

# Nested Identification of Subjective Probabilities<sup>1</sup>

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## **Abstract**

The theory of games against nature relies on complete preferences among all conceivable acts, i.e. among all potential assignments of consequences to states of nature (case 1). Yet most decision problems call for choosing an element from a limited set of acts. And in games of strategy, the set of strategies available to a player is given and not amenable to artificial extensions. In “Assessing Strategic Risk” (CORE DP 2005/20), R.J Aumann and J.H. Drèze extend the basic result of decision theory (maximisation of subjectively expected utility) to situations where preferences are defined only for a given set of acts, and for lotteries among these and sure consequences (case 2). In this paper, we provide a similar extension for two other situations: those where only the *set of optimal elements* from a given set of acts is known (case 3); and those where only *a single optimal act* is known (case 4). To these four cases correspond four nested sets of admissible subjective probabilities over the states or the opponent’s strategies, namely a singleton in case 1 and increasing sets in cases 2-4. The results for cases 3 and 4 also define the extent to which

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<sup>1</sup>This paper is an outgrowth of joint research with R.J. Aumann (2005), and I regard it as joint work. Because I wrote up the paper on my own (after some e-mail exchanges), Aumann tactfully declined to appear as co-author. I had to agree, reluctantly, and I thank him warmly for the stimulating cooperation. I have also benefitted from helpful discussions with Jean-Francois Mertens. I assume sole responsibility for the contents.

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subjective probabilities must be specified in order to solve a given decision problem or play a given game.

# 1 Introduction

The standard model of decision theory, as used e.g. by Savage (1954) or Anscombe and Aumann (1963), proceeds from preferences on a comprehensive set of acts. Specifically, let  $S$  be the set of states of nature  $s$  and  $C$  be the set of pure consequences  $c$ . In Savage, the set of acts  $F$  is the set of mappings  $f$  of  $S$  into  $C$ . In Anscombe-Aumann, it is the set of probability distributions over  $F$ , say  $\Delta(F)$ . Yet, a standard decision problem calls for the choice of some element from a proper subset of  $F$ , say the set  $R$  of alternatives really open to choice. Additional acts, elements of  $F \setminus R$  or  $\Delta(F) \setminus \Delta(R)$ , are introduced for analytical convenience, and for the strength of conclusions thereby reached: preferences are represented by subjectively expected utility, with utility  $u$  defined uniquely up to positive linear transformations and *subjective probability  $p$  defined uniquely*.

More recently, Aumann and Drèze (2005) – hereafter ASR – have presented a parallel analysis for decision in games of strategy (GoS). They look at a game from the viewpoint of a single player, called “the protagonist”; all other players are combined into a single “opponent”. Let then  $S$  be the set of the opponent’s strategies  $s$ ,  $R$  be the set of the protagonist’s strategies  $r$ , and  $C$  be the set of possible outcomes of the game for the protagonist. Each strategy  $r \in R$  defines a mapping  $h_r$  of  $S$  into  $C$ . ASR proceeds from complete preferences over  $\Delta(R \cup C)$  and derives a subjective-expected-utility representation of these preferences. Utility is still unique up to positive linear transformations. Subjective probability is in general not unique: there may exist several probabilities, like  $p$  and  $p'$ , such that the expected utilities  $u_p(r) = \sum_s p_s u(h_r(s))$  and  $u_{p'}(r) = \sum_s p'_s u(h_r(s))$  are equal, for each  $r \in R$  – a property labeled “effective uniqueness” in ASR (section 6.2). In such cases, preferences over  $\Delta(R \cup C)$  do not permit discrimination between  $p$  and  $p'$ . And such cases arise when the matrix  $[u(h_r(s)), r \in R, s \in S]$ , has rank less than  $S$  – a situation avoided under a comprehensive set of acts.<sup>1</sup>

The reason for entertaining preferences over  $\Delta(R \cup C)$  is twofold: (i) introducing hypothetical strategies  $\tilde{r} \notin R$  changes the game, with potential consequences for preferences and their expected-utility representation; (ii)  $R$  will typically fail to include constant strategies,

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<sup>1</sup> $F$  includes acts that “stake a prize” on a single state, and this feature applies to every  $s$  in  $S$ .

with  $h_r(s) = c \forall s \in S$ , some  $c \in C$ ; accordingly,  $u(c)$  cannot be inferred from preferences over  $\Delta(R)$  alone; elements of  $\Delta(R \cup C)$ , called “hybrid lotteries” in ASR, are introduced to that end.<sup>2</sup>

