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LOT-SIZING WITH PRODUCTION AND DELIVERY TIME WINDOWS

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Abstract

We study two different lot-sizing problems with time windows that have been proposed recently. For the case of production time windows, in which each client specific order must be produced within a given time interval, we derive tight extended formulations for both the constant capacity and uncapacitated problems with Wagner-Whitin (non-speculative) costs. For the variant with nonspecific orders, known to be equivalent to the problem in which the time windows can be ordered by time, we also show equivalence to the basic lot-sizng problem with upper bounds on the stocks. Here we derive polynomial time dynamic programming algorithms and tight extended formulations for the uncapacitated and constant capacity problems with general costs.

For the problem with delivery time windows, we use a similar approach to derive tight extended formulations for both the constant capacity and uncapacitated problems with Wagner-Whitin (non-speculative) costs.
Keywords: production time windows, lot-sizing, mixed integer programming

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1 Introduction

Recently two different lot-sizing problems with time windows have been studied. In both cases the demand data consists of a set of orders $k = 1, \ldots, K$ consisting of a quantity D^k and a time interval $[b^k, e^k]$ lying within the time horizon $[1, n]$. In Brahimi, Dauzère-Pérès and al. [3, 2, 6] the *production* time window is the interval during which the order must be produced. For this problem, two variants are considered: in the first each order is *distinct* (client-specific), whereas in the second orders are *indistinguishable* (non-specific). On the other hand in Lee et al. [9] the *delivery* time window for an order is the time interval in which the order must be delivered to the client.

For the client-specific problem with production time windows Brahimi et al. derive an $O(n^2)$ dynamic programming (DP) algorithm for the uncapacitated case with Wagner-Whitin (non-speculative) costs, denoted $WW-U-TWP$, and a pseudopolynomial DP algorithm for the general cost case, denoted $LS-U-TWP$. For the indistinguishable order problem, they show equivalence to the *noninclusive* time window problem in which no time window is strictly contained in another ($b^k < b^\kappa$ and $e^k > e^\kappa$ is not allowed). For this latter problem, denoted $LS-U-TWP(I)$ they present an $O(n^4)$ DP algorithm for the general cost case. In addition various mixed integer programming formulations for the two variants are presented, and then the algorithms for the single item problem are used as subproblems in a Lagrangian relaxation approach to solve multi-item problems with linking machine capacity constraints.

For the single item problem with delivery time windows Lee et al. derive polynomial time dynamic programming (DP) algorithms for the uncapacitated case when there are Wagner-Whitin costs, denoted $WW-U-TWD$, as well as a similar result in the presence of backlogging.

The approach taken here is to look both for polynomial time optimization algorithms and also tight mixed integer programming formulations possibly with additional variables, where *tight* means that the linear programming relaxation solves the problem, which in certain cases means that we have a description of the convex hull of feasible solutions. Then in tackling hard multi-item problems, one can either use column generation or Lagrangian relaxation approaches in which one requires the optimization algorithms to solve the subproblems, or one can use a direct mixed integer programming approach and provide an initial MIP formulation including the tight formulations of the subproblems.

Our main results are are as follows:

i) The presentation of several mixed integer programming formulations and the relationship between them, including those of Brahimi et al. and Lee et al.

ii) For the production time window problem with constant capacities and Wagner-Whitin costs, $WW-CC-TWP$, we derive a tight $O(n^2) \times O(n^2)$ extended formulation. For the uncapacitated version $WW-U-TWP$, we obtain a tight formulation in the original production, stock and set-up variables with $O(n^2)$ constraints.

iii) We show that the restricted production time window problem with non-inclusive time windows, or equivalently the production time window problem with indistinguishable orders, is also equivalent to the standard lot-sizing problem with upper bounds on stocks. For the problem with general cost structure, we derive an $O(n^2)$ DP algorithm and an $O(n^2) \times O(n^2)$ tight extended formulation for the uncapacitated problem $LS-U-TWP(I)$ by using the restricted time window structure. On the other hand for the constant capacity version $LS-CC-TWP(I)$ we derive an $O(n^3)$ DP algorithm and an $O(n^3) \times O(n^3)$ tight extended formulation by using the stock upper bound viewpoint.

iii) For the delivery time windows problem with constant capacities and and Wagner-Whitin costs, denoted $WW - CC - TWD$, we also derive a tight polynomial size extended formulation. Again for the uncapacitated case, it is of polynomial size in the original variables.

We now describe the contents of the rest of the paper. As production and delivery time window problems have very similar structure, we present some basic mathematical results in Section 2 that can be applied to both problems.

In Section 3 we treat the problem with production time windows and client specific orders. First we present different MIP formulations. Then we present the tight extended formulation for the constant capacity problem with Wagner-Whitin costs, and its simpler form when the problem is uncapacitated. In Section 4 we show that the problems with indistinguishable orders, noninclusive time windows and stock upper bounds are equivalent, and then derive two algorithms: first we present a DP algorithm and formulation for the noninclusive case when there is a natural ordering on the time intervals, and then an algorithm for lot-sizing with stock upper bounds and constant capacities. In Section 5 the presentation for the problem with delivery time windows follows much the same format as in Chapter 3. We terminate in Section 6 with a discussion of some open questions.

2 Useful Results

In this section we present first a simple result on feasible flows in transportation networks when the neighbors adjacent to the nodes form an interval. Specifically the nodes in the bipartition will correspond to time periods and orders respectively. This will be used later to show that one formulation can be obtained from another by projection. We then introduce mixing sets, and a polyhedral result for "generalized constant capacity lot-sizing with Wagner-Whitin costs", described in terms of mixing sets, that will be used to show that for a given problem, a particular formulation is "tight", i.e. it describes the convex hull of the feasible region.

2.1 Feasibility in Convex Transportation Networks

Definition 1 *A bipartite graph $(V_1 \times V_2, E)$ with $|V_1| = n, |V_2| = K$ is* convex *if for all $j \in V_2$, its set of neighbors $N(j) = \{i \in V_1 : (i,j) \in E\}$ forms an interval $[b^j, e^j] \subseteq [1, n]$, where $[p,q]$ denotes the interval $\{p, p+1, \ldots, q-1, q\}$.*
It is doubly convex *if the neighbors of nodes in V_1 also form intervals in V_2.*

Using the max flow/min cut theorem, it is straightforward to derive the following two Propositions. Proofs can be found in Cezik and Günlük [4].

