How to share when context matters: The Möbius Value As a Generalized Solution for Cooperative Games*

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Abstract

All quasivalues rest on a set of three basic axioms (efficiency, null player, and additivity), which are augmented with positivity for random order values, and with positivity and partnership for weighted values. We introduce the concept of Möbius value associated with a sharing system and show that this value is characterized by the above three axioms. We then establish that (i) a Möbius value is a random order value if and only if the sharing system is stochastically rationalizable and (ii) a Mö bius value is a weighted value if and only if the sharing system satisfies the Luce choice axiom.

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1 Introduction

The general question raised by any cooperative game can be described as follows: how should the utility sets available to all coalitions be used to determine an outcome from the set of feasible solutions? So far, no single solution-concept has emerged that satisfies everyone's sense of equity (Moulin, 1988). Yet, there seems to be a large agreement to consider the Shapley value as one of the most appealing solutions (Shapley, 1953). However, when players do not stand behind the veil of ignorance, this solution is no longer valid. Various concepts have then been proposed to deal with social and economic contexts in which players have idiosyncratic rights in sharing the final outcome (see Monderer and Samet, 2002, for a recent survey). All these solutions rest on a common set of three basic axioms (efficiency, null player, and additivity), which are augmented with positivity by Weber (1988) in the case of random order values, and with positivity and partnership by Kalai and Samet (1987) for weighted values. In this paper, we restrict ourselves to these three basic axioms only and characterize the set of corresponding values that we call Möbius values.

The extensions of the Shapley value allow for a redistribution of the total worth according to two dimensions: the marginal contribution of each player within all possible coalitions and a *sharing system* which is given a priori. The idea behind the sharing system is that the reward of a player may be related to her marginal contribution to each coalition in various ways. This aims at capturing the fact that a society may be governed according to a large variety of distributive rules, which are themselves based on various principles of justice (Bentham, Rawls, etc.). For example, in the theory of cooperative values, Kalai and Samet (1987) attribute a given weight to each player that expresses her power within each coalition whereas, in Weber (1988), the weight depends on the relative place of the player in society endowed with different orderings. As will be shown in this paper, the additional axioms that have been introduced in the literature (positivity and partnership) do actually restrict in a fairly strong manner the admissible sharing systems. More precisely, we will see that *the existing values*, called quasivalues, *are such that the sharing rule within a particular coalition is constrained by the way the sharing rule is defined within all broader coalitions.* Put differently, saying how to share within the grand coalition tells us how to share within all coalitions. In practice, the existence of such a master sharing rule may be problematic because it requires the implicit agreement of all players about it. By contrast, *our approach allows for a completely arbitrary sharing system and is, therefore, more general.* In other words, the way the worth of a coalition is shared does not constrain a priori the sharing of the worth of other coalitions. Our point is that sharing within a particular coalition need not be related to the way the worth any subcoalition or supercoalition is distributed among its members. Stated differently, by being the members of a coalition, the corresponding players find themselves in a particular sharing context that leads them to a distribution which depends only upon this coalition. It is in that sense that, for us, the M öbius value allows for sharing when context matters.

In our paper, the Möbius value of a player is given by a linear combination of the pure contribution of her cooperation within all coalitions including her; the coefficient associated with each coalition is the share that this player can claim in this coalition. By "pure contribution of cooperation" (**PCC**), we mean the net reward of cooperation within a coalition after having discounted for what cooperation brings about in all possible proper subcoalitions. Formally, the **PCC** of a coalition is the Möbius inverse of the characteristic function of the game. Focussing on the **PCC** of a coalition, instead of the marginal contribution of its members, concurs with our idea that a coalition defines a specific sharing context, which is a priori independent of all possible subcoalitions. Moreover, the coefficients of the linear combination define a probability over the corresponding coalition, but they need not be "consistent" across coalitions. By contrast, we identify two forms of consistency of the sharing system used in quasivalues. First, *a random order value is such that the sharing system is stochastically rationalizable*, that is, there exists a probability distribution defined over all orderings on the set of players which yields the sharing system (Block and Marschack, 1960). Second, *a weighted value*, as introduced by Kalai and Samet (1987), *is such that the sharing system satisfies the more demanding condition given by the Luce choice axiom used in discrete choice theory* (Luce, 1959). This axiom says that each sharing rule may be viewed as the Bayesian restriction of a master distribution defined on the set of players. Hence, our approach to cooperative values allows us to characterize each quasivalue by means of restrictions imposed on the corresponding sharing system.[1]

The remainder of this paper is organized as follows. Definitions and notation are given in Section 2. The concept of a Möbius value is defined and axiomatically characterized in Section 3 (Theorem 1). The relationships with quasivalues are explored in Section 4 where the following results are proven: (i) a Möbius value is a random order value if and only if the sharing system is stochastically rationalizable (Theorem 4) and (ii) a Möbius value is a weighted value if and only if the sharing system satisfies the Luce choice axiom (Theorem 5). In Section 4, we prove that a Möbius value is positive if and only if the game is monotone (Theorem 6) and that the set of Möbius values is the core if and only if the game is convex (Theorem 8). Section 6 concludes.

[1]These results also uncover some new connections between cooperative values and probabilistic discrete choices, a topic which has already been under investigation (Monderer, 1992; Gilboa and Monderer, 1992).

2 The Pure Contribution of Cooperation in a TU-Game

A *cooperative game with transferable utility* (TU-game) is a pair (Z, ν) where Z, the *grand coalition* with $\sharp Z = n$, is defined by a finite set of *players* and ν, the *characteristic function*, is defined by a mapping from 2^Z to $\mathbb{R}$ such that $\nu(\emptyset) = 0$. Any subset Y of Z is called a *coalition* and for any nonempty coalition Y, we denote $Z \backslash Y$ by $\overline{Y}$, $Y \backslash \{i\}$ by Y_{-i}, $Y \cup \{i\}$ by Y_{+i} and $2^Y \backslash \emptyset$ by $2^Y_{-\emptyset}$.

The set of TU-games whose set of players is Z is given by the vector space $\mathbb{R}^{2^Z_{-\emptyset}}$. A characteristic function ν is *monotone* if $\nu(X) \leq \nu(Y)$ for every $X \subset Y$ and *convex* if $\nu(X \cup Y) + \nu(X \cap Y) \geq \nu(X) + \nu(Y)$ for every pair $X, Y \in Z$. For convenience, all properties that are satisfied by ν on Z are said to be satisfied by the TU-game itself.

A *solution* of the game (Z, ν) is a mapping $\varphi : \mathbb{R}^{2^Z_{-\emptyset}} \rightarrow \mathbb{R}^n$. A solution $\varphi(\nu)$ is said to be *positive* when $\varphi_i(\nu) \geq 0$ for all $i \in Z$.

The above concepts are standard and we now introduce one of the new tools of this paper.

Consider any TU-game (Z, ν). Then, for any nonempty coalition Y, following Shapley (1953), there exists a unique set of coefficients $(\Gamma_\nu(X)$ $:X \in 2^Y_{-\emptyset})$ such that:

$$\nu(Y) = \sum_{X \in 2^Y_{-\emptyset}} \Gamma_\nu(X) \tag{1}$$

that are given by

$$\Gamma_\nu(Y) = \sum_{X \in 2^Y_{-\emptyset}} (-1)^{y-x} \nu(X) \tag{2}$$

where y and x stand for the cardinalities of Y and X, respectively. These coefficients may be interpreted as follows. Set $\nu(i) \equiv \nu(\{i\})$. If $Y = \{i, j\} \subset Z$, the worth $\nu(Y)$ may be different from $[\nu(i) + \nu(j)]$. In such a context, two cases may arise. In the first, the cooperation is "negative" because the two players are worse off when they cooperate. In the second, the cooperation is "positive" because the two players are better off when they cooperate. In both cases, it is natural to express the *pure contribution of cooperation* **(PCC)** *of* Y, denoted $\Gamma_{\nu}(Y)$, by the difference

$$\Gamma_{\nu}(Y) = \nu(Y) - [\nu(i) + \nu(j)]. \tag{3}$$

In other words, $\Gamma_{\nu}(Y)$ measures the exact contribution of the cooperation inside of Y because we have accounted for the individual worths. When $Y = \{i, j, k\}$, one might think that $\Gamma_{\nu}(Y)$ would be given by $\Gamma_{\nu}(Y) = \nu(Y) - [\nu(i) + \nu(j) + \nu(k)]$. However, this expression already includes the **PCC** of each pair $\{i, j\}$, $\{i, k\}$ and $\{j, k\}$ to the **PCC** of Y. Given (3), the **PCC** of $Y = \{i, j, k\}$ should instead be defined as follows:

$$\begin{aligned} \Gamma_{\nu}(Y) &= \{\nu(Y) - [\nu(i) + \nu(j) + \nu(k)]\} \\ &\quad - \{\Gamma_{\nu}(i,j) + \Gamma_{\nu}(i,k) + \Gamma_{\nu}(j,k)\}. \end{aligned} \tag{4}$$

