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Affine recourse for the robust network design problem: between static and dynamic routing[3]

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Abstract

Affinely-Adjustable Robust Counterparts provide tractable alternatives to (two-stage) robust programs with arbitrary recourse. We apply them to robust network design with polyhedral demand uncertainty, introducing the affine routing principle. We compare the affine routing to the well-studied static and dynamic routing schemes for robust network design. All three schemes are embedded into the general framework of two-stage network design with recourse. It is shown that affine routing can be seen as a generalization of the widely used static routing still being tractable and providing cheaper solutions. We investigate properties on the demand polytope under which affine routings reduce to static routings and also develop conditions on the uncertainty set leading to dynamic routings being affine. We show however that affine routings suffer from the drawback that (even totally) dominated demand vectors are not necessarily supported by affine solutions. Uncertainty sets have to be designed accordingly. Finally, we present computational results on networks from SNDlib. We conclude that for these instances the optimal solutions based on affine routings tend to be as cheap as optimal network designs for dynamic routings. In this respect the affine routing principle can be used to approximate the cost for two-stage solutions with free recourse which are hard to compute.

Keywords: Robust optimization; Network design; Recourse; Affine Adjustable Robust Counterparts; Affine Routing; Demand polytope.

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1 Introduction

In the classical deterministic network design problem, a set of point-to point commodities with known demand values is given, and capacities have to be installed on the network links at minimum cost such that the resulting capacitated network is able to accommodate all demands simultaneously by a multi-commodity flow. When demands are known with precision, this problem has been studied extensively in the literature, involving decompositions algorithms, extended formulations and strong cutting planes, see [5, 18, 22, 23, 25, 41], among others.

In practice however, exact demand values are usually not known at the time the design decisions must be made. In telecommunications, demands are estimated which can be done for instance by using traffic measurements (see [45], among others) or population statistics, see [19, 24]. These estimations allow the problem to be formulated and solved using classical tools of deterministic mathematical programming. However, the actual traffic forecast is strongly overestimated in order to yield robust networks capable of routing potential traffic peaks. Such overestimation results in overprovisioned networks wasting capital as well as operational expenditures such as energy resources.

Robust optimization overcomes this problem by explicitly taking into account the uncertainty of the data already in the modeling introducing so-called *uncertainty sets*. A solution is said to be feasible if it is feasible for all realizations of the data in a predetermined uncertainty set $\mathcal{D}$, see [10, 11, 17, 44]. The original framework of Soyster [44] assumes that all decisions must be identical for all values of the uncertain parameters. Introducing more flexibility, two-stage robust optimization allows to adjust a subset of the problem variables only after observing the actual realization of the data, see [12]. In fact, it is natural to apply this two-stage approach to network design since very often first stage capacity design decisions are made in the long term while the actual routing is adjusted based on observed user demands. This second stage adjusting procedure is called *recourse* which in the context of (telecommunications) network design relates to what is known as traffic engineering. Unrestricted second stage recourse in robust network design is called *dynamic routing*, see [20, 28], and [34]. Given a fixed design, the commodity routing can be changed arbitrarily for every realization of the demands. Chekuri et al. [20] and Gupta et al. [28] show that allowing for dynamic routing makes robust network design intractable. Already deciding whether or not a fixed capacity design allows for a dynamic routing of demands in a given polytope is $\mathcal{NP}$-complete (on directed graphs).

Even more general, Ben-Tal et al. [12] observe that two-stage robust linear programming with free recourse is computationally intractable and suggest to limit the flexibility in the second stage by affine functions which makes the problem tractable. Chen and Zhang [21] extend this idea by using extended formulations of uncertainty sets and by applying affine recourse in the resulting higher-dimensional variable spaces.

Interestingly, this limitation in the flexibility of the second stage recourse has been used earlier in robust network design without relating it to two-stage optimization. Ben-Ameur and Kerivin [8, 9] introduce the concept of *static routing* (or oblivious routing).After fixing the design, the routing of a commodity is allowed to change but the actual flow has to be a linear function of the observed demand of the same commodity. Static routing results in a fixed set of paths for every commodity and also a fixed percental splitting of flow among these paths independent of the realization of the commodity demand. In this context we speak of a *routing template* used by all demand realizations. Robust network design with static routing can be handled as a single-stage problem introducing template variables, see the next section. Static routing has been used extensively since then. Ordóñez and Zhao [37] study structural properties of the robust network design problem when cost and demand values belong to conic uncertainty sets. Altin et al. [1] develop a compact integer linear programming model for virtual private network design with continuous capacities and single path routing using. Altin et al. [2, 3] and Koster et al. [32] study the network design problem assuming splittable flow and integer capacities. Polyhedral investigations and computational evaluations of the models are carried out.

The restriction of routing templates for every commodity makes the problem tractable but it is of course very conservative in terms of capacity cost compared to dynamic routing. Recently there have been several attempts to handle less restrictive routing principles. These could be shown to be intractable just like the dynamic case. Ben-Ameur [7] partitions the demand uncertainty set into two (or more) subsets using hyperplanes and devises specific routings for each subset. The resulting optimization problem is $\mathcal{NP}$-hard when no assumptions is made on the hyperplanes. Ben-Ameur thus discusses simplifications where either the entire hyperplane or its direction is given. Scutellà [42] allows for two (or more) routing templates to be used conjointly. These routing templates are devised iteratively, that is, given a routing template and a capacity allocation, Scutellà [42] allows some commodities to use a second routing template in order to reduce the overall capacity allocation. The procedure is proved to be $\mathcal{NP}$-hard in [43]. Mattia [34] provides a branch-and-cut algorithm for robust network design with dynamic routing together with a computational comparison to the static version.

As an alternative to these $\mathcal{NP}$-hard approaches, Ouorou and Vial [40] and Babonneau et al. [6] only recently apply directly the affine recourse from [12] to network design problems using particular uncertainty sets. The resulting restrictive routing scheme is refereed to as *affine routing* in the following. We will introduce affine routing as a generalization of static routing. In this context affine routing provides an alternative in between static and dynamic routing yielding tractable robust counterparts in contrast to the schemes used in [7, 43].

The contributions of this paper consists of a theoretical and empirical study of network design under the affine routing principle for general polyhedral demand uncertainty sets $\mathcal{D}$. We embed affine routing into the context of two-stage network design with recourse and compare it to its natural counterparts, static and dynamic routings. Section 2 introduces the mathematical models and defines formally static, affine and dynamic routings. In Section 3 we show that, when $\mathcal{D}$ is full-dimensional, affine routings decompose into a combination of cycles and paths, that is, a routing template for every commodity can be affinely adjusted using a set of cycles, whenever a different commodity is perturbed within the feasible demand region. We describe then necessary and sufficient conditions on $\mathcal{D}$ under which affine routing is equivalent to static or dynamic routing. As a bi-product, we obtain that static and dynamic routings are equivalent under certain assumptions on $\mathcal{D}$. We show that dominated demand vectors are not automatically supported by affine solutions in contrast to static and dynamic solutions. In particular very small demand scenarios have to be included in the uncertainty set. Finally, Section 4 presents numerical comparisons of dynamic, affine and static routings carried out on instances from SNDlib, see [39]. It turns out that for these instances the affine routing principle is numerically very close to the dynamic second stage recourse rule. In fact, it provides enough flexibility to approximate the cost for optimal two-stage solutions with full flexibility.

2 Robust network design with recourse

We are given a directed graph $G = (V, A)$ and a set of commodities K. A commodity $k \in K$ has source $s(k) \in V$, destination $t(k) \in V$, and demand value $d^k \geq 0$. A *flow* for k is a vector $f^k \in \mathbb{R}^A_+$ satisfying:

$$\sum_{a \in \delta^+(v)} f^k_a - \sum_{a \in \delta^-(v)} f^k_a = d^k \psi_{vk} \quad \text{for all } v \in V, \tag{1}$$

where $\delta^+(v)$ and $\delta^-(v)$ denote the set of outgoing arcs and incoming arcs at node v, respectively. For node $v \in V$ and commodity $k \in K$ we set $\psi_{vk} := 1$ if $v = s(k)$, $\psi_{vk} := -1$ if $v = t(k)$, and $\psi_{vk} := 0$ else. Flows are non-negative. A *multi-commodity flow* is a collection of flows, one for each commodity in K. A *circulation* (or cycle-flow) is a vector $g \in \mathbb{R}^A$ satisfying

$$\sum_{a \in \delta^+(v)} g_a - \sum_{a \in \delta^-(v)} g_a = 0 \quad \text{for all } v \in V. \tag{2}$$

A circulation is not necessarily non-negative. A value $g_a < 0$ can be seen as a flow from the head of arc a to its tail. We call a circulation g *non-negative* if $g \geq 0$ and *positive* if additionally $g \neq 0$.

Notice that any two flows $\hat{f}^k, f^k$ for k only differ by a circulation, that is, there always exists a circulation g such that $\hat{f}^k = f^k + g$.

In many practical situations, the demand vector $d \in \mathbb{R}_+^K$ is uncertain. In the sequel we assume that $d \in \mathcal{D} \subset \mathbb{R}^K$ where $\mathcal{D}$ is a bounded convex set. We often call $\mathcal{D}$ the *uncertainty set*. Any $d \in \mathcal{D}$ is said to be a realization of the demand. A *routing* is a function $f : \mathcal{D} \to \mathbb{R}_+^{A \times K}$ that assigns a multi-commodity flow to every realization of the demand. We say that f *serves* $\mathcal{D}$ and call f a *dynamic routing* if there is no further restriction on the routing. A capacity allocation $x \in \mathbb{R}_+^A$ is said to *support* the set $\mathcal{D}$ if there exists a routing f serving $\mathcal{D}$ such that for every $d \in \mathcal{D}$ the corresponding multi-commodity flow $f(d)$ is not exceeding the arc-capacities given by x. Robust network design now aims at providing the cost minimal capacity allocation supporting $\mathcal{D}$. In this respect, robust network design is a two-stage robust program with recourse, following the more general framework described by [12]. The capacity design has to be fixed in the first stage, and, observing a demand realization $d \in \mathcal{D}$, we are allowed to adjust the routing $f(d)$ in the second stage. The problem can be written as the following (infinite) linear program denoted by (RND) in the following:

$$
(RND) \qquad \begin{aligned}
\min \ & \sum_{a \in A} \kappa_a x_a && && (3)\\
\text{s.t.} \ & \sum_{a \in \delta^+(v)} f_a^k(d) - \sum_{a \in \delta^-(v)} f_a^k(d) = d^k \psi_{vk}, && v \in V, k \in K, d \in \mathcal{D} && (4)\\
& \sum_{k \in K} f_a^k(d) \le x_a, && a \in A, d \in \mathcal{D} && (5)\\
& f_a^k(d) \ge 0, && a \in A, k \in K, d \in \mathcal{D} && (6)\\
& x_a \ge 0, && a \in A,
\end{aligned}
$$

where $\kappa_a \in \mathbb{R}_+$ is the cost for installing one unit of capacity on arc $a \in A$. Ben-Tal et al. [12] point out that allowing for arbitrary recourse very often makes robust optimization problems intractable. In fact, this is true for two-stage robust network design with free recourse. It is known that already deciding whether a given capacity vector x supports $\mathcal{D}$ is $\mathcal{NP}$-complete for general polytopes $\mathcal{D}$, see [20, 28]. It follows from this $\mathcal{NP}$-completeness result that it is impossible (unless $\mathcal{P} = \mathcal{NP}$) to derive a compact formulation for (RND) given a general uncertainty polytope $\mathcal{D}$ if there is no restriction on the second stage routing decision, contrasting with the reformulation discussed in Section 3.1. Using a branch-and-cut approach to solve (RND) based on Benders decomposition ([14]), Mattia [34] shows how to solve the $\mathcal{NP}$-hard separation problem for robust metric inequalities ([30, 36]) using bilevel and mixed integer programs.

