The Stability Threshold and Two Facets of Polarization

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May 10, 2005

Abstract

In this paper we introduce the *stability threshold* that quantifies the *minimal* returns to size sufficient to prevent credible secession threats by regions of the country. Severity of internal tension has been linked to degree of *polarization* of citizens' preferences and characteristics. We show that the increasing degree of polarization does not, in general, raise the stability threshold, even though this hypothesis holds in some asymptotic sense. We also examine the question of the number of smaller countries to be created if the unity of the large country is not sustainable, and investigate the link between this number and the degree of the country polarization.

Key Words: Polarization, Secession, Stability threshold, Clusters.

JEL Classification Numbers: H20, D70, D73.

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1 Introduction

Internal conflicts over government policies often threaten the stability of a country. Indeed, dissatisfied regions of the country may attempt to secede, if the economies of scale brought by unity are overweighed by the benefits of forming a separate entity (in which the level of internal confrontation is reduced, or eliminated altogether). In many cases, these conflicts are created by the lack of uniformity in citizens' preferences over the range of government policy choices and/or distinctions across ethnic, religious, historical or linguistic lines. Thus, stability of the country is linked to the distribution of its citizens' preferred policies or characteristics.

How stability of the country should be measured? We suggest to measure it by means of *stability threshold*. This index quantifies the *minimal* returns to size that are sufficient to prevent credible secession threats. An alternative interpretation of the stability threshold is that of the *minimal* burden that can be imposed on the country (and all regions, provided they decide to secede) which still guarantees its unity.

Our notion of stability, that requires "secession-proofness" of the country in the face of internal conflicts, effectively ties stability with the precise form of the society's conflict-inducing diversity, represented by the distribution of citizens' preferences and characteristics. Thus, the stability threshold is also a measure of *severity of internal conflicts*. It is indeed natural to define severity of a conflict as the strength of secession threats that this conflict generates (and, in turn, this strength is faithfully represented by the stability threshold or the size of overall resources that can prevent or at least mitigate internal conflicts[1]).

Severity of internal tension has been linked in the literature to *polarization* of the distribution of citizens' preferences and characteristics. The common belief (Esteban and Ray (1994), (1999), (2004)) is that raising the degree of polarization increases the probability of

[1]For the existing literature on "greed-based" conflicts motivated by competition over resources see Grossman (1991), Gershenson and Grossman (1999), Caselli and Coleman (2002).

internal conflicts, and thus makes secession threats by the country's dissatisfied regions more severe. It thus seems proper to check this hypothesis, by enquiring into the relation between the degree of country's polarization and its stability threshold (which as was said is also a measure of severity of internal conflicts).

Our main finding is that, somewhat counter-intuitively, the relation between polarization and the stability threshold is ambiguous. Recall (Esteban and Ray (1994)), that the concept of polarization is based on the existence of several population clusters with relative homogeneity of preferences within clusters and substantial heterogeneity across clusters. The overall measure of polarization is then determined by the following two factors. The first is the level of heterogeneity inside each cluster (for a given number of clusters), that represents the degree of polarization and conflict between existing population groups: the less heterogeneous each cluster is, the more polarized is the society at large. The second is reflected by the number of clusters in the society, when a smaller (but greater than one) number of clusters represents a higher degree of confrontation (and polarization) in the society. Dependence of the polarization index on the first factor will be called *fixed-clusters polarization effect (FCPE)*, and on the second factor – *variable-clusters polarization effect (VCPE)*. Our basic conclusions are as follows:

- The stability threshold of a country is positively correlated with the FCPE.
- The link between the stability threshold and VCPE is ambiguous.
- The impact of VCPE is sufficiently strong so that the combined effect of FCPE and VCPE is ambiguous as well.
- However, there is positive correlation between the stability threshold and the polarization when the polarization is low, which happens when the number of clusters is sufficiently large and each cluster is sufficiently heterogeneous.

The somewhat unexpected behavior of the stability threshold with respect to VCPE is due to the following reason. Existence of a "centrally-located" cluster (the one where the preferences fall in or close to the center of the political map) can make secessions more difficult compared to the situation when the center is "vacant". This is because the central cluster benefits the most from being in a united country (since the chosen policy would typically be geared towards the "median" citizen). It may therefore be costly to persuade this cluster to join a seceding region (if it is too small to profit from secession by itself), or the citizens of that region may actually favor unity because then they can demand compensation from the politically-satisfied central cluster. Thus, the stability threshold may *increase* when the country's population undergoes a division into more clusters (although located closer to each other) and the center becomes occupied, despite a *decrease* in polarization. This, as was said, cannot happen in the case of FCPE, and also not when the population preferences are distributed very uniformly across their range.

In the second part of the paper we examine the situation where the stability threshold has not been achieved and the break-up of the country is imminent. We then examine the *stable number* of countries, i.e., the number of independent entities into which the given united country should be broken in order to eliminate credible threats of secession.[2] We find that the stable number of countries also behaves non-monotonically with respect to polarization indices. However, monotonicity does appear when the stable number is large, and the stable number decreases when polarization rises.

