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THE MIXING SET WITH FLOWS

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Abstract

We consider here the mixing set with flows:

$$s + x_t \geq b_t,\ x_t \leq y_t \text{ for } 1 \leq t \leq n;\ \ s \in \mathbb{R}^1_+, x \in \mathbb{R}^n_+, y \in \mathbb{Z}^n_+.$$

It models the "flow version" of the basic mixing set introduced and studied by Günlük and Pochet, as well as the most simple stochastic lot-sizing problem with recourse, and more generally is a relaxation of certain mixed integer sets that arise in the study of production planning problems.

We study the polyhedron obtained by convexifying the above set. Specifically we provide a system of inequalities that gives its external description and characterize its vertices and rays.

Keywords: Mixing Set, Mixed Integer Sets, Convex Hull, Lot-Sizing

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1 Introduction

We give an external and internal description for the convex hull of the *mixing set with flows* X^{FM}:

$$s + x_t \geq b_t \text{ for } 1 \leq t \leq n$$
$$x_t \leq y_t \text{ for } 1 \leq t \leq n$$
$$s \in \mathbb{R}^1_+, x \in \mathbb{R}^n_+, y \in \mathbb{Z}^n_+$$

where $0 < b_1 \leq \ldots \leq b_n$.

This set is a relative of the *mixing set*, X^{MIX}:

$$s + y_t \geq b_t \text{ for } 1 \leq t \leq n$$
$$s \in \mathbb{R}^1_+, y \in \mathbb{Z}^n_+$$

introduced formally by Günlük and Pochet [3] and studied by Pochet and Wolsey [5] and Miller and Wolsey [4] for which an internal and external description is known.

So the "flow version" of the mixing set that we study here models the fact that the (production) variables are continuous, but are bounded from above by the installment of capacities, that can take only discrete values. In fact, conv(X^{MIX}) is a face of conv(X^{FM}). Notice that for each of these sets, the matrix associated with the defining constraints is totally unimodular (indeed a very special one): so the problem studied here is a special case of the more general question of characterizing the internal and external description of a mixed integer sets that can be formulated using a totally unimodular system of linear inequalities. Miller and Wolsey [4], and Van Vyve [6] have introduced and studied a different extension of the mixing set: their model is again a mixed integer set which satisfies a totally unimodular system of constraints.

Now we show how it models the simple lot-sizing problem with recourse. Specifically suppose that the demand forecast of sales for some item for the coming season are b_t with probability ϕ_t for $t = 1, \ldots, n$. The problem is to select the stock level s that will be available already at the beginning of the season, and then if the demand b_t is realized, the second stage is to satisfy the demand by producing a quantity x_t so as to satisfy the demand where the demand is in batches of capacity ($C = 1$). If the unit preseason production and storage cost is h, and then the unit production costs and batch costs during the season are p and q respectively, the problem of minimizing the total expected cost of satisfying the demand is:

$$\min\{hs + \sum_{t=1}^{n} \phi_t[px_t + qy_t] : (s, x, y) \in X^{FM}\}.$$

Note that the production costs during the season can also be treated as uncertain. The uncapacitated case when $b_t \leq 1$ for all t has been treated in Guan et al. [2].

The *unrestricted mixing set with flows* X^{UFM} is defined by the same constraint set as the mixing set with flows, except that the nonnegativity requirements on x are dropped. That is, X^{UFM} is the following set:

$$s + x_t \geq b_t \text{ for } 1 \leq t \leq n$$
$$x_t \leq y_t \text{ for } 1 \leq t \leq n$$
$$s \in \mathbb{R}^1_+, x \in \mathbb{R}^n, y \in \mathbb{Z}^n_+$$

where $0 < b_1 \leq \ldots \leq b_n$.

The next proposition shows that the unrestricted mixing set with flows and the mixing set are essentially the same set.

Proposition 1 *For an unrestricted mixing set with flows X^{UFM} and the mixing set X^{MIX} defined on the same vector b, we have:*

$$\mathrm{conv}(X^{UFM}) = \{(s,x,y) : (s,y) \in \mathrm{conv}(X^{MIX});\ b_t - s \leq x_t \leq y_t,\ 1 \leq t \leq n\}.$$

Proof: Let $P = \{(s,x,y) : (s,y) \in \mathrm{conv}(X^{MIX});\ b_t - s \leq x_t \leq y_t,\ 1 \leq t \leq n\}$. The inclusion $\mathrm{conv}(X^{UFM}) \subseteq P$ is obvious. In order to show that $P \subseteq \mathrm{conv}(X^{UFM})$, we prove that the extreme rays (resp. vertices) of P are rays (resp. feasible points) of $\mathrm{conv}(X^{UFM})$.

Since the recession cone of a mixed integer set and of its linear relaxation coincide, the cone: $\{s \in \mathbb{R}^1_+, x \in \mathbb{R}^n, y \in \mathbb{R}^n_+ : -s \leq x_t \leq y_t,\ 1 \leq t \leq n\}$ is the recession cone of both P and $\mathrm{conv}(X^{UFM})$. (Incidentally, its extreme rays are: $(1, 0\ldots 0, 0\ldots 0)$, $(0, 0\ldots 0, e_i)$, $(0, e_i, e_i)$, $(1, e_S, 0\ldots 0)$, where e_S is a representative vector of a subset S of $N = \{1,\ldots,n\}$).