The main theorem in ASR, hereafter MTASR, which asserts existence of a subjective-expected-utility representation of preferences over  $\Delta(R \cup C)$  verifying effective uniqueness, is of course applicable to games against nature (GAN) as well – although the motivation for restricting attention there to preferences over  $\Delta(R \cup C)$  instead of the full  $\Delta(F)$  is less compelling.<sup>3</sup> Still, it is a useful result in that context, because the decision maker “might have difficulty in forming meaningful preferences between highly hypothetical options” or “might be reluctant to evaluate carefully acts that are clearly irrelevant”.<sup>4</sup>

In games of strategy, preferences over mixed strategies are meaningful: these are precisely the objects of choice open to the protagonist. Yet, these preferences are not “observable”, in particular not subject to (potentially) observable binary choices. The only observable choices concern optimal strategies: the subset, say  $M \subset R$ , or  $\Delta(M) \subset \Delta(R)$ , some element of which the protagonist will *actually play*. Indeed, the protagonist must choose some mixed strategy, and the set of preferred choices is  $\Delta(M)$ . Thus the definition of  $M$ , and preferences over  $\Delta(M \cup C)$  are “operational” concepts. In fact, they are the very concepts entertained, for a different context, in the “revealed-preferences” theory of Samuelson (1950) and his followers.

Under a more restrictive notion of “operationalism”, one might regard a single element of  $M$  as “observable”, namely the pure strategy, say  $r^*$ , *actually played* by the protagonist.

The present paper develops this revealed-preferences approach to decision theory for both GAN and GoS (section 2), and relates it to standard decision theory as well as to ASR (section 3). Our main result, theorem 1, provides a subjective-expected-utility representation based on axiomatisation of  $M$  and of preferences that are complete only on  $\Delta(M \cup C)$ . Corollary 1 treats the case where preferences are complete only on  $\Delta(r^* \cup C)$ .

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<sup>2</sup>That preferences over  $\Delta(R) \cup \Delta(C) \subset \Delta(R \cup C)$  would not quite do is a subtle point not relevant to our purpose here.

<sup>3</sup>In particular, adding hypothetical acts does not affect nature’s choices.

<sup>4</sup>ASR, section 6.1. In particular, existence of a dominant strategy eliminates the need to assess alternatives.

## 2 Main Result

We adopt the notation of ASR, and interpret it indifferently for GAN's or for GoS's. A game  $G$  consists of

- a finite set  $R$  with elements  $r$  (the pure strategies of the protagonist or the acts of the decision maker),
- a finite set  $S$  with elements  $s$  (the pure strategies of the opponent or the states of nature; states for short),
- a finite set  $C$  with elements  $c$  (pure consequences),
- a function  $h : R \times S \rightarrow C$  (the outcome function of the protagonist in a GoS or the definition of the acts in a GAN).

Thus,  $G = (R, S, C, h)$ . We write  $h_r(s)$  for the consequence associated with the pair  $(r, s) \in R \times S$ .

For a finite set  $A$ , the set of probability distributions on  $A$  is denoted  $\Delta(A)$ , with elements  $\alpha$ . Thus,  $\gamma \in \Delta(C)$  is a mixed consequence, and  $\rho \in \Delta(R)$  is a mixed strategy in a GoS or a lottery over acts in a GAN. By a slight abuse of notation, we write  $\rho_s$  for the mixed consequence associated by  $\rho$  with state  $s$ . As for  $\Delta(R \cup C)$ , with elements  $\lambda$ , it is a set of hybrid lotteries defined by triplets  $(\rho^\lambda, \gamma^\lambda, t^\lambda) \in \Delta(R) \times \Delta(C) \times [0, 1]$ . In state  $s$ , the hybrid lottery  $\lambda$  entails the mixed consequence  $\lambda_s$  yielding  $\rho_s^\lambda$  with probability  $t^\lambda$  and  $\gamma^\lambda$  with probability  $(1 - t^\lambda)$ ; so, we write  $\lambda_s = t\rho_s^\lambda + (1 - t)\gamma^\lambda \in \Delta(C)$ .

In order to develop our “revealed preference” analysis, we start from a *partial ordering*  $\succsim$  on  $\Delta(R \cup C)$ , which in particular separates a set of preferred mixed strategies  $\Delta(M)$ ,  $M \subseteq R$ , from the remaining mixed strategies,  $\Delta(R) \setminus \Delta(M)$ . The interpretation is that *the protagonist in the game  $G$  is indifferent between playing any strategy  $\rho \in \Delta(M)$  but will not play any  $\rho' \in \Delta(R) \setminus \Delta(M)$ .*

Four assumptions will define fully our *partial ordering*  $\succsim$  on  $\Delta(R \cup C)$ . By definition,  $\succsim$  is transitive and reflexive, but not necessarily complete; it embodies the usual definitions of indifference ( $\sim$ ) and strict preference ( $\succ$ ).

