An Algebraic Index Theorem for Non-smooth Economies

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Abstract:

In this paper, we prove an existence theorem for equilibria in production economies with increasing returns, which generalizes the classic results on this topic. In particular, we eliminate both the free-disposal assumptions and any smoothness requirements on the boundary of the production sets. For this purpose, we propose a new definition of the topological degree for non-convex-valued correspondences defined on non-smooth topological manifolds.

Résumé : On démontre l'existence d'un équilibre dans une économie de production avec rendements croissants, qui généralise la plupart des résultats classiques consacrés à cette question. En particulier, on élimine l'hypothèse de libre-disposition et de différentiabilité des bords des ensembles de production. La preuve est fondée sur une nouvelle construction du degré topologique adaptée aux correspondances à valeurs non-convexes, définies sur des variétés non-lisses.

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1 Introduction

1.1. Since the middle of the 1970's, many papers have addressed the question of existence of equilibria in economies with increasing returns in the production sector. Nonconvex producers are then assumed to follow various pricing set-valued maps which formalize standard behavioural rules like the marginal pricing rule, the average cost pricing, the voluntary trading pricing, etc. (see Cornet (1988) for a survey). In all these papers, the production sets Y_j are assumed to satisfy the free-disposal assumption

$$\forall y_j \in Y_j,\ y_j - \mathbb{R}^L_+ \subset Y_j,$$

i.e. to be such that, given a feasible production plan, all the production plans with larger or equal quantities of inputs and smaller or equal quantities of outputs are also possible. As emphasized by Jouini (1992b) and Hamano (1994) however, this assumption merely plays a technical role, and is not realistic. Already in the *Theory of Value*, when ordering by decreasing plausibility the classical assumptions made on the production sets, Debreu put the free-disposal hypothesis in the last position. Nuclear technologies of production of electricity are an example of increasing returns to scales technologies where, at a human scale, the destruction of undesirable by-products is very hard and costly, if not impossible, hence, which do not satisfy the free-disposal assumption. R & D and human capital are other examples of inputs for which the free-disposal assumption is very disputable.

On a more technical level, there is also a need for droping the free-disposal assumption due to purely mathematical reasons: *existence* problems are usually thought of as topological problems related to some fixed point argument. In this kind of argument (typically, Brouwer or Kakutani's theorem), no assumption of differentiability plays a role. Consequently, the tools of differential topology introduced by Debreu (1970) in order to provide a rigorous formalization of Walras' "counting equations and unknowns" should not be necessary as long as one is solely concerned with proving existence, though they definitely provide powerful instruments for proving local determinacy. In a similar vein, it is well-known that convexity is a geometric property which is not intrinsically related to the question of existence, and can usually be replaced by some (weaker) topological assumption, like acyclicity. But a careful reading of the existence proofs available reveals that this is not so. It has indeed been proven by Bonnisseau & Cornet (1988, Lemma 5.1) that the free-disposal assumption endows the boundary of each production set with the structure of a smooth manifold. It is, in fact, this implicit smoothness assumption on production sets which permits the usual existence proofs to go through. This is made explicit in Jouini's paper (1992b), where the free-disposal assumption is replaced by the weak free elimination hypothesis

$$\exists y \in Y_j\ :\ y - \mathbb{R}^L_+ \subset Y_j$$

and by the assumption that each boundary ∂Y_j be a differentiable manifold. On the other hand, when returns to scale are not increasing, the free-disposal assumption is not necessary in order to prove existence, but then production sets are of course convex. Clearly, none of these two requirements — convexity or smoothness of the boundary — should be necessary for just an existence proof, and both look rather like artifacts of the way we usually prove existence than a property intrinsically needed for the model to make sense.

This paper is devoted to obtaining a degree formula analogous to the one obtained by Jouini (1992b) for a broad class of economies with non-convex production sets, *without free-disposal* and *without any smoothness assumption.* As a by-product, we get an existence result which seems to be weaker than all the previous ones (the unique exception being the paper by Hamano (1994) which is discussed below). The advantage of the degree-like approach, as opposed to an existence proof in terms of fixed-point, is that, as already alluded to, degree theory is the key for the subsequent study of issues related to local uniqueness. The virtue of our topological viewpoint is that it enables to investigate these issues in categories for which the differentiable point of view is helpless, such as the class of piecewise-linear economies. The formula obtained here is indeed applied, in a companion paper, in order to prove that finitely-subanalytic economies (to which belong, e.g., the piecewise-linear ones) generically have a uniformly bounded, odd number of equilibria (Giraud (1999)).

1.2. Most of the recent work devoted to proving existence in non-convex production economies relies on an extension, due to Cellina & Lasota (1969), of the standard topological degree theory from smooth maps to correspondences. They prove, *lato sensu*, that, given an upper semi-continuous correspondence with convex and compact values F, for every $\varepsilon > 0$, there exists a continuous map f_ε whose graph belongs to an ε-ball around the graph of F. After having approximated f_ε itself by a smooth function, this property allows to define the degree of F as that of any smooth approximation for ε small enough. Although it constitutes a significant progress for set-valued analysis, this approach heavily rests on the smoothness of the domain of the correspondences at hand and the convex-valuedness of these set-valued maps, hence does not fit our purposes. We therefore have to propose a new construction of the topological degree of a correspondence. In comparison with the route taken by Cellina and Lasotta, the point of view adopted in this paper is purely topological and intrinsic, in the sense that it does not rely upon any approximation property by any smooth object.

The analogy with fixed-point theory may help the reader in understanding the differences between our standpoint and the previous one. Some versions of Kakutani's fixed point theorem can be deduced from Brouwer's theorem by using the kind of (continuous) approximation techniques mentioned above. Mas-Colell (1974) and McLennan (1991) extend these approximation tools to the case of upper semi-continuous correspondences with compact, contractible values, defined on compact sources, and deduce from it several generalizations of Kakutani's fixed point theorem. Of course, one could be tempted to try to generalize Cellina & Lasota's degree theory along the same line of proof as the one used by McLennan in order to generalize Kakutani's theorem. However, whether this can solve our problem is unclear to us. It ultimately requires that the domain of the correspondence under study (typically, the boundary of a production set) be a smooth manifold, which is precisely what we want to avoid in this paper. Therefore, not only is the correspondence to be approximated by a smooth function, but also the domain, in a sense to be made precise, by a smooth manifold. We know, however, since the late seventies, that there exist $\mathcal{C}^0$-manifolds (of dimension greater than 4) which cannot have any differentiable structure (see Kirby & Siebenmann (1977)). Moreover, even the fact that we actually focus on the *boundary* ∂Y_j of each production set Y_j does not facilitate the task of endowing it with a differentiable structure, since there exist subsets X of $\mathbb{R}^{L-1}$ (e.g. the *continuum* of Whitehaed) which are such that $X \times \mathbb{R}$ is a smooth submanifold of $\mathbb{R}^L$, without X being a manifold[1]. On the other hand, the theory of approximation of non-smooth and non-convex sets by smooth manifolds is still burgeoning (e.g., Czarnecki (1996)), and seems to hold water only for the class of epi-lipschitzian sets. Fortunately, an alternative approach to fixed point theory can be taken, using the (homological) Lefschetz fixed point theorem for continuous map. When extended to set-valued maps, this leads to the more powerful Eilenberg-Montgomery's fixed-point theorem. It is the analogue of this last viewpoint that we follow here, when extending the definition of the topological degree from maps to correspondences.

