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Fertility Choice and Semi-Endogenous Growth: Where Becker Meets Jones*

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Abstract

Introducing fertility choice into an R&D-based semi-endogenous growth model makes it possible for the economy's long-run growth rate to be again fully endogenously determined. A *positive* growth rate along the balanced growth path requires a certain knife-edge assumption, though. In the usual framework, it would be the assumption that the intertemporal elasticity of substitution in consumption be exactly unity (IES=1). We argue that such an assumption constitutes the ultimate source of long-run growth in these models; thus, we analyze the alternatives. If one relaxes the IES=1 assumption, and introduces a minimum "subsistence" fertility level to the model, there may (but may not) emerge an asymptotic balanced growth path with positive growth rates, to which the economy eventually converges as levels of variables diverge to infinity. This balanced growth path is either saddle-path stable or completely stable. We also address the issue of the economy's invariance towards fertility-promoting policy within the semi-endogenous growth framework. We conclude that such policy can bring long-run effects only in the knife-edge case of IES=1 type. Jones' policy invariance result is typically consistent with endogenous fertility.

Keywords: fertility choice, semi-endogenous growth, R&D, long-run dynamics, knife-edge conditions

JEL Classification Codes: J13, O41

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1 Introduction

One of the most often discussed features of R&D-based semi-endogenous growth models is their prediction, that long-run economic growth rates cannot be positive if population growth rates are not. Although this prediction has been questioned for many reasons (e.g. "people become skillful researchers by education rather than birth" – Strulik, 2005), the semi-endogenous theory remains one of the most prominent contemporary growth theories. However, ever since Jones' (1995) pathbreaking article, authors of semi-endogenous growth models usually assumed an exponential and *exo*genous population growth. It seemed to be the correct assumption, since "it is a biological fact of nature, that people reproduce in proportion to their number" (Jones, 2003); and since policy and the economy are believed not to affect people's fertility much. On the other hand, it effectively pushed the *endo*genous growth mechanism out of the models.

Moreover, the assumption of an exogenous population growth may soon be at odds with evidence. Modern demographic trends, notably the Second Demographic Transition, already present in all developed countries (see van de Kaa, 1997), put the exponential population growth assumption into severe doubt. On the other hand, the inevitable growth slowdown predicted by the semi-endogenous growth literature has not yet materialized, which made many researchers question their assumptions, and return to models where growth is fully endogenous.[1]

In the meantime, influential ideas on how population growth can be endogenized, already appeared in the literature. Becker was probably the first to doubt the Malthusian (1798) claim, that "the passion between the sexes has appeared in every age to be so nearly the same, that it may always be considered, in algebraic language as a given quantity", and his ideas have influenced many economic theories (see e.g. Becker, 1981; and more notably, the growth theory of Barro and Becker, 1989) – the semi-endogenous growth theories as well. However, existence of long-run growth in models, which include both semi-endogenous growth and endogenous fertility, typically relies upon the weakly motivated knife-edge assumption, that intertemporal elasticity of substitution (IES) in consumption be exactly unity (Jones, 2001, 2003).[2]

In this paper, a generalized framework for analyzing the long-run dynamics of

[1] The research on "new" fully-endogenous growth models, free of strong scale effects, began with the article of Young (1998).

[2] Another question is whether perpetual population growth is, in fact, a desirable outcome. Clearly, semi-endogenous growth theories do not explicitly consider finiteness of Earth, and the fact that production of goods depends on various kinds of natural resources, *exhaustible* resources in particular. If one believes that finiteness of Earth will ultimately put a limit to population growth (see Pimentel et al., 1999, for a survey on this "interdisciplinary" strand of literature), she will not be pleased with the prediction of the semi-endogenous theory, that growth in per capita wealth will also cease. Here, however, we only point at this problem, and do not consider it any further.

semi-endogenous growth models with endogenous fertility is presented. The components of an economic framework, which are not decisive for its long-run behavior, have been herein reduced to the necessary minimum. The households' optimization problem, where consumption and fertility are the only sources of utility (the Barro-Becker approach), has been given the most attention. A detailed discussion of all possible cases, together with long-run dynamics, and the policy invariance property, that arises in some of them, and does not arise in others, is the most important contribution of this paper. In particular, we identify and characterize an "asymptotic", policy-invariant balanced growth path, yet unnoticed in literature but for a very specific case in Jones (2001).

Summarizing: it is argued herein, that there exist

- *knife-edge* cases, where it is possible for the population growth rate to stabilize at a positive steady-state value. This value can be altered by fertility policy;
- cases, in which an "asymptotic" balanced growth path exists, to which there is convergence only as levels of variables such as per capita consumption diverge to infinity. This steady state is either saddle-path stable or completely stable. It is invariant to fertility policy;
- (economically implausible) cases of complete instability.

The value of the IES in consumption is decisive for the long-run outcome of semi-endogenous growth models with endogenous fertility, as we shall see shortly.

All results obtained herein continue to hold also if one expands the basic model – so that it allows for human capital accumulation, endogenous labor allocation, imperfect competition in capital goods production, etc. – to take an example, to the form of Jones' (2005) model with an addition of endogenous fertility.[3] Long-run behavior of R&D-based semi-endogenous growth models, where population growth is exogenous, is typically qualitatively different to the behavior of models with endogenous fertility. And on the other hand, the Jones' (1995) policy invariance result typically prevails.

To reinforce our argumentation, let us also point out, that the knife-edge character of the "IES=1"-type assumptions consists not only in the fact, that the set of parameter values satisfying them is of Lebesgue measure zero (or more generally, has an empty interior) in the set of all possible parameter values, but also in the fact, that they bound away from each other cases of qualitatively different dynamic behavior of the model (e.g. explosive cases from convergent cases).

In section 2, the basic model is laid out and used to present the main point of the paper. The households' problem and their long-run fertility choice are particularly emphasized. In section 3, long-run dynamics of the model are analyzed, and the policy invariance issue is addressed. Section 4 concludes.

[3]Derivation of this result is available from the author upon request.