The paper is organized as follows. Section 2 contains the formal model of a country with heterogeneous citizens and the definition of stability threshold. In Section 3 we discuss the notion of polarization. Our results on the link between stability and polarization are presented in Section 4, whereas Section 5 studies stability in the multi-country framework and its relation to polarization. The proofs are relegated to the Appendix.

[2]See Alesina and Spolaore (1997) in the case of the uniform distribution of citizens' characteristics.

2 Model

We consider a country $\mathcal{W}$ with a population of total mass 1, whose citizens have preferences over the unidimensional policy space given by the interval $I = [0, 1]$. Citizens have symmetric single-peaked preferences over the set I, and we identify each citizen with her ideal point (and thus $\mathcal{W} \equiv I$). The distribution of all ideal points (and, thus, of all citizens' preferences) is given by a cumulative distribution function F with density f, defined over I.

The country $\mathcal{W}$ chooses a policy in the policy space I. In this paper, as in Alesina and Spolaore (1997) and Le Breton and Weber (2003), we adopt a spatial interpretation of the model by identifying a policy with the physical location of the government, so we do not distinguish between geographical and preference dimensions. The country $\mathcal{W}$ has to cover the cost of provision of public good, or government cost, c. We assume that the government costs are the same for all regions, and if a region secedes from $\mathcal{W}$, it will have to cover the same cost c. For simplicity, we restrict our analysis of possible secessions to those subsets of $\mathcal{W}$ that consist of the union of a finite number of intervals and we will use the term *region* only for such subsets of citizens.

Suppose now that an individual t belongs to the set S, which could be either the unified country ($S = \mathcal{W}$) or a seceding region ($S \subsetneq \mathcal{W}$), and whose government chooses a location $p \in I$. Then the disutility or "transportation" cost $d(t, p)$, incurred by individual t from the choice of p, is determined by the distance between t and the government location p and we shall assume that:

$$d(t, p) = |t - p|.$$

Now denote

$$D(S, p) = \int_S d(t, p) f(t) dt.$$

Then the value

$$D(S) = \min_{p \in I} \int_S d(t, p) f(t) dt$$

represents the minimal aggregate transportation cost incurred by the citizens of S.[3]

Under the linearity assumption, the aggregate transportation cost for every set S is minimized when the government location chooses its location at the ideal point of its "median citizen", $m(S)$, that satisfies $\int_{\{t\in S|t\leq m(S)\}} f(t)dt = \int_{\{t\in S|t\geq m(S)\}} f(t)dt$. Note that if S is an interval and f is positive on S, then its median citizen is uniquely defined. However, if S consists of a several disjoint intervals, the median of S is not necessarily unique.

We now introduce the notion of *S-cost allocation* that determines the monetary contribution of each individual t towards the cost of government c.

Definition 2.1: A bounded measurable function x defined on the set $S \subseteq \mathcal{W}$ is called an S-cost allocation if it satisfies the budget constraint:

$$\int_S x(t)f(t)dt = c.$$

When the government location of S is at p, the total disutility of citizen $t \in S$ under S-cost allocation x is:

$$d(t,p) + x(t).$$

We allow for lump sum transfers and do not restrict the mechanism of sharing costs in any way. Thus every region S that contemplates secession, could take into account only its *total* cost of being a separate country, given by the sum of government and transportation costs, in estimating its future gains:

$$c + D(S).$$

If region S can make its members better off than under the central government, then S is said to be *prone to secession*:

[3]There always exists an optimal location of the government (see the next paragraph) and, therefore, the cost function is well defined.

Definition 2.2: Consider a pair (p, x), where p is the location of national government and x is an $\mathcal{W}$-cost allocation. We say that region S is prone to secession (given (p, x)) if

$$\int_S (d(t,p) + x(t))f(t)dt > D(S) + c.$$

If no region is prone to secession, then the pair (p, x) is called *secession-proof.* The country is called *stable* if there exists a secession-proof pair (p, x).

We now introduce *stability threshold* or *unity index* that quantifies the minimal returns to size that are sufficient to prevent credible secession threats. As we mentioned in the introduction, this threshold can be viewed as the *minimal* burden on the country which still guarantees its unity. It is quite easy to observe that the notions of stability and secession-proofness are closely linked to the cost of public good. Indeed, a high cost of public good may facilitate regional cooperation and mitigate a threat to instability posed by regions. On the other hand, a low cost of public goods could reduce incentives for economic unity and raise the intensity of secession threats. Formally,

Proposition 2.3: For a given distribution of ideal points F, there is a cut-off value of government costs $c^{st}(F)$ such that the country is stable if and only if $c \geq c^{st}(F)$. The value $c^{st}(F)$ is called the *stability threshold* of F.

The natural question would be concerning the link between stability and polarization. In the next section we proceed with examination of polarization index.