Proposition 1 *Given a convex bipartite graph with supplies $a^i \in \mathbb{R}^1_+$ for $i \in V_1$ and demands $d^k \in \mathbb{R}^1_+$ for $k \in V_2$ such that $\sum_{i \in V_1} a^i = \sum_{k \in V_2} d^k$, there exists a nonnegative feasible flow satisfying all the supplies and demands at equality if and only if*

$$\sum_{i=t}^{l} a^i \geq \sum_{\{k: t \leq b^k \leq e^k \leq l\}} d^k \text{ for } 1 \leq t \leq l \leq n.$$

Proposition 2 *Given a doubly convex bipartite graph with supplies* $a^i \in \mathbb{R}^1_+$ *for* $i \in V_1$ *and demands* $d^k \in \mathbb{R}^1_+$ *for* $k \in V_2$ *such that* $\sum_{i \in V_1} a^i = \sum_{k \in V_2} d^k$*, there exists a nonnegative feasible flow satisfying all the supplies and demands at equality if and only if*

$$\begin{gathered}\sum_{i=1}^l a^i \geq \sum_{\{k: e^k \leq l\}} d^k \text{ for } 1 \leq l \leq n \\ \sum_{i=1}^l a^i \leq \sum_{\{k: b^k \leq l\}} d^k \text{ for } 1 \leq l \leq n.\end{gathered}$$

2.2 Mixing Sets

Consider the *mixing set* $X^M(s, z, b)$ consisting of the points (s, z) satisfying

$$s + z_l \geq b_l \text{ for } l = 1, \ldots, n \tag{1}$$

$$s \in \mathbb{R}^1_+, \; z \in \mathbb{Z}^n_+. \tag{2}$$

The following results can be found in Günlük and Pochet [8] and Miller and Wolsey [11], see also Pochet and Wolsey [14].

Let $f_l = b_l - \lfloor b_l \rfloor$ for $l = 1, \ldots, n$, and let $f_0 = 0$.

Theorem 3 *The convex hull of the mixing set* $\text{conv}(X^M(s, z, b))$ *is obtained by taking the initial constraints (1), the nonnegativity constraints* $s, z \geq 0$ *and two classes of inequalities*

$$s \geq \sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - z_{i_t}) \tag{3}$$

and

$$s \geq \sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - z_{i_t}) + (1 - f_{i_r})(\lfloor b_{i_1} \rfloor - z_{i_1}) \tag{4}$$

where $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ *and* $0 \leq f_{i_1} < f_{i_2} < \ldots < f_{i_r} < 1$.

Theorem 4 *A tight extended formulation for* $\text{conv}(X^M(s, z, b))$ *is:*

$$\begin{gathered}s = \sum_{i=1}^n f_i \delta_i + \mu \\ z_t + \mu + \sum_{\{i: f_i \geq f_t\}} \delta_i \geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \ldots, n \\ \sum_{i=0}^n \delta_i = 1 \\ \delta \in \mathbb{R}^{n+1}_+, \mu \in \mathbb{R}^1_+, z \in \mathbb{R}^n_+.\end{gathered}$$

2.3 "Generalized" Constant Capacity Lot-Sizing with Wagner-Whitin Costs

Suppose that we are given two vectors $\alpha, \beta \in \mathbb{R}^{n(n+1)/2}_+$ where each coordinate corresponds to an interval $[t, l] \subseteq [1, n]$. In addition suppose that α is *nondecreasing* ($\alpha_{tl} \leq \alpha_{\tau\lambda}$ whenever $\tau \leq t \leq l \leq \lambda$), β is nondecreasing and $\beta \leq \alpha$.

Consider now a "generalized" constant capacity lot-sizing set X in which s_t denotes the stock at the end of period t and $y_t \in \{0,1\}$ is the set-up variable in period t. X is given by:

$$\begin{gathered} s_{t-1} + C\sum_{u=t}^{l} y_u \geq \alpha_{tl} \text{ for } 1 \leq t \leq l \leq n \\ C\sum_{u=t}^{l} y_u \geq \beta_{tl} \text{ for } 1 \leq t \leq l \leq n \\ s \in \mathbb{R}_+^n, y \in \{0,1\}^n. \end{gathered}$$

Let $Y^t = (y_t, y_t + y_{t+1}, \ldots, y_t + \ldots + y_n)$ and $\alpha^t = (\alpha_{tt}, \ldots, \alpha_{tn})$. Now the set X can be rewritten as:

$$X = \cap_{t=1}^{n} X^M(s_{t-1}/C, Y^t, \alpha^t/C) \cap \{(y \in \{0,1\}^n : \sum_{u=t}^{l} y_u \geq \beta_{tl}/C \text{ for } 1 \leq t \leq l \leq n\}.$$

Theorem 5

$$\begin{aligned} \text{conv}(X) \quad &= \quad \bigcap_{t=1}^{n} \text{conv}(X^M(s_{t-1}/C, Y^t, \alpha^t/C)) && (5) \\ & \quad\quad \bigcap \{(y \in [0,1]^n : \sum_{u=t}^{l} y_u \geq \lceil \beta_{tl}/C \rceil \text{ for } 1 \leq t \leq l \leq n\}. && (6) \end{aligned}$$

The proof is obtained by taking the proof based on the extended formulation for constant capacity lot-sizing with Wagner-Whitin costs in Pochet and Wolsey [14], and observing that the addition of the constraints (6) (a consecutive 1's matrix in the space of the y variables) does not invalidate the proof.

It is also interesting to consider what happens when C is large, in particular when $\alpha_{1n} < C$. Either by specializing the above result, or by observing that the inequalities:

$$s_{t-1} \geq \sum_{u=t}^{l} (\alpha_{tu} - \alpha_{t,u-1})(1 - y_t \ldots - y_l)$$

are valid for $1 \leq t \leq l \leq n$, one obtains:

Proposition 6 *If* $\alpha_{1n} < C$,

$$\begin{gathered} \text{conv}(X) = \{(s,y) \in \mathbb{R}_+^n \times [0,1]^n : \\ s_{t-1} + \sum_{u=t}^{l} (\alpha_{tl} - \alpha_{t,u-1}) y_u \geq \alpha_{tl} \text{ for } 1 \leq t \leq l \leq n \\ \sum_{u=t}^{l} y_u \geq 1 \text{ for all } 1 \leq t \leq l \leq n \text{ with } \beta_{tl} > 0\}. \end{gathered}$$

3 Production Time Windows

3.1 MIP Formulations

We consider first the problem with production time windows, constant capacities and general costs, denoted $LS{-}CC{-}TWP$. Here the problem data consist of a time horizon n, production costs $p' \in \mathbb{R}^n$, storage costs $h' \in \mathbb{R}^n$, set-up costs $f \in \mathbb{R}^n$, and a list of K orders each consisting of a time window $[b^k, e^k]$ with $1 \le b^k \le e^k \le n$, and a positive order quantity D^k. Order k must be produced within the time interval and delivered in period e^k.