More generally, in view of these expressions, we define the **PCC** of a TU-game as a mapping $\Gamma_{\nu} : 2^{Z}_{-\emptyset} \to \mathbb{R}$ such that, for each coalition $Y \subset Z$, such that (1) and (2) hold. In words, $\Gamma_{\nu}(Y)$ can be interpreted as *the contribution of cooperation within the coalition* Y *independently of what cooperation brings about in all possible subcoalitions that could have been formed before the coalition* Y *is determined*. Stated differently, $\Gamma_{\nu}(Y)$ measures the total benefit generated by the coalition Y once we have accounted for all the possible subcoalitions formed by any proper subset of players.[2]

The **PCC** Γ_{ν} is equivalent to the Möbius transform of the characteristic function ν (Rota, 1964; Chateauneuf and Jaffray, 1992). Note that, for any

[2] The **PCC** of a coalition is the game-theoretic counterpart of the "contextual utility" as defined by Billot and Thisse (1999) in discrete choice theory and of the "evidence of an event" in Dempster-Shafer's theory of belief functions.

$Y \in 2^{Z}_{-\emptyset}$, we have:

$$\nu(Y) = \sum_{i \in Y} \nu(i) + \sum_{\substack{X \subset Y \\ x \geq 2}} \Gamma_{\nu}(X) \tag{5}$$

which means that the worth of a coalition is equal to the sum of the individual worths plus the sum of the PCCs of all possible subcoalitions. In particular, for the grand coalition, we have:

$$\nu(Z) = \sum_{Y \subset Z} \Gamma_{\nu}(Y)$$

that is, the worth of the grand coalition is equal to the sum of the pure contributions of all possible coalitions.

In what follows, we show that the **PCC** of a coalition may be negative even when the TU-game is monotone. The same example is used throughout the paper.

Example 1: Consider the TU-game (Z, ν) such that $Z = \{1, 2, 3\}$ whereas its characteristic function v is defined by

$$\begin{cases} \nu(Z) = 8, \\ \nu(Z_{-i}) = 7 - i, \qquad \forall i \in Z \\ \nu(i) = i, \qquad \forall i \in Z. \end{cases}$$

This characteristic function is monotone and convex. The associated **PCCs** can be computed as follows:

$$\begin{cases} \Gamma_{\nu}(Z) = \Gamma_{\nu}(123) = 8 - (6 + 5 + 4) + (1 + 2 + 3) = -1, \\ \Gamma_{\nu}(Z_{-1}) = \Gamma_{\nu}(23) = 6 - (2 + 3) = 1, \\ \Gamma_{\nu}(Z_{-2}) = \Gamma_{\nu}(13) = 5 - (1 + 3) = 1, \\ \Gamma_{\nu}(Z_{-3}) = \Gamma_{\nu}(12) = 4 - (1 + 2) = 1, \\ \Gamma_{\nu}(1) = 1, \\ \Gamma_{\nu}(2) = 2, \\ \Gamma_{\nu}(3) = 3. \end{cases}$$

This implies that (i) the **PCC** of a pair Z_{-i} is greater than that of the grand coalition Z, (ii) the **PCC** of a pair is constant whoever is in the pair, and (iii) the **PCC** of the grand coalition is negative. Note also that

$$\begin{aligned}\nu(Z) &= \sum_{Y \subset Z} \Gamma_\nu(Y) \\ &= -1 + (1+1+1) + (1+2+3) \\ &= 8\end{aligned}$$

while

$$\begin{cases} \nu(Z_{-1}) = \nu(23) = 1+2+3 = 6, \\ \nu(Z_{-2}) = \nu(13) = 1+1+3 = 5, \\ \nu(Z_{-3}) = \nu(12) = 1+1+2 = 4. \end{cases}$$

3 Möbius Values

3.1 Definition

The *sharing rule* of a coalition $Y \in 2^Z_{-\emptyset}$ is a mapping $p_Y : 2^Y \to [0,1]$ where $p_Y(i)$ corresponds to the share player $i \in Y \in 2^Z_{-\emptyset}$ may claim in coalition Y. A sharing rule p_Y is supposed to satisfy the probability axioms. We also assume a *negligible player condition*: if $p_{\{i,j\}}(i) = 0$ for some $i, j \in Y$, then for all coalitions $X \subset Y \in 2^Z_{-\emptyset}$, $p_Y(X) = p_{Y_{-i}}(X_{-i})$. In words, when a player gets a zero share in a 2-person coalition, she gets the same share in any other coalition. This condition implies that $p_Y(i) = 0$ for each negligible player, whereas $p_{\{i\}}(i) = 1$ for each player $i \in Z$. A *sharing system*, denoted $(Z, \mathcal{P})$, is then defined by the set Z of players and by a mapping which associates each coalition $Y \in 2^Z_{-\emptyset}$ with a sharing rule p_Y satisfying the negligible player condition: $\mathcal{P} = \left(p_Y : Y \in 2^Z_{-\emptyset}\right)$.

Apart from the negligible player condition, the sharing rule p_Y depends only upon the particular redistribution context defined by the coalition. In other words, for any coalition X different from Y, p_X need not be related to p_Y in the system $(Z, \mathcal{P})$. As will be seen below, this makes our approach to cooperative values more general than standard quasivalues (Monderer and Samet, 2002).

Consider a TU-game (Z, ν). We define the *Möbius value* of the player $i \in Z$ associated with the sharing system $(Z, \mathcal{P})$, denoted $\varphi_i(\nu, \mathcal{P})$ by

$$\varphi_i(\nu, \mathcal{P}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \ni i}} p_Y(i) \Gamma_\nu(Y) \,. \tag{6}$$

In words, the Möbius value of player i is given by a linear combination of the **PCCs** of all nonempty coalitions Y including i , where the coefficient $p_Y(i)$ associated with the coalition Y is the share that player i can claim in this coalition. When the expression above holds for all sharing systems, we discard $\mathcal{P}$ in $\varphi_i(\nu, \mathcal{P})$; similarly, we denote p_Z by p.

Remark. For any negligible player i, we have $\varphi_i(\nu) = \nu(i)$.

Example 2: Consider a sharing rule $\mathcal{P}^*$ given by $p^*(1) = 0$ while $p^*(2) = p^*(3) = 1/2$, $p^*_{12}(1) = 0$ while $p^*_{12}(2) = 1$, $p^*_{13}(1) = 0$ while $p^*_{13}(3) = 1$ and $p^*_{23}(2) = 2/3$ while $p^*_{23}(3) = 1/3$. Then, the associated Möbius value $\varphi_i(\nu, \mathcal{P}^*)$ defined by (9) leads to

$$\begin{aligned}
\varphi_1(\nu, \mathcal{P}^*) &= \Gamma_\nu(1)\, p^*_1(1) + \Gamma_\nu(12)\, p^*_{12}(1) + \Gamma_\nu(13)\, p^*_{13}(1) \\
&\quad + \Gamma_\nu(Z)\, p^*(1) \\
&= \Gamma_\nu(1) = \nu(1) = 1,
\end{aligned}$$

$$\begin{aligned}
\varphi_2(\nu, \mathcal{P}^*) &= \Gamma_\nu(2)\, p^*_2(2) + \Gamma_\nu(12)\, p^*_{12}(2) + \Gamma_\nu(23)\, p^*_{23}(2) \\
&\quad + \Gamma_\nu(Z)\, p^*(2) \\
&= 2 + 1 + \frac{2}{3} - \frac{1}{2} \cong 3.15,
\end{aligned}$$

and

$$\begin{aligned}\varphi_3(\nu,\mathcal{P}^*) &= \Gamma_\nu(3)\,p^*_3(3)+\Gamma_\nu(13)\,p^*_{13}(3)+\Gamma_\nu(23)\,p^*_{23}(3)\\ &\quad +\Gamma_\nu(Z)\,p^*(3)\\ &= 3+1+\frac{1}{3}-\frac{1}{2}\cong 3.85.\end{aligned}$$

The Möbius solution of our example is therefore given by the triplet

$$(1, 3.15, 3.85)\,.$$

Note that, in this example, p^*_{23} is *not* defined as the Bayesian restriction of p^* onto the subset $\{2,3\}$. In other words, the sharing rule p^*_{23} is independent of p^*.

