix}.$$

Following the results from Arrow-Debreu theory it is known that in each $(K+1)$-modulo exchange economy there exists an equilibrium. Application of Lemma 9.6 allows to get the following result for $L=1$:

Lemma 9.8
If $L=1$, the price system $\tilde{p} = (1,\ldots,1)^T \in \mathbb{R}_{++}^{K+1}$ is equilibrium price system of the modulo exchange economy.

Proof: I have to prove

$$\sum_{k\in\tilde{K}}\sum_{i\in I} x_g^{i_k}(1,\ldots,1) = \sum_{n\in\tilde{N}}\sum_{i\in I}\omega_n^i$$

for all $g \in \tilde{K}$. However, applications of Lemma 9.6 show

$$x_g^{i_k}(1,\ldots,1) = x_{s_K^{-k}(g)}^{i_0}(1,\ldots,1).$$

Therefore, for all $g \in \tilde{K}$:

$$\begin{aligned}\sum_{k\in\tilde{K}} x_g^{i_k}(1,\ldots,1) &= \sum_{k\in\tilde{K}} x_{s_K^{-k}(g)}^{i_0}(1,\ldots,1)\\ &= \sum_{\tilde{g}\in\tilde{K}} x_{\tilde{g}}^{i_0}(1,\ldots,1)\\ &= \sum_{\tilde{g}\in\tilde{K}}\tilde{\omega}_{\tilde{g}}^{i_0}\\ &= \sum_{n\in\tilde{N}}\omega_n^i\end{aligned}$$

where the third equality follows from consumer i_0's budget constraint.[13]

□

[13]Compare Tuinstra and Weddepohl (1999, Proposition 2 for $I=1$ and Section 2.6).

Remark 9.9
The following additional observations can be made:

1. *An equilibrium price system with equal prices for all goods means that the marginal rates of substitution are equal to one, independently of the goods. In such an equilibrium consumers of different basic types $i \in I$ do not trade with each other since — because of the symmetries in the considered economy — the marginal rates of substitution can be balanced out within the single types.*

2. *In the $(N+1)$- and hence in each $(K+1)$-modulo exchange economy for all $K \geq N$ Tuinstra and Weddepohl (1999, Section 2.6) also prove under the restriction $I = 1$ the existence of an equilibrium of which the price vector is a replication of an L-dimensional vector. The idea of the proof is to solve*

$$\max_{x \in \mathbb{R}_+^{(N+1)L}} U^1(x) \text{ s.t. } \sum_{n \in \tilde{N}} x_n = \sum_{n \in \tilde{N}} \omega_n^1$$

or equivalently

$$\max_{x \in \mathbb{R}_+^{NL}} U^1\Big(\sum_{n \in \tilde{N}} \omega_n^1 - \sum_{n \in N} x_n, x_1, \ldots, x_N\Big)$$

and

$$x_0 = \sum_{n \in \tilde{N}} \omega_n^1 - \sum_{n \in N} x_n.$$

Choosing for all $n \in \tilde{N}, l \in L$

$$\tilde{p}_{nl} = \partial_{x_{0l}} U^1(\hat{x})$$

where $\hat{x}$ solves the maximization problem, defines a price system displaying the equilibrium properties and being replication of an L-dimensional vector.

In general, the following result on the equilibrium set of the $(K+1)$-modulo exchange economy holds.

Lemma 9.10
If $(\tilde{p}^*, (x^{*i_k})_{k \in \tilde{K}, i \in I})$ is an equilibrium of the $(K+1)$-modulo exchange economy, this is also true for $(\tilde{p}', (x'^{i_k})_{k \in \tilde{K}, i \in I})$, defined by

$$\tilde{p}'_g = \tilde{p}^*_{s_K^m(g)} \tag{9.9}$$

$$x'^{i_k}_g = x^{*i_{s_K^m(k)}}_{s_K^m(g)} \tag{9.10}$$

for all $m \in \tilde{K}$.

Proof: Follows directly by several applications of Lemma 9.6.[14]

□

It would be interesting to relate equilibrium sets for different modulo exchange economies to each other. Therefore, consider a $(K+1)$- and a $(K'+1)$-modulo exchange economy with $K' > K$.

Lemma 9.11
If $\tilde{p}^* = (\tilde{p}_0^*, \ldots, \tilde{p}_K^*) \in \mathbb{R}_{++}^{(K+1)L}$ is an equilibrium price system of the $(K+1)$-modulo exchange economy, then its m-repetition is always equilibrium price system in the $(K'+1)$-modulo exchange economy where $K'+1 = (K+1)m$.

Proof: The result is quite obvious when comparing the extended system for the $(K+1)$- and the $(K'+1)$-modulo exchange economy. The extended system is the system of equations of first order conditions for consumers' maximization problems and market clearing conditions, see equations (2.7), (2.8), and (2.9).[15]

□

For $L = N = 1$ a stronger result can be proved. Its importance will become clear when I examine the number of cycles in OLG-economies later on.

Lemma 9.12
Let $L = N = 1$. If $\tilde{p}^* = (\tilde{p}_0^*, \ldots, \tilde{p}_{K'}^*) \in \mathbb{R}_{++}^{K'+1}$ is an equilibrium price system of the $(K'+1)$-modulo exchange economy and if it is $\tilde{p}_{b'}^* = \tilde{p}_b^*$ for some $b', b \in \tilde{K}'$ with $b' < b$, then:

1. *if $b - b' > 1$, then $(\tilde{p}_{b'}^*, \ldots, \tilde{p}_{b-1}^*)$ is an equilibrium price system of the $b-b'$-modulo exchange economy;*
2. *if $K' > b - b'$, then $(\tilde{p}_b^*, \ldots, \tilde{p}_{K'}^*, \tilde{p}_0^*, \ldots, \tilde{p}_{b'-1}^*)$ is an equilibrium price system of the $(K' - (b - b') + 1)$-modulo exchange economy.*

Proof: See Appendix B.2.

9.3 Identification of Arrow-Debreu Equilibria and Periodic OLG-Equilibria

In this section I relate the periodic equilibria of OLG-economies to the Arrow-Debreu equilibria in the corresponding $(K+1)$-modulo exchange economies.

[14] Compare Tuinstra and Weddepohl (1999, Proposition 1 for $I = L = 1$ and Section 2.6 for $L = 1$ or $I = 1$).

[15] Compare Tuinstra and Weddepohl (1999, p. 191).

Especially, the relations between the number of cycles and the number of Arrow-Debreu equilibria allow to exploit the theory of uniqueness of equilibria in Arrow-Debreu theory for OLG-theory. This will be done in the next chapter.

At first, I have a look on monetary stationary equilibria in the OLG-economies.

Theorem 9.13 (Equivalence for Monetary Stationary Equilibria)
$((c^{*i})_{i\in I}, p^*)_{t\in\mathbb{Z}}$ is a monetary stationary OLG-equilibrium if and only if $((c^{*i}_0, \ldots, c^{*i}_N), (c^{*i}_1, \ldots, c^{*i}_N, c^{*i}_0), \ldots, (c^{*i}_N, c^{*i}_0, \ldots, c^{*i}_{N-1}), (p^*, \ldots, p^*))$ is an equilibrium of the $(N+1)$-modulo exchange economy.

Proof: The equivalence of the market clearing properties is obvious. The equivalence of the maximization problems follows by application of Lemma 9.6.[16]

□

The following corollary is used to apply results on uniqueness of equilibria in Arrow-Debreu theory. Notice that concerning the number of monetary stationary equilibria as well as the number of Arrow-Debreu equilibria in the modulo exchange economies there is one degree of price normalization left.[17] Therefore, I have to do a price normalization.[18]

Corollary 9.14 (Number of Monetary Stationary Equilibria)
The number of monetary stationary OLG-equilibria with $p^{*t} \cdot \sum_{n\in\tilde{N}} \omega_n = 1$ is equal to the number of Arrow-Debreu equilibria in the $(N+1)$-modulo exchange economy satisfying $\tilde{p}_0 = \ldots = \tilde{p}_N$ and $\tilde{p}_0 \cdot \omega_0 + \ldots + \tilde{p}_N \cdot \omega_N = 1$ where $\omega_n = \sum_{i\in I} \omega^i_n$ for all $n \in \tilde{N}$.

The proof is obvious.

Remark 9.15
Theorem 9.13 and its Corollary 9.14 can be reformulated for an arbitrary $(K+1)$-modulo exchange economy with $K \geq N$. I choose this presentation because the $(N+1)$-modulo exchange economy is the simplest one.

What concerns periodic OLG-equilibria, I obtain the following result.

---

[16] This result is contained in Tuinstra and Weddepohl (1999, Theorem 1 for $I = L = 1$ and Section 2.6 for $L = 1$ and $I = 1$).

[17] Compare Definition 3.3 for uniqueness in Chapter 3.

[18] In the following, I always pass over this necessary normalization in silence and do not mention it each time. For a discussion of price normalization in the OLG-framework consider Geanakoplos (1987b, p. 769).

Theorem 9.16 (Equivalence for Periodic Equilibria)
The price system $(\ldots, p_0^*, \ldots, p_K^*, \ldots)$ *and the allocations* $(\ldots, (c^{*i,0})_{i \in I}, \ldots, (c^{*i,K})_{i \in I}, \ldots)$ *constitute an OLG-equilibrium of period* $(K+1) \geq (N+1)$ *if and only if* $((x^{*i_k})_{k \in \tilde{K}, i \in I}, (p_0^*, \ldots, p_K^*))$ *is an Arrow-Debreu equilibrium in the* $(K+1)$*-modulo exchange economy where*

$$x^{*i_k}_{\ g} = 0, \quad \text{if} \quad g \notin \{s_K^k(0), \ldots, s_K^k(N)\}, \tag{9.11}$$

$$x^{*i_k}_{\ g} = c^{*i,k}_{s_K^{-k}(g)}, \quad \text{otherwise.} \tag{9.12}$$

Proof: Again, the equivalence of the maximization problems follows by application of Lemma 9.6. Market clearing is a simple computation.[19]

□

Comparing the number of Arrow-Debreu equilibria and cycles I obtain:

Corollary 9.17 (Number of Periodic Equilibria)
$(K+1)$*-times the number of OLG-equilibria with exact period* $K+1$ *and* $(p_0^*, \ldots, p_K^*) \cdot \mathbb{1} = 1$ *is equal to the number of Arrow-Debreu equilibria in the* $(K+1)$*-modulo exchange economy with* $\tilde{p}_k \neq \tilde{p}_{k'}$ *for all* $k, k' \in \tilde{K}$ *with* $k \neq k'$ *and* $\tilde{p} \cdot \mathbb{1} = 1$.

Corollary 9.17 looses its validity when deviating from the assumption of exact periods.

Corollary 9.18 (Necessary Condition for Periodic Equilibria)
A necessary condition for the existence of OLG-equilibria with exact period $K+1$, $K \in \mathbb{N}$ *are multiple equilibria in the* $(K+1)$*-modulo exchange economy.*

With Corollary 9.17 and Lemma 9.10 the proof is trivial.

Notice that this approach via modulo exchange economies for the study of periodic equilibria only is possible for those cycles with length equal or bigger than life-cycle length or being a divisor of this. These are those mainly studied in OLG-theory. For cycles of other lengths compare Aiyagari (1989).

The results of this chapter are applied in the following one to obtain explicit conditions on OLG-economies which, for example, guarantee a unique monetary stationary equilibrium or exclude cycles.

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[19] Compare Tuinstra and Weddepohl (1999, Theorem 1 for $I = L = 1$ and Section 2.6 for $L = 1$ or $I = 1$).

Chapter 10

On Monetary Stationary and Periodic Equilibria in OLG-Models

Before I use the relations between Arrow-Debreu equilibria in $(K+1)$-modulo exchange economies and periodic OLG-equilibria which I proved before, recall the Index Theorem 3.6 (Dierker (1972)) which assures that as soon as a $(K+1)$-modulo exchange economy is regular[1], there is an odd number of normalized equilibrium price systems and the indices[2] sum up to 1.[3]

The first results are on monetary stationary OLG-equilibria.

Proposition 10.1 (Odd Number of Mon. Stationary Equilibria) *If $N = 1$ and the two-modulo exchange economy is regular, then there exists an odd number of monetary stationary OLG-equilibria.*

---

[1]An Arrow-Debreu economy is regular if the normalized Jacobian matrix of market demand has maximal rank for all equilibrium price systems (compare Definition 3.4).

[2]The index of an equilibrium (price system) is $+1$ if the determinant of the normalized Jacobian matrix of market demand for this price system has negative sign and -1 if it is positive (compare page 45).

[3]According to Tuinstra and Weddepohl (1999, p. 190), $(K+1)$-modulo exchange economies are generically regular. However, I think that this point has to be clarified: normally, generic regularity is proved by disturbing one consumer's endowment in an arbitrary direction. Here one is restricted to perturbations within a special class because one has to assure to stay in the class of modulo exchange economies. Some notes on regularity in OLG-models can be found in Kehoe and Levine (1984b), but they are not applicable to my context. The problem of suitable perturbations in modulo exchange economies is similar to this problem in sunspot economies (for example, Cass (1992)) and economies without aggregate risk (for example, Siconolfi and Villanacci (1991)) where endowments cannot be disturbed arbitrarily.

Proof: If $N = 1$, according to Lemma 9.10 to each equilibrium $(\tilde{p}_0, \tilde{p}_1)$ in the modulo exchange economy which does not satisfy $\tilde{p}_0 = \tilde{p}_1$, there exists another equilibrium, namely the equilibrium with price system $(\tilde{p}_1, \tilde{p}_0)$. However, since the total number of equilibria has to be odd, there exists at least 1, but surely an odd number of equilibria which are replications of L-dimensional vectors and allow to construct a monetary stationary OLG-equilibrium (compare Theorem 9.13).

□

Notice that this result implies for $N = 1$ that as soon as the 2-modulo exchange economy is regular, the OLG-economy has a monetary stationary equilibrium.

This result is weaker than the corresponding result in Kehoe and Levine (1984b, Proposition 3.2) because they do not need the restriction to regularity for their existence result. Nevertheless, my proof is quite simple.

A further result on the existence and even uniqueness of monetary stationary equilibria is given by the following proposition.

Proposition 10.2 (Sufficient Cond. for Unique Stat. Equil.)
As soon as the $(N+1)$-modulo exchange economy has a unique equilibrium with a price system being replication of an L-dimensional vector, there exists a unique monetary stationary OLG-equilibrium.

Proof: Follows from Corollary 9.14.

□

Therefore, in the next step I will formulate conditions such that the $(N+1)$-modulo exchange economy has a unique equilibrium (or at least a unique one where the price system is replication of an L-dimensional vector).

Theorem 10.3 (Sufficient Cond. for Unique Stat. Equil.)
The OLG-economy $OLG = \{I, N, L; (U^i, \omega^i)_{i \in I}\}$ has a unique monetary stationary OLG-equilibrium if one of the following conditions is satisfied (conditions formulated for some $i \in I$ should be valid for all $i \in I$):

(1) $L = 1$;

(2) $I = 1$;

(3) $U^i(x) = \sum_{n \in \tilde{N}} \sum_{l \in L} \rho^i_{nl} u^i(x_{nl})$ *and* $RRA^i(y) := -\frac{(u^i)''(y) \cdot y}{(u^i)'(y)} \leq 1$ *for all* $y \in \mathbb{R}_{++}$ *where* $u^i : \mathbb{R}_+ \to \mathbb{R}$ *satisfies standard assumptions and* $\rho^i \in \mathbb{R}^{(N+1)L}_{++}$;

(4) $U^i(x) = \sum_{n\in\tilde{N}} \delta^n V(x_n)$ *where* $\delta \in \mathbb{R}_{++}$ *and* $V : \mathbb{R}^L_+ \to \mathbb{R}$ *is an additively separable homothetic utility function* $V(x_n) = \sum_{l=1}^L v_l(x_{nl})$ *where* $v_l : \mathbb{R}_+ \to \mathbb{R}$ *satisfy standard assumptions and* v_l' *are homogeneous of degree* $d \neq 0$[4]*;*

(5) $U^i(x) = \sum_{n\in\tilde{N}} \delta^n V^i(x_n)$ *where* $\delta \in \mathbb{R}_{++}$ *and* $V^i : \mathbb{R}^L_+ \to \mathbb{R}$ *is an additively separable homothetic utility functions* $V^i(x_n) = \sum_{l=1}^L v_l^i(x_{nl})$ *where* $v_l^i : \mathbb{R}_+ \to \mathbb{R}$ *satisfy standard assumptions and* $v_l^{i\prime}$ *are homogeneous of degree* $d \neq 0$ *and the* $\sum_{n\in\tilde{N}} \omega_n^i, i \in I$*, are collinear;*

(6) $U^i(x) = \sum_{n\in\tilde{N}} V(x_n)$ *is a homothetic function where* $V : \mathbb{R}^L_+ \to \mathbb{R}$ *satisfies standard assumptions;*

(7) $U^i : \mathbb{R}^{(N+1)L}_+ \to \mathbb{R}$ *is a homothetic function satisfying standard assumptions,* $\omega_0^i = \ldots = \omega_N^i$ *for all* $i \in I$*, and the* $\omega_n^i, i \in I$*, are collinear;*

(8) $U^i : \mathbb{R}^{(N+1)L}_+ \to \mathbb{R}$ *satisfies standard assumptions, is strictly concave with* $MP_{U^i}(x) := -\frac{x^T D^2 U^i(x) x}{\nabla U^i(x) x} < 4$ *for all* $x \in \mathbb{R}^{(N+1)L}_{++}$, $\omega_0^i = \ldots = \omega_N^i$ *for all* $i \in I$*, and the* $\omega_n^i, i \in I$*, are collinear;*

(9) $U^i(x) = \sum_{n\in\tilde{N}} V^i(x_n)$ *and* $V^i(y) = \sum_{l\in L} \bar{\rho}_l v^i(y_l)$ *with* $v^i : \mathbb{R}_+ \to \mathbb{R}$ *satisfying standard assumptions,* $\bar{\rho} \in \mathbb{R}^L_{++}$*, and* $\sum_{i\in I}\sum_{n\in\tilde{N}} \omega_{nl}^i$ *is constant for all* $l \in L$*;*

(10) $U^i(x) = \sum_{n\in\tilde{N}} V^i(x_n)$ *and* $V^i(y) = \sum_{l\in L} \bar{\rho}_l(y_l - \frac{1}{2}\gamma^i y_l^2)$ *where* $\bar{\rho} \in \mathbb{R}^L_{++}$ *and* $\gamma^i \in \mathbb{R}_+$ *with* $1 - \gamma^i \sum_{i\in I}\sum_{n\in\tilde{N}} \omega_{nl}^i > 0$ *for all* $l \in L$*.*

Proof: See Appendix B.2.