We now prove that if (s^*, x^*, y^*) is a vertex of P, then (s^*, x^*, y^*) belongs to $\mathrm{conv}(X^{UFM})$. It is enough to show that y^* is integer. We do so by proving that (s^*, y^*) is a vertex of $\mathrm{conv}(X^{MIX})$. If not, there exists a nonzero vector $(u,w) \in \mathbb{R}^{n+1}$ such that $(s^*, y^*) \pm (u,w) \in \mathrm{conv}(X^{MIX})$ and $w_t = -u$ whenever $y^*_t = b_t - s^*$. Define a vector $v \in \mathbb{R}^n$ as follows: If $x^*_t = b_t - s^*$, set $v_t = -u$ and if $x^*_t = y^*_t$, set $v_t = w_t$. (Since x^*_t satisfies at least one of these 2 equations, this assignment is indeed possible). It is now easy to check that, for $\epsilon > 0$ sufficiently small, $(s^*, x^*, y^*) \pm \epsilon(u,v,w) \in P$, a contradiction. Therefore (s^*, y^*) is a vertex of $\mathrm{conv}(X^{MIX})$ and thus $(s^*, y^*) \in X^{MIX}$. Then $(s^*, x^*, y^*) \in X^{UFM}$ and the result is proved. $\square$

So an internal or external description of $\mathrm{conv}(X^{MIX})$ yields a corresponding description for $\mathrm{conv}(X^{UFM})$. The problem here is to study what happens when the polyhedron $\mathrm{conv}(X^{UFM})$ is intersected with the set of inequalities $x_t \geq 0,\ 1 \leq t \leq n$.

2 Some Equivalences of Polyhedra

We seek to relate the polyhedra $\mathrm{conv}(X^{FM})$ and $\mathrm{conv}(X^{MIX})$. However, the relation will not be as simple as the one stated in Proposition 1: we will need some polyhedral equivalences that we introduce here.

For a polyhedron P in $\mathbb{R}^n$ and $a \in \mathbb{R}^n$, let $\mu_P(a)$ be the value $\min\{ax,\ x \in P\}$ and $M_P(a)$ be the face $\{x \in P : ax = \mu_P(a)\}$, where $M_P(a) = \emptyset$ whenever $\mu_P(a) = -\infty$.

Lemma 2 *Let $P \subseteq Q$ be two nonempty polyhedra in $\mathbb{R}^n$ and let a be a nonzero vector in R^n. Then the following conditions are equivalent:*

1. *$\mu_P(a) = \mu_Q(a)$;*
2. *$M_P(a) \subseteq M_Q(a)$.*

Proof: Suppose $\mu_P(a) = \mu_Q(a)$. Since $P \subseteq Q$, every point in $M_P(a)$ belongs to $M_Q(a)$. So if 1. holds, then 2. holds as well. The converse is obvious. $\square$

Lemma 3 *Let $P \subseteq Q$ be two nonempty polyhedra in $\mathbb{R}^n$, where P is not an affine variety. Suppose that for every inequality $ax \geq \beta$ that is facet-inducing for P, at least one of the following holds:*

1. $\mu_P(a) = \mu_Q(a)$
2. $M_P(a) \subseteq M_Q(a)$.

Then $P = Q$.

Proof: We prove that if $M_P(a) \subseteq M_Q(a)$ for every inequality $ax \geq \beta$ that is facet-inducing for P, then every facet-inducing inequality for P is a valid inequality for Q and every hyperplane containing P also contains Q. This shows $Q \subseteq P$ and therefore $P = Q$. By Lemma 2, the conditions $\mu_P(a) = \mu_Q(a)$, $M_P(a) \subseteq M_Q(a)$ are equivalent and we are done.

Let $ax \geq \beta$ be a facet-inducing inequality for P. Since $M_P(a) \subseteq M_Q(a)$, then $\beta = \mu_P(a) = \mu_Q(a)$ and $ax \geq \beta$ is an inequality which is valid for Q. Now let $cx = \delta$ be a hyperplane containing P. If $Q \not\subseteq \{x : cx = \delta\}$, then there exist $\bar{x} \in Q$ such that $c\bar{x} \neq \delta$. We assume w.l.o.g. $\sigma = c\bar{x} - \delta > 0$. Since P is not an affine variety, there exists an inequality $ax \geq \beta$ which is facet-inducing for P (and so it is valid for Q). Choose $\lambda > 0$ such that $\lambda(a\bar{x} - \beta) < \sigma$. Then the inequality $(\lambda a - c)x \geq \lambda\beta - \delta$ is also facet-inducing for P, so it is valid for Q. This is a contradiction, as $(\lambda a - c)\bar{x} = \lambda a\bar{x} - c\bar{x} < \lambda\beta + \sigma - c\bar{x} = \lambda\beta - \delta$. □

If P is not full-dimensional, for each facet F of P there are infinitely many distinct inequalities that define F. (Two inequalities are distinct if their associated halfspaces are distinct: i.e. if one is not the positive multiple of the other). Observe that the hypotheses of the lemma must be verified for *all* distinct facet-defining inequalities (not just one facet-defining inequality for each facet), otherwise the result is false. For instance, consider the polyhedra $P = \{(x, y) : 0 \leq x \leq 1,\ y = 0\} \subsetneq Q = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$. The hypotheses of Lemma 2 are satisfied for the inequalities $x \geq 0$ and $x \leq 1$, which define all the facets of P.

Also notice that the assumption that P is not an affine variety cannot be removed: indeed, in such case P does not have proper faces, so the hypotheses of the lemma are trivially satisfied, even if $P \neq Q$.

Corollary 4 *Let $P \subseteq Q$ be two pointed polyhedra in $\mathbb{R}^n$, with the property that every vertex of Q belongs to P. Let $Cx \geq d$ be a system of inequalities that are valid for P such that for every inequality $cx \geq \delta$ of the system, $P \not\subseteq \{x \in \mathbb{R}^n : cx = \delta\}$.*

If for every $a \in \mathbb{R}^n$ such that $\mu_P(a)$ is finite but $\mu_Q(a) = -\infty$, $Cx \geq d$ contains an inequality $cx \geq \delta$ such that $M_P(a) \subseteq \{x \in \mathbb{R}^n : cx = \delta\}$, then $P = Q \cap \{x \in \mathbb{R}^n : Cx \geq d\}$.