We start with a first formulation as a mixed integer program. We use the standard variables
x_t is the amount produced in period t
s_t is the stock at the end of period t
$y_t \in \{0,1\}$ is the set-up variable for period t.
In addition
z_t^k is the amount of order k produced in period t for $t \in [b^k, e^k]$.

We also introduce some additional notation:
$D_{tl} = \sum_{k:t \le b^k, e^k \le l} D^k$ is the minimum amount that must be produced in the interval $[t,l]$,
$\Delta_t = D_{1t} - D_{1,t-1} = \sum_{k:e^k=t} D^k$ is the amount that must be delivered in the period t,
$\Delta_{tl} = \sum_{u=t}^{l} \Delta_u$ is the amount that must be delivered in the interval $[t,l]$,
$\Gamma_t = D_{tn} - D_{t+1,n} = \sum_{k:b^k=t} D^k$ is the amount that becomes available for production in the period t,
$\Gamma_{tl} = \sum_{u=t}^{l} \Gamma_u$ is the amount that becomes available for production in the interval $[t,l]$.

With the above variables a natural formulation for $LS-CC-TWP$ is:

$$\min \sum_t p_t' x_t + \sum_t h_t' s_t + \sum_t q_t y_t \tag{7}$$
$$s_{t-1} + x_t = \Delta_t + s_t \quad \text{for } t = 1, \dots, n \tag{8}$$
$$\sum_{u=b^k}^{e^k} z_u^k = D^k \quad \text{for } k = 1, \dots, K \tag{9}$$
$$\sum_{\{k:u \in [b^k,e^k]\}} z_u^k = x_u \quad \text{for } u = 1, \dots, n \tag{10}$$
$$x_u \le C y_u \quad \text{for } u = 1, \dots, n \tag{11}$$
$$s, x, z \ge 0, y \in \{0,1\}^n \tag{12}$$

where we suppose that $s_0 = 0$.

Now if one prefers to work in the (s,x,y) or (x,y) space, we look for a formulation in which the z_t^k variables have been eliminated.

Observation 1 *Applying Proposition 1 to (9)-(10) with $a^t = x_t$ for $t = 1, \dots, n$, there exist variables $z_t^k \ge 0$ satisfying (9)-(10) if and only if*

$$\sum_{u=t}^{l} x_u \ge D_{tl} \text{ for } 1 \le t \le l \le n. \tag{13}$$

This immediately gives a formulation in the (x,s,y) space, proposed by Brahimi [3].

Proposition 7 *Let $Q^{TWP} = \{(s,x,y,z)$ satisfying $(8)-(11), s,x \ge 0, y \in [0,1]^n, z \ge 0\}$. The projection of Q^{TWP} is given by*

$$\sum_{u=t}^{l} x_u \ge D_{tl} \text{ for } 1 \le t \le l \le n$$
$$x_u \le C y_u \quad \text{for } u = 1, \dots, n$$
$$s_{t-1} + x_t = \Delta_t + s_t \quad \text{for } t = 1, \dots, n$$
$$s \in \mathbb{R}^n, x \in \mathbb{R}_+^n, y \in [0,1]^n.$$

Observation 2 *i) It is possible to completely eliminate the s variables, or completely eliminate the x variables using the flow balance constraints (8). Specifically*

$$\sum_t p'_t x_t + \sum_t h'_t s_t = \sum_t p_t x_t + C_1 = \sum_t h_t s_t + C_2$$

where $p_t = p'_t + \sum_{u=t}^n h'_u$ *and* $h_t = h'_t + p'_t - p'_{t+1}$ *for all* t, *and* C_1, C_2 *are constants with* $C_1 = -\sum_t h'_t \Delta_{1t}$ *and* $C_2 = \sum_t p'_t \Delta_t$.

Observation 3 *As* $s_{t-1} = \sum_{u=1}^{t-1} x_u - \Delta_{1,t-1}$ *and* $\Delta_{1t} = D_{1t}$ *for all* t, *the inequality* $\sum_{u=1}^{l} x_u \geq D_{1l}$ *can be rewritten as*

$$s_{t-1} + \sum_{u=t}^{l} x_u \geq D_{1l} - D_{1,t-1} = \Delta_{tl}. \tag{14}$$

3.2 The Uncapacitated Case with General Costs: $LS - U - TWP$

In the uncapacitated case when C is large, the original formulation (8)-(12) can be tightened with the inequalities

$$z_u^k \leq D^k y_u \quad \text{for } k = 1, \ldots, K,\ u = 1, \ldots, n. \tag{15}$$

giving

$$\min \textstyle\sum_t p_t x_t + \sum_t q_t y_t \tag{16}$$
$$\textstyle\sum_{u=b^k}^{e^k} z_u^k = D^k \quad \text{for } k = 1, \ldots, K \tag{17}$$
$$z_u^k \leq D^k y_u \quad \text{for } k = 1, \ldots, K,\ u = 1, \ldots, n \tag{18}$$
$$x_u = \textstyle\sum_{\{k: u \in [b^k, e^k]\}} z_u^k \quad \text{for } u = 1, \ldots, n \tag{19}$$
$$s, x, z \geq 0, y \in \{0,1\}^n. \tag{20}$$

Unfortunately the linear programming relaxation of this extended formulation is not in general tight. However if the production costs $p \in \mathbb{R}^n$ are *unimodal*, $p_1 \geq p_2 \geq \ldots \geq p_r \leq p_{r+1} \leq \ldots \geq p_n$ for some r, more can be said.

Theorem 8 *With unimodal production costs, the linear programming relaxation of the problem (16)-(20) has an optimal solution with y integer.*

Proof: We indicate the main steps. Substituting $w_t^k = z_t^k / D^k$ and $c_t^k = p_t D^k$ for all $t \in [b^k, e^k]$ and all k, and introducing the variables w_t^k with a large weight $c_t^k = M$ for $t \notin [b^k, e^k]$, one obtains the facility location problem:

$$\min \textstyle\sum_{k,t} c_t^k w_t^k + \sum_t q_t y_t \tag{21}$$
$$\textstyle\sum_{u=1}^{n} w_u^k = 1 \quad \text{for } k = 1, \ldots, K \tag{22}$$
$$w_u^k \leq y_u \quad \text{for } k = 1, \ldots, K,\ u = 1, \ldots, n \tag{23}$$
$$w \geq 0, y \in \{0,1\}^n. \tag{24}$$

In Cornuéjols et al. [5], it is shown how by modifying the set of clients and the cost/profit matrix, it is possible to obtain a reformulation of (21)-(24) such that the linear programming relaxations of (21)-(24) and of the reformulation also have the same value, and the dual of the new linear program takes the form:

$$\max \textstyle\sum_{T\subseteq N} u_T \tag{25}$$
$$\textstyle\sum_{\{T:j\in T\}} u_T \leq q_j \text{ for } j=1,\ldots,n \tag{26}$$
$$0 \leq u_T \leq r_T \text{ for } T\subseteq N. \tag{27}$$

where the data r_T are integer if the original costs c_j^k are integral.