Remark 10.4

1. *The cases (1)* $L = 1$ *and (2)* $I = 1$ *are often used in OLG-theory.*
2. *In cases (3) and (6)–(10) the* $(N+1)$*-modulo exchange economy is characterized by a unique equilibrium. Hence, any kind of OLG-equilibrium with period* P *where* $P = \frac{N+1}{m}$ *for some element* $m \in \{1, \ldots, N\}$ *is excluded (compare Theorem 9.16 and Lemma 9.11).*

[4]This is especially satisfied if v_l is homogeneous of degree unequal to one, but the requirement in the theorem is more general because, for example, $v_l(y) = \rho_l \ln y$ is "only" homothetic, but its derivative is homogeneous: $v_l{}'(\lambda y) = \frac{\rho_l}{\lambda y} = \lambda^{-1} v_l{}'(y)$.

For $I = N = L = 1$ this point is very important: cycles of length two are excluded. As presented by Grandmont (1985, Theorem 4.3) Sarkowski's Theorem then assures that there does not exist an OLG-equilibrium of any period greater than two. He makes Assumptions Grandmont 9.3.[5]

3. *Case (9) for $I = N = L = 1$ clarifies why Balasko and Ghiglino (1995, Proposition 2) have to assume a discount factor less than 1 to assure that there are endowment vectors for which OLG-equilibria with exact period two exist. If the discount factor is equal to 1, the proof of case (9) via uniqueness in the two-modulo exchange economy makes clear that there cannot exist a two-cycle, no matter what is the endowment vector.*

4. *What concerns (10), Example 3 in Gale (1973) shows that to exclude cycles it is necessary to assume the same factor of risk aversion γ^i for all commodities because otherwise the $(N + 1)$-modulo exchange economy does not necessarily have a unique equilibrium.*[6]

Before considering some general results on periodic OLG-equilibria I present two classes of economies for which, even though there is no restriction such as $I = N = L = 1$, only a unique monetary stationary equilibrium and no kind of cycle exist or at least cycles up to a prespecified length are excluded. For the first class Ghiglino and Tvede (1995) assure a unique monetary stationary equilibrium under the further restriction to $N = 1$, that is their result holds only in a two-period framework. Notice that using the modulo exchange economy enables only statements on monetary stationary equilibria and not on balanced ones.

Proposition 10.5 (Implications of Pareto-efficient Endowments)
Let $(c^{i,t} = \omega^i)_{i\in I,t\in\mathbb{Z}}$ be a Pareto-efficient allocation in the economy $OLG = \{I, N, L; (U^i, \omega^i)_{i\in I}\}$. Then this is the unique monetary stationary equilibrium and there exist no $(K + 1)$-cyclical equilibria with $K \geq N$. For given $\bar{K} \geq N$ there is a neighborhood of $(\omega^i)_{i\in I}$ such that all elements $(\hat{\omega}^i)_{i\in I}$ of this neighborhood generate other economies $OLG = \{I, N, L; (U^i, \hat{\omega}^i)_{i\in I}\}$

[5]From Tuinstra and Weddepohl (1999, Proposition 3) it becomes clear that the Assumptions Gale 9.4 are also sufficient.

Tuinstra and Weddepohl (1999) themselves do not name these explicit restrictions on the utility functions and the endowments which guarantee the fulfillment of the requirements on excess demand in their Proposition 3.

[6]Gale's (1973) example demonstrates that two-cycles might exist in OLG-economies with $I = N = L = 1$.

which also display a unique monetary stationary OLG-equilibrium and for none of these economies there exists any periodic equilibrium with period $K+1 \in [N+1, \bar{K}+1]$.

Pareto-efficiency in an OLG-model is defined analogously to the Arrow-Debreu model.

Definition 10.6 (Pareto-Efficiency)
An allocation $(c^{i,t})_{i\in I, t\in\mathbb{Z}}$ *with* $\sum_{n\in\tilde{N}}\sum_{i\in I} c_n^{i,t-n} \leq \sum_{n\in\tilde{N}}\sum_{i\in I}\omega_n^i$ *for all* $t \in \mathbb{Z}$ *is Pareto-efficient in the economy* $OLG = \{I, N, L; (U^i, \omega^i)_{i\in I}\}$ *if there is no other allocation* $(\bar{c}^{i,t})_{i\in I, t\in\mathbb{Z}}$ *such that*

1. $\sum_{n\in\tilde{N}}\sum_{i\in I} \bar{c}_n^{i,t-n} \leq \sum_{n\in\tilde{N}}\sum_{i\in I}\omega_n^i$ *for all* $t \in \mathbb{Z}$ *and*
2. $U^i(\bar{c}^{i,t}) \geq U^i(c^{i,t})$ *for all* $i \in I$ *and* $t \in \mathbb{Z}$

where at least one of the last inequalities is strict.

The proposition can be simply proved by the following lemma.

Lemma 10.7
Given an economy $OLG = \{I, N, L; (U^i, \omega^i)_{i\in I}\}$ *such that* $(c^{i,t} = \omega^i)_{i\in I, t\in\mathbb{Z}}$ *is Pareto-efficient, for all* $K \geq N$ *the allocation* $(\tilde{\omega}^{i_0}, \ldots, \tilde{\omega}^{i_K})_{i\in I}$ *is Pareto-efficient in the corresponding* $(K+1)$*-modulo exchange economy.*

Proof: See Appendix B.2.

Then the proof of the proposition works as follows.

Proof of Proposition 10.5: Since $(c^{i,t} = \omega^i)_{i\in I, t\in\mathbb{Z}}$ is Pareto-efficient in the OLG-economy, Lemma 10.7 implies Pareto-efficiency of $(\tilde{\omega}^{i_0}, \ldots, \tilde{\omega}^{i_K})_{i\in I}$ in the $(K+1)$-modulo exchange economy for all $K \geq N$. However, Pareto-efficient endowments in an Arrow-Debreu economy imply a unique regular Arrow-Debreu equilibrium (compare Proposition 3.12 and Lemma 4.37). The Implicit Function Theorem can be applied to obtain a neighborhood $\mathcal{V}_K$ of Arrow-Debreu endowments for which the Arrow-Debreu economy with the same utility functions and endowments out of this neighborhood displays a unique equilibrium. Choosing a sufficiently small neighborhood of $(\omega^i)_{i\in I}$ such that the corresponding constructed endowments for the modulo exchange economy are elements of $\mathcal{V}_K$ and taking the finite intersection over all $K \in \{N, \ldots, \bar{K}\}$, it only remains to apply Proposition 10.2 and Corollary 9.18.

□

For the second class of OLG-economies for which I can exclude each kind of cycle I have already shown in Theorem 10.3 that there exists a unique monetary stationary equilibrium. For the following result compare also Kehoe et al. (1991) whose proof is much more tedious.

Proposition 10.8 (Implications of Small Rel. Risk Aversion)
If all consumers are characterized by utility functions

$$U^i(x) = \sum_{n \in \tilde{N}} \sum_{l \in L} \rho^i_{nl} u^i(x_{nl})$$

where $\rho^i \in \mathbb{R}^{(N+1)L}_{++}$, $u^i : \mathbb{R}_+ \to \mathbb{R}$ *satisfies standard assumptions and* $RRA^i(y) \leq 1$ *for all* $y \in \mathbb{R}_{++}$, *then the OLG-economy does not have any OLG-equilibrium of exact period* $P > 1$.

Proof: See Appendix B.2.

Now, I will have a look at general results on periodic OLG-equilibria. The first result concerns periodic equilibria of odd length.[7]

Proposition 10.9 (Even Number of Odd Cycles)
If the $(2P+1)$*-modulo exchange economy is regular for some* $P \in \mathbb{N}$, *then OLG-equilibria of exact period* $2P+1$ *always occur in even numbers when the OLG-economy satisfies either Assumptions Grandmont 9.3 or Assumptions Gale 9.4.*[8]

Note that Assumptions Grandmont 9.3 as well as Assumptions Gale 9.4 require $I = N = L = 1$.

Before I can prove the proposition, I need to verify the following two lemmas.

Lemma 10.10
If the OLG-economy satisfies Assumptions Grandmont 9.3 or Assumptions Gale 9.4, then each equilibrium of a $K'+1$*-modulo exchange economy corresponds either to one cyclical OLG-equilibrium of exact period* $K+1$ *such that* $\frac{K'+1}{K+1} \in \mathbb{N}$ *or to the monetary stationary equilibrium.*

Proof: See Appendix B.2.

A decisive difference to the results of Tuinstra and Weddepohl (1999) consists in the fact that I can exclude Arrow-Debreu equilibria in the modulo exchange economy which correspond to a mixture of OLG-equilibria of different exact period.[9]

[7]The idea was developed in a fruitful discussion together with Julio Dávila.

[8]As mentioned beforehand, Grandmont (1985) gives this result under Assumptions Grandmont 9.3 for $P = 1$.

[9]For example, in a three-modulo exchange economy equilibria with price systems such as $(1, 1, 2)$ are excluded.

Lemma 10.11 refers to regularity.

Lemma 10.11
If the OLG-economy satisfies Assumptions Grandmont 9.3 or Assumptions Gale 9.4 and the $(K'+1)$-modulo exchange economy is regular, i.e. for each Arrow-Debreu equilibrium the normalized Jacobian of excess demand has rank K', then any $(K+1)$-modulo exchange economy with $\frac{K'+1}{K+1} \in \mathbb{N}$ is regular.

Proof: See Appendix B.2.

Now, let me prove the proposition by induction.

Proof of Proposition 10.9: The proposition is proved by induction. The induction hypothesis is established by proving the proposition for $P = 1$.

Since the three-modulo exchange economy is regular, the Index Theorem implies an odd number of equilibria. Exactly one of them exhibits constant prices, i.e. $(\tilde{p}_0, \tilde{p}_1, \tilde{p}_2)$ with $\tilde{p}_0 = \tilde{p}_1 = \tilde{p}_2$. Besides, by Theorem 9.16 and Lemma 9.10 each OLG-equilibrium of exact period three, which is not stationary, corresponds to three different equilibria in the three-modulo exchange economy. So, if there were an odd number of three-cycles not equal to the monetary stationary equilibrium, these cycles would correspond to an odd number of Arrow-Debreu equilibria in the three-modulo exchange economy. According to Lemma 10.10 all equilibria in this economy have to correspond to OLG-equilibria of exact period three or monetary stationary equilibria. Therefore, the three-modulo exchange economy would have an even number of equilibria which contradicts the Index Theorem 3.6.

Lemma 10.11 assures regularity of all $(2P'+1)$-cyclical exchange economies with $P' \in \{1, \ldots, P\}$ and $\frac{2P+1}{2P'+1} \in \mathbb{N}$. Therefore, I get the following induction hypothesis:

> OLG-equilibria of exact period $2P'+1, P' \in \{1, \ldots, P-1\}$ with $\frac{2P+1}{2P'+1} \in \mathbb{N}_{>1}$, always appear in even numbers (even also including zero if there is no such cycle).

Now, look on the $(2P+1)$-modulo exchange economy with $P > 1$. This economy is regular. According to Lemma 10.10 each equilibrium of the $(2P+1)$-modulo exchange economy corresponds to an OLG-equilibrium of exact period k' with $\frac{2P+1}{k'} \in \mathbb{N}$. Therefore, $k' = 2P'+1$ for some $P' \in \{0, \ldots, P\}$.

$P' = 0$ corresponds to the monetary stationary equilibrium which accounts as one equilibrium in the $(2P+1)$-modulo exchange economy.

For $P' \in \{1, \ldots, P-1\}$ with $\frac{2P+1}{2P'+1} \in \mathbb{N}$ by induction hypothesis there is an even number of OLG-equilibria of exact period $2P'+1$ which corresponds to an even number of equilibria in the $(2P+1)$-modulo exchange economy.

So, up-to-now there is an odd number of equilibria in the $(2P+1)$-modulo exchange economy. Therefore, the number of Arrow-Debreu equilibria in this economy, generated by OLG-equilibria of exact period $2P+1$, has to be even. This requires an even number of such cycles.

$\square$

It would be nice to obtain similar results as in Proposition 10.9 for OLG-equilibria of exact even period. At least for two-cycles this is possible.

Proposition 10.12 (On the Number of Two-Cycles)
If Assumptions Grandmont 9.3 are satisfied and the two-modulo exchange economy is regular, then the OLG-economy is characterized by an even (including zero) number of OLG-equilibria with exact period two if and only if the slope of the consumer's offer curve in the monetary stationary equilibrium is (strictly) greater than one or less than zero.[10]

If Assumptions Gale 9.4 are satisfied and the two-modulo exchange economy is regular, then the OLG-economy is characterized by an even (including zero) number of OLG-equilibria with exact period two if and only if the slope of the consumer's offer curve in the monetary stationary equilibrium is (strictly) less than one.

This proposition implies that under Assumptions Grandmont 9.3 a positive slope less than one in the monetary stationary equilibrium is equivalent to an odd number of OLG-equilibria of exact period two. Under Assumptions Gale 9.4 the same is true for a slope greater than one.

I get the following corollary concerning the existence of two-cycles which — as mentioned beforehand — for Assumptions Grandmont 9.3 is given in Grandmont (1985) and Balasko and Ghiglino (1995). The proof is obvious from the proposition.

Corollary 10.13 (Sufficient Condition for Two-Cycles)
Under Assumptions Grandmont 9.3 a sufficient condition for the existence of an OLG-equilibrium of exact period two is a consumer's offer curve with slope less than one and greater than zero at the monetary stationary equilibrium.

Under Assumptions Gale 9.4 a sufficient condition for the existence of an OLG-equilibrium of exact period two is a consumer's offer curve with slope greater than one at the monetary stationary equilibrium.

To prove the proposition I have to apply the following lemmas. Note that I give the proofs only for Assumptions Grandmont 9.3. The proofs for Assumptions Gale 9.4 are similar.

[10] If it were equal to one, the two-modulo exchange economy would not be regular.

Lemma 10.14
Under Assumptions Grandmont 9.3 the slope of consumer's offer curve is greater than one or less than zero if and only if the Arrow-Debreu equilibrium with price system $(1,1)$ *in the two-modulo exchange economy is regular and has a positive index.*

Under Assumptions Gale 9.4 the slope of consumer's offer curve is less than one if and only if the Arrow-Debreu equilibrium with price system $(1,1)$ *in the two-modulo exchange economy is regular and has a positive index.*

Proof: See Appendix B.2.

Lemma 10.15
If $I = N = L = 1$*, then the Arrow-Debreu equilibria in the two-modulo exchange economy which correspond to the same periodic OLG-equilibrium are characterized by the same index.*

Proof: See Appendix B.2.

From the proof of this lemma it is clear that it remains valid for more general OLG-models and corresponding modulo exchange economies.

Now, let me prove the proposition.

Proof of Proposition 10.12: Since the two-modulo exchange economy is regular, the Index Theorem can be applied to obtain that the sum of the indices of all equilibria is equal to $+1$. Now assume Assumptions Grandmont 9.3.

If the slope of the consumer's offer curve at the monetary stationary equilibrium is greater than one or less than zero, following Lemma 10.14 this is equivalent to a positive index of the Arrow-Debreu equilibrium with price system $(1,1)$ in the two-modulo exchange economy.

Lemma 10.15 together with Lemma 9.10 implies that the sum of all indices of Arrow-Debreu equilibria in the two-modulo exchange economy which correspond to the same OLG-equilibrium of exact period two is equal to ± 2.

Let M_+ resp. $M_- \in \mathbb{N}_0$ be the number of OLG-equilibria of exact period two which correspond to a positive resp. negative index in the two-modulo exchange economy. Then, the before-mentioned results including the Index Theorem imply

$$\begin{aligned}
& \text{slope of the offer curve in the mon. stationary equilibrium } \notin [0,1] \\
\Leftrightarrow\ & \text{index}(1,1) = +1 \\
\Leftrightarrow\ & 2M_+ + (-2)M_- + 1 = +1 \\
\Leftrightarrow\ & M_+ = M_- \\
\Leftrightarrow\ & \text{even number of OLG-equilibria of exact period 2}
\end{aligned}$$

The proof for Assumptions Gale 9.4 is similar.

□

Besides it is worth mentioning that examples on the existence of any periodic OLG-equilibrium (compare, for example, Benhabib and Day (1982)) can be used to construct explicit examples of Arrow-Debreu economies with an arbitrary number of Arrow-Debreu equilibria.

Let me finish this part with some remarks on the implications which the results presented in this part have on the existence of stationary sunspot equilibria. This again emphasizes the relations between multiple equilibria in static general equilibrium models, the existence of sunspot equilibria in general equilibrium models, and the existence of cycles in OLG-models.

As indicated before, the existence of sunspot equilibria is not only analyzed in finite dimensional models as described in Part III, but also in infinite dimensional models. Many authors consider two-period OLG-models, extended by extrinsic uncertainty, which can be embedded in a more general infinite sunspot framework, as it is described by Chiappori and Guesnerie (1991).

Extension of the two-period OLG-model to include extrinsic uncertainty can be done in analogy to the one in finite dimensional models: agents are faced with an expected utility for their "old" consumption; expectations are formed with respect to objective probability measures which depend on the current state and are summarized in a Markov matrix. However, many economists (for example, Guesnerie (1986)) describe the "certainty" OLG-model and its sunspot extension in terms of excess demand functions which is more general.