Proof: We first show that $\dim(P) = \dim(Q)$. If not, there exists a hyperplane $ax = \beta$ containing P but not Q. W.l.o.g we can assume that $\mu_Q(a) < \beta = \mu_P(a)$. So $\mu_Q(a) = -\infty$, otherwise there would exist an a-optimal vertex $\bar{x}$ of Q such that $a\bar{x} < \beta$, contradicting the fact that $\bar{x} \in P$. Now the system $Cx \geq d$ must contain an inequality $cx \geq \delta$ such that $P = M_P(a) \subseteq \{x \in \mathbb{R}^n : cx = \delta\}$, a contradiction.

Let $Q' = Q \cap \{x \in \mathbb{R}^n : Cx \geq d\}$. Notice that $P \subseteq Q' \subseteq Q$, thus $\dim(P) = \dim(Q') = \dim(Q)$. Let $ax \geq \beta$ be a facet-inducing inequality for P. If $\mu_Q(a)$ is finite, then Q contains an a-optimal vertex which is in P and therefore $\beta = \mu_P(a) = \mu_{Q'}(a) = \mu_Q(a)$. If $\mu_Q(a) = -\infty$, the system $Cx \geq d$ contains an inequality $cx \geq \delta$ such that $M_P(a) \subseteq \{x \in \mathbb{R}^n : cx = \delta\}$ and

$P \not\subseteq \{x \in \mathbb{R}^n : cx = \delta\}$. It follows that $cx \geq \delta$ is a facet-inducing inequality for P and that it defines the same facet of P as $ax \geq \beta$ (that is, $M_P(a) = M_P(c)$). This means that there exist $\nu > 0$, a vector λ and a system $Ax = b$ which is valid for P such that $c = \nu a + \lambda A$ and $\delta = \nu\beta + \lambda b$. Since $\dim(P) = \dim(Q')$ and $P \subseteq Q'$, the system $Ax = b$ is valid for Q', as well. As $cx \geq \delta$ is also valid for Q', it follows that $ax \geq \beta$ is valid for Q' (because $a = \frac{1}{\nu}c - \frac{\lambda}{\nu}A$ and $\beta = \frac{1}{\nu}\delta - \frac{\lambda}{\nu}b$). Therefore $\beta = \mu_P(a) = \mu_{Q'}(a)$.

Now assume that P consists of a single point and $P \neq Q$. Then Q is a cone having P as apex. Chosen a ray a of Q, $\mu_P(a)$ is finite while $\mu_Q(a) = -\infty$, so the system $Cx \geq d$ contains an inequality $cx \geq \delta$ such that $P \subseteq \{x \in \mathbb{R}^n : cx = \delta\}$, a contradiction. So we can assume that P is not a single point and thus P is not an affine variety, as it is pointed. Now we can conclude by applying Lemma 3 to P and Q'. $\square$

We remark that in the statement of Corollary 4 the condition that the two polyhedra are pointed is not necessary: if we replace the property "every vertex of Q belongs to P" with "every minimal face of Q belongs to P", the proof needs a very slight modification to remain valid. (However, in this case we should assume that P is not an affine variety, so that we can apply Lemma 3 in the proof.)

We also observe that the condition "for every inequality $cx \geq \delta$ of the system, $P \not\subset \{x \in \mathbb{R}^n : cx = \delta\}$" is necessary. For instance, consider the polyhedra $P = \{(x, y) : 0 \leq x \leq 1,\ y = 0\} \subsetneq Q = \{(x, y) : x \geq 0,\ y = 0\}$ and the system consisting of the single inequality $y \geq 0$.

3 An external description of X^{FM}

3.1 A relaxation of X^{FM}

Consider the set Z:

$$
\begin{aligned}
s + y_t \geq b_t & \text{ for } 1 \leq t \leq n \\
s + x_k + y_t \geq b_t & \text{ for } 1 \leq k < t \leq n \\
s + x_t \geq b_t & \text{ for } 1 \leq t \leq n \\
s \in \mathbb{R}^1_+, x \in R^n, y \in \mathbb{Z}^n_+ &.
\end{aligned}
$$

Proposition 5 *Let X^{FM} and Z be defined on the the same vector b. Then $X^{FM} \subseteq Z$ and $X^{FM} = Z \cap \{(s, x, y) :\ 0 \leq x \leq y\}$.*

Proof: To see that $X^{FM} \subseteq Z$, observe that for $(s, x, y) \in X^{FM}$, $s + y_t \geq s + x_t \geq b_t$, so $s + y_t \geq b_t$ is a valid inequality. Also as $s + y_t \geq b_t$ and $x_k \geq 0$, $s + x_k + y_t \geq b_t$ is a valid inequality. The only inequalities that define X^{FM} but do not appear in the definition of Z are the inequalities $0 \leq x \leq y$. $\square$

Observation 1 *The extreme rays of* $\text{conv}(X^{FM})$ *are the following* $2n+1$ *vectors:* $(1, 0 \ldots 0, 0 \ldots 0)$, $(0, 0 \ldots 0, e_i)$, $(0, e_i, e_i)$. *The* $2n + 1$ *extreme rays of* $\text{conv}(Z)$ *are* $(0, 0 \ldots 0, e_i)$, $(0, e_i, 0 \ldots 0)$, $(1, -1 \ldots -1, 0 \ldots 0)$. *Therefore both recession cones of* $\text{conv}(X^{FM})$, $\text{conv}(Z)$ *are full-dimensional simplicial cones, thus showing that* $\text{conv}(X^{FM})$ *and* $\text{conv}(Z)$ *are both full-dimensional polyhedra.*

Observation 2 *Let* (s^*, x^*, y^*) *be a vertex of* $\text{conv}(Z)$*. Then*

$$s^* = \max \left\{ \begin{array}{l} 0 \\ b_t - y_t^*,\ 1 \le t \le n \\ b_t - x_t^*,\ 1 \le t \le n \\ b_t - y_t^* - x_k^*,\ 1 \le k < t \le n \end{array} \right\}$$

$$x_k^* = \max \left\{ \begin{array}{l} b_k - s^* \\ b_t - s^* - y_t^*,\ k < t \le n. \end{array} \right\}$$

Lemma 6 *Let* (s^*, x^*, y^*) *be a vertex of* $\text{conv}(Z)$*. Then* $0 \le x^* \le y^*$*.*

Proof: Assume $x_k^* < 0$ for some index k. Then $s^* > 0$, otherwise, if $s^* = 0$, the constraints $s + x_k \ge b_k$, $b_k \ge 0$ imply $x_k^* \ge 0$.