3.2 Axioms

We introduce the following three axioms to characterize the Möbius value as defined by (6).

Axiom 1 (φ-Efficiency) : *Let (Z,ν) be any TU-game. Then,*

$$\varphi_Z(\nu)=\sum_{i\in Z}\varphi_i(\nu)=\nu(Z).$$

This means that the solution of the grand coalition is equal to its worth.

Axiom 2 (φ-Null Player) : *For each player $i\in Z$, if for each coalition $Y\subset Z_{-i}$ we have $\Gamma_\nu(Y_{+i})=0$, then*

$$\varphi_i(\nu)=0.$$

This axiom says that the solution of an individual is zero when her **PCC** of any coalition she belongs to is always zero. Note that $\nu(i)=0$ and $\nu(Y_{+i})=\nu(Y)$ when i is a null player.[3]

[3] This axiom is weaker than the dummy axiom used by Shapley (1953) and Weber (1988). See Nowak and Radzik (1994) and Monderer and Samet (2002) for a discussion of the null player axiom *vs* the dummy axiom.

Axiom 3 (φ-Linearity) : *Let (Z,ν) and (Z,μ) any two TU-games and $\alpha \in \mathbb{R}$. Then,*

$$\varphi(\alpha\nu + \mu) = \alpha\varphi(\nu) + \varphi(\mu).$$

For any $X \subset Z$, consider a *X-unanimity TU-game* $\left(Z, \nu^X\right)$ for which the characteristic function ν^X is defined as follows:

$$\nu^X(Y) = \begin{cases} 1 & \text{if } X \subset Y, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Lemma 1 : *For any coalition $X \subset Z$, the **PCC** Γ_{ν^X} associated with the unanimity TU-game $\left(Z, \nu^X\right)$ is such that:*

$$\Gamma_{\nu^X}(Y) = \begin{cases} 1 & \textit{if } X = Y, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2 : *For any TU-game (Z,ν), we have:*

$$\nu = \sum_{X \in 2^Z_{-\emptyset}} \Gamma_\nu(X)\, \nu^X.$$

Proofs are straightforward and omitted. Note that Shapley (1953, p. 311) already proved that every characteristic function can be decomposed in a unique way as a linear combination of unanimity games (our Lemma 2).

Lemma 3 : *Let j be any player of Z. Under* A1-A3, *for any nonempty Y, we have:*

$$\varphi_i\left(\alpha\nu^Y\right) = \begin{cases} \alpha p_Y(i) & \textit{if } i \in Y, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: By A3, we may assume without loss of generality that $\alpha = 1$. Consider a player j and a coalition $Y \subset Z_{-j}$. If $i \in Z_{-j}$, then, for each coalition $X \subset Z_{-i}$, (7) implies that $\nu^{Y_{+j}}(X_{+i}) = \nu^{Y_{+j}}(X) = 0$. Thus, any player $i \in Z_{-j}$ is a null player, that is, $\Gamma_{\nu^{Y_{+j}}}(X_{+i}) = 0$. By A2, we then have $\varphi_i\left(\nu^{Y_{+j}}\right) = 0$ for all $i \notin Y_{+j}$.

By construction, it follows from A1 that

$$\varphi_Z\left(\nu^{Y_{+j}}\right) = \sum_{X \subset Z} \Gamma_{\nu^{Y_{+j}}}(X).$$

Because $p_X(Z) = 1$ for all $X \subset Z$, we have:

$$\varphi_Z\left(\nu^{Y_{+j}}\right) = \sum_{X \subset Z} \Gamma_{\nu^{Y_{+j}}}(X)\, p_X(Z).$$

Since $\varphi_i\left(\nu^{Y_{+j}}\right) = 0$ for all $i \neq j$, we also have $\varphi_j\left(\nu^{Y_{+j}}\right) = \varphi_Z\left(\nu^{Y_{+j}}\right)$ so that

$$\varphi_j\left(\nu^{Y_{+j}}\right) = \sum_{X \subset Z} \Gamma_{\nu^{Y_{+j}}}(X)\, p_X(j).$$

Lemma 1 implies that $\Gamma_{\nu^{Y_{+j}}}(X) = 0$ for all $X \neq Y_{+j}$ and $\Gamma_{\nu^{Y_{+j}}}(X) = 1$ for $X = Y_{+j}$. Consequently, we obtain:

$$\varphi_j\left(\nu^{Y_{+j}}\right) = \Gamma_{\nu^{Y_{+j}}}(Y_{+j}) p_{Y_{+j}}(j) = p_{Y_{+j}}(j).$$

$\square$

This result also shows the existence of a one-to-one correspondence between the Möbius values and the sharing systems.

We may now state one of our main results.

Theorem 1 : *Any solution $\varphi(\nu)$ of the TU-game (Z, ν) is a Möbius value if and only if $\varphi(\nu)$ satisfies the axioms* A1-A3.

Proof: (Sufficiency) Using Lemma 2, A1 and A2, we have:

$$\varphi_i(\nu) = \sum_{Y \in 2^Z_{-\emptyset}} \varphi_i\left[\Gamma_\nu(Y)\, \nu^Y\right].$$

Hence, from Lemma 3 and A3, it follows that

$$\begin{aligned}\varphi_i(\nu) &= \sum_{Y \subset Z_{-i}} \sum_{X \subset Y} \varphi_i\left[\Gamma_\nu(X_{+i})\,\nu^{X_{+i}}\right] \\ &= \sum_{X \in 2^Z_{-\emptyset}} \Gamma_\nu(X)\,p_X(i)\end{aligned}$$

which is identical to (6).

(Necessity) The proof is straightforward.

□

3.3 The Shapley Value as a Uniform Möbius Value

The Shapley value of a TU-game (Z, ν), denoted $S(\nu)$, allocates the worth $\nu(Z)$ among all players $i \in Z$ as follows:

$$S_i(\nu) = \frac{1}{n!} \sum_{\substack{X \subset Z \\ i \in X}} (x-1)!\,(n-x)![\nu(X) - \nu(X_{-i})]. \qquad (8)$$

The standard interpretation of the Shapley value is as follows. Assume that the players in Z are randomly ordered as $(i_1, i_2, \cdots, i_n)$ such that each ordering is equally probable. The Shapley value $S_i(\nu)$ is then the average of player i's marginal contributions $\nu(X) - \nu(X_{-i})$ taken over all coalitions $X \subset Z$. The probability of any coalition X is defined by the probability that the predecessors of i in the random ordering $(i_1, i_2, \cdots, i_n)$ are the elements of X.

Our next result suggests another interpretation: when player i cooperates within a coalition X whose **PCC** equals $\Gamma_\nu(X)$, player i gets the same "share" from this coalition than any other member of X. In other words, the sharing of $\Gamma_\nu(X)$ is *uniform* within X. Hence, the Shapley value of player i is the unweighted and normalized sum of all coalition worths. The associated sharing system is denoted $(Z, \mathcal{U})$ where $\mathcal{U} = \left(u_Y : Y \in 2^Z_{-\emptyset}\right)$ and u_Y the uniform probability distribution over Y. This result can be proven by using symmetry but, for future use, we propose a proof that does not rely on symmetry.