Economists mainly concentrate on stationary sunspot equilibria and do not examine arbitrary sunspot equilibria in this infinite model for two reasons: "stationary sunspot beliefs are more likely to emerge from learning processes [...] and the understanding of stationary sunspot equilibria may be a decisive step for the understanding of the phenomenon in its full generality" (Guesnerie (1986, p. 105)).

A lot of research has been done to find out which are sufficient conditions in terms of periodic OLG-equilibria for the existence of stationary sunspot equilibria.

Azariadis and Guesnerie (1986) prove under some weak regularity assumptions that in the simple OLG-model ($N = I = L = 1$, Assumptions Grandmont 9.3) the existence of two-cycles is equivalent to the existence of a two-dimensional Markov matrix such that there is a stationary sunspot equilibrium in the associated sunspot economy.[11] Therefore, from Theorem

[11] With reference to the dimension of the Markov matrix such an equilibrium is called a

10.3 and its proof, one can conclude that as soon as the consumer's characteristics in the certainty economy satisfy the conditions (3), (6)–(10) of the theorem, then there does not exist a two-dimensional Markov matrix which allows for the existence of sunspot equilibria. Similar results can be obtained for Proposition 10.5.

Guesnerie (1986) extends the result of two-cycles implying two-stationary sunspot equilibria to OLG-models with $I = N = 1$, but without the restriction $L = 1$. Under some mild further assumptions he even proves that the existence of two-cycles implies the existence of K-stationary sunspot equilibria for arbitrary $K \in \mathbb{N}_{>2}$. Although the existence of cycles in the OLG-model is generically sufficient for the existence of stationary sunspot equilibria[12], it is not necessary. Even worse: Chiappori and Dávila (1996) show that whenever a two-cycle can be found by means of a specific, but nevertheless quite general method, then for any even $K \in \mathbb{N}_{>1}$ there exists a stationary sunspot equilibrium of order K. Therefore, the Markov matrix can be chosen arbitrarily close to a degenerated Markov matrix describing a K-cycle and it is not necessary that the certainty economy really displays a K-cycle.

As a consequence, for more general OLG-models ($L > 1, I = N = 1$) the results of this Part IV cannot be used to exclude the existence of sunspot equilibria. But nevertheless, one observes that as soon as the modulo exchange economy approach proves the existence of cycles, this can be applied to infer the existence of stationary sunspot equilibria. Besides economists, for example Guesnerie (1986), formulate sufficient conditions for the existence of stationary sunspot equilibria[13]. Some of these conditions contain the assumption of a unique stationary equilibrium in the certainty economy which — as explained before — can be analyzed by Arrow-Debreu theory.

2-stationary sunspot equilibrium.

[12]Cycles can be seen as degenerated stationary sunspot equilibria such that some continuity arguments can be applied under regularity assumptions.

[13]For an overview compare Chiappori and Guesnerie (1991).

Part V

General Conclusion

Chapter 11

On the Number of Monetary Stationary Equilibria in OLG-Models with Incomplete Markets

Before finishing with a global overview of results given in this thesis and the prospects for further research in the different fields, in this chapter I want to briefly present an example of an economy — the OLG-model with incomplete markets described by Gottardi (1996) — for which close relationships, whether direct or indirect, to all the models described before can be established. I illuminate the different aspects to give a starting-point for further research on this topic which could be built on the foundations laid in Parts II to IV.

Gottardi's (1996) model is a stochastic overlapping generations model where there is a set of finitely lived assets on the one hand and fiat money (i.e. an asset which does not pay any dividend) on the other hand being in non-zero net supply. In contrast to many other stochastic overlapping generations models, Gottardi (1996) allows multiple consumers per generation to coexist which vastly extends the importance of asset trading. Within this framework, under the assumption of outside money, the generic existence and local uniqueness of monetary stationary equilibria with a non-zero price of money are proved. With only inside money there is a continuum of stationary equilibria.

Considering this chapter as an outlook, the presentation will be brief. I want to encourage further research on analyzing the number of monetary stationary equilibria. With focus on this aspect, I start with a description of the model[1] and then give an example of an economy displaying multiple monetary stationary equilibria. This example can be interpreted in the light of simple OLG-models (Part IV) and sunspots (Part III). Since it is built on quadratic utility functions, it can be contrasted to incomplete markets models with finite horizon guaranteeing a unique equilibrium under these assumptions (Chapter 4). Nevertheless, under some very limiting restrictions uniqueness of monetary stationary OLG-equilibria under incomplete markets can be proved by means of results given in Chapter 4.

11.1 The Model

Time evolves from the infinite past into the infinite future. Uncertainty is incorporated such that at each time $t \in \mathbb{Z}$, one out of a finite number of states $s, s = 1 \ldots, S$, occurs. The realization influences endowments and preferences of the I agents, $i = 1, \ldots, I$, born at each time in each state. These agents live for two periods and their characteristics only depend on states s_t, s_{t+1} realized during their life, but not on time or past history. Assuming the existence of only a single good, consumer i born at time t when state s was realized is characterized by endowments

$$\omega_s^i := (\omega_{s0}^i, \omega_{s1}^i, \ldots, \omega_{sS}^i)^T$$

and the utility function

$$U_s^i : \mathbb{R}_+^{S+1} \to \mathbb{R}$$

which satisfies standard assumptions, i.e. Assumption 2.4.

There is a set of spot markets which open at each date after the realization of the state and where the commodity and J assets $j = 1, \ldots, J$, can be exchanged among consumers. One of these assets, asset J, pays no interest and can be stored without any costs. All the others disappear after one period of time. The payoff denominated in money is described by a matrix $R \in \mathbb{R}^{S \times J}$, i.e.

$$R = (R_{\cdot 1}, \ldots, R_{\cdot (J-1)}, R_{\cdot J} = \mathbb{1})$$

where $rank\ R = J$ is assumed (i.e. Assumption 2.21). Assets $1, \ldots, J-1$ are in zero net supply, but there is a constant total stock M of fiat money J.

Following Gottardi (1996) I directly restrict attention to stationary equilibria such that I only have to consider time-independent consumption plans $c_s^i := (c_{s0}^i, c_{s1}^i; \ldots, c_{sS}^i)^T \in \mathbb{R}_+^{S+1}$ where upper indices mark the type of consumer, the first lower index denotes the realized state at date of birth, and

[1] The introduction of notation is in some kinds incomplete, but the missing parts should be clear from the main part of this thesis.

the second lower index refers to the states realized in the second period. The same is true for portfolio plans $y_s^i := (y_{s1}^i, \ldots, y_{sJ}^i) \in \mathbb{R}^J$, but here the second lower index marks the asset. Commodity prices are called $p := (p_1, \ldots, p_S)^T$ and asset prices

$$q := \begin{pmatrix} q_1 \\ \vdots \\ q_S \end{pmatrix} \in \mathbb{R}^{S \times J}$$

where $q_s := (q_{s1}, \ldots, q_{sJ}) \in \mathbb{R}^{1 \times J}$. q_{sJ} denotes the money price in state s and y_{sJ}^i money demand of individual i born in state s.

The stationary stochastic OLG-economy with incomplete markets therefore is defined by

$$SOLG = \{I, S; (U_s^i, \omega_s^i)_{i \in I, s \in S}; R, M\}.$$

Consumer (i, s)'s maximization problem is given by

$$\max_{c_s \in \mathbb{R}_+^{S+1}, y_s \in \mathbb{R}^J} U_s^i(c_s) \quad \text{s.t.} \quad p_s(c_{s0} - \omega_{s0}^i) + q_s y_s = 0$$
$$\wedge\ p_{s'}(c_{ss'} - \omega_{ss'}^i) = q_{s'J} R_{s'}.y_s \text{ for all } s' \in S$$

where equality in the budget constraints follows from the assumptions. It is obvious that "there is then, for each s, one degree of normalization left unexploited" (Gottardi (1996, p. 80)) such that without loss of generality $p_s = 1, s = 1, \ldots, S$, can be normalized.

Definition 11.1 (Stationary OLG-Equilibrium)
Given $SOLG = \{I, S; (U_s^i, \omega_s^i)_{i \in I, s \in S}; R, M\}$ a stationary OLG-equilibrium is a tuple of consumption plans $(c_s^{*i})_{i \in I, s \in S} \in \mathbb{R}^{(S+1) \times IS}$, of portfolio plans $(y_s^{i*})_{i \in I, s \in S} \in \mathbb{R}^{J \times IS}$, and of price systems $q^* \in \mathbb{R}^{S \times J}$ such that

1. *for all $i \in I$ and $s \in S$:*

$$(c_s^{*i}, y_s^{*i}) = arg \max_{c_s \in \mathbb{R}_+^{S+1}, y_s \in \mathbb{R}^J} U_s^i(c_s) \text{ s.t. } c_{s0} - \omega_{s0}^i + q_s y_s = 0$$
$$\wedge\ c_{ss'} - \omega_{ss'}^i = q_{s'J} R_{s'}.y_s \text{ for all } s' \in S;$$

2. *for all $s, s' \in S$:*

$$\sum_{i=1}^{I} c_{s0}^{*i} + \sum_{i=1}^{I} c_{s's}^{*i} = \sum_{i=1}^{I} \omega_{s0}^i + \sum_{i=1}^{I} \omega_{s's}^i;$$

3. for all $s \in S$:

$$\sum_{i=1}^{I} y_{sJ}^{*i} = M;$$

4. for all $j \in \{1, \ldots, J-1\}$ and $s \in S$:

$$\sum_{i=1}^{I} y_{sj}^{*i} = 0.$$

If in addition $q_{\cdot J}^{*} \neq 0$, the equilibrium is called monetary.

Note that conditions 3. and 4. imply 2. Within this framework Gottardi (1996, Theorem 1) proves the following theorem.

Theorem 11.2 (Gottardi (1996))
Given $SOLG = \{I, S; (U_s^i, \omega_s^i)_{i \in I, s \in S}; R, M\}$ with $M \neq 0$, generically in endowments there exists a stationary monetary equilibrium (with money prices of the same sign in all states) and this equilibrium is generically regular and locally unique.

However, Gottardi (1996) in no way makes any statement on the global number of monetary stationary equilibria. The remaining part of this chapter concentrates on that point.

11.2 An Example of Multiple Monetary Stationary Equilibria

The following example constitutes the simplest molding of Gottardi's (1996) OLG-model with incomplete markets. Assume $I = 1$, that is one representative consumer for each period such that intragenerational trade is excluded and it constitutes no further restriction to assume $J = 1$, i.e. fiat money is the only asset. For simplicity assume $M = 1$. Uncertainty is only of dimension 2, i.e. $S = 2$. The single consumer's characteristics do not depend on the state of nature realized at his date of birth. Utility is of the quadratic type such that

$$U(c_s) = \bar{k}\left(c_{s0} - \frac{1}{2}\gamma(c_{s0})^2\right) + \sum_{s'=1}^{2} \bar{\rho}_{s'}\left(c_{ss'} - \frac{1}{2}\gamma(c_{ss'})^2\right)$$

where $\gamma > 0$, $\bar{\rho} \in \mathbb{R}^2_{++}$ with $\bar{\rho}_1 + \bar{\rho}_2 = 1$ and $\bar{k} \in \left(\frac{13}{7}, \frac{17}{3}\right)$. Endowments are given by

$$\omega = \left(\frac{8}{17\gamma}, \frac{8}{17\gamma}, \frac{8}{17\gamma}\right)^T =: (\bar{\omega}, \bar{\omega}, \bar{\omega})^T .$$

This implies that there is neither risk concerning endowment distribution between first and second period of life nor concerning different states.

The utility function does not satisfy standard assumptions (Assumption 2.4) as required by Gottardi (1996) because it is neither strictly monotonic nor are the indifference curves bounded away from the boundary of $\mathbb{R}^{S+1}_+$, i.e. $\partial\mathbb{R}^{S+1}_+$. However, since total endowment at some point of time is given by $2\bar{\omega} = \frac{16}{17\gamma}$ and $(c_{ss'} - 1/2\ \gamma(c_{ss'})^2)' = 1 - \gamma c_{ss'} \geq 1 - \gamma\frac{16}{17\gamma} > 0$, U is strictly monotonic in the relevant domain. What concerns the indifference curves, tedious calculations could be used to show irrelevance of corner solutions for my considerations.

A stationary OLG-equilibrium within this simplified framework requires a constant money demand $\bar{y} = 1$ of the young consumer at each point of time, i.e. one has to find equilibrium money prices $q_1, q_2 \in \mathbb{R}$ such that $\bar{y}$ solves

$$\max_{y \in \mathbb{R}} \bar{k} \cdot \left[\bar{\omega} - q_s y - \frac{1}{2}\gamma(\bar{\omega} - q_s y)^2\right] + \sum_{s'=1}^{2} \bar{\rho}_{s'} \left[\bar{\omega} + q_{s'} y - \frac{1}{2}\gamma(\bar{\omega} + q_{s'} y)^2\right]$$

for $s = 1$ and $s = 2$.

This is equivalent to the following first order conditions for $s = 1, 2$:

$$\text{(11.1)} \qquad \bar{k} \cdot [-q_s + \gamma q_s(\bar{\omega} - q_s)] + \sum_{s'=1}^{2} \bar{\rho}_{s'} \left[q_{s'} - \gamma q_{s'}(\bar{\omega} + q_s')\right] = 0.$$

One solution $\bar{q}$ is given by $\bar{q}_1 = \bar{q}_2 = 0$ which describes a non-monetary stationary equilibrium. A second solution $\hat{q}$ is a non-stochastic monetary stationary equilibrium with non-zero money prices in each state, i.e.

$$\hat{q}_1 = \hat{q}_2 = \frac{9(1 - \bar{k})}{17\gamma(1 + \bar{k})}.$$

Substituting $\hat{q}$ into first order equations (11.1) for both s gives

$$\begin{aligned}
& \bar{k}\left[-1 + \gamma\left(\tfrac{8}{17\gamma} - \tfrac{9(1-\bar{k})}{17\gamma(1+\bar{k})}\right)\right] + \left[1 - \gamma\left(\tfrac{8}{17\gamma} + \tfrac{9(1-\bar{k})}{17\gamma(1+\bar{k})}\right)\right] && = 0 \\
\Leftrightarrow \quad & \bar{k} \cdot \left(-\tfrac{9}{17}\right)\left[1 + \tfrac{1-\bar{k}}{1+\bar{k}}\right] + \tfrac{9}{17}\left[1 - \tfrac{1-\bar{k}}{1+\bar{k}}\right] && = 0 \\
\Leftrightarrow \quad & \tfrac{9}{17} \cdot \tfrac{2\bar{k}}{1+\bar{k}} && = \tfrac{9}{17} \cdot \tfrac{2\bar{k}}{1+\bar{k}}
\end{aligned}$$

which proves the equilibrium properties as soon as $\bar{\omega} \pm \hat{q}_s > 0$ for both s which follows because of the restrictions on $\bar{k}$.[2]

However, except for these two solutions there is a further one $\tilde{q} \neq 0$ with $\tilde{q}_1 \neq \tilde{q}_2$, namely

$$\tilde{q}_1 = -\frac{27}{170\gamma},\ \tilde{q}_2 = -\frac{63}{170\gamma}$$

(or equivalently because of symmetry the reverse pair).

For $s = 1$ the left term in equation (11.1) is given by

$$\begin{aligned} & \bar{k}\left[\frac{27}{170\gamma} + \gamma\left(-\frac{27}{170\gamma}\right)\cdot\left(\frac{8}{17\gamma} + \frac{27}{170\gamma}\right)\right] \\ = \ & \bar{k}\frac{27}{170\gamma}\left[1 - \frac{107}{170}\right] \\ = \ & \frac{\bar{k}\cdot 27\cdot 63}{170^2\gamma} \end{aligned}$$

which is equal to the left term of equation (11.1) for $s = 2$:

$$\begin{aligned} & \bar{k}\left[\frac{63}{170\gamma} + \gamma\left(-\frac{63}{170\gamma}\right)\cdot\left(\frac{8}{17\gamma} + \frac{63}{170\gamma}\right)\right] \\ = \ & \bar{k}\frac{63}{170\gamma}\left[1 - \frac{143}{170}\right] \\ = \ & \frac{\bar{k}\cdot 63\cdot 27}{170^2\gamma}. \end{aligned}$$

The right term for both s is equal to

$$\begin{aligned} & \bar{\rho}_1\left[-\frac{27}{170\gamma} - \gamma\left(-\frac{27}{170\gamma}\right)\cdot\left(\frac{8}{17\gamma} - \frac{27}{170\gamma}\right)\right] \\ & +\bar{\rho}_2\left[-\frac{63}{170\gamma} - \gamma\left(-\frac{63}{170\gamma}\right)\cdot\left(\frac{8}{17\gamma} - \frac{63}{170\gamma}\right)\right] \\ = \ & \bar{\rho}_1\left(-\frac{27}{170\gamma}\right)\left[1 - \frac{80}{170} + \frac{27}{170}\right] + \bar{\rho}_2\left(-\frac{63}{170\gamma}\right)\left[1 - \frac{80}{170} + \frac{63}{170}\right] \\ = \ & -\frac{27\cdot 117\cdot\bar{\rho}_1}{170^2\gamma} - \frac{63\cdot 153\cdot\bar{\rho}_2}{170^2\gamma}. \end{aligned}$$

Therefore, equation (11.1) is satisfied for both s whenever

$$\frac{\bar{k}\cdot 63\cdot 27}{170^2\cdot\gamma} = \frac{27\cdot 117\cdot\bar{\rho}_1 + 63\cdot 153\cdot\bar{\rho}_2}{170^2\gamma}$$

---

[2] $\bar{\omega} \pm \hat{q}_s = \frac{1}{17\gamma}\frac{8+8\bar{k}\pm(9-9\bar{k})}{1+\bar{k}} > 0.$

or equivalently

$$1701\bar{k} = 3159\bar{\rho}_1 + 9639\bar{\rho}_2 \quad = \quad 9639 - 6480\bar{\rho}_1. \tag{11.2}$$

The restrictions on $\bar{k}$ assure $\bar{\rho}_1 \in (0,1)$.

