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A QUASI-VARIATIONAL INEQUALITY APPROACH TO THE FINANCIAL EQUILIBRIUM PROBLEM

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Abstract

This paper presents the time-dependent, multi-agent and multi-activity financial equilibrium problem when budget constraints are implicitly defined. Specifically, we assume that total wealth is elastic with respect to the optimal investment. Such a problem is formulated as an infinite-dimensional quasi-variational inequality for which an existence result is given.

Keywords: Portfolio optimization, equilibrium condition, quasi-variational inequality.

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1 Introduction

We are concerned with investigating the time-dependent multi-agent portfolio optimization problem formulated in terms of an infinite-dimensional quasi-variational inequality. The problem of portfolio optimization, since the seminal works by Markowitz (1952, 1959), has been extensively studied. Recently, it has been shown that the variational inequality theory may be fruitfully exploited in order to describe and solve such a problem. Thus, valuable results in both the formulation and the computation of competitive financial equilibria have been achieved. In particular, the static multi-agent and multi-activity financial equilibrium model was thoroughly examined by Nagurney et al. (1992), Nagurney (1992, 1993), and Nagurney and Dong (2002), who showed that the equilibrium may be characterized as the solution of an appropriate finite-dimensional variational inequality. The time-dependent model was discussed in Daniele (2003, 2005), and Daniele at el. (2005), where the equivalence with an infinite-dimensional variational inequality problem was proved.

It is by now well-known that several problems are characterized by quasi-equilibria so that they may be formulated as quasi-variational inequalities (see Mosco (1976), Chan and Pang (1982), Baiocchi and Capelo (1984), Harker and Pang (1990), Harker (1991), Yao (1991), and Scrimali (2004) for a survey on theoretical and applicative aspects). Since the early seventies, when quasi-variational inequalities were introduced by Bensoussan et al. (1973) in the context of impulse control, their applications in numerous different areas were studied and, particularly, in economics and finance. The reader interested in some economics problems may, for instance, refer to Pang and Fukushima (2005), who, exploiting the connection between generalized Nash equilibria and quasi-variational inequalities, discussed some oligopolistic electricity models. For financial applications, we refer, among the others, to Korn (1998), who examined the portfolio optimization with transaction costs and suggested an impulse control approach.

Our aim is to deal with the time-dependent financial equilibrium problem in the most general case of non quadratic utility functions. The simplest financial model is described by the mean value and the variance surrounding the mean, which, in turn, reflects the agent's risk assessment. Therefore, the criteria underlying the behavioral choices of investors are given, on the one hand, by the maximization of the profit and, on the other, by the minimization of the risk associated with the portfolio selection. However, many practical situations can not be completely described by mean and variance and thus the mean-variance model fails to represent the portfolio choice model.

For this reason, as also suggested by Yao (1991) and Nagurney (1992) for Euclidean spaces and, subsequently, by Daniele (2005) for Hilbert spaces we were prompted to study a model which involves a more general operator.

The novelty of our viewpoint lies in the fact that we do not take into consideration agents' utility as a potential whose gradient represents the operator of the quasi-variational inequality expressing the financial problem. In fact, we directly focus on an operator which reflects the maximization goal of agents, and only require monotonicity assumptions. Clearly, it is possible to convert our general model to the model with gradient known in the literature. However, the main advantage of our new perspective is that it enables us to state clearly the equilibrium conditions underlying the model. In other words, the equilibrium conditions which govern the trend of the market are not deduced as in Nagurney et al. (1992), Daniele (2003, 2005), and Daniele et al. (2005) by having recourse to the Lagrangian theory, but are achieved in a simpler and more direct way.

Moreover, we assume that wealth is not fixed, as often appears in the literature, but fluctuates as activity prices change. As a consequence, budget constraints of agents are implicitly defined and depend on the equilibrium activity price itself. Therefore, the resulting problem is formulated as an infinite-dimensional quasi-variational inequality.

The paper is organized as follows. After introducing the model and the optimality conditions, we prove the equivalence of the financial equilibrium with a solution to an opportune quasi-variational inequality. We then provide a theorem for the existence of solutions which is of independent interest, as it can be applied to numerous quasi-variational inequality problems.