The strength of our angle of attack is illustrated by the extension of Jouini's (1992b) existence result to the case where production sets are topological manifolds with non-smooth boundary while pricing rules and demand correspondences may fail to take convex values. In addition, we replace the assumption of weak free elimination by the following (hopefully more natural) hypothesis: for each firm j, there exist a compact $K_j \subset \mathbb{R}^L$ and a convex, pointed cone $\Gamma_j \subset \mathbb{R}^L_+$ containing the unit vector in its interior, and such that:

(a) $\big(\partial Y_j \setminus K_j\big) \cap (-\mathbb{R}^L_{++}) = \big(\partial Y_j \setminus K_j\big) \cap (+\mathbb{R}^L_{++}) = \emptyset$,
(b) $\exists y_j \in Y_j \ : \ y_j - \Gamma_j \subset Y_j$.

The price to pay is that we now need to assume that the graphs of the set-valued maps we are dealing with be Euclidean Neighborhood Retracts. This is, however, a very weak condition, and it is *a priori* hardly conceivable that correspondences whose graph would not satisfy such a property could be of any economic relevance. It is in any case sufficient for the applications we have in mind (see Giraud (1999)). As far as we know, this is the first time the need for such an hypothesis is felt in general equilibrium theory.

The existence result given in this paper is not directly comparable with Hamano (1994), though an existence theorem for non-convex economies without free-disposal is also obtained there. Indeed, in Hamano's paper, traders' demand correspondences are not assumed to "tend to infinity" as the price vector tends to the boundary of the simplex. Consequently, Hamano must allow for negative equilibrium prices, which forces him to assume that production sets are "strictly star-shaped." Here, we follow a well-established tradition by imposing the classical boundary behavior on demands, so that, even if firms are able to quote negative prices, equilibrium prices must be positive. As a consequence, we can deal with non-convex production sets without free-disposal, which are not strictly star-shaped, so that, to our knowledge, the existence obtained in this paper is the strongest, available theorem for economies satisfying the boundary condition mentioned above.

1.3. As our (purely topological) approach of the degree of a correspondence may be useful in many other contexts (for instance, in incomplete markets, game-theoretical study of the refinements of Nash equilibrium, etc.), we isolate the background material needed for this approach in the second section. The third section is then devoted to proving our main existence result.

[1] I owe this remark to Stefano de Michelis.

2 The topological degree of a correspondence

In this section, we provide the mathematical background needed for proving the existence of equilibria in non-convex and non-smooth production economies.

2.1 From maps...

Before embarking in the construction of the topological degree of a correspondence, let us recall some facts borrowed from the homological degree theory of a continuous map. (For details, see Dold (1972, p. 267, ff.)).[2]

Consider M, N two topological n-dimensional, oriented submanifolds (without boundary) of some Euclidean space $\mathbb{R}^L$, and a continuous, surjective, proper map $f : M \to N$.

For any compact subset $C \subset M$, the relative (singular) homology group $H_n(M, M - C)$ is called the local homology group of M at C. (All the absolute homology groups we consider are reduced homology groups, taken with integral coefficients.) It is a "local" invariant in the sense that it depends only on the behavior of X in a neighborhood of C. Indeed, for any neighborhood V of C, we have the following: $H_n(V, V - C) \simeq H_n(M, M - C)$ (Dold (1972, Prop. 3.2, p. 59)). Moreover, when C is a point, say $P \in M$, the local homology groups $H_k(M, M - P)$ are easy to compute: they are trivial for $k \neq n$ and isomorphic to $\mathbb{Z}$ for $k = n$. A generator o_P of $H_n(M, M - P)$ is called an orientation of M at P. There are exactly two possible orientations at a given point P, say O_P and $-O_P$. We associate with M a new manifold, denoted by $\tilde{M} \otimes \mathbb{Z}$, and a map $\gamma : \tilde{M} \otimes \mathbb{Z} \to M$ s.t. $\gamma^{-1}(P) = H_n(M, M - P)$ for every $P \in M$. Hence, as a set, $\tilde{M} \otimes \mathbb{Z}$ is defined by: $\tilde{M} \otimes \mathbb{Z} := \cup_{P \in M} H_n(M, M - P, \mathbb{Z}) = \cup_{P \in M} H_n(M, M - P)$. We topologize $\tilde{M} \otimes \mathbb{Z}$ by defining a base as the set of all the V_z defined by:

$$V_z = \{[z]_P \in H_n(M, M - P),\ P \in V\},$$

where V is any open subset of M and $z \in Z_n(M, M - V)$ is any cycle modulo $M - V$. With respect to this topology, the map γ is locally homeomorphic and the next map:

$$\beta : \tilde{M} \otimes \mathbb{Z} \to \mathbb{Z} \quad \beta(u) = |u|, \forall u \in H_n(M, M - P) \simeq \mathbb{Z},$$

is continuous. (Here, $|u|$ is the absolute value of u, viewed as an element of $\mathbb{Z}$.) Let $A \subset M$ and $s : A \to \tilde{M} \otimes \mathbb{Z}$ a (continuous) map. If $\gamma \circ s(P) = P$ for all $P \in A$, s is called a section of γ over A. The sections form an abelian group, denoted $\Gamma(A)$.