Most authors ([2, 4, 9, 32, 35, 37], among others) use a simpler version of (RND) introducing a restriction on the second stage recourse known as *static routing* (also called oblivious routing). Each component $f^k : \mathcal{D} \to \mathbb{R}_+^A$ is forced to be a linear function of d^k:

$$
f_a^k(d) := y_a^k d^k \qquad a \in A, k \in K, d \in \mathcal{D}. \tag{7}
$$

Notice that by (7) the flow for k is not changing if we perturb the demand for $h \neq k$. By combining (4) and (7) it follows that the multipliers $y \in \mathbb{R}_+^{A \times K}$ define a multi-commodity (percentage) flow. For every $k \in K$, the vector $y^k \in \mathbb{R}_+^A$ satisfies (1) setting $d^k = 1$. The flow y is called a *routing template* since it decides, for every commodity, which paths are used to route the demand and what is the percental splitting among these paths. The routing template has to be used by all demand scenarios $d \in \mathcal{D}$ under the static routing scheme.

Ben-Tal et al. [12] introduce Affine Adjustable Robust Counterparts restricting the recourse to be an affine function of the uncertainties. Applying this framework to (RND) means restricting f^k to be an affine function of *all* components of d giving

$$
f_a^k(d) := f_a^{0k} + \sum_{h \in K} y_a^{kh} d^h \ge 0, \qquad a \in A, k \in K, d \in \mathcal{D}, \tag{8}
$$

where $f_a^{0k}, y_a^{kh} \in \mathbb{R}$ for all $a \in A, k, h \in K$, see also [40]. In what follows, a routing f serving $\mathcal{D}$ and satisfying (8) for some vectors f^0 and y is called an *affine routing.*

The following lemma formalizes the relation between optimal solutions of robust network design using dynamic, affine, or static routings. Affine routing generalizes static routing allowing for more flexibility in reacting to demand fluctuations, but it is not as flexible as dynamic routing.

Lemma 1. *Given an arbitrary demand uncertainty set $\mathcal{D}$, let opt_{dyn}, opt_{aff}, and opt_{stat} be the cost of the optimal solution to (RND) where f is allowed to be dynamic, affine, or static, respectively. It holds that*

$$opt_{dyn} \leq opt_{aff} \leq opt_{stat}.$$

Proof. Trivially any routing f is a dynamic routing. The number of possible routings serving $\mathcal{D}$ is restricted by imposing (8) hence $opt_{dyn} \leq opt_{aff}$. Moreover, we see immediately that static routing can be obtained from (8) by setting $f_a^{0k} = 0$ and $y_a^{kh} = 0$ for each $a \in A$ and all $k, h \in K$ with $k \neq h$ yielding $opt_{aff} \leq opt_{stat}$. □

Note that there is a proven (tight) worst-case optimality gap of $O(log|V|)$ between the dynamic and static routing principle, see [27]. For special graphs it can be shown that $opt_{dyn} \in O(|V|)$ while $opt_{stat} \in \Omega(|V| \log|V|)$. In this paper we do not establish optimality gaps between the three routing principles. We rather focus on studying properties of the demand scenarios $\mathcal{D}$ that either yield $opt_{stat} = opt_{aff}$ or $opt_{aff} = opt_{dyn}$. These properties are independent of the actual graph. Our computational experiments in Section 4 suggest that for realistic networks and demand scenarios the static/dynamic optimality gap is small in practice (also see [34]), and if there is a gap, the cost for affine solutions tends to be very close to the cost for dynamic solutions. In fact in most cases $opt_{aff} \approx opt_{dyn}$.

Given a demand polytope $\mathcal{D}$, a static routing f is completely described by the vector $y \in \mathbb{R}_+^{A \times K}$. Similarly, an affine routing is completely described by fixing the vectors $f^0 \in \mathbb{R}^{A \times K}$ and $y \in \mathbb{R}^{A \times K \times K}$. Extending the previous definitions, any routing template $y \in \mathbb{R}_+^{A \times K}$ is said to serve $\mathcal{D}$ if it yields a (static) routing f serving $\mathcal{D}$. Similarly, any pair of vectors $f^0 \in \mathbb{R}^{A \times K}$ and $y \in \mathbb{R}^{A \times K \times K}$ that satisfies (4) and (8) are said to *serve* $\mathcal{D}$. Given a capacity allocation $x \in \mathbb{R}_+^A$, the pair (x, y) with y serving $\mathcal{D}$, or the triplet (x, f^0, y) with (f^0, y) serving $\mathcal{D}$ are said to *support* $\mathcal{D}$ if the corresponding routings satisfy (5).

When $\mathcal{D}$ is not finite, model (RND) contains an infinite number of variables and inequalities. However, when $\mathcal{D}$ is convex, we can replace $\mathcal{D}$ by the set of its extreme points, which is finite whenever $\mathcal{D}$ is a polytope.

Lemma 2. *Let $\mathcal{D} \subset \mathbb{R}^K$ be a bounded set and x be a capacity allocation $x \in \mathbb{R}^A$.*

(a) x supports $\mathcal{D}$ if and only if x supports $\text{conv}(\mathcal{D})$.

(b) (x, y) supports $\mathcal{D}$ if and only if (x, y) supports $\text{conv}(\mathcal{D})$.

(c) (x, f^0, y) supports $\mathcal{D}$ if and only if (x, f^0, y) supports $\text{conv}(\mathcal{D})$.

Proof. $(c) \Leftarrow$: Trivial.
$\Rightarrow$: Consider (x, f^0, y) that satisfies (4), (5) and (8) for each $d \in \mathcal{D}$. Any $d^* \in \text{conv}(\mathcal{D})$ can be defined as follows: let $d_i \in \mathcal{D}$ and $\lambda_i \geq 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \lambda_i = 1$ and $d^* = \sum_{i=1}^n \lambda_i d_i$. In order to satisfy (8), $f(d^*)$ must be defined by

$$f_a^k(d^*) = f_a^{0k} + \sum_{h \in K} y_a^{kh} d^{h*} = \sum_{i=1}^n \lambda_i (f_a^{0k} + \sum_{h \in K} y_a^{kh} d_i^h), \qquad a \in A, k \in K, d \in \mathcal{D}.$$

It is easy to see that $f_a^k(d^*)$ defined above together with x satisfy (4) and (5). (a) and (b) can be proved similarly. □

When $\mathcal{D}$ is a polytope, Lemma 2 implies that (RND) can be discretized by restricting the model to the extreme demand scenarios that correspond to vertices of $\mathcal{D}$. This yields a linear programming reformulation of (RND) with potentially an exponential number of variables and constraints. However, not all extreme points must be considered. For instance, if $0 \in \mathcal{D}$, it is an extreme point of $\mathcal{D}$ that any capacity allocation supports. This intuitive idea can be formalized using the domination concepts introduced by Oriolo [38]. Given two demands vectors d_1 and d_2, we say that d_1 *dominates* d_2 if any capacity allocation $x \in \mathbb{R}_+^A$ supporting d_1 also supports d_2. Moreover, d_1 *totally dominates* d_2 if any pair (x, y) supporting d_1 also supports d_2. Thus, the optimal solution to (RND) with dynamic routing (resp. static routing) for a polytope $\mathcal{D} = \text{conv}\{d_1, \ldots, d_n\}$ stays unchanged if we solve the problem without considering the dominated (resp. totally dominated) extreme points of $\mathcal{D}$. We refer to [38] for interesting characterizations of domination. We point out that the situation is different for affine routing, that is, it is impossible to define d_1 and d_2 such that any triplet (x, f^0, y) supporting d_1 also supports d_2, see Proposition 6.

In what follows, we always consider that $\mathcal{D}$ is either a polytope or a finite set of demand vectors, which is equivalent in view of Lemma 2. The large linear program coming from the discretization of $\mathcal{D}$ can be sensibly simplified for affine and static routings. Namely, given a polytope $\mathcal{D}$ defined by a polynomial number of inequalities, Soyster [44] shows how to use the linear programming duality to obtain a formulation that contains the deterministic variables and constraints plus an additional polynomial number of variables and constraints, and that is equivalent to the robust infinite program. Altin et al. [4] extend this idea to (RND) with static routing and we show in Proposition 4 how this can be done for (RND) with affine routing.

We close this section by pointing out some very trivial cases for which $opt_{stat} = opt_{aff} = opt_{dyn}$. First, if $\mathcal{D}$ is a singleton (or even more general if $\mathcal{D}$ contains a demand vector that totally dominates all $d \in \mathcal{D}$) then $opt_{stat} = opt_{aff} = opt_{dyn}$ since given any dynamic solution, the routing for the single (dominating) demand vector defines a static routing template at the same cost. Moreover, in this case every routing corresponding to an optimal solution is a shortest path routing, that is, the flow for commodity k is sent on a shortest path between $s(k)$ and $t(k)$ with respect to the arc capacity cost κ_a. To see this, observe that (RND) decomposes into $|K|$ shortest path problems, one for every commodity, if $\mathcal{D}$ is a singleton. In such a case we speak of a *shortest path template* for K.

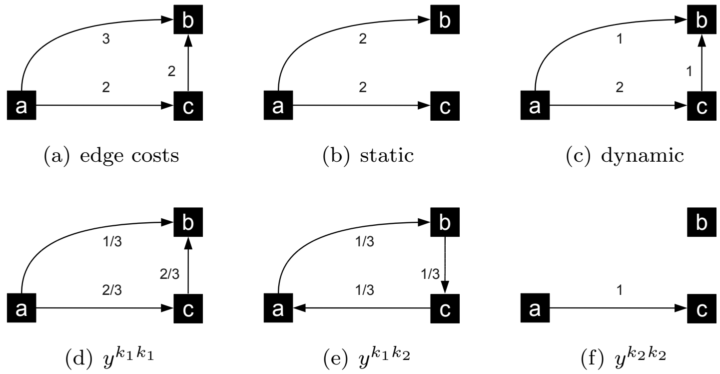
Secondly, in the single commodity case, that is $|K| = 1$, we get that $opt_{stat} = opt_{dyn}$ since in this case polytope $\mathcal{D} \subset \mathbb{R}_+$ is an interval, and we can solve the problem for the maximum single demand instead. Also observe that (8) reduces to (7).

