# Balance of Power and Divergence of Policies in a Model of Electoral Competition*

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**Abstract**

Two incumbent parties choose their platforms in a unidimensional policy space while facing a credible threat of an entry by the third party. Relative electoral support is the predominant objective of each party, and the third party enters only if it can displace one of the incumbents. In an equilibrium the two incumbents choose to prevent the entry and achieve the balance of power, i.e., splitting the electorate equally. The incumbents' positions might diverge more as compared to a system in which the parties seek to solely maximize the voters' support. Therefore, rank preoccupation under the threat of entry might contribute to more polarized political platforms of the two leading parties.

**Keywords:** Incumbent parties, threat of entry, entry-deterrence, rank concerns, balance of power.

**JEL Classification Numbers:** C62, C72, D72.

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# 1 Introduction

In many political and economic environments, success is measured by a relative, rather than absolute, standard. In situations where rank matters, potential entrants are often required to surpass the rank of at least one incumbent to be deemed successful. This phenomenon is especially prevalent in an electoral context. In the first round of presidential elections in many countries, including e.g. France, Russia, Poland, Indonesia, and Argentina, candidates must guarantee themselves at least the second place in order to advance to the next round. In Britain and Canada, the second largest party has the status of "official opposition" which entitles it to certain perks and privileges. Louisiana holds an "open primary" for governor,[1] with the top two candidates facing each other in a general election thereafter.

Under these circumstances, incumbents must consider not only their position relative to the current competition, but also the possibility of displacement by an entrant. Should an incumbent take steps to improve its position at the expense of the existing competition, the field may become open to a new competitor. In such an environment, the "balance of power" among incumbents must be maintained in such a way that no established party can improve its position with respect to a current competitor without running the risk of being displaced altogether by a new entrant.

In this paper we consider a model of spatial competition with two incumbents and a potential entrant. Competing parties choose their positions in the unidimensional policy space, and each voter supports the party that proposes a platform closest to her ideal point.[2] Similar to Palfrey (1984), the incumbents in this model behave as Nash players with respect to one another, but as Stackelberg leaders with respect to the entrant. In other words, the incumbents choose their platforms simultaneously, but in full anticipation of the third party's location along the spectrum of political issues. For entry to be deemed successful, the third party must garner more votes than at least one of the established parties; otherwise, the potential entrant stays out of the race altogether. Thus, becoming one of the top two is crucial for a party.

The focus of the paper is to show existence and to characterize an equi-

[1] All candidates, regardless of party run against each other followed by a run-off election in case none of the winners in the first round got more than 50% of the votes.

[2] One could view this voting behavior as a desire to associate oneself with a certain platform, defining one's identity. This motif might prevail in large elections, in which pivotalness of an individual voter is minuscule. See Green, Palmquist, and Schickler (2002) for a related overview.

Moreover, there are reasons to believe that sincere voting is a good "positive" assumption in the view of the recent empirical findings, see, for example, Alvarez and Nagler (2000).

librium of this game, referred to as $\mathcal{R}$-equilibrium. This allows us to provide an interesting comparison with the outcomes in different environments, in which absolute support of voters, rather than the relative performance, is the main indicator of success.

We find that a third party entry is not sustainable in an $\mathcal{R}$-equilibrium, i.e., the potential entrant can never become one of the top two. This immediately implies that the set of $\mathcal{R}$-equilibria is a subset of incumbent strategies which prevent entry by a third party, a notion introduced by Greenberg and Shepsle (1987). This is true even though in our setting the incumbents are forward looking; in particular, each of them might want to trigger an entry to displace the other incumbent from being among the top two, thereby improving her own rank or increasing the share of own supporters, while preserving its relative standing.

Greenberg and Shepsle (1987) pointed out that a profile of entry-preventing incumbents' strategies, which we refer to as $\mathcal{D}$-strategies, may fail to exist. We derive quite general conditions which yield existence and uniqueness of $\mathcal{D}$-strategies. We go on to demonstrate that $\mathcal{R}$-equilibrium exists if, and only if incumbents achieve a balance of power whereby the electorate is shared equally among them.

The positions of the incumbents in $\mathcal{R}$-equilibrium diverge and, interestingly, more so than under electoral systems based solely on the size of electoral support. Palfrey (1984) and Weber (1992) developed equilibrium notions (*limit equilibrium* and *hierarchical equilibrium*, respectively) for models of electoral competition where two vote-maximizing established parties are challenged by a new third party. The incumbents in such a system choose their positions along the political spectrum, taking as given the entrant's vote-maximizing response. The third party always enters the race in these models, and we find that the two established parties are generally separated by a greater percentage of the constituency when rank matters. Whether absolute or relative performance matters, the established parties will always have an incentive to position themselves closer to their current rival in an effort to increase voter support. However, the threat of third party entry forces incumbents to choose sharply differentiated positions in the issue space, otherwise each of the established parties risks being "squeezed" between its current rival and a new party. Our results suggest that the degree to which the established parties separate themselves from one another is greater under rank-related objectives than under a system of sheer vote maximization. Therefore, motivation of the politicians, driven either by rewards built into the electoral system or by traditional values imprinted in personal preferences might reflect in the divergence of chosen platforms. In particular, our results demonstrate that the same fundamental preferences of the voters, or distribution of their ideal points, can translate into different political platforms depending on the electoral system.

In their recent contribution, Faulí-Oller, Ok, and Ortuño-Ortín (2003) show that ability to commit (delegate) by the two competing parties might lead to polarization of the proposed platforms, which holds under rather general conditions, thus, providing another way to understand the gap between Downsian predictions and the observed phenomena. The threat of the entry is also known to generate diversion of the two established parties, and to account for the related contributions in spatial competition we will refer the reader to an overview by Osborne (1995). In contrast to the existing literature, however, we describe scenarios in which neither of the incumbents can commit to a chosen policy, thus, expecting to react to a potential entry by a third party. Moreover, we consider environments in which each of the competitors strives to be either no.1 or no.2, i.e., rank objectives are predominant.

The paper is organized as follows. The following section describes the spatial competition game. Section 3 demonstrates that an equilibrium of the game has to be entry-deterrent. In Section 4 we consider the effects of polarization of the electorate on equilibrium separation between the platforms and compare the platform divergence under different electoral systems. Section 5 contains concluding remarks.

# 2 The Model

We consider a model of electoral competition with two incumbent parties, 1 and 2, and one potential entrant, $e$. The competing parties choose their positions in the issue space $I = [0, 1]$. After the two incumbents chose their positions, the entrant can either stay out, the option denoted by $N$, or enter the race and choose a platform in $I$.

Payoffs to the parties are based on voters' support. Each voter has symmetric single peaked preferences with the most preferred alternative, or *ideal point*, in $I$. Voters' ideal points are described by a cumulative distribution function $F$, defined over the issue space $I$, with $F(0) = 0$ and $F(1) = 1$.

Given positions of the competing parties, each voter supports the party whose position is closest to her ideal point, and she randomly picks one of closest by, in case there are many. No abstention is allowed – each voter identifies herself with a party. Let $x = (x_1, x_2)$ be a pair of positions (not necessarily different) chosen by the established parties 1 and 2 and assume that $x_1 \leq x_2$, let $X = (x, x_e)$ be the choices made by all three parties, where the entrant can either choose a platform or decide not to enter: $x_e \in I \cup \{N\}$.

All three parties have lexicographic preferences. Each of them first considers the rank and then the fraction of votes they garner. Given the choice $X$ of the parties, we denote by $r_1(X)$, $r_2(X)$, $r_e(X)$ their corresponding ranks. Rank $r_i(X)$ can obtain one of the six values:
$A1$ the sole possession of the first place,
$A2$ sharing the first place with one of the other parties,
$B1$ the sole possession of the second place,
$A3$ sharing the first place with two other parties,
$B2$ sharing the second place with one of the other parties,
$C1$ the sole possession of the third place.

We assume the following natural preferences for all parties:

$$A1 \succ A2 \succ B1 \succ A3 \succ B2 \succ C1.$$

If the entrant decides to stay out, we say that $r_e(X) = N$, and her ranking reflects the desire to enter only if she can become at least the second (attaining rank $B1$) in the electoral competition.[3] That is, the preferences of the entrant are given by

$$A1 \succ A2 \succ B1 \succ N \succ A3 \succ B2 \succ C1.$$

If two different outcomes generate the same rank, a party prefers the one that yields a higher vote share. This share is also determined by the positions of the three parties.