A section $O : A \to \tilde{M} \otimes \mathbb{Z}$ is called an orientation of M along A if $\beta \circ O(P) = 1$ for all $P \in A$, i.e., if $O(P)$ is a generator of $H_n(M, M - P)$. M is orientable along A if

[2]**Notations**: Throughout this paper, coX [resp. $\overline{co}X$] denotes the convex hull [closed convex hull] of X, FrX its topological frontier, int$X$ its interior, cl$X$ its closure, and 2^X its power set. If M is a manifold with boundary (for short, a ∂-manifold), ∂M denotes its boundary. For any map g and any subset F of its domain, $g|_F$ is the restriction of g to F. We denote by $\text{proj}|_K$ the projection operator over K. e is the vector of $\mathbb{R}^N$ whose coordinates are all equal to 1 and, for any $i = 1, ..., N$, e_i is the i^{th} vector $e_i = (0, ..., 1, ..., 0)$ of the basis of $\mathbb{R}^L$. For any positive integer n, $\mathbb{N}_n$ designates the set $\{1, ..., n\}$. If S is a subset of $\mathbb{N}_L$, we write $\mathbb{R}^S = \{x \in \mathbb{R}^L \ :\ x_h = 0, \forall h \notin S\}$. If $y \in \mathbb{R}^L$, then $y_S = \text{proj}_{\mathbb{R}^S}(y) \in \mathbb{R}^S$. Finally, $\Delta^L = \{p \in \mathbb{R}^L \ :\ \sum |p_k| = 1\}$, $\Delta^L_+ = \Delta^L \cap \mathbb{R}^L_+$, $B_L = \{x \in \mathbb{R}^L : \|x\| \leq 1\}$, and $S_L = \{x \in \mathbb{R}^L : \|x\| = 1\}$, where $\|.\|$ is the Euclidean norm.

such an orientation exists. If $A = M$, M is orientable. If $O \in \Gamma(M)$ is an orientation of M and $K \subset M$ is a compact set, then the restriction $O|K$ provides an orientation of K. It turns out that $\Gamma(K)$ is then isomorphic to $H_n(M, M-K)$ (see Dold, *op. cit.*, p. 260, Prop. 3.3). Thus, for M oriented, O its orientation and $K \subset M$ compact, there exists a unique element $o_K \in H_n(M, M-K)$ corresponding to $O|K \in \Gamma(K)$. The element o_K is called the fundamental class around K which allows for generalizing the orientation of M around sets K which are not connected. Of course, if K is connected and non-empty, then $H_n(M, M-K) \simeq \mathbb{Z}$ and o_K is a generator.

Then, for each non-empty, connected, compact subset $K \subset N$, the homomorphism in homology $f_* : H_n(M, M - f^{-1}(K)) \to H_n(N, N-K)$ induced by f takes the fundamental class $o_{f^{-1}K}$ of $H_n(M, M - f^{-1}K)$ around K into an integral multiple of o_K. This multiple is called the degree of f over K and is denoted $\deg_K f$. By convention, if $K = \emptyset$, then $\deg_K f = \mathbb{Z}$.

For instance, if M is a compact, connected, oriented, n-dimensional, differentiable manifold, $Y = \mathbb{R}^n$, f is smooth, and $K = \{y\} \in \mathrm{Im}(f)$ a regular value of f, then $\deg_K$ reduces to the familiar definition of the degree in terms of Jacobians (Dold (*op. cit.*, Ex. 5.13, p.71)). If $f^{-1}(K) = \emptyset$, then $\deg_K f = 0$. If f is the inclusion map of an open subset $M \subset N$ into N, then $\deg_K f = 1$ for every $K \subset M$. The two next properties will also be helpful: if $K' \subset M$ is any compact set containing $f^{-1}(K)$, then $H(f)_{K'} : H_n(M, M-K') \to H_n(N, N-K)$ takes the fundamental class $o_{K'}$ into $(\deg f_K) o_K$ (Dold (*op. cit.*, p.267, Prop. 4.3)). In other words, we can safely replace $f^{-1}K$ by any larger compact set without affecting the degree of f over K. On the other hand, if N is connected, the equality $f_*(o_{f^{-1}K}) = (\deg_K(f)) o_K$ still holds for all non-empty compacts $K \subset N$, whether they are connected or not.

2.2 ...to correspondences

Before going any further, let us recall the following result, which is the crux of the matter when generalizing topological properties of maps to correspondences (see Vietoris (1950) for a treatment in the language of Čech-homology with compact supports or, for a Čech-cohomological treatment, Spanier (1966, p. 344)).[3]

Theorem 2.1 *(A variant of Vietoris-Begle mapping theorem)*

If X, Y are non-empty, compact, topological spaces, $f : X \to Y$ is surjective and such that $f^{-1}(y)$ is connected, and if there exists some integer n with $H_k(f^{-1}(y)) = 0$, for every $n \geq k \geq 0$ and every $y \in N$, then the homology homomorphism $f_k : H_k(M) \to H_k(N)$ is an isomorphism for every $n \geq k \geq 0$.

We will apply this result to the class of *Euclidean Neighborhood Retracts* (ENRs), i.e., to subsets $M \subset \mathbb{R}^L$ such that there exists a subset $N \subset \mathbb{R}^K$ which is homeomorphic with X and admits a neighborhood V and a continuous map (the retract) $r : V \to N$ whose restriction to M, $r_{|M}$, is the identity over M.

[3]The variant given here can be deduced from Vietoris (1950) by using the fact that any compact manifold is a Euclidean Neighborhood Retract, so that, in our context, Čech-homology with compact supports reduces to (reudced) singular homology (Dold (*op. cit.*, p. 340, Prop. 13.17)).

Any compact, topological ∂-manifold is an ENR. Any locally compact and locally contractible subset of $\mathbb{R}^L$ is an ENR (see Dold (1980), Prop. 8.12, p. 83). Thus, this class of sets potentially covers a great number of situations. Semi-algebraic and finitely-sub-analytic sets are also ENRs (even if they are not compact), a fact which will prove useful when studying the determinacy problem for non-smooth economies (see Giraud (1999)).

