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COMPACT FORMULATIONS AS A UNION OF POLYHEDRA

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Abstract

We explore one method for finding the convex hull of certain mixed integer sets. The approach is to break up the original set into a small number of subsets, find a compact polyhedral description of the convex hull of each subset, and then take the convex hull of the union of these polyhedra. The resulting extended formulation is then compact, its projection is the convex hull of the original set, and optimization over the mixed integer set is reduced to solving a linear program over the extended formulation.

The approach is demonstrated on three different sets: a continuous mixing set with an upper bound and a mixing set with two divisible capacities both arising in lot-sizing, and a single node flow model with divisible capacities that arises as a subproblem in network design.

Keywords: Mixed Integer Sets, Convex Hull, Unions of Polyhedra, Lot-sizing

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1 Introduction

Given a set of the form $P = \{(x, y) : \; Ax + By \leq d\}$, we address the question of finding a formulation for the mixed-integer set $Z^{MIP} = \{(x, y) \in P, \; y \text{ integer}\}$ associated with P.

In this paper a *formulation* in the (x, y)-space is a polyhedral description of conv(Z^{MIP}) in the original space. It consists of a finite set of inequalities such that conv(Z^{MIP}) = $\{(x, y) : \; A'x + B'y \leq d'\}$. A formulation of Z^{MIP} is *extended* whenever it gives a polyhedral description of conv(Z^{MIP}) in a space that uses variables (x, y, w) and includes the original space, so that conv(Z^{MIP}) is the projection of this polyhedral description onto $w = 0$. An extended formulation is *compact* if the size of its polyhedral description is polynomial in the size of the description of P. The size of numbers will not play a role in this paper. So this means that the dimension of the constraint matrix provided by the formulation is polynomial in the dimension of the matrix $[A|B]$.

Finding an extended formulation for a mixed-integer set Z^{MIP} which is compact is important. For instance, tight formulations for relaxations allow us to strengthen the linear programming representations of hard MIPs, and theoretically a proof that a problem has a compact extended formulation implies that one can optimize a linear objective over Z^{MIP} using linear programming, and thus demonstrates that this problem is in $\mathcal{P}$.

In this paper we find extended formulations that are compact for generalizations of certain mixed-integer sets that arise as relaxations of lot-sizing and network design problems and have been studied in the last decade.

For multi-item production planning problems in which these sets typically arise as single-item relaxations, these compact formulations provide an a priori strengthening of the original representation. The effectiveness of this approach has been demonstrated in Eppen and Martin [4], Miller and Wolsey [6] and Wolsey [11] among others.

For the mixed-integer sets that we study, the formulations in their original space are known to have exponential size, and they have only been partially characterized so far. Furthermore the convex hulls of these mixed-integer sets have an exponential number of vertices.

Given a mixed-integer set Z^{MIP}, the approach that we use here to compute an extended formulation for conv(Z^{MIP}) is as follows. We study the sets $V_{Z^{MIP}}$, $R_{Z^{MIP}}$ of vertices and extreme rays of conv(Z^{MIP}). We then find a small number of subsets $V^i \subseteq$ conv(Z^{MIP}) and $R^i \subseteq R_{Z^{MIP}}$ whose union contains $V_{Z^{MIP}}$ and $R_{Z^{MIP}}$ respectively. For each of the pairs (V^i, R^i) we compute a compact formulation for conv(V^i) + cone(R^i). This compact formulation will typically be an extended formulation. The last step is to derive a compact formulation which is extended for the convex hull of the union of these polyhedra. For this we use a classical result of Balas [2].

The idea of breaking the set $V_{Z^{MIP}} \cup R_{Z^{MIP}}$ into a small number of subsets has been used before: one approach found in Pochet and Wolsey [7] and developed systematically in the thesis of Van Vyve [9] is to develop an extended MIP representation for such problems explicitly including all the extreme points, and then tighten with valid inequalities until a extended formulation is obtained; another is to generate an extended formulation based on an explicit or implicit representation of all the extreme points and rays as in Miller and Wolsey [5]. A simple example of the approach studied here has appeared very recently in Atamturk [1].