Finally, note that if there exists only one path from $s(k)$ to $t(k)$ for a commodity $k \in K$, then static, affine and dynamic routings coincide for that commodity. In the sequel we assume that for all $k \in K$ there exist at least two distinct paths p_1, p_2 in G from $s(k)$ to $t(k)$, that is, two paths that differ by one arc at least.

3 Affine Routings

In this section, we study properties and consequences of the affine routing principle. First, we remark that when $\mathcal{D}$ is full-dimensional, then affine routing can be expressed as a routing template (just like in the static case) plus a set of circulations.

Lemma 3. *Let $\mathcal{D}$ be a demand polytope and let $(f^0, y) \in \mathbb{R}^{A \times K} \times \mathbb{R}^{A \times K \times K}$ be an affine routing serving $\mathcal{D}$. If $\mathcal{D}$ is full-dimensional, then $y^{kk} \in \mathbb{R}^A$ is a routing template for $k \in K$ and $f^{0k} \in \mathbb{R}^A, y^{kh} \in \mathbb{R}^A$ are circulations for every $k, h \in K$ with $k \neq h$.*



(d) $y^{k_1 k_1}$ (e) $y^{k_1 k_2}$ (f) $y^{k_2 k_2}$

Figure 1: Optimal solutions for static, dynamic, and affine recourse, also see Example 1

Proof. Plugging (8) into (4) and grouping together coefficients of each d^h, we obtain

$$\sum_{a \in \delta^+(v)} f_a^{0k} - \sum_{a \in \delta^-(v)} f_a^{0k} + d^k \left(\sum_{a \in \delta^+(v)} y_a^{kk} - \sum_{a \in \delta^-(v)} y_a^{kk} \right)$$
$$+ \sum_{h \in K \setminus k} d^h \left(\sum_{a \in \delta^+(v)} y_a^{kh} - \sum_{a \in \delta^-(v)} y_a^{kh} \right) = d^k \psi_{vk} \tag{9}$$

for each $v \in V$, $k \in K$. Let e^h denote the unit vector in $\mathbb{R}^K$ corresponding to commodity h. Since $\mathcal{D}$ is full-dimensional there exists a vector $d_0 \in \mathcal{D}$ and $\varepsilon > 0$ such that $d_0 + \varepsilon e^h \in \mathcal{D}$ for all $h \in K$. Subtracting (9) written for $d_0 + \varepsilon e^h$ from (9) written for d_0 gives

$$\varepsilon \left(\sum_{a \in \delta^+(v)} y_a^{kk} - \sum_{a \in \delta^-(v)} y_a^{kk} \right) = \varepsilon \psi_{vk} \tag{10}$$

for $h = k$ and similarly

$$\varepsilon \left(\sum_{a \in \delta^+(v)} y_a^{kh} - \sum_{a \in \delta^-(v)} y_a^{kh} \right) = 0 \tag{11}$$

for $h \neq k$. Hence $y^{kk} \in \mathbb{R}^A$ is a routing template for $k \in K$ and $y^{kh} \in \mathbb{R}^A$ is a circulation if $k \neq h$. Plugging (10) and (11) into (9) also shows that $f^{0k} \in \mathbb{R}^A$ is a circulation for all $k \in K$. □

Just like in the static case, the flow for commodity k changes linearly with d^k on the paths described by the template y_a^{kk}. However, the flow for commodity k may change also if the demand for $h \neq k$ changes which is described by circulations y^{kh}. In addition there is a constant circulation shift described by variables f^{0k}.

As already mentioned, a dynamic routing for commodity k could also be described by one (representative) routing and circulations depending on the demand fluctuations. In the dynamic case however, the circulations can be chosen arbitrarily while in the affine case the actual flow changes according to (8). We illustrate this concept in Example 1 and Figure 1 which shows that affine routing can be as good as dynamic routing in terms of the cost for capacity allocation and that f^0 and y^{kh} may not describe circulations when $\mathcal{D}$ is not full-dimensional

Example 1. Consider the network design problem for the graph depicted in Figure 1(a) with two commodities $k_1 : a \to b$ and $k_2 : a \to c$. The uncertainty set $\mathcal{D}$ is defined by the extreme points $d_1 = (2, 1), d_2 = (1, 2)$, and $d_3 = (1, 1)$, and the capacity unitary costs are the edge labels of Figure 1(a). Edge labels from Figure 1(b) and 1(c) represent optimal capacity allocations with static and dynamic routing, respectively. They have costs of 10 and 9, respectively. Then, Figure 1(d)-1(f) describes coefficients y^{kh} for an affine routing feasible for the capacity allocation 1(c). If we remove $d_3 = (1,1)$ from the set of extreme points, the dimension of the uncertainty set reduces to 1. The affine routing prescribed by $y_{ac}^{k_2k_2} = 1$, $f_{ab}^{0k_1} = 3$ and $y_{ab}^{k_1k_2} = -1$ serves all demands in the convex hull of $d_1 = (2,1)$ and $d_2 = (1,2)$ but f^{0k_1} and $y^{k_1k_2}$ do not describe a circulation.

3.1 A compact reformulation

Because of Lemma 2, a compact linear formulation for the robust network design problem with affine routing can be prescribed whenever the number vertices of $\mathcal{D}$ is polynomial in the number of nodes, arcs, and commodities. This is achieved by writing (RND) with (8) for the vertices of $\mathcal{D}$. In the following we provide a compact linear reformulation for the case that $\mathcal{D}$ has a compact inequality description, that is, the polytope $\mathcal{D}$ has a linear description in $\mathbb{R}^K$ where the number of defining inequalities is polynomial in the number of node, arcs, and commodities. The reformulation extends the one from Altin et al. [4] for static routing. Let $\mathcal{D}$ be given by

$$\mathcal{D} := \{d \in \mathbb{R}^K : \mathcal{A}d \leq b, d \geq 0\},$$

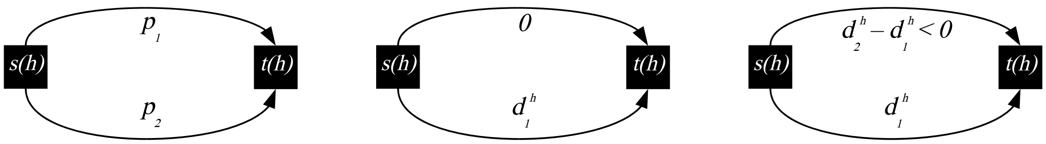
where $\mathcal{A} = (\alpha^{ik}) \in \mathbb{R}^{m \times K}$ and $b \in \mathbb{R}^m$, $m \geq 1$ and let $\mathcal{D}$ be full-dimensional.

Proposition 4. *The robust network design problem* (RND) (3)-(6) *under the affine routing principle* (8) *respecting the uncertainty polytope* $\mathcal{D}$ *is equivalent to the following linear program denoted by* $(AARND)$ *in the following:*

$$
\begin{array}{llllll}
 & \min & \sum_{a\in A} \kappa_a x_a & & & \\
(AARND) & s.t. & \sum_{a\in\delta^+(v)} y_a^{kk} - \sum_{a\in\delta^-(v)} y_a^{kk} & = \psi_{vk} & v\in V, k\in K & (12)\\
 & & \sum_{a\in\delta^+(v)} y_a^{kh} - \sum_{a\in\delta^-(v)} y_a^{kh} & = 0 & v\in V, k\neq h\in K & (13)\\
 & & \sum_{a\in\delta^+(v)} f_a^{0k} - \sum_{a\in\delta^-(v)} f_a^{0k} & = 0 & v\in V, k\in K & (14)\\
 & & \sum_{k\in K} f_a^{0k} + \sum_{i=1}^m b_i\mu_a^i & \leq x_a, & a\in A & (15)\\
 & & \sum_{i=1}^m \alpha^{ih}\mu_a^i & \geq \sum_{k\in K} y_a^{kh}, & a\in A, h\in K & (16)\\
 & & \sum_{i=1}^m b_i\lambda_a^{ik} & \leq f_a^{0k}, & a\in A, k\in K & (17)\\
 & & \sum_{i=1}^m \alpha^{ih}\lambda_a^{ik} & \geq -y_a^{kh}, & a\in A, k\in K, h\in K & (18)\\
 & & x, f^0, y, \mu, \lambda & \geq 0 & &
\end{array}
$$

Proof. It has been shown in Lemma 3 that the flow balance constraints (4) reduce to (12)-(14) using (8). The capacity constraint (5) can be rewritten as

$$\sum_{k\in K} f_a^{0k} + \max_{d\in\mathcal{D}} \sum_{k\in K}\sum_{h\in K} d^h y_a^{kh} \leq x_a.$$



(a) paths (b) routing for d_1 (c) unfeasible routing for d_2

Figure 2: Example of affine routing feasible for d_1 but not for d_2.

Dualizing this constraint for every $a \in A$ using the inequality description of $\mathcal{D}$ gives (15) and (16), where μ_a^i is the dual variable of the $i-th$ inequality in $\mathcal{A}d \leq b$. The non-negativity constraints (8) can be rewritten as

$$-f_a^{0k} + \max_{d \in \mathcal{D}} \sum_{h \in K} -y_a^{kh} d^h \leq 0. \tag{19}$$

Dualizing this constraint for every $a \in A$ and every $k \in K$ gives (17) and (18), where this time λ_a^{ik} corresponds to the dual variable of the $i-th$ inequality in the description of $\mathcal{D}$. □

We note that reformulating by dualizing constraints is a standard technique in robust optimization, see for instance [11]. Formulation $(AARND)$ generalizes the formulation of Ouorou and Vial [40] which considers affine routing for a specific polyhedral demand polytope $\mathcal{D}$. It also generalizes the model in [4] for general polytopes $\mathcal{D}$ but static routing. The model is obviously compact if m is a polynomial function of $(|A|, |V|, |K|)$, that is, the description of $\mathcal{D}$ is compact. We remark that $\mathcal{D}$ being compact is not necessary to obtain a compact robust counterpart of (RND). It suffices to provide a compact *extended* formulation of $\mathcal{D}$ as shown in Section 4.

Since static routing is a special case of affine routing, formulation (12)–(18) generalizes the reformulation for static routing given in [4]. The latter is obtained from (12)–(18) by withdrawing (13), (14) and (17) and fixing to zero the vectors f^0, λ as well as the vectors y^{kh} for all $k \neq h$.

Corollary 5. *If $\mathcal{D}$ is a full-dimensional polytope and either the number of its vertices or the number of its facets is polynomial in $(|A|, |V|, |K|)$, then (RND) with the affine recourse (8) can be solved in polynomial time in $(|A|, |V|, |K|)$.*

Corollary 5 implies that given a capacity allocation x, the existence of an affine routing can be answered in polynomial time for every demand polytopes with either polynomial many vertices or polynomial many facets contrasting with the $\mathcal{NP}$-complete results using dynamic routing, see [20, 28].