Consider, now, X, Y two non-empty, closed ENRs of $\mathbb{R}^L$, a continuous and proper map $f : X \to Y$ and a non-empty, compact $K \subset Y$. Since X and Y are ENRs, there exist two compact subsets W, V of X and Y respectively, s.t. $K \subset \operatorname{int}V \subset Y$, $f^{-1}(K) \subset \operatorname{int}W \subset X$, and $f^{-1}(V) = W$. Denote by $g = f_{|W} : W \to V$ the restriction of f on W. Suppose that $H_k(f^{-1}(y)) = 0$ for any $n \geq k \geq 0$ and any $y \in Y$. Then, $H_k(g^{-1}(y)) = 0$ for any $n \geq k \geq 0$ and any $y \in V$. Take any non-empty, compact connected subset $K \subset N$ and apply Vietoris-Begle mapping theorem to g and to $g_{|W-g^{-1}(K)} : W - g^{-1}(K) \to V - K$. The exact homology sequence of the pairs $\big(W, W - g^{-1}(K)\big)$ and $(V, V - K)$ provides the following commutative diagram:

$$\begin{array}{ccccccc}
H_k\big(W - g^{-1}(K)\big) & \longrightarrow & H_k(W) & \longrightarrow & H_k\big(W, W - g^{-1}(K)\big) & \longrightarrow & H_{k-1}\big(W - g^{-1}(K)\big) \\
\simeq\big\downarrow & & \simeq\big\downarrow & & \big\downarrow & & \big\downarrow\simeq \\
H_k\big(V - K\big) & \longrightarrow & H_k(V) & \longrightarrow & H_k\big(V, V - K\big) & \longrightarrow & H_{k-1}\big(V - K\big)
\end{array}$$

which, together with the five-lemma (e.g., Spanier (1966, p. 185)) yields that the relative homology homomorphism

$$g_* : H_n\big(W, W - g^{-1}(K)\big) \to H_n\big(V, V - K\big)$$

is an isomorphism. Since

$$\operatorname{cl}(X - W) \subset \operatorname{int}\big(X - f^{-1}(K)\big) \text{ and } \operatorname{cl}(Y - V) \subset \operatorname{int}(Y - K),$$

the excision property (Dold (*op. cit.*, p.44, Corollary 7.4)) leads now to the following commutative diagram:

$$\begin{array}{ccc}
H_n\big(X, X - f^{-1}(K)\big) & \xrightarrow{\simeq} & H_n\big(W, W - g^{-1}(K)\big) \\
f_*\big\downarrow & & \big\downarrow g_* \\
H_n\big(Y, Y - K\big) & \xleftarrow{\simeq} & H_n\big(V, V - K\big)
\end{array}$$

The vertical arrow on the left $f_* : H_n(X, X - f^{-1}(K)) \to H_n(Y, Y - K)$ therefore represents an isomorphism.

We are now ready to deal with set-valued mappings. Consider $F : X \to Y$ a correspondence with non-empty, compact values, X and Y two n-dimensional, topological manifolds. The set-valued map F is said upper hemi-continuous (u.h.c.) if, for every

$p \in \mathbb{R}^L$, the function $x \mapsto \sup_{y \in F(x)} y \cdot p$ is upper semi-continuous on X.[4] F is said *acyclic* if it takes acyclic values, i.e., if $F(x)$ is *non-empty* and verifies $H_k(F(x)) = 0$ for every $k \geq 0$ and every $x \in X$. (Non-empty convex or contractible sets are examples of acyclic sets.) It is said to be *proper* if, for every compact $K \subset Y$, the set $F^{-1}(K) = \{x \in X \ : \ F(x) \cap K \neq \emptyset\}$ is compact. Let $G =$Graph$(F) \subset X \times Y$ denote the graph of F. F is said G-closed if its graph is closed. A correspondence whose graph is a ENR shall be called a ENR-correspondence.

Suppose, therefore, that F is an u.h.c., G-closed, proper, acyclic- and compact-valued ENR-correspondence. Let $p : G \to X$, and $q : G \to Y$ be the canonical projections. Since p is a surjective, continuous and proper map between two ENRs, and has acyclic preimages, we deduce from the preceding remarks that, for every non-empty, compact $C \subset X$, p induces an isomorphism $p_* : H_n\big(G, G - p^{-1}(C)\big) \to H_n(X, X - C)$. Take, now, any non-empty, connected, compact $K \subset Y$ and denote $C = F^{-1}(K)$. Since F is proper, u.h.c. with bounded values, $F(C)$ is bounded (Aubin & Cellina (1984), Prop. 3, p. 64). Since the graph of F is closed, $F(C)$ is, in fact, compact. Hence, both sets $q^{-1}(K)$ and $p^{-1}(C)$ are compact. Moreover, $q^{-1}(K) \subset p^{-1}(C)$. The homomorphism $H(q) : H_n\big(G, G - p^{-1}(C)\big) \to H_n(Y, Y - K)$ has the same degree as $q_* : H_n(G, G - q^{-1}(K)) \to H_n(Y, Y - K)$.

The homomorphism $H_K(F) = H(q) \circ p_*^{-1} : H_n(X, X - C) \to H_n(Y, Y - K)$ will be considered as the homomorphism of homology induced by F relative to K. The degree of F relative to K, $\deg_K F$, is then simply defined as the degree of $H_K(F)$. Of course, our definition does reduce to the familiar one when F turns out to be a function, but we stress that, in general, it will not be possible to prove, in the contexts we have in mind, that F admits any continuous selector.

> **Remarks.** Notice that the restriction to acyclic correspondences (hence the use of Vietoris-Begle mapping theorem) is made only for the sake of emphasis, and in order, later on, to highlight the link with the usual convex-valuedness hypothesis made on demand (or tarification) correspondences in economic theory. Were we to strive for the utmost generality, we would have pictured the preceding construction for the more general class of set-valued maps for which only the *conclusion* of Vietoris-Begle mapping theorem applies on p_* as defined above. I also would like to stress that, even if the correspondence F admits a continuous selection f, it is not true, in general, that F and f have the same degree. Moreover, in the contexts we have in mind, most of the correspondences at hand do not admit any continuous selector at all.

2.3 Properties of the degree

Let denote by $\mathbf{C}$ the set of 4-tuples (φ, X, Y, K) where X and Y are n-dimensional topological manifolds, $K \subset Y$ is non-empty, connected, compact, and $\varphi : X \to Y$ is an

[4] For convex-valued correspondences the change between semi- and hemi-continuity is nit-picking, since every u.s.c correspondence is, in any case, u.h.c, while every convex- and compact-valued u.h.c. correspondence is u.s.c. But this is not so for maps which may take non-convex values.

u.h.c., G-closed, proper, ENR-correspondence with compact and acyclic values. When necessary, we write $\deg_K(F, X, Y)$ in order to emphasize the target and the source of F. One derives from Granas & Jaworowski (1958) and Jaworowski (1958) that the map deg: $\mathbf{C} \to \mathbb{Z}$ fulfills the next axioms:[5]

Normalization: If φ is a multivalued identity mapping (i.e. $x \in \varphi(x), \forall x$), then $\deg_K(\varphi) = 1$.