Theorem 2 : *Let $(Z,\mathcal{U})$ be the uniform sharing system. Then, the corresponding Möbius value of the TU-game (Z,ν) is the Shapley value:*

$$\varphi_i(\nu,\mathcal{U}) = \sum_{\substack{Y\in 2^Z_{-\emptyset}\\ Y\ni i}} \frac{\Gamma_\nu(Y)}{y} = S_i(\nu) \quad \textit{for all players } i \in Z.$$

Proof: The uniform Möbius value $\varphi(\nu,\mathcal{U})$ is defined for each nonempty coalition $X \subset Z$ by

$$\varphi_X(\nu,\mathcal{U}) = \sum_{\substack{Y\in 2^Z_{-\emptyset}\\ Y\supset X}} \Gamma_\nu(Y)\, u_Y(X) \tag{9}$$

where

$$u_Y(X) = \frac{x}{y},$$

x and y being the cardinalities of X and Y, respectively. Hence, by definition of the **PCC**, for each player $i \in Z$, (9) becomes

$$\begin{aligned} \varphi_i(\nu,\mathcal{U}) &= \sum_{Y\in 2^Z_{-\emptyset}} \Gamma_\nu(Y)\, u_Y(i) \qquad (10)\\ &= \sum_{Y\in 2^Z_{-\emptyset}} \Gamma_\nu(Y)\, \frac{1}{y} \\ &= \sum_{\substack{Y\subset Z\\ i\in Y}} \sum_{X\subset Y} \frac{(-1)^{y-x}\,\nu(X)}{y} \\ &= \sum_{X\subset Z} \sum_{\substack{Y\subset Z\\ X_{+i}\subset Y}} \frac{(-1)^{y-x}}{y}\nu(X). \end{aligned}$$

Set

$$\lambda(i,X) \equiv \sum_{\substack{Y\subset Z\\ X_{+i}\subset Y}} \frac{(-1)^{y-x}}{y}.$$

When the player $i \in X$, there are $\begin{pmatrix} n-x \\ y-x \end{pmatrix}$ coalitions Y such that $X \subset Y$. Consequently, we have:

$$\begin{aligned} \lambda(i,X) &= \sum_{\substack{Y \subset Z \\ X_{+i} \subset Y}} \frac{(-1)^{y-x}}{y} && (11) \\ &= \sum_{y=x}^{n} (-1)^{y-x} \begin{pmatrix} n-x \\ y-x \end{pmatrix} \frac{1}{y} \\ &= \sum_{y=x}^{n} (-1)^{y-x} \begin{pmatrix} n-x \\ y-x \end{pmatrix} \int_0^1 t^{y-1} dt \\ &= \int_0^1 t^{x-1} \sum_{y=x}^{n} (-1)^{y-x} \begin{pmatrix} n-x \\ y-x \end{pmatrix} t^{y-x} dt \\ &= \int_0^1 t^{x-1} (1-t)^{n-x} \, dt. \end{aligned}$$

It is well known that

$$\int_0^1 t^{x-1} (1-t)^{n-x} \, dt = \frac{(x-1)! \, (n-x)!}{n!} = \lambda(i,X). \tag{12}$$

Note that, in (10), if the player $i \in X$, then $\lambda(i, X_{-i}) = -\lambda(i,X)$. Hence, (10) may be rewritten as follows:

$$\varphi_i(\nu, \mathcal{U}) = \sum_{\substack{X \subset Z \\ i \in X}} \lambda(i,X) \left(\nu(X) - \nu(X_{-i})\right). \tag{13}$$

Using (12) and (13), we then get the desired expression, i.e.

$$\varphi_i(\nu, \mathcal{U}) = \frac{1}{n!} \sum_{\substack{X \subset Z \\ i \in X}} (x-1)! \, (n-x)! \left(\nu(X) - \nu(X_{-i})\right) = S_i(\nu).$$

□

In other words, the Shapley value corresponds to a sharing of the **PCC** s which is uniform across players. This interpretation is perfectly consistent with the axiom of anonymity (or symmetry) which defines the Shapley value

(Shapley, 1953): players are a priori given the same share in *all* possible coalitions. This should not come as a surprise since, on the one hand, we know from Kalai and Samet (1987) that the Shapley value is a weighted value with identical weights and, on the other hand, that the uniform distribution satisfies the Luce choice axiom that characterizes weighted values (see our Theorem 5 below).

Example 3: Consider the uniform sharing rule given by $u_X(i) = 1/x$ for all $i \in X$, all $X \subset Z$. Then, the Shapley value $S_i(\nu)$ defined by (9) leads to

$$\begin{aligned}
S_1(\nu) &= \Gamma_\nu(1)\,u_1(1) + \Gamma_\nu(12)\,u_{12}(1) + \Gamma_\nu(13)\,u_{13}(1) \\
&\quad + \Gamma_\nu(Z)\,u(1) \\
&= 1 \times 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{2} - 1 \times \frac{1}{3} \\
&= 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{5}{3} \simeq 1.66 > 1,
\end{aligned}$$

$$\begin{aligned}
S_2(\nu) &= \Gamma_\nu(2)\,u_2(2) + \Gamma_\nu(12)\,u_{12}(2) + \Gamma_\nu(23)\,u_{23}(2) \\
&\quad + \Gamma_\nu(Z)\,u(2) \\
&= 2 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{8}{3} \simeq 2.66 > 2
\end{aligned}$$

and

$$\begin{aligned}
S_3(\nu) &= \Gamma_\nu(3)\,u_3(3) + \Gamma_\nu(13)\,u_{13}(3) + \Gamma_\nu(23)\,u_{23}(3) \\
&\quad + \Gamma_\nu(Z)\,u(3) \\
&= 3 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{11}{3} \simeq 3.66 > 3.
\end{aligned}$$

The Shapley solution of our example is therefore given by the triplet

$$(1.66, 2.66, 3.66)\,.$$

4 Relationships between Möbius Values and Quasivalues

4.1 Random Order Values

Weber (1988) has introduced a generalization of the Shapley value, called *random order values*, by weighting the marginal contributions $\nu(Y_{+i}) - \nu(Y)$ of player i by the probability π_Y^i of joining any coalition Y in Z_{-i}:

$$\phi_i(\nu) = \sum_{Y \subset Z_{-i}} \pi_Y^i [\nu(Y_{+i}) - \nu(Y)]. \tag{14}$$

Then, Weber (1988) has proved that a solution is a quasivalue if and only if it is a random order value.

When comparing (6) and (14), we first note that the coefficients π_Y^i in (14) are interpreted by Weber as the probability for i to become a member of Y (or to join Y) while, in the present paper, $p_Y(i)$ in (6) is defined as the share attributable to player i when i is a member of the coalition Y. The two interpretations are therefore different. Second, the marginal contribution $\nu(Y_{+i}) - \nu(Y)$ differs from the pure contribution $\Gamma_\nu(Y_{+i})$ of coalition Y_{+i}. So, the connection between the two values is not clear (at least to us). Hence, our research strategy is naturally to uncover the relationships between (6) and (14) through their respective coefficients. More precisely, we are interested in determining *the connections between the share a player may obtain within a particular coalition and the probability she has to join this coalition.*

Definition (6) may be rewritten in terms of marginal contribution as follows (this is proven in the sufficiency part of Theorem 6):

$$\varphi_i(\nu) = \sum_{Y \subset Z_{-i}} \left\{ \sum_{X \supset Y_{+i}} (-1)^{x-(y+1)} p_X(i) \right\} [\nu(Y_{+i}) - \nu(Y)]. \tag{15}$$

This shows that the Möbius value involves coefficients γ_Y^i of the marginal contributions of i to Y of the type

$$\gamma_Y^i \equiv \sum_{X \supset Y_{+i}} (-1)^{x-y-1} p_X(i), \text{ for all } Y \subset Z_{-i} \tag{16}$$

which are not here primitives of the game, as they are in the various extensions of the Shapley value (Monderer and Samet, 2002). Furthermore, γ_Y^i need not be a probability and may even be negative. As will been seen, all quasivalues are special cases of the Möbius value in which the coefficients γ_Y^i take a particular form. Stated differently*, all quasivalues are special Möbius values associated with specific sharing systems.* In particular, the Möbius value is a random order value if and only if the coefficients γ_Y^i are probabilities. In this case, whenever the game is monotone, the positivity axiom for quasivalues - which one can find from Kalai and Samet (1987) to Monderer and Samet (2002) through Weber (1988) - always holds. Hence, it remains to identify the restrictions to be imposed on the sharing system for a Möbius value to have probabilistic coefficients.