3.2 Domination of demand vectors

Next result shows that it is not possible to define an "affine domination" between two demands vectors, in opposition to the domination and total domination defined by Oriolo [38] for dynamic and static routings, respectively.

Proposition 6. *Let $d_1, d_2 \in \mathbb{R}_+^K$, $d_1 \neq d_2$. There exists (x, f^0, y) that supports d_1 but not d_2.*

Proof. Any (x, y) that supports d_1 also supports d_2 if and only if $d_1^k \geq d_2^k$ for each $k \in K$ [38, Theorem 2.5]. Since static routings are special cases of affine routing we can assume that $d_1^k \geq d_2^k$ for each $k \in K$.

Since $d_1 \neq d_2$, there exists $h \in K$ such that $d_1^h > d_2^h$. We describe next a capacity allocation and an affine routing that support d_1 but do not support d_2. Let $x_a = M$ for each $a \in A$ and M large enough; for each $k \in K$, let p_k be a path between $s(k)$ and $t(k)$ and let $y_a^{kk} = 1$ for each $a \in p_k$, 0 otherwise. Then, let $y^{k_1 k_2} = 0$ for each $k_1, k_2 \in K$ and $k_1 \neq k_2$, and $f^{0k} = 0$ for each $k \in K \backslash \{h\}$. The construction for h is illustrated in Figure 2. Let $p_h = p_1$ and p_2 be a path from $s(h)$ to $t(h)$ different from p_1, see Figure 2(a). Finally, set $f_a^{0h} = -d_1^h$ for each $a \in p_1 \backslash p_2$, $f_a^{0h} = d_1^h$ for each $a \in p_2 \backslash p_1$, and 0 otherwise. The triplet (x, f^0, y) just defined supports d_1 with the flow depicted in Figure 2(b) but prescribes a negative flow on arcs $a \in p_1 \backslash p_2$ for d_2 as depicted in Figure 2(c). □

Given a demand polytope $\mathcal{D}$ for problem (RND) with static (resp. dynamic) routing, domination (resp. total domination) among demands vectors enables to withdraw the dominated extreme points from $\mathcal{D}$, obtaining a smaller uncertainty set possibly easier to describe. For instance, 0 never needs to be considered. Proposition 6 shows that such simplification is not possible with affine routings.

Remark 1. Notice that it is possible to introduce an "affine domination" if G is such that some commodities can use only one path. Namely, if for each $k \in Q \subseteq K$, there exists only one path from $s(k)$ to $t(k)$, then any triplet (x, f^0, y) that supports d_1 also supports d_2 if and only if $d_1^k = d_2^k$ for $k \in K \backslash Q$ and $d_1^k \geq d_2^k$ otherwise.

3.3 Relation to static routing

Proposition 6 implies that all extreme points of $\mathcal{D}$ must be considered when using affine routing, in particular very small demand realization. Also $0 \in \mathcal{D}$, if required, cannot be removed from the uncertainty set. This comes from the non-negativity constraints (19), which impose important restrictions on the circulation variables f^{0k} and y^{kh}. Proposition 6 is a negative result since in the following we will show that whenever $\mathcal{D}$ contains very small demand realization affine routings reduce to static routings. In what follows let e^k be the k-th unit vector in $\mathbb{R}_+^K$.

Lemma 7. *Let $\mathcal{D}$ be full-dimensional. If $0 \in \mathcal{D}$ then $f^{0k} \in \mathbb{R}_+^A$ describes a non-negative circulation in G. If $\epsilon e^h \in \mathcal{D}$ for some $\epsilon > 0$ and $h \in K$ and $f^{0k} = 0$ for $h \neq k \in K$ then $y^{kh} \in \mathbb{R}_+^A$ describes a non-negative circulation in G.*

Proof. Writing (8) for $0 \in \mathcal{D}$ gives $f_a^{0k} \geq 0$ for all $a \in A$ and $k \in K$ and hence by Lemma 3 f^{0k} is a non-negative circulation. Similarly, writing (8) for $\epsilon e^h \in \mathcal{D}$ we get $f_a^{0k} + \epsilon y_a^{kh} \geq 0$ for all $a \in A$ and $k \in K$. If $f^{0k} = 0$ for $k \neq h$ we get that y_a^{kh} is a non-negative circulation by Lemma 3. □

It is clear that the non-negative circulations mentioned in Lemmas 7 do not yield useful affine routings because they increase the capacity requirement without providing additional flexibility. If a flow f^k for k contains a positive circulation, that is, there exists a positive circulation g such that $f^k - g$ is a flow for k then f^k can be reduced to $f^k - g$ without changing the flow balance at $s(k)$ and $t(k)$. In this spirit we call any routing f *cycle-free* if for all $d \in \mathcal{D}$ and all commodities $k \in K$ the commodity flows do not contain positive circulations. Of course every optimal capacity allocation has a *cycle-free* (static, affine, or dynamic) routing.

In the case of cycle-free affine routings, Lemma 7 provides conditions under which $f^{0k} = 0$ for $k \in K$ and $y^{kh} = 0$ for all $k, h \in K$, $k \neq h$ and thus, the corresponding affine routing is static:

Proposition 8. *Let $\mathcal{D}$ be a demand polytope. If $0 \in \mathcal{D}$ and for each $k \in K$ there is $\epsilon_k > 0$ such that $\epsilon_k e^k \in \mathcal{D}$, then all cycle-free affine routings serving $\mathcal{D}$ are static and hence $opt_{aff} = opt_{stat}$.*

Proposition 8 also highlights that affine routing suffers from a drawback related to Proposition 6. Adding dominated or totally dominated vectors to $\mathcal{D}$ might restrict the set of feasible affine routings. The condition in Proposition 8 can be weakened yielding a necessary and sufficient condition for acyclic graphs and a necessary condition for general graphs. Sufficiency and Necessity are shown in Proposition 9 and Theorem 10, respectively. In the rest of this section, let $\mathcal{D}_0^k$ denote the set obtained from $\mathcal{D}$ by removing $d \in \mathcal{D}$ with $d^k > 0$, that is, $\mathcal{D}_0^k := \{d \in \mathcal{D} : d^k = 0\}$.

Proposition 9. *Let $\mathcal{D}$ be a demand polytope and let G be acyclic. If $\dim(\mathcal{D}_0^k) = |K| - 1$ for all $k \in K$, then all cycle-free affine routings serving $\mathcal{D}$ are static and hence $opt_{aff} = opt_{stat}$.*

Proof. Let $k \in K$ be a given commodity. For all $d \in \mathcal{D}_0^k$, any flow $f^k(d)$ must either be equal to 0 or describe a positive circulation. The latter is impossible because G is acyclic, so that $f^k(d) = 0$ for all $d \in \mathcal{D}_0^k$. Let $\{d_1, \dots, d_{|K|}\}$ be a set of affinely independent vectors spanning $\mathcal{D}_0^k$. Any affine flow for k must satisfy

$$f_a^{0k} + \sum_{h \in K \setminus \{k\}} y_a^{kh} d_i^h = 0, \qquad 1 \le i \le |K|, \tag{20}$$

which is a system of $|K|$ independent linear equations with $|K|$ variables. Therefore, its unique solution is 0 and the affine routing must be static. □

In the following we show that for general graphs the condition $\dim(\mathcal{D}_0^k) = |K| - 1$ for all $k \in K$ is also necessary for all cycle-free affine routing to be static. This means that in all other cases there exists cycle-free affine routings that are not static. But notice that from the latter does not follow $opt_{aff} < opt_{stat}$, see Example 3 in the next section.

Theorem 10. *Let $\mathcal{D}$ be a demand polytope. If all cycle-free affine routings serving $\mathcal{D}$ are static then* $\dim(\mathcal{D}_0^k) = |K| - 1$ *for all* $k \in K$.

Proof. Let $k \in K$ such that $\dim(\mathcal{D}_0^k) < |K| - 1$. We construct an affine routing for commodity k (all other commodities are routed arbitrarily). Consider two distinct paths p_1 and p_2 in G from $s(k)$ to $t(k)$. Let y^{kk} be the routing template that splits the flow equally between p_1 and p_2, that is, y_a^{kk} is equal to 0.5 for $a \in p_1 \cup p_2 \backslash (p_1 \cap p_2)$, 1 for $a \in p_1 \cap p_2$, and 0 otherwise. We will construct an affine perturbation for k using the cycles formed by $p_2 \backslash p_1$ and $p_1 \backslash p_2$ where arcs in $p_1 \backslash p_2$ are taken in the reverse direction. If $\mathcal{D}_0^k$ is non-empty, let $p - 1$ be its dimension, with $1 \le p \le |K| - 1$, and let $\{d_1, \dots, d_p\}$ be a set of affinely independent vectors spanning $\mathcal{D}_0^k$. It follows that the system of equations

$$\lambda^0 + \sum_{h \in K \backslash k} d_i^h \lambda^h = 0, \qquad 1 \le i \le p \tag{21}$$

has a solution $\overline{\lambda} \neq 0$. In case $\mathcal{D}_0^k$ is empty we chose the vector $\overline{\lambda} \neq 0$ arbitrarily. We construct next an affine perturbation for the flow of commodity k based on $\overline{\lambda}$.

Given $\epsilon > 0$, let $f_a^{0k} = \epsilon \overline{\lambda}^0$ for $a \in p_2 \backslash p_1$, $f_a^{0k} = -\epsilon \overline{\lambda}^0$ for $a \in p_1 \backslash p_2$, and $f_a^{0k} = 0$ otherwise. Similarly, given $h \neq k$, we set $y_a^{kh} = \epsilon \overline{\lambda}^h$ for all arcs a in $p_2 \backslash p_1$, $y_a^{kh} = \epsilon \overline{\lambda}^h$ for $a \in p_2 \backslash p_1$ and $y_a^{kh} = 0$ otherwise. Combining these cycle variables with the routing template y^{kk} defined above yields a cycle-free affine routing f^k serving $\mathcal{D}$.

It remains to show that $\epsilon > 0$ can be chosen such that the flow $f^k(d)$ is non-negative on all arcs for all $d \in \mathcal{D}$. Because of Lemma 2 we can restrict ourselves to the finite set E of extreme points of $\mathcal{D}$. Consider first $d \in E$ with $d^k = 0$. By definition, $f_a^k(d) = 0$ for each $a \in A \backslash (p_1 \cup p_2) \cup (p_1 \cap p_2)$. Assume $a \in p_2 \backslash p_1$. The vector d can be written as an affine combination of the vectors in $\{d_1, \dots, d_p\}$, that is, $d = \sum_{i=1}^p \mu_i d_i$ for some multipliers $\mu_i \in \mathbb{R}$ with $\sum_{i=1}^p \mu_i = 1$. Hence the flow on arc $a \in p_2 \backslash p_1$ for demand d satisfies

$$f_a^k(d) = \epsilon \left(\overline{\lambda}^0 + \sum_{h \in K \backslash k} d^h \overline{\lambda}^h \right) = \epsilon \sum_{i=1}^p \mu_i \left(\overline{\lambda}^0 + \sum_{h \in K \backslash k} d_i^h \overline{\lambda}^h \right) = 0,$$

where the last equation follows from (21). Similarly, the flow can be shown to be zero for $a \in p_1 \backslash p_2$.