2 The portfolio selection model and the equilibrium conditions

We start with defining the financial structure of the model. We consider an economy with a set $I = \{1, \ldots, m\}$ of agents and a set $J = \{1, \ldots, n\}$ of activities. The typical agent is denoted by i and the typical activity by j. Agents may operate financial transfers of wealth during the time interval $[0, T]$, investing in the activities that can be held as assets or liabilities. For technical reasons the functional setting of our model is the Hilbert space of the real square-integrable functions defined on the closed interval $[0, T]$. On the lines of the notations introduced by Nagurney et al. (1992), we assume that $x_j^i(t)$ represents the amount of assets associated with activity j in the portfolio of the ith agent and $y_j^i(t)$ denotes the amount of liabilities associated with activity j in the portfolio of the ith agent. To the end of simplifying notations, we group the assets into the column vector

$$x^i(t) = (x^i_1(t), \dots, x^i_n(t))^T \in L^2(0, T; \mathbb{R}^n),$$

and the liabilities into the column vector

$$y^i(t) = (y^i_1(t), \dots, y^i_n(t))^T \in L^2(0, T; \mathbb{R}^n).$$

We further group the assets $x^i(t)$ into the matrix

$$x(t) = ((x^1(t))^T, \dots, (x^m(t))^T)^T \in L^2(0, T; \mathbb{R}^{mn}),$$

and, analogously, we introduce the matrix of liabilities

$$y(t) = ((y^1(t))^T, \dots, (y^m(t))^T)^T \in L^2(0, T; \mathbb{R}^{mn}).$$

Let us also suppose that some lower and upper bounds to the investments held by the agents as both assets and liabilities are imposed. This is a reasonable assumption since agents naturally try to control their expenses as well as expected returns, forcing themselves not to invest less than a minimum quantity and more than a fixed maximum one.

Hence, for each agent i, we introduce the functions $\underline{x}^i(t)$, $\overline{x}^i(t)$, $\underline{y}^i(t)$, $\overline{y}^i(t) \in L^2(0, T; \mathbb{R}^n)$, $0 \le \underline{x}^i(t) \le \overline{x}^i(t)$, $0 \le \underline{y}^i(t) \le \overline{y}^i(t)$, a.e. $t \in [0, T]$, $i = 1, \dots, m$.

Let $p(t) = (p_1(t), \dots, p_n(t))^T \in L^2(0, T; \mathbb{R}^n)$ be the per-unit market price vector of the activities, which is assumed to be exogenous to the individual agent optimization problem. We also suppose that there is a fixed minimum activity price $\underline{p}(t) \in L^2(0, T; \mathbb{R}^n)$ and a fixed maximum one $\overline{p}(t) \in L^2(0, T; \mathbb{R}^n)$. Bounds imposed to prices deserve attention. In fact, if we refer to assets, it means that a minimum profit is guaranteed and, at the same time, a fixed maximum price value can not be exceeded. In addition, agents are led to incur in a minimum amount of liabilities, but it is not admissible to go beyond the fixed maximum price (see equilibrium condition (3)).

As we assume that there is free disposal, it results that $\underline{p}(t) \ge 0$ a.e. $t \in [0, T]$. Let $s^i(t, p(t)) : [0, T] \times L^2(0, T; \mathbb{R}^n)$ denote the total wealth available for the ith agent, which depends on time as well as on the activity price vector $p(t)$.

Let us consider the set

$$\begin{aligned} E = \quad & \Big\{ (x(t), y(t), p(t)) \in L^2(0, T; \mathbb{R}^{2mn+n}) : \underline{x}^i_j(t) \le x^i_j(t) \le \overline{x}^i_j(t), \\ & \underline{y}^i_j(t) \le y^i_j(t) \le \overline{y}^i_j(t),\ \underline{p}_j(t) \le p_j(t) \le \overline{p}_j(t), \\ & i = 1, \dots, m,\ j = 1, \dots, n,\ \text{a.e. } t \in [0, T] \Big\}. \end{aligned}$$

It is easy to verify that E is a convex, bounded and closed subset of $L^2(0,T;\mathbb{R}^{2mn+n})$.