3 Polarization Index

Indices of polarization, introduced in Esteban and Ray (1994), Duclos, et al. (2004) Tsui and Wang (2000), are based on the notions of identification within one's own group and alienation towards the others. The axioms that allow to derive polarization indices require, in particular, that a mean preserving reduction in the spread of the distribution (weakly) reduces the degree of polarization. For a continuous cumulative distribution function F on $[0,1]$, Duclos et al. (2004) have derived the following polarization index $\gamma_\alpha(F)$:

$$\gamma_\alpha(F) = \int_0^1 \int_0^1 |x-y| \, f(x)^{1+\alpha} f(y) dx dy, \tag{1}$$

where f is the density function of F and $0 < \alpha < 0.5$. If F is a discrete distribution supported on the set $\{x_0, \ldots, x_n\}$, and p_i is the probability of x_i, the index (derived by Esteban and Ray (1994)) is given by

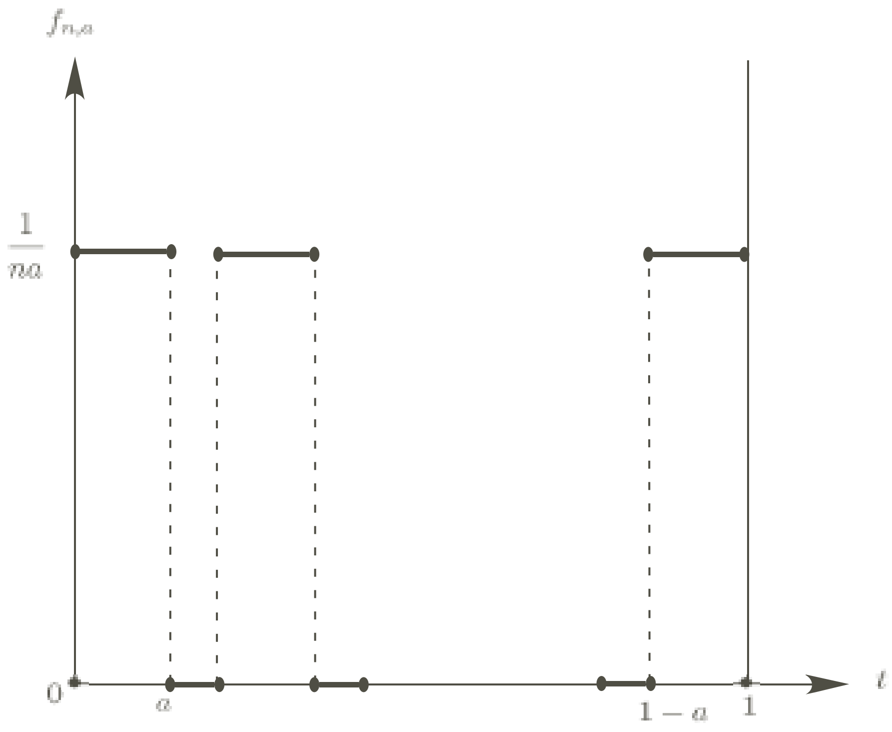
$$\gamma_\alpha(F) = \sum_{i=0}^{n} \sum_{j=0}^{n} p_i^{1+\alpha} p_j \left| x_i - x_j \right|. \tag{2}$$

As alluded to in the introduction, our analysis of conflicts will be performed under the assumption that citizens' ideal points form several disjoint clusters (that represent geographical regions or groups with similar political views). This will highlight the following two attributes of conflict situations (in addition to the existence of clusters). The first is heterogeneity of preferences within clusters, which represents conflicts within each region or group. The second is reflected by the number of distinct groups within the society, when a smaller (but greater than one) number of clusters represents a higher degree of confrontation. In order to focus solely on these two factors and eliminate other effects, we shall consider a family of step distribution functions with the support over a finite number of equal intervals (clusters). We shall also assume complete uniformity of the distribution of citizens' ideal points within each cluster. Thus, all distributions in our class $\mathcal{F}$ will be characterized by two parameters, the number of clusters, n and their length, a.

Formally, let an integer $n \geq 2$ and the parameter $a \in (0, \frac{1}{n}]$ be given. Consider a function $f_{n,a}$ on the unit interval $[0,1]$:

$$f_{n,a}(t) = \begin{cases} \frac{1}{na} & if \;\; t \in [j\frac{1-a}{n-1}, j\frac{1-a}{n-1} + a] \;\; \text{for} \;\; j = 0, 1, \ldots, n-1 \\ 0 & otherwise \end{cases}$$

That is, $f_{n,a}$ is the density function of the distribution which is supported and uniform on the n intervals of length a, removed from each other by the same distance. Denote the corresponding distribution by $F_{n,a}$. We also introduce $\{F_{n,0}\}$ for $n \geq 2$, which is a discrete limiting distribution of $\{F_{n,a}\}$ for $a \in (0, \frac{1}{n}]$. That is, $F_{n,0}$ is supported, and is uniform, on the finite set that consists of n equidistant points $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-1}{n-1} = 1\}$. (See Figure 1.)



