Divergence of Perturbation Theory and Resummation

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Current determinations of fundamental constants [1] and the comparison of theory and expriment in high-precision experiments are based on perturbative expansions which can at best be regarded as divergent, asymptotic series in the coupling constant [2]. The first terms of the series decrease in absolute magnitude, before the factorial growth of the perturbative coefficients overcompensates the additional coupling factors of higher orders in perturbation theory, and the perturbation series ultimately diverges. Dyson's related argument [3] has given rise to much discussion and confusion, until recent explicit 30–loop calculations of perturbation series pertaining to ϕ^3 and Yukawa theories have firmly established the factorially divergent character of the perturbative expansion [4]. These considerations naturally lead to the question of how complete, nonperturbative results can be obtained from a finite number of perturbative coefficients, and how the nonperturbative result is related to the partial sums of the perturbation series.

We have investigated this problem [5] in connection to the Euler-Heisenberg-Schwinger effective Lagrangian which describes the quantum electrodynamic corrections to Maxwell's equations. The forward scattering amplitude of the vacuum ground state is described by a factorially divergent asymptotic series, $S_{\rm B} \propto \text{const.} \times g_{\rm B} \sum_{n=0}^{\infty} c_n \, g_{\rm B}^n$. The expansion coefficients

$$c_n = \frac{(-1)^{n+1} \, 4^n \, |\mathcal{B}_{2n+4}|}{(2n+4)(2n+3)(2n+2)} \,,$$

where $\mathcal{B}_{2n+4}$ is a Bernoulli number and $g_{\rm B}$ is the coupling, display an alternating sign pattern and grow factorially in absolute magnitude. The process by which a finite, nonperturbative result is ascribed to a divergent perturbation series is known as *resummation*. The resummation to the complete nonperturbative result for $S_{\rm B}$ is accomplished by employing the delta transformation,

$$\delta_n^{(0)}(\beta, s_0) \;=\; \frac{\displaystyle\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(\beta+j)_{n-1}}{(\beta+n)_{n-1}} \frac{s_j}{a_{j+1}}}{\displaystyle\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(\beta+j)_{n-1}}{(\beta+n)_{n-1}} \frac{1}{a_{j+1}}} \,,$$

where $s_n = \sum_{k=0}^{n} a_k$ is the partial sum of the input series and $a_j = c_j \, g^j$ is the jth term in the perturbation series; $(a)_m = \Gamma(a+m)/\Gamma(a)$ is a Pochhammer symbol. It has been observed that the delta transformation can be used under rather general assumptions for the extrapolation of the perturbation series, i.e. the prediction of unknown perturbative coefficients, and various applications to phenomenologically important quantum field theoretic perturbation series have been presented in [5, 6]. In many cases, the delta transformation leads to better results than Padé approximants which have been discussed abundantly in the literature (see e.g. [7]).

A perturbation series often misses physically important physical effects when interpreted "at face value". For example, the perturbative expansion describing the energy shift of hydrogenic levels in an electric field (known as the Stark effect) has purely real coefficients, whereas the complete energy eigenvalue (more precisely, pseudoeigenvalue or resonance) also has an imaginary component. The imaginary part of the resonance, which describes the autoionization width, can be obtained from the purely real perturbation series by a transformation which can be characterized as a generalized Borel summation [8, 9]. First, the factorial growth of the perturbation series is divided out by calculating the Borel transform, and the perturbation series $f(g) = \sum_n c_n g^n$ is replaced by its Borel transform $\mathcal{B}(g) = \sum_n c_n g^n / n!$. Then, the autoionization width is obtained by integrating the Borel transform in the complex plane along integration contours specified in [9, 10]. The same resummation procedure can be used to obtain the quantum electrodynamic pair production amplitude for the case of an electric field background; the pair-production amplitude is related to the imaginary part of the effective action [10].

Recently, techniques have been investigated to acclerate the convergence of resummation procedures in order to obtain results even at large coupling, which are paradoxically based on the weak-coupling perturbative expansions [11]. The extrapolation from the weak-coupling limit to the regime of strongly coupled systems by analytic continuation of the perturbation series via Borel or delta transformations has wide applicability.

References

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