Let $K : E \to 2^E$ be the following set-valued map

$$K(p) = \left\{ \begin{bmatrix} x(t) \\ y(t) \\ p(t) \end{bmatrix} \in E : \sum_{j=1}^{n} \varphi_{ij} x_j^i(t) = s^i(t,p(t)), \right.$$
$$\sum_{j=1}^{n} \varphi_{ij} y_j^i(t) = s^i(t,p(t)),\ i = 1,\dots,m,$$
$$\left. j = 1,\dots,n,\ \text{a.e. } t \in [0,T] \right\},$$

where φ_{ij} is such that φ_{ij} is 1 if the ith agent invests in activity j and 0 otherwise.

We observe that conditions

$$\begin{aligned} \sum_{j=1}^{n} \varphi_{ij} x_j^i(t) &= s^i(t,p(t)), \\ \sum_{j=1}^{n} \varphi_{ij} y_j^i(t) &= s^i(t,p(t)), \end{aligned}$$

which depend on price, represent the implicit budget constraints of agents. Thus, wealth is supposed to be subject to the trend of activity prices evaluated by agents, who change their investment strategy according to the greater or less possibility of profit.

In view of presenting a general financial equilibrium problem, without involving agent utility functions and their gradients explicitly (as in Nagurney et al. (1992), Daniele (2003, 2005), and Daniele et al. (2005)), we introduce, $\forall i = 1,\dots,m$, the following functions

$$U^i(t,x^i(t),y^i(t),p(t)) = u^i(t,x^i(t),y^i(t)) - p(t),$$
$$V^i(t,x(t)^i,y^i(t),p(t)) = v^i(t,x^i(t),y^i(t)) + p(t),$$

where $u^i(t,x^i(t),y^i(t))$, defined on $[0,T] \times L^2(0,T;\mathbb{R}^{2n})$, is measurable in t, continuous and monotone in x^i, whereas $v^i(t,x^i(t),y^i(t))$, defined on $[0,T] \times L^2(0,T;\mathbb{R}^{2n})$, is measurable in t, continuous and monotone in y^i.

We group $U^i(t,x^i(t),y^i(t),p(t))$ into the vector

$$U(t,x(t),y(t),p(t)) = (U^1(t,x^1(t),y^1(t),p(t)),\dots,U^m(t,x^m(t),y^m(t),p(t)))^T,$$

and $V^i(t,x^i(t),y^i(t),p(t))$ into the vector

$$V(t,x(t),y(t),p(t)) = (V^1(t,x^1(t),y^1(t),p(t)),\dots,V^m(t,x^m(t),y^m(t),p(t)))^T.$$

Remark 1 *The advantage of our approach is that it allows us to see the mappings describing the financial model under a new perspective, so that we can directly state the equilibrium conditions, without applying the Lagrangian theory. The relationship with the model with gradient known in the literature is immediate. In fact, let us introduce the utility functions*

$$\Omega^i(t, x^i(t), y^i(t), p(t)) = \omega^i(t, x^i(t), y^i(t)) + p(t)(x^i(t) - y^i(t)),$$

where $\omega^i(t, x^i(t), y^i(t))$ *is defined on* $[0, T] \times L^2(0, T; \mathbb{R}^{2n})$*, measurable in* t*, concave and continuous differentiable with respect to* x^i *and* y^i*. Then it results*

$$\begin{aligned} u^i(t, x^i(t), y^i(t)) &= -\nabla_{x^i}\omega^i(t, x^i(t), y^i(t)), \\ v^i(t, x^i(t), y^i(t)) &= -\nabla_{y^i}\omega^i(t, x^i(t), y^i(t)), \end{aligned}$$

and hence

$$\begin{aligned} U^i(t, x^i(t), y^i(t), p(t)) &= -\nabla_{x^i}\Omega^i(t, x^i(t), y^i(t), p(t)), \\ V^i(t, x^i(t), y^i(t), p(t)) &= -\nabla_{y^i}\Omega^i(t, x^i(t), y^i(t), p(t)). \end{aligned}$$