Localization: If U and V are open subsets of X and Y respectively, with $\varphi(U) \subset V$ and $\varphi^{-1}(K) \subset U$, then $\deg_K(F, X, Y) = \deg_K(F_{|U}, U, V)$.

The next property shows that the degree can be determined locally as "number of counter-images of a point," each counter-image counted with its multiplicity. (For a definition of "local degrees" for maps, see e.g., Dold (1972), p. 66).

Additivity: If X_1 and X_2 are two connected disjoint open sets such that $X = X_1 \cup X_2$ and $\left(\varphi_{|X_i}, X_i, Y, K\right) \in \mathbf{C}$ for each $i = 1, 2$, let denote by $N_i = \varphi^{-1}(K) \cap X_i$ for each i, [6] and $\varphi_i = \varphi_{|N_i}$. Then:

$$\deg_K(\varphi, X, Y) = deg_K(\varphi_1, X_1, Y) + \deg_K(\varphi_2, X_2, Y).$$

Homotopy invariance: $\deg_K(H(t, .), X, Y)$ is independent from $t \in [0, 1]$, where $H : [0, 1] \times X \to \mathbb{R}^n$ is a compact- and acyclic-valued u.h.c., G-closed correspondence s.t. $\cup_{t\in[0,1]} H^{-1}(t, K) \subset X$ is compact.

Put in other terms, two set-valued mappings $F, G : X \to Y$ are *acyclically homotopic* if there exists a correspondence H as above, such that, for each $x \in X$:

$$H(0, x) = F(x) \text{ and } H(1, x) = G(x).$$

Hence, two acyclically homotopic correspondences have the same degree.

Independence from K: If Y is connected, $\deg_K F$ is the same for all compact and non-empty parts $K \subset Y$, whether K is connected or not. It is written $\deg F$.

Chain rule: If (f, X, Y) and (F, Y, Z) are in $\mathbf{C}$, f is a function and Y is connected, then $\deg_K(F \circ f) = \deg_K F \cdot \deg f$.

The last property is the key which allows for deducing an existence proof from any Index theorem.

Non-triviality: If $\deg_K(\varphi) \neq 0$, then $\varphi^{-1}(y) \neq \emptyset$ for every $y \in K$.

3 Existence of equilibria

In this section, we apply the preceding material to the problem of existence of an equilibrium in production economies with increasing returns.

[5]The mentioned authors proved similar results for the Lefshetz index in the context of absolute simplicial homology on the sphere. The material gathered in the previous section makes the task of obtaining the next properties in the same vein rather easy.

[6]Note that each N_i is compact, since $f^{-1}(K)$ is the topological sum of the N_i's.

3.1 The economy

We consider economies with $L \in \mathbb{N}^*$ commodities, $m = \#I \geq 1$ consumers and $n = \#J \in \mathbb{N}$ producers. For all $j = 1, ..., n$, $Y_j \subset \mathbb{R}^L$ is the production set of the j^{th} producer. We formally define the possibly differing pricing rules of producers into non-empty-valued pricing rule correspondences:

$$\forall j,\ \varphi_j : \partial Y_j \to (\Delta_+^L),$$

which assigns to each production plan $y_j \in \partial Y_j$ the set of prices $q \in \varphi_j(y_j) \subset (\Delta_+^L)$ for outputs. (This is standard, for examples, see Cornet (1988) and Remark 3.1, *infra.*)

The price-taking behavior of the i^{th} consumer is described by her demand correspondence $D^i : \Delta_{++}^L \times \mathbb{R}_{++}$, which assigns to a price vector $p \in \Delta_{++}^L$ and a wealth $w_i \in \mathbb{R}_{++}$ the set $D_i(p, w_i) \subset \mathbb{R}_+^L$ of desirable consumption plans for agent i.

The revenue function $r_i : \Delta_+^L \times \prod_{j=1}^m Y_j \to \mathbb{R}$ associates with every price vector p and every n-tuple of production programs $(y_j) \in \prod_j Y_j$ an income $r_i\big(p, (y_j)\big) \in \mathbb{R}$ for trader i, whose wealth is then defined by: $w_i = p \cdot \omega_i + r_i(p, (y_j))$. An economy is summarized by $\mathcal{E} = \langle((D_i, r_i)_i, (Y_j, \varphi_j)_j\rangle)$.

Definition 3.1 *(i) A production equilibrium of* $\mathcal{E}_p = \langle(Y_j, \varphi_j)_j\rangle$ *is an element* $((y_j)_j, p) \in \prod_{j=1}^n \partial Y_j \times \Delta_+^L$ *such that* $p \neq 0$ *and* $\forall j,\ p \in \varphi_j(y_j)$ *(i.e., the pricing rule holds for each firm).*

(ii) A general equilibrium (GE) of $\mathcal{E}$ *is a collection* $((y_j)_j, (x_i)_i, p) \in \prod_{j=1}^n Y_j \times \mathbb{R}^{Lm} \times \Delta_+^L$ *such that* $((y_j)_j, p)$ *is a production equilibrium of* $\mathcal{E}_p$ *and*

(a) $\sum_{i=1}^n (x_i - \omega_i) = \sum_{j=1}^n y_j$ *(market clearing),*

(b) $\forall i, x_i \in D_i(p, p \cdot \omega_i + r_i(p, (y_j))$ *(individual optimization).*

$GE(\mathcal{E})$ (resp. $PE(\mathcal{E}_p)$) is the set of general equilibria (production equilibria) of the economy $\mathcal{E}$ (resp. $\mathcal{E}_p$). For any vector $z \in \mathbb{R}^L$, we adopt the now classical convention consisting in denoting by $\overline{z}$ the vector $(z_1 - z_L, ..., z_{L-1} - z_L) \in \mathbb{R}^{L-1}$.

Assumption 3.2 *(i) For each* i, D_i *is an u.h.c., G-closed, ENR-correspondence with compact values in* $\mathbb{R}_+^L$*. Moreover, for each* $(p, w) \in \Delta_{++}^L \times \mathbb{R}_{++}$, $\overline{D_i(p, w)}$ *is acyclic.*

(ii) For every i *and every* $(p, w) \in \Delta_{++}^L \times \mathbb{R}_{++}$*, we have:*

$$p \cdot D_i(p, w) = w. \ \textit{(Walras' law)}$$

(iii) If $(p_n, w_n)_n$ *is a sequence in* $\Delta_{++}^L \times \mathbb{R}_{++}$ *converging to* $(p, w) \in \partial\Delta_+^L \times \mathbb{R}_{++}$*, then* $d\big(0, D_i(p_n, w_n)\big) \to +\infty$.