2 The basic model

2.1 Demographics

To give the demographics of the model an explicit treatment, we shall use a continuous-time overlapping-generations model with indeterministic lifespan. For mathematical simplicity, we assume that N_t – population at time $t \geq 0$ – is in fact not the (integer) number of individuals, but rather the measure of an interval, populated by a continuum of agents. Thus, although the lifespan of each individual is random, the Law of Large Numbers enables us to treat the death rate at each instant of time as deterministic. For each individual, we shall introduce a survival function $m : \mathbb{R}_+ \longrightarrow [0, 1]$, such that $m(0) = 1, \lim_{t\to\infty} m(t) = 0$, and m is decreasing. Total number of births at time t is denoted b_t. The size of generation t at time $z \geq t$ is equal to

$$S_{z,t} = b_t m(z - t), \tag{1}$$

and the total population at time t is

$$N_t = \int_0^t b_z m(t - z) dz. \tag{2}$$

The population growth rate can be calculated as

$$n_t = \frac{\dot{N}_t}{N_t} = \frac{b_t m(0) + \int_0^t b_z m'(t - z) dz}{N_t} \equiv \underbrace{\frac{b_t}{N_t}}_{\text{birth rate}} - \underbrace{d_t}_{\text{death rate}}. \tag{3}$$

Instead of maintaining the general form of the survival function m throughout the paper, we shall simplify the analysis by limiting it to the case of "perpetual youth". Namely, we shall take the exponential function m implying a constant probability of death β at all ages $x \geq 0$, conditional on having reached the age x. This assumption reads:

$$m(x) = e^{-\beta x} \quad \Rightarrow \quad d_x \equiv \beta, \quad \text{where } \beta > 0. \tag{4}$$

It is possible for some individuals to live forever, although such probability can well be neglected. For simplicity, we also neglect the impact of technological progress and increasing per capita wealth on the survival function m.[4]

2.2 Production technology

The production function of the single consumption good is assumed to be Cobb-Douglas, with constant returns to scale in physical capital K and labor N, and

[4] Provided that we rule out the possibility, that the expected lifespan be growing without bound, this assumption does not change our results qualitatively.

increasing returns to scale, once the technology level A is included as well, which reflects the non-rivalry of ideas:

$$Y_t = A_t^\sigma K_t^\alpha N_t^{1-\alpha}, \qquad \sigma > 0, \quad 0 < \alpha < 1. \tag{5}$$

Ideas are accumulated according to the Jones' (1995) R&D equation

$$\dot{A}_t = \nu N_t^\lambda A_t^\phi, \qquad 0 < \lambda < 1, \quad 0 < \phi < 1, \quad \nu > 0. \tag{6}$$

Thus, the spillovers in idea production are positive but not sufficiently strong for fully-endogenous R&D-driven growth ($\phi < 1$).

The above Cobb-Douglas assumptions are standard in the associated literature (see Jones, 2005, for a justification). At the same time, they greatly facilitate obtaining steady-state growth. In this paper, we think of this property as desirable, because we aim to emphasize *different* (population-side, not production-side) barriers to endogenous long-run growth.

It is assumed that the whole population works both in R&D and in the production sector. To keep things as simple as possible, we do not consider allocation of labor between these two sectors explicitly. In our setup, people receive remuneration for their production work, and not for research. Thus, R&D is considered here an inevitable side-effect of production, rather than a distinct sector of the economy.[5] And despite the fact that this model may be interpreted a model of "learning by doing" in the Arrow's (1962) tradition (or better: "inventing by doing"), all the long-run results we are discussing would have clearly gone through if we had endogenized labor allocation and allowed for an explicit treatment of the R&D outlay. We abstract from these issues only to simplify exposition.

Physical capital is accumulated according to the familiar equation of motion:

$$\dot{K}_t = Y_t - C_t - \delta K_t, \qquad \delta \geq 0. \tag{7}$$

Since all markets are perfectly competitive in this somewhat simplified setup, we obtain that the real interest rate r_t and the real wage w_t are for all t given by

$$r_t = \alpha \frac{Y_t}{K_t} - \delta, \tag{8}$$

$$w_t = (1-\alpha) \frac{Y_t}{N_t}. \tag{9}$$

The real interest rate is expected to be constant along the balanced growth path.

[5] This is an admittedly heroic assumption. However, it does not change the results qualitatively, because in the long run, the ratio of researchers is expected to approach a constant. We assure it by trivially setting it to a constant – i.e. endowing each individual with a constant amount of time for production work and for research. Then, we say that the long-run research/production effort ratio is already included in ν and A_0.

2.3 Households

We assume that the preferences of individual households may be proxied by the preferences of a representative agent.

The representative agent maximizes discounted utility of the whole dynasty, which is born at time 0. There is perfect bequest motive: the representative agent does not have to take into account the fact that some members of the dynasty are born, and some die at each instant of time. Utility is derived from consumption and the number of children (as in Barro and Becker, 1989):

$$\max_{\{c_t,b_t\}_{t=0}^{\infty}} \int_0^{\infty} N_t u(c_t, b_t) e^{-\rho t} dt, \qquad \rho > 0. \tag{10}$$

Moreover, we shall assume that the kernel utility function is of the argument-separable CRRA form:

$$u(c_t, b_t) = (1-\mu)\frac{c_t^{1-\gamma}}{1-\gamma} + \mu \frac{\left(\frac{b_t}{N_t^{\kappa}} - \bar{b}\right)^{1-\eta}}{1-\eta}, \tag{11}$$

where $0 \leq \mu \leq 1$; $\kappa > 0$; $\bar{b} \geq 0$ and γ, η are both positive.[6]

Although we adopt the usual CRRA form of the kernel utility function, we substantially generalize its fertility component by introducing κ and $\bar{b}$. Such functional form of u is versatile enough to present the multiplicity of long-run outcomes of the model, as we shall see shortly.

The parameter κ accounts for the potential spillover effects, brought about by an increasing population. These effects can be either negative if $\kappa < 1$, or positive if $\kappa > 1$. The "natural" value $\kappa = 1$ corresponds to the situation, in which no spillover effects are present and individuals derive utility from the undistorted number of their children. At this point, we shall stress that *constancy* of κ is also a simplifying assumption – in general, one could expect $\kappa(N)$ to be a decreasing function of N, so that for low N, $\kappa > 1$, which would reflect positive spillovers accrued thanks to an increased population density, and for large N, $\kappa < 1$ because of population congestion, *over*population, and all the problems associated with finiteness of Earth. Nevertheless, since we are interested in the long run exclusively, an assumption of $\kappa \leq 1$ seems reasonable. We emphasize that to our knowledge, introduction of a population spillover parameter like κ is novel to the literature.

The parameter $\bar{b}$ is an exogenous "subsistence" fertility level (introduced e.g. by Jones, 2001), which is conjectured to reflect the Malthusian notion of physiological "passion between the sexes", further corrected for religious beliefs, access to contraceptives, etc., as well as some yet undefined fertility policy. We shall see shortly

[6] Assuming that one of them is unity calls for a replacement of the relevant CRRA function with a logarithm.

that in some cases, manipulating $\bar{b}$ may allow arbitrary fertility levels to be targeted in the long run.