Hence, since also $\bar{\omega} \pm \tilde{q}_s > 0$ for $s = 1,2$, as soon as $\bar{\rho}_1$ and $\bar{k}$ satisfy equation (11.2) there exists a third stationary (and consequently a second monetary stationary) OLG-equilibrium.

Notice that for $\bar{k} = \frac{79}{21} \approx 3.76$ it is $\bar{\rho}_1 = \bar{\rho}_2 = \frac{1}{2}$ such that the economy is totally symmetric between the different states of uncertainty, but nevertheless displays an asymmetric stationary equilibrium.

The example can be interpreted in the light of sunspots in simple OLG-models (compare Parts III and IV). Take a single consumer living for two periods with utility

$$U(c) = \bar{k}\left(c_0 - \frac{1}{2}\gamma(c_0)^2\right) + \left(c_1 - \frac{1}{2}\gamma(c_1)^2\right)$$

and endowments $\omega_0 = \omega_1 = \bar{\omega}$. If the agent believes in two states of extrinsic uncertainty with probabilities $\bar{\rho}_1$ and $\bar{\rho}_2$, there exists a monetary non-sunspot equilibrium defined by $\hat{q}$ and a monetary stationary sunspot equilibrium defined by $\tilde{q}$.

The existence of sunspot equilibria is well-known in the OLG-framework. As indicated before, Azariadis and Guesnerie (1986) mention sunspot equilibria to exist whenever the basic OLG-economy displays two-cycles.[3] However, note that not all standard-OLG-sunspot-equilibria correspond to a stationary monetary equilibrium (within the special framework described above) because an additional property of the probabilities is necessary, namely the probability for some state s in t has to be independent of the state realization in the period before, i.e. at $t-1$.[4] Therefore, especially those sunspot equilibria constructed as perturbations of two-cycles are worthless in the sense here, because they require high probabilities to go from $s = 1$ to $s = 2$ and from $s = 2$ to $s = 1$, but low probabilities to go from state 1 to 1 and from state 2 to 2.

Obviously, even though utility in the example is quadratic with an objective underlying probability measure, there exist multiple monetary stationary equilibria in the OLG-model with incomplete markets. This might be a hint

[3]Such conditions can be found in Part IV.

[4]Guesnerie (1986, p. 118) interprets stationary sunspot equilibria under such assumptions on the probability measure as "non-informative, or semi-trivial, or of pure multiplicity".

to have no dependencies between uniqueness in incomplete markets models with two periods (which is guaranteed for the special quadratic utility) and uniqueness of monetary stationary OLG-equilibria. Nevertheless, in the next section I prove that there is at least a little hope to get some relations, perhaps via an approach similar to Part IV.

11.3 Uniqueness of Monetary Stationary OLG-Equilibria under Incomplete Markets

In this section I prove that as soon as utility of each consumer is linear in consumption of his first period of life (or utility is of the expected utility type displaying a relative risk aversion less or equal to one and the only asset is money) and consumers' characteristics are independent of the states of nature at their date of birth, then there is a unique monetary stationary OLG-equilibrium.[5] The point is to apply monotonicity of demand functions derived in the Finance GEI-model (Chapter 4) to assure state-independence of asset prices and use this observation — again combined with the monotonicity property — to obtain a unique monetary stationary equilibrium. Recall from the last section that in general even though consumers' characteristics are state-independent, this is not necessarily true for asset prices in a stationary equilibrium.

Assume state-independence of consumers' characteristics, i.e. there are some $\bar{U}^i : \mathbb{R}_+^{S+1} \to \mathbb{R}$ and some $\bar{\omega}^i \in \mathbb{R}_+^{S+1}$ such that $U_s^i = \bar{U}^i$ and $\omega_s^i = \bar{\omega}^i$ for all $i \in I$ and $s \in S$. Besides, either $\bar{U}^i$ is quasi-linear for all $i \in I$, i.e.

$$\bar{U}^i(c) = c_0 + \tilde{U}^i(c_{-0})$$

with some $\tilde{U}^i : \mathbb{R}_+^S \to \mathbb{R}$ satisfying standard assumptions, i.e. Assumption 2.4, and $\bar{\omega}_0^i$ is sufficiently large, or the only asset is money such that $J = 1$ and $\bar{U}^i$ is for all $i \in I$ of the expected utility type, i.e.

$$\bar{U}^i(c) = \sum_{s=0}^{S} \rho_s^i u^i(c_s)$$

for some $u^i : \mathbb{R}_+ \to \mathbb{R}$ also guaranteeing Assumption 2.4 with a relative risk aversion less or equal to one and $\rho^i \in \mathbb{R}_{++}^{S+1}$. Remember from Chapter 4

[5] To be correct I should say that there exists at most one monetary stationary equilibrium because Gottardi (1996) only proves generic existence of monetary stationary equilibria where generic refers to endowments without restricting to endowments independent of the state at date of birth of the consumers. Since this chapter just constitutes an outlook, the reader will excuse this inaccuracy.

(Proofs of Propositions 4.20 and 4.23) that

$$\theta^i(a, A) := arg \max_{\theta \in \mathbb{R}^J} \bar{U}^i(\bar{\omega}_0^i - a \cdot \theta, \omega_{-0}^i + A\theta) \tag{11.3}$$

is strictly monotone in the differentiable version, i.e. $D_a\theta^i(a, A)$ is negative definite for all $a \in \mathcal{Q}_A := \left\{ \check{a} \in \mathbb{R}^J \mid \check{a} = A^T\pi \text{ for some } \pi \in \mathbb{R}_{++}^S \right\}$. According to Hildenbrand (1994, Appendix 1) this implies

$$(\bar{a} - \tilde{a}) \cdot \big(\theta^i(\bar{a}, A) - \theta^i(\tilde{a}, A)\big) < 0 \tag{11.4}$$

for all $\bar{a}, \tilde{a} \in \mathcal{Q}_A$ with $\bar{a} \neq \tilde{a}$.[6]

For the OLG-model with incomplete markets, following Gottardi (1996), individual demand for consumer (i, s) is defined by

$$y^i(q_s, q_{\cdot J}, R) := arg \max_{y \in \mathbb{R}^J} \bar{U}^i(\bar{\omega}_0^i - q_s y, \bar{\omega}_1^i + q_{1J}R_{1\cdot}y, \ldots, \bar{\omega}_S^i + q_{SJ}R_{S\cdot}y)$$

with each consumption positive, but this follows from the assumptions on the individual characteristics.

Therefore, individual demand satisfies

$$y^i(q_s, q_{\cdot J}, R) = \theta^i(q_s^T, \begin{pmatrix} q_{1J}R_{1\cdot} \\ \vdots \\ q_{SJ}R_{S\cdot} \end{pmatrix})$$

which implies for the demand of one generation born in state s

$$\begin{aligned} y(q_s, q_{\cdot J}, R) &:= \sum_{i=1}^I y^i(q_s, q_{\cdot J}, R) \\ &= \sum_{i=1}^I \theta^i(q_s^T, \begin{pmatrix} q_{1J}R_{1\cdot} \\ \vdots \\ q_{SJ}R_{S\cdot} \end{pmatrix}) =: \theta(q_s^T, \begin{pmatrix} q_{1J}R_{1\cdot} \\ \vdots \\ q_{SJ}R_{S\cdot} \end{pmatrix}). \end{aligned} \tag{11.5}$$

Lemma 11.3
Given SOLG $= \{I, S; (U_s^i, \omega_s^i)_{i \in I, s \in S}; R, M\}$ such that the functions θ^i for $i = 1, \ldots, I$ defined by (11.3) satisfy strict monotonicity (11.4), stationary OLG-equilibria fulfill

$$q_s = q_{s'}$$

for all $s, s' \in S$.

[6] Note that in Part II the non-differentiable version of strict monotonicity with respect to the first unit vector and under first period consumption is different in the sense that the inequality is required for those $\bar{a}, \tilde{a} \in \mathcal{Q}_A$ with $\theta^i(\bar{a}, A) \neq \theta^i(\tilde{a}, A)$ and/or $\bar{a} \cdot \theta^i(\bar{a}, A) \neq \tilde{a} \cdot \theta^i(\tilde{a}, A)$.

Proof: Let $\bar{q}$ be an asset price system generating a stationary equilibrium. Assume $\bar{q}_s \neq \bar{q}_{s'}$ for some $s, s' \in S$ with $s \neq s'$. The equilibrium properties require[7]

$$\theta(\bar{q}_s^T, \begin{pmatrix} \bar{q}_{1J}R_{1\cdot} \\ \vdots \\ \bar{q}_{SJ}R_{S\cdot} \end{pmatrix}) = \theta(\bar{q}_{s'}^T, \begin{pmatrix} \bar{q}_{1J}R_{1\cdot} \\ \vdots \\ \bar{q}_{SJ}R_{S\cdot} \end{pmatrix}) = \begin{pmatrix} 0 \\ M \end{pmatrix}.$$

This implies

$$\begin{aligned} (\bar{q}_s - \bar{q}_{s'}) \left[\theta(\bar{q}_s^T, \begin{pmatrix} \bar{q}_{1J}R_{1\cdot} \\ \vdots \\ \bar{q}_{SJ}R_{S\cdot} \end{pmatrix} - \theta(\bar{q}_{s'}^T, \begin{pmatrix} \bar{q}_{1J}R_{1\cdot} \\ \vdots \\ \bar{q}_{SJ}R_{S\cdot} \end{pmatrix} \right] &= \\ (\bar{q}_s - \bar{q}_{s'}) \left[\begin{pmatrix} 0 \\ M \end{pmatrix} - \begin{pmatrix} 0 \\ M \end{pmatrix} \right] &= 0, \end{aligned}$$

which contradicts the fact that (11.4) is satisfied for all $i \in I$.

□

This enables the proof of the following proposition.

Proposition 11.4 (Uniqueness of Mon. Stat. Equilibria)
Assume $SOLG = \{I, S; (U_s^i, \omega_s^i)_{i \in I, s \in S}; R, M\}$ with state-independent characteristics of the consumers such that utility is either quasi-linear and endowments when young sufficiently large or $J = 1$ and utility is of the expected utility type with a relative risk aversion less or equal to one. Then, the economy displays at most one monetary stationary OLG-equilibrium.

Proof: From Chapter 4 it follows that individual demand functions θ^i, defined by (11.3) for all $i \in I$ are strictly monotone in the sense of (11.4). This transfers to the function θ. Besides, from Lemma 11.3 it is known that monetary stationary OLG-equilibria have to satisfy state-independence $\bar{q}_s = \bar{q}_{s'}$ for all $s, s' \in S$. Assume there are two different monetary stationary equilibria with price systems represented by the J-dimensional vectors $\tilde{q}$ and $\hat{q}$.

The definitions of the θ^i, $i = 1, \ldots, I$, and θ, together with the equilibrium conditions imply

$$\begin{pmatrix} 0 \\ M \end{pmatrix} = y(\tilde{q}, \tilde{q}_J \mathbb{1}, R) = \theta(\tilde{q}^T, \tilde{q}_J R) = \frac{1}{\tilde{q}_J} \theta(\frac{1}{\tilde{q}_J} \tilde{q}^T, R) \tag{11.6}$$

[7] 0 here is a $(J-1)$-dimensional vector with all entries equal to zero.

and

$$\begin{pmatrix} 0 \\ M \end{pmatrix} = y(\hat{q}, \hat{q}_J \mathbb{1}, R) = \theta(\hat{q}^T, \hat{q}_J R) = \frac{1}{\hat{q}_J}\theta(\frac{1}{\hat{q}_J}\hat{q}^T, R). \tag{11.7}$$

If $\tilde{q} = \lambda\hat{q}$ for some $\lambda \neq 1$ it follows

$$\frac{1}{\tilde{q}_J}\tilde{q} = \frac{1}{\hat{q}_J}\hat{q}$$

which implies

$$\theta(\frac{1}{\tilde{q}_J}\tilde{q}^T, R) = \theta(\frac{1}{\hat{q}_J}\hat{q}^T, R)$$

and therefore because of (11.6) and (11.7) $\tilde{q}_J = \hat{q}_J$, i.e. $\lambda = 1$, which contradicts the assumptions.

Now assume $\tilde{q} \neq \lambda\hat{q}$ for all $\lambda \in \mathbb{R}$. (11.6) and (11.7) imply

$$\begin{aligned} & \left(\frac{1}{\tilde{q}_J}\tilde{q}^T - \frac{1}{\hat{q}_J}\hat{q}^T\right) \cdot \left[\theta(\frac{1}{\tilde{q}_J}\tilde{q}^T, R) - \theta(\frac{1}{\hat{q}_J}\hat{q}^T, R)\right] \\ = & \begin{pmatrix} \frac{1}{\tilde{q}_J}\tilde{q}^T_{-J} - \frac{1}{\hat{q}_J}\hat{q}^T_{-J} \\ 1-1 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 \\ \tilde{q}_J M \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{q}_J M \end{pmatrix}\right] \\ = & \ 0, \end{aligned}$$

which contradicts the strict monotonicity property (11.4). There is at most one stationary monetary equilibrium.

□

Chapter 12

Conclusion

This thesis gives an overview of conditions for the uniqueness of competitive equilibria in various kinds of pure exchange equilibrium models. Concentrating on individuals' characteristics, it relates *and* contrasts results on the number of equilibria in the different frameworks.

Not surprisingly, the basis is laid by the analysis of the Arrow-Debreu model which can be seen as the root of all competitive equilibrium models. Since this model is well-understood, it is natural to start from results on uniqueness of Arrow-Debreu equilibria and focus on extensions to more general and more realistic models.

Nonetheless, one point needed to be clarified in the Arrow-Debreu model because recently a variety of economic models focused on the related aspect of complete risk sharing: Arrow-Debreu equilibria are not necessarily unique when consumers are characterized by expected utility with respect to an objective probability measure as soon as risky total endowment is allowed. This is proved by means of an example. A question for further research is to find out whether additional restrictions similar to those of Gollier and Pratt (1996) on "risk vulnerability" or of Ross (1981) on decreasing risk aversion exist which guarantee a unique equilibrium.

The first step of generalizing the Arrow-Debreu model is to allow restrictions of markets which leads to incomplete markets models with finite horizon in its simplest form, to the Finance GEI-model. Even though for this framework a lot of conditions sufficient for a unique equilibrium have already been established, by no means this topic is completely understood. Via different kinds of examples I illuminate on the one hand the limits of transferability between Arrow-Debreu economies and the Finance GEI-model with actual incomplete markets: even though consumers' characteristics create a unique equilibrium when markets are complete, there is some incomplete asset structure for which they induce multiple equilibria. On the other hand I clarify that the actual incompleteness of markets might have positive effects in the sense that it generates uniqueness: consumers' characteristics create multiple equilibria with complete markets, but for some incomplete asset structure the equilibrium is unique. All these — even robust — examples induce a further comprehension of the consequences on uniqueness evoked by market incompleteness because really everything except for the market structure remains constant. Understanding this aspect, lays the foundation for understanding general differences between allocations in complete and in incomplete markets models. Besides, the examples might create some understanding on how to establish further conditions for uniqueness. Since most algorithms for computation of equilibria in the Finance GEI-model are extensions of corresponding algorithms in the Arrow-Debreu model, the examples might be helpful for testing these algorithms because of emphasizing the differences between the models.

Another important aspect for future research is to analyze not only the extreme cases complete markets respectively the non-existence of assets in contrast to some actual incompleteness in between, but also to understand the consequences of different degrees of incompleteness and its effects on uniqueness. What concerns constrained Pareto-efficient endowments, I present some new, even though not surprising relationships between Finance GEI-economies with different dimensions of the market space. For different market structures, but of the same dimension I establish some results on the robustness of the number of equilibria.

Surely there is some possibility of extending the Finance GEI-model and its positive results on uniqueness to more general incomplete markets models.

One extension and a yet more general model is given when asymmetric market participation is allowed for different agents. What concerns uniqueness of equilibria, simple examples emphasize the strong limits of transferability from the Finance GEI-model. Some — even though quite restrictive — conditions on individual characteristics and market structures are obtained which are sufficient to guarantee a unique equilibrium. At any rate, there is much room for further research on this aspect. Perhaps this finite dimensional model is even more suitable to understand the existence of monetary stationary and periodic equilibria in OLG-models than the modulo Arrow-Debreu economy described in Part IV: considering an OLG-economy, in principle there exists a market for each commodity at each point of time, but consumers have different access to these markets.

A second possibility of extending the Finance GEI-model to a more realistic one is to include extrinsic uncertainty and therefore in general enable the existence of sunspots. Proving the non-validity of Mas-Colell's (1992) result on the existence of sunspot equilibria under deviating to the assumption of two commodities when there is no intrinsic uncertainty, assets are sunspot-independent, and utility is additively separable with a non-decreasing relative risk aversion, is a further step on the way of understanding the emergence of sunspot equilibria: "potential multiplicity" in the underlying basic economy is not sufficient for the existence of sunspot equilibria, even though it is known to be necessary under the assumption of sunspot-independent asset structures. It is left for further research to clarify to what extent the additive separability of utility functions is necessary. This aspect is related to the question of "order crossing" of Lagrangian multipliers for different Arrow-Debreu equilibria.

Further research should try to establish more general restrictions to exclude sunspot equilibria in different kinds of models for which a priori the existence of such equilibria is possible. This would give a further understanding of sunspot equilibria in general. One focus could be to apply results on uniqueness of equilibria presented before, even though — as already explained — multiplicity/uniqueness and existence/non-existence of sunspot equilibria cannot be equated.

