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The Two-Component Camassa-Holm Equations CH(2,1) and CH(2,2): First-Order Integrating Factors and Conservation Laws

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The Two-Component Camassa-Holm Equations
CH(2,1) and CH(2,2): First-Order Integrating
Factors and Conservation Laws

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Abstract:
Recently, Holm and Ivanov, proposed and studied a class of multi-component generalisations of the Camassa-Holm equations [D D Holm and R I Ivanov, Multi-component generalisations of the CH equation: geometrical aspects, peakons and numerical examples, J. Phys A: Math. Theor 43, 492001 (20pp), 2010]. We consider two of those systems, denoted by Holm and Ivanov by CH(2,1) and CH(2,2), and report a class of integrating factors and its corresponding conservation laws for these two systems. In particular, we obtain the complete set of first-order integrating factors for the systems in Cauchy-Kovalevskaya form and evaluate the corresponding sets of conservation laws for CH(2,1) and CH(2,2).

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1 Introduction

It is well known that certain conservation laws of shallow water wave equations, such as the Camassa-Holm equation [4] and the the Degasperis-Procesi equation [8], are useful to prove blow-up, cf. the papers [5], [18] and [15]. Furthermore, conservation laws play a central role in the prove of the global existence (in time) for solutions evolving from certain initial data, cf. the paper [6], and for proving the stability of peakons for both model equations, cf. the papers [7], [12] and [13]. In the context of the Camassa-Holm equation they are instrumental in the set-up of a theory of global weak solutions for nonlinear nonlocal conservation laws, cf. the considerations in the papers [2], [3] and [10].

In the current paper we derive all first-order integrating factors and its corresponding conservation laws for some recently proposed multi-component generalizations of the Camassa-Holm equation [11]. We concentrate on two explicit systems, namely CH(2,1)
and \( \text{CH}(2,2) \), proposed by Holm and Ivanov in [11] (see 1.1a) – (1.1b) and (1.6a) – (1.6b) below).

We recently reported in [9] the complete set of first-order integrating factors and conservation laws for a class of Camassa-Holm type equations, which includes the Camassa-Holm equation [4] and the Degasperis-Procesi equation [8]. Our approach applied in this paper is based on the direct method described by Anco and Bluman in their paper [1], which can be applied to derive conservation laws of evolution equations that are in Cauchy-Kovalevskaya form. We also refer the reader to [16] and [17] for more details and alternate methods for computing conservation laws for partial differential equations and systems.

Consider the two-component Camassa-Holm equations introduced and denoted by Holm and Ivanov [11] as \( \text{CH}(2,1) \), which has the following form:

\[
\begin{align*}
\sigma_1 q_t + 2q u_x + u q_x + \sigma \rho \rho_x &= 0 \quad (1.1a) \\
\rho_t + \rho u_x + u \rho_x &= 0, \quad (1.1b)
\end{align*}
\]

where

\[
q = \sigma_1 u - u_{xx} + s
\]

and \( s, \sigma \) and \( \sigma_1 \) are arbitrary constants. The physically interesting cases are \( \sigma = \pm 1 \) and \( \sigma_1 = 1 \) or \( \sigma_1 = 0 \). By defining the new dependent variables

\[
\begin{align*}
u := U_1, \quad u_x := U_2 \\
u_{xx} := U_3, \quad \rho := U_4
\end{align*}
\]

and the change of independent variables,

\[
X := t, \quad T := x,
\]

we can write system (1.1a) – (1.1b) in the following Cauchy-Kovalevskaya form:

\[
\begin{align*}
E_1 := U_{1,T} - U_2 &= 0 \quad (1.5a) \\
E_2 := U_{2,T} - U_3 &= 0 \quad (1.5b) \\
E_3 := U_{3,T} - \sigma_1^2 U_1^{-1} U_{1,X} + \sigma_1 U_1^{-1} U_{3,X} - 3\sigma_1 U_2 + 2U_1^{-1} U_2 U_3 + \sigma U_1^{-2} U_4 U_{4,X} \\
&\quad + \sigma U_1^{-2} U_2 U_4^2 - 2s U_1^{-1} U_2 = 0 \quad (1.5c) \\
E_4 := U_{4,T} + U_1^{-1} U_{4,X} + U_1^{-1} U_2 U_4 &= 0. \quad (1.5d)
\end{align*}
\]