Now let $d \in E$ such that $d^k > 0$. Again $f_a^k(d) = 0$ for $a \in A \backslash (p_1 \cup p_2) \cup (p_1 \cap p_2)$ by definition. For $a \in p_2 \backslash p_1$ and $a \in p_1 \backslash p_2$ it holds that $f_a^k(d) = 0.5 d^k + \epsilon g(d)$ and $f_a^k(d) = 0.5 d^k - \epsilon g(d)$, respectively, where $g(d) = \overline{\lambda}^0 + \sum_{h \in K \backslash k} d^h \overline{\lambda}^h$. These flows are obviously positive if either $g(d) = 0$ or $\epsilon < d^k / |2g(d)|$ for all $d \in E$ with $g(d) \neq 0$. Such an ϵ exists since E is finite. □

Combining Proposition 9 with Theorem 10, we have complectly described polytopes for which cycle-free affine routings and static routings are equivalent, assuming that G is acyclic. However, Proposition 9 is wrong for general graphs because $f^k(d)$ for $d \in \mathcal{D}_0^k$ is not necessarily equal to 0, it can also be a positive circulation. Then, one can check that, when G has the required structure, a positive circulation can be decomposed into circulations that are not positive, thus yielding a cycle-free affine routing and a counter-example to Proposition 9.

3.4 Relation to dynamic routing

Proposition 8 identifies demand polytopes for which affine routing is no better than static routing. However, we saw in Example 1 that affine routing may also perform as well as dynamic routing does, yielding strictly cheaper capacity allocations. For general robust optimization problems, [13] and [15] show that affine recourse is equivalent to dynamic recourse when $\mathcal{D}$ is a simplex. Here we show that in the context of robust network design this condition is also necessary.

Proposition 11. *Given a demand polytope $\mathcal{D}$, all dynamic routings serving $\mathcal{D}$ are affine routings if and only if $\mathcal{D}$ is a simplex.*

Proof. Let $\{d_i, i = 1, \dots, n\}$, $1 \leq n \leq |K| + 1$, be the extreme points of $\mathcal{D}$.

Sufficiency (see also [13, 15]). Since $\mathcal{D}$ is a simplex, its vertices are affinely independent. It is enough to prove that for any routing f there exists $f^0 \in \mathbb{R}^{A \times K}$ and $y \in \mathbb{R}^{A \times K \times K}$ such that $f_a^k(d_i) = f_a^{0k} + \sum_{h \in K} d_i^h y_a^{kh}$, for each $k \in K, a \in A$ and $i = 1, \dots, n$. This is done by showing that the following system of linear equations has a solution for each $k = h_1 \in K$ and $a \in A$:

$$\begin{pmatrix} 1 & d_1^{h_1} & \cdots & d_1^{h_{|K|}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d_n^{h_1} & \cdots & d_n^{h_{|K|}} \end{pmatrix} \begin{pmatrix} f_a^{0h_1} \\ y_a^{h_1 h_1} \\ \vdots \\ y_a^{h_1 h_{|K|}} \end{pmatrix} = \begin{pmatrix} f_a^{h_1}(d_1) \\ \vdots \\ f_a^{h_1}(d_n) \end{pmatrix}. \tag{22}$$

The solution set of system (22) is non-empty for any right-hand side if and only if the rows of the matrix are linearly independent. Furthermore, these rows are linearly independent if and only if vectors in $\{d_i, i = 1, \dots, n\}$ are affinely independent.

Necessity. Consider $k \in K$. Assume first that $\mathcal{D}$ is full-dimensional. Let S be an arbitrary subset of $|K|+1$ affinely independent vectors in $\{d_i, i = 1, \dots, n\}$. Choose arbitrary routings $\tilde{f}^k(d_i)$ for $i \in S$. The system (22) for vectors in S and right-hand side given by $\tilde{f}^k$ has a unique solution, yielding an affine routing $(\overline{f^{0k}}, \overline{y^k})$. Notice that any affine routing for S can be constructed this way. Choose $1 \leq j \leq n$ such that $d_j \notin S$. Since there exists at least two paths between $s(k)$ and $t(k)$ we may construct a dynamic routing such that $\tilde{f}^k(d_j) \neq \overline{f^{0k}} + \sum_{h \in K} \overline{y^{kh}} d_j^h$.

If $dim(\mathcal{D}) = m < K$, any $d \in \mathcal{D}$ satisfies a set of $K - m$ independent linear equations. We can use these equations to substitute $K - m$ variables so that the affine recourse becomes an affine function of m variables. Therefore, let S be an arbitrary subset of $m + 1$ affinely independent vectors in $\{d_i, i = 1, \dots, n\}$ and the rest of the proof is similar. □

Example 2 shows that when $\mathcal{D}$ is not a simplex and does not contain the origin, capacity allocation costs required by static, affine, and dynamic routings can be strictly different.

Example 2. Consider the network design problem from Example 1 with the uncertainty set $\mathcal{D}$ defined by the extreme points $d_1 = (3, 0)$, $d_2 = (0, 3)$, $d_3 = (2, 2)$ and $d_4 = (0.5, 0.5)$. The optimal capacity allocation costs with static, affine, and dynamic routings are, respectively, $13 + \frac{1}{2}$, $13 + \frac{1}{3}$, and 13. Notice that moving d_4 along the segment $(0, 0) - (1, 1)$ leaves static and dynamic optimal capacity allocations unchanged while the affine solution cost moves between 13 and $13 + \frac{1}{2}$. In particular, if d_4 is set to $(0, 0)$, the affine and static costs are the same, which we knew already from Proposition 8. If d_4 is in $\text{conv}\{d_1, d_2, d_3, (1, 1)\}$, the affine and dynamic cots are the same.

Propositions 8 and 11 relate theoretically affine routing to the well-known static and dynamic routings. Combining these results we obtain conditions on the demand set $\mathcal{D}$ which yield that dynamic routings are static and which establish that $opt_{stat} = opt_{dyn}$.

Corollary 12. *Let the demand polytope $\mathcal{D}$ be the convex hull of $\{0\} \cup \{\epsilon_h e^h, h \in K\}$ with $\epsilon_h > 0$ for all $h \in K$. Then, all dynamic routings serving $\mathcal{D}$ are static routings.*

Corollary 13. *Let the demand polytope $\mathcal{D} \subset \mathbb{R}^K_+$ contain $\{\epsilon_h e^h, h \in K\}$ and be such that any $d \in \mathcal{D}$ is totally dominated by some vector in $\operatorname{conv}\{\epsilon_h e^h, h \in K\}$. The following are equivalent:*

- *x supports $\mathcal{D}$.*
- *there exists $y \in \mathbb{R}^{A \times K}_+$ such that (x, y) supports $\mathcal{D}$.*

Corollary 13 generalizes Theorem 3 from [26]. Note that there exists other situations in which dynamic routing is not better than static routing, see Example 3. These situations depend on the topology of G and the design cost function κ.

Example 3. Consider the network design problem from Example 1 with the uncertainty set $\mathcal{D}$ defined by the extreme points $d_1 = (2, 1), d_2 = (0, 2)$ and $d_3 = (1, 1)$. Although $\mathcal{D}$ does not satisfy the hypothesis from Corollary 13, we can see that the optimal solutions to this problem using static and dynamic routing have a cost of 9.

4 Computational experiments

In this section we investigate the objective gaps between optimal networks designs using static, affine, and dynamic routings, respectively, facing network topologies from the SNDlib ([39]). We start by defining the demand uncertainty polytope $\mathcal{D}$ our computations rely on.

4.1 Uncertainty sets and data

We use a variation of the well-known interval uncertainty set introduced by [16, 17], which we refer to as the Γ-model, also see [2, 3, 32, 40]. We assume that the traffic d^k for commodity k varies around a mean-demand $\bar{d}^k > 0$ with a maximal possible deviation of $0 \le \hat{d}^k \le \bar{d}^k$, that is,

$$d^k \in [\bar{d}^k - \hat{d}^k, \bar{d}^k + \hat{d}^k] \text{ for all } k \in K. \tag{23}$$

The Γ-model now limits the number of simultaneously deviations from the mean by the value $\Gamma \in \{0, \dots, |K|\}$, that is,

$$\sum_{k \in K} |d^k - \bar{d}^k| / \hat{d}^k \le \Gamma. \tag{24}$$

The corresponding uncertainty polytope $\mathcal{D}$ can be described in $\mathbb{R}^K$ directly using exponential many linear inequalities or alternatively using a compact extended formulation. For the latter, we rewrite $d^k = \bar{d}^k + (\sigma^k_+ - \sigma^k_-)\hat{d}^k$ for all $k \in K$ and let $(\sigma_+, \sigma_-) \in \mathcal{D}^\sigma$, where

$$\mathcal{D}^\sigma := \{(\sigma_+, \sigma_-) \in \mathbb{R}^{2|K|}_+ \ : \ \sum_{k \in K} (\sigma^k_+ + \sigma^k_-) \le \Gamma \text{ and } \sigma^k_+ + \sigma^k_- \le 1 \text{ for all } k \in K\}. \tag{25}$$

The set $\mathcal{D}^\sigma$ corresponds to all possible deviation scenarios from the nominal vector $\bar{d}$. The number of simultaneous deviations is restricted independent of being downward or upward deviations.