Suppose first that party $e$ does not enter. If the incumbents choose the same position in the issue space, each party is supported by one half of the electorate. If their positions are different, $x_1 < x_2$, then party 1 is supported by all those voters whose ideal points are to the left of the middle point $\frac{x_1+x_2}{2}$, whereas party 2 is supported by voters whose ideal points are to the right of $\frac{x_1+x_2}{2}$. That is, the support of party $i = 1, 2$ denoted by $s_i(X)$ is determined by:

$$s_1(X) = F(\frac{x_1 + x_2}{2}),$$
$$s_2(X) = 1 - F(\frac{x_1 + x_2}{2}).$$

Suppose now that party $e$ enters and chooses position $x_e \in I$ in the issue space. If all three parties choose the same position, then each is supported by one-third of the electorate, i.e., $s_1(X) = s_2(X) = s_e(X) = \frac{1}{3}$. If the

[3] As mentioned in the introduction, in many situations "winning" is associated with being one of the top two and under sufficient costs of entry, it will be prevented in case the entrant is not expecting to "win."

incumbents choose the same position but the entrant locates herself at a different point, say $x_1 = x_2 < x_e$, then the support of each incumbent is given by $F(\frac{x_1+x_e}{2})/2$, whereas party $e$ is supported by $1 - F(\frac{x_1+x_e}{2})$ voters. If all parties choose different positions, say $x_1 < x_2 < x_e$, then the support of parties $1, 2$ and $e$ is given by $F(\frac{x_1+x_2}{2})$, $F(\frac{x_2+x_e}{2}) - F(\frac{x_1+x_2}{2})$ and $1 - F(\frac{x_2+x_e}{2})$, respectively. That is, if party $i$ has rivals both to the left and to the right,[4] then its support covers the interval between two points: one equidistant from $x_i$ and its opponent from the left side, and the other equidistant from $x_i$ and its opponent from the right. If party has no rivals to its left, then its support covers the interval between 0 and the point which is equidistant from $x_i$ and its closest opponent from the right side. Similarly, if party $i$ has no rivals to its right,[5] then its support covers the interval between the point which is equidistant from $x_i$ and its closest opponent from the left side and the right endpoint of the issue space.

Unfortunately, subgame perfect equilibria of the game described above do not always exist. Let us first examine the reasons for the non-existence and then modify the game to avoid the problem.

Recall party $e$ enters only when it can win higher support than at least one of the incumbents. To describe these cases, fix the positions of the incumbent parties, $x = (x_1, x_2)$, and consider the following sets:

$$D_1(x) = \{x_e \in I | s_e(x, x_e) > \max[s_1(x, x_e), s_2(x, x_e)]\}$$

is the set of positions for the entrant that guarantee the entrant the sole possession of the first place,

$$D_{12}(x) = \{x_e \in I | s_e(x, x_e) = \max[s_1(x, x_e), s_2(x, x_e)] > \min[s_1(x, x_e), s_2(x, x_e)]\}$$

is the set of positions that yield the entrant the share of the first place with one of the incumbents, and

$$D_2(x) = \{x_e \in I | \max[s_1(x, x_e), s_2(x, x_e)] > s_e(x, x_e) > \min[s_1(x, x_e), s_2(x, x_e)]\}$$

is the set of positions where the entrant hold the second place. Also, let

$$D(x) = D_1(x) \cup D_{12}(x) \cup D_2(x) = \{x_e \in I | s_e(x, x_e) > \min[s_1^E(x, x_e), s_2^E(x, x_e)]\}.$$

The entrant decision process goes as follows: if the set $D(x)$ is empty, party $e$ does not enter. Otherwise, it considers the sets $D_1(x)$, $D_{12}(x)$, $D_2(x)$ (in

[4] Eaton and Lipsey (1975) call this location *interior*.

[5] A position which has no rivals either to the left or to the right is called *peripheral* in Eaton and Lipsey (1975).

this order) and makes its vote-maximizing choice over the first nonempty set in this sequence.

By using the arguments of Palfrey (1984), it is easy to see that the best response of the entrant over the sets $D_1(x)$ or $D_2(x)$ may fail to exist.[6] In this case we adopt the procedure offered by Palfrey (1984) and Weber (1992) and consider the average of the incumbents' payoffs over the set of "almost best" responses of the entrant. Specifically, for each incumbent $i$ and each positive $\varepsilon$ we determine the average of player $i$'s payoffs over the set of "$\varepsilon$-best" responses of the entrant in a subset $Y(x) \in \{D_1(x), D_2(x)\}$ of the set $D(x)$ and consider its limit when $\varepsilon$ approaches zero. For each pair of incumbent choices $x \in I^2$ and $\varepsilon \geq 0$, denote by $B_\varepsilon^Y(x)$ the set of $\varepsilon$-best responses over $Y \subset D(x)$. That is,

$$B_\varepsilon^Y(x) = \{x_e \in Y(x) | s_e(x, x_e) \geq s_e(x, y) - \varepsilon \ \ for \ \ all \ \ y \in Y(x)\}.$$

Let a pair of incumbents' strategies $x = (x_1, x_2)$ be such that the set $D(x)$ is nonempty. One can immediately observe that for $Y(x) \in \{D_1(x), D_2(x)\}$, while the set of best responses $B_0^Y(x)$ might be empty, the set $B_\varepsilon^Y(x)$ is nonempty for every strictly positive $\varepsilon$. It can be shown, using the argument in Weber (1997) that the set $B_\varepsilon^Y(x)$ is the union of a finite set of intervals and that if the set of best responses $B_0^Y(x)$ is nonempty, it consists of a unique element (see claim 3.1 in Weber (1997)).

If the entrant chooses to enter, so that for a given pair of incumbents' positions, $x$, the set $D(x)$ is nonempty, define by $E(x)$ the subset of $D(x)$ that the entrant considers, i.e., let

$$E(x) = \begin{cases} D_1(x) & if \quad D_1(x) \neq \varnothing \\ D_{12}(x) & if \quad D_1(x) = \varnothing, D_{12}(x) \neq \varnothing \\ D_2(x) & if \quad D_1(x) \cup D_{12}(x) = \varnothing, D_2(x) \neq \varnothing \end{cases}$$

Let

$$\mu_\varepsilon(x) \equiv \int_{B_\varepsilon^E(x)} dx, \quad \text{and} \quad u_i(x) \equiv \lim_{\varepsilon \to 0} \int_{B_\varepsilon^E(x)} \frac{1}{\mu_\varepsilon(x)} s_i(x, x_e) dx_e$$

for each party $i$, $i = 1, 2$. Palfrey (1984) has shown that the functions $u_i$ are well-defined. Roughly speaking, $u_i$ is a limit of incumbent $i's$ electoral support, provided the potential entrant is mixing across his $\varepsilon-$best responses with equal probability.

Now we can define the second component of the preferences for the incumbent parties. If the best response of the entrant exists, – either the entrant

[6] We shall show that if the set $D_{12}(x)$ is nonempty, it consists of a unique element.

does not enter, so that $x_e = N$, or her vote-maximizing position given is well defined, – this component is $s_i$, as defined before, and uniqueness of the best reply of the entrant in this case implies $x_e$ is a function of the positions of the incumbents $x = (x_1, x_2)$. If the set $B_0^E(x)$ is empty, then her payoff is set to be equal to the limit of the average support, $u_i$, which is also fully determined by the positions of the incumbents. To sum up, an incumbent $i$, $i = 1, 2$, derives payoff $\pi_i(x)$ from her electoral support:

$$\pi_i(x) \equiv \begin{cases} u_i(x) & if \quad D(x) \neq \varnothing, B_0^E(x) = \varnothing \\ s_i(x, x_e(x)) & otherwise \end{cases} .$$

We can now formally define game $\Gamma$ between the incumbents, who foresee that the third party, $e$, enters only if she can displace one of the incumbents and to guarantee at least the sole position of the second place. Formally,

**Definition 2.1** *In two-person game $\Gamma$ the incumbents have strategy set $I$. Players' preferences are lexicographic in (1) rank, $r_i$ and (2) payoff, $\pi_i$. A pure strategy equilibrium of the game $\Gamma$ is called an $\mathcal{R}$-equilibrium.*

It is important to distinguish our equilibrium notion from that introduced by Greenberg and Shepsle (1987). They refer to each pair of incumbent positions that prevent entry by a third party as 2-equilibrium. We will simply call these strategies *entry-deterrent*:

**Definition 2.2** *A pair of positions of established parties $x = (x_1, x_2)$ is called entry-deterrent ($\mathcal{D}$-*strategies*), if the set $D(x)$ is empty.*

Note that an entry-deterrent pair of incumbent strategies is not necessarily consistent with $\mathcal{R}$-equilibrium. Indeed, the latter requires the incumbents' positions to be immune to unilateral deviations by the incumbents, while correctly anticipating the response of a potential third party. Thus, we allow an incumbent to induce the entry, if it is in her interest. The next section offers a characterization of $\mathcal{R}$-equilibria. The key result is that in $\mathcal{R}$-equilibria neither of the incumbents will want to induce the entry.

# 3 Balance of Power in $\mathcal{R}$-equilibrium

In this section we derive conditions for the existence and uniqueness of both $\mathcal{R}$-equilibrium and $\mathcal{D}$-strategies, and study the relationship between the two. Throughout the remainder of the paper we consider only pairs of strategies $(x_1, x_2)$ where party 1 is located to the left of party 2, i.e., $x_1 \leq x_2$, so that the uniqueness of an equilibrium will be stated in terms of equilibrium configurations up to a permutation of incumbents' strategies. The proofs of all results in this section are relegated to the Appendix.