Assumption 3.3 *(i) For each* j*, the set* Y_j *is an* L*-dimensional, topological* ∂*-manifold whose boundary* ∂Y_j *is connected and orientable, and such that there exist a compact* $K_j \subset \mathbb{R}^L$ *and a convex, pointed cone* $\Gamma_j \subset \mathbb{R}_+^L$ *containing* e *in its interior, satisfying:*

(a) $\big(\partial Y_j \setminus K_j\big) \cap (-\mathbb{R}_{++}^L) = \big(\partial Y_j \setminus K_j\big) \cap (+\mathbb{R}_{++}^L) = \emptyset$.

(b) $\exists y_j \in Y_j\ :\ y_j - \Gamma_j \subset Y_j$.

(ii) For all $z \in \mathbb{R}^L$, *the set* $\{(y_j) \in \prod_j Y_j : \sum_j y_j \geq z\}$ *is bounded.*

(iii) For all j, φ_j *is an u.h.c., G-closed, acyclic-, compact-valued ENR-correspondence. Moreover, there exists a real number* α_j *s.t. for all* $y_j \in \partial Y_j$ *and all* $p \in \varphi_j(y_j)$, *we have:* $p \cdot y_j \geq \alpha_j$.

(iv) $\forall((y_j), p) \in \prod_j \partial Y_j \times \Delta^L_{++}$, $\sum_i r_i((y_j), p) = p \cdot \sum_j y_j$.

Remark 3.1. The first departure with Jouini's framework is that the convex-valuedness of each individual demand correspondence is replaced by the (strictly) weaker assumption that $\overline{D_i(p, w)}$ be acyclic. Notice that this last hypothesis is not implied, in general, by the alternative assumption that $D_i(.,.)$ takes only acyclic values (think of an helicoidal curve in $\mathbb{R}^3$). Secondly, the (weak) free elimination hypothesis is replaced by parts (a) and (b) of (P)-(i). They are, of course, satisfied for pur-exchange economies where $Y_j = -\mathbb{R}^L_+$ for every j. The readers who feel uncomfortable with this assumption should keep in mind that the (weak) free elimination hypothesis and this one are interchangeable (though not comparable) for our purposes. Notice, incidentally, that the boundary ∂Y_j may be orientable without Y_j being orientable (Möbius strip). On the other hand, the bounded-losses assumption (iii) is standard in the literature. It is compatible with the profit maximization when Y_j is convex. Pure-exchange economies can be viewed as production economies whose tarification rules are all loss-free, hence, have bounded losses. However, when the marginal pricing rule is captured by Clarke's normal cone, it has bounded losses if, and only if, the set Y_j is star-shaped (Bonnisseau & Cornet (1988)). Hence, this paper does not incorporate the marginal pricing rule, except if each Y_j is assumed, in addition, to be star-shaped ($j = 1, ..., n$).

The next survival assumptions have been introduced by Bonnisseau & Cornet (1988). A counter-example, due to Kamiya (1988), shows that some version of them is indispensable if we are to prove existence. Roughly speaking, they say that, at certain production equilibria, the economy can "survive", in the sense that the total wealth distributed among households is positive.

Assumption 3.4 $\forall\big((y_j), p\big) \in PE(\mathcal{E}_p)$ *s.t.* $\sum_j y_j + \sum_i \omega_i \geq 0$, *we have:* $p \cdot \big(\sum_j y_j + \sum_i \omega_i\big) > 0$.

Assumption 3.5 *The assumption 3.4 holds for all* ω' *in a connected set* W *containing* $\omega + \mathbb{R}^{Lm}_+$ *— where* ω *designates the "true" vector of initial endowments of the economy at hand.*

3.2 Preliminary constructions

If we want to apply degree theory to our economic model, we need a set-valued map, whose zeroes, when translated in economic terms, can be interpreted as equilibria. This subsection contains the material required for this "translation" for non-smooth and non-convex production economies. For the details, we refer to Jouini (1992a,b).

Let us consider the (connected and non-empty) set:

$$U := \Big\{ \big((y_j)_j, p, (\omega)_i\big)_i \in \prod_{j=1}^{n} \partial Y_j \times \Delta_{++}^L \times \mathbb{R}^{Lm} \ : \ p \cdot \Big(\sum_j y_j - \sum_i \omega_i \Big) > 0 \Big\},$$

and the next assumption, saying that, at some production equilibria, not only is the aggegated wealth of the economy positive, but *every* individual stays above her subsistence level:

Assumption 3.6 *$\forall \big((u_j), p\big) \in PE(\mathcal{E}_p)$ s.t. $p >> 0$ and $p \cdot \big(\sum_j y_j + \sum_i \omega_i\big) > 0$, we have: $p \cdot \omega_i + r_i\big((y_j), p\big) > 0$ for all i.*

Assumption 3.6 should be viewed as a strengthening of the above stated "survival assumptions", obviously fulfilled in the private-ownership case with loss-free pricing rules, hence in the pure-exchange case.

Since, except on the nice points focused on in assumptions 3.5 and 3.6, the revenue of some consumer may be zero, so that her demand correspondence may fail to be defined, we need to consider $F : U \to \mathbb{R}^L$ the "modified excess demand", defined by:

$$\big((y_j)_j, p, (\omega_i)_i\big) \mapsto \sum_i D_i\Big(p, p \cdot \omega_i + \tilde{r}_i\big((y_j)_j, p, (\omega_i)\big)\Big) - \sum_j y_j - \sum_i \omega_i,$$

where each $\tilde{r}_i : U \to \mathbb{R}$ is a "modified" revenue function, given by:

$$\tilde{r}_i := (1 - \theta(\rho))\Big(\frac{1}{m}\big(\sum_i \rho_i\big)\Big) + \theta(\rho)\rho_i - p \cdot \omega_i, \quad i \in I,$$

with $\rho := (\rho_i)_i := (p \cdot \omega_i + r_i((y_j), p))$ and $\theta(\rho) := 1$ if $\rho_i > 0$ for all i, and

$$\theta(\rho) := \frac{\sum_i \rho_i}{\sum_i \rho_i - m\mathrm{inf}_k \rho_k} \quad \text{otherwise.}$$

The following correspondence $\Lambda_0 : U \to (e^{\perp})^n \times \mathbb{R}^{L-1} \times \mathbb{R}^{Lm}$ will fit the bill, in the sense that it will provide us the tool on which we shall apply our degree theory:

$$\Lambda_0\big((y_j), p, (\omega_i)\big) := \prod_j (\varphi_j(y_j) - \{p\}) \times \overline{F}((y_j), p, (\omega_i)) \times \{(\omega_i)\}.$$

For the proof of the next Lemma can be easily adapted from Jouini (1992a).