Example In the quadratic utility function model, denoted by $Q^i(t) = \begin{pmatrix} Q^i_{11}(t) & Q^i_{12}(t) \\ Q^i_{21}(t) & Q^i_{22}(t) \end{pmatrix}$ the $2n \times 2n$ symmetric and positive definite variance-covariance matrix with entries in $L^\infty(0, T)$, it results that

$$u^i(t, x^i(t), y^i(t)) = -2[Q^i_{11}(t)]^T_j x^i(t) - 2[Q^i_{21}(t)]^T_j y^i(t),$$

$$v^i(t, x^i(t), y^i(t)) = -2[Q^i_{12}(t)]^T_j x^i(t) - 2[Q^i_{22}(t)]^T_j y^i(t).$$

Now we are ready to define the equilibrium concept which governs the trend of the market.

Definition 1 *A vector* $(x^*(t), y^*(t), p^*(t)) \in L^2(0, T; \mathbb{R}^{2mn+n})$ *is a financial equilibrium if and only if* $(x^*(t), y^*(t), p^*(t)) \in K(p^*)$ *and the following conditions are satisfied a.e. in* $[0, T]$

$$u^i(t, x^{*i}(t), y^{*i}(t)) - p^*_j(t) \begin{cases} \geq 0 & \text{if } x^{*i}(t) = \underline{x}^i(t) \\ = 0 & \text{if } \underline{x}^i(t) < x^{*i}(t) < \overline{x}^i(t) \\ \leq 0 & \text{if } x^{*i}(t) = \overline{x}^i(t), \end{cases} \tag{1}$$

$$v^i(t, x^{*i}(t), y^{*i}(t)) + p_j^*(t) \begin{cases} \geq 0 & \text{if } y^{*i}(t) = \underline{y}_j^i(t) \\ = 0 & \text{if } \underline{y}^i(t) < y^{*i}(t) < \overline{y}^i(t) \\ \leq 0 & \text{if } y^{*i}(t) = \overline{y}^i(t), \end{cases} \tag{2}$$

$$\sum_{i=1}^{m} (x_j^{*i}(t) - y_j^{*i}(t)) \begin{cases} \geq 0 & \text{if } p_j^*(t) = \underline{p}_j(t) \\ = 0 & \text{if } \underline{p}_j(t) < p_j^*(t) < \overline{p}_j(t) \\ \leq 0 & \text{if } p_j^*(t) = \overline{p}_j(t), \end{cases} \tag{3}$$

$\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$.

It is worth noting that (3) is nothing but the price equilibrium condition which regulates the activity market. Specifically, if the price of an activity is minimum, then there may be an excess supply of that activity or the market may clear. If the price of an activity is greater than the minimum and less than the maximum, then the market must clear for that activity. Finally, if the price of an activity reaches the maximum, then there may be an excess demand of that activity or the market may clear.

3 The quasi-variational inequality formulation

In this section we provide a characterization of the equilibrium vector as a solution to a suitable quasi-variational inequality. To this end, we introduce the operator $F : [0, T] \times L^2(0, T; \mathbb{R}^{2mn+n}) \to L^2(0, T; \mathbb{R}^{2mn+n})$

$$F(t, X(t)) = \begin{bmatrix} F_1(t, X(t)) \\ \vdots \\ F_{2mn+1}(t, X(t)) \\ \vdots \\ F_{2mn+n}(t, X(t)) \end{bmatrix} = \begin{bmatrix} U(t, x(t), y(t), p(t)) \\ V(t, x(t), y(t), p(t)) \\ \sum_{i=1}^m (x_1^i(t) - y_1^i(t)) \\ \vdots \\ \sum_{i=1}^m (x_n^i(t) - y_n^i(t)) \end{bmatrix}_{(2mn+n)\times 1},$$

where $X(t) = (x(t), y(t), p(t))^T$.