Now under the unimodal cost hypothesis on the production cost vector p, the sets $T\subseteq N$ with $r_T>0$ are all subintervals of $[1,n]$. It follows that the constraints (26) form a matrix with consecutive 1s. Thus this matrix is totally unimodular, and it follows from total dual integrality that the primal LP and thus the linear programming relaxations of (21)-(24) and (16)-(20) also have an optimal solution with y integer. □

For the case of general production and storage costs, the complexity of $LS-U-TWP$ is not known.

3.3 Constant Capacity and Wagner-Whitin Costs $WW-CC-TWP$

Definition 2 *A problem has* Wagner-Whitin costs *if $p'_t+h'_t\geq p'_{t+1}$, or equivalently $p_t\geq p_{t+1}$, or equivalently $h_t\geq 0$ for all t.*

Again using Observation 3, with Wagner-Whitin costs the objective function (7) can be rewritten as

$$\sum_t h_t s_t + \sum_t q_t y_t$$

with $h_t=p'_t+h'_t-p'_{t+1}\geq 0$ for all t.

Using (11) to replace x_u by its upper bound constraints Cy_u in (13) and (14), we obtain a relaxation

$$\min \textstyle\sum_t h_t s_t + \sum_t q_t y_t \tag{28}$$
$$s_{t-1} + C\textstyle\sum_{u=t}^{l} y_t \geq \Delta_{tl} \text{ for } 1\leq t\leq l\leq n \tag{29}$$
$$C\textstyle\sum_{u=t}^{l} y_t \geq D_{tl} \text{ for } 1\leq t\leq l\leq n \tag{30}$$
$$s\in\mathbb{R}^n_+, y\in\{0,1\}^n. \tag{31}$$

Let X^* be the set of (s,y) points satisfying (29)-(31).

Observation 4 *i) With $h\geq 0$, the relaxed problem $\min\{hs+gy:(s,y)\in X^*\}$ has an optimal solution in an extreme point of $\mathrm{conv}(X^*)$*
ii) In an extreme point of $\mathrm{conv}(X^*)$, $y\in\{0,1\}^n$ and

$$s_{t-1} = \max_{l=t,\ldots,n}(\Delta_{tl} - C\sum_{u=t}^{l} y_u)^+ \text{ for } t=2,\ldots,n.$$

iii) In an extreme point of $\mathrm{conv}(X^*)$, $0\leq \Delta_t + s_t - s_{t-1}\leq Cy_t$ for all t.

Proof: iii) follows from results concerning the constant capacity lot-sizing problem with Wagner-Whitin costs [14]. However it is easily verified as follows:
First we show that $\Delta_t + s_t - s_{t-1} \geq 0$. If $s_{t-1} = 0$, the result holds, so suppose that $s_{t-1} = \Delta_{tk} - CY_{tk}$ for some $k = t, \ldots, n$.
If $k > t$, then $s_t \geq \Delta_{t+1,k} - CY_{t+1,k}$, so $\Delta_t + s_t - s_{t-1} \geq \Delta_t + \Delta_{t+1,k} - CY_{t+1,k} - \Delta_{tk} + CY_{tk} = Cy_t \geq 0$.
If $k = t$, then $s_{t-1} = \Delta_t - Cy_t$, so $\Delta_t + s_t - s_{t-1} = \Delta_t + s_t - \Delta_t + Cy_t = s_t + Cy_t \geq 0$.

Now we show that $\Delta_t + s_t - s_{t-1} \leq Cy_t$.
If $s_t = 0$, as $s_{t-1} \geq \Delta_t - Cy_t$, we have that $\Delta_t + s_t - s_{t-1} \leq \Delta_t + 0 - \Delta_t + Cy_t = Cy_t$.
Otherwise $s_t = \Delta_{t+1,k} - CY_{t+1,k}$ for some $k = t+1, \ldots, n$. However $s_{t-1} \geq \Delta_{tk} - CY_{tk}$, so $\Delta_t + s_t - s_{t-1} \leq \Delta_t + \Delta_{t+1,k} - CY_{t+1,k} - \Delta_{tk} + CY_{tk} = Cy_t$. □

Setting $x_t = \Delta_t + s_t - s_{t-1}$, we see that (x, s, y) is feasible and optimal for the original problem.

Theorem 9 *i) The linear program*

$$\begin{gathered} \min hs + gy \\ (s, y) \in \text{conv}(X^*) \\ x_t = \Delta_t + s_t - s_{t-1} \text{ for all } t \end{gathered}$$

solves the lot-sizing problem $WW - CC - TWP$.
ii) conv(X^*) *is given by Theorem 5 with $\alpha_{tl} = \Delta_{tl}$ and $\beta_{tl} = D_{tl}$ for all t, l.*

Corollary 10 *For the uncapacitated problem $WW - U - TWP$, the formulation of* conv(X^*) *in the (s, y) space has $O(n^2)$ constraints*

$$\begin{gathered} s_{t-1} \geq \textstyle\sum_{\{k: b^k < t \leq e^k \leq l\}} D^k (1 - y_t - \ldots - y_{e^k}) \text{ for } 1 \leq t \leq l \leq n \\ \textstyle\sum_{u=b^k}^{e^k} y_u \geq 1 \text{ for } k = 1, \ldots, K \\ s \in \mathbb{R}^n_+, y \in [0,1]^n. \end{gathered}$$

Note that this linear program is more compact than the multicommodity formulation (16)-(20), though both solve the problem $WW - U - TWP$ as Wagner-Whitin costs are unimodal.

4 Indistinguishable Orders, Non-Inclusive Time Windows or Stock Upper Bounds

Here we discuss the indistinguishable order problem $LS - \{U, CC\} - TWP(I)$ and two other apparently distinct problems that are shown to be equivalent. The other two are the non-inclusive time window problem and the standard lot-sizing problem with stock upper bounds.