To this end, we introduce a new concept related to the Möbius inverse (2). If $Y = Z_{-i}$, then the coefficient (we give below a necessary and sufficient condition for this coefficient to be a probability) π_Y^i for player i to join the coalition Y is identical to her share $p(i)$. Consider now $Y = Z_{-ij}$. Once i has joined Y, either i belongs to the coalition Z_{-j} or to the coalition Z because Y_{+i} is a subset of both. Since $Y = Z_{-ij}$, the weight for i to join Y is therefore given by:

$$\pi_Y^i = p_{Z_{-j}}(i) - p(i). \tag{17}$$

In other words, π_Y^i is the coefficient of joining the coalition Y without being in the coalition Z. If $Y = Z_{-ijk}$, one might think that π_Y^i is such that

$$\pi_Y^i = p_{Z_{-jk}}(i) - p_{Z_{-j}}(i) - p_{Z_{-k}}(i) - p(i).$$

However, this expression does not account for the fact that, when i belongs to Z_{-j} (resp. Z_{-k}), this may be because she has joined Z_{-ij} (resp. Z_{-ik}).

Deleting these occurrences, we obtain:

$$\pi_Y^i = p_{Z_{-jk}}(i) - \left[p_{Z_{-j}}(i) - \pi_{Z_{-ij}}^i\right] - \left[p_{Z_{-k}}(i) - \pi_{Z_{-ik}}^i\right] - p(i).$$

Given (17), this may be rewritten as follows:

$$\pi_Y^i = p_{Z_{-jk}}(i) - p_{Z_{-j}}(i) - p_{Z_{-k}}(i) + p(i).$$

More generally, for all $i \in Z$ and all $Y, X \subset Z_{-i}$, the coefficient for i to join Y is given by:

$$\pi_Y^i = \sum_{X \supset Y} (-1)^{x-y} p_{X_{+i}}(i)$$

where π_Y^i may be interpreted as a *dual Möbius transform* of the share $p_{Y_{+i}}(i)$. It is readily verified that this expression can be also written as follows:

$$\pi_Y^i = p_{Y_{+i}}(i) - \sum_{X \supsetneq Y} \left[p_{X_{+i}}(i) - \pi_X^i\right]$$

where $\pi_Z^i \equiv 0$. The difference $p_{X_{+i}}(i) - \pi_X^i$ may be viewed as the *net share* of player i for being in X_{+i}, once π_X^i is interpreted as the (normalized) "cost" she bears to join the coalition X. Then, the coefficient for i to join the coalition Y is equal to her share in the coalition Y_{+i} minus the sum of the net shares that i belongs to all the supercoalitions $X_{+i} \supset Y$. Put differently, π_Y^i *is the coefficient to join* Y *directly and not through any of its supercoalitions.*

Theorem 3 : *For all* $i \in Z$*, all* $Y, X \subset Z_{-i}$*, we have:*

$$\pi_Y^i = \sum_{X \supset Y} (-1)^{x-y} p_{X_{+i}}(i) \tag{18}$$

if and only if

$$p_{Y_{+i}}(i) = \sum_{X \supset Y} \pi_X^i. \tag{19}$$

The proof is given in appendix.

Equation (19) may be given the following interpretation: the share of player i in Y_{+i} is equal to the sum of the coefficients that this player has to join all supercoalitions of Y, that is, *her share must cover exactly the sum of the costs that she would incur by joining all the supercoalitions of* Y.

Remark. In the special case where there exist some players i such that $p(i) = 0$, then (18) implies $\pi_Y^i = 0$ for all coalitions $Y \neq \emptyset$. In other words, all such players always stay alone because $\pi_\emptyset^i = 1$.

Equations (18) and (16) imply

$$\gamma_Y^i = \pi_Y^i.$$

However, for π_Y^i to be a probability, the sharing system $(Z, \mathcal{P})$ must satisfy some additional conditions that we now investigate. Following Block and Marschak (1960) and Falmagne (1978), we say that the sharing system $(Z, \mathcal{P})$ is *stochastically rationalizable* if and only if the Block-Marschak polynomials of $(Z, \mathcal{P})$ are all nonnegative. Recall that the Block-Marschak polynomials of $(Z, \mathcal{P})$ are defined for all subsets $Y \subset Z_{-i}$ by the expression:

$$K(i, Y) = \sum_{k=0}^{y} (-1)^k \sum_{X \in \mathcal{F}(Y, y-k)} p_{\overline{X}}(i)$$

where $\mathcal{F}(Y, y-k)$ is the family of subsets of Y whose cardinal is equal to $y - k$ and $\overline{X}$ the complement of X in Z. We thus have:

Theorem 4 : *For any TU-game* (Z, ν)*, the Möbius value is a random order value, i.e.*

$$\varphi_i(\nu) = \phi_i(\nu),$$

if and only if the sharing system $(Z, \mathcal{P})$ *is stochastically rationalizable.*

Proof: By Theorem 3, (18) and (19) define a one-to-one correspondence between the two sets of coefficients γ^i_Y and π^i_Y. To prove that the coefficients π^i_Y correspond to Weber's probabilities, it remains to show, on one hand, that they are all nonnegative and, on the other hand, that $\sum_{Y \subset Z_{-i}} \pi^i_Y = 1$.

Let X and Y be any two subsets of Z such that $i \notin X$ and $i \in Y$. We have

$$\begin{aligned}
K(i,Y) &= \sum_{k=0}^{y} (-1)^k \sum_{X \in \mathcal{F}(Y, y-k)} p_{\overline{X}}(i) \\
&= \sum_{X \subset Y} (-1)^{y-x} p_{\overline{X}}(i) \\
&= \sum_{\overline{X} \supset \overline{Y}} (-1)^{\overline{x}-\overline{y}} p_{\overline{X}}(i) \\
&= \pi^i_{\overline{Y}}. \qquad \text{by (18)}
\end{aligned}$$

Since Y is arbitrary, π^i_Y is nonnegative if and only if the sharing system (Z,p) is stochastically rationalizable. Moreover, it is readily verified that $\sum_{Y \subset Z_{-i}} \pi^i_Y = p_i(i) = 1$, which ends the proof.

□

Corollary 1 : *Any Block-Marschak polynomial $K(i,Y)$ of a choice probability system $(Z,\mathcal{P})$ corresponds to the coefficient $\pi^i_{\overline{Y}}$ as defined by (18).*

Theorem 4 is consistent with the following result derived by Monderer (1992): for any random order value, there exists a rationalizable system of choice probabilities defined on Z consistent with the probabilities π^i_Y in (14). Note also that the stochastic rationality of the sharing system $(Z,\mathcal{P})$ is equivalent to the positivity axiom. Then, *a solution satisfying A1-A3 whose sharing system is stochastically rationalizable is a quasivalue.*

Observe that (18) allows for the computation of the coefficients used by Weber from the individual shares. This, in turn, permits the study of the likelihood of various coalitions and, therefore, to analyze the occurrence of coalition formation and to perform some "comparative statics" on the sharing rule. Everything else equal, *the smaller* (resp. the larger) *a player's share, the higher* (resp. the lower) *her probability to stand alone*, a situation which involves no coalitional cost. Likewise, the smaller (resp. the larger) a player's share, the higher (resp. the lower) her probability to be joined by players with larger shares. Unfortunately, it seems hard to say something about players with intermediate shares without specifying the connections between the sharing system $\mathcal{P}$ and the characteristic function ν.