We shall now introduce two assumptions, (**A.1**) and (**A.2**), which will hold throughout the rest of the paper. The first is quite standard and requires the distribution of voters' ideal points to be unimodal and the density function $f$ to be continuous:

**Assumption (A.1)** $f(\cdot)$ is continuous and strictly positive on $[0,1]$. Moreover, there exists $\hat{x} \in I$, such that $f(\cdot)$ is strictly increasing on the interval $[0,\hat{x}]$ and strictly decreasing on the interval $[\hat{x},1]$.

The second assumption assures that the ideal points of the voters are not too concentrated at any given interval. Following Haimanko, Le Breton and Weber (2005) we will refer to this assumption as *gradually escalating median (GEM).*[7] Let $l : [0,1] \to [0,1]$ be the median of $[0,t]$ and $r : [0,1] \to [0,1]$ be the median of $[t,1]$ under $F$. Given the first assumption both functions are continuously differentiable.

**Assumption (A.2)** $l'(t) < 1, r'(t) < 1$.

Assumptions (**A.1**) and (**A.2**) will allow us to compare $\mathcal{D}$-strategies and strategies of the incumbents under $\mathcal{R}$-equilibria. The definition of $\mathcal{D}$-strategies rules out a move by the entrant, while $\mathcal{R}$-equilibrium requires an incumbent's position to be immune against a unilateral deviation of another incumbent that can "invite" an entry by the third party.

As defined, $\mathcal{R}$-equilibrium does not preclude third party entering the race and ranking at least the second. It is important to establish, therefore, whether there exists $\mathcal{R}$-equilibrium in which the established parties allow for the entry of party $e$. Proposition 3.1 demonstrates that the answer is negative, implying that the notion of $\mathcal{R}$-equilibrium is no less restrictive than the notion of entry-deterrent strategies.

**Proposition 3.1** *Assume that (**A.1**) and (**A.2**) hold. Then in any $\mathcal{R}$-equilibrium, party e does not enter. That is, every pair of incumbents' $\mathcal{R}$-equilibrium strategies is also a pair of $\mathcal{D}$-strategies.*

[7] This assumption is weaker than log-concavity, which is rather mild on its own, see Bergstrom and Bagnoli (2005) for the discussion and connection to other properties, including monotone hazard ratio. More precisely, a stronger assumption than (**A.2**) would require function $F(t)$ to be log-concave on the interval $[0,\hat{x}]$ and the function $1-F(1-t)$ to be log-concave on the interval $[\hat{x},1]$.

Greenberg and Shepsle (1987) concluded that, in general, the set of $\mathcal{D}$-strategies might be empty. Providing sufficient conditions for existence of these strategies remained open. Cohen (1985) has shown that $\mathcal{D}$-strategies exists for the special case where the distribution of voters' ideal points is given by a normal density function. The following proposition demonstrates that the condition of normality, and even symmetry, of the distribution can be dropped. Unimodality and GEM yield existence and uniqueness of $\mathcal{D}$-strategies.

**Proposition 3.2** *Under (**A.1**) and (**A.2**), there is a unique pair of $\mathcal{D}$-strategies $x^d = (x_1^d, x_2^d)$. Moreover, $x^d$ satisfies*[8]

$$F(x_1^d) = \frac{1}{2}F(\frac{x_1^d + x_2^d}{2}), \tag{1}$$

$$1 - F(x_2^d) = \frac{1}{2}(1 - F(\frac{x_1^d + x_2^d}{2})), \tag{2}$$

$$\frac{1}{3} < F(\frac{x_1^d + x_2^d}{2}) < \frac{2}{3}. \tag{3}$$

Our next proposition derives necessary and sufficient conditions for the existence of $\mathcal{R}$-equilibrium. Note that by Proposition 3.1, the set of $\mathcal{R}$-equilibria is a subset of the set of $\mathcal{D}$-strategies. We show that, in general, the converse is not true. Since, by Proposition 3.2, for a given distribution of ideal points a pair $\mathcal{D}$-strategies is unique, it follows that the set of $\mathcal{R}$-equilibria might be empty. That is, even though (under (**A.1**) and (**A.2**), there is always a unique entry-deterring pair strategies for established parties, one of the incumbents could be better off by deviating from it, thus *allowing for entry* of party $e$. Given that we impose an additional requirement of Nash behavior on incumbents, it is not surprising to find out that $\mathcal{R}$-equilibrium may fail to exist in circumstances which guarantee existence of $\mathcal{D}$-strategies. Our result shows that $\mathcal{R}$-equilibrium exists only in the case where the established parties, while locating themselves at quartiles of the distribution, achieve a balance of power by equally splitting the total electoral vote.

**Proposition 3.3** *Assume that (**A.1**) and (**A.2**) hold and let the pair $x^d = (x_1^d, x_2^d)$ be a $\mathcal{D}$-strategies. Then a pair of incumbents' strategies $x^d$ is an $\mathcal{R}$-equilibrium if and only if*

$$F(\frac{x_1^d + x_2^d}{2}) = \frac{1}{2}. \tag{4}$$

[8] These equations were formulated in Greenberg and Shepsle (1987) as necessary conditions for the existence of a $\mathcal{GS}$-equilibrium.

That is, if

$$F^{-1}(\frac{1}{4}) + F^{-1}(\frac{3}{4}) = 2F^{-1}(\frac{1}{2}), \tag{5}$$

then the set of $\mathcal{R}$-equilibria consists of the unique element, $(F^{-1}(\frac{1}{4}), F^{-1}(\frac{3}{4}))$.

Otherwise the set of $\mathcal{R}$-equilibria is empty.

The intuition behind Proposition 3.3 is quite simple. When power is balanced between incumbents, in the sense that each garners 50% of the vote, neither party can improve its standing by altering its position. Take, for example, the candidate located on the left. Moving further to the left will reduce her support relative to the other incumbent as well as make it possible for the entrant to locate "very close" on her right and displace her by garnering slightly more than 25% of the vote. On the other hand, should the left-most incumbent attempt to increase her support by moving closer to the current rival, the entrant will displace her by locating "very close" on her left and garnering slightly more than 25% of the vote. By the same argument, the right-most candidate cannot improve her standing when the incumbents choose platforms in such a manner that the electorate is divided equally between the established parties.

Balance of power is essential for $\mathcal{R}$-equilibrium to exist. If power is not shared equally between the two incumbents, the second place incumbent can improve its standing by moving slightly closer to the incumbent who ranks first. In doing so, the entrant can now garner more votes than the top-ranked incumbent by entering "very close" on her outside. The incumbent formerly in second place will now win the election, the third party will come in second place, and the incumbent formerly in first place will now be ranked third as it is "squeezed" between its old rival and the new third party.

Implicitly, Proposition 3.3 characterizes societies (described by distributions of voters' ideal points) for which $\mathcal{R}$-equilibrium exists. In particular, any symmetric density function satisfies condition 5. Clearly, symmetry is not necessary for that condition to hold.

# 4 Rank Preoccupation and Policy Divergence

In this section we will show that although more polarized societies might have more extreme winning platforms, polarization of preferences is not the only contributor to the divergence of platforms. Two different electoral systems might affect the degree of policy divergence through its effect on the objectives of the politicians.

Greenberg and Shepsle (1987) distinguished between two election procedures – one, based on *fixed-standard method* and the other on *fixed-number method.* Under the first one candidates which garner a certain number of votes are deemed elected, thus the appropriate objective for candidates is to maximize votes in an effort to surpass the predetermined quota (see Greenberg and Weber (1985)). In a system characterized by a *fixed-number method*, the number of winners is exogenously determined and candidates are deemed elected based on relative performance. In this case a candidate's rank relative to the existing and potential competition is more important than the sheer number of votes a candidate receives.

In the prior section, we found that power between established parties is balanced when candidates have rank-related objectives such as those under a fixed-number method. The results, however, give no indication as to how these platforms compare to those chosen under an electoral system where the objective of candidates is to maximize votes. In this section, we compare the equilibrium platforms selected under rank-related objectives to alternative equilibrium notions where candidates seek to maximize votes. We find that, in general, incumbent positions are more extreme when rank matters.

Palfrey (1984) consider a game between two incumbents anticipating a third party's entry, where the candidates' objective is to maximize its share of the total vote. Palfrey demonstrates the existence of a unique non-cooperative equilibrium, called a *limit equilibrium*, assuming the distribution of voters' ideal points is symmetric. Weber (1992) points out that the notion of a limit equilibrium cannot be directly extended to a broader class of asymmetric distributions, and therefore demonstrates the existence of a unique non-cooperative equilibrium, referred to as a *hierarchical equilibrium*, for the general case of unimodal distributions. In a hierarchical equilibrium, the number of votes available to the entrant to the left of party 1 is the same as the number of votes available to the right of party 2, and moreover, is equal to the maximal number of votes the entrant could garner by locating between the two established parties. In the case of symmetric distributions the notions of limit and hierarchical equilibria yield the same pair of strategies.