**Assumption 1** *There exists  $M \subseteq R, M \neq \emptyset$ , such that:  $\rho \sim \rho' \forall \rho, \rho' \in \Delta(M)$  and  $\rho \succ \rho' \forall \rho \in \Delta(M), \rho' \in \Delta(R) \setminus \Delta(M)$ .*

Assumption 1 amounts to the assertion that the protagonist is willing to play the game ( $M \neq \emptyset$ ), and reveals her *full set of preferred strategies*  $\Delta(M)$ .

Next, we define a *complete preference ordering*  $\succsim$  on a set  $\Delta(A)$  to be an  $N - M$  *preference ordering* if it satisfies the standard axioms of utility theory, as stated for instance in von Neumann and Morgenstern (1944) or Luce and Raiffa (1957). And we define an  $N - M$  *utility* on  $\Delta(A)$  to be a real-valued  $u$  on  $\Delta(A)$  such that,  $\forall \alpha, \alpha' \in \Delta(A)$  and  $\forall t \in [0, 1]$ ,

- $\alpha \succsim \alpha'$  iff  $u(\alpha) \geq u(\alpha')$ ;
- $u(t\alpha + (1 - t)\alpha') = tu(\alpha) + (1 - t)u(\alpha')$ .

As is well known, and  $N - M$  preference admits an  $N - M$  utility representation.

**Assumption 2** *The restriction of  $\succsim$  to  $\Delta(M \cup C)$  is an  $N - M$  preference.*

Thus, on  $\Delta(M \cup C)$ , the preference ordering is complete and admits an  $N - M$  utility representation.

**Assumption 3** *For  $\lambda, \lambda' \in \Delta(R \cup C)$ , if  $\lambda_s \succsim (\succ) \lambda'_s \forall s \in S$ , then  $\lambda \succsim (\succ) \lambda'$ .*

Assumption 3 introduces a condition of monotonicity which extends our partial preference ordering to *some* elements of  $\Delta(R \cup C) \setminus \Delta(M \cup C)$ , namely those elements among which a preference domination holds. Note that  $\lambda_s \succsim \lambda'_s$  is well defined in view of assumption 2, applied to  $\Delta(C) \subset \Delta(M \cup C)$ . Assumption 3 embodies the “reversal of order” condition of Anscombe and Aumann (1963) and a weak form of the “sure-thing principle” of Savage (1954).<sup>5</sup>

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<sup>5</sup>Beside GAN and GoS, there exist one-person games where the occurrence of the “states” is influenced by the strategy choices of the decision maker. (Think about record-breaking performances in sports or athletics.) Such games are called “games of strength and skill” by von Neumann and Morgenstern (1944); they are called “games with moral hazard” by Drèze (1987), where strategies are not observable. In such situations, “reversal of order” fails, and is replaced in Drèze (1987) by the weaker assumption of “non-negative value of information”. No doubt, the developments in the present paper have a counterpart under moral hazard; but we have not attempted to spell out that counterpart, which is apt to be complex. The same remark applies for state-dependent preferences, the other generalisation covered in Drèze (1987).

Assumptions 1-3 have an important implication, worth stating formally.

**Proposition 1** *Let  $\rho, \rho', \rho'' \in \Delta(R)$  be such that, for some  $t \in (0, 1)$ ,  $\rho_s \sim t\rho'_s + (1-t)\rho''_s$  for all  $s \in S$ ; then,  $\rho \in \Delta(M)$  if and only if  $\rho' \in \Delta(M)$  and  $\rho'' \in \Delta(M)$ .*

**Proof** By assumption 3,  $\rho$  is indifferent to  $\rho' + (1-t)\rho''$ . If  $\rho \in \Delta(M)$ , then  $\rho' + (1-t)\rho'' \in \Delta(M)$ . Accordingly, by assumption 2,  $\rho \sim \rho'$  and  $\rho \sim \rho''$ , so that  $\rho' \in \Delta(M)$  and  $\rho'' \in \Delta(M)$ . Conversely, either  $\rho' \notin \Delta(M)$  or  $\rho'' \notin \Delta(M)$  implies  $\rho \notin \Delta(M)$ .  $\square$

This is a natural property: in GAN,  $\rho$  could not be part of the preferred set  $\Delta(M)$  if it is a convex combination (preference wise) of a preferred and a discarded strategy, or of two discarded strategies. The status of this property in *GoS* is discussed in section 4.2.