4.2 Weighted Values

Kalai and Samet (1987) have considered a subset of quasivalues defined as follows. Set a *weight system* $w = (w_X)_{X \in 2^Z_{-\emptyset}}$ such as

$$w_X(i) = \frac{w_Y(i)}{w_Y(X)}$$

for all $i \in X \subset Y \subset Z$ and $w_Y(X) > 0$. It is worth noting that a weight system w is *strictly* positive. The associated *weighted value* ϕ^w is then defined for any unanimity game ν^X by

$$\phi_i^w(\nu^X) = \begin{cases} w_X(i) & \text{if } i \in X, \\ 0 & \text{otherwise.} \end{cases}$$

In words, a player belonging to coalition X receives her weight within this coalition. Moreover, a coalition Y is said to be a coalition of partners or a *p-type coalition* in (Z, ν) if, for every subcoalition $X \subset Y$ and each $W \subset \overline{Y}$, $\nu(W \cup X) = \nu(W)$. In other words, players are called partners when they refuse to cooperate outside the coalition of partners. A value ϕ satisfies the *partnership axiom* if, whenever Y is a p-type coalition:

$$\phi_i(\nu) = \phi_i(\phi_Y(\nu)\, \nu^Y) \qquad \text{for all } i \in Y. \tag{20}$$

This axiom, introduced by Kalai and Samet (1987), requires that if sub-coalitions of Y are irrelevant, then it makes no difference either players of Y receive their individual shares in ν, or they altogether receive their group share in ν and determine their individual shares later. Kalai and Samet (1987) then proves that a weighted value is a quasivalue that satisfies the partnership axiom. Hence, we need to identify the properties of the sharing rules which characterize a Möbius value as a weighted value, i.e. to interpret the partnership axiom in terms of shares.

Lemma 4 *: For any TU-game (Z,ν), a Möbius value satisfies the partnership axiom if and only if the sharing system $(Z,\mathcal{P})$ satisfies the Luce choice axiom: for all $Y \in 2^Z_{-\emptyset}$*

$$p(i) = p(Y) \times p_Y(i) \qquad \textit{for all } i \in Z \textit{ such that } 0 < p(i) < 1.$$

Proof: As noticed by Chun (1991, p.186), it is always possible to define the weight system w by $w_i = \varphi_i\left(\nu^Z\right)$ where ν^Z is the characteristic function of the unanimity game $\left(Z,\nu^Z\right)$. Accordingly, since $w_Y(X) > 0$ for all nonempty coalitions $X \subset Y \subset Z$, we have $\varphi_i\left(\nu^Z\right) > 0$ for all $i \in Z$. Furthermore, A1 implies that $\sum_{i\in Z}\varphi_i\left(\nu^Z\right) = 1$. As a result, we can identify the weight system w with a strictly positive sharing rule p such that $\varphi_i\left(\nu^Z\right) = p(i) > 0$ for all players $i \in Z$. Let $\left(Z,\nu^Y\right)$ be a unanimity game such that $Y \subset Z$ and $Y \neq Z$. The coalition Y being a p-type coalition for ν^Z, we have for any player $i \in Y$: $\varphi_i\left(\nu^Z\right) = \varphi_i(\varphi_Y\left(\nu^Z\right)\nu^Y)$. Using A3, this expression becomes $\varphi_i\left(\nu^Z\right) = \varphi_Y\left(\nu^Z\right) \times \varphi_i(\nu^Y)$, i.e.

$$\varphi_i(\nu^Y) = \frac{\varphi_i\left(\nu^Z\right)}{\varphi_Y\left(\nu^Z\right)} = \frac{p(i)}{p(Y)}.$$

Now, by Lemma 3, we have $\varphi_i(\nu^Y) = p_Y(i)$ and, then, the Luce choice axiom holds.

□

We are now able to establish the following result:

Theorem 5 : *For any TU-game (Z,ν), the Möbius value is a weighted value, i.e.*

$$\varphi_i(\nu) = \phi_i(\nu)$$

if and only if the sharing system $(Z, \mathcal{P})$ *satisfies the Luce choice axiom.*

Monderer and Samet (2002, Th. 5) have proved that a weighted value is a random order value that satisfies the partnership axiom, a result consistent with our Theorem 5. Hence, since a random order value is a Möbius value with a stochastically rationalizable sharing system (our Theorem 4), we know, using Luce and Suppes (1965), that the necessary and sufficient condition for the sharing system $(Z, \mathcal{P})$ to satisfy the Luce choice axiom is (1) to be stochastically rationalizable and (2) to satisfy the following condition:

$$\pi_Y^i = p_{Y_{+i}}(i) \times p_{Y_{+ij}}(j) \times p_{Y_{+ijk}}(k) \ldots$$

which always holds for weighted values.

Remark. Example 2 in Section 3.1 is associated with a sharing system that does not satisfy stochastic rationality (because $\gamma_\emptyset^2 = -1/6$) nor the Luce choice axiom (because $p^*(2) \neq p^*(23) \times p_{23}^*(2)$). Hence, it is neither a random order value nor a weighted value, but a M öbius value.

5 Properties of the Möbius Value

5.1 Monotone TU-Games

Most variations on the Shapley value assume that the positivity axiom holds: whenever the game is monotone, each individual value is positive (Monderer and Samet, 2002). Hence, the literature seems to focus on values for which the monotonicity of the game would be a sufficient condition for positivity. We show below that monotonicity is both a necessary and sufficient condition for any Möbius value to be positive. This implies that the positivity axiom may be replaced by the assumption of game monotonicity in the study of Möbius values.

Theorem 6 : *Any Möbius value* $\varphi(\nu)$ *is positive if and only if the TU-game* (Z, ν) *is monotone.*

The proof is given in appendix.

Since a quasivalue is defined by a solution characterized by the axioms A1-A3 as well as by positivity (Weber, 1988), it then follows from Theorem 1 that *a quasivalue is a Möbius value that satisfies the positivity axiom.* This proves our claim that quasivalues are special cases of Möbius values.

5.2 Convex TU-Games

We know from Shapley (1971) that the core of a convex game is nonempty. The following result shows that all the Möbius values belong to the core for a convex game.

Theorem 7 : *Any Möbius value $\varphi^p(\nu)$ is in the core of the TU-game (Z, ν) if and only if this game is convex.*

The proof is given in appendix.

We may now show that the set of Möbius values is identical to the core of a convex game. Indeed, when the game is convex, all the Möbius values belong to the core as shown by Theorem 7. Hence, for a nonconvex game, the set of random order values is a proper subset of Möbius values and, when the game is convex, we have the following result:

Theorem 8 : *For any TU-game (Z, ν), the set of all Möbius values is equal to its core if and only if the game is convex.*

Theorems 4 and 7 together with Weber's Theorem 14 imply that the core of a convex game being equal to the set of random order values, then the set of stochastically rationalizable Möbius values is equal to that of Mö bius values, i.e. is equal to the core itself.

6 Concluding Remarks

Our approach to cooperative values allows us to shed new light on cooperative game theory. Indeed, we have shown that the weighted values correspond to the most constrained class of solutions. They are axiomatically characterized by Kalai and Samet (1987) through efficiency (A1), null-player (A2), additivity, positivity and partnership. Since positivity and additivity imply homogeneity (as shown by Kalai and Samet, 1987, p.213), the first two axioms may be replaced by linearity (A3) while positivity may be replaced by the stochastic rationality of the sharing system and partnership by the Luce choice axiom. Hence, our main results may be summarized as follows.

- For *any sharing system*, a solution that satisfies A1-A3 is *a Möbius value*.
- For *any stochastically rationalizable sharing system*, a solution that satisfies A1-A3 is *a random order value* (i.e. a solution that satisfies A1, A2, additivity and positivity).
- For *any sharing system satisfying the Luce choice axiom*, a solution that satisfies A1-A3 is *a weighted value* (i.e. a solution that satisfies A1, A2, additivity, positivity and partnership).

Some questions remain open. First, is there always an element in the nonempty core of a nonconvex game that can be represented by a Mobius value? If yes, what are the restrictions that the corresponding sharing system satisfies? And more generally, can Theorem 8 be extended to the case of nonconvex games with a nonempty core?

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Appendix

Proof of Theorem 3: In (15), note that i belongs to X. This is consistent with (18) and (19) where $i \notin X$ since the subset X used in (15) is replaced by X_{+i} in (18) and (19).

The proof involves three steps.

Step 1. We show that

$$\sum_{X \supset Y} (-1)^x = \begin{cases} (-1)^n & \text{if } Y = Z, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for $y < n$, we have:

$$\begin{aligned} \sum_{X \supset Y} (-1)^x &= (-1)^y + (-1)^{y+1} + ... + (-1)^n \\ &= (-1)^y [(-1)^0 + ... + (-1)^{n-y}] \\ &= (-1)^y (1-1)^{n-y} = 0. \end{aligned}$$

For $y = n$, we have $\sum_{X \supset Y} (-1)^x = (-1)^n$ because $x = n$.