The second 2-component Camassa-Holm equation that we study in the current paper, denoted by \( \text{CH}(2,2) \), has the form [11]

\[
\begin{align*}
q_{1,t} + u_0 q_{1,x} + 2q_1 u_{0,x} + u_1 q_{2,x} + 2q_2 u_{1,x} &= 0 \quad (1.6a) \\
q_{2,t} + u_0 q_{2,x} + 2q_2 u_{0,x} &= 0. \quad (1.6b)
\end{align*}
\]
where
\[ q_1 = u_1 - u_{1,xx} + s_1 \]  
\[ q_2 = u_0 - u_{0,xx} + 3u_x^2 - u_{1x} - 2u_1u_{1,xx} + 4s_1u_1 + s_2. \]  

(1.7a) \hspace{1cm} (1.7b)

Here \( s_1, \ s_2 \) are arbitrary constants. By defining the new dependent variables

\[ u_0 := U_1, \ u_{0,x} := U_2, \ u_{0,xx} := U_3 \]  
\[ u_1 := U_4, \ u_{1,x} := U_5, \ u_{1,xx} := U_6 \]  

(1.8a) \hspace{1cm} (1.8b)

and the change of independent variables (1.4), we can present (1.6a) - (1.6b) in the following Cauchy-Kovalevskaya form:

\[ E_1 := U_1,T - U_2 = 0 \]  
\[ E_2 := U_2,T - U_3 = 0 \]  

(1.9a) \hspace{1cm} (1.9b)

\[ E_3 := U_3,T + 12U_1^{-1}U_4^3U_5 - 4U_1^{-1}U_4U_4,X + 2U_1^{-1}U_5U_5,X - 4s_1U_1^{-1}U_4,X \]
\[ + 4U_5U_6 - 4s_1U_5 + 2U_1^{-1}U_2U_3 - 6U_1^{-1}U_2U_4 + 2U_1^{-1}U_2U_5^2 - 2s_2U_1^{-1}U_2 \]
\[ - 4s_1U_1^{-1}U_2U_4 - 12U_1^{-2}U_2U_4^4 + 2U_1^{-1}U_6U_4,X - 8U_1^{-1}U_2U_5U_6 \]
\[ + 16s_1U_1^{-1}U_2U_5^2 + 4U_1^{-2}U_2U_4U_4,X - 8s_1U_1^{-2}U_3U_5^2U_4,X + 4U_1^{-2}U_4^2U_2U_3 \]
\[ + 4U_1^{-2}U_2U_5U_5,U_3 - 8U_1^{-1}U_2U_4U_5 + 4s_2U_1^{-1}U_4U_5 - 12U_1^{-2}U_2U_4^2 \]
\[ + 2U_1^{-2}U_2U_3,X + 4U_1^{-2}U_2U_6,X - 4U_1^{-1}U_4U_5 - 2U_1^{-2}U_4U_1,X \]
\[ - U_1^{-1}U_1,X + U_1^{-1}U_3,X - 3U_2 = 0 \]  

(1.9c)

\[ E_4 := U_4,T - U_5 = 0 \]  
\[ E_5 := U_5,T - U_6 = 0 \]  