2 Extended formulations and the union of polyhedra

A *polyhedron* P is the intersection of a finite number of half-spaces. Equivalently, $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. We use Minkowski-Weyl's theorem which asserts that a pointed polyhedron P has a finite set of vertices V_P and a finite set of extreme rays R_P and $P = \text{conv}(V_P)+\text{cone}(R_P)$. Conversely, for every pair of finite families V and R, there is a matrix $[A|b]$ such that $\{x \in \mathbb{R}^n : Ax \geq b\} = \text{conv}(V) + \text{cone}(R)$. We also use the fact that $\text{cone}(R) = \{x \in \mathbb{R}^n : Ax \geq 0\}$.

Lemma 1 *(Balas [2]) Assume $P^i = \{x \in \mathbb{R}^n : Q^i x \geq q^i\}$ are m polyhedra. For $1 \leq i \leq m$, let V^i, R^i be the sets of vertices and extreme rays of P^i, so $P^i = \text{conv}(V^i) + cone(R^i)$. Let*

$$P = \text{conv}(\cup_{i=1}^m V^i) + cone(\cup_{i=1}^m R^i).$$

Then the following set of inequalities provides an extended description of P:

$$\begin{array}{c} x = \sum_{i=1}^m x^i \\ Q^i x^i \geq q^i \delta^i,\ 1 \leq i \leq m \\ \sum_{i=1}^m \delta^i = 1 \\ \delta^i \geq 0,\ 1 \leq i \leq m. \end{array}$$

Proof: If $x \in P$, there are vectors in $v^i \in \text{conv}(V^i)$ and $r^i \in \text{cone}(R^i)$ such that:

$$\begin{array}{c} x = \sum_{i=1}^m (\delta^i v^i + \lambda^i r^i) \\ \sum_{i=1}^m \delta^i = 1 \\ \delta^i \geq 0,\ \lambda^i \geq 0,\ 1 \leq i \leq m. \end{array}$$

Since $v^i \in \text{conv}(V^i)$, $r^i \in \text{cone}(R^i)$, then $Q^i(v^i + \mu r^i) \geq q^i$ and $Q^i(\mu r^i) \geq 0$ for every $\mu \geq 0$: Therefore if we define $x^i = \delta^i v^i + \lambda^i r^i$, then $Q^i x^i \geq q^i \delta^i$. This shows that for every $x \in P$, the above system is feasible.

For the other direction, let (x, x^i, δ^i) be a solution of the above system. For $\delta^i > 0$, $Q^i \frac{x^i}{\delta^i} \geq q^i$ and therefore $\frac{x^i}{\delta^i} \in \text{conv}(V^i) + \text{cone}(R^i)$. For $\delta^i = 0$, $Q^i x^i \geq 0$ and therefore $x^i \in \text{cone}(R^i)$. Since $x = \sum_{\delta^i>0} \delta^i \frac{x^i}{\delta^i} + \sum_{\delta^i=0} x^i$, the result follows. □

Lemma 2 *The polyhedron P defined in Lemma 1 is the closure of the set $\text{conv}(\cup_{i=1}^m P^i)$. If all the polyhedra P^i have the same recession cone, i.e. $R^i = R^j$ for $i, j \in \{1, \ldots, m\}$, then $\text{conv}(\cup_{i=1}^m P^i)$ is a closed set and $P = \text{conv}(\cup_{i=1}^m P^i)$.*

Proof: Since P is a polyhedron, P is a closed set. We now show that for every $x \in P$, there is a sequence of vectors $x(\epsilon) \in \text{conv}(\cup_{i=1}^m P^i)$ that converges to x when $\epsilon \to 0$.

If $x \in P$, there are vectors in $v^i \in \text{conv}(V^i)$ and $r^i \in \text{cone}(R^i)$ such that:

$$x = \sum_{i=1}^m (\delta^i v^i + \lambda^i r^i),\ \sum_{i=1}^m \delta^i = 1,\ \delta^i \geq 0,\ \lambda^i \geq 0,\ 1 \leq i \leq m.$$