Step 2. We now show that

$$\sum_{X \supset W \supset Y} (-1)^w = \begin{cases} (-1)^x & \text{if } Y = X, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, since $X \cup \overline{X} = Z \supset W \cup \overline{X} \supset Y \cup \overline{X}$, then $Z \supset T \supset Y \cup \overline{X}$ where $T \equiv W \cup \overline{X}$. Hence, $w = t - \overline{x}$, and we have:

$$\begin{aligned} \sum_{X \supset W \supset Y} (-1)^w &= \sum_{T \supset Y \cup \overline{X}} (-1)^{t - \overline{x}} \\ &= (-1)^{-\overline{x}} \sum_{T \supset Y \cup \overline{X}} (-1)^t \\ &= (-1)^{x-n} \sum_{T \supset Y \cup \overline{X}} (-1)^t \\ &= (-1)^{x-n} (-1)^n \qquad \text{(by Step 1)} \\ &= (-1)^x. \end{aligned}$$

Otherwise, the argument used in Step 1 still applies.

Step 3. Consider two functions f and g defined on 2^Z. We first show that (18) implies (19) for f and g. We have:

$$\begin{aligned}
\sum_{X \supset Y} (-1)^{x-y} f(X) &= (-1)^{-y} \sum_{X \supset Y} (-1)^{x} f(X) \\
&= (-1)^{-y} \sum_{X \supset Y} (-1)^{x} \sum_{W \supset X} g(W) \\
&= (-1)^{-y} \sum_{W \supset Y} g(W) \sum_{W \supset X \supset Y} (-1)^{x} \\
&= (-1)^{-y} g(Y) (-1)^{y} \qquad \text{(by Step 2)} \\
&= g(Y).
\end{aligned}$$

It remains to prove that (19) implies (18). We have:

$$\begin{aligned}
\sum_{X \supset Y} g(X) &= \sum_{X \supset Y} \sum_{W \supset X \supset Y} (-1)^{x-y} f(W) \\
&= \sum_{W \supset Y} (-1)^{-y} f(W) \sum_{W \supset X \supset Y} (-1)^{x} \\
&= (-1)^{-y} f(Y) (-1)^{y} \qquad \text{(by Step 2)} \\
&= f(Y).
\end{aligned}$$

So, we have the desired implications once $f(Y)$ (resp. $f(X)$) is replaced by $p_{Y_{+i}}(i)$ (resp. $p_{X_{+i}}(i)$) and $g(Y)$ (resp. $g(X)$) by π_Y^i (resp. π_X^i).

□

Proof of Theorem 6: (Necessity) Assume that the TU-game (Z, ν) is not monotone. Then, there exists a coalition $Y \subset Z$, with $y \geq 2$, and one player $i \in Y$ such that $\nu(Y) - \nu(Y_{-i}) < 0$. Then, by (2), it follows that

$$\sum_{X \subset Y} \Gamma_\nu(X) - \sum_{X \subset Y_{-i}} \Gamma_\nu(X) = \sum_{\substack{X \subset Y \\ i \in X}} \Gamma_\nu(X) < 0. \tag{21}$$

It remains to prove that there exists a sharing rule, i.e. a probability p , such that $\varphi_i^p(\nu) < 0$. From (6), we have:

$$\varphi_i(\nu) = \sum_{X \subset Y} \Gamma_\nu(X) p_X(i) + \sum_{\substack{X \subset Z \\ X \not\subset Y}} \Gamma_\nu(X) p_X(i). \tag{22}$$

Two cases may then arise. In the first one, we have $Y = Z$. Then, replace $\varphi(\nu)$ in (22) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability p_ε in which

$$p_\varepsilon(j) = \frac{\varepsilon}{n-1}$$

for each player $j \neq i$ and $p_\varepsilon(i) = 1 - \varepsilon$. As a result, we obtain

$$\lim_{\varepsilon \to 0} \varphi_i^{p_\varepsilon}(\nu) = \sum_{\substack{X \subset Y \\ i \in X}} \Gamma_\nu(X). \tag{23}$$

From (21), $\lim_{\varepsilon \to 0} \varphi_i^{p_\varepsilon}(\nu) < 0$, i.e., there exists a positive ε such that $\varphi_i^{p_\varepsilon}(\nu) < 0$.

In the second case, we have $Y \subsetneq Z$. Then, replace $\varphi(\nu)$ in (22) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability p_ε in which $p_\varepsilon(j) = \varepsilon^2$ for all player $j \in Y_{-i}$:

$$p_\varepsilon(j) = \frac{1 - p_\varepsilon(Y)}{n-y}$$

for all player $j \in Z \backslash Y$ and $p_\varepsilon(i) = \varepsilon$. Then, (23) also holds. Hence, when the TU-game (Z, ν) is not monotone, the Möbius value $\varphi^p(\nu)$ is not necessarily positive.

(Sufficiency) For the proof, it is sufficient to show that $\varphi_i(\nu) \geq 0$ for any player $i \in Z$ when the TU-game (Z, ν) is monotone. Using (6), we obtain:

$$\begin{aligned}
\varphi_i(\nu) &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y)\, p_Y(i) \\
&= \sum_{\substack{Y \subset Z \\ i \in Y}} p_Y(i) \sum_{X \subset Y} (-1)^{y-x} \nu(X) \\
&= \sum_{X \subset Z} \nu(X) \sum_{Y \supset X_{+i}} (-1)^{y-x} p_Y(i) \\
&= \sum_{X \subset Z_{-i}} \underbrace{[\nu(X_{+i}) - \nu(X)]}_{(A)} \underbrace{\sum_{Y \supset X_{+i}} (-1)^{y-(x+1)} p_Y(i)}_{(B)}.
\end{aligned}$$

Since ν is monotone, each term (A) is positive. To sign (B), we use an argument developed by Sundberg and Wagner (1992, Lemma 3). Set $p_Y(i) =$

$\alpha/p\,(Y)$, $\alpha \geq 0$, $\beta \equiv p\,(X_{+i})$, $p_j \equiv p(j)$ for $j \in Z \backslash X_{+i} \equiv \{j_1, \cdots, j_m\}$, and let $\sharp J$ be the cardinal of J. Since $\beta > 0$ and $p_j \geq 0$ for all $j \in \{1, \cdots, m\}$, each term (B) is identical to

$$\begin{aligned} \alpha \sum_{J \subset \{1,\cdots,m\}} (-1)^{\sharp J} \left(\beta + \sum_{j \in J} p_j\right)^{-1} &= \alpha \int_0^{\infty} \left(\sum_{J \subset \{1,\cdots,m\}} (-1)^{\sharp J} e^{-\sum_{j \in J} p_j t}\right) e^{-\beta t} dt \\ &= \alpha \int_0^{\infty} \left(\sum_{J \subset \{1,\cdots,m\}} \prod_{j \in J} \left(-e^{-p_j t}\right)\right) e^{-\beta t} dt. \end{aligned}$$

Hence,

$$\alpha \sum_{J \subset \{1,\cdots,m\}} (-1)^{\sharp J} \left(\beta + \sum_{j \in J} p_j\right)^{-1} = \alpha \int_0^{\infty} \left(\prod_{j=1}^{m} \left(1 - e^{-p_j t}\right)\right) e^{-\beta t} dt \geq 0 \tag{24}$$

since $\int_0^{\infty} e^{-xt} dt = x^{-1}$ if $x > 0$. Thus, $\varphi_i\,(\nu)$ is positive when the TU-game is monotone.

□

Proof of Theorem 7: (Sufficiency) If the TU-game (Z, ν) is convex, then we must show that $\sum_{i \in Y} \varphi_i^p\,(\nu) \geq \nu\,(Y)$ for all nonempty coalitions $Y \subset Z$, i.e. $\varphi_Y^p\,(\nu) \geq \nu\,(Y)$.