Now, as in Duclos et al. (2004), let $0 < \alpha < 0.5$, and denote $\gamma_\alpha(n, a) \equiv \gamma_\alpha(F_{n,a})$. We have the following expression for the polarization index:

Proposition 3.1:

$$\gamma_\alpha(n, a) = \begin{cases} \left(\frac{1}{na}\right)^\alpha \frac{n+1-na}{3n}, & if \quad a > 0 \\ \left(\frac{1}{n}\right)^\alpha \frac{n+1}{3n}, & if \quad a = 0 \end{cases} \tag{3}$$

Obviously, the distribution $F_{n,a}$ becomes less polarized when a or n increase:

Corollary 3.2: The polarization index $\gamma_\alpha(n, a)$ declines in each of its two variables.

According to our interpretation in the introduction, the dependence of $\gamma_\alpha(n, a)$ on a describes the fixed-clusters polarization effect *(FCPE)*, while its dependence on n reflects the variable-clusters polarization effect *(VCPE)*. Thus, both effects reduce the polarization index.

It is worth pointing out that the index γ_α exhibits discontinuity in the transition from continuous distributions $F_{n,a}$ for $a > 0$ to $F_{n,0}$: $\lim_{\alpha \searrow 0} \gamma_\alpha(n,a) = \infty$. The reason is that according to this index discrete distributions are infinitely more polarized than continuous ones (due to the presence of infinitely dense clusters in former). The index still allows comparisons of discrete distributions $\{F_{n,0}\}$, via (3), but they belong to a different (higher) league of polarization when it comes to comparing them with continuous distributions $\{F_{n,a}\}$. The index should not therefore be used for comparisons across these two subfamilies of distributions, but only for comparisons inside each subfamily.

4 The Linkage between the Stability Threshold and Polarization

In this section we study how the stability threshold reacts to changes in polarization. First, we explicitly calculate the stability threshold for the distributions in our class. For every function $F_{n,a} \in \mathcal{F}$ we shall use a notation $c^{st}(n,a)$ instead of $c^{st}(F_{n,a})$.

Proposition 4.1: For $n \geq 2, a \in [0, \frac{1}{n}]$, the stability threshold $c^{st}(n,a)$ is given by:

$$c^{st}(n,a) = \frac{1}{8}(1 + (1 - an)\frac{1 + \frac{4}{n}([\frac{n+2}{4}] - [\frac{n+1}{4}])}{2[\frac{n-1}{2}] + 1}).$$

We now turn to our conclusions:

Proposition 4.2: (i) The stability threshold is positively correlated with FCPE. That is, the increase in a for fixed n reduces both the polarization index and the stability threshold.

(ii) The link between the stability threshold and VCPE is ambiguous. That is, while an increase in n reduces the polarization index $\gamma_\alpha(n,a)$, it does not necessarily reduce, or increase, the stability threshold $c^{st}(n,a)$ for fixed a.

(iii) The VCPE is strong enough to make the combined effect of FCPE and VCPE on the stability threshold ambiguous as well. That is, while the simultaneous increase in both n and a reduces the polarization index $\gamma_\alpha(n,a)$, it does not necessarily reduce, or increase, the stability threshold $c^{st}(n,a)$.

Thus, in general, the relationship between polarization and stability is not monotone.

According to Proposition 4.2, the stability threshold of $F_{n,a}$ decreases with the increase of a (and the implied fall in the distribution's polarization) for fixed n, but occasionally fails to be monotonic in $\gamma(n,a)$ for fixed a. For instance, as is pointed out in the proof of Proposition 4.2, the first deviation from monotone decline of $c^{st}(n,0)$ in n occurs when $n=6$ which follows from the fact that $c^{st}(6,0)=\frac{1}{6}>c^{st}(5,0)=\frac{3}{20}$. The reason is the one already mentioned in the introduction: when $n=6$, the "central cluster" $\frac{1}{2}$ (which does not exist when $n=5$) makes secessions difficult. Indeed, in the united country scenario the optimally chosen government location is also at the center[4]. The existence of a relatively big central cluster (which incurs zero transportation cost) has a mitigating effect on the aggregate transportation cost burden. However, if we consider a subinterval of I which contains one of the endpoints of I and $\frac{1}{2}$, or the subinterval that complements it,[5] none has a "central block" with zero transportation cost. This means that these intervals would incur quite high transportation costs in the case of secession, which makes secession less likely and the country more stable compared to the more polarized distribution $F_{5,0}$.

It is worthwhile to note that, for a positive fixed a, the decline of $c^{st}(n,a)$ in n is restored if the value of n is large enough (and thus polarization is low):

Proposition 4.3: For every $0<a<1$, there exists a value $n(a)$ such that $c^{st}(n_1,a)\leq$

[4]It is easily to verify that, under the linearity assumption, in a secession-proof pair (p,x) the government location p must be the ideal point of the "median citizen" $m(I)=\frac{1}{2}$.

[5]Our proofs indicate that only these intervals play a role in the determination of $\mathcal{W}$'s stability – see Lemma A.2 in the Appendix.