Two aspects distinguish the notions of limit and hierarchical equilibrium from $\mathcal{D}$-strategies and $\mathcal{R}$-equilibrium. First, the candidates seek to maximize votes, rather than rank, in a limit or hierarchical equilibrium. Second, the third party always enters the race regardless of whether it can displace one of the incumbents. Because the candidates have different objectives and entry is prevented altogether under $\mathcal{D}$-strategies and $\mathcal{R}$-equilibrium, the incumbents' platforms diverge from those chosen in a limit and hierarchical equilibrium. We will start with the case of symmetric distributions.

## 4.1 Symmetric Distributions

Consider a distribution function $F(x)$, satisfying (**A.1**)-(**A.2**), with symmetric density around its mode $\frac{1}{2}$, i.e., $f(x) = f(1-x)$ for all $x \in [0,1]$. By Propositions 3.2 and 3.3, there exists a unique $\mathcal{R}$-equilibrium $x^d = (x_1^d, x_2^d)$. It is easy to verify that in equilibrium the incumbents choose positions which are equidistant from the mode $\frac{1}{2}$ and, moreover, $F(x_1^d) = \frac{1}{4}$, $F(x_2^d) = \frac{3}{4}$. Palfrey has shown the limit equilibrium $x = (\tilde{x}_1, \tilde{x}_2)$, is also symmetric around $\frac{1}{2}$, and, moreover, satisfies

$$F(\tilde{x}_1) = 1 - 2F(\frac{\tilde{x}_1 + \frac{1}{2}}{2})$$

**Example 4.1** *If the distribution is triangular and the density function is given by*

$$f(x) = \begin{cases} 4x & if \quad x \leq \frac{1}{2} \\ 4 - 4x & if \quad x \geq \frac{1}{2} \end{cases}$$

*the $\mathcal{R}$-equilibrium is given by* $(\frac{1}{\sqrt{8}}, 1 - \frac{1}{\sqrt{8}}) = (.35, .65)$ *whereas the limit equilibrium is* $(\frac{\sqrt{10}-1}{6}, \frac{5-\sqrt{10}}{6}) = (.36, .64)$. *Note that in this example, the incumbents' positions are closer to the mode of the distribution in a limit equilibrium than in $\mathcal{R}$-equilibrium.*

The above example suggests that the gap between the incumbents' positions will be larger when candidates have rank-related objectives. Recall that the notion of a hierarchical equilibrium is a generalization of Palfrey's limit equilibrium. A comparison between $\mathcal{R}$-equilibrium and hierarchical equilibrium immediately yields the following result:

**Proposition 4.2** *Let $x^d = (x_1^d, x_2^d)$ and $x = (\tilde{x}_1, \tilde{x}_2)$ denote the $\mathcal{R}$-equilibrium and the hierarchical equilibrium, respectively. Then we have*

$$x_1^d < \tilde{x}_1 < \tilde{x}_2 < x_2^d$$

Proposition 4.2 demonstrates that, in general, the positions of the established parties are more extreme in an $\mathcal{R}$-equilibrium than in a hierarchical equilibrium. In an electoral system where absolute performance is more important than relative performance, established parties are just as likely to be challenged by a moderate third party (i.e. a third party which enters between the incumbents), as they are by a third party which is slightly more extreme than one of the incumbents.

However, in an electoral system where rank matters more, established parties are likely to be "mimicked" by a third party which is only slightly more moderate or slightly more extreme. When rank matters more than absolute performance, established parties are equally, if not more, concerned with maintaining the support of fringe voters (i.e. those voters whose ideal point is more extreme than that of the candidate) as they are with maintaining the support of moderate voters. The gap between the positions of the established parties is sharper in an $\mathcal{R}$-equilibrium than in a hierarchical equilibrium due to the incumbents' efforts to prevent a third party from capturing the support of its fringe voters.

Notably, the gap between the positions of the two incumbents will increase in more heterogeneous or polarized societies. This is true for both hierarchical and $\mathcal{R}$-equilibria. To formalize this statement, let us introduce a way to compare two distributions based on how "peaked" they are. The following definition is adopted from Shaked and Shanthikumar (1994), p.77.

**Definition 4.3** *Consider two unimodal distributions, $F$ and $H$, symmetric about $\mu$. $F$ is more peaked than $H$, if $H(x) \geq F(x)$ for all $x \leq \mu$, i.e., if $F(x|x \leq \mu)$ first order stochastically dominates $H(x|x \leq \mu)$.*

We will apply this order to distributions describing the ideal points of the voters, thus a less peaked distribution will correspond a less homogeneous society. It is easy to see that if $H$ is less peaked than $F$, then $F^{-1}\left(\frac{1}{4}\right) > H^{-1}\left(\frac{1}{4}\right)$ and $F^{-1}\left(\frac{3}{4}\right) < H^{-1}\left(\frac{3}{4}\right)$, therefore in the view of Proposition 3.3, we have the following

**Remark 4.4** *Let $F$ and $H$ be two symmetric distributions of the ideal points of different societies, such that $H, F$ satisfy assumptions $(\mathbf{A.1})$ and $(\mathbf{A.2})$ with $H$ being less peaked than $F$. Then the $\mathcal{R}$-equilibrium under $H$ is more polarized than that under $F$, i.e.,*

$$x_1^d < \hat{x}_1^d < \hat{x}_2^d < x_2^d,$$

*where $\left(\hat{x}_1^d, \hat{x}_2^d\right)$ is the $\mathcal{R}$-equilibrium under $F$ and $\left(x_1^d, x_2^d\right)$ is the $\mathcal{R}$-equilibrium under $H$.*

In the view of the necessary condition $(18)$, the same statement is true for the hierarchical equilibrium.

To sum up, more spread ideal points of the voters lead to a divergence of equilibrium positions of the incumbents in the presence of a credible threat of entry, keeping the rank-concerned politicians always at more extreme positions as compared to those chosen by the incumbents solely driven by the absolute size of the electoral support.

## 4.2 Monotone Distributions

To extend the comparison of the gap between incumbents positions under different to a wider range of environments, we can consider the class of distribution functions satisfying (**A.1**)-(**A.2**) which do not possess $\mathcal{R}$-equilibrium, although allowing for existence of $\mathcal{D}$-strategies. Specifically, consider a family of distributions whose density is monotone on the issue space, which would amount to either concavity or convexity of cumulative distribution functions. Since the case of decreasing density could be examined in a similar manner, we shall restrict our attention to strictly convex distribution functions $F(x)$, satisfying (**A.1**)-(**A.2**). By Proposition 3.2, there exists a unique $\mathcal{D}$-strategies $x^d = (x_1^d, x_2^d)$. However, the set of $\mathcal{R}$-equilibria is empty. Indeed, we can rearrange (1) and (2) to obtain

$$F(\frac{x_1^d + x_2^d}{2}) = 2(F(\frac{x_1^d + x_2^d}{2}) - F(x_1^d)),$$
$$1 - F(\frac{x_1^d + x_2^d}{2}) = 2(F(x_2^d) - F(\frac{x_1^d + x_2^d}{2})).$$

Since the density function $f$ is increasing, we have $F(\frac{x_1^d+x_2^d}{2}) - F(x_1^d) < F(x_2^d) - F(\frac{x_1^d+x_2^d}{2})$ or $F(\frac{x_1^d+x_2^d}{2}) < \frac{1}{2}$, violating condition (4) in the statement of Proposition 3.3.

We will now show that the assertion regarding the positions of established parties in Proposition 4.2 does not hold in general if we replace $\mathcal{R}$-equilibrium by $\mathcal{D}$-strategies. However, we further demonstrate that the gap between the incumbents' positions is still larger under rank-related objectives. In other words, regardless of the equilibrium notion, platforms are more extreme when rank matters.

More specifically, in the case of convex cumulative distribution functions, the incumbents' positions in a hierarchical equilibrium are shifted to the right relative to their positions in a $\mathcal{D}$-strategies:

**Proposition 4.5** *If the distribution $F$ is convex, then*

$$x_1^d < \tilde{x}_1 < x_2^d < \tilde{x}_2.$$

**Example 4.6** *Consider linear density function given by $f(x) = 2x$ for all $x \in [0,1]$. Then the $\mathcal{D}$-strategies is the pair $(\frac{\sqrt{2+\sqrt{2}}}{4}, \frac{\sqrt{10+\sqrt{2}}}{4}) = (.46, .84)$ whereas the hierarchical equilibrium is $(\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}) = (.51, .86)$.*

Note that in the above example of a linear density function, the gap between the incumbents' positions is again wider under a system where rank matters more than absolute performance. We conclude this section by stating that, though it is difficult to derive general conclusions on the relative location of incumbents' positions in $\mathcal{D}$-strategies and hierarchical equilibrium, the *fraction* of voters whose ideal points are between the incumbents' positions is always larger in $\mathcal{D}$-strategies than in hierarchical equilibrium:

**Proposition 4.7** *Let $x^d = (x_1^d, x_2^d)$ and $x = (\tilde{x}_1, \tilde{x}_2)$ denote the $\mathcal{R}$-equilibrium and the hierarchical equilibrium, respectively. Then*

$$F(\tilde{x}_2) - F(\tilde{x}_1) < F(x_2^d) - F(x_1^d)$$

**Proof.** >From (1) and (2) we have $F(x_2^d) - F(x_1^d) = \frac{1}{2}$. Moreover, (19) implies that $F(\tilde{x}_1) = 1 - F(\tilde{x}_2) > \frac{1}{4}$. Thus, $F(\tilde{x}_2) - F(\tilde{x}_1) < F(x_2^d) - F(x_1^d)$. ■

# 5 Conclusions

The model offers an analysis of electoral competition in the presence of rank concerns. If two dominant parties face a threat of entry by a third competitor aiming at displacing one of the top two, they should (if possible) keep the balance of power by choosing distinct positions and sharing the electoral support equally. This provides another possible reason for divergence of platforms in an electoral competition model. Moreover, we find that candidates' preoccupation with their rank leads them to take more extreme stands under the threat of entry, thus widening the gap between the proposed platforms as compared to the candidates who focus on the size of the electoral support.