Lemma 1 *The correspondence $\Lambda_0 : U \to e^{\perp} \times \mathbb{R}^{L-1} \times \mathbb{R}^{Lm}$ is u.h.c., G-closed, acyclic-, and compact-valued, and verifies, for all $\omega \in \mathbb{R}^{Lm}$:*

$$\big(GE(\omega) \times \{\omega\}\big) \cap U \subset \Lambda_0^{-1}(0, 0, \omega),$$

and such that, if $RR(\omega)$ is in force, then the converse inclusion holds.

The small twist between Jouini's proof and this one is that, here, the correspondences under study (1) are u.h.c. and G-closed, instead of being u.s.c, (2) fail to take convex values. But the unique property of upper semi-continuity needed is that it is preserved by any finite sum or product — a feature which is also shared by upper hemi-continuity (Aubin & Cellina *op. cit*, Prop. 4, p. 64) and G-closedness. On the other hand, 3.2-(i) and 3.3-(iii) provide exactly what is needed in order to insure that Λ_0 takes acyclic values.

3.3 The existence result

Theorem 3.7 *Under 3.2, 3.3, and 3.5,*

$$deg_{(0,0,\omega)}\big(\Lambda_0\big) = (-1)^{L-1}.$$

If, moreover, the assumption 3.6 holds, $GE(\mathcal{E}) \neq \emptyset$.

Since our definition of the degree of a correspondence shares properties identical to that of the degree as defined by Cellina & Lasota (1969), the first steps of the proof look very much like a combination of the original proofs of Jouini (1992c,d).

Proof: Assumption 3.3-(ii) ensures that the set of admissible production plans $\Lambda_0^{-1}(0,0,\omega)$ is bounded. On the other hand, the localization property makes the definition of the degree local in the sense that $\deg_K(F)$ does not depend upon the behavior of F outside of a compact neighborhood of $F^{-1}(K)$. As a consequence, one can modify the pricing rules out of a compact neighborhood of $\Lambda_0^{-1}(0,0,\omega)$ without any modification of the set of equilibria of $\mathcal{E}$ and without any modification of $\deg_{(0,0,\omega)}\big(\Lambda_0\big)$.

If we denote by $s_j = \text{proj}_{e^\perp}(y_j)$ and $\lambda_j(y_j) = -\frac{y_j \cdot e}{L}$, then $y_j = s_j - \lambda_j(y_j)e$, and it follows from part (a) of 3.3-(i) that, for every $y_j \in \big(\partial Y_j \setminus K_j\big)$, one has:

$$\lambda_j(y_j) \leq \|s_j\|.$$

Therefore, there also exists a real number B_j such that, for every $y_j \in \partial Y_j$:

$$\lambda_j(y_j) \leq \max\big\{\|s_j\|, B_j\big\}.$$

On the other hand, the boundedness assumption (Pr)-(ii) insures that $\text{proj}_{e^\perp}$ is proper.

Consider, now, a continuous function $\beta : e^\perp \to \mathbb{R}$, which is equal to zero on $B(0,R)$ and to one out of $B(0,R+1)$ for some sufficiently large R. Following Jouini (1992b), we construct the following artificial pricing rule defined on $e^\perp$ and taking values in $e^\perp + \frac{1}{L}e$:

$$\psi_j(y_j) = (1-\beta(s_j))\varphi_j(y_j) + \beta(s_j)\big(s_j + \frac{1}{L}e\big).$$

Denote by $\tilde{\mathcal{E}}$ the economy defined in the same way as $\mathcal{E}$, but where the pricing rules φ_j are replaced by ψ_j $(j = 1, ..., n)$.

Mutatis mutandis, one can deduce from Jouini (1992a, proof of Thm. 5.1 together with Thm. 4.1) the following Index formula for the auxiliary economy $\tilde{\mathcal{E}}$:

$$\deg_{(0,0,\omega)}(\Lambda_0) = (-1)^{L-1} \prod_j \deg\psi_j.$$

Here, since each ψ_j takes values in a connected manifold, its degree does not depend upon the choice of any connected, compact subset $K \subset e^{\perp} + \frac{1}{L}e$. Furthermore, for $y_j \in \partial Y_j$ and $\|y_j\|$ large enough, we have $\psi_j(y_j) = \mathrm{proj}_{e^{\perp}}(y_j) + \frac{1}{L}e$. Thus, $\deg\psi_j = \deg \mathrm{proj}_{e^{\perp}|\partial Y_j}$.

Now, following 3.3-(iii), there exists $\mu_j \in \mathbb{R}_{++}$ s.t. $\left(\partial\mathbb{R}^L_+ + \mu_j e\right) \cap Y_j = \emptyset$. Let M_j be the connected, L-dimensional, topological ∂-submanifold of $\mathbb{R}^L$ given by:

$$M_j := \mathrm{cl}\{y \in \mathbb{R}^L \ : \ y \notin Y_j \text{ and } y \notin \mu_j e + \mathbb{R}^L_+\}.$$

The boundary $\partial M_j = \partial Y_j \cup \left(\partial\mathbb{R}^L_+ + \mu_j e\right)$ is a $(L-1)$-dimensional submanifold of $\mathbb{R}^{L-1}$. The continuous map $f_j := \mathrm{proj}_{e^{\perp}|\partial M_j}$ can be extended continuously to $g_j := \mathrm{proj}_{e^{\perp}|M_j}$. Moreover, following 3.3-(iii), it is proper. Therefore, the degrees of its induced homomorphisms in relative homology are well-defined. Now, observe that $H_{L-1}(\partial M_j, \partial M_j - a_k) \simeq \mathbb{Z}$, so that there are exactly two possible orientations at each point a_k of ∂M_j, which are o_{a_k} and $-o_{a_k}$, where o_{a_k} is the generator of $H_{L-1}(\partial M_j, \partial M_j - a_k)$ (cf. Dold *op. cit* Prop-def. 2.1 p. 252). Moreover, the boundary of the orientation class of M_j (cf. Dold *op. cit*, Prop. 2.19, p. 257) is the difference between the orientation class of ∂Y_j and that of $\partial\mathbb{R}^L_+ + \mu_j e$. These two classes must be mapped to cohomologous classes under f_j, so that the degree of the restrictions of the projection on these two components, $f^1_j := f_{j|\partial Y_j}$ and $f^2_j := f_{j|\partial\mathbb{R}^L_+ + \mu_j e}$, must be equal. But the restriction f^2_j is a homeomorphism. Thus, we can choose the orientation of this last boundary in such a way that

$$\deg\ f^2_j = 1.$$

Consequently, $\deg\Lambda_0 = (-1)^{L-1}$. Hence, by the non-triviality property of the degree, there exists $\left((y_j), p\right) \in \prod_j \partial Y_j \times S$ s.t. $\Lambda_0\left((y_j), p, \omega\right) = (0, 0, \omega)$. Together with assumption **(RR)**(ω) and Lemma 3.1, we conclude that $\mathrm{GE}(\tilde{\mathcal{E}}) \neq \emptyset$, hence that $\mathrm{GE}(\mathcal{E}) \neq \emptyset$. □