In contrast to the Arrow-Debreu model, the OLG-model allows the economy to evolve over time. Nevertheless, as presented in Part IV, its monetary stationary and periodic equilibria can be characterized by means of a special static Arrow-Debreu economy. This allows application of the results on uniqueness of Arrow-Debreu equilibria to understand the emergence of multiple monetary stationary and periodic OLG-equilibria. I am able to establish restrictions on consumers' characteristics guaranteeing a unique monetary stationary equilibrium, a lot of them also excluding each kind of cycle. Besides, in the simple OLG-model I prove that cycles with a given odd length always appear in even numbers and I define conditions assuring an even respectively odd number of two-cycles.

However, not only the existence, also the stability of such OLG-equilibria is important. Therefore, a point of future research is to clarify whether stability properties in the modulo Arrow-Debreu economy can be applied to understand the stability of OLG-equilibria. Besides it is promising to extend this approach of comprehending dynamical via static models: what could be done with OLG-models displaying sequentially incomplete markets and/or extrinsic uncertainty?

As presented in Chapter 11, there exist economic models relating all aspects analyzed in this thesis:

- the number of monetary stationary equilibria;
- the existence of sunspot equilibria which itself — depending on the underlying model — is related to the existence of cycles in simple OLG-models, and
- the uniqueness of equilibria under incomplete markets which is also related to the uniqueness of equilibria in the Arrow-Debreu model and in RPGEI-economies.

It is no question that the relations between the different aspects require a further clarification, whether by focusing on two of them alone as it is done in each of the Parts II to IV or by consideration of a combining economy as it is indicated by Chapter 11.

To summarize: this thesis provides several devices which can be used to analyze the equilibria of a variety of equilibrium models — static and dynamic ones, with complete and incomplete market structures — constituting a sound foundation for the understanding of financial markets. The power of these tools is strengthened by demonstrating the interrelations between the different models.

Part VI

Appendix

Appendix A

Some Mathematics

A.1 Notations

In this section I give an overview of mathematical notations applied in this thesis. I categorize them as concerning sets, matrices and vectors, correspondences and functions, or those that cannot be assigned to one of the other categories. Note that a symbol struck out (for example $\neq$) means negation.

A.1.1 Notations for Sets

$\mathbb{N}$	$\mathbb{N} = \{1, 2, 3, \ldots\}$, i.e. the set of natural numbers (without zero).
$\mathbb{N}_0$	$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, i.e. the set of natural numbers including zero.
$\mathbb{N}_{>N}$	The set of natural numbers larger than $N \in \mathbb{N}$: $\mathbb{N}_{>N} = \{N+1, N+2, N+3, \ldots\}$.
N	For given $N \in \mathbb{N}$ the set of the first N natural numbers unequal to zero: $N = \{1, \ldots, N\}$.
$\tilde{N}$	For given $N \in \mathbb{N}$ the set of the first $N+1$ natural numbers including zero: $\tilde{N} = \{0 \ldots, N\}$.
$\mathcal{P}(N)$	The power set of N, i.e. the set of all subsets of $N = \{1, \ldots, N\}$: $\mathcal{P}(N) = \{J \mid J \subseteq N\}$.
$\mathbb{Z}$	The set of integers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non-negative real numbers.
$\mathbb{R}_{++}$	The set of strictly positive real numbers.
$[a, b]$	Closed intervall: the set of real numbers $r \in \mathbb{R}$ with $a \leq r \leq b$.

(a,b)	Open intervall: the set of real numbers $r \in \mathbb{R}$ with $a < r < b$.
$\mathbb{R}^{N\times M}$	The set of $N \times M$-dimensional matrices with real entries.
$\mathbb{R}^{N\times M}_{+}$	The set of $N \times M$-dimensional matrices with non-negative real entries.
$\mathbb{R}^{N\times M}_{++}$	The set of $N \times M$-dimensional matrices with strictly positive real entries.
$\hat{\mathcal{P}}_{c,n} \subseteq \mathbb{R}^N$	For $c \in \mathbb{R}_{++}$ and $n \in \mathbb{R}^N_+\backslash\{0\}$: $\hat{\mathcal{P}}_{c,n} = \{x \in \mathbb{R}^N_{++} \mid x{\cdot}n = c\}$.
$\hat{\mathcal{P}}_{1,\mathbb{1}} \subseteq \mathbb{R}^N$	The interior of the $(N-1)$-dimensional unit simplex.
$\hat{\mathcal{P}} \subseteq \mathbb{R}^N$	The set of vectors in $\mathbb{R}^N_{++}$ with the first coordinate equal to 1, i.e. $\hat{\mathcal{P}} = \hat{\mathcal{P}}_{1,e^1}$.
$\mathcal{X}$	In general sets are denoted by calligraphical letters.
$\mathcal{X} \times \mathcal{Y}$	The Cartesian product of two sets $\mathcal{X}, \mathcal{Y}$.
$\mathcal{X}^N$	$\mathcal{X}^N = \mathcal{X} \times \ldots \mathcal{X}$ denotes the N-times Cartesian product of $\mathcal{X}$.
$\times_{n\in N}\mathcal{X}^n$	$\times_{n\in N}\mathcal{X}^n = \mathcal{X}^1 \times \ldots \times \mathcal{X}^N$.
$cl\mathcal{X}$	The closure of a set $\mathcal{X}$.
$int\mathcal{X}$	The interior of a set $\mathcal{X}$.
$\partial\mathcal{X}$	The boundary of a set $\mathcal{X}$.
$conv\mathcal{X}$	The convex hull of a set $\mathcal{X}$, i.e. the smallest convex set which contains $\mathcal{X}$.
$x \in \mathcal{X}$	x is element of $\mathcal{X}$.
$\sharp\mathcal{X}$	The number of elements in $\mathcal{X}$ where $\mathcal{X}$ is a discrete finite set.
$\mathcal{X} \subseteq \mathcal{Y}$	$\mathcal{X}$ is a subset of $\mathcal{Y}$.
$\mathcal{X} \subset \mathcal{Y}$	$\mathcal{X}$ is a proper subset of $\mathcal{Y}$.
$\mathcal{X} \cup \mathcal{Y}$	The union of the sets $\mathcal{X}$ and $\mathcal{Y}$.
$\bigcup_{n\in N} \mathcal{X}^n$	The union of the sets $\mathcal{X}^1, \ldots, \mathcal{X}^N$.
$\mathcal{X} \cap \mathcal{Y}$	The intersection of the sets $\mathcal{X}$ and $\mathcal{Y}$.
$\bigcap_{n\in N} \mathcal{X}^n$	The intersection of the sets $\mathcal{X}^1, \ldots, \mathcal{X}^N$.
$\mathcal{X} + \mathcal{Y}$	The sum of two sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^N$: $\mathcal{X} + \mathcal{Y} = \{z \in \mathbb{R}^N \mid z = x + y$ for some $x \in \mathcal{X}, y \in \mathcal{Y}\}$.
$\mathcal{X}\backslash\mathcal{Y}$	The complement of $\mathcal{Y}$ in $\mathcal{X}$: $\mathcal{X}\backslash\mathcal{Y} = \{x \in \mathcal{X} \mid x \notin \mathcal{Y}\}$.

A.1.2 Notations for Vectors and Matrices

$x \in \mathbb{R}^N = \mathbb{R}^{N\times 1}$	N-dimensional column vector.
$x \in \mathbb{R}^{1\times N}$	N-dimensional row vector.

$e^n \in \mathbb{R}^N$	The nth unit vector.
$1\!\!\mathrm{I} \in \mathbb{R}^N$	The N-dimensional vector with all entries equal to 1.
$0 \in \mathbb{R}^N$	The N-dimensional vector with all entries equal to 0.
$x \leq y$ $(x \geq y)$	The coordinates of $x \in \mathbb{R}^N$ (resp. $x \in \mathbb{R}^{1\times N}$) are less or equal (bigger or equal) than the corresponding coordinates of $y \in \mathbb{R}^N$ (resp. $y \in \mathbb{R}^{1\times N}$).
$x < y$ $(x > y)$	The coordinates of $x \in \mathbb{R}^N$ (resp. $x \in \mathbb{R}^{1\times N}$) are less or equal (bigger or equal) than the corresponding coordinates of $y \in \mathbb{R}^N$ (resp. $y \in \mathbb{R}^{1\times N}$) and $x \neq y$.
$x \ll y$ $(x \gg y)$	Each coordinate of $x \in \mathbb{R}^N$ (resp. $x \in \mathbb{R}^{1\times N}$) is strictly less (strictly bigger) than the corresponding coordinate of $y \in \mathbb{R}^N$ (resp. $y \in \mathbb{R}^{1\times N}$).
$x \parallel y$	$x, y \in \mathbb{R}^N$ (resp. $\in \mathbb{R}^{1\times N}$) are collinear.
$(x^m)_{m\in M}$	Collection of the M vectors $x^1, \ldots, x^M$.
$A \in \mathbb{R}^{N\times M}$	Matrix with N rows and M columns: A_{nm} is the real entry in the nth row and mth column.
$< A > \subseteq \mathbb{R}^N$	The set of N-dimensional vectors which are a linear combination of the columns of A: the span of A.
$x_n, A_{n\cdot}, A_{\cdot n}$	The nth row resp. column of the vector x resp. matrix A.
$x_R, A_{R\cdot}, A_{\cdot R}$	Vector x resp. matrix A restricted to rows resp. columns with index $r \in R$.
$x_{-n}, A_{(-n)\cdot}, A_{\cdot(-n)}$	Vector x resp. matrix A without the nth row resp. column.
$x_{-R}, A_{(-R)\cdot}, A_{\cdot(-R)}$	Vector x resp. matrix A without the rows resp. columns with index $r \in R$.
$x \cdot y$	The scalar product of two column vectors $x, y \in \mathbb{R}^N$.
$x \cdot A$	The row vector consisting of the scalar products $x \cdot A_{\cdot 1}, \ldots, x \cdot A_{\cdot M}$ for $x \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N\times M}$.
AB	The matrix product of $A \in \mathbb{R}^{N\times M}$ and $B \in \mathbb{R}^{M\times L}$ (A might be a row vector and B a column vector).
$\lVert.\rVert$	An arbitrary vector resp. matrix norm.
$\lVert.\rVert_{\max}$	The maximum norm for vectors resp. matrices.
$rank\ A$	The maximal number of linearly independent vectors contained in $< A >$ for $A \in \mathbb{R}^{N\times M}$.
$\det A, \lvert A\rvert$	The determinant of a quadratric matrix $A \in \mathbb{R}^{N\times N}$.
A^{-1}	The inverse of an invertible quadratric matrix $A \in \mathbb{R}^{N\times N}$.
x^T, A^T	The transposed of the vector x resp. the matrix A.
$x^\perp$	The orthogonal space with respect to $x \in \mathbb{R}^N$: $x^\perp := \{v \in \mathbb{R}^N \mid x \cdot v = 0\}$.

$A\mathcal{X}$	The set of vectors given as product of $A \in \mathbb{R}^{N\times M}$ and an element of $\mathcal{X} \subseteq \mathbb{R}^M$, i.e. $A\mathcal{X} = \{y \in \mathbb{R}^N \mid \exists\, x \in \mathcal{X}$ with $y = Ax\}$.

A.1.3 Notations for Correspondences and Functions

$\mathcal{F} : \mathcal{X} \to \mathcal{Y}$	$\mathcal{F}$ is the correspondence mapping elements of $\mathcal{X}$ on subsets of $\mathcal{Y}$: $\mathcal{F}(x) \subseteq \mathcal{Y}$ for $x \in \mathcal{X}$.
$f : \mathcal{X} \to \mathcal{Y}$	f is a function mapping elements of $\mathcal{X}$ on elements of $\mathcal{Y}$: $f(x) \in \mathcal{Y}$ for $x \in \mathcal{X}$.
f^{-1}	f^{-1} is the inverse function of a bijective function f.
$f \circ g : \mathcal{X} \to \mathcal{Z}$	The composition of two functions $g : \mathcal{X} \to \mathcal{Y}$ and $f : \mathcal{Y} \to \mathcal{Z}$.
$f_r(x), f_R(x)$	For a function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^N, \mathcal{Y} \subseteq \mathbb{R}^M$ the restriction of $f(x)$ to coordinate $r \in M$ respectively to those coordinates $r \in R \subset M$.
$f_{-r}(x), f_{-R}(x)$	For a function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^N, \mathcal{Y} \subseteq \mathbb{R}^M$ the restriction of $f(x)$ to those coordinates $\tilde{r} \in M$ not equal to r respectively not in $R \subset M$.
$f'(x)$	For a differentiable function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}, \mathcal{Y} \subseteq \mathbb{R}$ $f'(x) \in \mathbb{R}$ is the first derivative at point x.
$f''(x)$	For a twice differentiable function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}, \mathcal{Y} \subseteq \mathbb{R}$ $f''(x) \in \mathbb{R}$ is the second derivative at point x.
$\nabla f(x)$	For a differentiable function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^N, \mathcal{Y} \subseteq \mathbb{R}$ $\nabla f(x) \in \mathbb{R}^{1\times N}$ is the gradient at point x, i.e. the row vector consisting of the first derivatives of f to each variable at point x.
$\partial_{x_R} f(x)$	For a differentiable function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^N, \mathcal{Y} \subseteq \mathbb{R}^M$ $\partial_{x_R} f(x) \in \mathbb{R}^{M\times\sharp R}$ is the matrix consisting of the partial derivatives of f at point x with respect to the variables $x_r, r \in R$.
$Df(x)$	For a differentiable function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^N, \mathcal{Y} \subseteq \mathbb{R}^M$ $Df(x) \in \mathbb{R}^{M\times N}$ is the Jacobian matrix at point x, i.e. the matrix consisting of the first derivatives of f at point x.
$D^2 f(x)$	For a twice differentiable function $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^N, \mathcal{Y} \subseteq \mathbb{R}$ $D^2 f(x) \in \mathbb{R}^{N\times N}$ is the Hessian matrix at point x, i.e. the matrix consisting of the second derivatives of f at point x.
ln	The natural logarithm.

mod_N for $N \in \mathbb{N}$	The modulo function, i.e. the function which maps an arbitrary natural number on the natural number remaining after division by N.

A.1.4 Other Notations

$:=$	Definition.
$\Leftrightarrow$	Equivalence.
$\sum_{n=1}^{N}$	Sum over different terms indexed by $n = 1, \ldots, N$.
$\wedge$	Conjunction: and...
$\exists$	Existential quantifier: there exists...
$\forall$	Universal quantifier: for all...
∞	Infinity.
$\lfloor x \rfloor$	The largest natural number less or equal to $x \in \mathbb{R}$.
$\|x\|$	The absolute value of $x \in \mathbb{R}$.
$sign(x)$	For $x \in \mathbb{R}$: $sign(x) = \begin{cases} +1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$
lim	The limit.

A.2 Mathematical Definitions

A.2.1 Diagonal Dominance

For this topic compare Murata (1977, Section 1.3).

Definition A.1 (Diagonal Dominance)
A quadratic matrix $M = (M_{nl})_{n \in N, l \in N} \in \mathbb{R}^{N \times N}, N \in \mathbb{N}$, *has a row dominant diagonal if there are positive scalars* $d_1, \ldots, d_N$ *such that for all* $n \in N$

$$d_n |M_{nn}| > \sum_{\substack{l=1 \\ l \neq n}}^{N} d_l |M_{nl}|.$$

A quadratic matrix $M = (M_{nl})_{n \in N, l \in N} \in \mathbb{R}^{N \times N}, N \in \mathbb{N}$, *has a column dominant diagonal if there are positive scalars* $\tilde{d}_1, \ldots, \tilde{d}_N$ *such that for all* $n \in N$

$$\tilde{d}_l |M_{ll}| > \sum_{\substack{n=1 \\ n \neq l}}^{N} \tilde{d}_n |M_{nl}|.$$

A.2.2 Continuity of Correspondences

In this appendix I only want to give a brief recall of the definitions concerning continuity of correspondences (compare Mas-Colell, Whinston, and Green (1995) and Takayama (1974)).

Definition A.2 (Correspondence)
Given a set $\mathcal{X} \subseteq \mathbb{R}^N$, a correspondence $\mathcal{F} : \mathcal{X} \to \mathbb{R}^K$ is a rule that assigns a set $\mathcal{F}(x) \subseteq \mathbb{R}^K$ to each $x \in \mathcal{X}$.

Definition A.3 (Continuity of Correspondences)
The correspondence $\mathcal{F} : \mathcal{X} \to \mathbb{R}^K$ is called upper hemi-continuous at $\bar{x} \in \mathcal{X}$ if for each open set $\mathcal{O}$ containing $\mathcal{F}(\bar{x})$ there is some neighborhood $\mathcal{V}_\epsilon(\bar{x})$ such that

$$x \in \mathcal{V}_\epsilon(\bar{x}) \Rightarrow \mathcal{F}(x) \subseteq \mathcal{O}.$$

The correspondence $\mathcal{F} : \mathcal{X} \to \mathbb{R}^K$ is called lower hemi-continuous at $\bar{x} \in \mathcal{X}$ if for each open set $\mathcal{O}$ containing $\mathcal{F}(\bar{x})$ there is some neighborhood $\mathcal{V}_\epsilon(\bar{x})$ such that

$$x \in \mathcal{V}_\epsilon(\bar{x}) \Rightarrow \mathcal{F}(x) \cap \mathcal{O} \neq \emptyset.$$

The correspondence $\mathcal{F} : \mathcal{X} \to \mathbb{R}^K$ is called continuous at $\bar{x} \in \mathcal{X}$ if it is upper and lower hemi-continuous at $\bar{x}$.

A.3 Mathematical Theorems

In this appendix I recall some mathematical theorems applied by myself in the thesis or used for some quoted related results. I just present the pure theorems and refrain from giving further details. For this, I refer to the given literature. Nonetheless, those theorems directly applied in this work do not need any further explanation.