**Theorem 1** *Under assumptions 1, 2 and 3, there exist:*

- *an  $N - M$  utility  $u$  on  $\Delta(C)$ ,*
- *a non-empty convex set  $\Gamma \subset \Delta(C)$  such that  $\gamma \in \Gamma$  and  $\rho \in \Delta(M)$  imply  $\gamma \sim \rho$ ,*
- *a non-empty convex set  $P_3$  of probabilities on  $S$  such that, for all  $p \in P_3$ :*

(i)  $u_p(\rho) := \sum_{s \in S} p_s u(\rho_s) = u(\gamma) \forall \rho \in \Delta(M), \gamma \in \Gamma;$

(ii)  $u_p(\rho) > u_p(\rho') \forall \rho \in \Delta(M), \rho' \in \Delta(R) \setminus \Delta(M).$

This theorem establishes that the choice by the protagonist of the set  $M$  of preferred strategies is sustained by a subjective expected utility analysis, where probabilities are in general not unique, but satisfy effective uniqueness over  $\Delta(M)$ . Indeed, (i) implies  $u(\gamma) = u_p(\rho) = u_{p'}(\rho) \forall p, p' \in P_3$  and  $\rho \in \Delta(M)$ . Theorem 1 covers case 3 in the abstract (hence the notation  $P_3$ ).

**Proof of theorem 1** Assume w.l.o.g. that  $C$  contains  $\gamma, \gamma'$  with  $\gamma \succ \gamma'$ . (Otherwise, take  $u$  to be identically 0, and  $P_3$  the set of all probability distributions on  $S$ .)

From assumption 2, we obtain directly the utility  $u$ , which we normalize (arbitrarily), and the set  $\Gamma$ . Next, we eliminate temporarily from consideration any state  $\hat{s} \in S$  such

that there exist  $\rho \in \Delta(R), \rho' \in \Delta(M)$  with  $\rho_s \succsim \rho'_s \forall s$  and  $\rho_{\bar{s}} \succ \rho'_{\bar{s}}$ .<sup>6</sup> Indeed, it will be the case that  $p_{\bar{s}} = 0 \forall p \in P_3$ . Denote by  $\bar{S}$  the set of remaining states, i.e. those not thereby eliminated.

**Convention:** for the rest of this proof, we represent a hybrid lottery  $\lambda$  by the  $\bar{S}$ -vector of the (expected) utilities  $u(\lambda_s)$  of its mixed consequences in states  $s \in \bar{S}$ . Thus,  $\lambda$  is a point in Euclidean  $\bar{S}$ -space  $\mathfrak{R}^{\bar{S}}$ .

Preference relations among consequences are thus replaced by inequalities among utilities; every hybrid lottery in  $\Delta(R \cup C)$  is defined by a point in  $\mathfrak{R}^{\bar{S}}$ ; and every  $\gamma \in \Delta(C)$  is a point on the main diagonal of  $\mathfrak{R}^{\bar{S}}$ . We henceforth write  $\tilde{\gamma}$  for the unique point of the main diagonal of  $\mathfrak{R}^{\bar{S}}$  corresponding to  $\Gamma$ . As for  $\Delta(R)$ , it is a convex compact set in  $\mathfrak{R}^{\bar{S}}$ .

The proof of theorem 1 rests on the concept of “admissibility”, introduced in Arrow et al. (1953).

**Definition 1** *Let  $\mathcal{A}, \mathcal{B}$  be convex sets in  $\mathfrak{R}^{\bar{S}}$  with  $\mathcal{A} \subseteq \mathcal{B}$ . Then  $\mathcal{A}$  is an admissible set for  $\mathcal{B}$  if and only if there do not exist  $\lambda \in \mathcal{A}, \lambda' \in \mathcal{B}$  with  $\lambda' > \lambda$ .*<sup>7</sup>

**Lemma 1**  *$\Delta(M)$  is an admissible set for  $\Delta(R)$ .*

**Proof** Using assumptions 1 and 3, and the definition of  $\bar{S}$ , if  $\rho \in \Delta(M)$ , then there does not exist  $\rho' \in \Delta(R)$  with  $\rho' > \rho$ . Thus  $\Delta(M)$  is an admissible set for  $\Delta(R)$ .  $\square$

An alternative statement of lemma 1 is: for every  $\rho \in \Delta(M), \{\rho + \mathfrak{R}_+^{\bar{S}}\} \cap \Delta(R) = \{\rho\}$ .