Remark 2 *For the sake of generality, we change our notations as follows*

$$K(p^*) = K(X^*) \text{ and } s(t, p^*(t)) = s(t, X^*(t)).$$

Thus, the quasi-variational inequality associated with the financial equilibrium problem with implicit budget constraints is given by

$$X^*(t) \in K(X^*) : \int_0^T \langle F(t, X^*(t)), X(t) - X^*(t) \rangle dt \geq 0, \forall X(t) \in K(X^*), \tag{4}$$

or equivalently

$$\int_0^T \Big\{ \sum_{i=1}^m U^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) \times (x^i(t) - x^{*i}(t))$$
$$+ \sum_{i=1}^m V^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) \times (y^i(t) - y^{*i}(t))$$
$$+ \sum_{j=1}^n \Big(\sum_{i=1}^m (x_j^{i*}(t) - y_j^{i*}(t)) \Big) \times (p_j(t) - p_j^*(t)) \Big\} dt \geq 0,$$
$$\forall (x(t), y(t), p(t)) \in K(X^*).$$

The subsequent result shows the equivalence of the equilibrium vector with a solution to (4).

Theorem 1 *A vector $(x^*(t), y^*(t), p^*(t)) \in K(X^*)$ is a financial equilibrium if and only if it solves quasi-variational inequality (4).*

Proof. First we suppose that $(x^*(t), y^*(t), p^*(t))$ is an equilibrium vector and show that

$$U^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) \times (x^i(t) - x^{*i}(t)) \geq 0, \tag{5}$$

$\forall x^i(t)$ and $\forall i = 1, \ldots, m$.

Three situations may occur:

1. $x^{*i}(t) = \underline{x}^i(t)$, $\forall i = 1, \ldots, m$, a.e. $t \in [0, T]$.

   According to (1), it follows that $U^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) \geq 0$ and $x^i(t) - x^{*i}(t) \geq 0$, hence (5) is verified.

2. $\underline{x}^i(t) \leq x^{*i}(t) \leq \overline{x}^i(t)$, $\forall i = 1, \ldots, m$, a.e. $t \in [0, T]$.

   By (1), it results that $U^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) = 0$.

3. $x^{*i}(t) = \overline{x}^i(t)$, $\forall i = 1, \ldots, m$, a.e. $t \in [0, T]$.

   In this case, again by the equilibrium conditions, we have that the factors are non positive, thus relationship (5) is proved.

Analogously, involving (2) and (3), respectively, we prove that

$$V^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) \times (y^i(t) - y^{*i}(t)) \geq 0,$$
$$\sum_{i=1}^m (x_j^{*i}(t) - y_j^{*i}(t)) \times (p_j(t) - p_j^*(t)) \geq 0.$$

Conversely, we suppose that the quasi-variational inequality is verified and prove that equilibrium conditions (1), (2), (3) hold.

Let us fix $i \in \{1, \ldots, m\}$ and suppose that $p(t) = p^*(t)$, $y(t) = y^*(t)$ and $x^q(t) = x^{*q}(t)$, $\forall q \neq i$. Therefore quasi-variational inequality (4) becomes

$$\int_0^T U^i(t, x^{*i}(t), y^{*i}(t), p^*(t)) \times (x^i(t) - x^{*i}(t))dt$$
$$= \int_0^T (u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t)) \times (x^i(t) - x^{*i}(t))dt \geq 0.$$

In order to prove condition (1), we consider the following cases:

1. We assume that $x^{*i}(t) = \underline{x}^i(t)$ a.e. $t \in [0, T]$ and show that
$$u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t) \geq 0.$$
We argue by contradiction and suppose that there exists a set of positive measure $G \subset [0, T]$ so that $u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t) < 0$ a.e. $t \in G$. Let us choose $x^i(t)$ as follows
$$x^i(t) \begin{cases} > \underline{x}^i(t) & \text{if } t \in G \\ = x^{*i}(t) & \text{if } t \in [0, T] \setminus G, \end{cases}$$
then we have
$$\int_0^T (u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t)) \times (x^i(t) - x^{*i}(t))dt$$
$$= \int_G (u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t)) \times (x^i(t) - \underline{x}^i(t))dt < 0.$$

2. We suppose that $x^{*i}(t) = \overline{x}^i(t)$ a.e. $t \in [0, T]$ and prove that
$$u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t) < 0.$$
We argue by contradiction and consider a set of positive measure $G \subset [0, T]$ so that $u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t) > 0$ a.e. $t \in G$. By choosing $x^i(t)$ as follows
$$x^i(t) \begin{cases} < \overline{x}^i_j(t) & \text{if } t \in G \\ = x^{*i}(t) & \text{if } t \in [0, T] \setminus G, \end{cases}$$
it results that
$$\int_0^T (u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t)) \times (x^i(t) - x^{*i}(t))dt$$
$$= \int_G (u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t)) \times (x^i(t) - \overline{x}^i(t))dt < 0.$$

3. Finally, if we suppose that $\underline{x}^i(t) < x^{*i}(t) < \overline{x}^i(t)$ a.e. $t \in [0,T]$, by using the same technique as in the previous cases, it can be easily proved that $u^i(t, x^{*i}(t), y^{*i}(t)) - p^*(t)$ can not be either positive or negative on any set of positive measure.

