The g factor of an electron bound in hydrogenlike ions - status of the theoretical predictions

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For only few fundamental physical quantities experiment [1] and theory [2] agree that well as for the g factor of the free electron. It can therefore be considered as a precision test of quantum electrodynamics (QED) for free particles. To test QED also in the presence of strong electric fields, measurements on the g factor of an electron bound in a hydrogenlike system are one possible way. In $^{12}C^{5+}$, a value of $g = 2.001\,041\,596(5)$ was measured [3] which has to be compared to the theoretical prediction of $g = 2.001\,041\,591(7)$ [4]. This measurement is therefore the most stringent comparison of QED theory and experiment in any system heavier than hydrogen up to now. The major uncertainty of the experimental value results from the mass ratio $m_e/m_{^{12}C^{5+}}$, taken from [5], and an only slight improvement in the theoretical precision would allow to determine the electron mass more accurately. Here, the current limits to theory are presented and discussed.

The value given in [4] includes the g factor of the free electron including all QED corrections to that value, and in addition the corrections due to the binding to a heavy nucleus:

1. the binding correction itself which can be characterized as a transition from the spin quantum number to the total angular-momentum quantum number that is the only observable in a central field. It describes the deviation of g from the Dirac value of 2 for the free electron and is given by $g_j = (2/3)[1 + 2\sqrt{1-(Z\alpha)^2}]$ for the 1s state. This applies only to point-like nuclei. For extended nuclei, the wave function of the electron is slightly modified.
2. The finite nuclear-size correction which takes into account the extension of the nucleus is about 4×10^{-10} for carbon but amounts up to 1×10^{-3} for uranium where the uncertainty of the nuclear radius itself affects the prediction already on the 10^{-7} level.
3. The not-infinite nuclear mass causes the nucleus to move itself when orbited by the electron. A correct relativistic treatment has to consider nuclear recoil to all orders in the coupling constant $Z\alpha$ where Z is the charge of the nucleus. The exact form of this correction is not yet known and an existing expansion in $Z\alpha$ [6,7] yields reliable results only for light systems. For carbon it amounts to 87.5×10^{-9} with an estimated uncertainty of 1 % because of the expansion. This uncertainty should be considered to be 10 % of the value already for calcium. The complete relativistic recoil correction was calculated only for the Lamb shift up to now [8] and it seems to be much more complex for the g factor and the hyperfine structure splitting.
4. Another quantity connected with nuclear properties is that of nuclear polarization, i.e, the virtual excitation of nuclear degrees of freedom by exchange of at least two virtual photons with the electron. For the g factor, no investigations were carried out up to now. The works of G. Plunien and G. Soff [9] deal with the influence on the Lamb shift in the approximation of Coulomb-photon exchange, and only recently the problem of transverse-photon exchange which is crucial for magnetic interactions was considered at least for the Lamb shift case [10]. However, we expect this effect to be even weaker than for the Lamb shift because the typical matrix element for g factor measurements is $\langle r \rangle$, compared to $\langle 1/r \rangle$ for the Lamb shift, and therefore the inner parts of the electronic wave function that contribute most to the nuclear polarization are less pronounced. It should be mentioned that this is *not* the case for the hyperfine structure splitting where the typical matrix element is given by $\langle 1/r^2 \rangle$. In that case, however, the effect is screened by other nuclear uncertainties (for a recent overview see [11]).
5. The most interesting quantities related to the g factor are the bound-state QED corrections. Those of first order in (α/π) (i.e. one virtual photon line in the corresponding Feynman diagram) are depicted in Fig. 1. They were eval-

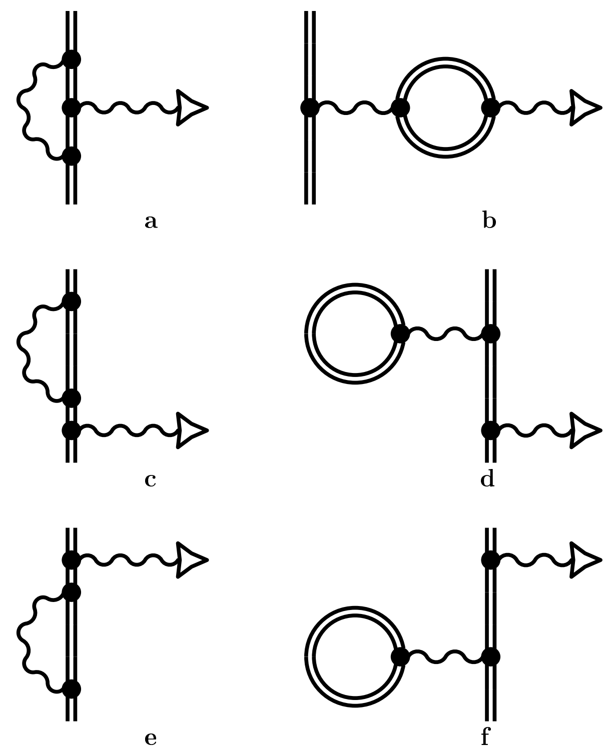

Fig. 1. Feynman diagrams representing the QED contributions of order (α/π) to the g factor of a bound electron. The wavy lines denote photons, which mediate the interaction with the external magnetic field represented by a triangle. In each diagram there is also one virtual photon. The solid double line indicates the electron and on the right side also virtual leptons in the electron-positron loops. The diagrams on the left are the self-energy-like corrections, those on the right the vacuum-polarization-like corrections. For the free electron, only the diagram similar to diagram a contributes.