Let S be the subset of $M = \{1, \ldots, m\}$ such that $\lambda_i > 0$ while $\delta_i = 0$ and let $v^* = \sum_{i \in S} \frac{1}{|S|} v^i$. Define $x(\epsilon) = (1 - \epsilon)x + \epsilon v^*$. Then $x(\epsilon)$ converges to x when $\epsilon \to 0$. Furthermore we have:

$$x(\epsilon) = (1-\epsilon)\sum_{i\in M\setminus S}^{m} \delta^i(v^i + \frac{\lambda^i}{\delta^i} r^i) + \epsilon \sum_{i\in S} \frac{1}{|S|}(v^i + \frac{\lambda^i |S|(1-\epsilon)}{\epsilon} r^i)$$

So for $\epsilon > 0$, $x(\epsilon)$ is the convex combination of vectors in P_i, hence $x(\epsilon) \in \mathrm{conv}(\cup_{i=1}^m P^i)$. This shows that P is the closure of the set $\mathrm{conv}(\cup_{i=1}^m P^i)$.

We now prove the last part of the lemma by showing that $P \subseteq \mathrm{conv}(\cup_{i=1}^m P^i)$. Given an extended description of $x \in P$ as in Lemma 1, let $T = \{i \in M : \delta^i > 0\}$. Now $\frac{x^i}{\delta^i} \in P^i$ for $i \in T$, and $x^i \in \mathrm{cone}(R)$ for $i \in M \setminus T$. It follows that $w = \sum_{i\in M\setminus T} x^i \in \mathrm{cone}(R)$, and thus $\frac{x^i}{\delta^i} + w \in P^i$ for all $i \in T$. Now $x = \sum_{i\in T}[\frac{x^i}{\delta^i} + w]\delta^i$ with $\sum_{i\in T}\delta^i = 1$ and $\delta^i \geq 0$ for all $i \in T$, and so $x \in \mathrm{conv}(\cup_{i=1}^m P^i)$.

□

We will need the following straightforward "extended" version of Lemma 1:

Remark 3 *If an extended formulation* $\{(x,w) \in \mathbb{R}^{n+p} : \ A^i x + B^i w \geq d^i\}$ *is given for each of the polyhedra* $P^i = \mathrm{conv}(V^i) + cone(R^i)$*, then the following set of inequalities provides an extended formulation of the polyhedron* $P = \mathrm{conv}(\cup_{i=1}^m V^i) + cone(\cup_{i=1}^m R^i)$*:*

$$\begin{gathered} x = \textstyle\sum_{i=1}^m x^i \\ A^i x + B^i w \geq d^i \delta^i, \ 1 \leq i \leq m \\ \textstyle\sum_{i=1}^m \delta^i = 1 \\ \delta^i \geq 0, \ 1 \leq i \leq m. \end{gathered}$$

Given a set $P = \{(x,y) : \ Ax + By \leq d\}$, we use the above results to obtain an (extended) formulation for the polyhedron $P_I = \mathrm{conv}((x,y) \in P, \ y \text{ integer})$. Specifically we study the sets V_{P_I}, R_{P_I} of vertices and extreme rays of P_I. We then find subsets $V^1, \ldots, V^m$ of P_I such that $V_{P_I} \subseteq \cup_{i=1}^m V^i$. We also find sets $R^1, \ldots, R^m$ such that $R_{P_I} = \cup_{i=1}^m R^i$ and we compute a formulation for each of the m polyhedra $P^i = \mathrm{conv}(V^i) + \mathrm{cone}(R^i)$.

The formulations of the polyhedra P^i in their natural space typically involve an exponential number of inequalities: For each of the polyhedra P^i, we increase the dimension of the space by adding extra variables and find an extended formulation, which is compact. We finally use Remark 3 to obtain an extended formulation for the polyhedron P_I. The number m will be small, and thus the extended formulation will be compact.

3 Three examples

We apply the approach to derive compact extended formulations for three sets.

3.1 The Continuous Mixing Set with Upper Bound

The first set studied is the continuous mixing set with upper bound. We first introduce the continuous mixing set:

$$X^{CMIX} = \{(s, r, y) \in \mathbb{R}^1_+ \times \mathbb{R}^m_+ \times \mathbb{Z}^m_+ : s + r_t + y_t \geq b_t \text{ for } \ t = 1, \dots, m\}.$$

Miller and Wolsey [5] have given an extended formulation for X^{CMIX} which is compact and have characterized the vertices and rays. It follows from their work that both the external (inequality) representation and the internal (extreme point and ray) representation of X^{CMIX} in their original space have exponential size. Van Vyve [10] has provided a new more compact extended formulation which only involves $O(m)$ additional variables, and has shown that the separation problem in the original space can be solved by flow techniques.