Theorem A.4 (Brouwer's Fixed Point Theorem)[1]
Suppose that $\mathcal{K} \subseteq \mathbb{R}^N$ is a non-empty, compact, and convex set. If $f : \mathcal{K} \to \mathcal{K}$ is a continuous function from $\mathcal{K}$ in itself, it has a fixed point, i.e. there is some $x \in \mathcal{K}$ such that $f(x) = x$.

[1]Compare Mas-Colell, Whinston, and Green (1995, Theorem M.I.1).

Theorem A.5 (Chaos Theorem) [2]
Suppose $\mathcal{K} \subseteq \mathbb{R}$ is an interval and $f : \mathcal{K} \to \mathcal{K}$ is a continuous function from $\mathcal{K}$ in itself. If there is some $x \in \mathcal{K}$ with $y = f(x), z = f(y)$, and $a = f(z)$ satisfying[3]

$$a \leq x < y < z \ (\textit{or } a \geq x > y > z),$$

then

1. *for every $n = 1, 2, \ldots$ there is a periodic point in $\mathcal{K}$ with period n;*
2. *there is an uncountable set $\mathcal{K}' \subset \mathcal{K}$ (containing no periodic points) with*
   (a) *for all $b, c \in \mathcal{K}'$ with $b \neq c$*

   $$\lim_{n\to\infty} \sup |f^n(b) - f^n(c)| > 0$$

   and

   $$\lim_{n\to\infty} \inf |f^n(b) - f^n(c)| = 0;$$

   (b) *for all $b \in \mathcal{K}'$ and a periodic point $c \in \mathcal{K}$*

   $$\lim_{n\to\infty} \sup |f^n(b) - f^n(c)| > 0.$$

Theorem A.6 (Diagonal Dominance) [4]
If $M \in \mathbb{R}^{N\times N}$ is column diagonal dominant, it is invertible.

Theorem A.7 (Implicit Function Theorem) [5]
Suppose $\mathcal{K}^1 \subseteq \mathbb{R}^N$ and $\mathcal{K}^2 \subseteq \mathbb{R}^M$ are open sets and

$$\begin{array}{rll} f: & \mathcal{K}^1 \times \mathcal{K}^2 & \to \mathbb{R}^M \\ & (x, y) & \mapsto f(x, y) \end{array}$$

is a continuously differentiable function. If $(\bar{x}, \bar{y}) \in \mathcal{K}^1 \times \mathcal{K}^2$ with $f(\bar{x}, \bar{y}) = 0$ and $\partial_y f(\bar{x}, \bar{y})$ is invertible, then there exist open sets $\mathcal{X} \subseteq \mathcal{K}^1$ and $\mathcal{Y} \subseteq \mathcal{K}^2$ with $\bar{x} \in \mathcal{X}$ and $\bar{y} \in \mathcal{Y}$ and a continuously differentiable function $g : \mathcal{X} \to \mathcal{Y}$ with $f(x, g(x)) = 0$ for all $x \in \mathcal{X}$. If $(x, y) \in \mathcal{X} \times \mathcal{Y}$ such that $f(x, y) = 0$, then $y = g(x)$. Besides

$$D_x g(x) = -\left[\partial_y f(x, g(x))\right]^{-1} \partial_x f(x, g(x)).$$

---

[2] Compare Li and Yorke (1975, Theorem 1).
[3] This condition is satisfied if there is a periodic point of period three.
[4] Compare Murata (1977, Theorems 21 and 24).
[5] Compare Forster (1991, §8, Satz 1 and 2).

Theorem A.8 (Maximum Theorem)[6]
Suppose $\mathcal{K} \subseteq \mathbb{R}^N$ is a compact set and

$$\begin{aligned} f: \ \mathcal{K} &\to \mathbb{R} \\ x &\mapsto f(x) \end{aligned}$$

is a continuous function. Then the function f is bounded from above and from below on $\mathcal{K}$ and takes its maximum as well as its minimum.

Theorem A.9 (Parametric Transversality Theorem)[7]
Suppose $\mathcal{K}^1 \subseteq \mathbb{R}^N$ and $\mathcal{K}^2 \subseteq \mathbb{R}^S$ are open sets of full dimension and

$$\begin{aligned} f: \ \mathcal{K}^1 \times \mathcal{K}^2 &\to \mathbb{R}^M \\ (x,y) &\mapsto f(x,y) \end{aligned}$$

is a function which is as many times differentiable as necessary. If for all $(\bar{x},\bar{y}) \in \mathcal{K}^1 \times \mathcal{K}^2$ with $f(\bar{x},\bar{y}) = 0$ it is rank $D_{(x,y)}f(\bar{x},\bar{y}) = M$, then there exists a set $\tilde{\mathcal{K}} \subseteq \mathcal{K}^2$ with $\mathcal{K}^2 \backslash \tilde{\mathcal{K}}$ a set of measure zero, such that rank $\partial_x f(x,y) = M$ whenever $y \in \tilde{\mathcal{K}}$ and $f(x,y) = 0$.

Theorem A.10 (Poincaré-Hopf Theorem)[8]
Suppose $\mathcal{N}$ is a compact N-dimensional C^1 manifold with boundary and f a continuous vectorfield on $\mathcal{N}$. If

1. *f points outward at $\mathcal{N}$ (when $\mathcal{N} \subseteq \mathbb{R}^N$ this means $f(x) \cdot g(x) > 0$ for all $x \in \partial\mathcal{N}$ where g is the Gauss map); and*

2. *f has a finite number of zeros,*

then the sum of the indices of f at the different zeros equals the Euler characteristic of $\mathcal{N}$.

Theorem A.11 (Sarkowski's Theorem)[9]
Consider the ordering of integers

$$\begin{array}{lllllllll} & 3 & >_S & 5 & >_S & 7 & >_S & \ldots & \\ >_S & 2\cdot 3 & >_S & 2\cdot 5 & >_S & 2\cdot 7 & >_S & \ldots & \\ \ldots & & & & & & & & \\ >_S & 2^n\cdot 3 & >_S & 2^n\cdot 5 & >_S & 2^n\cdot 7 & >_S & \ldots & \\ \ldots & & & & & & & & \\ >_S & \ldots & >_S & 2^m & >_S & \ldots & >_S & 2 & >_S \ 1. \end{array}$$

[6] Compare Forster (1991, §3, Satz 7).
[7] Compare Mas-Colell, Whinston, and Green (1995, Proposition 17.D.3).
[8] Compare Mas-Colell (1985, Theorem J.3.2).
[9] Compare Grandmont (1986, Theorem 1).

That is, first the odd integers greater than or equal to 3 forward, then the powers of 2 times these odd integers, and then the powers of 2 backward. Suppose $\mathcal{K} \subseteq \mathbb{R}$ *is a closed interval and* $f : \mathcal{K} \to \mathcal{K}$ *is a continuous function from* $\mathcal{K}$ *in itself. If* f *has a cycle with period* k*, then it has a cycle of period* k' *for every integer* k' *such that* $k >_S k'$*.*

Theorem A.12 (Separation Theorem)[10]
Suppose $\mathcal{X}$ *is a finite-dimensional vector space and* $\mathcal{K}^1, \mathcal{K}^2$ *are non-empty, convex subsets of* $\mathcal{X}$ *with* $\mathcal{K}^1 \cap \mathcal{K}^2 = \emptyset$*. If* $\mathcal{K}^1$ *is compact and* $\mathcal{K}^2$ *is closed, then there exists* $\pi \in \mathcal{X}, \pi \neq 0$ *such that*

$$\sup_{x \in \mathcal{K}^2} \pi \cdot x < \inf_{x \in \mathcal{K}^1} \pi \cdot x.$$

[10] Compare Magill and Quinzii (1996b, Theorem 9.4).

Appendix B

Proofs

B.1 Proofs for Part II

B.1.1 Proof of Lemma 4.27

Distinguish the three types of matrices.

(I) Since multiplication by a matrix is a continuous function, equation (4.17) makes clear that for each $\tilde{\epsilon}_2 > 0$ there is some $\epsilon_2 > 0$ such that $A \in \mathcal{V}_{\epsilon_2}(\bar{A})$ implies $\tilde{A}(A) \in \mathcal{V}_{\tilde{\epsilon}_2}(\tilde{A}(\bar{A}))$.

(II) Modeling first period consumption, the matrix $\bar{A}$ always satisfies Assumption 4.25.

(III) For given $\tilde{\epsilon}_2$ define ϵ_2 in analogy to case (I). Then those ϵ_2-perturbations of $\bar{A}$ have to be considered which are elements of the set

$$\{A \in \mathcal{V}_{\epsilon_2}(\bar{A})|\ A\theta_{\bar{A}} > 0\}.$$

This is a quite inconvenient restriction. A much stronger, but more convenient restriction is given by

$$\{A \in \mathcal{V}_{\dot{\epsilon}_2}(\bar{A})|\ A_{sj} = \bar{A}_{sj} \text{ if } \bar{A}_{s\cdot}\theta_{\bar{A}} = 0\}.$$

where

$$\dot{\epsilon}_2 < \min\left\{\epsilon_2, \min\left\{\frac{\bar{A}_{s\cdot}\theta_{\bar{A}}}{\sum_{j=1}^{J}|\theta_{\bar{A}j}|}|\ s \in \dot{S}\right\}\right\}$$

with

$$\dot{S} = \{s \in S|\ \bar{A}_{s\cdot}\theta_{\bar{A}} > 0\}.$$

$\square$

B.1.2 Proof of Lemma 4.28

Since $\mathcal{K}$ is compact, from the different representations of $\mathcal{Q}_{norm}(A)$ given on page 90 f. the existence of $l_s, u_s \in (0,1), s \in \mathcal{S}^+(\bar{A}_{\cdot 1})$, and $l_s, u_s \in (0,\infty), s \in \mathcal{S}^0(\bar{A}_{\cdot 1})$ is known, such that for all $\hat{q} \in \mathcal{K}$ there is some $\mu \in \prod_{s=1}^{S}[l_s, u_s]$ with

$$\begin{pmatrix}1\\ \hat{q}\end{pmatrix} = \sum_{s\in\mathcal{S}^+(\bar{A}_{\cdot 1})} \mu_s \begin{pmatrix}1\\ \frac{1}{\bar{A}_{s1}}\bar{A}^T_{s(-1)}\end{pmatrix} + \sum_{s\in\mathcal{S}^0(\bar{A}_{\cdot 1})} \mu_s \begin{pmatrix}0\\ \bar{A}^T_{s(-1)}\end{pmatrix}.$$

Because of $rank\ \bar{A} = J$ there is some subset $G \subseteq \{1,\ldots,S\}$ with $\sharp G = J$ such that

$$\left(\begin{pmatrix}1\\ \frac{1}{A_{s1}(C)}A^T_{s(-1)}(C)\end{pmatrix}_{s\in G\cap\mathcal{S}^+(\bar{A}_{\cdot 1})}, \begin{pmatrix}0\\ A^T_{s(-1)}(C)\end{pmatrix}_{s\in G\cap\mathcal{S}^0(\bar{A}_{\cdot 1})}\right)$$

is invertible for all $C \in \mathcal{V}_{\epsilon_2}(\bar{C})$.[1] The inverse is called $I^{-1}(C)$. Define

$$\mu_G: \quad \mathcal{K} \times \prod_{s\in\{1,\ldots,S\}\setminus G} [l_s, u_s] \times cl(\mathcal{V}_{\frac{\epsilon_2}{2}}(\bar{C})) \quad \to \mathbb{R}^J$$

by

$$\begin{aligned}
&\mu_G(\hat{q}, (\mu_s)_{s\in\{1,\ldots,S\}\setminus G}, C) := \\
&\quad I^{-1}(C)\left[\begin{pmatrix}1\\ \hat{q}\end{pmatrix} - \sum_{s\in\mathcal{S}^+(\bar{A}_{\cdot 1})\setminus G} \mu_s \begin{pmatrix}1\\ \frac{1}{A_{s1}(C)}A(C)^T_{s(-1)}\end{pmatrix}\right. \\
&\quad \left. - \sum_{s\in\mathcal{S}^0(\bar{A}_{\cdot 1})\setminus G} \mu_s \begin{pmatrix}0\\ A^T_{s(-1)}(C)\end{pmatrix}\right].
\end{aligned}$$

This function is continuous and — because defined on a compact set — even uniformly continuous. Therefore, for some

$$\bar{\delta} \in (0, \min\{l_s|\ s \in G\})$$

there is an $\epsilon_3 > 0$ such that $\|C - \bar{C}\| < \epsilon_3$ implies

$$\|\mu_G(\hat{q}, (\mu_s)_{s\in\{1,\ldots,S\}\setminus G}, C) - \mu_G(\hat{q}, (\mu_s)_{s\in\{1,\ldots,S\}\setminus G}, \bar{C})\| < \bar{\delta}$$

[1] To be correct in notation assume without loss of generality that $s_1 \in \mathcal{S}^+(\bar{A}_{\cdot 1})$ and $s_2 \in \mathcal{S}^0(\bar{A}_{\cdot 1})$ implies $s_2 > s_1$. Otherwise renumerate the states.

for all $(\hat{q}, (\mu_s)_{s\in\{1,\dots,S\}\setminus G}) \in \mathcal{K} \times \prod_{s\in\{1,\dots,S\}\setminus G}[l_s, u_s]$. Hence, for all $\hat{q} \in \mathcal{K}$ one gets

$$\begin{aligned}\begin{pmatrix}1\\ \hat{q}\end{pmatrix} &= \sum_{s\in G\cap\mathcal{S}^+(\bar{A}_{\cdot 1})} \underbrace{\mu_s(\hat{q}, (\mu_{s\in\{1,\dots,S\}\setminus G}, C)}_{\geq\ \mu_s(\hat{q},(\mu_{s\in\{1,\dots,S\}\setminus G},\bar{C})-\bar{\delta}>0} \begin{pmatrix}1\\ \frac{1}{A_{s1}(C)}A^T_{s(-1)}(C)\end{pmatrix}\\ &+ \sum_{s\in G\cap\mathcal{S}^0(\bar{A}_{\cdot 1})} \underbrace{\mu_s(\hat{q}, (\mu_{s\in\{1,\dots,S\}\setminus G}, C)}_{\geq\ \mu_s(\hat{q},(\mu_{s\in\{1,\dots,S\}\setminus G},\bar{C})-\bar{\delta}>0} \begin{pmatrix}0\\ A^T_{s(-1)}(C)\end{pmatrix}\\ &+ \sum_{s\in\mathcal{S}^+(\bar{A}_{\cdot 1})\setminus G} \underbrace{\mu_s}_{>0} \begin{pmatrix}1\\ \frac{1}{A_{s1}(C)}A^T_{s(-1)}(C)\end{pmatrix}\\ &+ \sum_{s\in\mathcal{S}^0(\bar{A}_{\cdot 1})\setminus G} \underbrace{\mu_s}_{>0} \begin{pmatrix}0\\ A^T_{s(-1)}(C)\end{pmatrix}\end{aligned}$$

such that $\hat{q} \in \tilde{\mathcal{Q}}_{norm}(C)$ for all $C \in \mathcal{V}_{\epsilon_3}(\bar{C})$.

□

B.1.3 Proof of Lemma 4.33

The set of attainable and individually rational consumption plans is defined as

$$\mathcal{C}^i(\omega^i, \omega) := \{x \in \mathbb{R}^S_+ \mid U^i(x) \geq U^i(\omega^i) \ \wedge x \leq \omega\}$$

which obviously is a compact subset of $\mathbb{R}^S_+$.[2] Equilibrium portfolio allocations $(\theta^{*i})_{i=1,\dots,I}$ for asset structure A have to satisfy $\omega^i + A\theta^{*i} \in \mathcal{C}^i(\omega^i, \omega)$ for all $i = 1, \dots, I$.

Let $\epsilon_7 \in (0, \epsilon_6)$. Define

$$\begin{aligned} h^i:\ \ \mathcal{C}^i(\omega^i, \omega) \times cl(\mathcal{V}_{\epsilon_7}(\bar{C})) &\to \mathbb{R}^{J-1}\\ (x, C) &\mapsto \left[\frac{1}{\nabla U^i(x) A_{\cdot 1}(C)} \nabla U^i(x) A_{\cdot(-1)}(C)\right]^T. \end{aligned}$$

Because of $A_{\cdot 1}(C) > 0$ and $\nabla U^i(x) \gg 0$ this function is continuous and even uniformly continuous since it is defined on a compact set. Images of compact sets under continuous functions are compact such that

$$\mathcal{H}(C) := \bigcap_{i=1}^{I} h^i(\mathcal{C}^i(\omega^i, \omega), C)$$

[2]This is implied by the properties of U^i.

as finite intersection of compact sets is compact for $C \in cl(\mathcal{V}_{\epsilon_7}(\bar{C}))$. Since first order conditions (2.7) have to be satisfied for all consumers $i = 1, \ldots, I$ one obtains

$$\mathcal{E}_{norm}(A(C)) \subseteq \mathcal{H}(C)$$

for all $C \in cl(\mathcal{V}_{\epsilon_7}(\bar{C}))$. There has to be some $\delta > 0$ with $\mathcal{H}(\bar{C}) + cl(\mathcal{V}_\delta(0)) \subseteq \tilde{\mathcal{Q}}_{norm}(\bar{C})$[3] because $\mathcal{H}(\bar{C}) \subseteq \tilde{\mathcal{Q}}_{norm}(\bar{C})$ is compact and $\tilde{\mathcal{Q}}_{norm}(\bar{C})$ is open. Application of Lemma 4.28 delivers the existence of some $\epsilon_8 \in (0, \epsilon_6]$ with

$$\mathcal{H}(\bar{C}) + cl(\mathcal{V}_\delta(0)) \subseteq \tilde{\mathcal{Q}}_{norm}(C)$$

for all $C \in \mathcal{V}_{\epsilon_8}(\bar{C})$. On the other hand there has to be some $\epsilon_9 > 0$ such that for all $C \in \mathcal{V}_{\epsilon_9}(\bar{C})$

$$\mathcal{H}(C) \subseteq \mathcal{H}(\bar{C}) + \mathcal{V}_\delta(0).$$

This follows from uniform continuity of h^i because given $\delta > 0$ there is some $\check{\epsilon}^i > 0$ such that

$$\|(y, C) - (x, \tilde{C})\|_{\max} < \check{\epsilon}^i \Rightarrow \|h^i(y, C) - h^i(x, \tilde{C})\|_{\max} < \frac{\delta}{I}$$

for all $(y, C), (x, \tilde{C}) \in \mathcal{C}^i(\omega^i, \omega) \times cl(\mathcal{V}_{\epsilon_7}(\bar{C}))$ and therefore especially

$$C \in \mathcal{V}_{\check{\epsilon}^i}(\bar{C}) \Rightarrow \|h^i(x, C) - h^i(x, \bar{C})\|_{\max} < \frac{\delta}{I}$$

for all $x \in \mathcal{C}^i(\omega^i, \omega)$. Therefore for all $\hat{q} \in \mathcal{H}(C)$ there is some $\bar{\hat{q}} \in \mathcal{H}(\bar{C})$ with

$$\hat{q} \in \{\bar{\hat{q}}\} + \mathcal{V}_\delta(0)$$

when $\epsilon_9 := \min\{\check{\epsilon}^i |\ i = 1, \ldots, I\}$. If $\epsilon_{10} := \min\{\frac{1}{2}\epsilon_8, \frac{1}{2}\epsilon_9\}$, $C \in cl(\mathcal{V}_{\epsilon_{10}}(\bar{C}))$ implies

$$\mathcal{H}(C) \subseteq \mathcal{H}(\bar{C}) + cl(\mathcal{V}_\delta(0)) \subseteq \tilde{\mathcal{Q}}_{norm}(C) \cap \tilde{\mathcal{Q}}_{norm}(\bar{C}).$$

The set

$$\mathcal{Q}^H := \left[\mathcal{H}(\bar{C}) + cl(\mathcal{V}_\delta(0))\right] \setminus \bigcup_{n=1}^{N} \mathcal{V}_{\delta^n}(\hat{q}^{*n})$$

is a compact subset of $\mathbb{R}^{J-1}$. The construction of this set assures that as soon as $C \in cl(\mathcal{V}_{\epsilon_{10}}(\bar{C}))$ allows that additional equilibrium price systems unequal to $\hat{q}^1_*(C), \ldots, \hat{q}^N_*(C)$ exist, these systems have to be elements of $\mathcal{Q}^H$.