We now consider the sets of hybrid lotteries  $\Delta(M \cup \Gamma)$  and  $\Delta(R \cup \Gamma) \in \mathfrak{R}^{\bar{S}}$ . These subsets of  $\Delta(M \cup C)$  and  $\Delta(R \cup C)$  respectively correspond to the convex hulls of the union  $M$ , respectively  $R$ , with the point  $\tilde{\gamma}$  on the main diagonal of  $\mathfrak{R}^{\bar{S}}$  defining the utility of a mixed consequence indifferent to playing the game  $G$ .

The properties of  $\Delta(M) \subseteq \Delta(R)$  also hold for  $\Delta(M \cup \Gamma) \subseteq \Delta(R \cup \Gamma)$ , in particular proposition 1 and lemma 1.

**Lemma 2** *Let  $\lambda, \lambda', \lambda'' \in \Delta(R \cup \Gamma)$  be such that, for some  $\tau \in (0, 1)$ ,*

<sup>6</sup>Of course,  $\rho$  as defined also belongs to  $\Delta(M)$ .

<sup>7</sup>Notation:  $\rho' \geq \rho$  if  $\rho'_s \geq \rho_s \forall s$ ;  $\rho' > \rho$  if  $\rho'_s \geq \rho_s \forall s$  and  $\rho' \neq \rho$ ;  $\rho' \gg \rho$  if  $\rho'_s > \rho_s \forall s$ .



$\lambda_s = \tau\lambda'_s + (1 - \tau)\lambda''_s \ \forall s \in \bar{S}$ ; then,  $\lambda \in \Delta(M \cup \Gamma)$  if and only if  $\lambda' \in \Delta(M \cup \Gamma)$  and  $\lambda'' \in \Delta(M \cup \Gamma)$ .

**Proof** With  $\lambda = t\rho + (1 - t)\tilde{\gamma}$ ,  $\lambda' := t'\rho' + (1 - t')\tilde{\gamma}$  and  $\lambda = \tau\lambda' + (1 - \tau)\lambda''$ , let  $\hat{\tau} := \tau t' + (1 - \tau)t'' \in (0, 1)$  and  $\hat{\rho} = \frac{\tau t' \rho' + (1 - \tau)t'' \rho''}{\hat{\tau}}$ , so that  $\lambda = \hat{\tau}\hat{\rho} + (1 - \hat{\tau})\tilde{\gamma}$ . If  $\lambda \in \Delta(M \cup \Gamma)$  so that  $\lambda \sim \rho \sim \tilde{\gamma}$ , then  $\lambda \sim \hat{\tau}\hat{\rho} + (1 - \hat{\tau})\tilde{\gamma} \sim \hat{\tau}\hat{\rho} + (1 - \hat{\tau})\tilde{\gamma}$  implying  $\rho \sim \hat{\rho}$ . By proposition 1,  $\rho \in \Delta(M)$  if and only if  $\rho', \rho'' \in \Delta(M)$  hence  $\lambda', \lambda'' \in \Delta(M \cup \Gamma)$ .  $\square$

**Lemma 3**  $\Delta(M \cup \Gamma)$  is an admissible set for  $\Delta(R \cup \Gamma)$ .

**Proof** Let  $\lambda \in \Delta(M \cup \Gamma)$ ,  $\lambda' \in \Delta(R \cup \Gamma) \setminus \Delta(M \cup \Gamma)$  with  $\lambda' = t'\rho' + (1 - t')\tilde{\gamma} > \lambda = t\rho + (1 - t)\tilde{\gamma} \sim t'\rho + (1 - t')\tilde{\gamma}$ . This implies  $\rho' \succ \rho$ , contradicting  $\lambda' \in \Delta(R \cup \Gamma) \setminus \Delta(M \cup \Gamma)$ .  $\square$   
An alternative statement of lemma 3 is: for every  $\lambda \in \Delta(M \cup \Gamma)$ ,  $\{\lambda + \mathfrak{R}_+^{\bar{S}}\} \cap \Delta(R \cup \Gamma) = \{\lambda\}$ .

**Lemma 4** Either  $\Delta(M) = \Delta(R)$  or  $\Delta(M \cup \Gamma) \cap ri\Delta(R \cup \Gamma) = \emptyset$ .

where *ri* stands for “relative interior”.