We consider here the continuous mixing set with upper bound on s:

$$X^{CMIX-UB} = X^{CMIX} \cap \{(s, r, y) : s \leq u\}.$$

This set provides a relaxation motivated by the problem of treating upper bounds on stocks in lot-sizing models. Let $f_t = b_t - \lfloor b_t \rfloor$ for $t = 1, \dots, m$, $f_0 = 0$ and $f_{m+1} = u$.

Remark 4 *The extreme rays (s, y, z) of* $\text{conv}(X^{CMIX-UB})$ *are:* $(0, e_j, 0)_{j=1}^m$ *and* $(0, 0, e_j)_{j=1}^m$. *At a vertex of* $\text{conv}(X^{CMIX-UB})$, *$s = 0$, $s = f_i \bmod 1$ for some $i \in \{1, \dots, m\}$ or $s = u$.*

This shows that the set of vertices $V_{X^{CMIX-UB}}$ of $X^{CMIX-UB}$ lie in the union of the $m+2$ sets $V^i = V_{X^{CMIX-UB}} \cap \{(s, r, y) : s = f_i \bmod 1\}$.

For $0 \leq i \leq m+1$, let $P^i = \text{conv}(X^{CMIX-UB} \cap \{(s, r, y) : s = f_i \bmod 1\})$.

Remark 5 *The set V^i is contained in P^i and the recession cones of $X^{CMIX-UB}$ and P^i coincide.*

The following theorem gives a formulation for P^i that is compact. Let $f_{ti} = (b_t - f_i) - \lfloor b_t - f_i \rfloor$ for all $t = 1, \dots, m$ and $i = 0, \dots, m+1$.

Theorem 6 *The following set of inequalities gives a formulation for P^i:*

$$\begin{gathered} s \geq f_i \\ s + r_t + y_t \geq b_t \text{ for } t = 1, \dots, m \\ r_t + f_{ti}(y_t + s) \geq f_{ti}(\lceil b_t - f_i \rceil + f_i) \text{ for } t = 1, \dots, m \\ s \leq \lfloor u - f_i \rfloor + f_i \\ r \in R^m_+, s \in \mathbb{R}^1_+, y \in R^m_+. \end{gathered}$$

Proof: We reformulate the set $X^{CMIX-UB} \cap \{(s, r, y) : s = f_i \bmod 1\}$. We model the condition $s = f_i \bmod 1$ with $s = f_i + \sigma$, $\sigma \in \mathbb{Z}^1_+$. Substituting for s in the set of inequalities defining $X^{CMIX-UB}$, we obtain:

$$\begin{gathered} \sigma + r_t + y_t \geq b_t - f_i \text{ for } t = 1, \dots, m \\ \sigma \leq u - f_i \\ r \in R^m_+, \sigma \in \mathbb{Z}^1_+, y \in Z^m_+. \end{gathered}$$

Now observe that in a vertex of the convex hull of this set, either $r_t = 0 \bmod 1$ or $r_t = f_{ti} \bmod 1$ for $t = 1, \ldots, m$. This leads us to write $r_t = f_{ti}\delta_t + \mu_t$ with $\delta_t \in \{0,1\},\ \mu_t \in \mathbb{Z}^1_+$. Substituting for r_t in the above system, we obtain :

$$\sigma + f_{ti}\delta_t + \mu_t + y_t \geq b_t - f_i \text{ for } t = 1, \ldots, m$$
$$\sigma \leq u - f_i$$
$$\delta \in \{0,1\}^m, \sigma \in \mathbb{Z}^1_+, \mu \in Z^m_+, y \in Z^m_+.$$

Applying Chvátal-Gomory rounding to the above system, an equivalent, but tighter, set of inequalities is:

$$\sigma + \delta_t + \mu_t + y_t \geq \lceil b_t - f_i \rceil \text{ for } t = 1, \ldots, m$$
$$\sigma \leq \lfloor u - f_i \rfloor$$
$$\delta \in \{0,1\}^m, \sigma \in \mathbb{Z}^1_+, \mu \in Z^m_+, y \in Z^m_+.$$