$\square$

[3]In other words: $\mathcal{H}(\bar{C})$ can be extended a little bit without leaving $\tilde{\mathcal{Q}}_{norm}(\bar{C})$.

B.2 Proofs for Part IV

B.2.1 Proof of Lemma 9.6

Part (1):

$$
\begin{aligned}
& x^{i_k}(\tilde{p}_0, \ldots, \tilde{p}_K) \\
&= \arg\max_{x \in \mathbb{R}_+^{(K+1)L}} \tilde{U}^{i_k}(x) \text{ s.t. } \tilde{p} \cdot x = \tilde{p} \cdot \tilde{\omega}^{i_k} \\
&= \arg\max_{x \in \mathbb{R}_+^{(K+1)L}} \tilde{U}^{i_0}(x_{s_K^k(0)}, \ldots, x_{s_K^k(K)}) \\
&\quad \text{s.t. } \sum_{g \in \tilde{K}} \tilde{p}_g x_g = \tilde{p} \cdot \begin{pmatrix} \tilde{\omega}^{i_0}_{s_K^{-k}(0)} \\ \ldots \\ \tilde{\omega}^{i_0}_{s_K^{-k}(K)} \end{pmatrix} \\
&= \arg\max_{x \in \mathbb{R}_+^{(K+1)L}} \tilde{U}^{i_0}(x_{s_K^k(0)}, \ldots, x_{s_K^k(K)}) \\
&\quad \text{s.t. } \sum_{g \in \tilde{K}} \tilde{p}_g x_g = \sum_{g \in \tilde{K}} \tilde{p}_{s_K^k(g)} \tilde{\omega}^{i_0}_g \\
&= \arg\max_{x \in \mathbb{R}_+^{(K+1)L}} \tilde{U}^{i}(x_{s_K^k(0)}, \ldots, x_{s_K^k(N)}) \\
&\quad \text{s.t. } \sum_{g \in \tilde{K}} \tilde{p}_{s_K^k(g)} x_{s_K^k(g)} = \sum_{n \in \tilde{N}} \tilde{p}_{s_K^k(n)} \omega^i_n
\end{aligned}
$$

This formula allows to conclude that

$$x_g^{i_k}(\tilde{p}_0, \ldots, \tilde{p}_K) = x^i_{s_K^{-k}(g)}(\tilde{p}_{s_K^k(0)}, \ldots, \tilde{p}_{s_K^k(N)})$$

for all $g \in \{s_K^k(0), \ldots, s_K^k(N)\}$ because the other elements of x^{i_k} are zero as a consequence of strict monotonicity of U^i.

Part (2): According to Part (1) of this lemma it is

$$
\begin{aligned}
x_g^{i_k}(\tilde{p}_{s_K(0)}, \ldots, \tilde{p}_{s_K(K)}) &= \begin{cases} 0, \text{ if } g \notin \{s_K^k(0), \ldots, s_K^k(N)\}, \\ x^i_{s_K^{-k}(g)}(\tilde{p}_{s_K^{k+1}(0)}, \ldots, \tilde{p}_{s_K^{k+1}(N)}), \text{ otherwise.} \end{cases} \\
x^{i_{s_K(k)}}_{s_K(g)}(\tilde{p}_0, \ldots, \tilde{p}_K) &= \begin{cases} 0, \text{ if } s_K(g) \notin \{s_K^{k+1}(0), \ldots, s_K^{k+1}(N)\}, \\ x^i_{s_K^{-k-1+1}(g)}(\tilde{p}_{s_K^{k+1}(0)}, \ldots, \tilde{p}_{s_K^{k+1}(N)}), \text{ otherwise.} \end{cases} \\
&= \begin{cases} 0, \text{ if } g \notin \{s_K^k(0), \ldots, s_K^k(N)\}, \\ x^i_{s_K^{-k}(g)}(\tilde{p}_{s_K^{k+1}(0)}, \ldots, \tilde{p}_{s_K^{k+1}(N)}), \text{ otherwise.} \end{cases}
\end{aligned}
$$

$\square$

B.2.2 Proof of Lemma 9.12

I only prove the first part, the other part follows by application of Lemma 9.10. Besides Lemma 9.10 allows to assume $b' = 0$. Therefore take $H := b-1$ with $\tilde{H} = \{0, \dots, b-1\}$.

Obviously, from Lemma 9.6, it is known for the $H+1 = b$-modulo exchange economy for all $i \in I, k \in \tilde{H}$ it is

$$\begin{aligned} x_g^{i_k}(p_0^*, \dots, p_{b-1}^*) &= 0, \text{ if } g \notin \{s_H^k(0), s_H^k(1)\}, \\ x_{s_H^k(0)}^{i_k}(p_0^*, \dots, p_{b-1}^*) &= x_0^i(p^*_{s_H^k(0)}, p^*_{s_H^k(1)}), \\ x_{s_H^k(1)}^{i_k}(p_0^*, \dots, p_{b-1}^*) &= x_1^i(p^*_{s_H^k(0)}, p^*_{s_H^k(1)}). \end{aligned}$$

The market clearing conditions which have to be verified are as follows:

$$\sum_{i \in I} \sum_{k=0}^{b-1} x_g^{i_k}(p_0^*, \dots, p_{b-1}^*) = \sum_{i \in I} (\omega_0^i + \omega_1^i) \text{ for all } g \in \tilde{H}.$$

This is equivalent to

$$\begin{aligned} \sum_{i \in I} x_0^i(p^*_{s_H^g(0)}, p^*_{s_H^g(1)}) + \sum_{i \in I} x_1^i(p^*_{s_H^{g-1}(0)}, p^*_{s_H^{g-1}(1)}) \\ = \sum_{i \in I} (\omega_0^i + \omega_1^i) \text{ for all } g \in \tilde{H}. \end{aligned} \tag{2.1}$$

Since

$$\begin{aligned} p^*_{s_H^g(0)} &= p^*_{s_{K'}^g(0)} \text{ for } g = 0, \dots, b-1, \\ p^*_{s_H^g(1)} &= p^*_{s_{K'}^g(1)} \text{ for } g = 0, \dots, b-1, \\ p^*_{s_H^{g-1}(0)} &= p^*_{s_{K'}^{g-1}(0)} \text{ for } g = 1, \dots, b-1, \\ p^*_{s_H^{g-1}(1)} &= p^*_{s_{K'}^{g-1}(1)} \text{ for } g = 0, \dots, b-1, \end{aligned}$$

equations (2.1) are satisfied for $g = 1, \dots, b-1$ because the corresponding equations are satisfied for the equilibrium prices $(\tilde{p}_0, \dots, \tilde{p}_{b-1}, \tilde{p}_b, \dots, \tilde{p}_{K'})$ in the $(K'+1)$-modulo exchange economy. Besides, from the budget constraints it is known for $k = 0, \dots, b-1$ that

$$\begin{aligned} & x_1^i(p^*_{s_H^k(0)}, p^*_{s_H^k(1)}) - \omega_1^i \\ &= -\frac{p^*_{s_H^k(0)}}{p^*_{s_H^k(1)}} \left(x_0^i(p^*_{s_H^k(0)}, p^*_{s_H^k(1)}) - \omega_0^i \right). \end{aligned} \tag{2.2}$$

It remains to prove equation (2.1) for $g = 0$. Therefore, apply alternately equations (2.2, $k = 0, \ldots, b-1$) and equations (2.1, $g = 1, \ldots, b-1$):

$$\begin{aligned}
\sum_{i \in I} (x_1^i(p_{b-1}^*, p_0^*) - \omega_1^i) &= -\frac{p_{b-1}^*}{p_0^*} \sum_{i \in I} \left(x_0^i(p_{b-1}^*, p_0^*) - \omega_0^i \right) \\
&= \frac{p_{b-1}^*}{p_0^*} \sum_{i \in I} \left(x_1^i(p_{b-2}^*, p_{b-1}^*) - \omega_1^i \right) \\
&= -\frac{p_{b-2}^*}{p_0^*} \sum_{i \in I} \left(x_0^i(p_{b-2}^*, p_{b-1}^*) - \omega_0^i \right) \\
&= \ldots \\
&= -\frac{p_0^*}{p_0^*} \sum_{i \in I} \left(x_0^i(p_0^*, p_1^*) - \omega_0^i \right) .
\end{aligned}$$

□

B.2.3 Proof of Theorem 10.3

(1): Follows from Lemma 9.8 (existence) and Proposition 10.2 (uniqueness).

(2): From Remark 9.9 in the $(N+1)$-modulo exchange economy there exists an equilibrium of which the price system is a replication of an L-dimensional vector. Therefore, the only point to prove is uniqueness.

Assume there are two different such equilibria with allocations $(x^{*i_0}, \ldots, x^{*i_N})$ and $(\hat{x}^{i_0}, \ldots, \hat{x}^{i_N})$ for $i = 1$. Then, the allocation

$$\bar{x}^{i_k} = \lambda x^{*i_k} + (1-\lambda)\hat{x}^{i_k} \text{ for some } \lambda \in (0,1)$$

is also attainable in the $(N+1)$-modulo exchange economy. Besides

$$\begin{aligned}
\tilde{U}^{i_k}(\bar{x}^{i_k}) &> \min\{\tilde{U}^{i_k}(x^{*i_k}), \tilde{U}^{i_k}(\hat{x}^{i_k})\} \\
&= \tilde{U}^{i_k}(x^{*i_k})
\end{aligned}$$

where the inequality follows from strict quasi-concavity of $\tilde{U}^{i_k}$. The last equation can be assumed w.l.o.g. since symmetry and $I = 1$ assure the same utility for all consumers in an equilibrium. This contradicts Arrow-Debreu theory because all equilibria in the $(N+1)$-modulo exchange economy have to be Pareto-efficient (utility of the $^-$-allocation is higher than utility of the *-allocation). It remains to apply Proposition 10.2.

(3): Existence of an equilibrium in the $(N+1)$-modulo exchange economy follows from Arrow-Debreu theory. According to the proof of Proposition 3.19 (Hens and Löffler (1995)) market demand in this economy satisfies the strict gross substitution property and therefore the economy has a unique equilibrium. Apply Proposition 10.2.

(4),(5): A monetary stationary OLG-equilibrium is defined via the following extended system:

$$\delta^n \nabla V^i(c_n^i) - \lambda^i p^T = 0, \text{ for all } n \in \tilde{N}, i \in I, \tag{2.3}$$

$$p \cdot \sum_{n \in \tilde{N}} c_n^i = p \cdot \sum_{n \in \tilde{N}} \omega_n^i, \text{ for all } i \in I, \tag{2.4}$$

$$\sum_{i \in I} \sum_{n \in \tilde{N}} c_n^i = \sum_{i \in I} \sum_{n \in \tilde{N}} \omega_n^i. \tag{2.5}$$

If $V^i(y) = \sum_{l \in L} v_l^i(y_l)$ for some functions $v_l^i : \mathbb{R}_+ \to \mathbb{R}$, equation (2.3) is equivalent to

$$\delta^n v_l^{i\prime}(c_{nl}^i) - \lambda^i p_l = 0 \text{ for all } n \in \tilde{N}, l \in L, i \in I.$$

Substituting the equations for $n = 0$ in the equations for $n \in N$ gives

$$\delta^n v_l^{i\prime}(c_{nl}^i) = v_l^{i\prime}(c_{0l}^i) \text{ for all } n \in N, l \in L, i \in I,$$

which because of homogeneity of $v_l^{i\prime}$ can be rewritten as

$$\delta^n \left(\frac{c_{nl}^i}{c_{0l}^i}\right)^d v_l^{i\prime}(c_{0l}^i) = v_l^{i\prime}(c_{0l}^i) \text{ for all } n \in N, l \in L, i \in I,$$

or equivalently (since $d \neq 0$)

$$c_{nl}^i = \delta^{\frac{-n}{d}} c_{0l}^i \text{ for all } n \in N, l \in L, i \in I.$$

Therefore, equations (2.3)–(2.5) can be rewritten as

$$v_l^{i\prime}(c_{0l}^i) - \lambda^i p_l = 0 \text{ for all } l \in L, i \in I, \tag{2.6}$$

$$c_{nl}^i = \delta^{\frac{-n}{d}} c_{0l}^i \text{ for all } n \in N, l \in L, i \in I, \tag{2.7}$$

$$p \cdot c_0^i = p \cdot \left(\frac{1}{\sum_{n \in \tilde{N}} \delta^{\frac{-n}{d}}} \sum_{n \in \tilde{N}} \omega_n^i \right) \text{ for all } i \in I, \tag{2.8}$$

$$\sum_{i \in I} c_0^i = \sum_{i \in I} \left(\frac{1}{\sum_{n \in \tilde{N}} \delta^{\frac{-n}{d}}} \sum_{n \in \tilde{N}} \omega_n^i \right). \tag{2.9}$$

Obviously, this system of equations has the same number of solutions as the system of equations (2.6), (2.8), and (2.9). However, this reduced system describes the extended system for the Arrow-Debreu economy

$$GE = \{I, L; (\sum_{l \in L} v_l^i(c_l), \frac{1}{\sum_{n \in \tilde{N}} \delta^{\frac{-n}{d}}} \sum_{n \in \tilde{N}} \omega_n^i)_{i \in I}\}.$$

This economy has an equilibrium and by application of Proposition 3.16 (i.e. the results given in Shafer and Sonnenschein (1982, Section 3) on the existence of a representative consumer) under assumptions (4) or (5) it follows uniqueness of the equilibrium in this new Arrow-Debreu economy. Therefore the system (2.3)–(2.5) has a unique solution.

(6),(7): Existence of an equilibrium in the $(N+1)$-modulo exchange economy follows from Arrow-Debreu theory. Again application of Proposition 3.16 (i.e. applications of the results given in Shafer and Sonnenschein (1982)) is possible which guarantees the existence of a representative consumer in the $(N+1)$-modulo exchange economy and therefore uniqueness of the equilibrium. Apply Proposition 10.2.

(8): Existence of an equilibrium in the $(N+1)$-modulo exchange economy follows from Arrow-Debreu theory. Uniqueness in this economy is obtained by Proposition 3.18 (i.e. via the results from Mitjushin and Polterovich (1978)) because of the special structure of the endowments. Apply Proposition 10.2.

(9): Existence of an equilibrium in the $(N+1)$-modulo exchange economy follows from Arrow-Debreu theory. Application of Proposition 3.13 on complete risk sharing in this economy is possible because the aggregate endowment is riskless: all Pareto-efficient allocations in the modulo exchange economy are riskless. Since all equilibria have to be Pareto-efficient, this uniquely determines the equilibrium. Apply Proposition 10.2.

(10): Existence of an equilibrium in the $(N+1)$-modulo exchange economy follows from Arrow-Debreu theory. It is unique because all consumers are characterized by quadratic utility functions with an objective weighting for all commodities (compare Proposition 3.14). Apply Proposition 10.2.

□

B.2.4 Proof of Lemma 10.7

Assume $(\bar{x}^{i_0}, \ldots, \bar{x}^{i_K})_{i \in I}$ is a Pareto-improvement in the $(K+1)$-modulo exchange economy. Without loss of generality it is

(2.10) $$\bar{x}_g^{i_k} = 0 \text{ if } g \notin \{s_K^k(0), \ldots, s_K^k(N)\}$$

for all $g, k \in \tilde{K}$ and $i \in I$ and

$$\sum_{i \in I} \sum_{k \in K} \bar{x}_g^{i_k} = \sum_{i \in I} \sum_{k \in K} \tilde{\omega}_g^{i_k} = \sum_{i \in I} \sum_{n \in \tilde{N}} \omega_n^i \tag{2.11}$$

for all $g \in \tilde{K}$ because otherwise the monotonicity properties of $(\tilde{U}^{i_k})_{k \in \tilde{K}, i \in I}$ would make further improvements possible. Besides for all $i \in I$ and $k \in \tilde{K}$

$$\tilde{U}^{i_k}(\bar{x}^{i_k}) \geq \tilde{U}^{i_k}(\tilde{\omega}^{i_k}) \tag{2.12}$$

with at least one strict inequality. This implies for all $i \in I$ and $k \in \tilde{K}$

$$U^i(\bar{x}^{i_k}_{s^k_K(0)}, \ldots, \bar{x}^{i_k}_{s^k_K(N)}) \geq U^i(\omega^i) \tag{2.13}$$

with at least one strict inequality.