**Proof** Let  $\Delta(R) \setminus \Delta(M) \neq \emptyset$ . If  $\Delta(M \cup \Gamma)$  is a singleton ( $\gamma$ ), then it is an extreme point of  $\Delta(R \cup \Gamma)$  and the lemma is true. Otherwise, let  $\lambda \in \Delta(M \cup \Gamma)$ ,  $\lambda' \in \Delta(R \cup \Gamma) \setminus \Delta(M \cup \Gamma)$ . If  $\lambda \in ri\Delta(R \cup \Gamma)$ , there exists  $\mu > 1$  such  $\mu\lambda + (1 - \mu)\lambda' := \lambda'' \in \Delta(R \cup \Gamma)$  (Rockafellar, 1970, theorem 6.4); and there exists  $\tau = \frac{1}{\mu} \in (0, 1)$  such that  $\lambda = \tau\lambda' + (1 - \tau)\lambda''$ . But lemma 2 then implies  $\lambda' \in \Delta(M \cup \Gamma)$ , a contradiction.  $\square$

To prove theorem 1, let then  $M^* := \cup_{\lambda \in \Delta(M \cup \Gamma)} \{\lambda + \mathfrak{R}_+^{\bar{S}}\}$ , a convex set. By lemma 3 and lemma 4,  $ri M^* \cap ri \Delta(R \cup \Gamma) = \emptyset$ . Accordingly there exists a hyperplane, say  $B$  separating  $M^*$  from  $\Delta(R \cup \Gamma)$  (Rockafellar, 1970, theorem 11.3), and containing  $\Delta(M \cup \Gamma) = M^* \cap \Delta(R \cup \Gamma)$ . Accordingly, there exists a normal vector to  $B$ , say  $\bar{p}$ , with  $\bar{p} > 0$ ,  $\sum_{s \in \bar{S}} \bar{p}_s = 1$ ,  $\bar{p}\lambda = \bar{p}\lambda' \ \forall \lambda, \lambda' \in \Delta(M \cup \Gamma)$  and  $\bar{p}\lambda > \bar{p}\lambda' \ \forall \lambda \in \Delta(M \cup \Gamma)$ ,  $\lambda' \in \Delta(R \cup \Gamma) \setminus \Delta(M \cup \Gamma)$ .

Denote by  $\bar{P}_3$  the set of all vectors in  $\mathfrak{R}_+^{\bar{S}}$  verifying these four properties. We have just shown that  $\bar{P}_3$  is non-empty. Also,  $\bar{P}_3$  is convex because each of the four defining properties is preserved under convex combinations.<sup>8</sup>

<sup>8</sup>Actually  $\bar{P}_3$  is (i) the union of the normal cones to the set of hyperplanes containing  $\Delta(M)$  and separating  $\Delta(R)$  from  $\{x + R_+^{\bar{S}}, x \in \Delta(M)\}$ , (ii) intersected with the unit simplex of  $\mathfrak{R}_+^{\bar{S}}$ .

Abandoning our convention, let  $P_3 \subset \mathfrak{R}^S$  be the set of vectors  $p$  defined by :  
 $\exists \bar{p} \in \bar{P}_3, p_{\bar{S}} = \bar{p}; p_s = 0 \forall s \in S \setminus \bar{S}$ . That is, the restriction of  $\hat{P}_3$  to  $\mathfrak{R}^{\bar{S}}$  is given by  $\bar{P}_3$  and the restriction of  $P_3$  to  $\mathfrak{R}^{S \setminus \bar{S}}$  is the zero vector. Then  $P_3$  satisfies conclusions (i) and (ii) of theorem 1.  $\square$

**Remark** It is not claimed that  $P_3$  is closed. For instance, if  $\Delta(M)$  is an extreme point of  $\Delta(R)$ ,  $P_3$  does not include the vectors normal to the faces of  $\Delta(R)$  adjacent to the extreme point.

### 3 Nested Identification of Subjective Probabilities

#### 3.1

To cover case 4 in the abstract, we now state and prove the corollary to theorem 1 holding when assumptions 1 and 2 are weakened as follows.

**Assumption 1\*** *There exists  $r^* \in \mathfrak{R}$  such that  $r^* \succsim \rho \forall \rho \in \Delta(R)$ .*

**Assumption 2\*** *The restriction of  $\succsim$  to  $\Delta(r^* \cup C)$  is an  $N$ - $M$  preference.*

**Corollary 1** *Under assumptions 1\*, 2\* and 3 there exist:*

- *an  $N$ - $M$  utility  $u$  on  $\Delta(C)$ .*
- *a non-empty convex set  $\Gamma \subset \Delta(C)$  such that  $\gamma \in \Gamma$  implies  $\gamma \sim r^*$ ,*
- *a non-empty convex set  $P_4$  of probabilities on  $S, P_4 \supseteq P_3$ , such that, for all  $p \in P_4$  :*

(i)  $u_p(r^*) = \sum_{s \in S} p_s u(r_s^*) = u(\gamma) \forall \gamma \in \Gamma :$

(ii)  $u_p(r^*) \geq u_p(\rho) \forall \rho \in \Delta(R)$ .