Equilibrium conditions (2) and (3) may be analogously deduced. $\square$

The most general case in which, for each agent i, equilibrium solutions do not have the same behavior during the whole time interval, but take different values in different subsets of $[0,T]$ with positive measures, can be equally examined. It suffices to consider a partition of the interval $[0,T]$ and combine together the above three cases.

4 An existence result

Before showing our result, for reader's convenience, we recall the following theorem adapted to our case (see Tan (1985)), which will be useful to prove our main achievement (see also De Luca (1997)).

Theorem 2 *Let Y be a topological linear locally convex Hausdorff space and let $E \subset Y$ be a convex, compact and nonempty subset. Let $F : E \to Y'$ be a continuous function and let $K : E \to 2^E$ be a closed lower semicontinuous set-valued map with $K(X), X \in E$ convex, compact and nonempty. Then, there exists $X^* \in K(X^*)$*

$$\langle F(X^*), X - X^* \rangle \geq 0, \qquad \forall X \in K(X^*).$$

Now we are able to prove the following result.

Theorem 3 *Let us assume that the functions*

$$F : [0,T] \times \mathbb{R}^{2mn+n} \to \mathbb{R}^{2mn+n}$$
$$s : [0,T] \times \mathbb{R}^{2mn+n} \to \mathbb{R}^{m}$$

satisfy the following conditions

a) *$F(t,v)$ is measurable in t $\forall v \in \mathbb{R}^{2mn+n}$, continuous in v for t a.e. in $[0,T]$,*

$$\exists \gamma \in L^2(0,T) : |F(t,v)| \leq \gamma(t) + |v|;$$

b) *$s(t,v)$ is measurable in t $\forall v \in \mathbb{R}^{2mn+n}$, continuous in v for t a.e. in $[0,T]$,*

$$\exists \xi \in L^2(0,T) : |s(t,v)| \leq \xi(t) + |v|;$$

c) $\exists \nu \in L^2(0,T)$, $\nu(t) \geq 0$ a.e. $t \in [0,T]$:

$$|s(t,v_1) - s(t,v_2)| \leq \nu(t)|v_1 - v_2|, \quad \forall v_1, v_2 \in \mathbb{R}^{2mn+n};$$

d) $F(t,v)$ is convex in v for t a.e. in $[0,T]$ and upper semicontinuous with respect to the weak topology in $v \in E$ for t a.e. in $[0,T]$;

e) $s(t,v)$ is convex in v for t a.e. in $[0,T]$ and upper semicontinuous with respect to the weak topology in $v \in E$ for t a.e. in $[0,T]$.

Then the quasi-variational inequality problem (4) admits a solution.

Proof. We proceed using arguments of weak topology. First we observe that under hypotheses a), b) and since $X^*(t) \in L^2(0,T;\mathbb{R}^{2mn+n})$, it results that

$$F(t, X^*(t)) \in L^2(0,T;\mathbb{R}^{2mn+n}) \text{ and } s(t, X^*(t)) \in L^2(0,T;\mathbb{R}^n).$$

Moreover, by a) and b) it follows F and s belong to the class of Nemytskii operators, therefore if $\{X^{*k}\}$ stronly converges to X^* in L^2, then

$$\|F(t, X^{*k}(t)) - F(t, X^*(t))\|_{L^2} \to 0, \ \ \|s(t, X^{*k}(t)) - s(t, X^*(t))\|_{L^2} \to 0,$$

and the functions F and s are continuous in L^2 with respect to the strong topology.