We assume out the costs of child rearing here. Thus, the only reason for which infinite fertility is never optimal is the fact that with greater fertility today, future consumption will have to be distributed among more people.

To solve the households' optimization problem, we set up the usual Hamiltonian

$$\mathbb{H} = N_t u(c_t, b_t) e^{-\rho t} + \Lambda_t \Big(y_t - c_t - (\delta + n_t) k_t \Big), \tag{12}$$

where $c_t = C_t/N_t$, $k_t = K_t/N_t$, $y_t = A_t^{\sigma} k_t^{\alpha}$, and Λ_t is the shadow price of physical capital. Here, c_t and b_t are control variables, and k_t is the only state variable.

Note that in such simple setup, the R&D equation (6) does not enter the household optimization problem; and that the households solve the same problem, as the social planner would do.

Calculating the first order conditions, we obtain the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{\alpha \frac{y_t}{k_t} - \delta - \rho}{\gamma}, \tag{13}$$

which determines the evolution of per capita consumption, given its initial level.

We also have that

$$\frac{N_t u_b(c_t, b_t)}{u_c(c_t, b_t)} = k_t, \tag{14}$$

or, after substituting the utility function (11) into the above equation,

$$\frac{b_t}{N_t} = \bar{b} N_t^{\kappa - 1} + \left(\frac{\mu}{1-\mu} \frac{c_t^{\gamma} N_t^{(\kappa-1)(\eta-1)}}{k_t} \right)^{\frac{1}{\eta}}. \tag{15}$$

We shall consider equation (15), which describes the optimal fertility rate, central to this paper. From this equation, we see that the fertility level excess of $\bar{b} N_t^{\kappa-1}$ is proportional to the ratio c_t^{γ}/k_t, corrected by the term $N_t^{(\kappa-1)(\eta-1)}$, responsible for the possible population spillover effects. The "subsistence" fertility level $\bar{b}$ is also corrected for these spillovers. Both correction terms disappear if $\kappa = 1$, i.e. no spillover effects of increased population are present.[7] Long-run consequences of equation (15) are inspected in more detail in the following sections.

Finally, we also have to take care of the standard transversality condition

$$\lim_{t \to \infty} \Lambda_t k_t = 0. \tag{16}$$

From the third first-order condition, we know that the shadow price Λ_t must grow (or decline) at an exponential rate $\delta + n_t - \alpha \frac{y_t}{k_t}$. Thus, equation (16) is equal to the

[7]Note that that the second correction term disappears also if $\eta = 1$ – i.e. if the intertemporal elasticity of substitution in fertility is exactly unity.

condition that the capital stock be asymptotically growing at a rate smaller than minus the rate at which its shadow price grows. This translates into the well-known condition that population growth rate n_t be smaller than the discount rate ρ, at least from some point in time t_0 on (Barro, Sala-i-Martin, 1995). Formally, it is the condition that there exists a constant $n^* > 0$ for which, for all times $t > t_0$

$$n_t \leq n^* < \rho, \tag{17}$$

that is sufficient for the transversality condition to be satisfied and for the representative agent's total utility to remain finite.

2.4 Balanced growth path

We shall define the balanced growth path (BGP) as a sequence of time paths $\{A_t, C_t, K_t, N_t, Y_t\}_{t=0}^{\infty}$, along which all economic variables grow at a constant non-negative rate, possibly zero. This definition is maintained hereafter. From the R&D equation (6), it is obtained that along the BGP (which may or may not exist), necessarily

$$\frac{\dot{A}_t}{A_t} = \frac{\lambda n_t}{1-\phi}, \tag{18}$$

where n_t is the endogenous steady-state population growth rate.

Analogously, the growth rates of per capita product, physical capital and consumption are given by

$$g_t \equiv \frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t} = \frac{\dot{c}_t}{c_t} = \frac{\sigma}{1-\alpha}\frac{\lambda}{1-\phi}n_t \equiv \xi n_t. \tag{19}$$

After straightforward algebraic manipulations, we obtain that the steady-state product/capital ratio equals

$$\frac{y_t}{k_t} = \frac{\gamma\xi n_t + \delta + \rho}{\alpha}, \tag{20}$$

and hence, the long-run product/capital ratio depends positively on the population growth rate.

The steady-state savings rate is given by

$$s_t = \frac{y_t - c_t}{y_t} = \alpha\frac{(1+\xi)n_t + \delta}{\gamma\xi n_t + \delta + \rho}. \tag{21}$$

Hence, the savings rate s_t is a homographic function of the population growth rate n_t, and their bilateral relation is positive if and only if $\delta + \rho + \xi(1 + \delta - \gamma) > 0$. In particular, it is always positive if $\gamma \leq 1$ (IES ≥ 1).

2.5 The steady-state population growth rate

Now let us find the steady-state population growth rate (the population growth rate along the BGP). First, we note that the death rate is constant by assumption: $d_t \equiv \beta$. So pass on directly the fertility issue. The outcome of the households' optimization problem implies that

$$n_t = \frac{\dot{N}_t}{N_t} = \frac{b_t}{N_t} - d_t = \bar{b}N_t^{\kappa-1} - \beta + \left(\frac{\mu}{1-\mu}\frac{c_t^{\gamma}N_t^{(\kappa-1)(\eta-1)}}{k_t}\right)^{\frac{1}{\eta}}. \qquad (22)$$

We shall take take logs and time derivatives of all components of the above equation to see that n_t can be constant (and thus define a BGP) in the three following cases exclusively:

$$\textbf{(A)} \qquad \kappa = 1 \quad \wedge \quad \gamma = 1; \qquad (23)$$

$$\textbf{(B)} \qquad \bar{b} = 0 \quad \wedge \quad \xi(\gamma-1) + (\kappa-1)(\eta-1) = 0; \qquad (24)$$

$$\textbf{(C)} \qquad N \equiv \text{const.} \qquad (25)$$

We shall also consider case **(B')**, where $\xi(\gamma-1)+(\kappa-1)(\eta-1) = 0$, but $\bar{b} > 0$; and the general case **(D)** where $\xi(\gamma-1)+(\kappa-1)(\eta-1) \neq 0$. In the two latter cases, there may exist an "asymptotic" BGP – defined as a sequence of paths $\{A_t, C_t, K_t, N_t, Y_t\}_{t=0}^{\infty}$, within which constant ("steady-state") growth rates are approached only as levels of these five variables diverge to infinity. We shall consider these two cases in greatest detail, because we consider their emergence to be an important novel result, which helps to push the semi-endogenous theory forward and yet confirms the Jones' (1995) result of growth being invariant to policy in the long run.