(1.9d) \hspace{1cm} (1.9e)

\[ E_6 := U_6,T + 4U_1^{-1}U_4U_5 - 8s_1U_1^{-1}U_4U_5 + 2U_1^{-1}U_3^3 - 3U_5 - U_1^{-1}U_4,X \]
\[ + U_1^{-1}U_6,X - 2U_1^{-2}U_2U_6,X + 6U_1^{-2}U_2U_4 + U_1^{-2}U_4U_3,X + U_1^{-2}U_4U_1,X \]
\[ + 6U_1^{-2}U_4U_4,X - 2U_1^{-2}U_4U_6U_4,X + 4s_1U_1^{-2}U_4U_4,X - 2U_1^{-2}U_2U_3U_4 \]
\[ - 2U_1^{-2}U_2U_5U_5^2 - 4U_1^{-2}U_2U_3^2U_6 + 8s_1U_1^{-2}U_3U_4^2 + 2s_2U_1^{-2}U_2U_4 \]
\[ - 2U_1^{-2}U_1U_5U_5,U_3 + 2U_1^{-1}U_5U_5 - 2s_2U_1^{-1}U_5 + 2U_1^{-1}U_2U_6 \]
\[ - 2s_1U_1^{-1}U_2 - 6U_1^{-1}U_2U_5 = 0. \]  

(1.9f)
The above first-order Cauchy-Kovalevskaya systems can now be investigated for integrating factors to derive conservation laws for the systems; which then leads to conservation laws of the systems \( \text{CH}(1,1) \) and \( \text{CH}(2,2) \) in the original variables.

## 2 General description

In this section we briefly describe the direct method [1] of integrating factors (or multipliers) for the general first-order Cauchy-Kovalevskaya system of six equations:

\[
E_j := U_{j,T} - F_j(U_1,\ldots,U_6,U_{1,X},\ldots,U_{6,X}) = 0, \quad j = 1,2,\ldots,6.
\]  

(2.1)

Every conserved density, \( \Phi^T \), and conserved flux, \( \Phi^X \), of system (2.1) must satisfy

\[
D^T \Phi^T + D^X \Phi^X \bigg|_{\vec{E}=\vec{0}} = 0,
\]  

(2.2)

where, in general, both \( \Phi^T \) and \( \Phi^X \) are functions of \( X,T,U \) as well as \( X \)-derivatives of \( U_j \).

Moreover, every \( \Phi^T \) requires six integrating factors, \( \{\Lambda_1, \Lambda_2,\ldots,\Lambda_6\} \), which are directly related to the conserved density by the relation [1]

\[
\Lambda_k = \hat{E}[U_k] \Phi^T, \quad k = 1,2,\ldots,6.
\]  

(2.3)

Here \( \hat{E} \) is the Euler Operator,

\[
\hat{E}[U_k] := \frac{\partial}{\partial U_k} - D_T \circ \frac{\partial}{\partial U_{k,T}} + \sum_{j=1}^{q} (-1)^j D^j_X \circ \frac{\partial}{\partial U_{k,jX}},
\]  

(2.4)

where we use the notation

\[
U_{k,jX} := \frac{\partial^j U_k}{\partial X^j}.
\]

The conditions on the integrating factors, \( \{\Lambda_j\} \), of system (2.1) are

\[
\hat{E}[U_k] (\Lambda_1 E_1 + \Lambda_2 E_2 + \cdots + \Lambda_6 E_6) = 0, \quad k = 1,2,\ldots,6.
\]  

(2.5)

However, since all integrating factors of system (2.1) are adjoint symmetries of the system (2.1), we can calculate \( \{\Lambda_j\} \) by the condition

\[
\begin{bmatrix}
L_{E_1}^* \left[ U_1 \right] & L_{E_2}^* \left[ U_1 \right] & \cdots & L_{E_6}^* \left[ U_1 \right] \\
L_{E_1}^* \left[ U_2 \right] & L_{E_2}^* \left[ U_2 \right] & \cdots & L_{E_6}^* \left[ U_2 \right] \\
\vdots & \vdots & \ddots & \vdots \\
L_{E_1}^* \left[ U_6 \right] & L_{E_2}^* \left[ U_6 \right] & \cdots & L_{E_6}^* \left[ U_6 \right]
\end{bmatrix}
\begin{bmatrix}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_6
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]  

\[
\left|_{\vec{E} = \vec{0}}
\right.
\]  