We now claim that there is an index t such that $s^* = b_t - y_t^*$. If not, $s^* > b_t - y_t^*,\ 1 \le t \le n$, and there is an $\epsilon \ne 0$ such that $(s^*, x^*, y^*) \pm \epsilon(1, -1 \ldots -1, 0 \ldots 0)$ belong to $\text{conv}(Z)$, a contradiction.

So there is an index t such that $s^* = b_t - y_t^* > 0$. Since $b_t - y_t^* \ge b_t - y_t^* - x_k^*,\ 1 \le k < t$, this implies $x_k^* \ge 0,\ 1 \le k < t$. Observation 2 also implies $b_t - y_t^* \ge b_k - x_k^*,\ 1 \le k \le n$. Together with $y_t^* \ge 0$ and $b_t \le b_k,\ k \ge t$, this implies $x_k^* \ge y_t^* \ge 0,\ k \ge t$. This completes the proof that $x^* \ge 0$.

Assume $x_k^* > y_k^*$ for some index k. Then $y_k^* \ge 0$ implies $x_k^* > 0$. Assume $x_k^* = b_k - s^*$. Then $y_k^* \ge b_k - s^*$ implies that $x_k^* \le y_k^*$, a contradiction. Therefore by Observation 2, $x_k^* = b_t - s^* - y_t^*$ for some $t > k$. Since $x_k^* > 0$, then $b_t - s^* - y_t^* > 0$, a contradiction to $s^* + y_t^* \ge b_t$. This shows $x^* \le y^*$ and the proof is complete. □

We now can state the main theorem of this section:

Theorem 7 *Let* X^{FM} *and* Z *be defined on the the same vector* b*. Then* $\text{conv}(X^{FM}) = \text{conv}(Z) \cap \{(s, x, y) :\ 0 \le x \le y\}$*.*

Proof: By Proposition 5, $\text{conv}(X^{FM}) \subseteq \text{conv}(Z)$. By Lemma 6 and Proposition 5, every vertex of $\text{conv}(Z)$ belongs to $\text{conv}(X^{FM})$.

Let $a = (h, p, q),\ h \in \mathbb{R}^1,\ p \in \mathbb{R}^n,\ q \in \mathbb{R}^n$ be such that $\mu_{\text{conv}(X^{FM})}(a)$ is finite and $\mu_{\text{conv}(Z)}(a) = -\infty$. Since by Observation 1, the extreme rays of $\text{conv}(Z)$ that are not rays of $\text{conv}(X^{FM})$ are $(0, e_i, 0 \ldots 0)$ and $(1, -1 \ldots -1, 0 \ldots 0)$, then either $p_k < 0$ for some index k or $h < \sum_{t=1}^n p_t$.

If $p_k < 0$, then $M_{\text{conv}(X^{FM})}(a) \subseteq \{(s, x, y),\ x_k = y_k\}$.

If $h < \sum_{t=1}^n p_t$, let $N^+ = \{j : p_j > 0\}$ and $k = \min\{j : j \in N^+\}$. We show that $M_{\text{conv}(X^{FM})}(a) \subseteq \{(s, x, y) : x_k = 0\}$. Suppose that $x_k > 0$ in some optimal solution. As the solution is optimal and $p_k > 0$, we cannot just decrease x_k and remain feasible. Thus $s + x_k = b_k$, which implies that $s < b_k$. However this implies that for all $j \in N^+$, we have $x_j \ge b_j - s > b_j - b_k \ge 0$ as $j \ge k$. Now as $x_j > 0$ for all $j \in N^+$, we can increase s by $\epsilon > 0$ and decrease x_j by ϵ for all $j \in N^+$. The new point is feasible in X^{FM} and has lower objective value, a contradiction.

To complete the proof, since $\text{conv}(X^{FM})$ is full-dimensional, the system $0 \le x \le y$ does not contain an improper face of $\text{conv}(X^{FM})$. So we can now apply Corollary 4 to $\text{conv}(X^{FM})$, $\text{conv}(Z)$ and the system $0 \le x \le y$. □

3.2 The intersection set

The following set is the *intersection set* X^{INT}:

$$\sigma_k + y_t \geq b_t - b_k \text{ for } 0 \leq k < t \leq n$$
$$\sigma \in \mathbb{R}_+^{n+1}, y \in \mathbb{Z}_+^n.$$

where $0 = b_0 \leq b_1 \leq \ldots \leq b_n$.

Notice that X^{INT} is the intersection of the following $n+1$ mixing sets X_k^{MIX}, each one associated with a single variable σ_k:

$$\sigma_k + y_t \geq b_t - b_k \text{ for } k < t \leq n$$
$$\sigma_k \in \mathbb{R}_+^1, y \in \mathbb{Z}_+^{n-k+1}.$$

Theorem 8 *Let X^{INT} be an intersection set and X^{FM} be defined on the same vector b. The linear transformation $\sigma_0 = s$ and $\sigma_t = s + x_t - b_t$, $1 \leq t \leq n$, maps* $\mathrm{conv}(X^{FM})$ *into* $\mathrm{conv}(X^{INT}) \cap \{(\sigma, y) : 0 \leq \sigma_k - \sigma_0 + b_k \leq y_k,\ 1 \leq k \leq n\}$.