- **Case (A)** appears most frequently in literature. Population spillover effects are neglected. Then, to obtain positive long-run growth, a weakly motivated knife-edge assumption, that $\gamma = 1$, is called for – i.e. that the intertemporal elasticity of substitution (IES) in consumption is *exactly* one (IES=1). We have to assume that consumption enters the utility function in logs and this precise IES value is of critical importance here! However, the issue of IES in consumption has been widely discussed in literature, both theoretical and empirical, and it still remains controversial. There is no sign of consensus whatsoever, that the IES in consumption is precisely one. A brief review of the empirical literature on this issue is given in the following section, to emphasize the fragility of the IES=1 assumption.

- **Case (B)** disposes of the "subsistence" fertility level $\bar{b}$. In such case, however, a positive long-run economic growth rate requires the knife-edge assumption of

$$\xi(\gamma-1) + (\kappa-1)(\eta-1) = 0 \qquad (26)$$

to hold. If (26) does not hold, population explodes to infinity in finite time, and thus eventually violates the transversality condition, or gradually dies off. Condition (26) reduces to the above mentioned request, that IES = 1 ($\gamma = 1$), if we eliminate population spillovers as well, assuming $\kappa = 1$. In this specific situation, we are brought back again to the widely-discussed problem of IES in consumption. The general equality condition (26) is arbitrary, and apparently makes very little sense: it lumps both technological and preference parameters in a single *equation*, which makes it no easier to justify than the IES=1 condition.

- **Case (B')** is a knife-edge case in which (26) continues to hold, but $\bar{b} > 0$. Then, an "asymptotic" BGP with positive growth emerges whenever $\kappa \in (0, 1)$, but only as the level of population N_t diverges to infinity. (The term $\bar{b}N_t^{\kappa-1}$ becomes negligible as $N_t \to \infty$.) If $\kappa > 1$, the model explodes.

- **Case (C)**. A steady state such that $N \equiv \text{const}$ does not exist. Namely, from (18), we have that $\dot{A} = 0$; and hence, from (6), either $A = 0$ or $N = 0$. In both cases, the economy does not exist. However, there is a possibility of obtaining an *asymptotically* constant population within the general case (D) discussed below.

- **Case (D)**. In this general case, there may be convergence to an asymptotic BGP, in which $\bar{n} = \bar{b} - \beta$ (if $\kappa = 1$ and $\bar{b} \geq \beta$), or otherwise $\bar{n} = 0$ (if $\kappa \in (0, 1)$ and $\eta \geq 1$, or $\kappa = 1$ and $\bar{b} < \beta$). The necessary condition for convergence to take place is that
$$\xi(\gamma - 1) + (\kappa - 1)(\eta - 1) < 0. \tag{27}$$
The LHS of this inequality consists of two components. The former one accounts for agents' impatience towards consumption. The latter accounts for impatience towards having children. They are corrected for the long-run consumption growth rate, and the magnitude of the population spillover effect, respectively. If we assume out population spillovers (impose $\kappa = 1$), then this inequality reduces to the familiar condition that $\gamma < 1$ – i.e., that the IES in consumption is greater than 1.[8] We shall find out later on, that this steady state is either saddle-path stable, or completely stable.

  Let us now emphasize the two following points. First, $\kappa = 1$ (no population spillover effects present, neither positive nor negative) *is also a knife-edge condition*. The behavior of the model is qualitatively different with $\kappa < 1$, $\kappa = 1$, or $\kappa > 1$. Second, provided that $\kappa = 1$, then asymptotically, the rate of economic growth is pinned down by the two arbitrary values $\bar{b}$ and

[8] Otherwise, the model exhibits explosive behavior and comes to violate the transversality condition. Empirical literature tends to suggest an explosive $\gamma > 1$. A brief review of this strand of literature can be found in section 3.3.

β, both of an entirely demographic nature: the long-run growth rate of the economy equals $\xi(\bar{b}-\beta)$. Moreover, if $\kappa < 1$ (and $\eta \geq 1$) – so there exist negative population congestion effects (these could be arbitrarily weak!), then the long-run population growth rate is zero.

We see clearly that some knife-edge assumption on parameter values is indeed necessary if one wants to obtain *positive* long-run growth.

2.6 Some knife-edge cases quoted in literature

- **$\kappa = 1$, $\gamma = 1$ and $\bar{b} = 0$.** Satisfies (A) and (B). A variant followed by Jones (2003). In such case, equation (22) reduces to

$$n_t = -\beta + \left(\frac{\mu}{1-\mu}\frac{c_t}{k_t}\right)^{\frac{1}{\eta}}. \tag{28}$$

  Plugging into the above equation the $c_t = (1-s_t)y_t$ identity, (20) and (21), yields a single equation of a single variable n_t, which is thus implicitly determined. Population growth rate in the steady state is endogenously determined, and this result is obtained only thanks to the double knife-edge assumption $\gamma = 1; \kappa = 1$.

  If we additionally assume $\eta = 1$, an explicit solution for the steady-state population growth rate (and hence, for the growth rate of the economy g) can be found:

$$\bar{n} = \frac{\frac{\mu}{\alpha(1-\mu)}[\delta(1-\alpha)+\rho]-\beta}{1-\frac{\mu}{\alpha(1-\mu)}[\xi(1-\alpha)-\alpha]}. \tag{29}$$

- **$\kappa = 1$, $\gamma = 1$, $\eta = 1$ and $\bar{b} = \beta$.** Satisfies (A). A variant followed by Connolly and Peretto (2003). All above reasoning applies. Moreover, let us point out that introduction of a kernel utility function

$$u(c_t, n_t) = (1-\mu)\ln c_t + \mu \ln n_t \tag{30}$$

  automatically rules out the possibility of zero population growth (which would yield infinite negative utility to the agent), as well as negative population growth rates. These are observed currently in many developed countries, though.

- **$\kappa = 1$ and $\bar{b} = \beta$.** Satisfies (D). A variant followed by Jones (2001). This variant delivers zero population growth in the long run, but retains the potential of generating a positive long-run growth rate. After obvious substitutions, we obtain that in the long run, the savings rate approaches $\bar{s} = \frac{\alpha\delta}{\delta+\rho}$; the capital/output ratio approaches $\overline{k/y} = \frac{\alpha}{\delta+\rho}$; and the savings/capital ratio approaches δ. In the following section, we shall prove that this asymptotic steady state is saddle-path stable.