$c^{st}(n_2, a)$ whenever $n_1 > n_2 > n(a)$ and $n_1 a \leq 1$.

5 The Stable Number of Countries and Polarization Indices

When the government cost is low, $\mathcal{W}$ is no longer stable (Proposition 2.3) and could be broken up into smaller entities. The question we analyze in this section is what is the number of smaller countries that could guarantee the stability of partition of $\mathcal{W}$:

Definition 5.1: Consider a partition $(S_1, \ldots, S_m)$ of $\mathcal{W}$ into m countries, an m-tuple of pairs $((p_1, x_1), \ldots, (p_m, x_m))$, where p_i is the government location in S_i and x_i is an S_i-cost allocation. We say that region S is prone to secession (given $((p_1, x_1), \ldots, (p_m, x_m))$) if

$$\sum_{i=1}^{m} \int_{S \cap S_i} (d(t, p_i) + x_i(t)) f(t) dt > D(S) + c.$$

If no region is prone to secession, then the m-tuple $((p_1, x_1), \ldots, (p_m, x_m))$ is called *secession-proof*. The partition $(S_1, \ldots, S_m)$ is called *stable* if there exists a secession-proof m-tuple $((p_1, x_1), \ldots, (p_m, x_m))$.

Proposition 5.2 below follows from the main result in Haimanko et al. (2004):

Proposition 5.2: For a given distribution of ideal points $F \in \mathcal{F}$ and the government cost $c > 0$, there exists a stable partition $(S_1, \ldots, S_n)$ of $\mathcal{W}$.

In particular, when $c \geq c^{st}(n, a)$, the trivial partition of $\mathcal{W}$ (consisting of $\mathcal{W}$ itself) is stable.

Denote by $\overline{K}(c, n, a)$ the maximal number of countries in a stable partition of I (when the distribution of ideal points is $f_{n,a}$ and the government cost is c), and by $\underline{K}(c, n, a)$ – the minimal number of countries. For simplicity, we will focus attention on $K(c, n, a) = \underline{K}(c, n, a)$; all our observations apply to $\overline{K}(c, n, a)$ just as well. We shall call $K(c, n, a)$ the *stable number* of countries. It is natural to ask how it is affected by the change in $\gamma_\alpha(n,a)$, the polarization degree of $f_{n,a}$.

First, it turns out that K does not, in general, behave monotonically in the polarization degree. Indeed, pick $c_0 \in \left(\frac{3}{20}, \frac{1}{6}\right)$. Then, since $c^{st}(4,0) = c^{st}(6,0) = \frac{1}{6} > c_0$, and $c^{st}(5,0) = \frac{3}{20} < c_0$ (these computations were made in the proof of Proposition 4.2), we have

$$K(c_0, 4, 0), K(c_0, 6, 0) > 1, \text{ and } K(c_0, 5, 0) = 1.$$

Moreover, since $c^{st}(n,a)$ is continuous in a for a fixed n, for all positive and sufficiently small a_4, a_5, and a_6

$$K(c_0, 4, a_4), K(c_0, 6, a_6) > 1, \text{ and } K(c_0, 5, a_5) = 1.$$

Consequently:

Corollary 5.3: The stable number of countries is not monotone in the polarization degree. That is, while a simultaneous increase of both n and a reduces the polarization index $\gamma_\alpha(n,a)$, it does not necessarily decrease, or increase, the stable number $K(c,n,a)$ for a given c.

The example on which this corollary is based uses relatively high values of c. It turns out that for low values of c the stable number *does* behave monotonically in the polarization index: it decreases with polarization, as we show in Proposition 5.4. Intuitively, this reflects the fact that in a very polarized society each cluster is relatively uniform, and hence, when separated from others, can exist as a separate and stable country even when the government cost is very low. Thus, for a wide range of low c, highly polarized I should not be split into more countries than there are clusters, which keeps the stable number bounded. However, when the society is not polarized, and its members' preferences are spread uniformly, low c necessitates a very fine partition to achieve stability, because of the wide spread of preferences.

Proposition 5.4: Given two integers $2 \leq n_1 \leq n_2$ and $0 \leq a_1 \leq a_2 \leq \frac{1}{n_2}$, there exists $c(n_1, n_2, a_1, a_2) > 0$ such that for every $0 < c \leq c(n_1, n_2, a_1, a_2)$,

$$K(c, n_1, a_1) \leq K(c, n_2, a_2).$$

6 Appendix

We start with the following lemma:

Lemma A.1: If k, n are integers with $1 \leq k \leq n-1$, then for the distribution $F_{n,0}$

$$D([0, \frac{k}{n-1}]) = \frac{(\left[\frac{k}{2}\right] + 1)(k - \left[\frac{k}{2}\right])}{n(n-1)}.$$