There are two important consequences of this result. First, as motivation of the candidates (parties) is partly driven by electoral system, the result contributes to the comparison between different election procedures, for example, *fixed-standard method* and *fixed-number method.* In particular, one could use our results to evaluate the merits of introducing (or abandoning) the system of open primaries.

Second, under the assumptions of the model, the same fundamental preferences of an electorate will be "reflected" differently in the views of its representatives depending on the electoral institution. Analyzing this result empirically can be an interesting direction for future investigation.

# 6 Appendix

Let a pair of incumbents' strategies $x = (x_1, x_2)$ be given. Let $\alpha \equiv \frac{x_1+x_2}{2}$, $\mathbf{I}_1 \equiv F(x_1)$, $\mathbf{I}_2 \equiv F(\alpha) - F(x_1)$, $\mathbf{I}_3 \equiv F(x_2) - F(\alpha)$, $\mathbf{I}_4 \equiv 1 - F(x_2)$.

Assume, without loss of generality, that

$$\mathbf{I}_1 \geq \mathbf{I}_4. \tag{6}$$

**Lemma 6.1** *If $I_1 \geq I_2$ then the support of an entrant choosing a policy $z \in (x_1, x_2)$ will not exceed that of the first party and if $I_3 \leq I_4$ the support of an entrant choosing a policy $z \in (x_1, x_2)$ will not exceed that of the second party. If both inequalities $I_1 \geq I_2$ and $I_3 \leq I_4$ hold, then no entry will occur between $x_1$ and $x_2$, i.e., the set*

$$D^m(x) \equiv \{z \in D(x) | x_1 < z < x_2\}$$

*is empty.*

**Proof.** Consider the entry of party $e$ between $x_1$ and $x_2$, say, at $z$. Let $I_1 \geq I_2$ hold. Then the support of the entrant is equal to $F(z+r_2) - F(z-r_1)$ where $r_2(z) \equiv (x_2 - z)/2$, $r_1(z) \equiv (z - x_1)/2$. This implies $r_1'(z) = -r_2'(z) = \frac{1}{2}$ and then by assumption (**A.2**), for $r_2 \in [0, (x_2 - x_1)/2] = [0, \alpha - x_1]$

$$\begin{aligned} l(z + r_1) - l(\alpha) &< z + r_2 - \alpha \\ &= z - r_1 - x_1, \end{aligned}$$

thus

$$l(z + r_b) < z - r_a,$$

as $l(\alpha) \leq x_1$. Then $F(z+r_2) < 2F(z-r_1)$, so that the first party has higher support than the entrant. Similarly, if $I_3 \leq I_4$ $G(1-z+r_1) < 2G(1-z-r_2)$ and the second party has a higher support than the entrant. If both inequalities $I_1 \geq I_2$ and $I_3 \leq I_4$ hold,

$$F(z+r_1) - F(z-r_2) \leq \min[F(z-r_1), 1 - F(z+r_2)].$$

Hence the support of the entrant does not exceed that of any of the two established parties. ■

In order to prove Proposition 3.2 we shall use the following lemma:

**Lemma 6.2** *For each $y, 0 < y < 1$, define the values of $a(y)$ and $b(y)$ by:*

$$2F(a(y)) = F(y) \text{ and } 2(1 - F(b(y))) = 1 - F(y). \tag{7}$$

*Then there is a unique $y^d$, satisfying*

$$a(y^d) + b(y^d) = 2y^d. \tag{8}$$

*Moreover, the value of $y^d$ is such that $\frac{1}{3} < F(y^d) < \frac{2}{3}$.*

**Proof.** It is easy to verify that the functions $a(\cdot)$ and $b(\cdot)$ are well-defined, increasing and differentiable on the interval $[0,1]$, and so is their sum, $h(y) \equiv 2y - a(y) - b(y)$. Since $a(0) = 0, b(0) > 0, a(1) < 1, b(1) = 1$, it follows that $h(0) = -b(0) < 0$ and $h(1) = 1 - a(1) > 0$. Hence, there exists a value $y$ that solves $(8)$.

For uniqueness rewrite the necessary conditions as a system of two equations,

$$\begin{cases} 2F(a) - F\left(\frac{a+b}{2}\right) = 0 \\ 2F(b) - F\left(\frac{a+b}{2}\right) = 1 \end{cases}$$

for $(a,b) \in [0,1]^2$. Its Jacobian is

$$\begin{bmatrix} 2f(a) - \frac{1}{2}f\left(\frac{a+b}{2}\right) & -\frac{1}{2}f\left(\frac{a+b}{2}\right) \\ -\frac{1}{2}f\left(\frac{a+b}{2}\right) & 2f(b) - \frac{1}{2}f\left(\frac{a+b}{2}\right) \end{bmatrix},$$

the principal minors of which are positive by (**A.2**), which implies

$$\begin{aligned} l'(x) &= \frac{1}{2}\frac{f(x)}{f(l(x))} < 1 \\ r'(x) &= \frac{1}{2}\frac{f(x)}{f(r(x))} < 1. \end{aligned} \tag{9}$$

This leads to the following conditions:

$$\begin{aligned} 2f(a) - \frac{1}{2}f\left(\frac{a+b}{2}\right) > \frac{1}{2}f\left(\frac{a+b}{2}\right) > 0 \\ 2f(b) - \frac{1}{2}f\left(\frac{a+b}{2}\right) > \frac{1}{2}f\left(\frac{a+b}{2}\right) > 0 \end{aligned} \tag{10}$$

and, therefore,

$$\left[2f(a) - \frac{1}{2}f\left(\frac{a+b}{2}\right)\right]\left[2f(b) - \frac{1}{2}f\left(\frac{a+b}{2}\right)\right] - \left[\frac{1}{2}f\left(\frac{a+b}{2}\right)\right]^2 > 0 \tag{11}$$

Conditions $(10, 11)$ assure the positiveness of the principal minors of the Jacobian and, thus, uniqueness by the Fundamental Global Univalence Theorem (Parthasarathy (1983), p.20).[9]

Finally, we need to verify the inequalities

$$\frac{1}{3} < F(y^d) < \frac{2}{3}.$$

[9] The initial formulation is due to Gale and Nikaido (1965).

By the necessary conditions $(7)$, we know $l\left(y^d\right) = a$. By assumption $(\mathbf{A}.\mathbf{2})$ for any $x \in \left[y^d, b\right]$, $l'(x) < 1$, or for $r \in \left[0, b - y^d\right]$

$$l\left(y^d + r\right) - r < l\left(y^d\right),$$

which implies

$$l(b) < l\left(y^d\right) + b - y^d = y^d$$

thus

$$\frac{F(b(y^d))}{F(y^d)} < \frac{F(y^d)}{F(a(y^d))} = 2$$

and, moreover,

$$F(y^d) > F(b(y^d)) - F(y^d) = \frac{1 - F(y^d)}{2},$$

yielding $F(y^d) > \frac{1}{3}$. Similarly,

$$\frac{1 - F(a(y^d))}{1 - F(y^d)} < \frac{1 - F(y^d)}{1 - F(b(y^d))} = 2,$$

which implies

$$1 - F(y^d) > F(y^d) - F(a(y^d)) = \frac{F(y^d)}{2},$$

or $F(y^d) < \frac{2}{3}$. ∎

Before proceeding with Proposition 3.1, we provide the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Greenberg and Shepsle (1987) have shown that conditions $(1, 2, 3)$ are necessary for a pair of strategies to be $\mathcal{D}$-strategies. By Lemma 6.2, there is a unique pair of strategies which satisfies those conditions. Thus, it remains to show that this pair of strategies is indeed a pair of $\mathcal{D}$-strategies under $(\mathbf{A}.\mathbf{1})$ and $(\mathbf{A}.\mathbf{2})$.

Lemma 6.2 yields the existence of a real number $y \in (0, 1)$ such that the pair $w = (a(y), b(y))$ is the unique pair of strategies which satisfies $(1,\ 2,\ 3)$. To demonstrate that the set $D(w)$ is empty, let us consider options available for party $e$, given the incumbents' locations at points $a(y)$ and $b(y)$, respectively. If party $e$ enters to the left of $a(y)$, it would not displace party 1. Moreover, since $F(a(y)) \leq \frac{1}{3} \leq F(y)$, party 2 would not be displaced either. By using similar arguments one can show that neither of the established parties would be displaced if party $e$ enters to the right of $b(y)$. In the view of conditions $1, 2$ lemma 6.1 guarantees that no entrant will choose a position in between $a(y)$ and $b(y)$. ∎

We shall turn now to the proof of Proposition 3.1.