Observe now that the matrix associated to the first block of m constraints is a totally unimodular matrix, and the requirements vector and bounds are integer. It follows form the theorem of Hoffman and Kruskal that we can substitute the integrality requirements with variable bounds, and obtain a formulation for the above set. This yields the following extended formulation for P^i:

$$s = \sigma + f_i$$
$$r_t = f_{ti}\delta_t + \mu_t \text{ for } t = 1, \ldots, m$$
$$\sigma + \delta_t + \mu_t + y_t \geq \lceil b_t - f_i \rceil \text{ for } t = 1, \ldots, m$$
$$\sigma \leq \lfloor u - f_i \rfloor$$
$$\delta \in [0,1]^m, \sigma \in \mathbb{R}^1_+, \mu \in R^m_+, y \in R^m_+.$$

Projecting back into the original s, r, y space using Fourier-Motzkin elimination, it is easily checked that one obtains the set of inequalities in the statement of the theorem.

□

The formulations for the polyhedra P^i obtained in Theorem 6 are in the original space and are compact. Therefore applying Lemma 1 we obtain an extended formulation for $X^{CMIX-UB}$ which is compact.

Proposition 7 *The following set of inequalities provides an extended formulation for* $\text{conv}(X^{CMIX-UB})$*:*

$$s = \sum_{i=0}^{m+1} s^i$$
$$r_t = \sum_{i=0}^{m+1} r^i_t \text{ for } t = 1, \ldots, m$$
$$y_t = \sum_{i=0}^{m+1} y^i_t \text{ for } t = 1, \ldots, m$$
$$\sum_{i=0}^{m+1} \delta^i = 1$$
$$s^i \geq f_i\delta^i \text{ for } i = 0, \ldots, m+1$$
$$s^i + r^i_t + y^i_t \geq b_t\delta^i \text{ for } t = 1, \ldots, m,\ i = 0, \ldots, m+1$$
$$r^i_t + f_{ti}(y^i_t + s^i) \geq f_{ti}(\lceil b_t - f_i \rceil + f_i)\delta^i \text{ for } t = 1, \ldots, m,\ i = 0, \ldots, m+1$$
$$s^i \leq \lfloor (u - f_i \rfloor + f_i)\delta^i \text{ for } i = 0, \ldots, m+1$$
$$r^i \in R^m_+, s^i \in \mathbb{R}^1_+, \delta^i \in \mathbb{R}^1_+, y^i \in R^m_+ \text{ for } i = 0, \ldots, m+1.$$

Remark 8 *The cases when $i = 0$ and $i = m + 1$ ($s = 0$ and $s = u$) are simpler as one can take $\sigma = 0$. This would lead to a slightly simpler formulation for P^0, but the same formulation for P^{m+1}.*

3.2 A Mixing Set with Two Divisible Capacities

Here we consider the set

$$X^{2DIV} = \{(s, y, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^m_+ \times \mathbb{Z}^m_+ : s + y_t + Cz_t \geq b_t,\ t = 1, \dots, m\}$$

where $C \in \mathbb{Z}^1$ with $C \geq 2$. This set is a relaxation for lot-sizing problems in which two machines with different capacities can produce the same item, and is close to a model treating lower bounds on production studied recently by Constantino, Miller and Van Vyve [3] and Van Vyve [9].
Let $f_t = b_t - \lfloor b_t \rfloor$ for $t = 1, \dots, m$, and $f_0 = 0$.

Remark 9 *The extreme rays (s, y, z) of $\operatorname{conv}(X^{2DIV})$ are : $(1, 0, 0), (0, e_j, 0)_{j=1}^m, (0, 0, e_j)_{j=1}^m$. At a vertex of $\operatorname{conv}(X^{2DIV})$, $s = f_t \bmod 1$ for some $t \in \{0, 1 \dots, m\}$.*

We again partition the set of vertices $V_{X^{2DIV}}$ of X^{2DIV} into $m + 1$ sets $V^i = V_{X^{2DIV}} \cap \{(s, r, y) : s = f_i \bmod 1\}$ and we define $P^i = \operatorname{conv}(X^{2DIV} \cap \{(s, r, y) : s = f_i \bmod 1\})$.