Proof: Clearly

$$\begin{aligned} D([0, \frac{k}{n-1}]) &= D([0, \frac{k}{n-1}], \frac{1}{2}\frac{k}{n-1}) = \frac{2}{n}(\frac{1}{2}\frac{k}{n-1} - 0) \\ &\quad + \frac{2}{n}(\frac{1}{2}\frac{k}{n-1} - \frac{1}{n-1}) + \ldots + \frac{2}{n}(\frac{1}{2}\frac{k}{n-1} - \frac{\left[\frac{k}{2}\right]}{n-1}) \\ &= \frac{2}{n}\frac{1}{2}\frac{k}{n-1}(\left[\frac{k}{2}\right] + 1) - \frac{2}{n(n-1)}\left(1 + 2 + \ldots . + \left[\frac{k}{2}\right]\right) \\ &= \frac{(\left[\frac{k}{2}\right] + 1)(k - \left\lfloor\frac{k}{2}\right\rfloor)}{n(n-1)}. \end{aligned}$$

□

Our second lemma provides a computational formula for the unity index of distributions in $\mathcal{F}$. Its proof relies on the result of Haimanko et al. (2004), stating that stability of the country is equivalent to its *efficiency* (the country is efficient if the total cost[6] incurred by its citizens is minimized when it is a united entity), and Proposition 3.3 of Haimanko et al. (2003), according to which the country is efficient if and only if splitting it into *two* independent regions does not decrease the total cost.

[6]Obviously, this cost has two components: the aggregate transportation cost, and the government cost.

Lemma A.2: For every distribution $F_{n,a} \in \mathcal{F}$

$$\begin{aligned} c^{st}(n,a) &= \max_{s\in[0,1]} [D(I) - D([0,s]) - D([s,1])] \\ &= D(I) - \min_{s\in[0,1]} [D([0,s]) + D([s,1])]. \end{aligned}$$

Proof of Proposition 3.1: Note that for $\alpha = 0$, the index γ_0 defined by (1) for $F_{n,a}$ with $a > 0$ and by (2) $F_{n,0}$ is precisely the Gini inequality index. It is not a polarization index but it would be useful in our derivations. The index $\gamma_0(n,a)$ is simply the expected distance between two random points in I, each chosen according to $F_{n,a}$ and independently of the other one. We claim that $\gamma_0(n,0) = \frac{n+1}{3n}$ for every n. Indeed, clearly

$$\begin{aligned} \gamma_0(n+1,0) &= \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{1}{(n+1)^2}\left|\frac{i}{n} - \frac{j}{n}\right| \\ &= 2\sum_{i=0}^{n} \frac{i}{n(n+1)^2} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{1}{(n+1)^2}\left|\frac{i}{n} - \frac{j}{n}\right| \\ &= \frac{1}{n+1} + \frac{n(n-1)}{(n+1)^2}\gamma_0(n,0). \end{aligned}$$

And

$$\gamma_0(n,0) = \frac{n+1}{3n} \tag{4}$$

obviously satisfies this recursive relation, with the initial condition $\gamma_0(2,0) = \frac{1}{2}$. It is also clear that

$$\gamma_0(n,\frac{1}{n}) = \frac{1}{3} \tag{5}$$

(recall that $F_{n,\frac{1}{n}}$ is the uniform distribution). Further, it follows from the definition of Gini index as the expected distance between two random points that $\gamma_0(n,\cdot)$ is an affine function of a for fixed n, and therefore (4) and (5) imply that

$$\gamma_0(n,a) = \frac{n+1-na}{3n}.$$

To shift from $\gamma_0(n,a)$ to $\gamma_\alpha(n,a)$ for positive values of α, notice that

$$\gamma_\alpha(n,a) = \left(\frac{1}{na}\right)^\alpha \gamma_0(n,a).$$

Thus,

$$\gamma_\alpha(n,a) = \left(\frac{1}{na}\right)^\alpha \frac{n+1-na}{3n}$$

and

$$\gamma_\alpha(n,0) = \left(\frac{1}{n}\right)^\alpha \frac{n+1}{3n}.$$

□

Proof of Proposition 4.1: Note that the assertion of the proposition can be restated as follows:

(i) if $n = 4m$ for $m \geq 1$, then

$$c^{st}(n,a) = \frac{m}{2(4m-1)}(1-a)\,; \tag{6}$$

(ii) if $n = 4m+1$ for $m \geq 1$, then

$$c^{st}(n,a) = \frac{2m+1}{4(4m+1)} - \frac{a}{8}; \tag{7}$$

(iii) if $n = 4m+2$ for $m \geq 0$, then

$$c^{st}(n,a) = \frac{2m^2+2m+1}{2(4m+1)(2m+1)} - a\frac{2m+3}{4(4m+1)}; \tag{8}$$