For each pair of positions of the established parties $x = (x_1, x_2)$ denote by $y^1 = y^1(\frac{x_2-x_1}{2})$ and $y^2 = y^2(\frac{x_2-x_1}{2})$ two locations in the issue space which satisfy

$$F(y^1 + \frac{x_2 - x_1}{2}) = 2F(y^1); \tag{12}$$

$$G(y^2 - \frac{x_2 - x_1}{2}) = 2G(y^2). \tag{13}$$

Let

$$z^1(x) \equiv 2y^1 - x_1; \;\; z^2(x) \equiv 2y^2 - x_2.$$

In the case of $z^1(x) > x_1$, if party $e$ enters at $z^1(x)$ it would generate the support equal to that of party 1. Assumption (**A**.**2**) can be used to show that if party $e$ enters between $x_1$ and $z^1(x)$ it would displace party 1, and if it enters to the right of $z^1(x)$ it would not displace party 1. Similarly, in case $z^2(x) < x_2$, if party $e$ enters to the left of $z^2(x)$ it would not displace party 2, if it enters at $z^2(x)$ it would generate the same number of votes as party 2, and if it enters between $z^2(x)$ and $x_2$ it would displace party 2. In addition, for each pair of positions of the established parties $x = (x_1, x_2)$, consider the function $s^e(x, t)$ which determines the support of party $e$ generated by entry at $t \in I$. It has been shown in Weber (1992) that assumptions (**A**.**1**) guarantee that $s^e(x, \cdot)$ is continuous and strictly quasi-concave on the interval $(x_1, x_2)$. It allows us to continuously extend this function to the closed interval $[x_1, x_2]$. Thus, there exists a unique value $z^\star(x)$, $x_1 \leq z^\star(x) \leq x_2$ such that

$$z^\star(x) = \arg \max_{x_1 \leq t \leq x_2} s^e(x, t).$$

$z^\star(x)$ determines the vote-maximizing location[10] of party $e$ between the positions of the two established parties.

We will show now that if one of the sets $D_1(x)$, $D_{12}(x)$ or $D_2(x)$ is nonempty then $x$ is not an $\mathcal{R}$-equilibrium.

**Lemma 6.3** *Let a pair of strategies $x = (x_1, x_2)$ be an $\mathcal{R}$-equilibrium. Then set $D(x) \backslash D_2\,(x) = D_1(x) \cup D_{12}(x)$ is empty.*

**Proof.** Let $x = (x_1, x_2)$ be an $\mathcal{R}$-equilibrium. Suppose first that $D_1\,(x)$ is nonempty. Since by assumption $(6)$, $\mathbf{I}_1 \geq \mathbf{I}_4$, party $e$ may enter either to

[10] When $z^\star$ is equal either to $x_1$ or $x_2$, it represents the limit of "almost" vote-maximizing positions of the entrant.

the left of $x_1$ or between $x_1$ and $x_2$.

**(i)**. Suppose that party $e$ enters to the left of $x_1$, which will happen, only if $\mathbf{I}_1 > \mathbf{I}_4$. The entrant gains the support of almost $I_1$ and wins the race, if $\mathbf{I}_1 > \mathbf{I}_4 + \mathbf{I}_3$. But then a move of player 2 to the left of $x_2$ by a small $\varepsilon$ would still hold party $e$ entering to the left of $x_1$. Thus, the payoff of party 2 will increase to $\pi_2(x_1, x_2 - \varepsilon) = 1 - F(\alpha - \frac{\varepsilon}{2}) > \pi_2(x)$, while the support of party 1 will drop to $F(\alpha - \frac{\varepsilon}{2}) - F(x_1) < \pi_1(x)$, a contradiction to the fact that $x$ is an $\mathcal{R}$-equilibrium.

**(ii)**. Suppose now that party $e$ enters between $x_1$ and $x_2$, which will only happen if $z_1(x) > z_2(x)$. Then $D_1(x) = (z_1(x), z_2(x))$, and party $e$ maximizes its support over this interval. Thus, if any of the established parties makes a "slight" move towards its competitor, party $e$ would still enter "in between" and, by Lemma 4.2 in Weber (1992), the party initiating the move, will increase its support, a contradiction. Thus, $D_1(x)$ is empty.

**(iii)**. Consider now the case where the set $D_1(x)$ is empty, whereas the set $D_{12}(x)$ consists of a unique element, $x_1$. Indeed, the entrant cannot share the first place by entering either to the left of $x_1$ or to the right of $x_2$. If $z_1(x) \leq z_2(x)$, then there is no position between $x_1$ and $x_2$ that guarantees the entrant the share of the first place. If $z_1(x) > z_2(x)$, then, contrary to our assumption, the set $D_1(x)$ is nonempty. Thus, the entrant selects her position $x_e$ at $x_1$ and we have $\frac{1}{2}F(\alpha) > F(1-\alpha)$ and $x_1 = \frac{\alpha}{2}$. Then let party 2 can leapfrog to the immediate left of $x_1$, thus, allowing the entrant to move to the immediate right of $x_1$. This move completely squeezes party 1, thus guaranteeing party 2 the sole possession of the second place, rather than being the last before the move took place. ■

**Proof of Proposition 3.1.** Let a pair of strategies $x = (x_1, x_2)$ be an $\mathcal{R}$-equilibrium. In the view of Lemma 6.3, it remains to show that the set $D_2(x)$ is empty.

It suffices to demonstrate that $I_1 = I_2$ and $I_3 = I_4$ (Recall that $I_1 \geq I_4$.) Indeed, in this case, the same arguments as in the proof of Proposition 3.2 would yield the emptiness of the set $D_2(x)$.

Assume, in negation, that, at least, one of equalities $I_1 = I_2$ or $I_3 = I_4$ does not hold. There are several cases to consider.

**1) $\mathbf{I}_1 > \mathbf{I}_2 \& \mathbf{I}_3 \leq \mathbf{I}_4$.**

Lemma 6.1 implies that the entrant could not displace one of the established parties by entering between $x_1$ and $x_2$. Depending on the relationship between $I_1, I_2, I_3$ and $I_4$, party $e$ still may enter either to the left of $x_1$ or to the right of $x_2$.

**1a**) $\mathbf{I}_1 > \mathbf{I}_4$. Party $e$ would enter to the left of $x_1$, then $x$ is not an $\mathcal{R}$-equilibrium by the same argument as in case **(i)** in Lemma 6.3.

**1b)** $\mathbf{I}_1 = \mathbf{I}_4 > \mathbf{I}_3$. Party $e$ would enter with equal probability to the left of $x_1$

and to the right of $x_2$: as both actions leads to being ranked as the second with equal electoral support. Hence

$$\pi_1(x) = \frac{F(\alpha) - F(x_1)}{2} + \frac{F(\alpha)}{2} = F(\alpha) - \frac{F(x_1)}{2},$$

recall, $\alpha = \frac{x_1+x_2}{2}$. Then there exists $\varepsilon > 0$ such that a "slight" shift of party 1 to its left to $x_1 - \varepsilon$ would force party $e$ to enter to the right of $x_2$, yielding $\pi_1(x_1 - \varepsilon, x_2) = F(\alpha - \frac{\varepsilon}{2}) > \pi_1(x)$, again a contradicting $x$ being an $\mathcal{R}$-equilibrium.

**1c) $\mathbf{I}_1$= $\mathbf{I}_4$= $\mathbf{I}_3$.** Party $e$ would enter to the left of $x_1$, thus assuring the second place and with support of almost $I_1$. In this case $\pi_1(x) = F(\alpha) - F(x_1)$. Then a move of party 1 to the left to $x_1$ leads to an entry of party $e$ to the left of $x_2$, leaving the entrant in the second place with a higher support. Since the shift of party 1 can be chosen arbitrarily small, one can guarantee that party $e$ enters "very close" to $x_2$. Thus, there exists $\varepsilon > 0$ such that $\pi_1(x_1 - \varepsilon, x_2) > \pi_1(x)$, a contradiction.

**2) $\mathbf{I}_1 \geq \mathbf{I}_2$, $\mathbf{I}_3 < \mathbf{I}_4$.** Could be examined in the same manner as the case **1)**.