For the static and the dynamic routing principle, by exploiting domination, respectively total domination (see Section 2 and [38]), it is possible to withdraw all vertices from $\mathcal{D}^\sigma$ that correspond to downward deviations and to consider uncertainty in the space σ_+ only, see [2, 32]. In this case $d^k = \bar{d}^k + \sigma^k_+ \hat{d}^k$ for all $k \in K$ and we consider uncertain upward deviations $\sigma_+ \in \mathcal{D}^\sigma_+$, where

$$\mathcal{D}^\sigma_+ := \{\sigma_+ \in \mathbb{R}^K_+ \ : \ \sum_{k \in K} \sigma^k_+ \le \Gamma \text{ and } \sigma^k_+ \le 1 \text{ for all } k \in K\}.$$

Of course $\mathcal{D}^\sigma_+$ is a projection of $\mathcal{D}^\sigma$. It follows that for the static and dynamic routing principle it is equivalent to optimize against $\mathcal{D}^\sigma$ or $\mathcal{D}^\sigma_+$. Static and also dynamic solution supporting $\mathcal{D}^\sigma_+$ will automatically support $\mathcal{D}^\sigma$. Moreover, static and dynamic routings might even cover demand-scenarios outside the actual deviation polytope $\mathcal{D}^\sigma$ as long as these are (totally) dominated, e.g., demand vectors with $d^k = \epsilon$ for all $k \in K$ and $\epsilon > 0$ small enough. By Proposition 6, this is not true for the affine routing principle. It can not be said a priori whether or not affine solutions supporting $\mathcal{D}^\sigma_+$ will support deviations in $\mathcal{D}^\sigma$. All possible downward variations have to be included in the uncertainty definition since the feasibility of an affine routing can only be ensured if demand-deviations never leave the used uncertainty set. In the sequel we will use the set $\mathcal{D}^\sigma_+$ for static and dynamic scenarios. But in the case of affine recourse we will study optimal solutions for both the uncertainty set $\mathcal{D}^\sigma_+$ and $\mathcal{D}^\sigma$. Notice that for $\Gamma \geq 1$ both $\mathcal{D}^\sigma_+$ and $\mathcal{D}^\sigma$ are full-dimensional since they contain all unit-vectors and the origin.

Remark 2. A simple (also compact) alternative to the above Γ-uncertainty model is to use (23) plus a relaxation of (24):

$$\sum_{k \in K} \frac{d^k - \bar{d}^k}{\hat{d}^k} \leq \Gamma.$$

In this case there might be more than Γ many upward deviations if compensated by an appropriate number of downward deviations and vice versa. Obviously, this results in a relaxed version of the Γ-model potentially giving more conservative solutions.

We selected the three instances *janos-us*, *sun*, and *giul39* from SNDlib [39] which are feasible for a directed formulation as considered in this paper. These networks have 26/27/39 nodes and 84/102/172 arcs, respectively. More characteristics and statistics can be found on the SNDlib website [39]. All instances contain directed node-to-node demand values which we used as nominal demands $\bar{d}^k$. To reduce the size of the formulations and to be able to do a series of runs we considered the largest 10 to 50 commodities k with respect to the value $\bar{d}^k$, that is, $|K|$ takes values in $\{10, 20, 30, 50\}$. We fixed the maximum deviation $\hat{d}^k := 0.4\bar{d}^k$ and considered all values Γ in $\{1, \ldots, 7\}$.

4.2 Robust counterparts

The demand uncertainty polytope $\mathcal{D}^\sigma$ described by (25) has $2|K|$ variables and $|K|+1$ constraints (not counting nonnegativity constraints). For $\mathcal{D}^\sigma_+$ the number of variables reduces to $|K|$. Consequently, the corresponding static and affine robust counterparts are compact, as shown in [4] (static routing) and in Section 3.1 (affine routing).

For the static case a compact reformulation using the Γ-model can be found for instance in [2, 32]. It includes flow template conditions (12) and a dualization of the capacity constraints which results in $2|A||K| + 2|A|$ variables and $|K|(|V| - 1) + (|K| + 1)|A|$ constraints. Notice that always one flow conservation constraint per commodity can be omitted.

To set up the affine robust counterpart we dualized the capacity constraints and the flow nonnegativity constraints following model $(AARND)$ described in Section 3. Notice that $\mathcal{D}^\sigma$ is described in dimension $2|K|$ which increases the number of rows of the dualization and hence the number of rows of the model $(AARND)$. The resulting affinely adjustable robust network design formulation $(AARND)$ for the Γ-model with uncertainty set $\mathcal{D}^\sigma$ has $2|A||K|^2 + 3|A||K| + 2|A|$ variables and $(|K|^2 + |K|)(|V| - 1) + 2|A||K|^2 + 3|A||K| + |A|$ constraints. If the set $\mathcal{D}^\sigma_+$ is used instead, the number of constraints reduces to $(|K|^2+|K|)(|V|-1)+|A||K|^2+2|A||K|+|A|$. Again we can omit one flow conservation per commodity in the routing template (12). Also one flow conservation constraint per commodity of the flow cycle conditions (13) and (14) is redundant.

To calculate dynamic optimal solutions we used (RND) introduced in Section 2 restricted to those $d \in \mathcal{D}$ that correspond to the vertices of the demand uncertainty polytope. As explained above it suffices to consider non-dominated vertices corresponding to extreme upward deviations scenarios. These are the demand-vectors where Γ out of $|K|$ commodities are at their peak values $\bar{d}^k + \hat{d}^k$ and the remaining $|K| - \Gamma$ commodities are at their nominal values $\bar{d}^k$. More precisely,

model	rows	columns
static $\mathcal{D}^\sigma$	$\|K\|(\|V\|-1)+(\|K\|+1)\|A\|$	$2\|A\|\|K\|+2\|A\|$
affine $\mathcal{D}^\sigma$	$(\|K\|^2+\|K\|)(\|V\|-1)+2\|A\|\|K\|^2+3\|A\|\|K\|+\|A\|$	$2\|A\|\|K\|^2+3\|A\|\|K\|+2\|A\|$
affine $\mathcal{D}^\sigma_+$	$(\|K\|^2+\|K\|)(\|V\|-1)+\|A\|\|K\|^2+2\|A\|\|K\|+\|A\|$	$2\|A\|\|K\|^2+3\|A\|\|K\|+2\|A\|$
dynamic $\mathcal{D}^\sigma$	$\binom{\|K\|}{\Gamma}(\|K\|(\|V\|-1)+\|A\|)$	$\binom{\|K\|}{\Gamma}\|A\|\|K\|+\|A\|$

Table 1: Model sizes with respect to the number of nodes $|V|$, arcs $|A|$, and commodities $|K|$ and the value Γ.

in terms of deviation scenarios, for every subset $Q \subseteq K$ with $|Q| = \Gamma$ we have to consider the vertex $\sigma_+ \in \mathcal{D}^\sigma_+$ with $\sigma^k_+ = 1$ for $k \in Q$ and $\sigma^k_+ = 0$ for $k \in K \backslash Q$, which results in $\binom{|K|}{\Gamma}$ many vertices (totally dominating all other vertices of $\mathcal{D}^\sigma_+$). Consequently, the resulting exponential model to solve the dynamic robust network design problem has $\binom{|K|}{\Gamma}|A||K| + |A|$ variables and $\binom{|K|}{\Gamma}(|K|(|V|-1)+|A|)$ constraints. Notice that we can sensibly reduce the problem size for the dynamic case, by aggregating commodities with a common source (or a common destination) node [18, 33]. For our instances $|K|$ could be reduced from 50 to 18 (resp. 10 and 6) for *janos-us* (resp. *giul39* and *sun*). It is however easy to show that such aggregation is not compatible with static and affine routing principles.

For comparison purposes we summarize the size of the three different models in Table 1.

4.3 Results

Our numerical results are summarized in Table 2 and Table 3. For our computations we used the interior point (barrier) solver of Cplex 12.1 [29] on a 64bit 3.0GHz Quad-Core CPU with 8GB of memory allowing for 4 threads and 8 hours of CPU time for every individual run. Since we are only interested in the objective value of the optimal solution we switched off the crossover of Cplex. LP models have been set up using the modeling language Zimpl [31].

The first three columns of Table 2 state the instance name followed by the number of commodities $|K|$ and the size of Γ. The value Φ in column 4 indicating the largest number of commodities using the same arc in the optimal static solution is discussed below, see Lemma 14. Column 5 gives the static optimal objective value $opt_{stat}(K,\Gamma)$. The last three columns state the percentage affine gap $100(1 - opt_{aff}(K,\Gamma)/opt_{stat}(K,\Gamma))$, where $opt_{aff}(K,\Gamma)$ corresponds to the optimal solution using the uncertainty set $\mathcal{D}^\sigma$, the percentage upward affine gap $100(1-opt^+_{aff}(K,\Gamma)/opt_{stat}(K,\Gamma))$ using $\mathcal{D}^\sigma_+$, and the percentage dynamic gap $100(1-opt_{dyn}(K,\Gamma)/opt_{stat}(K,\Gamma))$, respectively. Time and memory hits are indicated by the letters T and M. Table 2 clearly shows the relation

$$opt_{dyn}(K,\Gamma) \leq opt^+_{aff}(K,\Gamma) \leq opt_{aff}(K,\Gamma) \leq opt_{stat}(K,\Gamma).$$

In general we observe that the dynamic gap is relatively small. It is below 11% for all scenarios. It seems that for practical networks with a modest number of demands the cost of static solutions is fairly close to the optimal (dynamic) design cost. In particular for small $|K|$ and larger Γ the dynamic gap is extremely small and is even 0% in many cases. Notice that it always holds that $opt_{stat}(K,|K|) = opt_{dyn}(K,|K|)$ since for $\Gamma = |K|$ there is only one non-dominated vertex of the uncertainty polytope $\mathcal{D}^\sigma$. In this case we solve the nominal problem for the single worst-case demand-matrix having all demands at their peak. However, it can also be clearly seen that the dynamic gap increases with the number of considered demands. Notice that Mattia [34], studying robust network design with dynamic routings, shows that also for larger commodity sets the dynamic gap is extremely small if discrete capacities are considered.