4.1 Non-Inclusive Time Windows: Description and Formulations

Definition 3 *A set of time windows $[b^k, e^k]_{k=1}^K$ are* non-inclusive *if there are no two time windows k and κ with $b^k < b^\kappa \leq e^\kappa < e^k$.*

Observation 5 *[3] i) A set of non-inclusive time windows can be ordered so that for all k either $b^k < b^{k+1}$ and $e^k \leq e^{k+1}$, or $b^k = b^{k+1}$ and $e^k < e^{k+1}$.*
ii) With non-inclusive time windows ordered as in i), there exists an optimal solution in which order k is produced before (or at the same time) as order $k+1$ for all k.
iii) In the uncapacitated case there exists an optimal solution in which each order k is produced in a single period.

A formulation (7)-(12) of this problem has already been given. We again eliminate the z_t^k variables. The difference now is that the bipartite graph, whose edges correspond to the z_t^k variables, is doubly convex.

Observation 6 *Applying Proposition 2 to (9)-(10) with $a^i = x_t$ for $t = 1, \dots, n$, there exist variables $z_t^k \geq 0$ satisfying (9)-(10) if and only if*

$$\textstyle\sum_{u=1}^{l} x_u \geq \sum_{\{k:e^k \leq l\}} D^k = \Delta_{1l} \text{ for } 1 \leq l \leq n$$

and

$$\textstyle\sum_{u=1}^{l} x_u \leq \sum_{\{k:b^k \leq l\}} D^k = \Gamma_{1l} \text{ for } 1 \leq l \leq n.$$

Now the formulation obtained in the original space is even simpler.

Proposition 11 *Let $Q^{TWP} = \{(s, x, y, z) \text{ satisfying } (8) - (11), s, x \geq 0, y \in [0, 1]^n, z \geq 0\}$. The projection of Q^{TWP} is given by*

$$\textstyle\sum_{u=1}^{l} x_u \geq \Delta_{1l} \text{ for } 1 \leq l \leq n \quad (32)$$

$$\textstyle\sum_{u=1}^{l} x_u \leq \Gamma_{1l} \text{ for } 1 \leq l \leq n \quad (33)$$

$$x_t \leq C y_t \text{ for } 1 \leq t \leq n \quad (34)$$

$$x \in \mathbb{R}_+^n, y \in [0, 1]^n. \quad (35)$$

This formulation was also by Brahimi [3] in a slightly different form.

4.2 Indistinguishable Orders

In this problem it is assumed that order k with time window $[b^k, e^k]$ means the arrival of D^k units of a standard input product in period b^k and then delivery of D^k units of a standard output product in period e^k. Note that the total arrival of the input product in t is Γ_t and the total demand for the output product in t is Δ_t. If s_t^2 indicates the stock of the input product (assumed to have zero storage cost), the problem can be formulated as:

$$\begin{aligned}
\min \textstyle\sum_{t=1}^{n} p_t' x_t + \sum_{t=1}^{n} h_t' s_t + \sum_{t=1}^{n} q_t y_t \\
s_{t-1}^2 + \Gamma_t = x_t + s_t^2 \text{ for } 1 \leq t \leq n \\
s_{t-1} + x_t = \Delta_t + s_t \text{ for } 1 \leq t \leq n \\
x_t \leq C y_t \text{ for } 1 \leq t \leq n \\
s^2, s, x \in \mathbb{R}_+^n, y \in \{0, 1\}^n.
\end{aligned}$$

The s_t^2 variables can be eliminated using the equation $s_t^2 = \Gamma_{1t} - \sum_{u=1}^{t} x_u$ and $s_t^2 \geq 0$, and we modify the objective function as in Observation 2 to obtain again the formulation (**??**)-(35). This equivalence of the non-inclusive time window problem and of the indistinguishable order problem, as well as a specific procedure to modify the time windows of the latter till they are non-inclusive has been established in Brahimi [3].

4.3 Lot-Sizing with Upper Bounds on Stock

Here we consider a standard lot-sizing problem with demands $(d_1, \ldots, d_n)$ and stock upper bounds $(u_1, \ldots, u_n)$, denoted $LS - \{U, CC\} - SUB$. A standard formulation is

$$\begin{gathered}\min \textstyle\sum_{t=1}^n p_t x_t + \sum_{t=1}^n h'_t s_t + \sum_{t=1}^n q_t y_t \\ s_{t-1} + x_t = d_t + s_t \text{ for } 1 \leq t \leq n \\ x_t \leq C y_t \text{ for } 1 \leq t \leq n \\ s_t \leq u_t \text{ for } 1 \leq t \leq n \\ s, x \in \mathbb{R}^n_+, y \in \{0,1\}^n.\end{gathered}$$

First we make the standard assumption that the demand data has been preprocessed so that $d_t \leq C$ for all t. In addition, as $s_{t-1} \leq d_t + u_t$, we can assume wlog that $u_{t-1} \leq d_t + u_t$ for all t. Taking $\Delta_t = d_t$ for all t, the demand constraints can be rewritten as $\sum_{u=1}^t x_u \geq \Delta_{1t}$. On the other hand the constraint $s_t \leq u_t$ can be written as $\sum_{u=1}^t x_u - \Delta_{1t} \leq u_t$, so we obtain

$$\sum_{u=1}^t x_u \leq \Delta_{1t} + u_t \text{ for } 1 \leq t \leq n.$$

Letting $\Gamma_{1t} = (\Delta_{1t} + u_t)$, Γ_{1t} is nondecreasing from our assumption that $u_{t-1} \leq \Delta_t + u_t$, so these constraints are precisely

$$\sum_{u=1}^t x_u \leq \Gamma_{1t} \text{ for } 1 \leq t \leq n,$$

and we again obtain the formulation (32)-(35).

We have seen that the three problems are identical. The crucial link is that all three formulations just depend on the arrival and demand vectors $\Gamma \in \mathbb{R}^n$ and $\Delta \in \mathbb{R}^n$ satisfying

$$\begin{gathered}\Gamma_{1t} \geq \Delta_{1t} \text{ for } 1 \leq t \leq n-1 \\ \Gamma_{1n} = \Delta_{1n} \\ \Gamma, \Delta \in \mathbb{R}^n_+.\end{gathered}$$

To complete the picture, we also show how to compute the "orders" with non-inclusive time windows from these two vectors.

Algorithm to Compute the Orders
Initialization Set $L_t = \Gamma_t, R_t = \Delta_t$ for all t. $k = 1$
While $L, R \neq 0$
  Set $\sigma = \min\{t : L_t > 0\}, \tau = \min\{t : R_t > 0\}$.
  Set $D^k = \min\{L_\sigma, R_\tau\}$, $b^k = \sigma, e^k = \tau$.
  $L_\sigma \leftarrow L_\sigma - D^k, R_\tau \leftarrow R_\tau - D^k$
  $k \leftarrow k + 1$
end-While

Clearly there are at most $2n - 1$ orders and they are uniquely defined.