3 Long-run dynamics and the possibility of policy invariance

3.1 Dynamics in case (D) with $\kappa = 1$ and $\bar{b} \geq \beta$

We shall limit the scope of our analysis of dynamics to the case (D) with $\kappa = 1$ and $\bar{b} \geq \beta$. We motivate this choice by two means. First, we choose the general case (D), because some versions of the knife-edge cases (A) and (B) have already been discussed in literature; and because we do not consider equation (26) as empirically justifiable. Second, we compromise over the knife-edge assumption $\kappa = 1$, and additionally assume $\bar{b} \geq \beta$, because we would like to concentrate on the cases, in which a positive economic growth rate can persist over the long run.

Dynamics of the semi-endogenous growth model in its general case (D), with $\kappa = 1$, are to a great extent shaped by the result, that fertility either diverges or approaches $\bar{b}$, and that the latter outcome can happen only asymptotically, as per capita variables, e.g. consumption, tend to infinity.

Whether the asymptotic BGP is (at least saddle-path) stable, i.e. whether there exists a path assuring convergence to it, remains an open question, though. This question is vital because of the asymptotic nature of this BGP: the economy cannot "jump" directly into it, and if a path assuring gradual convergence it does not exist, the eventual disaster is inevitable.

It is worthy to operate on ratios and not on levels of economic variables, since it is the ratios that are constant in the asymptotic steady state (i.e. along the asymptotic BGP). Thus, we shall make the following substitutions:

$$\omega_t = \frac{y_t}{k_t}, \qquad \chi_t = \frac{c_t}{k_t}, \qquad \tilde{n} = n - \bar{b} + \beta. \tag{31}$$

The asymptotic steady state is characterized by the following values:

$$\bar{\omega} = \frac{\gamma\xi(\bar{b} - \beta) + \delta + \rho}{\alpha}, \quad \bar{\chi} = \frac{\gamma\xi(\bar{b} - \beta) + \delta + \rho}{\alpha} - (1+\xi)(\bar{b} - \beta) - \delta, \quad \bar{\tilde{n}} = 0. \tag{32}$$

Moreover, along the asymptotic BGP, we have that $\frac{\dot{A}_t}{A_t} = \frac{\lambda n_t}{1-\phi}$. In the following, we shall assume this equality to hold for all analyzed times $t > t_0$, however. We can do so without much loss of generality, because we are interested in the long run only: we can assume that A_{t_0} be already arbitrarily large. Moreover, we would anyway impose such equality when linearizing our system of differential equations around the steady state. The main aim of making this step so early is to maintain analytical tractability.

Making all the necessary substitutions, taking logs and time derivatives of (15), and assuming that $\frac{\dot{A}_t}{A_t} = \frac{\lambda n_t}{1-\phi}$ holds at all times, yields the following system of first-order ordinary differential equations in the $(\omega_t, \chi_t, \tilde{n}_t)$ space:

$$\begin{cases} \dot{\omega}_t &= \Big((1-\alpha+\frac{\sigma\lambda}{1-\phi})(\tilde{n}_t+\bar{b}-\beta)+(\alpha-1)(\omega_t-\chi_t-\delta) \Big)\omega_t, \\ \dot{\chi}_t &= \Big((\frac{\alpha}{\gamma}-1)\omega_t - \frac{\delta+\rho}{\gamma}+\chi_t+\delta+(\tilde{n}_t+\bar{b}-\beta) \Big)\chi_t, \\ \dot{\tilde{n}}_t &= \frac{1}{\eta}\Big((\tilde{n}_t+\bar{b}-\beta)+(\alpha-1)\omega_t+\chi_t-\rho \Big)\tilde{n}_t. \end{cases} \quad (33)$$

The three-dimensional system (33) shall be now linearized around the steady state (32), so that its local dynamics can be studied. In the vicinity of the steady state, we have that

$$\begin{pmatrix} \dot{\omega}_t \\ \dot{\chi}_t \\ \dot{\tilde{n}}_t \end{pmatrix} = F'(\mathbf{x}) \begin{pmatrix} \omega_t-\bar{\omega} \\ \chi_t-\bar{\chi} \\ \tilde{n}_t-\bar{\tilde{n}} \end{pmatrix} + R, \quad (34)$$

where $\mathbf{x} = (\bar{\omega}, \bar{\chi}, \bar{\tilde{n}})$, and R denotes all further terms in the Taylor expansion of (33), which are now going to be ignored. The $F'(\mathbf{x}) \equiv M$ matrix is given by

$$M = \begin{pmatrix} (1-\alpha+\frac{\sigma\lambda}{1-\phi})(\bar{\tilde{n}}+\bar{b}-\beta)+(\alpha-1)(2\bar{\omega}-\bar{\chi}-\delta); & (1-\alpha)\bar{\omega}; & (1-\alpha+\frac{\sigma\lambda}{1-\phi})\bar{\omega} \\ (\frac{\alpha}{\gamma}-1)\bar{\chi}; & (\frac{\alpha}{\gamma}-1)\bar{\omega}-\frac{\delta+\rho}{\gamma}+2\bar{\chi}+\delta+(\bar{\tilde{n}}+\bar{b}-\beta); & \bar{\chi} \\ \frac{\alpha-1}{\eta}\bar{\tilde{n}}; & \frac{1}{\eta}\bar{\tilde{n}}; & \frac{1}{\eta}\{(2\bar{\tilde{n}}+\bar{b}-\beta)+(\alpha-1)\bar{\omega}+\bar{\chi}-\rho\} \end{pmatrix}. \quad (35)$$

To analyze the local stability of the $\mathbf{x}$ steady state, we have to check whether M is positive definite (complete instability), negative definite (complete stability), or indefinite (saddle-path stability).

The results can be summarized within the three following propositions.

Proposition 3.1 *In Jones' (2001) model, where $\bar{b} = \beta$, the asymptotic steady state is saddle-path stable.*

Proposition 3.2 *If $\bar{b} > \beta$ and either of the two following cases:*

$$\xi(\frac{\gamma}{\alpha}-1) \geq 1$$

$$\text{or} \quad \begin{cases} \xi(\frac{\gamma}{\alpha}-1) < 1 \\ 0 < \bar{b}-\beta < \frac{\frac{\rho}{\alpha}+\delta(\frac{1-\alpha}{\alpha})}{1-\xi(\frac{\gamma}{\alpha}-1)} \end{cases}$$

holds, then the asymptotic steady state is saddle-path stable.

Proposition 3.3 *If $\bar{b} > \beta$ and*

$$\begin{cases} \xi(\frac{\gamma}{\alpha}-1) < 1 \\ \bar{b}-\beta > \frac{\frac{\rho}{\alpha}+\delta(\frac{1-\alpha}{\alpha})}{1-\xi(\frac{\gamma}{\alpha}-1)}, \end{cases}$$

then the asymptotic steady state is completely stable.