Now, in order to prove that K is a weakly closed set-valued map, we show that it is strongly closed, *i.e.*

$$\forall \{X^{*k}\} \xrightarrow{L^2} X^*, \ \forall \{X^k\} \xrightarrow{L^2} X \text{ with } X^k \in K(X^{*k}) \ \forall k \in \mathbb{N},$$

then $X \in K(X^*)$.

Let $\{X^{*k}\}$, $\{X^k\}$ two arbitrary strongly convergent sequences. Since $X^k \in K(X^{*k})$, we have that

$$\begin{aligned}
&\underline{x}^i_j(t) \leq x^{ik}_j(t) \leq \overline{x}^i_j(t),\\
&\underline{y}^i_j(t) \leq y^{ik}_j(t) \leq \overline{y}^i_j(t),\\
&\underline{p}_j(t) \leq p^k_j(t) \leq \overline{p}_j(t),\\
&\forall i = 1, \ldots, m, j = 1, \ldots, n,\\
&\text{a.e. } t \in [0,T],
\end{aligned}$$

and the convergence of the sequence $\{X^k\}$ in L^2 implies that even X satisfies the bound constraints. Moreover, $\forall i = 1, \dots, m$ the following relationships hold

$$\begin{aligned}\sum_{j=1}^{n} \varphi_{ij} x_j^{ik}(t) &= s^i(t, X^{*k}(t)), \text{ a.e. } t \in [0,T], \\ \sum_{j=1}^{n} \varphi_{ij} y_j^{ik}(t) &= s^i(t, X^{*k}(t)), \text{ a.e. } t \in [0,T].\end{aligned}$$

The left-hand sides converge almost everywhere to $\sum_{j=1}^{n} \varphi_{ij} x_j^i(t)$ and $\sum_{j=1}^{n} \varphi_{ij} y_j^i(t)$ respectively. By applying c) it follows that

$$\|s^i(t, X^{*k}(t)) - s^i(t, X^*(t))\|_{L^2} \longrightarrow 0,$$

and thus we conclude that $X \in K(X^*)$. In addition, it is easy to show that $K(X)$ is a convex subset of E, $\forall X \in E$. Therefore, being $K(X^*)$ convex and strongly closed, it is also weakly closed.

In order to prove that $K(X^*)$ is a lower semi-continuous set-valued map with respect to the weak topology, we show that $\forall \{X^{*k}\}$ weakly convergent to X^* (briefly $\{X^{*k}\} \rightharpoonup X^*$), $\forall X \in K(X^*)$ there exists $\{X^k\}$ so that

$$\{X^k\} \rightharpoonup X \text{ with } X^k \in K(X^{*k}) \ \forall k \in \mathbb{N}.$$

Let us consider an arbitrary sequence $\{X^{*k}\} \rightharpoonup X^*$, $X \in K(X^*)$ and fix $k \in \mathbb{N}$, $t \in [0,T]$. For any $i = 1, \dots, m$ we introduce the following sets

$$\begin{aligned}A_i &= \{j \in \{1, \dots, n\} : \varphi_{ij} = 1\}, \\ B_i(k,t) &= \{j \in A_i : s^i(t, X^*(t)) - s^i(t, X^{*k}(t)) \leq 0\}, \\ C_i(k,t) &= \{j \in A_i : 0 < s^i(t, X^*(t)) - s^i(t, X^{*k}(t)) < x_j^i(t) - \underline{x}_j^i(t)\}, \\ D_i(k,t) &= \{j \in A_i : x_j^i(t) - \underline{x}_j^i(t) \leq s^i(t, X^*(t)) - s^i(t, X^{*k}(t))\}.\end{aligned}$$

Let us also construct the sequence $X^k(t) = (x^k(t), y^k(t), p^k(t))$, such that

$$x_j^{ik}(t) = \begin{cases} x_j^i(t) & \text{if } j \in B_i \cup D_i, \\ x_j^i(t) - \dfrac{s^i(t, X^*(t)) - s^i(t, X^{*k}(t))}{\displaystyle\sum_{l \in C_i} \varphi_{il}} & \text{if } j \in C_i, \end{cases}$$

$$y_j^{ik}(t) = \begin{cases} y_j^i(t) & \text{if } j \in B_i \cup D_i, \\ y_j^i(t) - \dfrac{s^i(t, X^*(t)) - s^i(t, X^{*k}(t))}{\displaystyle\sum_{l \in C_i} \varphi_{il}} & \text{if } j \in C_i, \end{cases}$$

and $p_j^k(t) = p_j(t)$, $j = 1, \dots, n$.