4.4 Algorithms for the Indistinguishable Order Problem

Brahimi [3] gives an $O(n^4)$ DP algorithm for the uncapacitated problem $LS-U-TWP(I)$. Here we give an alternative algorithm, and then use the algorithm to obtain a tight extended formulation.

4.4.1 A Dynamic Programming Algorithm and Formulation for $LS-U-TWP(I)$

Here we take the objective function in the form

$$\min \sum_t p_t x_t + \sum_t q_t y_t,$$

and we assume that the orders $k = 1, \dots, K$ are numbered from earliest to latest.

Using iii) of Observation 5, we define the following two value functions:

Let $H(t, k)$ be the value of an optimal solution for periods $1, \dots, t$ in which the demands $D^1, \dots, D^k$ are produced in or before period t. Note that $H(t, k) = \infty$ if $b^k > t$.

Let $G(t, k)$ be the value of an optimal solution for periods $1, \dots, t$ in which the demands $D^1, \dots, D^{k-1}$ are produced in or before period t and D^k is produced in t. Note that $G(t, k) = \infty$ if $e^k < t$ or $b^k > t$.

The recursion one obtains is

$$\begin{aligned} H(t,k) &= \min[H(t-1,k),\ G(t,k)] \text{ for all } k,t \text{ with } b^k \leq t \\ G(t,k) &= \min[H(t-1,k-1) + q_t + p_t D^k,\ G(t,k-1) + p_t D^k] \text{ for all } k,t \text{ with } t \in [b^k, e^k], \end{aligned}$$

where the first equation uses the fact that in an optimal solution of value $H(t, k)$ either k is produced before period t or k is produced in period t, and the second the fact that in an optimal solution of value $G(t, k)$ either order $k-1$ is produced before t and k in t, or both orders $k-1$ and k are produced in t.

Obviously, as $K \leq 2n - 1$, the recursion provides an $O(n^2)$ algorithm for the problem.

We now use the approach of Eppen and Martin [7] to get a tight extended formulation. Specifically the recursion suggests the linear program

$$\begin{gathered} \max H(n, K) \\ H(t,k) - H(t-1,k) \leq 0 \text{ for all } k,t \text{ with } b^k \leq t \\ H(t,k) - G(t,k) \leq 0 \text{ for all } k,t \text{ with } t \in [b^k, e^k] \\ G(t,k) - G(t,k-1) \leq p_t D^k \ \text{ for all } k,t \text{ with } t \in [b^k, e^k] \\ G(t,k) - H(t-1,k-1) \leq q_t + p_t D^k \text{ for all } k,t \text{ with } t \in [b^k, e^k]. \end{gathered}$$

Let the dual variables be $v_{tk}, w_{tk}, x_{tk}, z_{tk}$ respectively.

The dual of this linear program is then

$$\min \textstyle\sum_{t,k} [(q_t + p_t D^k) z_{tk} + p_t D^k x_{tk}] \tag{36}$$

$$z_{tk} + x_{tk} - x_{t,k+1} - w_{t,k} = 0 \text{ for all } k,t \text{ with } b^k \leq t \tag{37}$$

$$v_{tk} - v_{t+1,k} + w_{tk} - z_{t+1,k+1} = 0 \text{ for all } k,t \text{ with } t \in [b^k, e^k] \tag{38}$$

$$v_{n,K} + w_{n,K} = 1 \tag{39}$$

$$v, w, x, z \geq 0. \tag{40}$$

This can be seen as a shortest path problem. An interpretation of the variables is as follows:
$z_{tk} = 1$ if order k is the first order produced in period t (i.e order $k-1$ is produced earlier)
$x_{tk} = 1$ if order k is produced in t, but also order $k-1$ at least
$w_{tk} = 1$ if order k is the last order produced is period t
$v_{t,k} = 1$ if the last order produced in or before t was order k.

To obtain a complete formulation, we just need to add:

$$1 \geq y_t \geq \sum_k z_{tk} \text{ for all } t \tag{41}$$
$$x_t = \sum_k D^k(z_{tk} + x_{tk}) \text{ for all } t. \tag{42}$$

Theorem 12 *Let X^U be the set of feasible solutions of (32)-(35) of the uncapacitated problem $LS-U-TWP(W)$. A tight extended formulation for* $\text{conv}(X^U)$ *is given by the polyhedron (37)-(42).*

Both the DP algorithm and the extended formulation can be seen as generalizations of results of Ortega [12] for the problem with a perishable good and time windows $[t, t+\tau]$ for all t.

4.4.2 A Dynamic Programming Algorithm and Formulation for $LS-CC-SUB$

The standard approach for the constant capacity lot-sizing problem $LS-CC$ is to calculate the optimal cost of each $[t,l]$ regeneration interval, and then solve an $O(n^2)$ shortest path problem on an acyclic digraph with $n+1$ nodes and $O(n^2)$ arcs to find the optimal sequence of regeneration intervals. There a regeneration interval is an interval in which $s_{t-1} = s_l = 0$, but $s_u > 0$ for all $t \leq u < l$.

As observed explicitly by Atamtürk and Küçükyavuz [1], see also Love [10], this approach can be generalized for the problem with upper bounds on stocks.

Definition 4 *For $LS-\{U,CC\}-SUB$, the interval $[t,l]$ is an SUB*-regeneration interval *if $s_{t-1} \in \{0, u_{t-1}\}, s_l \in \{0, u_l\}$ and $0 < s_\tau < u_\tau$ for all $t \leq \tau < l$.*

If we can calculate the optimal cost of a $[t,l]$ SUB-regeneration interval in polynomial time, it is easy to see that we obtain a polynomial algorithm by constructing an appropriate shortest path problem with twice as many nodes and four times as many arcs as in the standard shortest path problem for $LS-CC$. In the uncapacitated case, Love [10] has given an $O(n^3)$ algorithm.