Proof of propositions 3.1-3.3. First, substitute (32) into (35) to see that $M_{31} = M_{32} = 0$, and therefore, $\det M = M_{33}(M_{11}M_{22} - M_{12}M_{21}) \equiv M_{33}D_2$. We would like to determine the signs of M_{11}, D_2, and $\det M$. Calculations yield

$$\left\{\begin{array}{lll} M_{11} & = (\bar{b}-\beta)(-\frac{\gamma\sigma\lambda}{\alpha(1-\phi)}) & -(\frac{1-\alpha}{\alpha})(\delta+\rho), \\ M_{12} & = (\bar{b}-\beta)(\frac{1-\alpha}{\alpha})\gamma\xi & +(\frac{1-\alpha}{\alpha})(\delta+\rho), \\ M_{21} & = (\bar{b}-\beta)(\frac{\alpha}{\gamma}-1)(\frac{\gamma\xi}{\alpha}-1-\xi) & +(\frac{\alpha}{\gamma}-1)(\frac{\delta+\rho}{\alpha}-\delta), \\ M_{22} & = (\bar{b}-\beta)(\frac{\gamma\xi}{\alpha}-1-\xi) & +\frac{\delta+\rho}{\alpha}-\delta, \\ M_{33} & = (\bar{b}-\beta)\frac{1}{\eta}\xi(\gamma-1). & \end{array}\right.$$

The steady state (32) would be saddle-path stable, if $D_2 < 0$, or if $D_2 > 0, M_{33} > 0, M_{11} < 0$, or if $D_2 > 0, M_{33} < 0, M_{11} > 0$. (Notice that $M_{33} < 0$ if and only if $\bar{b} > \beta$, and $M_{33} > 0$ if and only if $\bar{b} < \beta$.) It would be completely stable if $D_2 > 0, M_{33} < 0, M_{11} < 0$. It would be completely unstable if $D_2 > 0, M_{33} > 0, M_{11} > 0$.

In the Jones' (2001) case $\bar{b} = \beta$, after straightforward calculations, we obtain that $D_2 = (\frac{\alpha-1}{\alpha})(\delta+\rho)(\frac{\delta(1-\alpha)+\rho}{\gamma})$, and hence, unambiguously $D_2 < 0$. Hence, M is indefinite. This brings us to a conclusion, that the asymptotic steady state of Jones' (2001) model is saddle-path stable (despite the fact, that in this particular case, $M_{33} = \det M = 0$). This completes the proof of proposition 3.1.

Let us now take $\bar{b} > \beta$. For $\bar{b} \approx \beta$, the saddle-path stability property continues to hold by continuity of the matrix determinant, but does not have to (and in fact, does not) hold in general.

We have that $M_{21} = (\frac{\alpha}{\gamma} - 1)M_{22}$. It follows that

$$\mathrm{sgn} D_2 = \mathrm{sgn} M_{22} \cdot \mathrm{sgn}\Big(M_{11} - \Big(\frac{\alpha}{\gamma} - 1\Big)M_{12}\Big).$$

We obtain

$$\begin{aligned} \Big(M_{11} - \Big(\frac{\alpha}{\gamma}-1\Big)M_{12}\Big) &= (\bar{b}-\beta)\Big(-\frac{\sigma\lambda}{1-\phi}\Big) - \frac{1-\alpha}{\gamma}(\delta+\rho) < 0, \\ M_{11} &= (\bar{b}-\beta)(-\frac{\gamma\sigma\lambda}{\alpha(1-\phi)}) - (\frac{1-\alpha}{\alpha})(\delta+\rho) < 0, \\ M_{33} &= (\bar{b}-\beta)\frac{1}{\eta}\xi(\gamma-1) < 0. \end{aligned}$$

These equations hold because of our prior assumptions on parameter values and $\bar{b} > \beta$. They imply that the steady state can only be saddle-path stable, or completely stable, and the result depends on the sign of M_{22}.

So let us pass on to M_{22}. First, assume $\xi(\frac{\gamma}{\alpha} - 1) \geq 1$. Then, M_{22} becomes a sum of two positive numbers, so it is positive. It follows that $D_2 < 0$, and the steady state is saddle-path stable.

Second, assume $\xi(\frac{\gamma}{\alpha}-1)<1$ and $\bar{b}-\beta<\Xi$, where

$$\Xi=\frac{\frac{\rho}{\alpha}+\delta(\frac{1-\alpha}{\alpha})}{1-\xi(\frac{\gamma}{\alpha}-1)}.$$

We have that $M_{22}>0$, $D_2<0$ and again, the steady state is saddle-path stable. This completes the proof of proposition 3.2.

Analogously, if $\xi(\frac{\gamma}{\alpha}-1)<1$ and $\bar{b}-\beta>\Xi$, we obtain that $M_{22}<0$, $D_2>0$, and the steady state state is completely stable (proposition 3.3). ■

3.2 Invariance to fertility policy

To analyze the possibility of invariance of the long-run model outcome to fertility policy, we shall introduce a "government": an entity that conducts policy, aimed to increase (or decrease) the fertility rate, and thus, the growth rate of the economy. It is assumed to operate under a balanced-budget regime and without extra costs, and to return all levied taxes back to the households in the form of transfers. Without loss of generality, we shall from now on assume that the government intends to *increase* fertility (the other case is completely symmetric, and it appears if we switch signs of tax and transfer, respectively).

We shall maintain the assumptions of $\kappa=1$ and $\bar{b}\geq\beta$ throughout this subsection.

The households' budget constraint is modified in the following way, to include tax and transfer:

$$\dot{k}_t=y_t-c_t-(\delta+n_t)k_t+\tau_t\frac{b_t}{N_t}-T_t, \tag{36}$$

so that the lump-sum tax T_t is taken as given, but the per-child transfer τ_t is readily internalized by the families. Symmetry and the balanced-budget rule are imposed after optimization takes place. This simple "externality" trick is probably the easiest way to analyze the policy invariance issue in such framework.

We see that the equations (13), (16), (18), (19), (20) and (21) continue to hold. The only difference is that now the solution of the household's fertility problem (15) becomes

$$\frac{N_t u_b(c_t,b_t)}{u_c(c_t,b_t)}=k_t-\tau_t, \tag{37}$$

or, after substituting the utility function (11) into the above equation, and imposing $\kappa=1$,

$$\frac{b_t}{N_t}=\bar{b}+\left(\frac{\mu}{1-\mu}\frac{c_t^{\gamma}}{k_t-\tau_t}\right)^{\frac{1}{\eta}}. \tag{38}$$

From the above formula, it is quickly verified, that in the short run, an increase in the governmental subsidy rate τ_t unambiguously increases fertility.