(2.6)
and then require the self-adjointness condition on \( \{ \Lambda_j \} \) (as integrating factors are variational quantities), namely

\[
\begin{pmatrix}
L_{\Lambda_1}[U_1] & L_{\Lambda_1}[U_2] & \cdots & L_{\Lambda_1}[U_6] \\
L_{\Lambda_2}[U_1] & L_{\Lambda_2}[U_2] & \cdots & L_{\Lambda_2}[U_6] \\
\vdots & \vdots & \ddots & \vdots \\
L_{\Lambda_6}[U_1] & L_{\Lambda_6}[U_2] & \cdots & L_{\Lambda_6}[U_6]
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_6
\end{pmatrix}
= \begin{pmatrix}
L_{\Lambda_1}^*[U_1] & L_{\Lambda_1}^*[U_2] & \cdots & L_{\Lambda_1}^*[U_6] \\
L_{\Lambda_2}^*[U_1] & L_{\Lambda_2}^*[U_2] & \cdots & L_{\Lambda_2}^*[U_6] \\
\vdots & \vdots & \ddots & \vdots \\
L_{\Lambda_6}^*[U_1] & L_{\Lambda_6}^*[U_2] & \cdots & L_{\Lambda_6}^*[U_6]
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_6
\end{pmatrix},
\]

(2.7)

Here \( L \) is the linear operator and \( L^* \) its adjoint:

\[
L_P[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^{p} \frac{\partial P}{\partial U_{j,i}T} D^i_T + \sum_{k=1}^{q} \frac{\partial P}{\partial U_{j,k}X} D^k_X
\]

(2.9a)

\[
L^*_P[U_j] := \frac{\partial P}{\partial U^*_j} + \sum_{i=1}^{p} (-1)^i \frac{\partial P}{\partial U_{j,i}^*} D^i_T + \sum_{k=1}^{q} (-1)^k \frac{\partial P}{\partial U_{j,k}^*} D^k_X.
\]

(2.9b)

Note that the self-adjointness condition, (2.7), is independent of the form of the evolution system (2.1) and only depends on the functional arguments of \( \{ \Lambda_j \} \) as well as the number of equations in the system.

3 Integrating factors for system (1.5a) – (1.5d) and conservation laws for (1.1a) – (1.1b):

Solving conditions (2.6) and (2.7) for system (1.5a) – (1.5d), the complete set of first-order integrating factors \( \{ \Lambda_1, \ldots, \Lambda_4 \} \), of the form

\[ \Lambda_j = \Lambda_j(X, T, U_1, \ldots, U_4, U_{1,X}, \ldots, U_{4,X}), \quad j = 1, 2, \ldots, 4 \]
for arbitrary $\sigma, \sigma_1 \neq 0$ and $s$ is as follows:

$$\Lambda_1 = \lambda_1 U_4 + 2\lambda_2 \left(s + \frac{3}{2}\sigma_1 U_1 - \frac{1}{2} U_3\right)$$

$$-\lambda_3 \left(\sigma_1 U_{2,x} - 2s U_1 - \sigma U_4^2 - 3\sigma_1 U_1^2 + 2U_1 U_3\right) \sigma_1^{-1} \quad (3.1a)$$

$$\Lambda_2 = -\lambda_3 U_2 + \lambda_3 U_{1,x} \quad (3.1b)$$

$$\Lambda_3 = -\lambda_3 U_1 - \lambda_3 \sigma_1^{-1} U_1^2 \quad (3.1c)$$

$$\Lambda_4 = \lambda_1 U_1 + \lambda_2 \sigma U_4 + 2\lambda_3 \sigma_1^{-1} U_1 U_4 \quad (3.1d)$$