uated in detail to all orders in $Z\alpha$ in [4]. They contain also the contribution to the g factor of the free electron of the same order, given by $\alpha/\pi \approx 2.323 \times 10^{-3}$. For comparison, the effect of binding in C^{5+} amounts only to 8.442×10^{-7}. In uranium the binding effect is 3×10^{-3} and therefore in particular heavy systems form an excellent base for investigations of bound-state QED.
The QED corrections of second order in (α/π) were never investigated beyond the first term in the $Z\alpha$ expansion. It can be shown for *all* orders of (α/π) that the leading term of the corresponding $Z\alpha$ expansion is given by $2 \times A^{(n)} \times (Z\alpha)^2/6$ where $A^{(n)}$ is the expansion coefficient for the nth power of (α/π) in the series for $g_{\rm free}/2$ [12], i.e., $A^{(1)} = 1/2$. The next term in the $Z\alpha$ expansion is at least of the order $(Z\alpha)^4$, and therefore the $Z\alpha$ expansion allows to estimate the bound-state $(\alpha/\pi)^2$ contributions with an uncertainty of about 50 % for the case of carbon. This uncertainty increases rapidly for increasing Z, and we expect the error to be at least 100 % in the case of calcium already, where the leading term of the expansion for the first order in (α/π) already deviates for about 70 % from the non-perturbative value. The whole set of 50 diagrams for the order $(\alpha/\pi)^2$ is shown in Fig. 2. For the order $(\alpha/\pi)^3$, the number of diagrams exceeds 500. The 50 diagrams shown in Fig. 2 can be obtained by fixing the magnetic interaction to each point of the 10 diagams contributing to the Lamb shift of order α^2 in hydrogenlike atoms (e.g., [13]). The calculation is slightly more complex because for each diagram the magnetic interaction and one additional electron propagator with the corresponding integration has to be considered. As there are problems already for some of the Lamb-shift diagrams, an evaluation of the set shown in Fig. 2 can not be expected without considerable effort. In particular, the diagrams that contribute most to the g factor in lighter systems are those with two self-energy loops in the upper rows of Fig. 2, and unfortunately exactly their counterpart, the so-called two-loop self-energy graphs, cause the major problems in the recent calculations for the Lamb shift (e.g., [14] and references therein). The situation would be different in muonic atoms where the vacuum-polarization contributions are strongly enhanced compared to those from self-energy-like graphs. An additional experiment on a muonic system therefore could provide valuable additional information. However, in muonic systems the nuclear polarization can expected to be as large as the QED corrections of order (α/π).

All theoretical contributions to the g factor of the electron bound in hydrogenlike carbon are given in Table 1. Together with the experimental value, this leads to an independent new value for the electron mass [15], $m_e = 5.485\,799\,092 \times 10^{-3}$ u. A detailed discussion about the corresponding measurement and evaluation procedure is to be found elsewhere in this report [16].

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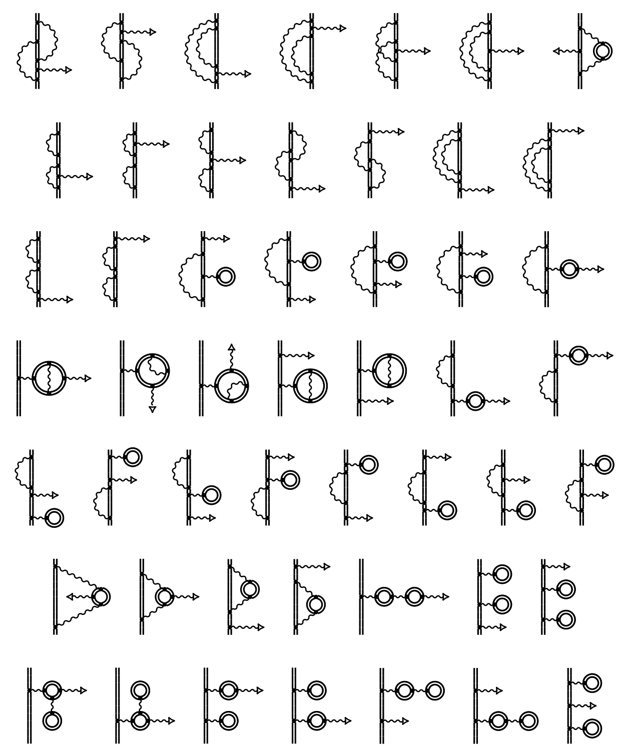

Fig. 2. Diagrams contributing to order $(\alpha/\pi)^2$ to the g factor of a bound electron. Only seven diagrams of this order have to be considered for the g factor of a free electron, similar to these of the first row.

Table 1. Known theoretical contributions to the g factor of an electron bound in the ground state of $^{12}C^{5+}$. All values are given in units of 10^{-9}. If no error is given, it is less than 0.5×10^{-10}. The error for the "total" value is a linear addition of the three errors given in order not to underestimate any systematic effect.

Contribution	numerical value (in 10^{-9})
binding	1 998 721 354.2
fin. nuc. size	0.4
recoil	87.5(9)
free QED, order (α/π):	2 322 819.6
bound QED, order (α/π):	844.3(12)
free QED, $(\alpha/\pi)^2$ to $(\alpha/\pi)^4$	−3 515.1
bound QED, $(\alpha/\pi)^2\,(Z\alpha)^2$	−1.1(5)
total:	2 001 041 589.8(26)