(iv) if $n = 4m+3$ for $m \geq 0$, then

$$c^{st}(n,a) = \frac{m+1}{2(4m+3)} - \frac{a}{8}. \tag{9}$$

We first prove the equalities for the case of $a = 0$. Start with (6), when $n = 4m$. Note that the minimum of $D([0,s]) + D([s,1])$ is attained at $s = \frac{1}{2}$ (or at any other point between $\frac{2m-1}{4m-1}$ and $\frac{2m}{4m-1}$). Indeed, if (say) $\frac{k-1}{4m-1} \leq s < \frac{k}{4m-1} < \frac{2m-1}{4m-1}$, then the following holds:

$$D([0,s]) + D([s,1]) \;=\; D([0,\frac{k-1}{4m-1}]) + D([\frac{k}{4m-1},1])$$

$$
\begin{aligned}
&= D([0, \frac{k-1}{4m-1}], \frac{k-1}{2(4m-1)}) + D([\frac{k}{4m-1}, 1], \frac{1}{2} + \frac{k-1}{2(4m-1)}) \\
&> D([0, \frac{k}{4m-1}], \frac{k-1}{2(4m-1)}) + D([\frac{k+1}{4m-1}, 1], \frac{1}{2} + \frac{k-1}{2(4m-1)}) \\
&\geq D([0, \frac{k}{4m-1}]) + D([\frac{k+1}{4m-1}, 1]) \\
&\geq \min_{s\in[0,1]} [D([0, s]) + D([s, 1])].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c^{st}(4m, 0) &= D(I) - \min_{s\in[0,1]} [D([0, s]) + D([s, 1])] \\
&= D(I) - D([0, \frac{1}{2}]) - D([\frac{1}{2}, 1]) \\
&= D(I) - D([0, \frac{2m-1}{4m-1}]) - D([\frac{2m}{4m-1}, 1]) \\
&= D(I) - 2D([0, \frac{2m-1}{4m-1}]) = \text{(using Lemma A.1)} \\
&= \frac{m}{4m-1} - 2\frac{m}{4(4m-1)} = \frac{m}{2(4m-1)},
\end{aligned}
$$

which establishes (6) for $n = 4m$ and $a = 0$.

Next, we consider the rest of the scenarios when $a = 0$. Similarly to the previous case,

$$
\begin{aligned}
c^{st}(4m+1, 0) &= D(I) - \min_{s\in[0,1]} [D([0, s]) + D([s, 1])] \\
&= D(I) - D([0, \frac{2m-1}{4m}]) - D([\frac{2m}{4m}, 1]) \\
&= D(I) - D([0, \frac{2m-1}{4m}]) - D([0, \frac{2m}{4m}]) \\
&= \frac{(2m+1)2m}{(4m+1)4m} - \frac{m^2}{(4m+1)4m} - \frac{(m+1)m}{(4m+1)4m} \\
&= \frac{2m+1}{4(4m+1)},
\end{aligned}
$$

and (7) is also established. Further,

$$
\begin{aligned}
c^{st}(4m+2, 0) &= D(I) - \min_{s\in[0,1]} [D([0, s]) + D([s, 1])] \\
&= D(I) - D([0, \frac{2m}{4m+1}]) - D([\frac{2m+1}{4m+1}, 1]) \\
&= D(I) - 2D([0, \frac{2m}{4m+1}])
\end{aligned}
$$

$$= \frac{(2m+1)^2}{(4m+2)(4m+1)} - 2\frac{m(m+1)}{(4m+2)(4m+1)}$$
$$= \frac{2m^2+2m+1}{2(4m+1)(2m+1)},$$

which shows (8). And finally,

$$\begin{aligned} c^{st}(4m+3,0) &= D(I) - \min_{s\in[0,1]}[D([0,s]) + D([s,1])] \\ &= D(I) - D([0,\frac{2m}{4m+2}]) - D([\frac{2m+1}{4m+2},1]) \\ &= D(I) - D([0,\frac{2m}{4m+2}]) - D([0,\frac{2m+1}{4m+2}]) \\ &= \frac{(2m+1)(2m+2)}{(4m+3)(4m+2)} - \frac{m(m+1)}{(4m+3)(4m+2)} - \frac{(m+1)^2}{(4m+3)(4m+2)} \\ &= \frac{m+1}{2(4m+3)}, \end{aligned}$$

and hence (9) is established as well.

It remains to prove the four equalities for $\{F_{n,a}\}_{n\geq 2, a\in(0,\frac{1}{n}]}$. Note that each such distribution is symmetric around $\frac{1}{2}$ and satisfies GEM (the "gradually escalating median" condition, set forth in Le Breton and Weber (2003). This condition requires that there be a (non-decreasing) selection of a median, $l(t)$, in every subinterval $[0,t]$, such that $l'(t) \leq 1$ for almost every t. And it obviously holds for every $F_{n,a}$ for $n \geq 2, a \in (0,\frac{1}{n}]$, since one can consider

$$l(t) = \begin{cases} \frac{t}{2}, & \text{if } t \in \left[k\frac{1-a}{n-1}, k\frac{1-a}{n-1}+a\right] \text{ and } k \text{ is even;} \\ \frac{a}{2} - \frac{1}{2}\frac{1-a}{n-1} + \frac{t}{2}, & \text{if } t \in \left[k\frac{1-a}{n-1}, k\frac{1-a}{n-1}+a\right] \text{ and } k \text{ is odd;} \\ \frac{k}{2}\frac{1-a}{n-1} + \frac{a}{2}, & \text{if } t \in \left[k\frac{1-a}{n-1}+a, (k+1)\frac{1-a}{n-1}\right] \text{ and } k \text{ is even;} \\ t - \frac{k+1}{2}\frac{1-a}{n-1}, & \text{if } t \in \left[k\frac{1-a}{n-1}+a, (k+1)\frac{1-a}{n-1}\right] \text{ and } k \text{ is odd.} \end{cases}$$