**3) $\mathbf{I}_1 < \mathbf{I}_2$, $\mathbf{I}_3 \leq \mathbf{I}_4$.** Coupled with assumption (6) it implies $\mathbf{I}_2 > \mathbf{I}_4$. Party $e$ would enter between $x_1$ and $x_2$. Indeed, by entering to the right of $x_1$, it could generate the support of "almost" $I_2$ voters, thus assuring the second place. On the other hand, outside of this region it could attract the support of no more than $I_1 < I_2$ voters, which gives her the second rank at best with less support. Moreover, since $I_4 \geq I_3$, Lemma 6.1 implies that the entrant is unable to attract more votes than party 2, by entering between $x_1$ and $x_2$, so that $D^m(x) = (x_1, z^1(x))$. Thus, the entrant should maximize the electoral support on that interval. The payoffs of both players in game $\Gamma$ will be determined by:

$$\pi_1(x) = \begin{cases} I_1 & if \quad z^\star = x_1 \\ F(\frac{x_1+z^\star(x)}{2}) & if \quad z^\star(x) \in D^m(x) \\ F(\frac{x_1+z^1(x)}{2}) & if \quad z^\star(x) \notin D^m(x) \end{cases}$$

$$\pi_2(x) = \begin{cases} I_3 + I_4 & if \quad z^\star(x) = x_1 \\ 1 - F(\frac{x_2+z^\star(x)}{2}) & if \quad z^\star(x) \in D^m(x) \\ 1 - F(\frac{x_2+z^1(x)}{2}) & if \quad z^\star(x) \notin D^m(x) \end{cases}$$

Suppose now that party 2 moves its position to $\bar{x}_2$, which is "slightly" to the left of $x_2$. This would shrink the set of potential "in between" entry positions of party $e$, i.e., $D^m(x_1, \bar{x}_2) \subset D^m(x)$, as the aforementioned move of party 2 would shift $y^1$ to the left by assumption (**A**.**2**) (combining (12) and (9)), and therefore $z^1(x_1, \bar{x}_2) < z^1(x)$. Thus, if $z^\star(x) \geq z^1(x)$, this move yields a higher payoff to party 2. If $z^\star(x) < z^1(x)$, then Lemma 4.2 in Weber (1992) implies that party 2 benefits from its move, a contradiction.

**4) $\mathbf{I}_1 \geq \mathbf{I}_2$, $\mathbf{I}_3 > \mathbf{I}_4$.** Could be examined in the same way as case **3)**.

The last case to be considered is:

**5) $\mathbf{I}_1 < \mathbf{I}_2$, $\mathbf{I}_3 > \mathbf{I}_4$.** Party $e$ enters in the interval $[x_1, x_2]$, as $D(x) \subset [x_1, x_2]$ in this case. By using the previous arguments, we have $D^m(x) = (x_1, z^1(x)) \bigcup (z^2(x), x_2)$, and, moreover, in the view of Lemma 6.3, we are left with the case $z^1(x) \leq z^2(x)$.

If $z^\star(x)$ is either less than $z_1(x)$ or greater than $z_2(x)$, then the consideration is the same as in the case **(ii)** of Lemma 6.3.

Assume, therefore, that $z_1(x) \leq z^\star(x) \leq z_2(x)$.

**5a).** Let us first consider the case of $s^e(x; z_1(x)) \neq s^e(x; z_2(x))$ and, without loss of generality, $s^e(x; z_1(x)) > s^e(x; z_2(x))$. Thus, party $e$ enters to the left of $z_1(x)$. Then, by moving to the left of $x_2$, party 2 would increase its support by forcing party $e$ to shift its vote-maximizing position to the left by Lemma 4.2 in Weber (1992).

**5b).** Let $s^e(x; z_1(x)) = s^e(x; z_2(x))$ and $z^\star(x) = z_1(x) = z_2(x)$, and so $s^e(x; z^*(x)) = 1/3$. and $D^m(x) = (x_1, z^\star(x)) \bigcup (z^\star(x), x_2)$. The best the entrant can do in this case is to displace one of the incumbents by entering as close as possible to $z^\star(x)$, and, being indifferent between entering slightly to the left of $z_1(x)$ or to slightly the right of $z_2(x)$, she chooses both actions with equal probability.

Let us show that $x$ is not a $\mathcal{R}$- equilibrium. Take a "small" $\varepsilon > 0$ and consider two alternative moves:

($\zeta$): Party 1 moves to the right to $x_1^{'} = x_1 + \varepsilon$. Denote $x_\zeta = (x_1^{'}, x_2) \in \mathbf{R}^2$.

($\eta$): Party 2 moves to the left to $x_2^{'} = x_2 - \varepsilon$. Denote $x_\eta = (x_1, x_2^{'}) \in \mathbf{R}^2$.

First consider $\zeta$. By assumption $(\mathbf{A}.\mathbf{2})$, there are $z_\zeta^1 = z^1(x_\zeta) < z^\star(x) < z_\zeta^2 = z^2(x_\zeta)$, such that the set of possible entry positions of party $e$ consists of two disjoint intervals: $D^m(x_\zeta) = (x_1, z_\zeta^1) \bigcup (z_\zeta^2, x_2) \subset \mathbf{R}$. In this case, again, only one of the established parties can be replaced, so provided party $e$ can only guarantee to be the second, it maximizes voters' support. The best position of party $e$ under $\zeta$, $z_\zeta^\star = z^\star(x_\zeta)$, should by to the right of $z^\star(x)$ by Lemma 4.2 in Weber (1992). Therefore, $z_\zeta^\star > z_\zeta^1$. If $s^e(x_\zeta; z_\zeta^1) < s^e(x_\zeta; z_\zeta^2)$, the entrant will replace party 2, thus benefiting party 1, a contradiction.

It is left to consider the case, in which $s^e(x_\zeta; z_\zeta^1) \geq s^e(x_\zeta; z_\zeta^2)$ after the move of the first party. Note that the last inequality could be rewritten as

$$s^e(x_\zeta; z_\zeta^1) = F(y^1(r - \frac{\varepsilon}{2})) \geq 1 - F(y^2(r - \frac{\varepsilon}{2})) = s^e(x_\zeta; z_\zeta^2). \qquad (14)$$

The examination of the move $\eta$ is similar. Again assumption $(\mathbf{A}.\mathbf{2})$ yields the existence of $z_\eta^1 = z^1(x_1, x_2)$ and $z_\eta^2 = z^2(x_\eta)$, such that $D^m(x_\eta) =$

$(x_1, z^1_\eta) \bigcup (z^2_\eta, x_2)$. Again we assume that $z^\star_\eta < z^2_\eta$ and $s^e(x_\eta; z^2_\eta) \geq s^e(x_\eta, z^1_\eta)$. Thus we have

$$s^e(x_\eta; z^2_\eta) = F(y^1(r - \frac{\varepsilon}{2})) \geq 1 - F(y^2(r - \frac{\varepsilon}{2})) = s^e(x_\eta; z^1_\eta). \qquad (15)$$

Combining (14) and (15), we conclude that party $e$ generates the support of the same number of voters, denoted by $s_0$, whenever it chooses one of the following positions $z^1_\zeta$ or $z^2_\zeta$ under $\zeta$ and $z^1_\eta$ or $z^2_\eta$ under $\eta$. Note also that since the functions $y^1(\cdot)$ and $y^2(\cdot)$ depend *only* on the distance between the positions of the first two parties, the set of voters who pick $z^1_\zeta$ ($z^2_\zeta$, respectively) under $\zeta$ is the same as of those whose best choice $z^1_\eta$ ($z^2_\eta$, respectively) under $\eta$. Hence party 1 would be better off under $\zeta$ than under $\eta$, whereas the opposite is true for party 2, i.e.,

$$\pi_1(x_\zeta) > \pi_1(x_\eta);\ \pi_2(x_\eta) > \pi_2(x_\zeta). \qquad (16)$$

As both moves generate the support of $s_0$ voters for party $e$, we also have

$$\pi_1(x_\zeta) + \pi_2(x_\zeta) = \pi_1(x_\eta) + \pi_2(x_\eta) = 1 - s_0. \qquad (17)$$

Combining (16) and (17) we obtain $\pi_1(x_\zeta) + \pi_2(x_\eta) > 1 - s_0$. However, by Assumption (**A.2**), both moves $\zeta$ and $\eta$ lead to a decline in support of party $e$, i.e., $s_0 < \frac{1}{3} = s^e(x; z^*(x))$. Thus

$$\pi_1(x_\zeta) + \pi_2(x_\eta) > \frac{2}{3}$$

It follows, therefore, that, at least, one of the numbers $\pi_1(x_\zeta)$ or $\pi_2(x_\eta)$ exceeds $\frac{1}{3}$, yielding either $\pi_1(x_\zeta) > \pi_1(x)$ or $\pi_2(x_\eta) > \pi_2(x)$. That is, at least one of the established parties would benefit by deviating, a contradiction.
**5c).** Finally, let $s^e(x; z_1(x)) = s^e(x; z_2(x))$ with $z_1(x) < z_2(x)$, in which case party $e$ enters with the equal probability to the left of $z_1(x)$ and to the right of $z_2(x)$. But then the similar arguments as in the consideration of the previous case show that $x$ is not an $\mathcal{R}$-equilibrium. ■

**Proof of Proposition 3.3.** By Proposition 3.2, there is a unique pair of $\mathcal{D}$-strategies strategies $(x^d_1, x^d_2)$. Then, by (1) and (2), $F(x^d_1) = \frac{1}{2}F(\frac{x^d_1+x^d_2}{2})$ and $1 - F(x^d_2) = \frac{1}{2}(1 - F(\frac{x^d_1+x^d_2}{2}))$. Assume that (3) holds, i.e., $F(\frac{x^d_1+x^d_2}{2}) = \frac{1}{2}$. Then the payoff of each player in game $\Gamma$ is equal to $\frac{1}{2}$. We shall show that if one of the players would choose a different strategy, her payoff will decline.