**Proof** Repeating step by step the reasoning in the proof of theorem 1, with  $M$  systematically replaced by  $\{r^*\}$ , we obtain successively the  $N - M$  utility  $u$  on  $\Delta(C)$ , the set  $\Gamma$  and a non-empty convex set of probabilities  $P_4$  satisfying conclusions (i) and (ii) of corollary 1. Furthermore,  $P_4 \supseteq P_3$  because every  $p \in P_3$  satisfies conclusions (i) and (ii) in corollary 1, and  $P_4$  is comprehensive.  $\square$

**Remark** When  $r^*$  is a dominant strategy,  $P_4$  is the unit simplex of  $\mathfrak{R}^S$  (and  $\bar{S}^* = S$ ).

### 3.2

Turning to case 2 in the abstract, ASR rests on assumption 3 and

**Assumption 2'** *There is an  $N - M$  preference  $\succsim$  on  $\Delta(R \cup C)$ .*

**Theorem 2** *Under assumptions 2' and 3, there exist*

- *an  $N$ - $M$  utility on  $\Delta(C)$ ,*
- *a non-empty convex set  $P_2 \subseteq P_3$  such that, for all  $\lambda, \lambda' \in \Delta(R \cup C)$ ,  $\lambda \succsim \lambda'$  iff, for each  $p \in P_2$ ,  $u_p(\lambda) \geq u_p(\lambda')$ ,*
- *for all  $\lambda \in \Delta(R \cup C)$ , for all  $p, p' \in P_2$ ,  $u_p(\lambda) = u_{p'}(\lambda)$ .*

**Proof** See main theorem and section 6.2 in ASR. That  $P_2$  is convex follows from  $u_p(\lambda) \equiv u_{p'}(\lambda)$ . That  $P_2 \subseteq P_3$  follows from the facts that every  $p \in P_2$  verifies conclusions (i) and (ii) of theorem 1, at unchanged  $u$  and  $\Gamma$ ; and  $P_3$  is comprehensive.  $\square$

**Theorem 3** *When  $\Delta(R) \subset \mathfrak{R}^{\bar{S}}$  owns an indifference class spanning an  $(\bar{S} - 1)$ -dimensional hyperplane, then  $P_2$  is a singleton; when the indifference class  $\Delta(M)$  has that property,  $P_3$  is a singleton, hence  $P_3 = P_2 = \{\tilde{p}\}$ .*

**Proof** An  $(\bar{S} - 1)$ -dimensional hyperplane in  $\mathfrak{R}^{\bar{S}}$  has a unique normal vector.  $\square$

**Remark** If  $R$  is comprehensive, i.e. owns every map of  $S$  into  $C$ , then  $\Delta(R)$  owns indifference classes spanning  $(\bar{S} - 1)$ -dimensional (parallel) hyperplanes. That sufficient condition is not necessary, however.

## 4 Conclusions

In GAN, one can apply the  $N - M$  axioms to preferences over four nested sets, namely  $\Delta(r^* \cup C)$ ,  $\Delta(M \cup C)$ ,  $\Delta(R \cup C)$  and  $\Delta(F) \equiv \Delta(F \cup C)$  – thereby obtaining four nested sets of subjective probabilities,  $P_4, P_3, P_2$  and the singleton  $\{\tilde{p}\}$ . There exist special situations where  $P_4 = P_3 = P_2 = \{p\}$ . In general, the set inclusions are proper:  $P_4 \supset P_3 \supset P_2 \supset \{\tilde{p}\}$ .

The standard analysis, based on  $\Delta(F)$ , brings out the logic of subjective-expected-utility analysis in the most demanding case: when  $P_3 = \{\tilde{p}\}$ , precise assessment of probabilities is needed to solve the decision problem. Yet, typical decision situations are less demanding. The sets  $P_4$  and  $P_3$  then define *the extent to which probabilities need be specified in order to sustain an optimal decision*.