According to Proposition 4.1 of Haimanko et al. (2003),

$$c^{st}(n,a) = \frac{1}{2} - 4\int_{l(\frac{1}{2})}^{\frac{1}{2}} t f_{n,a}(t)dt.$$

Due to the particular form of $f_{n,a}(t)$ and $l(\frac{1}{2})$, this implies that $c^{st}(n,a) = q(n)a + r(n) + s(n)\frac{1}{a}$. However, since $0 \leq c^{st}(n,a) \leq 1$ for all a, it follows that $c^{st}(n,a)$ has the form

$$c^{st}(n,a) = q(n)a + r(n), \tag{10}$$

i.e., it is an affine function of a for fixed n. Equality (10) also holds when $a = 0$, since the expression $\min_{s\in[0,1]}[D([0,s]) + D([s,1])]$ is continuous in the distribution F, as was established in Lemma A.7 in Haimanko et al. (2004).

Since $F_{n,\frac{1}{n}}$ is uniform on $[0,1]$,

$$q(n)\frac{1}{n} + r(n) = c^{st}(n, \frac{1}{n}) = D(I) - 2D([0, \frac{1}{2}]) = \int_0^1 \left|t - \frac{1}{2}\right| dt - 2\int_0^{\frac{1}{2}} \left|t - \frac{1}{4}\right| dt = \frac{1}{8}.$$

We also know that

$$r(n) = q(n) \cdot 0 + r(n) = c^{st}(n, 0),$$

and therefore

$$c^{st}(n, a) = n(\frac{1}{8} - c^{st}(n, 0))a + c^{st}(n, 0). \tag{11}$$

Substituting the values of $c^{st}(n, 0)$ that have been computed above into the above equality yields (6), (7), (8), and (9). □

Proof of Proposition 4.2: (i). Follows immediately from Proposition 4.1 and Corollary 3.2.

(ii) and **(iii)**. Consider the case where $a = 0$. When n increases, the distribution $F_{n,0}$ becomes less polarized and, in the limit, converges to the uniform distribution. The unity index $c^{st}(n, 0)$ clearly converges to $\frac{1}{8}$ as $n \to \infty$. By Proposition 2.3, it decreases for low values of $n : c^{st}(2, 0) = \frac{1}{2}$, $c^{st}(3, 0) = \frac{1}{6}$, $c^{st}(4, 0) = \frac{1}{6}$, $c^{st}(5, 0) = \frac{3}{20}$. However, $c^{st}(6, 0) = \frac{1}{6} > c^{st}(5, 0)$, and thus a spike in the unity index is observed on its way down to $\frac{1}{8}$, despite the decreasing polarization and increasing uniformity of the distribution. This spike is recurrent: clearly,

$$c^{st}(4m + 1, 0),\ c^{st}(4m + 3, 0) < c^{st}(4m + 2, 0), \tag{12}$$

and even

$$c^{st}(4m - 1, 0), c^{st}(4m, 0) \le c^{st}(4m + 2, 0) \tag{13}$$

(equality occurs only for $m = 1$). Moreover, if a_1, a_2, and a_3 are positive and sufficiently small, the inequality

$$c^{st}(4m+1, a_1),\ c^{st}(4m+3, a_2) < c^{st}(4m+2, a_3) \tag{14}$$

holds as well, due to continuity of $c^{st}(n, a)$ for a fixed n. This establishes (ii) and (iii) of the proposition.

□

Proof of Proposition 4.3: Fix $a > 0$. Note that $c^{st}(n+1, a) \leq c^{st}(n, a)$ for all feasible n, except possibly for those that have the form $n = 4m+1$. Consider the expression $c^{st}(4m+2, a) - c^{st}(4m+1, a)$. By Proposition 4.1, this difference is equal to

$$\frac{1}{8(4m+1)}\left(\frac{2}{2m+1} - 5a\right).$$

Thus, $c^{st}(n+1, a) - c^{st}(n, a) \leq 0$ for $n > n(a)$, where $n(a) = \frac{4}{5a}$.

□

Proof of Proposition 5.4: This follows immediately from Proposition 3.1 of Haimanko et al. (2004). Indeed, according to this proposition,

$$\lim_{c \to 0} K(c, n, a)\sqrt{c} = \frac{1}{2}\int_0^1 \sqrt{f_{n,a}(t)}dt = \frac{1}{2}\sqrt{na}$$

if $a > 0$, and clearly

$$\lim_{c \to 0} K(c, n, 0) = n$$

if $a = 0$.

□

7 References

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