where $\lambda_j$ are arbitrary constants. This leads to the following three sets of conserved density, $\Phi^t$, and conserved flux, $\Phi^x$, for the original system (1.1a) – (1.1b) (separated by means of the arbitrary $\lambda_1, \lambda_2$ and $\lambda_3$, respectively):

$$\Phi^t_1 = \rho \quad (3.2a)$$

$$\Phi^x_1 = u \rho \quad (3.2b)$$

$$\Phi^t_2 = \sigma_1^2 u - \sigma_1 u_{xx} \quad (3.3a)$$

$$\Phi^x_2 = 2su + \frac{1}{2}\sigma \rho^2 + \frac{3}{2}\sigma_1 u^2 - uu_{xx} - \frac{1}{2}u_x^2 \quad (3.3b)$$

$$\Phi^t_3 = \left(-\sigma_1 u_x^2 + \frac{1}{2}\sigma \rho^2 + \frac{1}{2}\sigma_1^2 u^2 - \sigma_1 uu_{xx} + \frac{1}{2}\sigma_1 u_x^2\right) \sigma_1^{-1} \quad (3.4a)$$

$$\Phi^x_3 = \left(\sigma_1 u_x u_t + su_x^2 + \sigma u \rho^2 + \sigma_1 u^3 - u^2 u_{xx}\right) \sigma_1^{-1} \quad (3.4b)$$

Some special must be considered:

**Special Case 1**: $\sigma = 0$ with $\sigma_1$ arbitrary, but nonzero, and $s$ arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{2U_3 - 3\sigma_1 U_1 - 2s}{(\sigma_1 U_1 - U_3 + s)^{1/2}}, \quad \Lambda_2 = 0 \quad (3.5)$$

$$\Lambda_3 = \frac{U_1}{(\sigma_1 U_1 - U_3 + s)^{1/2}}, \quad \Lambda_4 = 0. \quad (3.6)$$

and the corresponding conserved current for system (1.5a) – (1.5d) is

$$\Phi^t = \sigma_1 (\sigma_1 u - u_{xx} + s)^{1/2} \quad (3.7a)$$

$$\Phi^x = (\sigma_1 u - u_{xx} + s)^{1/2}. \quad (3.7b)$$
Special Case 2: \( \sigma = 0 \) with \( \sigma_1 = 1 \) and \( s \) arbitrary. The integrating factors are as follows:

\[
\Lambda_1 = -\frac{U_1 W^3 H'(W)}{U_4} + 2U_4 H(W), \quad \Lambda_2 = 0 \tag{3.8a}
\]

\[
\Lambda_3 = \frac{U_1 W^3 H'(W)}{U_4}, \quad \Lambda_4 = 2U_1 \left( WH'(W) + H(W) \right), \tag{3.8b}
\]

where \( H(W) \) is an arbitrary differentiable function with

\[
W := \frac{U_4}{(U_1 - U_3 + s)^{1/2}}. \tag{3.9}
\]

The conserved current for system (1.5a) – (1.5d) is then

\[
\Phi^t = H(w) \rho \tag{3.10a}
\]

\[
\Phi^x = H(w) u \rho. \tag{3.10b}
\]

Here the argument, \( w \), in the arbitrary function \( H \), is

\[
w := \frac{\rho}{(u - u_{xx} + s)^{1/2}}. \tag{3.11}
\]

Special Case 3: \( \sigma \) arbitrary, but nonzero, with \( \sigma_1 = 1 \) and \( s \) arbitrary. The integrating factors are as follows:

\[
\Lambda_1 = \frac{U_3 - 2U_1 - s}{\sigma U_4}, \quad \Lambda_2 = 0 \tag{3.12a}
\]

\[
\Lambda_3 = \frac{U_1}{\sigma U_4}, \quad \Lambda_4 = \frac{sU_1 - \sigma U_2^2 + U_1^2 - U_1 U_3}{\sigma U_4^2}. \tag{3.12b}
\]