Proof: Let Z be defined on the same vector b. It is straightforward to check that the linear transformation $\sigma_0 = s$ and $\sigma_t = s + x_t - b_t$, $1 \leq t \leq n$, maps $\mathrm{conv}(Z)$ into $\mathrm{conv}(X^{INT})$. By Theorem 7, $\mathrm{conv}(X^{FM}) = \mathrm{conv}(Z) \cap \{(s, x, y) :\ 0 \leq x \leq y\}$ and the result follows. □

The above theorem shows that an external description of $\mathrm{conv}(X^{FM})$ can be obtained from an external description of $\mathrm{conv}(X^{INT})$. Such a description is known:

Proposition 9 *(Günluk and Pochet [3]) Consider the mixing set X^{MIX}:*

$$s + y_t \geq b_t \text{ for } 1 \leq t \leq n$$
$$s \in \mathbb{R}_+^1, y \in \mathbb{Z}_+^n$$

For $t = 1, \ldots, n$ we define $f_t := b_t - \lfloor b_t \rfloor$. Let $T \subseteq \{1, \ldots, n\}$ and suppose that $i_1, \ldots, i_{|T|}$ is an ordering of T such that $0 = f_{i_0} \leq f_{i_1} \leq \cdots \leq f_{i_{|T|}}$. Then the mixing inequalities

$$s \geq \sum_{t=1}^{|T|} (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_{i_t})$$

and

$$s \geq \sum_{t=1}^{|T|} (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_{i_t}) + (1 - f_{i_{|T|}})(\lfloor b_{i_1} \rfloor - y_{i_1})$$

are valid for X^{MIX}. Moreover, adding all mixing inequalities to the linear constraints defining X^{MIX} gives the convex hull of X^{MIX}.

Proposition 10 *(Miller and Wolsey [4]) Let $X_k^{MIX}(n^k, s^k, y^k, b^k)$ for $k = 1, \ldots, n$ be n mixing sets with some or all y variables in common. Let $X^* = \cap_{k=1}^n X_k^{MIX}$. Then*

$$\mathrm{conv}(X^*) = \cap_{k=1}^n \mathrm{conv}(X_k^{MIX}).$$

Observation 3 *Günluk and Pochet [3] have shown that the polyhedron* $\mathrm{conv}(X^{MIX})$ *admits a compact formulation, see also [1]. Therefore it follows from Theorem 8 and Proposition 10 that a compact formulation of* $\mathrm{conv}(X^{FM})$ *can be obtained by writing the compact formulations of all the mixing polyhedra* $\mathrm{conv}(X_k^{MIX})$*, together with the inequalities $0 \leq \sigma_t - \sigma_0 + b_t \leq y_t$, $1 \leq t \leq n$ and then applying the transformation $s = \sigma_0$ and $x_t = -s + \sigma_t + b_t$, $1 \leq t \leq n$.*

4 An internal description of X^{FM}

Since the extreme rays of conv(X^{FM}) are described in Observation 1, in order to give a complete internal description of conv(X^{FM}) we have to characterize its vertices. First we state a result concerning the vertices of any mixed integer set.

Lemma 11 *Let $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{Z}^p : Ax + By \leq c\}$. If (x^*, y^*) is a vertex of* conv(P)*, then x^* is a vertex of the polyhedron $P(y^*) = \{x \in \mathbb{R}^n : Ax \leq c - By^*\}$.*

Proof: If x^* is not a vertex of $P(y^*)$, there exists a nonzero vector $\epsilon \in \mathbb{R}^n$, $\epsilon \neq (0 \ldots 0)$, such that $A(x^* \pm \epsilon) \leq c - By^*$. But then $(x^*, y^*) \pm (\epsilon, 0 \ldots 0)$ is in P and thus (x^*, y^*) is not a vertex of conv(P). □

In the following, given a point $p = (\bar{s}, \bar{x}, \bar{y})$ in conv(X^{FM}), we denote by $f_{\bar{s}}$ the fractional part of $\bar{s}$. Furthermore, we define T_p to be the set of inequalities $s + x_t \geq b_t$ which are tight for p: that is, $T_p = \{t : \bar{s} + \bar{x}_t = b_t\}$. Given $I \subseteq \{1, \ldots, n\}$, we define $e_I = \sum_{t \in I} e_t$.

Claim 12 *Let $v = (s^*, x^*, y^*)$ be a vertex of* conv(X^{FM})*. If $s^* > 0$ then $T_v \neq \emptyset$ and there exists an index $j \in T_v$ such that $f_{s^*} = f_j$.*

Proof: By Lemma 11, (s^*, x^*) is a vertex of the polyhedron $P(y^*)$ defined by

$$s + x_t \geq b_t \text{ for } 1 \leq t \leq n$$
$$x_t \leq y_t^* \text{ for } 1 \leq t \leq n$$
$$s \in \mathbb{R}^1_+, x \in \mathbb{R}^n_+$$

Then among the constraints defining $P(y^*)$ there exist $n + 1$ inequalities which are tight for (s^*, x^*) and whose left-hand sides form a nonsingular $(n + 1) \times (n + 1)$ matrix. Therefore, if $s^* > 0$ then there exists an index j such that $s^* + x_j^* = b_j$ (that is, $j \in T_v$) and either $x_j^* = y_j^*$ or $x_j^* = 0$. Therefore $x_j^* \in \mathbb{Z}$ and thus $f_{s^*} = f_j$. □

Claim 13 *Let $v = (s^*, x^*, y^*)$ be a vertex of* conv(X^{FM})*. Then for $1 \leq t \leq n$*

$$y_t^* = \max\{0, \lceil b_t - s^* \rceil\}. \tag{1}$$

Proof: Suppose $b_t - s^* < 0$. Then either $x_t^* = 0$ or $x_t^* = y_t^*$. Now if $y_t^* \geq 1$, in the first case both points $v \pm (0, 0, e_t)$ are in X^{FM}, in the second case both points $v \pm (0, e_t, e_t)$ are in X^{FM},a contradiction.