We now consider the problem of finding a minimum cost $[t,l]$ SUB-regeneration interval. Again we assume that $u_{t-1} \leq d_t + u_t$ for all t. There are four types of interval depending on whether the entering and leaving stocks are at 0 or their upper bound. We treat only one of the four cases, that with $s_{t-1} = u_{t-1}$ and $s_l = 0$. The other cases are similar

The total production in the interval is $d_{tl} - u_{t-1} + 0 = C\eta_{tl} + \rho_{tl}$ where $0 \leq \rho_{tl} < C$. Using the standard properties that, once the $y \in \{0,1\}^n$ variables have been fixed, the basic variables in the resulting flow problem must form an acyclic graph, we obtain

Observation 7 *There exists an optimal solution for the $[t,l]$ SUB-regeneration interval in which one produces η_{tl} times at full capacity C, and one produces the remaining quantity ρ_{tl} once.*

Now we can formulate the problem as an integer program using the variables:
$z_\tau = 1$ if $x_\tau = C$, and $z_\tau = 0$ otherwise
$w_\tau = 1$ if $x_\tau = \rho_{tl}$, and $w_\tau = 0$ otherwise.

The formulation, for the case where $\rho_{tl} > 0$, is then:

$$\min \textstyle\sum_{u=t}^{l} p_u(Cz_u + \rho_{tl}w_u) + \sum_{u=t}^{l} q_u(z_u + w_u) \tag{43}$$

$$s_k = \textstyle\sum_{\tau=t}^{k} Cz_\tau + \sum_{\tau=t}^{k} \rho_{tl}w_\tau - D_{tk} + u_{t-1} \text{ for } k = t, \dots, l-1 \tag{44}$$

$$0 \le s_k \le u_k \text{ for } k = t, \dots, l-1 \tag{45}$$

$$\textstyle\sum_{\tau=t}^{l} w_\tau = 1 \tag{46}$$

$$\textstyle\sum_{\tau=t}^{l} z_\tau = \eta_{tl} \tag{47}$$

$$z_k + w_k \le 1 \text{ for } k = t, \dots, l \tag{48}$$

$$z, w \in \{0, 1\} \tag{49}$$

This problem can be solved either by dynamic programming in $O((l-t+1)^2)$, or by linear programming. To explain the linear program, consider the constraint $s_k \le u_k$ rewritten in the form

$$\sum_{\tau=t}^{k} Cz_\tau + \sum_{\tau=t}^{k} \rho_{tl}w_\tau \le b_k$$

where $b_k = u_k + d_{tk} - u_{t-1} \ge 0$. Let $b_k = Cc_k + e_k$ with $0 \le e_k < C$.

Observation 8 *If $e_k \le \rho_{tl}$, the constraint (45) can be replaced by*

$$\sum_{\tau=t}^{k} z_\tau \le c_k$$

while if $e_k > \rho_{tl}$, the constraint (45) can be replaced by

$$\sum_{\tau=t}^{k} (z_\tau + w_\tau) \le c_k.$$

The constraints $s_k \ge 0$ also can be replaced by constraints having the same form, but with $\ge$ constraints, see [13].

After replacing these constraints, our IP for the $[t, l]$ regeneration interval has the form

$$\min \textstyle\sum_{u=t}^{l} p_u(Cz_u + \rho_{tl}w_u) + \sum_{u=t}^{l} q_u(z_u + w_u) \tag{50}$$

$$\begin{aligned} &\textstyle\sum_{\tau=t}^{k} z_\tau \le c_k \text{ for } k = t, \dots, l-1, \text{ or} \\ &\textstyle\sum_{\tau=t}^{k} (z_\tau + w_\tau) \le c_k \text{ for } k = t, \dots, l-1 \end{aligned} \tag{51}$$

$$\begin{aligned} &\textstyle\sum_{\tau=t}^{k} z_\tau \ge \tilde{c}_k \text{ for } k = t, \dots, l-1, \text{ or} \\ &\textstyle\sum_{\tau=t}^{k} (z_\tau + w_\tau) \ge \tilde{c}_k \text{ for } k = t, \dots, l-1 \end{aligned} \tag{52}$$

$$\textstyle\sum_{\tau=t}^{l} w_\tau = 1 \tag{53}$$

$$\textstyle\sum_{\tau=t}^{l} z_\tau = \eta_{tl} \tag{54}$$

$$z_k + w_k \le 1 \text{ for } k = t, \dots, l \tag{55}$$

$$z, w \in \{0, 1\}^{l-t+1}. \tag{56}$$

Observation 9 *The constraint matrix arising from the rows (51)-(55) is totally unimodular, so the IP can be solved as a linear program (or network flow), and the problem is polynomial.*

Using a standard technique, see [13], this $O(n) \times O(n)$ linear program can be embedded in a linear programming representation of the shortest path problem giving an $O(n^3) \times O(n^3)$ tight extended formulation.

Several families of valid inequalities for the problem with stock upper bounds have been proposed in Atamtürk and Küçükyavuz [1], and a complete formulation for the case with Wagner-Whitin costs was given in Pochet and Wolsey [14].

5 Delivery Time Windows

Here a time window $[b^k, e^k]$ for an order k of size D^k indicates that the client requires delivery at earliest in b^k and at latest in e^k. We denote the problem with general costs by by $LS - \{U, CC\} - TWD$.

5.1 MIP Formulations

To obtain a formulation for $LS - CC - TWD$, let x, s, y be the same as earlier, and let v_t^k be the amount of order k delivered to the client in period t with $t \in [b^k, e^k]$.

This leads to the formulation

$$\min \textstyle\sum_t p'_t x_t + \sum_t h'_t s_t + \sum_t q_t y_t \tag{57}$$
$$s_{t-1} + x_t = \textstyle\sum_{k: b^k \leq t \leq e^k} v_t^k + s_t \quad \text{for } t = 1, \ldots, n \tag{58}$$
$$\textstyle\sum_{u=b^k}^{e^k} v_u^k = d^k \quad \text{for } k = 1, \ldots, K \tag{59}$$
$$x_u \leq C y_u \quad \text{for } u = 1, \ldots, n \tag{60}$$
$$s, x, z \geq 0, y \in \{0,1\}^n. \tag{61}$$

Note that for this problem, $D_{tl} = \sum_{k: t \leq b^k \leq e^k \leq l} D^k$ is the amount that must be delivered in the interval $[t, l]$.