The conserved current for system (1.5a) – (1.5d) is then

\[
\Phi^t = \frac{u_{xx} - u - s}{\sigma \rho} \tag{3.13a}
\]

\[
\Phi^x = \frac{uu_{xx} - u^2 - \sigma \rho^2 - su}{\sigma \rho}. \tag{3.13b}
\]

Special Case 4: \( \sigma_1 = 0 \) with \( \sigma \) and \( s \) arbitrary. The integrating factors are as follows:

\[
\Lambda_1 = (U_3 - 2s)H(X, W), \quad \Lambda_2 = U_2 H(X, W) \tag{3.14a}
\]

\[
\Lambda_3 = U_1 H(X, W), \quad \Lambda_4 = -\sigma U_4 H(X, W), \tag{3.14b}
\]

where \( H \) is an arbitrary differentiable function and

\[
W := \frac{1}{2} \left( -4sU_1 - \sigma U_4^2 + 2U_1 U_3 + U_2^2 \right). \tag{3.15}
\]
The conserved current for system (1.5a) – (1.5d) is then

$$\Phi_t = 0$$ \hfill (3.16a)  

$$\Phi_x = wH(t,w),$$ \hfill (3.16b)

where

$$w := \frac{1}{2} \left(-4su - \sigma \rho^2 + 2\sigma u u_{xx} + u_{x}^2\right).$$ \hfill (3.17)

**Special Case 5:** $\sigma_1 = 0$ with $\sigma$ arbitrary, but nonzero, and $s$ arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{2}{\sigma} \left(sU_1 + \frac{\sigma}{2} U_2^2 - U_1U_3\right), \quad \Lambda_2 = 0$$ \hfill (3.18a)

$$\Lambda_3 = -\frac{U_1^2}{\sigma}, \quad \Lambda_4 = 2U_1U_4.$$ \hfill (3.18b)

The conserved current for system (1.5a) – (1.5d) is then

$$\Phi_t = \frac{\rho^2}{2}$$ \hfill (3.19a)

$$\Phi_x = \frac{1}{\sigma} \left(su^2 + \sigma u \rho^2 - u^2 u_{xx}\right).$$ \hfill (3.19b)

### 4 Integrating factors for system (1.9a) – (1.9f) and conservation laws for (1.6a) – (1.6b):

Solving conditions (2.6) and (2.7) for system (1.9a) – (1.9f), the complete set of first-order integrating factors $\{\Lambda_1, \ldots, \Lambda_6\}$, of the form

$$\Lambda_j = \Lambda_j(X,T,U_1,\ldots,U_6,U_{1,X},\ldots,U_{6,X}), \quad j = 1, 2, \ldots, 6$$
are the following:

\[ \Lambda_1 = \lambda_1 \left( 2U_6U_1 + 2U_3U_4 + 2U_4U_5^2 + 4U_4^2U_6 - 6U_4^3 - 2s_1U_1 - 8s_1U_4^2 - 6U_1U_4 \right) \]
\[ -2s_2U_4 + U_{5,X} \right) + \lambda_2 \left( U_3 + 2U_5^2 - 4s_1U_4 - 3U_1 - 2s_2 \right) \]
\[ + \lambda_3 (U_6 - 3U_4 - 2s_1) + \lambda_4 \left( -8s_1^2U_4 - 2s_1s_2 - s_1U_1 + 2s_1U_3 - 22s_1U_4^2 \right) \]
\[ + 12s_1U_4U_6 + 2s_1U_5^2 - 4s_2U_4 + 2s_2U_6 - 3U_1U_4 + U_1U_6 + 4U_3U_4 - 2U_3U_6 - 12U_4^2 \]
\[ + 14U_4^2U_6 + 4U_4U_5^2 - 4U_4^2U_6 - 2U_5U_6 \right) \]
\[ + \frac{\lambda_5}{2} \left( -8s_1U_4 - 2s_2 - 3U_1 + 2U_3 - 6U_4^2 + 4U_4U_6 + 2U_5^2 \right) \]
\[ Z^{-3/2} \] (4.1a)