Suppose $b_t - s^* \geq 0$. If $y_t^* \geq \lceil b_t - s^* \rceil + 1$ then, setting $\epsilon = \min\{x_t^* - (b_t - s^*), 1\}$, both points $v \pm (0, \epsilon e_t, e_t)$ are in X^{FM}, a contradiction. □

Claim 14 *Let $v = (s^*, x^*, y^*)$ be a vertex of* conv(X^{FM})*. Then for $1 \leq t \leq n$*

$$x_t^* = \begin{cases} 0 & \textit{if} \quad b_t - s^* < 0 \\ b_t - s^* \quad \textit{or} \quad \lceil b_t - s^* \rceil & \textit{if} \quad b_t - s^* \geq 0 \end{cases} \tag{2}$$

Proof: By Lemma 11, (s^*, x^*) is a vertex of the polyhedron $P(y^*)$ defined above and so, among the constraints defining $P(y^*)$, there exist $n+1$ inequalities which are tight for (s^*, x^*) and whose left-hand sides form a nonsingular $(n+1)\times(n+1)$ matrix. It is easy to verify that then for each t one of the following holds: either $s^* + x^*_t = b_t$ or $x^*_t = 0$ or $x^*_t = y^*_t = \max\{0, \lceil b_t - s^*\rceil\}$ (where the last equality follows from Claim 13). It follows that if $b_t - s^* < 0$ then $x^*_t = 0$ (otherwise inequality $x^*_t \geq 0$ would be violated) and that if $b_t - s^* \geq 0$ then $x^*_t \in \{b_t - s^*, \lceil b_t - s^*\rceil\}$ (otherwise inequality $s^* + x^* \geq b_t$ would be violated). □

Claim 15 *Let $v = (s^*, x^*, y^*)$ be a vertex of* $\mathrm{conv}(X^{FM})$. *Suppose $s^* > 0$ and $f_{s^*} = f_j$. Let $b^* = \max\{b_t : f_t = f_j,\ 1 \leq t \leq n\}$. Then $s^* \leq b^*$.*

Proof: By Claim 12 we may assume $j \in T_v$. Then $s^* \leq b_j \leq b^*$. □

Given a point $p = (\bar{s}, \bar{x}, \bar{y})$ in $\mathrm{conv}(X^{FM})$, we define the following subsets of $\{1, \ldots, n\}$:

$$\begin{aligned} N_p &= \{t : -1 < b_t - \bar{s} \leq 0\}, \\ P_p &= \{t : 0 < b_t - \bar{s} < 1\}. \end{aligned}$$

Claim 16 *Let $v = (s^*, x^*, y^*)$ be a vertex of* $\mathrm{conv}(X^{FM})$. *If $s^* \geq 1$ then $N_v \cup P_v \neq \emptyset$. Moreover, if $s^* \geq 1$ and $N_v = \emptyset$ then there exists $t \in P_v$ such that $0 < x^*_t < 1$.*

Proof: Suppose $s^* \geq 1$ and $N_v \cup P_v = \emptyset$. Then $|b_t - s^*| \geq 1,\ 1 \leq t \leq n$. Let I be the set of indices t such that $b_t - s^* \geq 1$. Notice that if $t \in I$ then $x^*_t \geq 1$ and that if $t \notin I$ then $s^* + x^*_t \geq b_t + 1$. It follows that both points $v \pm (1, -e_I, -e_I)$ are in X^{FM}, a contradiction. Now suppose $s^* \geq 1$ and $N_v = \emptyset$ and assume that for every $t \in P_v$ either $x^*_t = 0$ or $x^*_t \geq 1$. Then (2) implies that $x^*_t = 1$ for every $t \in P_v$. If $t \notin P_v$ then either $b_t - s^* \leq -1$ or $b_t - s^* \geq 1$, as $N_v = \emptyset$. Let I be the set of indices t such that $b_t - s^* \geq 1$. Notice that if $t \in I$ then $x^*_t \geq 1$ and that if $t \notin P_v \cup I$ then $s^* + x^*_t \geq b_t + 1$. From all these considerations it follows that both points $v \pm (1, -e_{P_v \cup I}, -e_{P_v \cup I})$ are in X^{FM}, a contradiction. □

We need the following Lemma.

Lemma 17 *Let $p = (\bar{s}, \bar{x}, \bar{y}) \in$* $\mathrm{conv}(X^{FM})$. *Suppose that the components of p satisfy both conditions (1) and (2). If for every convex combination of points in X^{FM} giving p, all the points appearing with nonzero coefficient have s-component equal to $\bar{s}$, then p is a vertex of* $\mathrm{conv}(X^{FM})$.