Remark 10 *The set V^i is contained in P^i and the recession cones of X^{2DIV} and P^i coincide.*

The following theorem shows the existence of an extended formulation for P^i which is compact. For this purpose, we denote by $X^{CMIX}(s, r, y, b)$ a continuous mixing set with variables (s, r, y) and requirement vector b.

Theorem 11 *Let β be an m-vector with components $\beta_t = \frac{\lceil b_t - f_i \rceil}{C}$. Then*

$$P^i = \operatorname{conv}(X^{CMIX}(\frac{s - f_i}{C}, \frac{y}{C}, z, \beta)).$$

Proof: We reformulate the set $X^{2DIV} \cap \{(s, r, y) : s = f_i \bmod 1\}$. The condition $s = f_i \bmod 1$ is again modeled using $s = f_i + \sigma,\ \sigma \in \mathbb{Z}^1_+$. Substituting for s in the set of inequalities defining X^{2DIV}, we obtain:

$$\sigma + y_t + Cz_t \geq b_t - f_i,\ t = 1, \dots, m$$
$$\sigma \in \mathbb{Z}^1_+, y \in \mathbb{Z}^m_+, z \in \mathbb{Z}^m_+.$$

As all the variables are integer, we can use Chvátal-Gomory rounding to round up the right hand sides, and we obtain the following alternative description:

$$\sigma + y_t + Cz_t \geq \lceil b_t - f_i \rceil,\ t = 1, \dots, m$$
$$\sigma \in \mathbb{Z}^1_+, y \in \mathbb{Z}^m_+, z \in \mathbb{Z}^m_+.$$

As the matrix $(1, I)$ is totally unimodular, we can drop the integrality constraints on σ and y. Let $\tau = \sigma/C$ and $r_t = y_t/C$. We now rewrite the above set as follows:

$$s = f_i + C\tau$$
$$y_t = Cr_t,\ t = 1, \ldots, m$$
$$\tau + r_t + z_t \geq \lceil b_t - f_i \rceil / C,\ t = 1, \ldots, m$$
$$\tau \in \mathbb{R}^1_+, r \in \mathbb{R}^m_+, z \in \mathbb{Z}^m_+.$$

This set is precisely the continuous mixing set $X^{CMIX}(\frac{s-f_i}{C}, \frac{y}{C}, z, \beta)$. □

Now one possibility is to use the result of the previous subsection to obtain a formulation of $\text{conv}(X^{CMIX}(\frac{s-f_i}{C}, \frac{y}{C}, z, \beta))$. Note that this formulation uses new variables and therefore is extended with respect to the original variables (s, y, z). So by Lemma 10, we can now apply Lemma 3 to obtain an extended formulation for $\text{conv}(X^{2DIV})$.

Van Vyve [10] has given a formulation of X^{CMIX} whose size is $O(m^2) \times O(m)$ and this is the smallest known. Using this formulation, we obtain the following:

Proposition 12 *There is an extended formulation for* $\text{conv}(X^{2DIV})$ *whose size is* $O(m^3) \times O(m^2)$.

3.3 A Divisible Capacity Single Node Flow Model

Here we consider the set:

$$X^{FDIV} = \{(s, x, y) \in \mathbb{R}^1_+ \times \mathbb{R}^n_+ \times \mathbb{Z}^n_+ : s + \sum_{j=1}^{n} x_j \geq b, x_j \leq C_j y_j \text{ for } j = 1, \ldots, n\},$$

where the capacities are divisible, i.e. $C_1 | C_2 | \ldots | C_n$. For $j = 1, \ldots, n$, define $\delta_j = b - C_j \lfloor \frac{b}{C_j} \rfloor$.

Lemma 13 *Let* (s, x, y) *be a vertex of* $\text{conv}(X^{FDIV})$. *Then* (s, x, y) *satisfies one of the following:*

1. $x_j = C_j y_j$ *for* $j = 1, \ldots, n$.
2. *There are indices* i, k, $1 \leq i \leq k \leq n$, *such that*
   $x_j = C_j y_j$ *for* $j \neq k$
   $x_k = C_k(y_k - 1) + \delta_i$ *or* $x_k = C_k y_k - C_i + \delta_i$.