\[ \Lambda_2 = -\lambda_1U_4 + \lambda_3 \]
\[ + 2\lambda_1U_1U_4 + \lambda_2 (U_1 - 2U_4^2) + \lambda_3U_4 \]
\[ + \lambda_4 \left( -s_1U_1 + 8s_1U_4^2 + 2s_2U_4 + U_1U_4 + U_1U_6 - 2U_3U_4 + 6U_4^3 \right) \]
\[ -4U_4^2U_6 - 2U_4U_5^2 \right) \]
\[ Z^{-3/2} + \frac{\lambda_5}{2} U_1Z^{-1/2} \] (4.1b)

\[ \Lambda_4 = \lambda_1 \left( 2U_6U_3 + 2U_6U_5 + 2U_4U_5, 2 \right) - 2s_2U_3 - 18U_1U_4^2 + U_{2,X} \]
\[ -16s_1U_4U_6 + 8U_1U_4U_6 \right) + \lambda_2 \left( 2U_4^3 - 4U_3U_4 - 4U_4U_5^2 - 12U_4^3U_6 - 2U_5,X \right) \]
\[ + 24s_1U_1^2 - 4s_1U_1 + 4s_2U_4 \right) + \lambda_3 \left( U_3 + 4U_4U_6 - 3U_1 + 2U_5^2 - 12U_4^2 \right) \]
\[ -12s_1U_4 - 2s_2 + 2s_4 \left( 2s_1U_1 - 8s_1^2U_4^2 - 20s_1s_2U_4 - 19s_1U_1U_4 - 3s_1U_1U_6 \right) \]
\[ + 20s_1U_3U_4 - 84s_1^3U_4 + 4U_3U_4^2U_6 + 20s_1U_4U_5^2 - 2s_2U_3 - 18s_2U_4^2 \]
\[ + 10s_2U_4U_6 + 4s_2U_5^2 - 3U_1^2 + 5U_1U_3 - 18U_1U_4^2 + 8U_1U_4U_6 + 5U_1U_5^2 + U_1U_6^2 - 2U_3^2 \]
\[ + 18U_3U_4^2 - 10U_3U_4U_6 - 4U_3U_5^2 - 36U_4^4 + 42U_4^3U_6 + 18U_4^2U_5^2 - 12U_4^2U_6^2 \]
\[ -10U_4^2U_5^2U_6 - 2U_5^2 \right) \]
\[ Z^{-3/2} + \lambda_5 U_1 \left( 2s_1 + 3U_4 + U_6 \right) \] (4.1c)

\[ \Lambda_5 = \lambda_1 \left( 4U_1U_4U_5 - U_{1,X} - 2U_4U_4,X \right) + \lambda_2 \left( 4U_1U_5 - 4U_4^2U_5 + 2U_4,X \right) \]
\[ + \lambda_3 \left( U_2 + 4U_4U_5 \right) + \lambda_4 \left( -2s_1 - 2U_3U_4 - U_1U_4 - U_1U_6 + 2U_3U_4 \right) \]
\[ - 6U_4^2 + 4U_4^2U_6 + 2U_4U_5^2 \right) \]
\[ Z^{-3/2} + \lambda_5 U_1U_5 \] (4.1d)

\[ \Lambda_6 = \lambda_1 \left( U_2^2 + 4U_1U_5 \right) - 4U_2U_4^2 + \lambda_3 \left( U_1 + 2U_4^2 \right) + \lambda_4 \left( 3s_1U_1U_4 + 8s_1U_4^3 + s_2U_1 \right) \]
\[ + 2s_2U_4^2 - U_1^2 - U_1U_3 - 4U_1U_4U_6 - U_1U_6^2 - 2U_3U_4^2 + 6U_4^4 \]
\[ - 4U_4^3U_6 - 2U_4^2U_5^2 \right) \]
\[ Z^{-3/2} + \lambda_5 U_1U_4 \] (4.1e)
where

\[ Z := s_1 U_4 - s_2 - U_1 + U_3 - 3 U_4^2 + 2 U_4 U_0 + U_0^2. \]  