However, we would like to analyze the long-run policy effects and not the short-run ones. We shall then concentrate on the asymptotic steady state.

Along the BGP where the population growth rate equals $\bar{n}$, the governmental per-child donation rate τ_t has to grow at a rate equal to $\xi\bar{n}$ if it is meant to remain operative.[9] Without loss of generality, we shall assume that the government conforms to this rule at all times, and thus substitute $\tau_t = \tau_0 e^{\xi n_t t}$. We shall now check whether a change in the initial donation rate τ_0 induces a long-run response in the population growth rate $\bar{n}$, and hence, in the economy's growth rate $\bar{g}$.

If the IES in consumption is smaller than one ($\gamma > 1$), the economy obviously diverges, regardless of τ_0. If it is greater than one, then the fertility level eventually approaches $\bar{b}$, again regardless of τ_0. This is the policy invariance result, obtained whenever $\gamma \neq 1$.

We have already shown that Jones' (1995) policy invariance result (of long-run growth being independent of policy variables) is typically consistent with endogenous fertility. Namely, provided that $\gamma \neq 1$, so that the specific growth-generating knife-edge condition does not hold, long-run growth is ultimately pinned down by exogenous "demographic" constants.

We shall now turn to the knife-edge case IES=1. In such case, things get more complicated as we have that

$$\bar{n} = \bar{b} - \beta + \left(\frac{\mu}{1-\mu} \frac{c_t}{k_t - \tau_t} \right)^{\frac{1}{\eta}} \Bigg|_{\text{BGP.}} \tag{39}$$

To our disappointment, it turns out that (39) contains – among other expressions – the steady-state per-child donation/capital ratio $\overline{\tau/k}$ which cannot be analytically determined.[10] Thus, instead of analyzing the impact of τ_0 on $\bar{n}$, we are forced to consider the impact of $\psi \in [0, 1)$, with ψ defined as the donation/capital ratio *targeted for the steady state* by the government. We shall use the Implicit Function Theorem to check in what manner can the steady-state population growth rate $\bar{n}$ be altered by changes in ψ that arrive unexpected by the households.

Rewritten in terms of $\bar{n}$ and ψ, equation (39) becomes

$$0 = F(\bar{n}, \psi) = -\bar{n} + \bar{b} - \beta + \left(\frac{\mu}{1-\mu} \bar{\chi}(\bar{n}) \left(\frac{1}{1-\psi} \right) \right)^{\frac{1}{\eta}}, \tag{40}$$

where $\bar{\chi}(\bar{n}) = \overline{c/k} = \frac{\xi\bar{n}+\delta+\rho}{\alpha} - (1+\xi)\bar{n} - \delta$.

[9] If it grows slower, it becomes asymptotically non-operative; and if it grows faster, government budget size eventually exceeds the size of the economy.

[10] That is, the donation/capital ratio cannot be determined without an explicit integration of the differential equations of the model; and such explicit integration is of course impossible because of numerous non-linearities.

After a fair amount of algebra, we obtain that an increase in the steady-state per-child donation/capital ratio ψ raises steady-state fertility $\bar{n}$ if and only if

$$0 \leq \psi < 1 - \left(\frac{1}{\eta}\right)^{\eta} \frac{\mu}{1-\mu} \bar{\chi}^{1-\eta} \left(\frac{d\bar{\chi}}{d\bar{n}}\right)^{\eta}, \tag{41}$$

where $\frac{d\bar{\chi}}{d\bar{n}} = \frac{\xi}{\alpha} - 1 - \xi$. As a corollary, we obtain that the value of ψ given by the right-hand side of (41) maximizes the long-run fertility rate and thus the long-run growth rate of the economy (provided that it falls between zero and one). Such value should be targeted in the long run by a benevolent government.

Summing up, we see that once the IES in consumption is set exactly to unity, the long-run fertility policy-invariance result, typical for the generic case IES $\neq 1$, disappears. It becomes again possible for fertility policy to influence the economy's long-run growth rate.

Let us note that this result is analogous to the one obtained when comparing fully-endogenous and semi-endogenous growth models (see Jones, 1995). When a certain knife-edge condition is relaxed, policy's ability to influence long-run growth rates disappears.

3.3 The IES in consumption

The two most prominent cases of the semi-endogenous growth model with endogenous fertility are: the knife-edge case (A) with $\gamma = \kappa = 1$, most frequently discussed in literature; and case (D), with $\kappa = 1$ and $\gamma \neq 1$. We deliberately rule out population spillover effects here, because we want to consider the cases, in which positive long-run growth remains possible. And once the spillover effects are assumed out, the critical question is whether the IES in consumption is *exactly* one or not.

The magnitude of the true IES in consumption has been estimated in a wide range of empirical works; for brevity, we shall name just a few contributions.

The discussion began with the work of Hall (1988), who concluded, that the IES in consumption in the United States was very small, and possibly zero – that is, that for sure $\gamma > 1$. In such case, the semi-endogenous growth model would inevitably explode.

Patterson and Pesaran (1992) upgraded Hall's methodology by assuming that the slope coefficient of the MA process, governing per capita consumption, is not known a priori, as Hall presumed. This modification helped them obtain the result of the American IES being around 0,213 and significantly different from zero. In such case, our model would continue to explode.

Hall's results have also been criticized on other grounds. It has been argued, that the Euler equation he estimated was misspecified, and unsuitable for the country-level aggregate data whatsoever. Beaudry and van Wincoop (1996) used a panel of U.S states instead (spanning 1953-1991, or in the other estimation round, 1978-1991), and modified the Hall's original Euler equation. They have achieved a large improvement in the estimation precision, and obtained a result of IES being around 0.7-1.1 (depending on the estimation method and the set of instrumental variables): clearly different from zero and not significantly different from one. In such cases, all possible scenarios of the dynamics of our model are possible.

Guvenen (2005) added another dimension to this discussion, pointing out, that in reality, as opposed to most theoretical approaches, agents are heterogenous. In particular, a large fraction of households does not participate in stock markets at all. Moreover, the IES in consumption varies significantly across individuals, increasing with income. Consumption is much more evenly distributed than wealth. This asymmetry accounts, according to Guvenen, for a serious underestimation of the IES in all previous studies. He concludes, that among non-stockholders, IES is indeed around 0.1 (as e.g. Hall suggested); but among stockholders, it is rather expected to fall into the interval (0.8, 1.2). And it is the stockholders who effectively determine the real interest rate of the economy. This makes the variant IES=1 again possible. However, Guvenen himself states that "a plausible range for this parameter is possibly (0,1)". Within this "plausible" range, the semi-endogenous growth model would explode.