instance	$\lvert K\rvert$	Γ	Φ	static opt_{stat}	affine $\mathcal{D}^{\sigma}$ gap in %	affine $\mathcal{D}^{\sigma}_{+}$ gap in %	dynamic gap in %
janos-us	10	1	2	3.149202e+05	4.9	5.7	5.7
	10	2	1	3.323827e+05	0.0	0.0	0.0
	20	1	3	4.657367e+05	6.4	7.2	7.2
	20	2	2	5.125317e+05	5.1	6.2	6.5
	20	3	2	5.125317e+05	0.1	0.9	2.5
	20	4	2	5.125317e+05	0.0	0.0	0.0
	20	5	2	5.125317e+05	0.0	0.0	M
	30	1	4	6.127240e+05	6.8	7.5	7.5
	30	2	5	6.722822e+05	7.5	8.3	8.7
	30	3	5	6.988964e+05	5.7	6.6	7.0
	30	4	3	6.992080e+05	1.4	2.4	M
	30	5	3	6.992080e+05	0.0	0.0	M
	40	1	5	6.729093e+05	7.5	8.2	8.2
	40	2	5	7.324675e+05	7.5	8.4	8.8
	40	3	5	7.631223e+05	5.6	6.7	M
	40	4	4	7.659107e+05	1.9	2.9	M
	40	5	4	7.659107e+05	0.0	0.2	M
	40	6	4	7.659107e+05	0.0	0.0	M
	50	1	5	7.311094e+05	7.7	8.4	8.4
	50	2	5	7.925296e+05	7.8	8.7	9.1
	50	3	5	8.266402e+05	6.3	7.4	M
	50	4	5	8.369683e+05	3.7	4.8	M
	50	5	4	8.386076e+05	1.1	2.1	M
	50	6	4	8.386076e+05	0.0	0.0	M
sun	10	1	3	2.616416e+02	7.5	9.8	9.8
	10	2	3	2.740441e+02	0.6	2.7	3.0
	10	3	2	2.753181e+02	0.0	0.0	0.4
	10	4	2	2.753181e+02	0.0	0.0	0.0
	10	5	2	2.753181e+02	0.0	0.0	0.0
	10	6	2	2.753181e+02	0.0	0.0	0.0
	10	7	2	2.753181e+02	0.0	0.0	0.0
	20	1	5	4.314919e+02	8.1	9.9	9.9
	20	2	5	4.666696e+02	6.1	8.9	9.2
	20	3	5	4.821624e+02	3.2	5.5	6.4
	20	4	5	4.867587e+02	0.8	2.0	3.5
	20	5	4	4.878762e+02	0.3	0.7	M
	20	6	4	4.878762e+02	0.0	0.2	M
	20	7	4	4.878762e+02	0.0	0.0	M
	30	1	7	5.563141e+02	8.0	9.2	9.2
	30	2	7	6.029896e+02	8.1	10.1	10.5
	30	3	8	6.303494e+02	6.6	8.8	9.6
	30	4	7	6.400667e+02	3.6	5.6	M
	30	5	7	6.465764e+02	1.8	3.1	M
	30	6	7	6.491593e+02	0.9	1.8	M
	30	7	6	6.500533e+02	0.4	0.8	M
	40	1	9	6.688681e+02	7.4	8.5	8.5
	40	2	10	7.230139e+02	8.6	10.3	10.6
	40	3	11	7.578679e+02	8.1	10.1	10.8
	40	4	9	7.783114e+02	6.7	8.8	M
	40	5	9	7.906376e+02	5.1	7.1	M
	40	6	9	7.964860e+02	3.5	5.5	M
	40	7	9	7.989951e+02	2.0	4.1	M
	50	1	10	7.342829e+02	T	8.0	8.0
	50	2	12	7.916815e+02	T	10.1	10.4
	50	3	12	8.299185e+02	T	10.4	M

	50	4	12	8.533162e+02	T	9.6	M
	50	5	12	8.704549e+02	T	8.6	M
	50	6	13	8.809769e+02	T	7.3	M
	50	7	10	8.862691e+02	T	6.2	M
giul39	10	1	3	2.682375e+01	2.6	2.6	2.6
	10	2	3	3.046875e+01	0.9	2.3	2.3
	10	3	3	3.063375e+01	0.0	0.0	0.5
	10	4	3	3.063375e+01	0.0	0.0	0.0
	20	1	5	4.996563e+01	4.3	4.7	4.7
	20	2	5	5.529969e+01	5.3	6.1	6.1
	20	3	5	5.787094e+01	3.6	4.4	5.1
	20	4	5	5.970656e+01	1.5	2.2	3.1
	20	5	5	6.052813e+01	0.0	0.2	M
	20	6	5	6.052813e+01	0.0	0.0	M
	30	1	7	7.894656e+01	5.6	6.0	6.0
	30	2	7	8.494313e+01	6.6	7.3	7.3
	30	3	7	8.976938e+01	6.5	7.6	7.8
	30	4	7	9.350938e+01	5.5	7.1	M
	30	5	7	9.565781e+01	3.1	5.0	M
	30	6	7	9.633750e+01	0.3	2.3	M
	30	7	7	9.668750e+01	0.0	0.0	M
	40	1	9	1.059966e+02	T	6.8	6.8
	40	2	10	1.136150e+02	T	8.2	8.3
	40	3	9	1.195538e+02	T	8.5	M
	40	4	9	1.235375e+02	T	8.0	M
	40	5	9	1.267484e+02	T	7.1	M
	40	6	8	1.283469e+02	T	5.2	M
	40	7	8	1.291969e+02	T	3.4	M
	50	1	10	1.233091e+02	M	7.9	7.9
	50	2	10	1.325416e+02	M	9.8	10.2
	50	3	10	1.385434e+02	M	9.7	M
	50	4	10	1.432656e+02	M	9.7	M
	50	5	10	1.468328e+02	M	9.0	M
	50	6	9	1.493625e+02	M	7.8	M
	50	7	9	1.502844e+02	M	5.8	M

Table 2: Comparing static, affine, and dynamic routing in terms of optimality gap. Γ-model with $\Gamma \in \{1, \ldots, 7\}$ and $|K| \in \{10, 20, 30, 40, 50\}$. We removed rows whenever objective values and gaps did not differ from the previous row.

Even more interesting, Table 2 shows that all affine solutions are almost optimal, i.e., the corresponding cost is very close to the dynamic cost. In particular the upward affine gap (considering $\mathcal{D}^\sigma_+$) in many cases even coincides with the dynamic gap. For $\Gamma = 1$ the uncertainty set $\mathcal{D}^\sigma_+$ is a simplex with $|K| + 1$ vertices, hence $opt^+_{aff}(K, 1) = opt_{dyn}(K, 1)$ by Proposition 11. But also for $\Gamma > 1$ it holds that $opt^+_{aff}(K, \Gamma) \approx opt_{dyn}(K, \Gamma)$. As mentioned above the corresponding affine routing does not necessarily support demands in $\mathcal{D}^\sigma \setminus \mathcal{D}^\sigma_+$. But even considering downward deviations by using the uncertainty set $\mathcal{D}^\sigma$ for affine recourse does not remarkably decrease the corresponding gap. Affine solutions for $\mathcal{D}^\sigma$ are still very close to the dynamic solutions and clearly improve on the static solutions in terms of capacity cost.

This result shows that the affine routing principle allows enough flexibility to almost capture dynamic routings. It also suggest the following general approach. Given a general uncertainty polytope $\mathcal{D}$, in order to calculate a cheap network together with a feasible dynamic routing, one may compute the cost-minimal affine solution (capacity and routing) for $\mathcal{D}$ instead. For an even cheaper still (dynamically) feasible capacity allocation one might remove all vertices from $\mathcal{D}$ that are (totally) dominated and then compute the cost-minimal affine solution. In the latter case the resulting affine routing is not necessarily feasible for $\mathcal{D}$ but there exists a dynamic routing for the computed capacity allocation, and this capacity allocation can be considered being almost optimal. In fact, if only the capacity allocation and its cost are of interest, affine recourse with such a reduced uncertainty set can be used to approximate the cost for free recourse. Notice that in our case, using the Γ-model, the affine robust counterpart ($AARND$) for the reduced set $\mathcal{D}^\sigma_+$ was much easier to solve than ($AARND$) for the original set $\mathcal{D}^\sigma$, see Table 3.

In the following we will prove special properties of the Γ-model which can be observed in Table 2. For this we have to introduce some notation. We define the numbers $\Phi(K,\Gamma), \Gamma^\star(K) \in \mathbb{Z}_+$ which depend on the structure of the network and commodities. For a scenario pair (K,Γ) and an optimal static solution $S = (x,y)$ with objective value $opt_{stat}(K,\Gamma)$ let $\Phi(K,\Gamma,S)$ denote the largest number of commodities using the same arc in the solution S. We set $\Phi(K,\Gamma) := \min\{\Phi(K,\Gamma,S) : S \text{ static optimal}\}$, that is, $\Phi(K,\Gamma)$ gives the smallest value $\Phi(K,\Gamma,S)$ among all optimal static solutions S. With this definition the value Φ in column 5 of Table 2 is an upper bound on $\Phi(K,\Gamma)$. For commodity set K we define $\Gamma^\star(K) := \min\{\Gamma : \Gamma \geq \Phi(K,\Gamma)\}$. Obviously, $\Gamma^\star(K) \leq |K|$ and also $\Gamma^\star(K) \geq 1$ as long as K is not empty. The value $\Phi(K,\Gamma)$ is non-decreasing if we fix Γ and increase the size of the commodity set (keeping the old commodities) since every solution for the smaller set of commodities can be extended to a solution of the larger set by keeping the old routing and arbitrarily route the new commodities. As a consequence $\Gamma^\star(K)$ is non-decreasing with K.

Lemma 14. *For a given network G, a commodity set $K \neq \emptyset$ and $\Gamma \geq \Gamma^\star(K)$ it holds*

- $\Phi(K,\Gamma) = \Phi(K,\Gamma^\star(K)) \leq \Gamma$,
- $opt_{stat}(K,\Gamma) = opt_{stat}(K,\Gamma^\star(K))$, *and*
- *every optimal static solution to scenario (K,Γ) admits a shortest path template.*

Proof. Let S be an optimal static solution for scenario $(K,\Gamma^\star(K))$ such that $\Phi(K,\Gamma^\star(K),S) = \Phi(K,\Gamma^\star(K))$. It follows that solution S is also optimal for scenario $(K,\Gamma^\star(K)+1)$ with the same objective since at most $\Phi(K,\Gamma^\star(K))$ commodities can be at their peak simultaneously for every individual arc. Hence $opt_{stat}(K,\Gamma^\star(K)) = opt_{stat}(K,\Gamma^\star(K)+1)$ (otherwise S was not optimal for scenario $(K,\Gamma^\star(K))$) and also $\Phi(K,\Gamma^\star(K)+1) = \Phi(K,\Gamma^\star(K))$ (otherwise $\Phi(K,\Gamma^\star(K),S)$ was not minimal for scenario $(K,\Gamma^\star(K))$). By induction we get that $opt_{stat}(K,\Gamma) = opt_{stat}(K,\Gamma^\star(K))$ and $\Gamma \geq \Phi(K,\Gamma) = \Phi(K,\Gamma^\star(K))$ for all $\Gamma \geq \Gamma^\star(K)$. To prove the third claim observe that $opt_{stat}(K,|K|) = opt_{stat}(K,\Gamma^\star(K))$. But every optimal static solution $S = (x,y)$ to scenario $(K,|K|)$ admits a shortest path template y. Hence every optimal static solution for scenario (K,Γ) with $\Gamma \geq \Gamma^\star(K)$ is a shortest path solution. □

Comparing the value Γ and the value Φ in columns 3 and 4 of Table 2 we get that $\Gamma^\star(K) = 2,2,4,4,5$ for network *janos-us* and $|K| = 10,20,30,40,50$, respectively. For network *sun* and *giul39* it holds that $\Gamma^\star(K) = 3,5,7$ if $|K| = 10,20,30$, respectively. For all networks the values $\Gamma^\star(K)$ are very small but increasing with the number of commodities. Notice that since $opt_{stat}(K,\Gamma^\star(K)) = opt_{stat}(K,|K|)$ the worst-case objective for the Γ-model (together with a shortest path solution) is obtained already early (for small Γ) with static recourse. Affine and dynamic solutions tend to admit the worst case later (for larger Γ).