Remark 3.2. Apart from the use of a new topological degree, the move from Jouini's proof (1992b) to this one simply amounts to observing that the conclusion of Milnor's Lemma (see Milnor (1965, Lemma 1, p. 28)) does, in fact, not depend upon the differentiability of ∂M_j. Suppose that ∂Y_j were smooth. In homological terms, Milnor's argument could be stated as follows: For any generic point P in the connected manifold $e^{\perp}$, g_j's properness implies that the counter-image $g_j^{-1}(P)$ consists in finitely many segments s_k $(k = 1, ..., \nu_{j,P})$, such that $g_j^{-1}(P) \cap \partial M_j = f_j^{-1}(P) = \cup_{k=1}^{\nu_{j,P}} \partial s_k = \cup_{k=1}^{\nu_{j,P}} \{a_k, b_k\}$. It is not difficult to see that, once an orientation of ∂M_j has been chosen, the orientations at a_k and b_k have to take opposite signs (the segments are smooth submanifolds whose orientation can be checked by the unit (tangent) vector). Taking the sum over all the segments s_k yields, by the additivity property of the degree of f_j, that $\deg f_j = 0$.

Remark 3.3. The definition of an economic equilibrium adopted in this paper is standard, and follows, in particular, Jouini (1992a,b) and Hamano (1994). It is, however, disputable on the following ground. When a production set Y_j no more satisfies the free-disposal assumption, the set of *efficient* production plans

$$\left\{y_j \in Y_j \ : \ \left(y_j + \mathbb{R}^L_{++}\right) \cap Y_j = \{y_j\}\right\}$$

no more coincides with the boundary ∂Y_j. Therefore, the definition of a production equilibrium should require that y_j belongs to the subset of efficient production plans, instead of merely imposing that it be on the boundary of the production set. Such a definition, however, would create new difficulties — already in the smooth case —, as there is no obvious topological property shared by the set of efficient production plans (which may itself have a boundary, several connected components, etc.). By relaxing several assumptions on the production sets and their boundaries, this paper, at least, partially paves the road towards such an inquiry.

References

[1] Aubin, J.-P. & A. Cellina (1984). *Differential Inclusions.* Springer-Verlag, Berlin.

[2] Beato, M. & A. Mas-Colell (1985). On Marginal Cost Pricing with given Tax-subsidy Rules *Journ. of Econ. Theory* 37, 356-365.

[3] Begle (1950). The Vietoris Mapping Theorem for Bi-compact Spaces. *Annals of Math.* 51, 534-544.

[4] Bonnisseau, J-M.& Cornet, B. (1988). Existence of Equilibria when Firms follow Bounded Losses Pricing Rules *Journ. of Math. Economics.* 17, 119-147.

[5] Cellina, A. & A. Lasota (1969). A New Approach to the Definition of Topological Degree for Multivalued Mappings *Atti della Academia Nazionale dei Lincei, Rendiconti. Classe de Scienze Fisiche, Mathematiche a Naturali.* 47, 434-440.

[6] Cornet, B. (1988). General Equilibrium Theory and Increasing Returns: a Presentation *Journ. of Math. Economics* 17, 103-118.

[7] Czarnecki, M.-O. (1996). Méthodes d'approximation pour des problèmes non-différentiables. Applications à l'existence de points fixes. Thèse de doctorat, Université Paris-1 Panthéon-Sorbonne

[8] Debreu, G. (1970) Economies with a Finite set of Equilibria *Econometrica.* 38, 387-392.

[9] Dold, A. (1972) *Lectures on Algebraic Topology.* Springer-Verlag, Berlin.

[10] Geanakoplos,J. & W. Shafer (1990). Solving Systems of Simultaneous Equations in Economics *mJourn. of Math. Economics.* 19, 69-93.

[11] Giraud, G. (1999). Non-smooth Economies with a Uniformly Bounded Set of Equilibria. mimeo.

[12] Granas, A. & J. W. Jaworowski (1959) Some Theorems on Multivalued Mappings of Subsets of the Euclidean Space.*Bulletin de l'Académie Polonaise des Sciences.* VII(5), 277-283.

[13] Hamano, T. (1994). On the Existence of Equilibria without Convexity or Free Disposal on Production Technologies. *Journ. of Math. Economics.* 23, 565-583.

[14] Jaworowski, J.W. (1958). Some Consequences of the Vietoris Mapping Theorem *Fund. Math.* 45, 261-272.

[15] Jouini, E. (1992a). An Index Theorem for Nonconvex Production Economies. *Journ. of Econ. Theory.* 57, 176-196.

[16] ——— (1992b). Existence of Equilibria in Nonconvex Economies without Free-disposal. *Economic Letters.* 38, 37-42.

[17] Kamiya, F. (1988). Existence and Uniqueness of Equilibria with Increasing Returns *Journ. of Math. Economics.* 17, 149-178.

[18] Kirby, R. & L. Siebermann (1977). Foundational Essays on Topological Manifolds, Smoothings and Triangulations.*Annals of Math. Studies.* 88, Princeton Univ. Press.

[19] Mas-Colell, A. (1974). A Note on a Theorem of F. Browder *Mathematical Programming.* 6, 229-233

[20] McLennan, A. (1991) Approximation of Contractible Valued Correspondences by Functions. *Journal of Mathematical Economics.* 20(6), 591-598.

[21] Milnor, J. (1965). *Topology from the Differentiable Viewpoint.* The University Press of Virginia, Charlottesville, VA

[22] Spanier, E. H. (1966) *Algebraic Topology.* New-York, McGraw Hill.

[23] Vietoris, L. (1927). Über den höheren Zusammenhang kompakter Raüme und eine Klaße von zusammenhangstreuen Abbildungen. *Math. Ann.* 97, 454-472.