Now define the $(K+1)$-cyclical OLG-allocation by

$$\Big(\ldots, (\bar{x}_0^{i_0}, \ldots, \bar{x}_N^{i_0}), (\bar{x}^{i_1}_{s^1_K(0)}, \ldots, \bar{x}^{i_1}_{s^1_K(N)}), \ldots, (\bar{x}^{i_K}_{s^K_K(0)}, \ldots, \bar{x}^{i_K}_{s^K_K(N)}), \ldots\Big)$$

which because of (2.10) and (2.11) satisfies OLG-market clearing for all $t \in \mathbb{Z}$. Besides (2.13) implies Pareto-improvement in the OLG-economy which constitutes a contradiction.

$\square$

B.2.5 Proof of Proposition 10.8

I will show that for all $P \geq 2$ the P-modulo exchange economy is characterized by a unique equilibrium. Therefore, fix some $\bar{P} \geq 2$ and $K = \bar{P} - 1$.

From Arrow-Debreu theory it is known that individual demand functions in the $(K+1)$-modulo exchange economy satisfy the weak gross substitution property for this special kind of utility functions. However, a priori weak gross substitution is not sufficient to guarantee a unique equilibrium.

At first, have a look on individual demand functions. It is

$$\partial_{p_{g'}} x_g^{i_k}(\tilde{p}) := \Big(\partial_{p_{g'l'}} x_{gl}^{i_k}(\tilde{p})\Big)_{l \in L, l' \in L}$$

where

$$\partial_{p_{g'}} x_g^{i_k}(\tilde{p}) \begin{cases} = 0, \text{ if } g \notin \{s_K^k(0), \ldots, s_K^k(N)\}, \\ = 0, \text{ if } g \in \{s_K^k(0), \ldots, s_K^k(N)\}, \text{but } g' \notin \{s_K^k(0), \ldots, s_K^k(N)\}, \\ \gg 0, \text{ if } g, g' \in \{s_K^k(0), \ldots, s_K^k(N)\} \text{ and } g \neq g', \\ \text{a matrix with strictly neg. diagonal and strictly pos.} \\ \quad \text{off-diagonal elements, if } g, g' \in \{s_K^k(0), \ldots, s_K^k(N)\} \\ \quad \text{and } g = g'. \end{cases}$$

Therefore, for market demand $x(\tilde{p}) = \sum_{i \in I} \sum_{k \in \tilde{K}} x^{i_k}(\tilde{p})$ observe

$$\partial_{p_{g'l'}} x_{gl}(\tilde{p}) \begin{cases} < 0, & \text{if } g = g' \text{ and } l = l', \\ = 0, & \text{if there is no } k \in \tilde{K} \text{ such that} \\ & \quad g, g' \in \{s_K^k(0), \ldots, s_K^k(N)\}, \\ > 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

Assume, there exist two linearly independent equilibrium price systems $\tilde{p}, \bar{p}$. W.l.o.g. there is a price normalization such that $\tilde{p} < \bar{p}$ and $\tilde{p}_{01} = \bar{p}_{01}$. Otherwise apply the round shift from Lemma 9.10 and renumerate the commodities $l = 1, \ldots, L$. Define a tuple $(\bar{g}, \bar{l}) \in \tilde{K} \times L$ such that

$$\begin{aligned} \bar{g} &= \min\{g \in \tilde{K} \mid \tilde{p}_{gl} \neq \bar{p}_{gl} \text{ for some } l \in L\}, \\ \bar{l} &= \min\{l \in L \mid \tilde{p}_{\bar{g}l} \neq \bar{p}_{\bar{g}l}\}. \end{aligned}$$

Now look on the sequence of price systems

$$\left((q^{g,l})_{l \in L}\right)_{g \in \tilde{K}}$$

defined by coordinatewise raising $\tilde{p}$ to $\bar{p}$. Then, for $\bar{l} > 1$ it is

$$q_{\bar{g}\bar{l}}^{\bar{g},\bar{l}-1} < q_{\bar{g}\bar{l}}^{\bar{g},\bar{l}} \text{ and } s_K^{\bar{g}}(0) = \bar{g}$$

and for $\bar{l} = 1$ it is

$$q_{\bar{g}1}^{\bar{g}-1,L} < q_{\bar{g}1}^{\bar{g},1} \text{ and } s_K^{\bar{g}-1}(0) = \bar{g} - 1, s_K^{\bar{g}-1}(1) = \bar{g}.$$

Then, a contradiction is obtained via (2.14) since for $\bar{l} > 1$ it is

$$0 = x_{\bar{g}(\bar{l}-1)}(\tilde{p}) \leq x_{\bar{g}(\bar{l}-1)}(q^{\bar{g}(\bar{l}-1)}) < x_{\bar{g}(\bar{l}-1)}(q^{\bar{g},\bar{l}}) \leq x_{\bar{g},\bar{l}-1}(q^{K,L}) = 0$$

and for $\bar{l} = 1$ it is

$$0 = x_{(\bar{g}-1)L}(\tilde{p}) \leq x_{(\bar{g}-1)L}(q^{\bar{g}-1,L}) < x_{(\bar{g}-1)L}(q^{\bar{g},1}) \leq x_{(\bar{g}-1)L}(q^{K,L}) = 0.$$

□

B.2.6 Proof of Lemma 10.10

W.l.o.g. assume $K' > 1$.

If $(\tilde{p}_0, \ldots, \tilde{p}_{K'})$ with pairwise different $\tilde{p}_0, \ldots, \tilde{p}_{K'}$ is an equilibrium in the $(K' + 1)$-modulo exchange economy, then according to Theorem 9.16 this corresponds to an OLG-equilibrium of exact period $K' + 1$.

If $(\tilde{p}_0, \ldots, \tilde{p}_{K'})$ is an equilibrium in the $(K'+1)$-modulo exchange economy with $\tilde{p}_0, \ldots, \tilde{p}_{K'}$ not pairwise different, then define

$$\begin{aligned} m \quad := \quad & \min\{\min\{g-g', K'+1-(g-g')\}| \\ & g, g' \in \tilde{K}', g > g', \tilde{p}_g = \tilde{p}_{g'}\}. \end{aligned} \tag{2.15}$$

Let $(\bar{g}, \bar{g}')$ be a minimizing tuple. Then, according to Lemma 9.10 w.l.o.g. look on the following cyclical permutations:

if $\bar{g} - \bar{g}' \leq K' + 1 - (\bar{g} - \bar{g}')$:

$$\hat{p} := (\hat{p}_0, \ldots, \hat{p}_{K'}) := (\tilde{p}_{\bar{g}'}, \tilde{p}_{\bar{g}'+1}, \ldots, \tilde{p}_{\bar{g}}, \tilde{p}_{\bar{g}+1}, \ldots, \tilde{p}_{K'}, \tilde{p}_0, \ldots, \tilde{p}_{\bar{g}'-1});$$

if $\bar{g} - \bar{g}' > K' + 1 - (\bar{g} - \bar{g}')$:

$$\hat{p} := (\hat{p}_0, \ldots, \hat{p}_{K'}) := (\tilde{p}_{\bar{g}}, \tilde{p}_{\bar{g}+1}, \ldots, \tilde{p}_{K'}, \tilde{p}_0, \ldots, \tilde{p}_{\bar{g}'}, \tilde{p}_{\bar{g}'+1}, \ldots, \tilde{p}_{\bar{g}-1}).$$

Besides $\bar{p} := \tilde{p}_{\bar{g}} = \tilde{p}_{\bar{g}'}$.

For $m = 1$, it is known that $(\ldots, \bar{p}, \bar{p}, \ldots)$ is monetary stationary equilibrium of the OLG-model.

For $m > 1$, Lemma 9.12 together with Theorem 9.16 implies that $(\ldots, \bar{p}, \hat{p}_1, \ldots, \hat{p}_{m-1}, \bar{p}, \hat{p}_1, \ldots)$ is OLG-equilibrium of exact period m (compare the definition of m).

Assumptions Grandmont 9.3 in the OLG-model assure

$$c_0^1(\bar{p}, \hat{p}_1) + c_1^1(q, \bar{p}) = \omega_0^1 + \omega_1^1$$

or equivalently

$$x_0^1(\bar{p}, \hat{p}_1) + x_1^1(q, \bar{p}) = \omega_0^1 + \omega_1^1$$

to have the uniquely determined solution $q = \hat{p}_{m-1}$. However, since $(\bar{p}, \hat{p}_1, \ldots, \hat{p}_{K'})$ is equilibrium in the $(K'+1)$-modulo exchange economy, it is known from Lemma 9.6:

$$\begin{aligned} & x_0^{1_{K'}}(\hat{p}) + x_0^{1_0}(\hat{p}) = \omega_0^1 + \omega_1^1 \\ \Leftrightarrow \quad & x_1^1(\hat{p}_{K'}, \bar{p}) + x_0^1(\bar{p}, \hat{p}_1) = \omega_0^1 + \omega_1^1. \end{aligned}$$

Therefore, $\hat{p}_{K'} = \hat{p}_{m-1}$ under Assumptions Grandmont 9.3 and in the same way by induction

$$\hat{p}_{K'-hm-u} = \hat{p}_{m-u-1} \tag{2.16}$$

for $u = 0, \ldots, m-1$ and $h = 0, \ldots, \lfloor \frac{K'+1}{m} \rfloor - 1$. On the other hand Assumptions Gale 9.4 in the OLG-model assure that

$$c_0^1(\bar{p}, q) + c_1^1(\hat{p}_{m-1}, \bar{p}) = \omega_0^1 + \omega_1^1$$

or equivalently

$$x_0^1(\bar{p}, q) + x_1^1(\hat{p}_{m-1}, \bar{p}) = \omega_0^1 + \omega_1^1$$

to have the uniquely determined solution $q = \hat{p}_1$. However, since $(\bar{p}, \hat{p}_1, \ldots, \hat{p}_{K'})$ is equilibrium in the $(K'+1)$-modulo exchange economy, it is known from Lemma 9.6 that

$$\begin{aligned} & x_m^{1_{m-1}}(\hat{p}) + x_m^{1_m}(\hat{p}) = \omega_0^1 + \omega_1^1 \\ \Leftrightarrow \quad & x_1^1(\hat{p}_{m-1}, \bar{p}) + x_0^1(\bar{p}, \hat{p}_{m+1}) = \omega_0^1 + \omega_1^1. \end{aligned}$$

Therefore, $\hat{p}_{m+1} = \hat{p}_1$ under Assumptions Gale 9.4 and in the same way by induction

$$\hat{p}_{hm+u} \quad = \quad \hat{p}_u \tag{2.17}$$

for $u = 0, \ldots, m-1$ and $h = 1, \ldots, \lfloor \frac{K'+1}{m} \rfloor - 1, 0$.

Now, by (2.16) as well as by (2.17) one obtains a contradiction to the minimizing property of m if not

$$\hat{p} = (\hat{p}_0, \ldots, \hat{p}_{m-1}, \hat{p}_0, \ldots, \hat{p}_{m-1}, \ldots, \hat{p}_0, \ldots, \hat{p}_{m-1}).$$

Following Theorem 9.16, it is therefore known that $\hat{p}$ corresponds to an OLG-equilibrium of exact period $m := K + 1$.

□

B.2.7 Proof of Lemma 10.11

From Lemma 9.11 it is known that all equilibria $\tilde{p} = (\tilde{p}_0, \ldots, \tilde{p}_K)$ of the $(K+1)$-modulo exchange economy by $\frac{K'+1}{K+1}$ replication become equilibria of the $(K'+1)$-modulo exchange economy.

Let $D\hat{z}_K(\tilde{p})$ be the normalized Jacobian of excess demand in the $(K+1)$-modulo exchange economy and $D\hat{z}_{K'}(\tilde{p}, \ldots, \tilde{p})$ in the $(K'+1)$-modulo exchange economy. Then simple, but tedious calculations show

$$D\hat{z}_{K'}(\tilde{p},\dots,\tilde{p}) =$$
$$\begin{pmatrix}
D\hat{z}_K(\tilde{p}) & \begin{matrix}0\\ \vdots\\ 0\\ *\end{matrix} & & & & & \\
0\dots0\,* & * & *\,0\dots0 & \begin{matrix}0\\ 0\end{matrix} & & & \\
& \begin{matrix}*\\ 0\\ \vdots\\ 0\end{matrix} & D\hat{z}_K(\tilde{p}) & \begin{matrix}\vdots\\ 0\\ *\end{matrix} & & & \\
& 0 & 0\dots0\,* & * & & & \\
& & & & \ddots & & \\
& & & & & * & *\,0\dots0 \\
& & & & & \begin{matrix}*\\ 0\\ \vdots\\ 0\end{matrix} & D\hat{z}_K(\tilde{p})
\end{pmatrix}$$

where $*$ is place-marker for arbitrary real numbers. $D\hat{z}_{K'}(\tilde{p},\dots,\tilde{p})$ is a K'-dimensional quadratic matrix. If the $(K+1)$-modulo exchange economy is not regular, then there is some $\tilde{p}=(\tilde{p}_0,\dots,\tilde{p}_K)$ such that rank $D\hat{z}_K(\tilde{p}) < K$. This implies

$$rank\begin{pmatrix}
D\hat{z}_K(\tilde{p}) & \begin{matrix}0\\ \vdots\\ 0\\ *\end{matrix} & \\
0\dots0\,* & * & *\,0\dots0 \\
& \begin{matrix}*\\ 0\\ \vdots\\ 0\end{matrix} & D\hat{z}_K(\tilde{p})
\end{pmatrix} < 2K+1.$$

Because of rank $D\hat{z}_K(\tilde{p}) < K$ there are elementary row transformations (transformations of the matrix which do not change the rank) which lead to one of the matrices of the following or similar forms:

$$\left(\begin{array}{ccc} 0\ldots 0 & 0 & \\ & \vdots & \\ H_1 & 0 & \\ & * & \\ 0\ldots 0\ * & * & *\ 0\ldots 0 \\ & * & \\ & 0 & \\ & \vdots & D\hat{z}_K(\tilde{p}) \\ & 0 & \end{array}\right) \text{ or } \left(\begin{array}{ccc} & 0 & \\ & \vdots & \\ D\hat{z}_K(\tilde{p}) & 0 & \\ & * & \\ 0\ldots 0\ * & * & *\ 0\ldots 0 \\ & * & \\ & 0 & H_2 \\ & \vdots & \\ & 0 & 0\ldots 0 \end{array}\right)$$

where $H_s, s = 1, 2$, are $(K-1) \times K$-dimensional matrices and $*$ place-marker for some arbitrary real numbers. These matrices have obviously rank less than $2K+1$. Now go on by induction.

$\square$

B.2.8 Proof of Lemma 10.14

The slope of the offer curve at the monetary stationary equilibrium is equal to

$$\frac{\partial_{p_0} x_1^1(1,1)}{\partial_{p_0} x_0^1(1,1)}$$

and hence under Assumptions Grandmont 9.3 — since according to Grandmont's (1985) Lemma 1.2 it is $\partial_{p_0} x_1^1(1,1) > 0$ — the slope is greater than one or less than zero if and only if

$$\partial_{p_0} x_0^1(1,1) < \partial_{p_0} x_1^1(1,1). \tag{2.18}$$

On the other hand by Lemma 9.6 in the two-modulo exchange economy it is obtained that

$$\begin{aligned} \text{index } (1,1) &= \text{sign } (-(\partial_{p_0} x_0^{1_0}(1,1) + \partial_{p_0} x_0^{1_1}(1,1))) \\ &= \text{sign } (-(\partial_{p_0} x_0^{1}(1,1) + \partial_{p_1} x_1^{1}(1,1))). \end{aligned}$$

Since demand functions are homogeneous of degree 0, it is known

$$D_p x^1(1,1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

which implies

$$\partial_{p_1} x_1^1(1,1) = -\partial_{p_0} x_1^1(1,1).$$

Therefore

$$\begin{array}{rcl} \text{index } (1,1) & = & +1 \\ \Leftrightarrow \partial_{p_0} x_0^1(1,1) - \partial_{p_0} x_1^1(1,1) & < & 0 \\ \Leftrightarrow \partial_{p_0} x_0^1(1,1) & < & \partial_{p_0} x_1^1(1,1) \end{array}$$

which is equivalent to (2.18) and the proof is finished for Assumptions Grandmont 9.3. For Assumptions Gale 9.4 the proof is similar.

□

B.2.9 Proof of Lemma 10.15

A possible definition (compare Mas-Colell (1985)) of the index at an Arrow-Debreu equilibrium in the two-modulo exchange economy is given by

$$\text{index } (\tilde{p}) = \text{sign } \left(\det \begin{pmatrix} -\sum_{k=0}^{1} \partial_{p_0} x_0^{1_k}(\tilde{p}) & -\sum_{k=0}^{1} \partial_{p_1} x_0^{1_k}(\tilde{p}) & \tilde{p}_0 \\ -\sum_{k=0}^{1} \partial_{p_0} x_1^{1_k}(\tilde{p}) & -\sum_{k=0}^{1} \partial_{p_1} x_1^{1_k}(\tilde{p}) & \tilde{p}_1 \\ -\tilde{p}_0 & -\tilde{p}_1 & 0 \end{pmatrix} \right)$$

which following Lemma 9.6 is independent of permutations according to Lemma 9.10.

□

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