By way of illustration, let  $S = \{s, t\}$  and  $R = \{r, r', r''\}$  where the three acts  $r, r', r''$  entail the respective utility vectors  $(\frac{4}{3}, \frac{1}{2}), (1, 1), (\frac{1}{2}, \frac{4}{3})$  in states  $(s, t)$ . Then  $r \succsim r'$  iff  $p_s \geq \frac{3}{5}$  and  $r' \succsim r''$  iff  $p_s \geq \frac{2}{5}$ . The information about  $p_s$  needed to choose an optimal act amounts to locating  $p_s$  relative to the interval  $[\frac{2}{5}, \frac{3}{5}]$ . This is less demanding than point estimation.

In the same illustration, assume that  $M = \{r'\} = \{(1, 1)\}$ . Then,  $P_3 = (\frac{2}{5}, \frac{3}{5})$ . Assume further (assumption 2') that  $r \sim (\frac{11}{12}, \frac{1}{12})$ . Then  $P_2 = \{\tilde{p}\} = \{\frac{1}{2}\}$ .

Assuming instead that  $M = \{r\}$ , and  $r \sim (\frac{9}{8}, \frac{9}{8})$ , then  $P_3 = P_2 = \{\tilde{p}\} = \frac{3}{4}$ . In GoS, there are only three nested sets over which preferences are meaningfully defined, namely  $\Delta(r^* \cup C)$ ,  $\Delta(M \cup C)$  and  $\Delta(R \cup C)$ ; the corresponding nested sets of subjective probabilities are  $P_4, P_3 \subseteq P_2$  and  $P_2 \subseteq P_3$ . But only  $P_4$  and  $P_3$  reflect *observable* preferences.

The nature of preferences among pure or mixed strategies in *GoS* is discussed at some length in ASR. A significant comment is related to assumption 4. Consider the simple two-person, zero-sum game of “matching pennies”, where  $r \in \{1, 2\}, s \in \{1, 2\}$ , and  $h_r(s) = 1$  for  $r = s$ ,  $h_r(s) = -1$  for  $r \neq s$ . What ultimately matters to each player in this game is that the opponent not be able to “guess” what he himself will play. That is, each player wants his opponent to assign equal probabilities to both of his own strategies. A simple way of achieving that goal is to adopt the mixed strategy  $(\frac{1}{2}, \frac{1}{2})$ . The clear decision by a player to play according to that mixed strategy might be construed as a violation of assumption 4, because the *pure* strategies “heads” or “tail” appear discarded in favor of the mixed strategy. It is simply

The proper interpretation of assumption 4 is different. It is simply claimed that a player adopting the mixed strategy  $(\frac{1}{2}, \frac{1}{2})$  thereby reveals indifference between *eventually* playing “head” or “tail”. Such indifference is consistent with the assignment of *equal*

probabilities to the opponent playing “head” or “tail”, and difficult to reconcile with any other assignment. Assumption 4 claims neither more nor less.

In every game situation, a full analysis of the game is needed to form reasonable expectations about the choice(s) of an opponent. We argue here that, under reasonable assumptions, these expectations admit a subjective probability representation, sustaining the retained strategy(ies) as maximising expected utility. The role of game theory in guiding expectations in *GoS* is then seen as logically equivalent to that of (Bayesian) statistics in GAN.

## References

Anscombe, F.J. and R.J Aumann (1963), “A Definition of Subjective Probability”, *Annals of Mathematical statistics*, 34, 199-205.

Arrow, K.J., E.W. Barankin and D. Blackwell (1953), “Admissible Points of Convex Sets”, pp 87-91 in H.W Kuhn and A.W Tucker, Eds, *Contributions to the Theory of Games II*, Princeton University Press, Princeton (*Annals of Math. Studies*, vol.28).

Aumann, R.J. and J.H Drèze (2005) “Assessing Strategic Risk”, CORE DP2005/20 (ASR in the text).

Drèze, J.H. (1987), “Decision Theory with Moral Hazard and State-Dependent Preferences”, pp.23-89 in J.H. Drèze, *Essays on Economic Decisions under Uncertainty*, Cambridge University Press, Cambridge (UK).

Luce, R.D. and H. Raiffa (1957), *Games and Decisions*, New York, John Wiley.

Samuelson, P.A. (1950), ‘The Problem of Integrality in Utility Theorem’, *Economica* NS 17, 355-385.

Rockafellar, R.T. (1970), *Convex Analysis*, Princeton University Press, Princeton.

Savage, L.J.(1954), *Foundations of Statistics*, Wiley, New York.

von Neumann, J. and O. Morgenstern (1944), *Theory of Games and Economic Behaviour*, Princeton University Press, Princeton.