Proof: Let (s^*, x^*, y^*) be a vertex of $\text{conv}(X^{FDIV})$.

Claim 1: *For* $1 \leq j \leq n$, $y^*_j = \lceil \frac{x^*_j}{C_j} \rceil$.

The integrality of y^*_j and the constraint $x_j \leq C_j y_j$ show $y^*_j \geq \lceil \frac{x^*_j}{C_j} \rceil$. This inequality must always be tight, else $(s^*, x^*, y^* \pm e_j) \in \text{conv}(X^{FDIV})$, and (s^*, x^*, y^*) cannot be a vertex, a contradiction.

Since (s^*, x^*, y^*) is a vertex of $\text{conv}(X^{FDIV})$, (s^*, x^*) is a vertex of the polyhedron defined by the system:

$$s + \sum_{j=1}^{n} x_j \geq b,\ x_j \leq C_j y^*_j,\ s \geq 0,\ x_j \geq 0,\ 1 \leq j \leq n.$$

This shows that if $s^* > 0$ or $s^* + \sum_j x_j^* > b$, then $x_j^* = 0$ or $x_j^* = C_j y_j^*$ for all j. In this case, by Claim 1, $x_j^* = 0$ implies $y*_j = 0$, so the second equality is always satisfied and therefore 1. holds. Thus we can assume that $s^* = 0$ and $\sum_j x_j^* = b$. Therefore $0 < x_j^* < C_j y_j^*$ for at most one index. Let k be such an index, so $x_k^* = C_k\beta + \alpha$ where β is integer and $0 < \alpha < C_k$.

Claim 2: *Let $i = argmin\{j : x_j^* > 0\}$. Then $\alpha = \delta_i \bmod C_i$.*

Since for $j \neq k$ we have $x_j^* = C_j y_j^*$ and $x_k^* = C_k\beta + \alpha$, the divisibility of the C_j and equation $\sum_j x_j^* = b$ imply $C_i K + \alpha = b$ for some nonnegative integer K. So $\alpha = b \bmod C_i$, or equivalently $\alpha = \delta_i \bmod C_i$.

Claim 3: *Either $\alpha < C_i$ or $\alpha > C_k - C_i$.*

If $i = k$, then $\alpha < C_i$. So assume $i < k$ and $C_i \leq \alpha \leq C_k - C_i$. Now both points $(s^*, x^* + C_i e_i - C_i e_k, y^* + e_i)$ and $(s^*, x^* - C_i e_i + C_i e_k, y^* - e_i)$ belong to $\mathrm{conv}(X^{FDIV})$, and (s^*, x^*, y^*) cannot be a vertex, a contradiction.

Claims 2 and 3 show that either $\alpha = \delta_i$ or $\alpha = C_k - C_i + \delta_i$. Finally, by Claim 1, we have $y_k^* - 1 = \beta$. So either $x_k = C_k(y_k - 1) + \delta_i$ or $x_k = C_k y_k - C_i + \delta_i$ and 2. holds. □

Let $Q^0 = \{(s,x,y) : x_j = C_j y_j,\ 1 \leq j \leq n\}$ and for $1 \leq i \leq k \leq n$, define $Q_1^{i,k} = \{(s,x,y) : x_j = C_j y_j,\ j \neq k;\ x_k = C_k(y_k - 1) + \delta_i,\ y_k \geq 1\}$ and $Q_2^{i,k} = \{(s,x,y) : x_j = C_j y_j,\ j \neq k;\ x_k = C_k y_k - C_i + \delta_i,\ y_k \geq 1\}$.

Finally let $P^0 = \mathrm{conv}(X^{FDIV} \cap Q^0)$, $P_1^{i,k} = \mathrm{conv}(X^{FDIV} \cap Q_1^{i,k})$ and $P_2^{i,k} = \mathrm{conv}(X^{FDIV} \cap Q_2^{i,k})$.