Table 3 reports on the computational complexity of the solved models. The first three columns of Table 2 again indicate the instance followed by the number of commodities $|K|$ and the size of Γ. Additionally, there are columns for every considered routing principle stating the number of nonzeroes (nonz) in the linear programming model and the time in seconds (time) to solve the problem. Since we were only interested in objective values we tried to solve all problems using the barrier method without crossover. It can be seen that all static models can be solved within 1 second of CPU time. The number of nonzeroes for the static and affine models is independent of the value Γ and increases polynomially with the number of considered commodities. The affine models however are very large already for small values of $|K|$ with a huge number of nonzeroes (in the order of 10^6 already for $|K| = 30, 40, 50$). We could still solve all affine counterparts corresponding to $\mathcal{D}_+^\sigma$ in less than one hour for *janos-us* and *sun* and in less than two hours for *giul39*. In contrast, we observed time and memory hits when solving $(AARND)$ for $\mathcal{D}^\sigma$ when $|K| \geq 40$ (*giul39*) and $|K| \geq 50$ (*sun*). As expected the affine robust counterpart for the reduced set $\mathcal{D}_+^\sigma$ is much easier to solve because of a smaller LP, also see Table 1.

As long as the number of nonzeroes is modest the dynamic models seem to be easier to solve than their affine counterparts. However, the number of nonzeroes is exponential both in the number of commodities and in the size of Γ. It exceeds to 10^9, 10^{10}, 10^{11} for $\Gamma = 7$ and $|K| = 30, 40, 50$, respectively such that we can provide dynamic solutions only for very small values of Γ or small values of $|K|$. In all other cases the memory limit was hit either already when setting up the LPs using Zimpl or later in the barrier algorithm.

instance	$\lvert K\rvert$	Γ	static nonz	static time	affine $\mathcal{D}^\sigma$ nonz	affine $\mathcal{D}^\sigma$ time	affine $\mathcal{D}_+^\sigma$ nonz	affine $\mathcal{D}_+^\sigma$ time	dynamic nonz	dynamic time
janos-us	10	1	5978	1	116998	27	81718	5	15540	1
	10	2	5978	1	116998	15	81718	5	69930	1
	10	3	5978	1	116998	16	81718	2	186480	1
	10	4	5978	1	116998	4	81718	4	326340	2
	10	5	5978	1	116998	3	81718	3	391608	2
	10	6	5978	1	116998	3	81718	5	326340	2
	10	7	5978	1	116998	2	81718	12	186480	1
	20	1	11786	1	450786	240	313026	32	45800	1
	20	2	11786	1	450786	336	313026	153	435100	4
	20	3	11786	1	450786	279	313026	79	2610600	38
	20	4	11786	1	450786	238	313026	22	11095050	111
	20	5	11786	1	450786	248	313026	163	35504160	M
	20	6	11786	1	450786	247	313026	180	88760400	M
	20	7	11786	1	450786	261	313026	192	177520800	M
	30	1	17594	1	1001534	1389	694094	88	76140	3
	30	2	17594	1	1001534	1847	694094	153	1104030	13
	30	3	17594	1	1001534	2161	694094	191	10304280	213
	30	4	17594	1	1001534	2490	694094	255	69553890	M
	30	5	17594	1	1001534	1802	694094	135	361680228	M
	30	6	17594	1	1001534	1693	694094	157	1507000950	M
	30	7	17594	1	1001534	1557	694094	141	5166860400	M
	40	1	23410	1	1769570	3021	1225250	265	150640	7
	40	2	23410	1	1769570	5362	1225250	407	2937480	49
	40	3	23410	1	1769570	7823	1225250	672	37208080	M
	40	4	23410	1	1769570	13192	1225250	802	344174740	M
	40	5	23410	1	1769570	6657	1225250	672	2478058128	M
	40	6	23410	1	1769570	6051	1225250	476	14455339080	M
	40	7	23410	1	1769570	5282	1225250	467	70211646960	M
	50	1	29220	1	2754420	6316	1906020	1031	225100	14
	50	2	29220	1	2754420	13331	1906020	1818	5514950	98
	50	3	29220	1	2754420	15007	1906020	2848	88239200	M
	50	4	29220	1	2754420	17181	1906020	3225	1036810600	M
	50	5	29220	1	2754420	19075	1906020	3413	9538657520	M
	50	6	29220	1	2754420	12814	1906020	2867	71539931400	M
	50	7	29220	1	2754420	11557	1906020	2333	449679568800	M
sun	10	1	7278	1	142278	5	99438	3	13020	1
	10	2	7278	1	142278	5	99438	3	58590	5
	10	3	7278	1	142278	4	99438	4	156240	1
	10	4	7278	1	142278	4	99438	2	273420	2
	10	5	7278	1	142278	3	99438	2	328104	2
	10	6	7278	1	142278	4	99438	3	273420	2
	10	7	7278	1	142278	3	99438	2	156240	1
	20	1	14338	1	547938	99	380658	33	32040	1

	20	2	14338	1	547938	100	380658	68	304380	5
	20	3	14338	1	547938	134	380658	61	1826280	47
	20	4	14338	1	547938	80	380658	59	7761690	292
	20	5	14338	1	547938	59	380658	67	24837408	M
	20	6	14338	1	547938	83	380658	38	62093520	M
	20	7	14338	1	547938	59	380658	50	124187040	M
	30	1	21404	1	1217384	627	844064	147	57000	4
	30	2	21404	1	1217384	677	844064	173	826500	14
	30	3	21404	1	1217384	722	844064	192	7714000	229
	30	4	21404	1	1217384	777	844064	175	52069500	M
	30	5	21404	1	1217384	741	844064	286	270761400	M
	30	6	21404	1	1217384	603	844064	205	1128172500	M
	30	7	21404	1	1217384	493	844064	153	3868020000	M
	40	1	28474	1	2150714	2480	1489754	470	76000	1
	40	2	28474	1	2150714	2796	1489754	512	1482000	31
	40	3	28474	1	2150714	2613	1489754	612	18772000	762
	40	4	28474	1	2150714	2866	1489754	591	173641000	M
	40	5	28474	1	2150714	3251	1489754	588	1250215200	M
	40	6	28474	1	2150714	3187	1489754	757	7292922000	M
	40	7	28474	1	2150714	2594	1489754	885	35422764000	M
	50	1	35550	1	3348150	*28000*	2317950	1212	95000	1
	50	2	35550	1	3348150	*28000*	2317950	1381	2327500	56
	50	3	35550	1	3348150	*28000*	2317950	1525	37240000	M
	50	4	35550	1	3348150	*28000*	2317950	1711	437570000	M
	50	5	35550	1	3348150	*28000*	2317950	1796	4025644000	M
	50	6	35550	1	3348150	*28000*	2317950	2133	30192330000	M
	50	7	35550	1	3348150	*28000*	2317950	2369	189780360000	M
	10	1	12640	1	240520	9	168280	7	30500	1
	10	2	12640	1	240520	9	168280	6	137250	12
	10	3	12640	1	240520	6	168280	6	366000	3
	10	4	12640	1	240520	6	168280	7	640500	6
	10	5	12640	1	240520	5	168280	6	768600	5
giul39	10	6	12640	1	240520	5	168280	6	640500	4
	10	7	12640	1	240520	4	168280	8	366000	2
	20	1	24594	1	925834	144	643754	91	71200	3
	20	2	24594	1	925834	185	643754	84	676400	6
	20	3	24594	1	925834	157	643754	111	4058400	98
	20	4	24594	1	925834	182	643754	147	17248200	717
	20	5	24594	1	925834	183	643754	95	55194240	M
	20	6	24594	1	925834	161	643754	118	137985600	M
	20	7	24594	1	925834	145	643754	87	275971200	M
	30	1	36564	1	2057124	1987	1427604	574	152520	15
	30	2	36564	1	2057124	1910	1427604	504	2211540	36
	30	3	36564	1	2057124	2160	1427604	539	20641040	942
	30	4	36564	1	2057124	1649	1427604	513	139327020	M
	30	5	36564	1	2057124	2619	1427604	463	724500504	M
	30	6	36564	1	2057124	1845	1427604	630	3018752100	M
	30	7	36564	1	2057124	1076	1427604	652	10350007200	M
	40	1	48524	1	3633804	*28000*	2519244	1791	203360	36
	40	2	48524	1	3633804	*28000*	2519244	1961	3965520	82
	40	3	48524	1	3633804	*28000*	2519244	2254	50229920	M
	40	4	48524	1	3633804	*28000*	2519244	2441	464626760	M
	40	5	48524	1	3633804	*28000*	2519244	2054	3345312672	M
	40	6	48524	1	3633804	*28000*	2519244	2657	19514323920	M
	40	7	48524	1	3633804	*28000*	2519244	2155	94783859040	M
	50	1	60478	1	5655778	M	3918578	4536	279500	61
	50	2	60478	1	5655778	M	3918578	5188	6847750	217
	50	3	60478	1	5655778	M	3918578	5360	109564000	M

50	4	60478	1	5655778	M	3918578	6858	1287377000	M
50	5	60478	1	5655778	M	3918578	4959	11843868400	M
50	6	60478	1	5655778	M	3918578	5044	88829013000	M
50	7	60478	1	5655778	M	3918578	4992	558353796000	M

Table 3: Comparing static, affine, and dynamic routing in terms of computational complexity

5 Concluding Remarks

In this paper we study two-stage robust network design problems with affine recourse for flow. We show that the resulting affine routing provides a reasonable alternative to the well-studied static and dynamic routing schemes. Similar to the static case, the corresponding robust counterpart is tractable but it turns out to provide solutions as cheap as the dynamic models.

All three routing principles are investigated with respect to their flexibility depending on the structure of the given demand uncertainty set $\mathcal{D}$. Fixing the demand polytope $\mathcal{D}$, the cost of optimal affine solutions is between the cost for optimal static and optimal dynamic solutions. In this work we develop necessary and sufficient conditions on $\mathcal{D}$ under which affine routings reduce to static routings and also develop properties of uncertainty sets leading to dynamic routings being affine.

We also consider the well-known concept of domination between demand-vectors and show that in contrast to the static and dynamic case there is no affine domination, that is, given two demand-vectors there is always an affine solution feasible for one but not for the other. In this respect affine routings suffer from the drawback that even totally dominated demand vectors are not necessarily supported by affine solutions. Uncertainty sets have to be designed accordingly in practice.

Finally, we compute the cost gap between static, affine, and dynamic solutions based on networks from SNDlib and the Bertsimas and Sim Γ-uncertainty-model. We conclude that for these instances the solutions based on affine routings tend to be as cheap as two-stage solutions with dynamic recourse. In this respect the affine routing principle allows enough flexibility to almost capture dynamic routings. Since it is in general $\mathcal{NP}$-hard to compute an optimal network design with dynamic routing, the affine principle can hence be used to approximate free recourse using tractable robust counterparts.

Affine models turn out to be attractive since polynomial reformulations are available. These formulations however tend to be very large such that they become hard to solve for large instances. For practical purposes one has to work on the formulations and methods to solve affine problems. In this context it might be wise to restrict the number of commodities in the affine recourse or apply decomposition methods (e.g. introducing cycle variables, see [6]).

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