Favero (2005) merged the Euler equation with the (linearized) budget constraint of the households, and used the resultant equation to estimate the IES. He obtained IES=0.78, with a standard deviation of 0.11 (i.e. clearly smaller than one).

Harashima (2005) went in a different direction. He overthrew the assumption, that in the estimated Euler equation, the real interest rate is taken as given (the "endowment" economy assumption). Instead, he proposed to consider a closed "production" economy, in which the IES is obtained directly from some version of equation (13). He concluded, that in such case, the IES would be again very low, around 0.09.[11]

The above literature review is by no means complete; it is included here only to point out, that the assumption of $\gamma = 1$ is not only knife-edge, but also disputable; and that even more troublesome is the assumption of $\gamma < 1$, made in the general case of the semi-endogenous growth model with endogenous fertility, but without population congestion effects. Empirical investigations bring somewhat convincing evidence, that $\gamma \geq 1$!

4 Conclusion

In this paper, we have studied the dynamic behavior of R&D-based semi-endogenous growth models with endogenous fertility. We have shown, that:

- theories discussed in literature usually rely on unmotivated knife-edge assump-

[11]Harashima just calibrated his theoretical growth model, and did not use any econometric methods.

tions, which guarantee positive steady-state growth. Among them, the assumption that the IES in consumption equals exactly unity is probably the most prominent;

- once one relaxes such knife-edge assumptions, one has to introduce a "subsistence" fertility level $\bar{b} > \beta$ and rule out population spillover effects to obtain a positive long-run population growth rate, and hence, a positive economic growth rate;
- in the general form of such models, there exists an asymptotic steady state, to which there may be convergence only as levels of variables diverge to infinity;
- the asymptotic steady state is either saddle-path stable or completely stable, depending on parameter values;
- in the general case, the long-run economic growth rate is policy-invariant. In the knife-edge cases, it is not,
- the dynamic behavior of the semi-endogenous growth models with endogenous fertility is typically qualitatively different from the behavior of their counterparts with exogenous fertility.

To sum up, we shall emphasize some of the facts, fundamental for this paper. The first fact is that all semi-endogenous growth models with endogenous fertility, found in literature and discussed herein, impose specific knife-edge conditions on the parameter values. These conditions are decisive for their long-run dynamics, because they guarantee steady-state growth; on the other hand, they make the models lose robustness to the smallest changes in values of some parameters. The second fact is that if population spillover effects are ruled out and a "subsistence" fertility level is introduced, a positive steady-state growth rate is again possible. In such case, as opposed to the other cases, it is policy-invariant. And last but certainly not least, the empirical plausibility of a given set of values of the model parameters is not sufficient for the model to exhibit convergence to the asymptotic steady state, not even along a saddle path. Indeed, in the empirically supported $\gamma > 1$ (IES < 1) case, semi-endogenous growth models with endogenous fertility explode.

References

[1] Arrow, K.J. (1962), The Economic Implications of Learning by Doing, *Review of Economic Studies* 29(3), pp. 155-173.

[2] Barro, R.J., G.S. Becker (1989), Fertility Choice in a Model of Economic Growth, *Econometrica* 57(2), pp. 481-501.

[3] Barro, R.J., X. Sala-i-Martin (1995), *Economic Growth* (McGraw-Hill, Inc.)

[4] Beaudry, P., E. van Wincoop (1996), The Intertemporal Elasticity of Substitution: An Exploration Using a U.S Panel of State Data, *Economica* 63, pp. 495-512.

[5] Becker, G.S. (1981), *A Treatise on the Family* (Harvard University Press, Cambridge).

[6] Connolly, M., P. Peretto (2003), Industry and the Family: Two Engines of Growth, *Journal of Economic Growth* 8(1), pp. 114-148.

[7] Favero, C.A. (2005), Consumption, Wealth, the Elasticity of Intertemporal Substitution and Long-Run Stock Market Returns, CEPR Working Paper No. 5110.

[8] Guvenen, M.F. (2005), Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective, forthcoming in *Journal of Monetary Economics.*

[9] Hall, R.E. (1988), Intertemporal Elasticity of Substitution in Consumption, *Journal of Political Economy* 96, pp. 339-357.

[10] Harashima, T. (2005), An Estimate of the Elasticity of Intertemporal Substitution in a Production Economy, mimeo, University of Tsukuba.

[11] Jones, C. I. (1995), R&D-Based Models of Economic Growth, *Journal of Political Economy* 103, pp. 759-784.

[12] __________ (2001), Was an Industrial Revolution Inevitable? Economic Growth Over the Very Long Run, *Advances in Macroeconomics* Vol. 1, No. 2, Article 1, http://www.bepress.com/bejm/advances/vol1/iss2/art1

[13] __________ (2003), Population and Ideas: A Theory of Endogenous Growth, in: P. Aghion, R. Frydman, J. Stiglitz, and M. Woodford, eds., *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps* (Princeton University Press).

[14] _________ (2005), Growth and Ideas, in: P. Aghion and S.N. Durlauf, eds., *Handbook of Economic Growth* (North-Holland, Amsterdam).

[15] van de Kaa, D. J. (1997), Options and Sequences: Europe's Demographic Patterns, *Journal of the Australian Population Association* 14 (1), pp. 1-30.

[16] Malthus, T.R. (1798), *An Essay on the Principle of Population* (printed for J. Johnson, in St. Paul's Church-Yard, London).

[17] Patterson K.D., B. Pesaran (1992), The Intertemporal Elasticity of Substitution in Consumption in the United States and the United Kingdom, *Review of Economics and Statistics* 74(4), pp. 573-584.

[18] Pimentel, D., O. Bailey, P. Kim, E. Mullaney, J. Calabrese, L. Walman, F. Nelson, X. Yao (1999), Will Limits of the Earth's Resources Control Human Numbers?, in: B. Nath, L. Hens and D. Pimentel, eds., *Environment, Development and Sustainability*, pp. 19-39 (Springer Science+Business Media B.V.).

[19] Strulik, H. (2005), The Role of Human Capital and Population Growth in R&D-Based Models of Economic Growth, *Review of International Economics* 13, pp. 129-145.

[20] Young, A. (1998), Growth Without Scale Effects, *Journal of Political Economy* 106, pp. 41-63.