Remark 14 *The extreme rays (s,x,y) of* $\mathrm{conv}(X^{FDIV})$ *are $(1,0,0)$, $(0, Ce_j, e_j)_{j=1}^n$ and $(0,0,e_j)_{j=1}^n$. The extreme rays of P^0, $P_1^{i,k}$, $P_2^{i,k}$ are $(1,0,0)$ and $(0, Ce_j, e_j)_{j=1}^n$.*

Denote by $X^{KDIV}(s,y,b)$ the knapsack set with divisible capacities:

$$\{(s,y) \in \mathbb{R}_+^1 \times \mathbb{Z}_+^n : s + \sum_{j=1}^n C_j y_j \geq b\}.$$

Finding a formulation for P^0 is obviously equivalent to finding a formulation for set $X^{KDIV}(s,y,b)$. The following lemma shows that finding a formulation for $P_1^{i,k}$ $P_2^{i,k}$ also reduces to finding a formulation for a set $X^{KDIV}(s,y,b')$ for some suitable b'.

Lemma 15 *Let $\alpha_1^{i,k} = \delta_i$ and $\alpha_2^{i,k} = C_k - C_i + \delta_i$ for $1 \leq i \leq k \leq n$. Then for $t = 1,2$*

$$P_t^{i,k} = \mathrm{conv}(X^{KDIV}(s, y - e_k, b - \alpha_t^{i,k})) \cap Q_t^{i,k}.$$

Proof: The set $X^{FDIV} \cap Q_t^{i,k}$ is the set of vectors (s,x,y), $s \in \mathbb{R}_+^1$, $x \in \mathbb{R}_+^n$, $y \in \mathbb{Z}_+^n$ satisfying the following system:

$$\begin{aligned} s + \textstyle\sum_{t=1}^n x_t &\geq b \\ x_k &= C_k(y_k - 1) + \alpha_t^{i,k} \\ x_j &= C_j y_j \text{ for } j \neq k \\ y_k &\geq 1. \end{aligned}$$

This system can be rewritten as:

$$x_k = C_k(y_k - 1) + \alpha_t^{i,k}$$
$$x_j = C_j y_j \text{ for } j \neq k$$
$$y_k \geq 1$$
$$s + \sum_{j=1}^n C_j y_j \geq b + C_k - \alpha_t^{i,k}.$$

The set of vectors (s, y), $s \in \mathbb{R}_+^1, y \in \mathbb{Z}_+^n$ that satisfy the system $y_k \geq 1$, $s + \sum_{j=1}^n C_j y_j \geq b + C_k - \alpha_t^{i,k}$ is precisely the set $X^{KDIV}(s, y - e_k, b - \alpha_t^{i,k}) \cap Q_t^{i,k}$. □

Let $V_{X^{FDIV}}$ be the set of vertices of $\text{conv}(X^{FDIV})$ and let V^0, $V_t^{i,k}$ be the set of vertices of P^0 and $P_t^{i,k}$. Now Lemma 13 and Remark 14 show the following:

Remark 16 *The extreme rays of P^0, $P_t^{i,k}$ are extreme rays of* $\text{conv}(X^{FDIV})$*. The extreme rays of* $\text{conv}(X^{FDIV})$ *that are not rays of any of the P^0, $P_t^{i,k}$ are $(0, 0, e_j)_{j=1}^n$. Furthermore $V_{X^{FDIV}} \subseteq V^0 \cup (\bigcup_{i \leq k}(V_1^{i,k} \cup V_2^{i,k}))$.*

Pochet and Wolsey (see Theorem 18 in [8]) give an extended formulation for $X^{KDIV}(s, y, b)$ of size $\text{O}(n^2) \times \text{O}(n^2)$. So by Lemma 15, each of the P^0, $P_1^{i,k}$, $P_2^{i,k}$ admits an extended formulation which is compact. The number of these polyhedra is $\text{O}(n^2)$.

Consider now the extended formulation obtained by applying Remark 3 to the compact extended formulations of all the polyhedra P^0, $P_1^{i,k}$, $P_2^{i,k}$. By Remark 16, the projection of this formulation in the s, x, y-space defines a polyhedron whose set of vertices is the set $V_{X^{FDIV}}$ and whose extreme rays are all the extreme rays of $\text{conv}(X^{FDIV})$ except $(0, 0, e_j)_{j=1}^n$. It is easy to modify the formulation given by Remark 3 so that these rays are included by transforming the equations defining vector y into inequalities. This argument shows the following:

Proposition 17 *There is an extended formulation for* $\text{conv}(X^{FDIV})$ *whose size is $O(n^4) \times O(n^4)$.*

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