By (6), we know that:

$$\begin{aligned} & \sum_{X \subset Y_{-\emptyset}} \sum_{T \subset Z \backslash Y} \Gamma_\nu\,(X \cup T)\, p_{X \cup T}(X) \qquad (25) \\ = & \sum_{X \subset Y_{-\emptyset}} \sum_{T \subset Z \backslash Y} \sum_{S \subset T} (-1)^{t-s} \sum_{W \subset X} (-1)^{x-w}\, \nu\,(W \cup S)\, p_{X \cup T}(X) \end{aligned}$$

whereas, by definition of a **PCC**,

$$\sum_{X \subset Y_{-\emptyset}} \sum_{T \subset Z_Y} \sum_{S \subset T} (-1)^{t-s} \sum_{W \subset X} (-1)^{x-w}\, \nu\,(W)\, p_{X \cup T}(X)$$

$$= \sum_{X \subset Y_{-\emptyset}} \Gamma_\nu\,(X)\, p_X(X) = \sum_{X \subset Y_{-\emptyset}} \Gamma_\nu\,(X) = \nu\,(Y)\,. \tag{26}$$

Hence, from (25) and (26), we obtain:

$$
\begin{aligned}
\varphi_Y^p(\nu) - \nu(Y) &= \sum_{X \subset Y_{-\emptyset}} \sum_{T \subset Z \backslash Y} \sum_{S \subset T} (-1)^{t-s} \sum_{W \subset X} (-1)^{x-w} \\
&\quad \times \left[\nu(W \cup S) - \nu(W)\right] p_{X \cup T}(X) \\
&= \sum_{X \subset Y_{-\emptyset}} \sum_{S \subset Z \backslash Y} \sum_{W \subset X} (-1)^{x-w} \left[\nu(W \cup S) - \nu(W)\right] \\
&\quad \times \sum_{S \subset T \subset Z \backslash Y} (-1)^{t-s} p_{X \cup T}(X) \\
&= \sum_{S \subset Z \backslash Y} \sum_{R \subset (Z \backslash Y) \backslash S} (-1)^{r} \sum_{X \subset Y_{-\emptyset}} \sum_{W \subset X} (-1)^{x-w} \\
&\quad \times \left[\nu(W \cup S) - \nu(W)\right] p_{X \cup S \cup R}(X) \\
&= \sum_{S \subset Z \backslash Y} \sum_{R \subset (Z \backslash Y) \backslash S} (-1)^{r} \\
&\quad \sum_{i \in Y} \underbrace{\sum_{\substack{X \subset Y \\ i \in X}} p_{X \cup S \cup R}(i) \times \sum_{W \subset X} (-1)^{x-w} \left[\nu(W \cup S) - \nu(W)\right].}_{(A)}
\end{aligned} \tag{27}
$$

We may rewrite (A) as follows:

$$
\begin{aligned}
&\sum_{X \subset Y_{-i}} \left\{ \sum_{W \subset X_{+i}} (-1)^{(x+1)-w} \left[\nu(W \cup S) - \nu(W)\right] \right\} \\
&\times p_{X_{+i} \cup S \cup R}(i).
\end{aligned}
$$

Hence, (A) is equivalent to:

$$\sum_{X \subset Y_{-i}} \left\{ \sum_{V \subset X} \sum_{W \subset V_{+i}} (-1)^{(v+1)-w} \left[\nu\left(W \cup S\right) - \nu\left(W\right) \right] \right\} \tag{28}$$

$$\times \left\{ \sum_{U \subset (Y_{-i}) \backslash X} (-1)^{u} p_{U \cup X_{+i} \cup S \cup R}\left(i\right) \right\}$$

$$= \sum_{\substack{X \subset Y \\ i \in X}} \underbrace{\sum_{\substack{V \subset X \\ i \in V}} \sum_{W \subset V} (-1)^{v-w} \left[\nu\left(W \cup S\right) - \nu\left(W\right) \right]}_{(B)}$$

$$\times \left\{ \sum_{U \subset Y \backslash X} (-1)^{u} p_{U \cup X \cup S \cup R}\left(i\right) \right\}.$$

By interchanging the summations, (B) becomes

$$\sum_{\substack{W \subset X \\ i \in W}} \left[\sum_{W \subset V \subset X} (-1)^{v-w} \right] \left[\nu\left(W \cup S\right) - \nu\left(W\right) \right]$$

$$+ \sum_{W \subset X_{-i}} \left[\sum_{W \subset V \subset X_{-i}} (-1)^{v+1-w} \right] \left[\nu\left(W \cup S\right) - \nu\left(W\right) \right]$$

$$= \nu\left(X \cup S\right) - \nu\left(X\right) - \nu\left(X_{-i} \cup S\right) + \nu\left(X_{-i}\right).$$

First, set

$$\sigma\left(i, X, S\right) \equiv \left[\nu\left(X \cup S\right) - \nu\left(X\right) \right] - \left[\nu\left(X_{-i} \cup S\right) - \nu\left(X_{-i}\right) \right].$$

Since $X \cap S = \emptyset$, the convexity of (Z, ν) implies that $\sigma\left(i, X, S\right) \geq 0$. Second, setting $W \equiv U \cup R$, we have

$$\rho\left(i, X, S\right) \equiv \sum_{W \subset Z \backslash (X \cup S)} (-1)^{w} p_{X \cup S \cup W}\left(i\right).$$

Using the same argument as for (24), we obtain $\rho\left(i, X, S\right) \geq 0$.

Therefore, using (27) leads to

$$\varphi_Y^p(\nu) - \nu(Y) = \sum_{S\subset Z\setminus Y}\sum_{i\in Y}\sum_{X\subset Y}\sigma(i,X,S)\times\rho(i,X,S) \geq 0.$$

(Necessity) The proof is by contradiction. Assume the TU-game (Z,ν) is not convex and show that there exists a Möbius value that does not belong to the core. First, applying Proposition 4 of Chateauneuf and Jaffray (1989) allows one to say that the **PCC** Γ_ν of ν satisfies:

$$\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X) \geq 0$$

for all pair of players $\{i,j\}$ belonging to each coalition $Y \subset Z$ if and only if the TU-game (Z,ν) is convex. Then, since our game is not convex, there exists a coalition $Y \subset Z$ and a pair of players $i, j \in Y$ such that:

$$\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X) < 0. \tag{29}$$

We now have to prove that there exists a Möbius value, $\varphi^p(\nu)$, which is not in the core, that is, $\varphi_{Y_{-i}}^p(\nu) - \nu(Y_{-i}) < 0$. Recall that $p_X(Y_{-i}) = 1$ when $X \subset Y_{-i}$. From (6), it follows that:

$$\begin{aligned}\varphi_{Y_{-i}}^p(\nu) - \nu(Y_{-i}) &= \sum_{\substack{X\subset Z\\ X\not\subset Y_{-i}}}\Gamma_\nu(X)\,p_X(Y_{-i}) \qquad (30)\\ &= \sum_{\substack{X\subset Y\\ i\in X}}\Gamma_\nu(X)\,p_X(Y_{-i})\\ &\quad + \sum_{\substack{X\subset Z\\ X\not\subset Y}}\Gamma_\nu(X)\,p_X(Y_{-i}).\end{aligned}$$

Two cases may then arise. In the first one, we have $Y = Z$. Then, replace $\varphi^p(\nu)$ in (30) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability $p = p_\varepsilon$ in which $p_\varepsilon(i) = p_\varepsilon(j) = (1-\varepsilon)/2$ and $p_\varepsilon(k) = \varepsilon/(n-2)$ for all player $k \in Z_{-i}$. Then:

$$\lim_{\varepsilon\to 0}\left[\varphi_{Y_{-i}}^p(\nu) - \nu(Y_{-i})\right] = \frac{1}{2}\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X), \tag{31}$$

which is negative by (29), i.e., there exists a positive ε such that $\varphi^{p}_{Y-_i}(\nu) - \nu(Y_{-i}) < 0$.

In the second case, we have $Y \subsetneq Z$. Then, replace $\varphi^{p}(\nu)$ in (30) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability $p = p_\varepsilon$ where $p_\varepsilon(i) = p_\varepsilon(j) = \varepsilon$, $p_\varepsilon(k) = \varepsilon^2$ for all player $k \in Y_{-i}$ and

$$p_\varepsilon(k) = \frac{1 - p_\varepsilon(Y)}{n - y}$$

for all player $k \in Z \backslash Y$. Again, (31) holds , i.e. there exists a positive ε such that $\varphi^{p}_{Y_{-i}}(\nu) - \nu(Y_{-i}) < 0$. Hence, if the TU-game (Z, ν) is not convex, the constructed Möbius value $\varphi^{p}(\nu)$ does not belong to the core.

□