Consider party 1. Suppose first that she moves to the right of $x^d_1$ by choosing $x_1 > x^d_1$. Since this move of party 1 shrinks the mass of voters located between the incumbents' positions, party $e$ could not displace an established party by entering between $x_1$ and $x_2^d$. Hence the entrant should come either to the left of $x_1$ or to the right of $x_2^d$. Since $F(x_1) > \frac{1}{4} = 1 - F(x_2^d)$, the optimal entrant's move would be to the left and "very close" to $x_1$. Then the payoff of player 1 in game $\Gamma$ would be less than $F(x_2^d) - F(x_1) < \frac{1}{2}$. Thus, player 1 would be worse off by moving to the right of $x_1^d$.

Suppose now that party 1 moves to the left of $x_1^d$ by choosing $x_1 < x_1^d$. Since the deviation of party 1 expands the mass of voters located between the incumbents' positions, party $e$ could not displace an established party by entering to the right of $x_2^d$. Moreover, by assumption (**A.2**) (and by the argument analogous to that in Lemma 6.1), $F(\frac{x_1+x_2^d}{2}) - F(x_1) > \frac{1}{4}$ so that party $e$ could not displace party one, and, clearly, not the second party whose support is over 1/2. Therefore, an entrant will not choose a platform to the left of $x_1$, either. Since, by Proposition 3.1, $D(x_1, x_2^d)$ is nonempty, the entrant could displace one of the established parties by entering between $x_1$ and $x_2^d$. Then the payoff of player 1 would be less than $F(\frac{x_1+x_2^d}{2}) < \frac{1}{2}$ making her worse off. This completes the "if" part of the Proposition.

Suppose that (3) does not hold and assume, without loss of generality, that $F(x_1^d) < 1 - F(x_2^d)$. Choose $\delta > 0$ such that $F(x_1^d + \delta) < 1 - F(x_2^d)$ and consider the move of party 1 to $x_1 = x_1^d + \delta$. By entering between $x_1$ and $x_2^d$, as well as by entering to the left of $x_1$ the entrant would get less than a quarter of the votes. However, by entering to the right and "very close" to $x_2^d$, party $e$ would displace party 2 and would receive more than 1/4 of the votes. Moreover, the support of the first party will increase, $\pi_1(x_1^d + \delta, x_2^d) = F(\frac{x_1^d+\delta+x_2^d}{2}) > F(\frac{x_1^d+x_2^d}{2}) = \pi_1(x_1^d, x_2^d)$. Thus, the party which gets less than 50% of the total vote in $\mathcal{D}$-strategies would be better off by moving towards its rival incumbent and allowing entry of the third party.

To conclude the proof of the proposition, note that by (1), (2), (3), the pair $(x_1, x_2)$ constitutes an $\mathcal{R}$-equilibrium if and only if $F(x_1) = \frac{1}{4}$, $F(x_2) = \frac{3}{4}$ and $F(\frac{x_1+x_2}{2}) = \frac{1}{2}$. ■

**Proof of Proposition 4.2.** Let $x^d = (x_1^d, x_2^d)$ and $x = (\tilde{x}_1, \tilde{x}_2)$ be $\mathcal{R}$-equilibrium and hierarchical equilibrium, respectively. Proposition 3.3 implies that $F(x_1^d) = 1 - F(x_2^d) = \frac{1}{4}$. Moreover, in a hierarchical equilibrium, the number of votes available for the entrant in three regions: to the left of $\tilde{x}_1$, between $\tilde{x}_1$ and $\tilde{x}_2$, and to the right of $\tilde{x}_2$ is the same. That is,

$$F(\tilde{x}_1) = \sup_{\tilde{x}_1 < y < \tilde{x}_2} s^e(x, y) = 1 - F(\tilde{x}_2) \tag{18}$$

Thus, $F(\tilde{x}_1) = 1 - F(\tilde{x}_2)$ and, since the third party can enter "very" close to $\tilde{x}_1$ and $\tilde{x}_2$, it follows that

$$F(\tilde{x}_1) \geq F(\frac{\tilde{x}_2 + \tilde{x}_1}{2}) - F(\tilde{x}_1)$$
$$1 - F(\tilde{x}_2) \geq F(\tilde{x}_2) - F(\frac{\tilde{x}_2 + \tilde{x}_1}{2})$$

Furthermore, assumption (**A.1**) implies that at least one of these two inequalities is strict, yielding

$$F(\tilde{x}_1) = 1 - F(\tilde{x}_2) > \frac{1}{4}. \tag{19}$$

Thus, $x_1^d < \tilde{x}_1 < \tilde{x}_2 < x_2^d$. ■

**Proof of Proposition 4.5.** Convexity of $F$ and condition (18) imply that

$$F(\tilde{x}_1) = F(\tilde{x}_2) - F(\frac{\tilde{x}_1 + \tilde{x}_2}{2}) = 1 - F(\tilde{x}_2) > F(\frac{\tilde{x}_1 + \tilde{x}_2}{2}) - F(\tilde{x}_1),$$

yielding $F(\tilde{x}_1) > \frac{1}{4}$. On the other hand, (1) and (2) yield $F(x_2^d) - F(x_1^d) = \frac{1}{2}$, and together with convexity of the function $F$ this gives rise to the following:

$$F(x_1^d) = F(\frac{x_1^d + x_2^d}{2}) - F(x_1^d) < F(x_2^d) - F(\frac{x_1^d + x_2^d}{2}) < \frac{1}{4}.$$

Hence $x_1^d < \tilde{x}_1$. Now consider the following equality

$$1 - F(\frac{x_1 + x_2}{2}) = 2(1 - F(x_2)) \tag{20}$$

which is satisfied for both pairs $(x_1^d, x_2^d)$ and $(\tilde{x}_1, \tilde{x}_2)$. Moreover, for each $x_1 \in [0, 1]$ there exists a unique $x_2 = x_2(x_1) > x_1$ for which (6) is satisfied. Furthermore, the function $x_2(\cdot)$ is differentiable on the interval $[0, 1]$. Since $x_2(x_1) > x_1$, the convexity of $F$ implies that

$$0 < \frac{dx_2}{dx_1} = -\frac{-\frac{1}{2}F'(\frac{x_1+x_2}{2})}{2F'(x_2) - \frac{1}{2}F'(\frac{x_1+x_2}{2})} < 1.$$

Thus, the established inequality $x_1^d < \tilde{x}_1$ yields $x_2^d < \tilde{x}_2$ and $\tilde{x}_2 - \tilde{x}_1 < x_2^d - x_1^d$.
■

## References


R. M. Alvarez and J. Nagler. A new approach for modeling strategic voting in multiparty elections. *British Journal of Political Science*, 30:57–75, 2000.

T. Bergstrom and M. Bagnoli. Log-concave probability and its applications. *Economic Theory*, 26:445–469, 2005.

R. Cohen. Symmetric 2-equilibria of unimodal voter distribution curves. mimeo, Harvard University, 1985.

B. Eaton and R. Lipsey. The principle of minimum product differentiation reconsidered: Some new developments in the theory of spatial competition. *Review of Economic Studies*, 42:27–49, 1975.

R. Faulí-Oller, E. A. Ok, and I. Ortuño-Ortín. Delegation and polarization of platforms in political competition. *Economic Theory*, 22:289–309, 2003.

D. Gale and H. Nikaido. The jacobian matrix and global univalence of mappings. *Mathematische Annalen*, 159:81–93, 1965.

D. Green, B. Palmquist, and E. Schickler. *Partisan Hearts and Minds: Political Parties and the Social Identities of Voters*. Yale University Press, 2002.

J. Greenberg and K. Shepsle. The effect of electoral rewards in multiparty competition with entry. *American Political Science Review*, 81:525–537, 1987.

J. Greenberg and S. Weber. Multiparty equilibria under proportional representation. *American Political Science Review*, 79:693–703, 1985.

O. Haimanko, M. Le Breton, and S. Weber. Transfers in a polarized country: Bridging the gap between efficiency and stability. *Journal of Public Economics*, 89(7):1277–1303, 2005.

M. J. Osborne. Spatial models of political competition under plurality rule: A survey of some explanations of the number of candidates and the positions they take. *Canadian Journal of Economics*, 27:261–301, 1995.

T. Palfrey. Spatial equilibrium with entry. *Review of Economic Studies*, 51: 139–156, 1984.

T. Parthasarathy. *On Global Univalence Theorems*. Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, 1983.

M. Shaked and J. Shanthikumar. *Stochastic Orders and Their Applications*. Academic Press, Boston, 1994.

S. Weber. On hierarchical spatial competition. *Review of Economic Studies*, 59:407–425, 1992.

S. Weber. Entry deterrence in electoral spatial competition. *Social Choice and Welfare*, 15:31–56, 1997.