Proof: Consider any convex combination of points in X^{FM} giving p and let C be the set of points in X^{FM} appearing with nonzero coefficient in such combination. Let $t \in \{1, \ldots, n\}$. Either $\bar{y}_t = 0$ or $\bar{y}_t = \lceil b_t - \bar{s}\rceil$. If $\bar{y}_t = 0$ then, since all points in C satisfy $y_t \geq 0$, they all satisfy $y_t = 0$. If $\bar{y}_t = \lceil b_t - \bar{s}\rceil$ then, since all points in C satisfy $y_t \geq \lceil b_t - \bar{s}\rceil$, they all satisfy $y_t = \lceil b_t - \bar{s}\rceil$. Thus all points in C have the same y-components. As to the x-components, either $\bar{x}_t = 0$ or $\bar{x}_t = b_t - \bar{s}$ or $\bar{x}_t = \lceil b_t - \bar{s}\rceil$. If $\bar{x}_t = 0$ then, since all points in C satisfy $x_t \geq 0$, they all satisfy $x_t = 0$. If $\bar{x}_t = b_t - \bar{s}$ then, since all points in C satisfy $x_t \geq b_t - \bar{s}$, they all satisfy $x_t = b_t - \bar{s}$. If $\bar{x}_t = \lceil b_t - \bar{s}\rceil$ then $\bar{x}_t = \bar{y}_t$ and so, since all points in C satisfy $x_t \leq y_t$, they all satisfy $x_t = y_t$. Thus all points in C have the same x-components. Therefore all points in C are identical, that is, the considered combination does not express p as convex combination of points *different from p*. Since this happens for every convex combination of points in X^{FM} giving p, such a point is a vertex of $\mathrm{conv}(X^{FM})$. □

Claim 18 *Let* $p = (\bar{s}, \bar{x}, \bar{y}) \in \mathrm{conv}(X^{FM})$. *Suppose that the components of* p *satisfy both conditions (1) and (2). If* $\bar{s} = 0$, *or* $\bar{s} = f_j$ *for some* j, *or* $\bar{s} = b_j$ *for some* j, *then* p *is a vertex of* $\mathrm{conv}(X^{FM})$.

Proof: Let us consider an arbitrary convex combination of points in X^{FM} giving p and let C be the set of points appearing with nonzero coefficient in such combination. Suppose $\bar{s} = 0$. Then all points in C satisfy $s = 0$. Thus, by Lemma 17, p is a vertex of $\mathrm{conv}(X^{FM})$. Suppose $\bar{s} = f_j$ for some j. Condition (1) implies that $\bar{s} + \bar{y}_j = b_j$. Then all points in C satisfy $s + y_j = b_j$ and thus they all have $f_s = f_j$, in particular $s \geq f_j$. It follows that they all satisfy $s = f_j$. The conclusion now follows from Lemma 17. Suppose $\bar{s} = b_j$ for some j. Then $\bar{x}_j = 0$, thus all points in C satisfy $x_j = 0$ and so they satisfy $s \geq b_j$. It follows that they all satisfy $s = b_j$. Again the conclusion follows from Lemma 17. □

Claim 19 *Let* $p = (\bar{s}, \bar{x}, \bar{y}) \in \mathrm{conv}(X^{FM})$. *Let* $\bar{s} = m + f_j$, *where* $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$. *Suppose that there exists an index* h *such that* $0 < b_h - \bar{s} < 1$. *Suppose that the components of* p *satisfy both conditions (1) and (2) and that* $\bar{x}_h = b_h - \bar{s}$. *Then* p *is a vertex of* $\mathrm{conv}(X^{FM})$.

Proof: Let us consider an arbitrary convex combination of points in X^{FM} giving p and let C be the set of points appearing with nonzero coefficient in such combination. Since $b_j - \bar{s} \geq 0$, condition (1) implies that $\bar{s} + \bar{y}_j = b_j$; then all points in C satisfy $s + y_j = b_j$ and thus they all have $f_s = f_j = f_{\bar{s}}$. Since $\bar{s} + \bar{x}_h = b_h$, all points in C satisfy $s + x_h = b_h$. Suppose that there exists a point in C satisfying $s \neq \bar{s}$. Then there exists a point in C satisfying $s > \bar{s}$, i.e. $s \geq \bar{s} + 1$. Therefore, for such point, $x_h = b_h - s \leq b_h - \bar{s} - 1 < 0$, a contradiction. Thus all points in C satisfy $s = \bar{s}$. Lemma 17 concludes the proof. □

Claim 20 *Let* $p = (\bar{s}, \bar{x}, \bar{y}) \in \mathrm{conv}(X^{FM})$. *Let* $\bar{s} = m + f_j$, *where* $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$. *Suppose that there exists an index* h *such that* $-1 < b_h - \bar{s} < 0$. *Suppose that the components of* p *satisfy both conditions (1) and (2). Then* p *is a vertex of* $\mathrm{conv}(X^{FM})$.

Proof: Let us consider an arbitrary convex combination of points in X^{FM} giving p and let C be the set of points appearing with nonzero coefficient in such a combination. Since $b_j - \bar{s} \geq 0$, condition (1) implies that $\bar{s} + \bar{y}_j = b_j$; then all points in C satisfy $s + y_j = b_j$ and thus they all have $f_s = f_j = f_{\bar{s}}$. Since $b_h - \bar{s} < 0$, condition (2) implies that $\bar{x}_h = 0$; then all points in C satisfy $x_h = 0$. Suppose that there exists a point in C satisfying $s \neq \bar{s}$. Then there exists a point in C satisfying $s < \bar{s}$, i.e. $s \leq \bar{s} - 1$. Therefore, for such point, $s + x_h = s \leq \bar{s} - 1 < b_h$, a contradiction. Thus all points in C satisfy $s = \bar{s}$. Lemma 17 concludes the proof. □

Theorem 21 *The point* $p = (s^*, x^*, y^*)$ *is a vertex of* $\mathrm{conv}(X^{FM})$ *if and only if its components satisfy one of the following conditions:*

(i) $s^* = 0$
$x_t^* = b_t$ *or* $x_t^* = \lceil b_t \rceil$ *for* $1 \leq t \leq n$
$y_t^* = \lceil b_t \rceil$ *for* $1 \leq t \leq n$

(ii) $s^* = f_j$ *for some* $1 \leq j \leq n$

$$x_t^* = \begin{cases} 0 & \text{if } b_t - f_j < 0 \\ b_t - f_j \ \text{ or } \ \lceil b_t - f_j \rceil & \text{if } b_t - f_j \geq 0 \end{cases}$$