(4.2)

This leads to the following set of three conserved densities and conserved flux for the system (1.6a) – (1.6b):

\[ \Phi_t^1 = u_1 u_{0,xx} + u_1^2 u_{1,xx} - u_0 u_1 - 2 s_1 u_1^2 - 2 u_1^3 \]  

(4.3a)

\[ \Phi_x^1 = \left( u_0 + u_1^2 \right) u_{1,xt} + 2 u_0 u_1 u_{0,xx} + 2 u_0 u_1 u_{1,xx} + \left( 4 u_0 u_1^2 + u_0^2 \right) u_{1,xx} \]  

\[ - \frac{1}{2} u_0^2 (6 u_1 + 2 s_1) - u_0 (6 u_1^3 + 2 s_2 u_1 + 8 s_1 u_1^2) - u_0 x u_{1,t} \]  

(4.3b)

\[ \Phi_t^2 = 2 u_1 u_{1,xx} + u_{0,xx} - u_0 - 2 u_1^2 + 2 u_1^2 x - 4 s_1 u_1 \]  

(4.4a)

\[ \Phi_x^2 = -2 u_1 u_{1,xt} + \left( u_0 - 2 u_1^2 \right) u_{0,xx} - 4 u_1^3 u_{1,xx} + \frac{1}{2} u_{0,x}^2 + 2 (u_0 - u_1^2) u_{1,xx} \]  

\[ -2 (s_2 + 2 s_1 u_1) u_0 - \frac{3}{2} u_0^2 + 2 u_1^2 (s_2 + 4 s_1 u_1 + 3 u_1^2) \]  

(4.4b)

\[ \Phi_t^3 = u_{1,xx} - u_1 \]  

(4.5a)

\[ \Phi_x^3 = (u_0 + 2 u_1^2) u_{1,xx} + u_1 u_{0,xx} + u_{0,x} u_{1,x} + 2 u_1 u_{1,xx} - (2 s_1 + 3 u_1) u_0 \]  

\[ -2 u_1 (s_2 + 3 s_1 u_1 + 2 u_1^2). \]  

(4.5b)

\[ \Phi_t^4 = 2 \left( s_1 - u_{1,xx} + u_1 \right) z^{-1/2} \]  

(4.6a)

\[ \Phi_x^4 = 2 \left( s_1 u_0 + 8 s_1 u_1^2 + 2 s_2 u_1 + 3 u_0 u_1 - u_0 u_{1,xx} - 2 u_0 u_{xx} u_1 + 6 u_1^3 \right) \]  

\[ -4 u_1^2 u_{1,xx} - 2 u_1 u_{1,xx} \right) z^{-1/2} \]  

(4.6b)

\[ \Phi_t^5 = z^{1/2} \]  

(4.7a)

\[ \Phi_x^5 = u_0 z^{1/2}, \]  

(4.7b)

where

\[ z := s_1 u_1 - s_2 - u_0 + u_{0,xx} - 3 u_1^2 + 2 u_1 u_{1,xx} + u_{1,xx}^2. \]  

(4.8)
5 Concluding remarks

We have derived the complete set of first-order integrating factors for the systems CH(2,1) and CH(2,2) in Cauchy-Kovalevskaya form. The corresponding sets of conservation laws related to these integrating factors have been derived for both these systems. It would certainly be interesting to calculate higher-order integrating factors, although the computations involved for such calculations appear to be rather challenging. We aim to report some results in a future paper.

We expect that the same method than was applied here could also be used to find conservation laws for more general CH-systems proposed in [11] and [14].

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References