Observation 10 *Using Proposition 1 applied to the constraints (58)-(59) with $a^t = s_{t-1} + x_t - s_t$, there exists $v_t^k \geq 0$ satisfying (58)-(59) if and only if*

$$s_{t-1} + \sum_{u=t}^{l} x_u - s_l \geq D_{tl} \text{ for } 1 \leq t \leq l \leq n.$$

Thus we obtain an equivalent formulation in the (x, s, y) space

$$\begin{gathered}
\min \textstyle\sum_t p'_t x_t + \sum_t h'_t s_t + \sum_t q_t y_t \\
\textstyle\sum_{u=1}^{n} x_u = \delta_{1n} + s_n \\
s_{t-1} + \textstyle\sum_{u=t}^{l} x_u \geq D_{tl} + s_l \text{ for } 1 \leq t \leq l \leq n \\
x_t \leq C y_t \text{ for } t = 1, \ldots, n \\
s, x \geq 0, y \in \{0,1\}^n.
\end{gathered}$$

5.2 The Constant Capacity Case with Wagner-Whitin Costs

Here we make s slightly stronger Wagner-Whitin cost assumption, namely that the production costs are constant and the storage costs non-negative, so the objective function can be taken in the form $\sum_t h_t s_t + \sum_t q_t y_t$ with $h_t \geq 0$ for all t. Now as there are no production costs, we can replace x_t by its upper bound Cy_t leading to the relaxation:

$$\min \textstyle\sum_t h_t s_t + \sum_t q_t y_t \tag{62}$$
$$s_{t-1} + C \textstyle\sum_{u=t}^{l} y_t \geq D_{tl} + s_l \text{ for } 1 \leq t \leq l \leq n \tag{63}$$
$$s \in \mathbb{R}_+^n, y \in \{0,1\}^n \tag{64}$$

having the same optimal value as the original problem (57)-(61) when $h \geq 0$.

Observation 11 *i) In a stock-minimal solution,*

$$s_{t-1} = \max_{l:l \geq t}(D_{tl} - C\sum_{u=t}^{l} y_t + s_l)^+, \text{ and}$$

ii) if $s_{t-1} = D_{tl} - C\sum_{u=t}^{l} y_t + s_l > 0$ *with* l *maximal, then* $s_l = 0$.

Proof: If not, then $s_l = D_{l+1,k} - \sum_{u=l+1}^{k} y_u + s_k$ for some $k > l$, and we have that $s_{t-1} = D_{tl} - C\sum_{u=t}^{l} y_t + D_{l+1,k} - \sum_{u=l+1}^{k} y_u + s_k$. However as $D_{tk} \geq D_{tl} + d_{l+1,k}$, and $s_{t-1} \geq D_{tk} - C\sum_{u=t}^{k} y_t + s_k$, it follows that $s_{t-1} \geq D_{tk} - C\sum_{u=t}^{k} y_t + s_k \geq D_{tl} - C\sum_{u=t}^{l} y_t + D_{l+1,k} - \sum_{u=l+1}^{k} y_u + s_k = s_{t-1}$, contradicting the maximality of l. □

It follows that the relaxation

$$\min \textstyle\sum_t h_t s_t + \sum_t q_t y_t \tag{65}$$
$$s_{t-1} + C \textstyle\sum_{u=t}^{l} y_t \geq D_{tl} \text{ for } 1 \leq t \leq l \leq n \tag{66}$$
$$s \in \mathbb{R}_+^n, y \in \{0,1\}^n. \tag{67}$$

also has the same optimal value as (57)-(61) when $h \geq 0$. Let Y^* be the feasible region (66)-(67).

Theorem 13 *i) The linear program*

$$\min hs + gy \tag{68}$$
$$(s, y) \in \text{conv}(Y^*) \tag{69}$$
$$(x, s, y, v) \text{ satisfy } (52) - (54) \tag{70}$$

solves the lot-sizing problem $WW - CC - TWD$ *(with* $p = 0, h \geq 0$*).*
ii) $\text{conv}(Y^*)$ *is given by Theorem 5 with* $\alpha_{tl} = D_{tl}$ *and* $\beta_{tl} = 0$ *for all* t, l.

Corollary 14 *For the uncapacitated problem* $WW - U - TWD$ *(with* $p = 0, h \geq 0$*), the formulation of* $\text{conv}(Y^*)$ *in the* (s, y) *space has* $O(n^2)$ *constraints:*

$$s_{t-1} \geq \textstyle\sum_{\{k:t \leq b^k \leq e^k \leq l\}} D^k(1 - y_t - \ldots - y_{e^k}) \text{ for } 1 \leq t \leq l \leq n$$
$$s \in \mathbb{R}_+^n, y \in [0,1]^n.$$

5.3 Indistinguishable Orders or Non-inclusive Time Windows

5.3.1 A Dynamic Programming Algorithm for $LS - U - TWD(I)$

Here we take the objective function in the form

$$\min \sum_t p'_t x_t + \sum_t h'_t s_t + \sum_t q_t y_t.$$

Let $H(t,k)$ be the value of an optimal solution for periods $1,\dots,t$ in which the orders $D^1,\dots,D^k$ are produced in or before period t.
Let $G(t,k)$ be the value of an optimal solution for periods $1,\dots,t$ in which demands $D^1,\dots,D^k$ are produced in or before period t and d^k is produced in t. $G(t,k)=\infty$ if $e^k < t$.

The recursion is

$$\begin{aligned} H(t,k) &= \min[H(t-1,k) + h'_{t-1} \sum_{\{\kappa:\kappa\le k, b^\kappa \ge t\}} D^\kappa,\ G(t,k)] \text{ for all } k,t \\ G(t,k) &= \min[H(t-1,k-1) + h'_{t-1} \sum_{\{\kappa:\kappa\le k-1, b^\kappa \ge t\}} D^\kappa + q_t + p'_t D^k,\ G(t,k-1) + p'_t D^k] \\ & \quad \text{for all } k,t \text{ with } t \le e^k. \end{aligned}$$

where the first equation treats the cases where order k is produced before period t or in period t, and the second the same cases but for order $k-1$. Obviously this provides an $O(n^2)$ algorithm for the problem.

Again an extended formulation can be obtained with the approach of Eppen and Martin [7] just as in Section 3.

6 Concluding Remarks

Several open questions remain, in particular the question whether there is a polynomial algorithm for $LS - U - TWP$. At present we have no clear ideas on how to approach this problem as the ordering of the demands implicit in most other polynomially solvable lot-sizing problems has been lost.

For $LS - U - TWP$, an obvious practical question is whether to use the Wagner-Whitin relaxation $WW - U - TWP$ or the slightly less compact indistinguishable order relaxation $LS - U - TWP(I)$. With constant capacities approximate versions of $WW - CC - TWP$ can be used, but the $O(n^3) \times O(n^3)$ version of $LS - CC - TWP(I)$ based on stock upper bounds appears to be impractically large.

Another immediate question concerns backlogging. Here the extension of the Wagner-Whitin uncapacitated case appears to be straightforward for production time windows using the extended formulation for $WW - U - B$ from [14], but for delivery time windows it is not clear, even though Lee et al. give an $O(n^2)$ algorithm for the problem.

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