$y_t^* = \max\{0, \lceil b_t - f_j \rceil\}$ *for* $1 \leq t \leq n$

(iii) $s^* = b_j$ *for some* $1 \le j \le n$

$$x_t^* = \begin{cases} 0 & \textit{if} \quad b_t - b_j < 0 \\ b_t - b_j \quad \textit{or} \quad \lceil b_t - b_j \rceil & \textit{if} \quad b_t - b_j \ge 0 \end{cases}$$

$y_t^* = \max\{0, \lceil b_t - b_j \rceil\}$ *for* $1 \le t \le n$

(iv) $s^* = m + f_j$ *for some* $1 \le j \le n$*, where* $0 < m < \lfloor b_j \rfloor$*,* $m \in \mathbb{Z}$*, and* $0 < b_h - s^* < 1$ *for some* $1 \le h \le n$

$$x_t^* = \begin{cases} 0 & \textit{if} \quad b_t - s^* < 0 \\ b_t - s^* \quad \textit{or} \quad \lceil b_t - s^* \rceil & \textit{if} \quad b_t - s^* \ge 0 \quad \textit{and} \quad t \ne h \\ b_t - s^* & \textit{if} \quad t = h \end{cases}$$

$y_t^* = \max\{0, \lceil b_t - s^* \rceil\}$ *for* $1 \le t \le n$

(v) $s^* = m + f_j$ *for some* $1 \le j \le n$*, where* $0 < m < \lfloor b_j \rfloor$*,* $m \in \mathbb{Z}$*, and* $-1 < b_h - s^* < 0$ *for some* $1 \le h \le n$

$$x_t^* = \begin{cases} 0 & \textit{if} \quad b_t - s^* < 0 \\ b_t - s^* \quad \textit{or} \quad \lceil b_t - s^* \rceil & \textit{if} \quad b_t - s^* \ge 0 \end{cases}$$

$y_t^* = \max\{0, \lceil b_t - s^* \rceil\}$ *for* $1 \le t \le n$

Proof: Claim 18 shows that points of types (i), (ii) and (iii) are vertices of $\mathrm{conv}(X^{FM})$. Claim 19 and Claim 20 show that points of types (iv) and (v) are vertices of $\mathrm{conv}(X^{FM})$. It remains to prove that there are not other vertices. If $p = (s^*, x^*, y^*)$ is a vertex of $\mathrm{conv}(X^{FM})$ then its components satisfy conditions (1) and (2). By Claim 12, either $s^* = 0$ or $f_{s^*} \in \{f_1, \ldots, f_n\}$. If $s^* = 0$, p satisfies the conditions of case (i). If $s^* = f_j$ for some j, p satisfies the conditions of case (ii). If $s^* = b_j$ for some j, then p satisfies the conditions of case (iii). Suppose that $f_{s^*} = f_j$ for some j, $s^* \ge 1$ and $s^* \ne b_t$, $1 \le t \le n$. We can choose j such that $b_j = \max\{b_t : f_t = f_j,\ 1 \le t \le n\}$. Then, by Claim 15, $s^* = m + f_j$, where $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$. Claim 16 implies that $N_p \cup P_p \ne \emptyset$. If $N_p \ne \emptyset$ then p satisfies the conditions of case (v). Otherwise $P_p \ne \emptyset$ and Claim 16 implies the existence of an index $h \in P_p$ such that $0 < x_h^* < 1$. But then necessarily $x_h^* = b_h - s^*$ and thus p satisfies the conditions of case (iv). □

Observation 4 *It follows from Theorem 21 that if* $(s^*, x^*.y^*)$ *is a vertex of* $\mathrm{conv}(X^{FM})$*, then* s^* *can take* $O(n^2)$ *possible values. Once* s^* *is fixed, then* y^* *is also fixed and a component of* x^* *can take at most 2 values. This shows that the problem* $\min hs + px + qy : (s, x, y) \in \mathrm{conv}(X^{FM})$ *can be solved as follows: If* $h < 0$ *or* $p_t + q_t < 0$ *or* $q_t < 0$ *for some* $1 \le t \le n$*, the solution is unbounded. Else enumerate the possible values of* s *(and corresponding* y*). Now for* $1 \le t \le n$*, the sign of* p_t *determines the value of* x_t*.*

5 A Final Remark

Consider the set:

$$W = \{s \in \mathbb{R}^1_+, x \in \mathbb{R}^n_+, y \in \mathbb{Z}^n : \ l_i \le y_i \le u_i, \alpha_{ij} \le y_i - y_j \le \beta_{ij} \text{ for all } 1 \le i, j \le n\}$$

where $l_i, u_i, \alpha_{ij}, \beta_{ij} \in \mathbb{Z} \cup \{+\infty, -\infty\}$ and assume that for every index i, W contains a vector with $y_i > 0$.

The set:

$$X^{BFM} = X^{FM} \cap W$$

models the mixing set with upper bounds and "dual network" constraints on the integer variables y_i. We now have the following:

Theorem 22 *Let X^{INT}, X^{BFM} be defined on the same vector b and W be a set satisfying the above conditions. The linear transformation $\sigma_0 = s$ and $\sigma_t = s + x_t - b_t,\ 1 \leq t \leq n$, maps $\text{conv}(X^{BFM})$ into $\text{conv}(X^{INT}) \cap \{(\sigma, y) : l_i \leq y_i \leq u_i, \alpha_{ij} \leq y_i - y_j \leq \beta_{ij},\ 1 \leq i, j \leq n\} \cap \{(\sigma, y) : 0 \leq \sigma_k - \sigma_0 + b_k \leq y_k,\ 1 \leq k \leq n\}$.*

The proof of Theorem 22 is identical to the proof of Theorem 8. The condition that for every index i, W contains a vector with $y_i > 0$, shows that none of the inequalities $0 \leq x_i \leq y_i$ defines an improper face of $\text{conv}(X^{BFM})$ and Corollary 4 can still be applied.

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