If $j \in B_i \cup D_i$, then $X^k(t) = X(t)$ and, since $X \in K(X^*)$, then the assertion is proved. If $j \in C_i$, it is easy to show that $X^k(t)$ satisfies the bound constraints.

Moreover,

$$\begin{aligned}\sum_{j=1}^{n} \varphi_{ij} x_j^{ik}(t) &= \sum_{j \in A_i} \varphi_{ij} x_j^{ik}(t) = \sum_{j \in B_i \cup D_i} \varphi_{ij} x_j^i(t) \\ &\quad + \sum_{j \in C_i} \varphi_{ij} \left(x_j^i(t) - \frac{s^i(t, X^*(t)) - s^i(t, X^{*k}(t))}{\sum_{l \in C_i} \varphi_{il}} \right) \\ &= \sum_{j \in A_i} \varphi_{ij} x_j^i(t) - \Big(s^i(t, X^*(t)) - s^i(t, X^{*k}(t)) \Big) \\ &= s^i(t, X^{*k}(t)).\end{aligned}$$

We proceed analogously for $y_j^i(t)$. As also budget balance is verified, we deduce that $X^k \in K(X^{*k})$ $\forall k \in \mathbb{N}$.

In order to prove that $\{X^k\}$ weakly converges to X, we show that

$$\forall f(t) \in L^2(0, T), \ \lim_{k \to \infty} \int_0^T f(t)(X^k(t) - X(t))dt = 0.$$

Due to the construction of the sequence, we have

$$\begin{aligned}&\left| \int_0^T f(t)(x^k(t) - x(t))dt \right| = \left| \int_0^T f(t) \sum_{i=1}^{m} \sum_{j \in A_i} (x_j^{ik}(t) - x_j^i(t))dt \right| \\ &= \left| \int_0^T f(t) \sum_{i=1}^{m} \Big[\sum_{j \in B_i \cup D_i} (x_j^{ik}(t) - x_j^i(t)) + \sum_{j \in C_i} (x_j^{ik}(t) - x_j^i(t)) \Big] dt \right| \\ &= \left| \int_0^T f(t) \sum_{i=1}^{m} \sum_{j \in C_i} \left(\frac{s^i(t, X^*(t)) - s^i(t, X^{*k}(t))}{\sum_{l \in C_i} \varphi_{il}} \right) dt \right| \\ &= \left| \int_0^T f(t) \sum_{i=1}^{m} \Big(s^i(t, X^{*k}(t)) - s^i(t, X^*(t)) \Big) dt \right|.\end{aligned}$$

Now, on the one hand strong continuity and convexity of s imply weak lower semicontinuity; on the other hand assumption e) ensures weak upper semicontinuity. Thus, s is continuous with respect to the weak topology and the last expression of the above equality chain converges to zero. Proceeding analogously for $y(t)$, we obtain that X^k weakly converges to X. Moreover, as E is convex, closed and bounded, it is weakly compact and hence $K(X)$ is also weakly compact for all $X \in E$. Finally, assumption d) and strong continuity of F imply that F is weakly continuous. Thus, by Theorem 2 the existence of at least one solution is ensured. □

5 Conclusion

In this paper, we introduced a time-dependent financial equilibrium model in the presence of implicit budget constraints, multiple agents and different activities that can be held as assets or liabilities. We first derived the equilibrium conditions without any recourse to the Lagrangian theory, then showed the equivalence with an infinite-dimensional quasi-variational inequality. It is worth noting that we were led to consider such a formulation since we were interested in a time-dependent setting and assumed budget constraints to be variable and dependent on the expected profit of investments.

We subsequently proved a theorem for the existence of solutions, which is of independent interest and can be applied to all the quasi-variational inequality problems cast in the form (4).

The model, which extends and improves other results in the literature, may have several important applications in the study of the decision-making process of agents and especially in order to obtain a more reliable market analysis.

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